Вычислительные схемы для Higher Order модели

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Дискретная сетка

В пространстве $\{x, y, z\} \in \mathbb{R}^3$ строится сетка

Перед введением схемы по z производится координатное отображение.

$$\xi|_{x,y} = \frac{z|_{x,y}}{s(x,y) - b(x,y)}$$

- равномерная сетка по x: $\{x_i\}: \quad x_i = i \cdot \Delta x$
- равномерная сетка по y: $\{y_j\}: \quad y_j = j \cdot \Delta y$
- неравномерная сетка по ξ : $\{\xi_k\}$
- некоторая сетка по времени t: $\{t_l\}$

Указанные индексные обозначения закрепим за координатами в пространстве и времени, т.е. i будет нумеровать узлы по координате x, j – по координате y, k – по координате z и l – по времени t.

Также для получения схем, включающих z нужны следующие соотношения.

$$\Delta\xi_{k-\frac{1}{2}} = \xi_k - \xi_{k-1}$$
$$\Delta\xi_{k+\frac{1}{2}} = \xi_{k+1} - \xi_k$$
$$\Delta\xi_k = \frac{\Delta\xi_{k+\frac{1}{2}} + \Delta\xi_{k-\frac{1}{2}}}{2} = \frac{\xi_{k+1} - \xi_{k-1}}{2}$$
$$f_{k-\frac{1}{2}} = \frac{f_k + f_{k-1}}{2}$$
$$f_{k+\frac{1}{2}} = \frac{f_{k+1} + f_k}{2}$$
$$f_k = \frac{\Delta\xi_{k-\frac{1}{2}}}{2\Delta\xi_k} f_{k+\frac{1}{2}} + \frac{\Delta\xi_{k+\frac{1}{2}}}{2\Delta\xi_k} f_{k-\frac{1}{2}}$$
$$\left(\frac{\partial f}{\partial \xi}\right)_k = \frac{\Delta\xi_{k-\frac{1}{2}}}{\Delta\xi_k \Delta\xi_{k+\frac{1}{2}}} f_{k+\frac{1}{2}} - \frac{\Delta\xi_{k+\frac{1}{2}}}{\Delta\xi_k \Delta\xi_{k-\frac{1}{2}}} f_{k-\frac{1}{2}} + 2\frac{\Delta\xi_{k+\frac{1}{2}} - \Delta\xi_{k-\frac{1}{2}}}{\Delta\xi_{k+\frac{1}{2}} \Delta\xi_{k-\frac{1}{2}}} f_k$$

Преобразование координат

Для моделирования ледников обычно производят преобразование по вертикальной координате, скалируя область ледника на единичную высоту. Для этого используется координатное отображение

$$(x, y, z, t) \mapsto (x', y', \xi, t)$$

Примером такого отображения может быть

$$\xi = \frac{z-b}{s-b} = \frac{z-b}{H}$$

Тогда нижняя граница ледника будет всегда находится на плоскости $\xi = 0$, а верняя граница на плоскости $\xi = 1$.

Рассмотрим, как происходит преобразование некоторой достаточно гладкой функции $f = f(x, y, z, t)$ при таком координатном отображении.

С небольшой погрешностью можно утверждать, что

$$\begin{cases} x \approx x' \\ y \approx y' \\ t \approx t' \end{cases}$$

Перед тем, как выводить вид функций в новых координатах приведем вспомогательные производные.

$$a_x = \frac{\partial \xi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{z-b}{H}\right) = \frac{1}{H}\frac{\partial}{\partial x}(z-b) + (z-b)\frac{\partial}{\partial x}\left(\frac{1}{H}\right) = -\frac{1}{H}\frac{\partial b}{\partial x} + (z-b)\cdot\frac{-1}{H}\frac{\partial H}{\partial x} =$$
$$= \frac{-1}{H}\left(\frac{\partial b}{\partial x} + \frac{z-b}{H}\frac{\partial H}{\partial x}\right) = -\frac{1}{H}\left(\frac{\partial b}{\partial x} + \xi\frac{\partial H}{\partial x}\right) \approx -\frac{1}{H}\left(\frac{\partial b}{\partial x'} + \xi\frac{\partial H}{\partial x'}\right)$$

