CONTINUITY

CLARK BAKER, ALEXA BAYANGOS, RA MUHAMMAD, AND GIGI VOSS

1. Introduction to Continuity

It is likely that you were first introduced to the concept of function's continuity in a standard beginners calculus course. While it is possible to understand visually and mathematically the basic meanings of continuity with this calculus knowledge, a deeper understanding emerges with the additional background in Real Analysis provided in this course.

Diving into concepts of continuity assumes a level of proficiency with the different concepts of limits discussed previously in this class. For the benefit of the reader, we will review the basics of limits here.

Definition 1.1. *Limits:* Let f(x) be a function defined on an interval containing the point a.

Then

$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

We say the *right-hand limit* is

$$\lim_{x \to a^+} f(x) = L$$

if for every $\epsilon > 0$ there is some $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $a < x < a + \delta$.

We say the *left-hand limit* is

$$\lim_{x \to a^-} f(x) = L$$

if for every $\epsilon > 0$ there is some $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $a - \delta < x < a$.

Using this background knowledge on limits, it is possible to begin defining continuity of functions at points.

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Definition 1.2. Let $A \subset \mathbb{R}$ and $a \in A$. Define a function, $f : A \to \mathbb{R}$. The function, f, is *continuous* at the point a if $\lim_{x\to a} f(x) = f(a)$.

This equation can be broken down into three conditions:

(1) The limit exists. $\lim_{x\to a} f(x) = L$ for some finite L.
(2) The function value $f(a)$ exists.
(3) The limit agrees with the function value. $L = f(a)$.

Example 1.1. Consider the function $f(x) = 2x + 5$.
We might ask, is this function continuous at $x = 4$? This can be assessed by considering the three conditions outlined above.

(1) The limit exists because $\lim_{x\to 4} f(x) = 13$.
(2) The function evaluated at $x = 4$ gives us $f(4) = 13$.
(3) The limit agrees with the function because $13 = 13$. Thus $f(x)$ is continuous at $x = 4$.

Definition 1.3. *Set Continuity:* Let $A \subset \mathbb{R}$. Define a function, $f : A \to \mathbb{R}$. This function f is continuous on A if it is continuous for every point $a \in A$.

Definition 1.4. Let $A \subset \mathbb{R}$ and $a \in A$. Define a function, $f : A \to \mathbb{R}$. The function, f, is *discontinuous* at the point a if it is not continuous, in other words, if at least one of the three continuity conditions does not hold.

From the concept of discontinuity come a number of different types of discontinuities including *jump discontinuities, infinite discontinuities,removable discontinuities*, and the concept of a function being *piecewise continuous* over an interval.

Definition 1.5. A function f on $[a, b]$ has a *jump discontinuity...*

(1) at x_0 if $f(x_0^+) \neq f(x_0^-)$.
(2) at a if $f(a^+) \neq f(a)$.
(3) at b if $f(b^-) \neq f(b)$.

Definition 1.6. A function f has *infinite discontinuity* at $x = a$ if

$$\lim_{x\to a} f(x) = \infty \text{ or } \lim_{x\to a} f(x) = -\infty.$$

This can be visualized as $f(x)$ having a vertical asymptote at $x = a$.

Definition 1.7. A function f has a *removable discontinuity* at $x = a$ if $\lim_{x\to a} f(x)$ exists but $f(a)$ does not exist or $\lim_{x\to a} f(x) \neq f(a)$. This can be visualized as $f(x)$ having a "hole" at $x = a$.

Definition 1.8. A function f is *piecewise continuous* if it can be broken up into a finite number of open subintervals on which the function is continuous.

Example 1.2. Consider the function, $f(x)$, defined on $[-2, 10]$ as such:

$$f(x) = \begin{cases} -x^2 + 5 & \text{if } -2 \leq x \leq 0 \\ \frac{1}{2}x + 7 & \text{if } 0 < x \leq 7 \\ -\frac{1}{2}x + 8 & \text{if } 7 < x \leq 10 \end{cases}$$

The graph of $f(x)$ can be seen in Figure 1. This function is piecewise continuous because for each of its three subintervals, all but the boundary points are continuous.

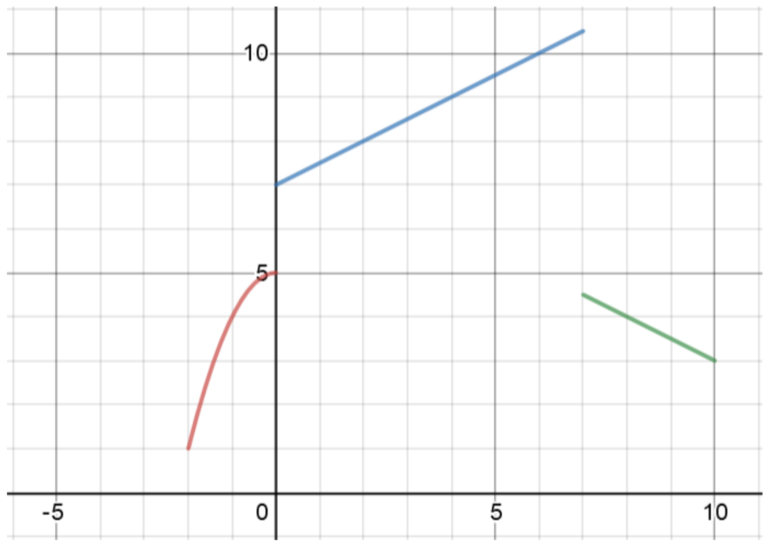

Figure 1. *The graph of f(x) for example 1.2.*

2. Continuity Using Epsilon Neighborhoods

While the calculus version of continuity may make perfect sense visually, the reasoning behind why certain definitions and theorems are true usually ranges from non-rigorous to completely unexplained. In order to understand and prove these ideas rigorously, an understanding of real analysis is necessary. To begin, it is simplest to consider how continuous functions in the reals behave according to principles of real analysis. To do so, we want to think about continuity in terms of *epsilon spaces*. So here we will redefine continuity.