Аналогично

$$a_y = \frac{\partial \xi}{\partial y} = -\frac{1}{H}\left(\frac{\partial b}{\partial y} + \xi\frac{\partial H}{\partial y}\right) \approx -\frac{1}{H}\left(\frac{\partial b}{\partial y'} + \xi\frac{\partial H}{\partial y'}\right)$$

$$a_z = \frac{\partial \xi}{\partial z} = \frac{\partial}{\partial z}\left(\frac{z-b}{H}\right) = \frac{1}{H}$$

$$b_x = \frac{\partial a_x}{\partial x} = \frac{\partial}{\partial x}\left[-\frac{1}{H}\left(\frac{\partial b}{\partial x} + \xi\frac{\partial H}{\partial x}\right)\right] = \frac{\partial}{\partial x}\left(-\frac{1}{H}\right)\left(\frac{\partial b}{\partial x} + \xi\frac{\partial H}{\partial x}\right) + \left(-\frac{1}{H}\right)\frac{\partial}{\partial x}\left(\frac{\partial b}{\partial x} + \xi\frac{\partial H}{\partial x}\right) =$$
$$= \frac{1}{H^2}\frac{\partial H}{\partial x}\left(\frac{\partial b}{\partial x} + \xi\frac{\partial H}{\partial x}\right) - \frac{1}{H}\left(\frac{\partial^2 b}{\partial x^2} + \frac{\partial \xi}{\partial x}\frac{\partial H}{\partial x} + \xi\frac{\partial^2 H}{\partial x^2}\right) =$$
$$= \frac{1}{H^2}\frac{\partial H}{\partial x}\left(\frac{\partial b}{\partial x} + \xi\frac{\partial H}{\partial x}\right) - \frac{1}{H}\left(\frac{\partial^2 b}{\partial x^2} + a_x\frac{\partial b}{\partial x} + \xi\frac{\partial^2 H}{\partial x^2}\right) =$$
$$= -\frac{1}{H}\frac{\partial H}{\partial x}a_x - \frac{1}{H}a_x\frac{1}{H} - \frac{1}{H}\left(\frac{\partial^2 b}{\partial x^2} + \xi\frac{\partial^2 H}{\partial x^2}\right) = -\frac{1}{H}\left(\frac{\partial^2 b}{\partial x^2} + \xi\frac{\partial^2 H}{\partial x^2} + 2a_x\frac{\partial H}{\partial x}\right) \approx$$
$$\approx -\frac{1}{H}\left(\frac{\partial^2 b}{\partial x'^2} + \xi\frac{\partial^2 H}{\partial x'^2} + 2a_x\frac{\partial H}{\partial x'}\right)$$

$$b_y = \frac{\partial a_y}{\partial y} = -\frac{1}{H}\left(\frac{\partial^2 b}{\partial y^2} + \xi\frac{\partial^2 H}{\partial y^2} + 2a_y\frac{\partial H}{\partial y}\right) \approx -\frac{1}{H}\left(\frac{\partial^2 b}{\partial y'^2} + \xi\frac{\partial^2 H}{\partial y'^2} + 2a_y\frac{\partial H}{\partial y'}\right)$$

$$b_z = \frac{\partial a_z}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{H}\right) = 0$$

$$
\begin{aligned}
c_{xy} = \frac{\partial a_y}{\partial x} = c_{yx} = \frac{\partial a_x}{\partial y} = \frac{\partial}{\partial x}\left[-\frac{1}{H}\left(\frac{\partial b}{\partial y} + \xi\frac{\partial H}{\partial y}\right)\right] = \\
= -\frac{\partial}{\partial x}\left(\frac{1}{H}\right)\left(\frac{\partial b}{\partial y} + \xi\frac{\partial H}{\partial y}\right) - \frac{1}{H}\frac{\partial}{\partial x}\left(\frac{\partial b}{\partial y} + \xi\frac{\partial H}{\partial y}\right) = \\
= \frac{1}{H^2}\frac{\partial H}{\partial x}\left(\frac{\partial b}{\partial y} + \xi\frac{\partial H}{\partial y}\right) - \frac{1}{H}\left(\frac{\partial^2 b}{\partial x\partial y} + \frac{\partial \xi}{\partial x}\frac{\partial H}{\partial y} + \xi\frac{\partial^2 H}{\partial x\partial y}\right) = \\
= -\frac{1}{H}a_y\frac{\partial H}{\partial x} - \frac{1}{H}\left(\frac{\partial^2 b}{\partial x\partial y} + a_x\frac{\partial H}{\partial y} + \xi\frac{\partial^2 H}{\partial x\partial y}\right) = -\frac{1}{H}\left(\frac{\partial^2 b}{\partial x\partial y} + a_x\frac{\partial H}{\partial y} + a_y\frac{\partial H}{\partial x} + \xi\frac{\partial^2 H}{\partial x\partial y}\right) \approx \\
\approx -\frac{1}{H}\left(\frac{\partial^2 b}{\partial x'\partial y'} + a_x\frac{\partial H}{\partial y'} + a_y\frac{\partial H}{\partial x'} + \xi\frac{\partial^2 H}{\partial x'\partial y'}\right)
\end{aligned}
$$

$$
c_{xz} = \frac{\partial a_z}{\partial x} = c_{zx} = \frac{\partial a_x}{\partial z} = \frac{\partial}{\partial x}\left(\frac{1}{H}\right) = -\frac{1}{H^2}\frac{\partial H}{\partial x} \approx -\frac{1}{H^2}\frac{\partial H}{\partial x'}
$$

$$
c_{yz} = \frac{\partial a_z}{\partial y} = c_{zy} = \frac{\partial a_y}{\partial z} = -\frac{1}{H^2}\frac{\partial H}{\partial y} \approx -\frac{1}{H^2}\frac{\partial H}{\partial y'}
$$

Перейдем к первым производным функции f

$$
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial x} \approx \frac{\partial f}{\partial x'} + a_x\frac{\partial f}{\partial \xi}
$$

$$
\frac{\partial f}{\partial y} \approx \frac{\partial f}{\partial y'} + a_y\frac{\partial f}{\partial \xi}
$$

$$
\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial z} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial z} + \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial z} \approx a_z\frac{\partial f}{\partial \xi}
$$

Далее нужно рассмотреть вторые производные

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x'} + a_x\frac{\partial f}{\partial \xi}\right) = \frac{\partial}{\partial x}\frac{\partial f}{\partial x'} + \frac{\partial}{\partial \xi}\frac{\partial f}{\partial x'}\frac{\partial \xi}{\partial x} + \frac{\partial a_x}{\partial x}\frac{\partial f}{\partial \xi} + a_x\frac{\partial}{\partial x}\frac{\partial f}{\partial \xi} + a_x\frac{\partial}{\partial \xi}\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} \approx \\
\approx \frac{\partial^2 f}{\partial x'^2} + 2a_x\frac{\partial^2 f}{\partial x'\partial \xi} + a_x^2\frac{\partial^2 f}{\partial \xi^2} + b_x\frac{\partial f}{\partial \xi}
\end{aligned}
$$

$$
\frac{\partial^2 f}{\partial y^2} \approx \frac{\partial^2 f}{\partial y'^2} + 2a_y\frac{\partial^2 f}{\partial y'\partial \xi} + a_y^2\frac{\partial^2 f}{\partial \xi^2} + b_y\frac{\partial f}{\partial \xi}
$$

$$
\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(a_z\frac{\partial f}{\partial \xi}\right) = a_z\frac{\partial}{\partial z}\frac{\partial f}{\partial \xi} \approx a_z^2\frac{\partial^2 f}{\partial \xi^2}
$$