Definition 2.1. Let $S \subset \mathbb{R}$, $x_0 \in S$, and let $f : S \to \mathbb{R}$ be a function. Then f is *continuous at* x_0 if for all $\epsilon > 0$ there exists a $\delta_{x_0} > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \text{ whenever } x \in S \text{ and } |x - x_0| < \delta_{x_0}.$$

Furthermore, f is a *continuous function* if it is continuous for all $x \in S$.

To understand this definition of continuity by epsilon neighborhoods, let us consider an example of the use of this definition as follows.

Example 2.1. Prove that the function $f(x) = \sin(x)$ is continuous.

For all $\epsilon > 0$, let $\delta = \epsilon$. Suppose $x \in S$ and $|x - x_0| < \delta = \epsilon$. Then

$$|f(x) - f(x_0)| = |\sin(x) - \sin(x_0)| = \left|2\sin\left(\frac{x - x_0}{2}\right)\cos\left(\frac{x + x_0}{2}\right)\right|$$

Knowing that $|\cos(y)| \leq 1$ and $|\sin(y)| \leq |y|$,

$$\left|2\sin\Big(\frac{x-x_0}{2}\Big)\cos\Big(\frac{x+x_0}{2}\Big)\right| \leq 2 \cdot |\frac{x-x_0}{2}| \cdot 1 = |x-x_0| < \delta = \epsilon.$$

Therefore, $|f(x) - f(x_0)| < \epsilon$, so $f(x)$ is continuous for any arbitrary x_0. □

In Calculus, one of the many things that are addressed when studying continuity are the extreme value theorem and intermediate value theorem. We can prove these common theorems by considering functions in the reals numbers and epsilon neighborhoods. In this chapter, we will work through the definitions and corollaries needed to prove the extreme value theorem. To begin, we must define the concept of boundedness.

Definition 2.2. Let $f : [a, b] \to \mathbb{R}$ be a function.
The function, f, is *bounded below* if there exists $m \in \mathbb{R}$ such that

$$f(x) \geq m \text{ for all } x \in [a, b].$$

The function, f, is *bounded above* if there exists $m \in \mathbb{R}$ such that

$$f(x) \leq m \text{ for all } x \in [a, b].$$

The function, f, is *bounded* on $[a, b]$ if f is bounded above and below on $[a, b]$.

This definition of boundedness is necessary to understand the following Theorems 2.1, 2.2, and 2.3. The Heine-Borel Theorem was proved during classtime and thus will be used here without a proof.

Theorem 2.1. *(Heine-Borel Theorem) A subset $S \subset \mathbb{R}$ is compact if and only if S is closed and bounded.*

Theorem 2.2. *(Bolzano-Weistrass Theorem) Suppose a sequence $\{x_n\}$ of real numbers is bounded. Then there exists a convergent subsequence $\{x_{ni}\}$.*

Using this background knowledge, we can prove Theorem 2.3.

Theorem 2.3. *If f is continuous on a finite closed interval $[a, b]$, then f is bounded on $[a, b]$.*

Proof. Since f is continuous at $x_0 \in [a, b]$, there exists an open interval O_{x_0} containing x_0 such that

$$|f(x) - f(x_0)| < \epsilon \text{ for } x \in O_{x_0} \cap [a, b].$$

Note that the interval $[a, b]$ is closed and bounded. Thus, by the Heine-Borel Theorem, $[a, b]$ is compact implying there are finitely many points $x_1, x_2, \ldots, x_n$ such that the open intervals $O_{x_1}, O_{x_2}, \ldots, O_{x_n}$ cover $[a, b]$.

Thus, for all x_i where $0 \leq i \leq n$,

$$|f(x) - f(x_i)| < \epsilon \text{ if } x \in O_{x_i} \cap [a, b].$$

Therefore,

$$|f(x)| = |f(x) - f(x_i) + f(x_i)| \leq |f(x) - f(x_i)| + |f(x_i)| \leq \epsilon + |f(x_i)|.$$

Let $M = \epsilon + \max_{1 \le i \le n} |f(x_i)|$.
Because $[a,b] \subset \bigcup_{i=1}^{n}(O_{x_i} \cap [a,b])$, we can conclude $|f(x)| \le M$ for all $x \in [a,b]$. Thus, f is bounded on $[a,b]$. [1][2] □

Now finally, this brings us to our original goal of proving the Extreme Value Theorem. Using the series of definitions and theorems that were just addressed, we can define and prove this theorem as follows.

Theorem 2.4. *(Extreme Value Theorem) Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then f achieves both an absolute minimum and maximum on $[a,b]$.*

Proof. Since f is continuous on a finite closed interval $[a,b]$, f is bounded on $[a,b]$ (Theorem 2.2).
Thus,

$$f([a,b]) = \{f(x) : x \in [a,b]\}$$

has a supremum and an infimum.
There must exist sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ such that $x_n, y_n \in [a,b]$ and

$$\lim_{n\to\infty} f(x_n) = \inf(f([a,b])) \text{ and } \lim_{n\to\infty} f(y_n) = \sup(f([a,b])).$$

Bolzano-Weistrass Theorem states that if a sequence of $\mathbb{R}$ is bounded, then there exists a convergent subsequence. Applying this theorem, there must exist convergence subsequences $\{x_{n_i}\}$ and $\{y_{m_i}\}$ such that

$$x := \lim_{i\to\infty} x_{n_i} \text{ and } y := \lim_{i\to\infty} y_{m_i}.$$

Thus, $a \le x \le b$ and $a \le y \le b$ implying $x, y \in [a,b]$.
Since the limit of a subsequence is equivalent to the limit of the sequence if the limit of the sequence exists,

$$\inf(f([a,b]) = \lim_{n\to\infty} f(x_n) = \lim_{i\to\infty} f(x_{n_i}) = f(\lim_{i\to\infty} x_{n_i}) = f(x)$$

and

$$\sup(f([a,b]) = \lim_{n\to\infty} f(y_n) = \lim_{i\to\infty} f(y_{m_i}) = f(\lim_{i\to\infty} y_{m_i}) = f(y).$$

Therefore, f achieves an absolute minimum at x and an absolute maximum at y. [1][3]

□

Thinking about continuity in terms of epsilon neighborhoods also allows for the expansion of the properties of continuous functions. In other words, we can assess which functions will be continuous in new ways.