Включая смешанные частные производные

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'} + a_y\frac{\partial f}{\partial \xi}\right) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y'} + \frac{\partial}{\partial \xi}\frac{\partial f}{\partial y'}\frac{\partial \xi}{\partial x} + \frac{\partial a_y}{\partial x}\frac{\partial f}{\partial \xi} + a_y\frac{\partial}{\partial x}\frac{\partial f}{\partial \xi} = \\
= \frac{\partial}{\partial x'}\frac{\partial f}{\partial y'}\frac{\partial x'}{\partial x} + a_x\frac{\partial^2 f}{\partial y'\partial \xi} + c_{xy}\frac{\partial f}{\partial \xi} + a_y\frac{\partial}{\partial x'}\frac{\partial f}{\partial \xi}\frac{\partial x'}{\partial x} + a_y\frac{\partial}{\partial \xi}\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} \approx \\
\approx \frac{\partial^2 f}{\partial x'\partial y'} + a_x\frac{\partial^2 f}{\partial y'\partial \xi} + a_y\frac{\partial^2 f}{\partial x'\partial \xi} + a_x a_y\frac{\partial^2 f}{\partial \xi^2} + c_{xy}\frac{\partial f}{\partial \xi}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x\partial z} = \frac{\partial^2 f}{\partial z\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial z} = \frac{\partial}{\partial x}\left(a_z\frac{\partial f}{\partial \xi}\right) = \frac{\partial a_z}{\partial x}\frac{\partial f}{\partial \xi} + a_z\frac{\partial}{\partial x}\frac{\partial f}{\partial \xi} = \\
= c_{xz}\frac{\partial f}{\partial \xi} + a_z\left(\frac{\partial}{\partial x'}\frac{\partial f}{\partial \xi}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial \xi}\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x}\right) \approx a_z\frac{\partial^2 f}{\partial x'\partial \xi} + a_x a_z\frac{\partial^2 f}{\partial \xi^2} + c_{xz}\frac{\partial f}{\partial \xi}
\end{aligned}
$$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} \approx a_z \frac{\partial^2 f}{\partial y' \partial \xi} + a_y a_z \frac{\partial^2 f}{\partial \xi^2} + c_{yz} \frac{\partial f}{\partial \xi}$$

Схемы основных операторов

В данном разделе приведены дискретные схемы основных дифференциальных операторов.

2D операторы

$$\frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial x}\right)_{i,j} = \frac{1}{\Delta x^2}\left\{f_{i-1,j}\left[g_{i-\frac{1}{2},j}\right] + f_{i,j}\left[-g_{i-\frac{1}{2},j} - g_{i+\frac{1}{2},j}\right] + f_{i+1,j}\left[g_{i+\frac{1}{2},j}\right]\right\}$$

$$\begin{aligned}\frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial y}\right)_{i,j} =& \frac{1}{4\Delta x\Delta y}\left\{f_{i-1,j-1}\left[g_{i-\frac{1}{2},j-\frac{1}{2}}\right] + f_{i-1,j}\left[-g_{i-\frac{1}{2},j-\frac{1}{2}} + g_{i-\frac{1}{2},j+\frac{1}{2}}\right] +\right.\\ &+f_{i-1,j+1}\left[-g_{i-\frac{1}{2},j+\frac{1}{2}}\right] + f_{i,j-1}\left[g_{i-\frac{1}{2},j-\frac{1}{2}} - g_{i+\frac{1}{2},j-\frac{1}{2}}\right] +\\ &+f_{i,j}\left[-g_{i-\frac{1}{2},j-\frac{1}{2}} + g_{i-\frac{1}{2},j+\frac{1}{2}} + g_{i+\frac{1}{2},j-\frac{1}{2}} - g_{i+\frac{1}{2},j+\frac{1}{2}}\right] +\\ &+f_{i,j+1}\left[-g_{i-\frac{1}{2},j+\frac{1}{2}} + g_{i+\frac{1}{2},j+\frac{1}{2}}\right] + f_{i+1,j-1}\left[-g_{i+\frac{1}{2},j-\frac{1}{2}}\right] +\\ &\left.+f_{i+1,j}\left[g_{i+\frac{1}{2},j-\frac{1}{2}} - g_{i+\frac{1}{2},j+\frac{1}{2}}\right] + f_{i+1,j+1}\left[g_{i+\frac{1}{2},j+\frac{1}{2}}\right]\right\}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial y}\left(g\frac{\partial f}{\partial x}\right)_{i,j} =& \frac{1}{4\Delta x\Delta y}\left\{f_{i-1,j-1}\left[g_{i-\frac{1}{2},j-\frac{1}{2}}\right] + f_{i-1,j}\left[g_{i-\frac{1}{2},j-\frac{1}{2}} - g_{i-\frac{1}{2},j+\frac{1}{2}}\right] +\right.\\ &+f_{i-1,j+1}\left[-g_{i-\frac{1}{2},j+\frac{1}{2}}\right] + f_{i,j-1}\left[-g_{i-\frac{1}{2},j-\frac{1}{2}} + g_{i+\frac{1}{2},j-\frac{1}{2}}\right] +\\ &+f_{i,j}\left[-g_{i-\frac{1}{2},j-\frac{1}{2}} + g_{i-\frac{1}{2},j+\frac{1}{2}} + g_{i+\frac{1}{2},j-\frac{1}{2}} - g_{i+\frac{1}{2},j+\frac{1}{2}}\right] +\\ &+f_{i,j+1}\left[g_{i-\frac{1}{2},j+\frac{1}{2}} - g_{i+\frac{1}{2},j+\frac{1}{2}}\right] + f_{i+1,j-1}\left[-g_{i+\frac{1}{2},j-\frac{1}{2}}\right] +\\ &\left.+f_{i+1,j}\left[-g_{i+\frac{1}{2},j-\frac{1}{2}} + g_{i+\frac{1}{2},j+\frac{1}{2}}\right] + f_{i+1,j+1}\left[g_{i+\frac{1}{2},j+\frac{1}{2}}\right]\right\}\end{aligned}$$