Theorem 2.5. *Suppose $A \subset \mathbb{R}$ and $c \in \mathbb{R}$ and define two functions, $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$. If f and g are continuous at point $x_0 \in A$, then $f + g$, fg and cf are continuous at x_0, and if $g(x_0) \neq 0$, then f/g is continuous at x_0.*

Proof. Statement 1: If f and g are continuous at x_0 in A then $f+g$ is continuous at x_0 in A.

Since f and g are continuous, then there exists $\delta_f, \delta_g > 0$ such that for $x \in A$ and $\epsilon > 0$,

$$|f(x) - f(x_0)| < \frac{\epsilon}{2} \text{ whenever } |x - x_0| < \delta_f$$

and

$$|g(x) - g(x_0)| < \frac{\epsilon}{2} \text{ whenever } |x - x_0| < \delta_g.$$

Let $\delta = \min\{\delta_f, \delta_g\}$.
Then for $|x - x_0| < \delta$,

$$\begin{aligned}
|(f(x)+g(x)) - (f(x_0)+g(x_0))| &= |f(x)+g(x)-f(x_0)-g(x_0)| \\
&= |(f(x)-f(x_0)) + (g(x)-g(x_0))| \\
&\leq |f(x)-f(x_0)| + |g(x)-g(x_0)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

So, $|(f(x)+g(x)) - (f(x_0)+g(x_0))| < \epsilon$ whenever $|x - x_0| < \delta$. Thus, $f(x)+g(x)$ is continuous.□

Statement 2: If f and g are continuous at x_0 in A then $f \cdot g$ is continuous at x_0 in A.

Since f and g are continuous, there exists $\delta_f, \delta_g > 0$ such that for $x \in A$ and $\epsilon_f, \epsilon_g, \epsilon > 0$,

$$|f(x) - f(x_0)| < \epsilon_f = \frac{\epsilon}{2|g(x_0)|+1} \text{ whenever } |x - x_0| < \delta_f$$

and

$$|g(x) - g(x_0)| < \epsilon_g = \frac{\epsilon}{2(\epsilon + |f(x_0)|)} \text{ whenever } |x - x_0| < \delta_g.$$

Note, that we strategically bound $|f(x) - f(x_0)|$ and $|g(x) - g(x_0)|$ with these epsilons so that some algebraic manipulation yields our desired inequality $|f(x)g(x) - f(x_0)g(x_0)| < \epsilon$.

Later on in the proof, it will also be important to establish that

$$\epsilon_f|g(x_0)| = \frac{\epsilon|g(x_0)|}{2|g(x_0)|+1} < \frac{\epsilon}{2}$$

and

$$\epsilon_g|f(x)| < \frac{\epsilon}{2(\epsilon + |f(x)|)}(\epsilon + |f(x)|) = \frac{\epsilon}{2}.$$

Now, we can consider the case when $f(x)$ is multiplied by $g(x)$.
Let $\delta = \min\{\delta_f, \delta_g\}$. Thus, for $|x - x_0| < \delta$,

$$\begin{aligned}
|f(x)g(x) - f(x_0)g(x_0)| &= |f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)| \\
&= |f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0))| \\
&\leq |f(x)||g(x) - g(x_0)| + |g(x_0)||f(x) - f(x_0)| \\
&< |f(x)|\epsilon_g + |g(x_0)|\epsilon_f \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Therefore, $|f(x)g(x) - f(x_0)g(x_0)| < \epsilon$, whenever $|x - x_0| < \delta$. So, $f(x)g(x)$ is continuous. $\square$

Statement 3: If f is continuous at x_0 in A then cf is continuous at x_0 where $c \in \mathbb{R}$.

Suppose $c \neq 0$. Since f is continuous, then there exists δ such that for $x \in A$ and $\epsilon > 0$,

$$|f(x) - f(x_0)| < \frac{\epsilon}{|c|} \text{ whenever } |x - x_0| < \delta.$$

Note that we have to take the absolute value of c to ensure that we are bounding $|f(x) - f(x_0)|$ with a positive epsilon.

We can now consider the case when $f(x)$ is multiplied by some c.

$$|f(x) - f(x_0)| < |c||f(x) - f(x_0)| < |c|\frac{\epsilon}{|c|} = \epsilon$$

So, when $c \neq 0$, $|cf(x) - cf(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

We must address the case where $c = 0$. Since f is continuous, then there exists δ_0 such that for $x \in A$ and $\epsilon_0 > 0$,

$$|f(x) - f(x_0)| < \epsilon_0 \text{ whenever } |x - x_0| < \delta_0.$$

Now, we can consider the case when $f(x)$ is multiplied by $c = 0$.

$$|cf(x) - cf(x_0)| = 0 \leq |f(x) - f(x_0)| < \epsilon_0$$

So, $|cf(x) - cf(x_0)| < \epsilon_0$ whenever $|x - x_0| < \delta_0$.

By showing all cases for $c \in \mathbb{R}$, we can conclude that $cf(x)$ is continuous at x_0 whenever f is continuous at x_0. $\square$

Statement 4: If f and g are continuous at x_0 in A then f/g is continuous at x_0 in A if $g(x_0) \neq 0$.

We will begin by first showing that if $g(x)$ is continuous at x_0, then its reciprocal $1/g(x)$ is also continuous at x_0. We will then apply Statement 2 to conclude that if f and $1/g$ are continuous at x_0, then f/g must be continuous at x_0.