$$\frac{\partial}{\partial y}\left(g\frac{\partial f}{\partial y}\right)_{i,j} = \frac{1}{\Delta y^2}\left\{f_{i,j-1}\left[g_{i,j-\frac{1}{2}}\right] + f_{i,j}\left[-g_{i,j-\frac{1}{2}} - g_{i,j+\frac{1}{2}}\right] + f_{i,j+1}\left[g_{i,j+\frac{1}{2}}\right]\right\}$$

3D операторы

$$\frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial x}\right)_{i,j,k} = \frac{1}{\Delta x^2}\left\{f_{i-1,j,k}\left[g_{i-\frac{1}{2},j,k}\right] + f_{i,j,k}\left[-g_{i-\frac{1}{2},j,k} - g_{i+\frac{1}{2},j,k}\right] + f_{i+1,j,k}\left[g_{i+\frac{1}{2},j,k}\right]\right\}$$

$$\begin{aligned}\frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial y}\right)_{i,j,k} =& \frac{1}{4\Delta x\Delta y}\left\{f_{i-1,j-1,k}\left[g_{i-\frac{1}{2},j-\frac{1}{2},k}\right] + f_{i-1,j,k}\left[-g_{i-\frac{1}{2},j-\frac{1}{2},k} + g_{i-\frac{1}{2},j+\frac{1}{2},k}\right] +\right.\\ &+f_{i-1,j+1,k}\left[-g_{i-\frac{1}{2},j+\frac{1}{2},k}\right] + f_{i,j-1,k}\left[g_{i-\frac{1}{2},j-\frac{1}{2},k} - g_{i+\frac{1}{2},j-\frac{1}{2},k}\right] +\\ &+f_{i,j,k}\left\{-g_{i-\frac{1}{2},j-\frac{1}{2},k} + g_{i-\frac{1}{2},j+\frac{1}{2},k} + g_{i+\frac{1}{2},j-\frac{1}{2},k} - g_{i+\frac{1}{2},j+\frac{1}{2},k}\right\} +\\ &+f_{i,j+1,k}\left[-g_{i-\frac{1}{2},j+\frac{1}{2},k} + g_{i+\frac{1}{2},j+\frac{1}{2},k}\right] + f_{i+1,j-1,k}\left[-g_{i+\frac{1}{2},j-\frac{1}{2},k}\right] +\\ &\left.+f_{i+1,j,k}\left[g_{i+\frac{1}{2},j-\frac{1}{2},k} - g_{i+\frac{1}{2},j+\frac{1}{2},k}\right] + f_{i+1,j+1,k}\left[g_{i+\frac{1}{2},j+\frac{1}{2},k}\right]\right\}\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial \xi}\right)_{i,j,k} =& \frac{1}{4\Delta x\Delta\xi_k\Delta\xi_{k-\frac{1}{2}}\Delta\xi_{k+\frac{1}{2}}}\times \\
&\times\left\{f_{i-1,j,k-1}\left[\Delta\xi^2_{k+\frac{1}{2}}g_{i-\frac{1}{2},j,k-\frac{1}{2}}\right]+f_{i-1,j,k}\left[-\Delta\xi^2_{k+\frac{1}{2}}g_{i-\frac{1}{2},j,k-\frac{1}{2}}+\Delta\xi^2_{k-\frac{1}{2}}g_{i-\frac{1}{2},j,k+\frac{1}{2}}\right]+\right.\\
&+f_{i-1,j,k+1}\left[-\Delta\xi^2_{k-\frac{1}{2}}g_{i-\frac{1}{2},j,k+\frac{1}{2}}\right]+f_{i,j,k-1}\left[\Delta\xi^2_{k+\frac{1}{2}}g_{i-\frac{1}{2},j,k-\frac{1}{2}}-\Delta\xi^2_{k+\frac{1}{2}}g_{i+\frac{1}{2},j,k-\frac{1}{2}}\right]+\\
&+f_{i,j,k}\left[-\Delta\xi^2_{k+\frac{1}{2}}g_{i-\frac{1}{2},j,k-\frac{1}{2}}+\Delta\xi^2_{k-\frac{1}{2}}g_{i-\frac{1}{2},j,k+\frac{1}{2}}+\Delta\xi^2_{k+\frac{1}{2}}g_{i+\frac{1}{2},j,k-\frac{1}{2}}-\Delta\xi^2_{k-\frac{1}{2}}g_{i+\frac{1}{2},j,k+\frac{1}{2}}\right]+\\
&+f_{i,j,k+1}\left[-\Delta\xi^2_{k-\frac{1}{2}}g_{i-\frac{1}{2},j,k+\frac{1}{2}}+\Delta\xi^2_{k-\frac{1}{2}}g_{i+\frac{1}{2},j,k+\frac{1}{2}}\right]+f_{i+1,j,k-1}\left[-\Delta\xi^2_{k+\frac{1}{2}}g_{i+\frac{1}{2},j,k-\frac{1}{2}}\right]+\\
&\left.+f_{i+1,j,k}\left[\Delta\xi^2_{k+\frac{1}{2}}g_{i+\frac{1}{2},j,k-\frac{1}{2}}-\Delta\xi^2_{k-\frac{1}{2}}g_{i+\frac{1}{2},j,k+\frac{1}{2}}\right]+f_{i+1,j,k+1}\left[\Delta\xi^2_{k-\frac{1}{2}}g_{i+\frac{1}{2},j,k+\frac{1}{2}}\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial y}\left(g\frac{\partial f}{\partial x}\right)_{i,j,k} =& \frac{1}{4\Delta x\Delta y}\left\{f_{i-1,j-1,k}\left[g_{i-\frac{1}{2},j-\frac{1}{2},k}\right]+f_{i-1,j,k}\left[g_{i-\frac{1}{2},j-\frac{1}{2},k}-g_{i-\frac{1}{2},j+\frac{1}{2},k}\right]+\right.\\
&+f_{i-1,j+1,k}\left[-g_{i-\frac{1}{2},j+\frac{1}{2},k}\right]+f_{i,j-1,k}\left[-g_{i-\frac{1}{2},j-\frac{1}{2},k}+g_{i+\frac{1}{2},j-\frac{1}{2},k}\right]+\\
&+f_{i,j,k}\left\{-g_{i-\frac{1}{2},j-\frac{1}{2},k}+g_{i-\frac{1}{2},j+\frac{1}{2},k}+g_{i+\frac{1}{2},j-\frac{1}{2},k}-g_{i+\frac{1}{2},j+\frac{1}{2},k}\right\}+\\
&+f_{i,j+1,k}\left[g_{i-\frac{1}{2},j+\frac{1}{2},k}-g_{i+\frac{1}{2},j+\frac{1}{2},k}\right]+f_{i+1,j-1,k}\left[-g_{i+\frac{1}{2},j-\frac{1}{2},k}\right]+\\
&\left.+f_{i+1,j,k}\left[-g_{i+\frac{1}{2},j-\frac{1}{2},k}+g_{i+\frac{1}{2},j+\frac{1}{2},k}\right]+f_{i+1,j+1,k}\left[g_{i+\frac{1}{2},j+\frac{1}{2},k}\right]\right\}
\end{aligned}
$$

$$
\frac{\partial}{\partial y}\left(g\frac{\partial f}{\partial y}\right)_{i,j,k} = \frac{1}{\Delta y^2}\left\{f_{i,j-1,k}\left[g_{i,j-\frac{1}{2},k}\right]+f_{i,j,k}\left[-g_{i,j-\frac{1}{2},k}-g_{i,j+\frac{1}{2},k}\right]+f_{i,j+1,k}\left[g_{i,j+\frac{1}{2},k}\right]\right\}
$$