Let $h(x) = 1/g(x)$ and $B = \{x \in A | g(x) \neq 0\}$. For $x_0 \in B$, let $g(x_0) = r$ for some $r \in \mathbb{R}$. By the definition of continuity, there exists $\delta_1 > 0$ such that

$$|g(x) - g(x_0)| < \frac{|r|}{2} \text{ whenever } |x - x_0| < \delta_1$$

Thus,

$$0 < \frac{|r|}{2} = g(x) - \frac{|r|}{2} < g(x_0) < g(x) + \frac{|r|}{2}$$

From this, we see $\frac{|r|}{2} < g(x_0)$ and thus, $\frac{1}{g(x_0)} < \frac{2}{|r|}$.
By the definition of continuity, there also exists $\delta_2 > 0$ and $\epsilon > 0$ such that

$$|g(x) - g(x_0)| < \frac{\epsilon r^2}{2} \text{ whenever } |x - x_0| < \delta_2$$

Now, we can let $\delta = \min\{\delta_1, \delta_2\}$.
Thus, for $|x - x_0| < \delta$ where $x \in A$,

$$|h(x_0) - h(x)| = \left| \frac{1}{g(x_0)} - \frac{1}{g(x)} \right| = \left| \frac{g(x) - g(x_0)}{g(x)g(x_0)} \right| =< \left(\frac{1}{r}\right)\left(\frac{2}{r}\right)\left(\frac{\epsilon r^2}{2}\right) = \epsilon$$

So, $|h(x_0) - h(x)| < \epsilon$ whenever $|x - x_0| < \delta$ meaning $h(x) = \frac{1}{g(x)}$ is continuous at x_0 in A.

We now know $f(x), g(x)$ and $\frac{1}{g(x)}$ are continuous at x_0 in A. From Statement 2, the product of two functions continuous at x_0, is also continuous at x_0. Thus, we can conclude that if f and $1/g$ are continuous at x_0, then f/g is continuous at x_0. □ □

Building upon Theorem 2.5, we can begin to consider specific types of functions such as polynomials, rational functions, and composite functions as is done here.

Theorem 2.6. *If $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ where f is a polynomial of the form $c_n x^n + c_{n-1} x^{n-1} + ... + c_0$, with c_i being real numbers, then f is continuous at all points $a \in A$.*

Proof. Consider first the arbitrary constant function $f(x) = c_0$.
Let $\delta = 1$.
Then for all $\epsilon > 0$ suppose that $x \in A$ and $|x - x_0| < 1$.
Then,

$$|f(x) - f(x_0)| = |c_0 - c_0| = 0 < \epsilon.$$

Since $|f(x) - f(x_0)| < \epsilon$, $f(x) = c_0$ is continuous.

Now consider the arbitrary function $f(x) = x$.
Let $\delta = \epsilon$.
Then, for all $\epsilon > 0$ suppose that $x \in A$ and $|x - x_0| < \delta = \epsilon$.
Then,

$$|f(x) - f(x_0)| = |x - x_0| < \delta = \epsilon.$$

Thus, $f(x) = x$ is continuous.

From Theorem 2.4 we know that the product of continuous functions is also continuous. Since any x^n can be written as the product of n continuous x's, $f(x) = x^n$ must also be continuous.

From Theorem 2.4 we also know that the addition of continuous functions is continuous and that continuous functions multiplied by constants are continuous. Therefore, all potential components of polynomials of this form and their combinations must be continuous. Thus all polynomials are continuous. □

Corollary 2.7. *If $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ is a rational function, then f is continuous on A.*

Proof. A rational function is any function that can be written as a fraction where both the numerator and denominator are polynomials. Now, it was proved in Theorem 2.5 that polynomial functions are continuous. Additionally, it was proved in Theorem 2.4 that a continuous function divided by a continuous function is also continuous. Thus, rational functions are continuous. □

Theorem 2.8. *Define two functions, f and g which have domains D_f and D_g respectively. If g is continuous at some x_0, $g(x_0)$ is an interior point of D_f, and f is continuous at $g(x_0)$, then $f(g(x)) = (f \circ g)x$ is continuous at x_0.*

Proof. Since f is continuous at $g(x_0)$, for all $\epsilon > 0$ there exist some δ_1 such that

$$|f(g(x)) - f(g(x_0))| < \epsilon \text{ whenever } |g(x) - g(x_0)| < \delta_1.$$

Since g is continuous at x_0, there exists some δ_2 such that

$$|g(x) - g(x_0)| < \delta_1 \text{ whenever } |x - x_0| < \delta_2$$

Thus, for all $\epsilon > 0$, there exists some $\delta_1, \delta_2 > 0$ such that if $|x - x_0| < \delta_2$, then $|g(x) - g(x_0)| < \delta_1$, which implies $|f(g(x)) - f(g(x_0))| < \epsilon$.
So, $f(g(x)) = (f \circ g)x$ is continuous at x_0. [1][2][3] □

3. Uniform Continuity

Applications of analysis often involve approximations and error estimates. If f is continuous, we can use a value of f to estimate nearby values. However, the error may vary greatly for different values of x. However, if f is uniformly continuous, then we can generate an error that is guaranteed to be less than a value ϵ provided no matter which x_0 we pick, provided that the distance between x and x_0 is less than a value, δ. In this section, we will explore the behavior of uniformly continuous functions.

Definition 3.1. A function f is *uniformly continuous* on a subset S of its domain if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \text{ whenever } x, x_0 \in S \text{ and } |x - x_0| < \delta.$$

This is different than our previous definition of epsilon continuity (Definition 2.1) because here δ is no longer dependent on one specific point but is independent.

Theorem 3.1. *Suppose $S \subset \mathbb{R}$ is compact and $f : S \to \mathbb{R}$ is continuous. Then f is uniformly continuous on S.*

Proof. Let $S \subset \mathbb{R}$, where $f : S \to \mathbb{R}$ is a continuous function and $x, x_0 \in S$. By definition, f is continuous at x_0 if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \text{ whenever } x \in S \text{ and } |x - x_0| < \delta_x.$$

So, for every $x, x_0 \in S$, choose δ_{x_0} such that

$$|f(x) - f(x_0)| < \frac{\epsilon}{2} \text{ whenever } |x - x_0| < \delta_{x_0}.$$

Now, let $I_{x_0} = (x_0 - \frac{\delta_{x_0}}{2}, x_0 + \frac{\delta_{x_0}}{2})$ for each $x_0 \in S$.
It then follows that $\{I_{x_0} | x_0 \in S\}$ is an open cover for S.
This implies that there there must exist finitely many points $x_1, x_2, x_3, ...x_n \in S$ such that $I_{x_1} \cup I_{x_2} \cup I_{x_3} ... \cup I_{x_n}$ is an open cover of S.

Let us define $\delta = \min\{\frac{\delta_1}{2}, \frac{\delta_2}{2}, \frac{\delta_3}{2}, ..., \frac{\delta_n}{2}\}$.

Consider two preimages $x, x' \in S$ such that $|x - x'| < \delta$. We know that $x \in I_{x_k}$ for one of the $x_1, x_2, ...x_n$. So, consider an arbitrary element, $x_k \in S$, where $|x - x_k| < \frac{\delta_{x_k}}{2}$ by definition of continuity.
We want to show that the associated images of x and x' will be separated by a distance less than ϵ.
By the triangle inequality,

$$|x' - x_k| \leq |x' - x| + |x - x_k| < \delta + \frac{\delta_{x_k}}{2} \leq \delta_{x_k}$$

because $\delta \leq \frac{\delta_{x_k}}{2}$ by definition.
Because $|x' - x_k| < \delta_{x_k}$, it follows that $|f(x') - f(x_k)| < \frac{\epsilon}{2}$ from the fact that f is continuous.
This means that,

$$|f(x) - f(x')| \leq |f(x) - f(x_a)| + |f(x_a) - f(x')| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, if the distance between any two preimages is less than δ, then the distance between their images will be less than ϵ. [3] □

Corollary 3.2. *If f is continuous on a closed and bounded interval $[a, b]$ then f is uniformly continuous on $[a, b]$.*

Proof. By the Heine-Borel Theorem, because $[a, b]$ is closed and bounded, it must be compact.
Thus, f is uniformly continuous because it is a continuous function on a compact subset, $S = [a, b]$. [2] □

Corollary 3.3. *If f is continuous on a set, S, then f is uniformly continuous on any finite closed interval contained in S.*

Example 3.1. The function $f(x) = 2x$ is uniformly continuous where the domain of f is $\mathbb{R}$.

This is true because

$$|f(x) - f(x_0)| = 2|x - x_0| < \epsilon \text{ whenever } |x - x_0| < \delta = \frac{\epsilon}{2}.$$

Counter Example 3.1. The function $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous on $(0, \infty)$.

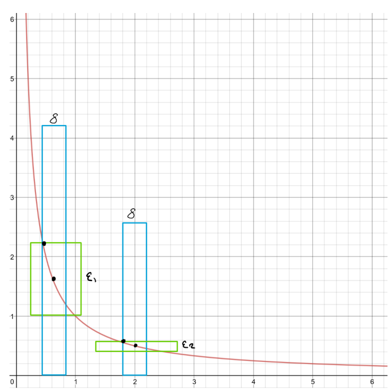

Figure 2. *For two different sets of points whose preimages are separated by the same δ, the images are separated by different ϵ's. Note $\epsilon_1 > \epsilon_2$. So, although the function is continuous, it is not uniformly continuous.*

Consider x as it approaches 0 from the right. No value of δ is small enough so that the images of two points are covered by the same value of ϵ.

Note that although f is not uniformly continuous on $(0, \infty)$, the function is uniformly continuous on $[1, \infty)$. This is because when elements of the domain do not approach a vertical asymptote, you can define a δ so that any preimages within δ of each other will output images within ϵ of each other.

Example 3.2. The function $f(x) = \sqrt{x}$ is uniformly continuous on the set $S = \{x | 0 \leq x \leq \infty\}$

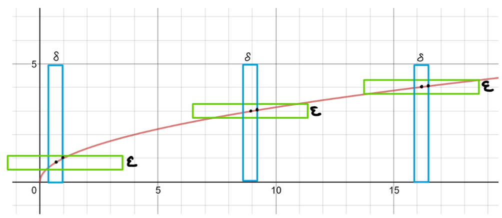

Figure 3. *Preimages separated by a value δ produce an output separated by a value ϵ. the same value of δ separating two preimages results in the images being separated by less than ϵ for all points in the function.*

Counter Example 3.2. The function $f(x) = \cos(\frac{1}{x})$ is continuous on the interval $(0, 1]$, but not uniformly continuous on $(0, 1]$.

Due to the rapid oscillation of the function, we can find preimages x, x_0 very close to 0 such that $|f(x) - f(x_0)| = 2$.

Thus, f cannot be uniformly continuous.

Definition 3.2. A sequence $\{x_n\}$ is a *Cauchy sequence* if for every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $a, b \geq N$, $|x_a - x_b| < \epsilon$.

Theorem 3.4. *Let $f : S \to \mathbb{R}$ be a uniformly continuous function. If $\{x_n\}$ is a Cauchy Sequence in S, then $f(x_n)$ is Cauchy.*

Proof. Let $f : S \to \mathbb{R}$ be a uniformly continuous function, where $x, x_0 \in S$. By the definition of uniform continuity, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \text{ whenever } |x - x_0| < \delta.$$

Because $\{x_n\}$ is Cauchy, for every $\delta > 0$ there exists a $N \in \mathbb{N}$ such that for all $a, b \geq N$, $|x_a - x_b| < \delta$.
Because f is uniformly continuous, it follows that for some $N \in \mathbb{N}$ and $a, b \geq N$

$$|f(x_a) - f(x_b)| < \epsilon \text{ whenever } |x_a - x_b| < \delta.$$

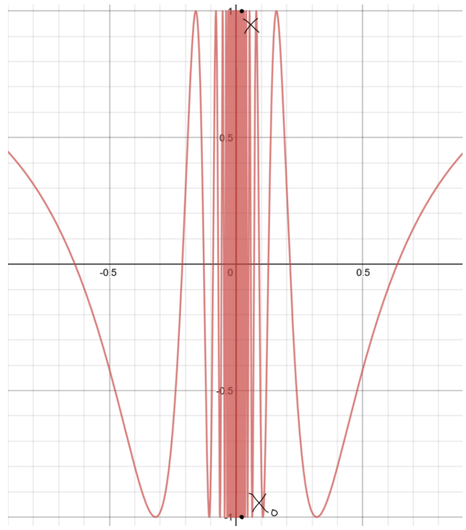

Figure 4. *Visual representation of how* $|f(x) - f(x_0)| = 2$ *while* $|x - x_0| < \delta$.