$$
\begin{aligned}
\frac{\partial}{\partial y}\left(g\frac{\partial f}{\partial \xi}\right)_{i,j,k} =& \frac{1}{4\Delta y\Delta\xi_k\Delta\xi_{k-\frac{1}{2}}\Delta\xi_{k+\frac{1}{2}}}\times \\
&\times\left\{f_{i,j-1,k-1}\left[\Delta\xi^2_{k+\frac{1}{2}}g_{i,j-\frac{1}{2},k-\frac{1}{2}}\right]+f_{i,j-1,k}\left[-\Delta\xi^2_{k+\frac{1}{2}}g_{i,j-\frac{1}{2},k-\frac{1}{2}}+\Delta\xi^2_{k-\frac{1}{2}}g_{i,j-\frac{1}{2},k+\frac{1}{2}}\right]+\right.\\
&+f_{i,j-1,k+1}\left[-\Delta\xi^2_{k-\frac{1}{2}}g_{i,j-\frac{1}{2},k+\frac{1}{2}}\right]+f_{i,j,k-1}\left[\Delta\xi^2_{k+\frac{1}{2}}g_{i,j-\frac{1}{2},k-\frac{1}{2}}-\Delta\xi^2_{k+\frac{1}{2}}g_{i,j+\frac{1}{2},k-\frac{1}{2}}\right]+\\
&+f_{i,j,k}\left[-\Delta\xi^2_{k+\frac{1}{2}}g_{i,j-\frac{1}{2},k-\frac{1}{2}}+\Delta\xi^2_{k-\frac{1}{2}}g_{i,j-\frac{1}{2},k+\frac{1}{2}}+\Delta\xi^2_{k+\frac{1}{2}}g_{i,j+\frac{1}{2},k-\frac{1}{2}}-\Delta\xi^2_{k-\frac{1}{2}}g_{i,j+\frac{1}{2},k+\frac{1}{2}}\right]+\\
&+f_{i,j,k+1}\left[-\Delta\xi^2_{k-\frac{1}{2}}g_{i,j-\frac{1}{2},k+\frac{1}{2}}+\Delta\xi^2_{k-\frac{1}{2}}g_{i,j+\frac{1}{2},k+\frac{1}{2}}\right]+f_{i,j+1,k-1}\left[-\Delta\xi^2_{k+\frac{1}{2}}g_{i,j+\frac{1}{2},k-\frac{1}{2}}\right]+\\
&\left.+f_{i,j+1,k}\left[\Delta\xi^2_{k+\frac{1}{2}}g_{i,j+\frac{1}{2},k-\frac{1}{2}}-\Delta\xi^2_{k-\frac{1}{2}}g_{i,j+\frac{1}{2},k+\frac{1}{2}}\right]+f_{i,j+1,k+1}\left[\Delta\xi^2_{k-\frac{1}{2}}g_{i,j+\frac{1}{2},k+\frac{1}{2}}\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \xi}\left(g\frac{\partial f}{\partial x}\right)_{i,j,k} =& \frac{1}{4\Delta x\Delta\xi_k\Delta\xi_{k-\frac{1}{2}}\Delta\xi_{k+\frac{1}{2}}}\times\left\{f_{i-1,j,k-1}\left[\Delta\xi^2_{k+\frac{1}{2}}g_{i-\frac{1}{2},j,k-\frac{1}{2}}\right]+\right.\\
&+f_{i-1,j,k}\left[2\Delta\xi_{k-\frac{1}{2}}\Delta\xi_{k+\frac{1}{2}}\left(g_{i-\frac{1}{2},j,k-\frac{1}{2}}-g_{i-\frac{1}{2},j,k+\frac{1}{2}}\right)-\Delta\xi^2_{k+\frac{1}{2}}g_{i-\frac{1}{2},j,k-\frac{1}{2}}+\Delta\xi^2_{k-\frac{1}{2}}g_{i-\frac{1}{2},j,k+\frac{1}{2}}\right]+\\
&+f_{i-1,j,k+1}\left[-\Delta\xi^2_{k-\frac{1}{2}}g_{i-\frac{1}{2},j,k+\frac{1}{2}}\right]+f_{i,j,k-1}\left[-\Delta\xi^2_{k+\frac{1}{2}}g_{i-\frac{1}{2},j,k-\frac{1}{2}}+\Delta\xi^2_{k+\frac{1}{2}}g_{i+\frac{1}{2},j,k-\frac{1}{2}}\right]+\\
&+f_{i,j,k}\left[-2\Delta\xi_{k-\frac{1}{2}}\Delta\xi_{k+\frac{1}{2}}\left(g_{i-\frac{1}{2},j,k-\frac{1}{2}}-g_{i-\frac{1}{2},j,k+\frac{1}{2}}-g_{i+\frac{1}{2},j,k-\frac{1}{2}}+g_{i+\frac{1}{2},j,k+\frac{1}{2}}\right)+\right.\\
&\qquad\left.+\Delta\xi^2_{k+\frac{1}{2}}g_{i-\frac{1}{2},j,k-\frac{1}{2}}-\Delta\xi^2_{k-\frac{1}{2}}g_{i-\frac{1}{2},j,k+\frac{1}{2}}-\Delta\xi^2_{k+\frac{1}{2}}g_{i+\frac{1}{2},j,k-\frac{1}{2}}+\Delta\xi^2_{k-\frac{1}{2}}g_{i+\frac{1}{2},j,k+\frac{1}{2}}\right]+\\
&+f_{i,j,k+1}\left[\Delta\xi^2_{k-\frac{1}{2}}g_{i-\frac{1}{2},j,k+\frac{1}{2}}-\Delta\xi^2_{k-\frac{1}{2}}g_{i+\frac{1}{2},j,k+\frac{1}{2}}\right]+f_{i+1,j,k-1}\left[-\Delta\xi^2_{k+\frac{1}{2}}g_{i+\frac{1}{2},j,k-\frac{1}{2}}\right]+\\
&+f_{i+1,j,k}\left[-2\Delta\xi_{k-\frac{1}{2}}\Delta\xi_{k+\frac{1}{2}}\left(g_{i+\frac{1}{2},j,k-\frac{1}{2}}-g_{i+\frac{1}{2},j,k+\frac{1}{2}}\right)+\Delta\xi^2_{k+\frac{1}{2}}g_{i+\frac{1}{2},j,k-\frac{1}{2}}-\Delta\xi^2_{k-\frac{1}{2}}g_{i+\frac{1}{2},j,k+\frac{1}{2}}\right]+\\
&\left.+f_{i+1,j,k+1}\left[\Delta\xi^2_{k-\frac{1}{2}}g_{i+\frac{1}{2},j,k+\frac{1}{2}}\right]\right\}
\end{aligned}
$$