Thus, for some Cauchy sequence $\{x_n\}$, $\{f(x_m)\}$ is Cauchy for a uniformly continuous function. [1] □

Proposition 3.5. *A function* $f : (a, b) \to \mathbb{R}$ *is uniformly continuous if and only if the limits* $L_a = \lim_{x\to a} f(x)$ *and* $L_b = \lim_{x\to b} f(x)$ *exist and the function defined:*

$$f^* = \begin{cases} f(x) & x \in (a,b) \\ L_a & x = a \\ L_b & x = b \end{cases} \quad \textit{is continuous.}$$

Proof. ($\Longleftarrow$)
Suppose the limits $L_a = \lim_{x\to a} f(x)$ and $L_b = \lim_{x\to b} f(x)$ exist and the function defined:

$$f^* = \begin{cases} f(x) & x \in (a,b) \\ L_a & x = a \\ L_b & x = b \end{cases} \quad \text{is continuous.}$$

By Corollary 3.2, f^* is uniformly continuous because it is continuous on a closed and bounded.
This implies that f is uniformly continuous on (a, b) by the definition of f^*.

($\Longrightarrow$)
We will not prove the forward direction of this proposition in this book, rather leave it as an exercise for the reader. We will provide guidelines as to how to perform this proof, though.

Suppose that f is uniformly continuous to show the limits exist.

Show that for a Cauchy sequence, $\{x_n\}$ that converges to a, $f(\{x_n\})$ must be Cauchy and converge to some value L_1 by Theorem 3.4.Show a similar process for another sequence $\{y_n\}$ that also converges to a, meaning $f(\{y_n\})$ converges to a number L_2.
If you show that $L_1 = L_2$, then L_a exists.
A similar process can prove that L_b exists.
If $L_a = \lim_{x \to x} f(x)$ exists, then $L_a = \lim_{x \to x} f^*(x)$ exists.
This would imply that f^* is continuous at a and b. [1] $\square$

4. Continuity in Topological Spaces

Definition 4.1. Let X be a set. We say that T is a *topology* on X if it is a family of open subsets of X. A topology on X will satisfy the following 3 axioms:

(1) The empty set and X itself are elements of T
(2) Any union of elements of T are themselves elements of T
(3) Any intersection of finitely many elements of T are themselves elements of T

We then define the ordered pair (X, T) as a *topological space* which is associated with the elements of set X and the open sets that the topology T defines.

Definition 4.2. A *function* between 2 topological spaces (X, T_1) to (Y, T_2) denoted $f : X \to Y$ is simply a function that maps the elements of set X to elements of set Y.

Example 4.1. Let $W = \{1, 2, 3, 4, 5\}$ and $V = \{6, 7, 8\}$. Let T_1 be the trivial topology which says that only the whole set and the empty set are open sets, and let T_2 be the discrete topology which defines all elements of a power set as open.

$(W, T_1) = (W, \{\emptyset, \{1, 2, 3, 4, 5\}\})$
$(V, T_2) = (V, \{\emptyset, \{6, 7, 8\}, \{6\}, \{7\}, \{8\}, \{6, 7\}, \{6, 8\}, \{7, 8\}\})$
Define a function $f : V \to W$ as the following:
$f(6) = 1$, $f(7) = 5$ and $f(8) = 3$.

Firstly, a function between topological spaces can be considered continuous solely at a single point in the domain.

Definition 4.3. Let X and Y be sets, and let T_1 and T_2 be topologies on the two sets respectively. We have a function $f : X$. Let $x \in X$ and $f(x) \in Y$, we say that the function f is said to be continuous at this specific point x if, for every open neighborhood $N \subseteq Y$ containing $f(x)$, there exists an open neighborhood $M \subseteq X$ where $x \in M$ such that $f(M) \subseteq N$

Complete continuity for a function between topological spaces (sets with topologies) is defined using the topology itself.

Definition 4.4. Let X and Y be sets, and let T_1 and T_2 be topologies on the two sets respectively. We have a function $f : X \to Y$ which maps elements of X to Y. We say f is continuous provided that for any open neighborhood O of Y as defined by T_2, the set of all elements X mapping into O (denoted $f^{-1}(O)$) form an open subset of X as defined by T_1

Example 4.2. Recall a function $f : V \to W$ as defined in (example 4.1). $V = \{6, 7, 8\}$ and uses the discrete topology. $f(6) = 1, f(7) = 5, f(8) = 3$.

We will show that $f : V \to W$ is continuous at 6.

The only open neighborhood in W containing $f(6) = 1$ is $\{1, 2, 3, 4, 5\}$. 6 itself is contained in the open neighborhoods $\{6\}, \{6, 7\}$, and $\{6, 7, 8\}$ in (V,discrete). We want to show that at least one of these neighborhoods has an image over f completely contained in $\{1, 2, 3, 4, 5\}$.

The following images of open neighborhoods of 6 are contained in $\{1, 2, 3, 4, 5\}$:
$f\{6\} \subseteq \{1, 2, 3, 4, 5\}$,
$f\{6, 7\} \subseteq \{1, 2, 3, 4, 5\}$ because $f(7) = 5 \in \{1, 2, 3, 4, 5\}$
$f\{6, 7, 8\} \subseteq \{1, 2, 3, 4, 5\}$ because f(8) $= 3 \in \{1, 2, 3, 4, 5\}$

There is definitely at least one open neighborhood in V that contains 6 and has an image completely contained in $\{1, 2, 3, 4, 5\}f$ is continuous at 6.

Observe that in (example 4.2) the domain uses the discrete topology. If we are to argue that the function f is a continuous function, we would have to show that for for both $\{1, 2, 3, 4, 5\}$ and $\{\emptyset\}$ their pre-image is also open. No elements in the domain map to the empty set so $f^{-1}(\emptyset) = \emptyset$. The empty set is open. For $\{1, 2, 3, 4, 5\}$, the elements of V: 6 ,7, and 8 all map into the it, so $\{6, 7, 8\}$ is the pre-image of $f^{-1}(\{1, 2, 3, 4, 5\})$, and by the topology on our domain, this pre-image is also open. Hence, f must be continuous.

We can observe that when the discrete topology is on a set and there is a function from that topological space to another, the function is continuous on all elements of the domain set because for each element in the domain, an open singleton neighborhood exists in the discrete topology whose image will always be a subset of any open neighborhood in the co-domain that contains the image of that specific element.

Lemma 4.1. *Let A be a subset of some domain of a function f. $A \subseteq f^{-1}(f(A))$*[4]

Proof. Let $x \in A$ then $f(x) \in f(A)$. Then, by definition of pre-image $x \in f^{-1}(f(A))$ so $A \subseteq f^{-1}(f(A))$ [5] □

Lemma 4.2. *Let A be a subset of a functions f's range, then $f(f^{-1}(A)) \subseteq A$.* [4]

Proof. Let $x \in f(f^{-1}(A))$, then there exists some b $\in f^{-1}(A)$ so that $f(b) = x$. Additionally, $f(b) \in A$ by definition of pre-image, and therefore, x $\in A$, Thus $f(f^{-1}(A)) \subseteq A$. [5] □

Lemma 4.3. *Let $f : X \to Y$ be a function between topological spaces and $B \subseteq C \subseteq Y$. We say $f^{-1}(C \setminus B) = f^{-1}(C) \setminus f^{-1}(B)$*

Proof. let $x \in f^{-1}(C \setminus B) \implies f(x) \in (C \setminus B)$. This means $f(x) \in C$ and $f(x) \notin B$ this further means that $x \in f^{-1}(C)$ and $x \notin f^{-1}(B)$ so $x \in f^{-1}(C) \setminus f^{-1}(B)$. Hence $f^{-1}(C \setminus B) = f^{-1}(C) \setminus f^{-1}(B)$. [5] □

Theorem 4.4. *Let X and Y be two topological spaces. Let $f\colon X \to Y$ be a function between the two spaces. We say f is a continuous function if and only if f is continuous at all points $x \in X$*

Proof.
$(\implies)$
Suppose that f is a continuous function from the topological space X to the topological space Y, and we have any $x \in X$. $f(x) \in Y$, and $O \subseteq Y$ is any open neighborhood containing $f(x)$. By the definition of continuity, the pre-image of O denoted $f^{-1}(O)$ is an open neighborhood of X as defined by its topology. Hence there exists an open neighborhood in X that contains x, $f^{-1}(O)$, and we know that by lemma 4.2 $f(f^{-1}(O)) \subseteq O$. Then If f is continuous then f is continuous on all points in its domain.

$(\impliedby)$
Assume that f is continuous at all points x in its domain X. Let O be any open set in the co-domain Y. Let $x \in f^{-1}(O) \implies f(x) \in O$. Given f is continuous at $x \in X$, there exists at least 1 open neighborhood, denoted $N \subseteq X$, where $x \in N$ and $f(N) \subseteq O$. There could be multiple neighborhoods N $\subseteq X$ that contain x and have images contained in O, so let the union of all N for that x be denoted as N_x. $N_x \subseteq X$ is open and $x \in N_x$. Because for each N, $f(N) \subseteq O$, all their elements collectively map into O, so $f(N_x) \subseteq O \implies N_x \subseteq f^{-1}(O)$ for all $x \in f^{-1}(O)$ given continuity at any point in the domain. So $\bigcup\limits_{x\in f^{-1}(O)} N_x \subseteq f^{-1}(O)$ and each N_x contains its respective x for all x in $f^{-1}(O)$ by definition of neighborhood meaning $f^{-1}(O) \subseteq \bigcup\limits_{x\in f^{-1}(O)} N_x$. So we have that $f^{-1}(O) = \bigcup\limits_{x\in f^{-1}(O)} N_x$ which is a union of open neighborhoods making $f^{-1}(O)$ open hence showing continuity. Hence provided that f is continuous at each point in its domain the function itself must be continuous. [5] [6] □

Example 4.3. Recall (example 4.2) which used the discrete topology on the domain, and any topology on the co-domain. We observed that provided a function between topological spaces maps elements from the discrete topological space to another, f is continuous at all points in the domain as the singleton sets will always be considered open neighborhoods. And this means that for any function between topological spaces where the domain has the discrete topology, f itself is considered continuous function. Hence $f : V \to W$ from example 4.2 is a continuous function.

Now we will consider continuity of functions between topological spaces from the perspective of closed sets.

Theorem 4.5. *Let $f : X \to Y$ be a function between topological spaces. f is continuous if and only if for all closed set $C \subseteq Y$, $f^{-1}(C) \subseteq X$ must also be closed*

Proof.
$(\Longrightarrow)$
f is continuous $\Longrightarrow$ for any open subset of Y denoted O, $f^{-1}(O)$ is open as defined by the topology on X. Because a topology is a family of open subsets we define a closed subset as the compliment of some open subset as defined by the topology. So give an open set O, $Y \setminus O$ is a closed set, and all closed subsets of Y can be defined this way. The same is the case for X as a topological space. So we let $C = Y \setminus O$, and we say that $f^{-1}(C) = f^{-1}(Y \setminus O)$ We want to show this pre-image is closed. By lemma 4.3 $f^{-1}(Y \setminus O) = f^{-1}(Y) \setminus f^{-1}(O)$. Assuming f is continuous, $f^{-1}(O)$ is an open set in X, and $f^{-1}(Y) = X$ by definition of pre-image, so $f^{-1}(Y) \setminus f^{-1}(O) = X \setminus f^{-1}(O)$. This set is a closed set, meaning provided f is continuous, any closed set $C \subseteq$ Y has a pre-image that is a closed subset in X.

$(\Longleftarrow)$
Assume that for all closed set C $\subseteq Y$, $f^{-1}(C) \subseteq X$ must also be closed. We want to show that provided an open neighborhood $O \subseteq Y$, its pre-image $f^{-1}(O)$ is also open in terms of the domains topology. Let C be any closed set in the co-domain and Let $O = Y \setminus C$ an open set that is the compliment of C. We denote the pre-image of this set as $f^{-1}(Y \setminus C) = f^{-1}(Y) \setminus f^{-1}(C) = X \setminus f^{-1}(C)$. We assumed that $f^{-1}(C)$ was also closed and the complement of a closed set in X is open. So we have shown that $f^{-1}(O)$ is an open set. Meaning that f is continuous. [7] $\square$

The following 3 lemmas provide results for functions and containment that will be used to prove the final bi-conditional statement about continuous functions between topological spaces.

Lemma 4.6. *Let $A \subseteq B$ which is a subset of some function f's range, Then $f^{-1}(A) \subseteq f^{-1}(B)$.*

Proof. Assume $A \subseteq B \subseteq Y$ where Y is the co-domain of some function f. Let x $\in f^{-1}(A)$ then $f(x) \in A \subseteq B$ therefor, $f(x) \in B$ thus $x \in f^{-1}(B)$ by definition. $\square$

Lemma 4.7. *Let A be a subset of B which is a subset of some function f's domain, Then $f(A) \subseteq f(B)$.*

Proof. Assume A $\subseteq$ B, let y $\in$ f(A) then there exists some x $\in A \subseteq B$ that maps to y. x$\in B$ so $f(x) \in f(B)$. So $f(A) \subseteq f(B)$. $\square$

Lemma 4.8. *Let A be a subset of B which is a subset of some topological space, then $\overline{A} \subseteq \overline{B}$.*

Proof. Assume $A \subseteq B$. We know that $B \subseteq \overline{B}$, and hence $A \subseteq \overline{B}$. The closure of B is closed and the closure of A is the smallest closed set containing all of A. Therefore, because $\overline{B}$ is closed and contains all of A it is either the closure of A or a set containing the closure of A, in both cases, $\overline{A} \subseteq \overline{B}$. $\square$

Theorem 4.9. *Let f:X $\to$ Y be a function between topological spaces. f is continuous if and only if for every subset S of the domain f($\bar{S}$) $\subseteq \overline{f(S)}$*

Proof.
($\Longrightarrow$)
Suppose f is a continuous function. Let S $\subseteq$ X and f(S) is the image of S in Y. $\overline{f(S)}$ is a closed subset of Y so its pre-image $f^{-1}(\overline{f(S)})$ must be closed by Theorem 4.5. We know $f(S) \subseteq \overline{f(S)}$ and by lemmas 4.1 and 4.7 $S \subseteq f^{-1}(f(S)) \subseteq f^{-1}(\overline{f(S)})$. Because $f^{-1}(\overline{f(S)})$ is closed we can say that $\overline{S} \subseteq f^{-1}(\overline{f(S)})$. This is by the definition of closure which says the closure of S is the smallest closed set containing all of S. Lastly if we take the image of both sides, by lemma 4.7 and 4.2 we have that $f(\overline{S}) \subseteq f(f^{-1}(\overline{f(S)}) \subseteq \overline{f(S)}$ hence if f is continuous, $f(\overline{S}) \subseteq \overline{f(S)}$.

($\Longleftarrow$)
Now suppose that for all subsets $T \subseteq X$, $f(\overline{T}) \subseteq \overline{f(T)}$. Let C be a closed subset of Y. We say that $f^{-1}(C) \subseteq X$, so we may apply our assumption, $f(\overline{f^{-1}(C)}) \subseteq \overline{f(f^{-1}(C))}$ By lemmas 4.2 and 4.8, $\overline{f(f^{-1}(C))} \subseteq \overline{C} = C$. Hence $f(\overline{f^{-1}(C)}) \subseteq C$; Now by lemma 4.6 and 4.1 $\overline{f^{-1}(C)} \subseteq f^{-1}(C)$ and a set is always contained in its closure so $\overline{f^{-1}(C)} = f^{-1}(C)$. Hence the pre-image of C is closed. This is the case for all closed sets C in Y hence f is continuous by theorem 4.5. [8] □

Because theorems 4.4, 4.5, and 4.9 are all bi-conditional with respect to a function being continuous, they are all equal definitions for a continuous function between topological spaces. Lastly for the basic properties of continuous functions between topological spaces, we will examine continuity of composition of continuous functions.

Theorem 4.10. *Let X, Y, Z be topological spaces. We have 2 continuous functions $f_1 : X \to Y$ and $f_2 : Y \to Z$; their composition $f_2 \circ f_1 : X \to Z$ is also continuous.*

Proof. Assume the f_1 and f_2 are both continuous in which case by the definition of continuity for functions between topological spaces, provided an open set T $\subseteq$ Y, $f_1^{-1}(T)$ is also open, and provided an open set S $\subseteq$ Z, $f_2^{-1}(S)$ is also open. In order for us to show that the composition $f_2 \circ f_1$ is continuous, we must show that for any open set $S \subseteq Z$, $U = [f_2 \circ f_1]^{-1}(S)$ is open in X. Let us have and open set $S \subseteq Z$ We may expand its pre-image under the composition as the following, $[f_2 \circ f_1]^{-1}(S) = f_1^{-1}(f_2^{-1}(S))$. We said $T = f_2^{-1}(S) \subseteq Y$ for any open set $S \subseteq Z$ will be open and we also said $U = f_1^{-1}(T) \subseteq X$ would be open for all open sets $T \subseteq Y$. Hence, $U = [f_2 \circ f_1]^{-1}(S)$ is open for any open set $S \subseteq Z$ and thus a composition $f_2 \circ f_1$ is continuous by definition of continuity. [7] [9] □

Lastly, we will examine a special kind of continuous function between topological spaces. While functions between non-topological spaces can be considered isomorphic, the topological equivalent of an isomorphism is a homeomorphism. The following is the definition.

Definition 4.5. Let X and Y be topological spaces, and f a continuous function from X to Y. $f : X \to Y$ is a Homeomorphism if $f : X \to Y$ is a continuous function between topological spaces that has a continuous inverse. If there is a homeomorphism between two topological spaces, those spaces are considered homeomorphic to one another. Two spaces that are homeomorphic share the same topological properties. [10]

Two types of properties that are preserved between topological spaces under a homeomorphism are connectedness and compactness. Now the following is a visual example of 2 surfaces that are homeomorphic to one another.

Example 4.4.

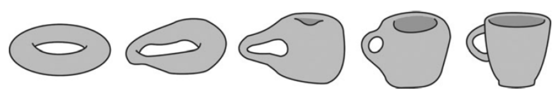

Figure 5. *The topological surfaces of a doughnut and mug are homeomorphic to one another as they both are forms of a torus in* $\mathbb{R}^3$*, and thus they are topologically equivalent.*[10]

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