$$\frac{\partial}{\partial\xi}\left(g\frac{\partial f}{\partial y}\right)_{i,j,k} = \frac{1}{4\Delta y\Delta\xi_k\Delta\xi_{k-\frac{1}{2}}\Delta\xi_{k+\frac{1}{2}}} \times \Bigg\{ f_{i,j-1,k-1}\left[\Delta\xi^2_{k+\frac{1}{2}} g_{i,j-\frac{1}{2},k-\frac{1}{2}}\right] +$$
$$+ f_{i,j-1,k}\left[2\Delta\xi_{k-\frac{1}{2}}\Delta\xi_{k+\frac{1}{2}}\left(g_{i,j-\frac{1}{2},k-\frac{1}{2}} - g_{i,j-\frac{1}{2},k+\frac{1}{2}}\right) - \Delta\xi^2_{k+\frac{1}{2}} g_{i,j-\frac{1}{2},k-\frac{1}{2}} + \Delta\xi^2_{k-\frac{1}{2}} g_{i,j-\frac{1}{2},k+\frac{1}{2}}\right] +$$
$$+ f_{i,j-1,k+1}\left[-\Delta\xi^2_{k-\frac{1}{2}} g_{i,j-\frac{1}{2},k+\frac{1}{2}}\right] + f_{i,j,k-1}\left[-\Delta\xi^2_{k+\frac{1}{2}} g_{i,j-\frac{1}{2},k-\frac{1}{2}} + \Delta\xi^2_{k+\frac{1}{2}} g_{i,j+\frac{1}{2},k-\frac{1}{2}}\right] +$$
$$+ f_{i,j,k}\Big[-2\Delta\xi_{k-\frac{1}{2}}\Delta\xi_{k+\frac{1}{2}}\left(g_{i,j-\frac{1}{2},k-\frac{1}{2}} - g_{i,j-\frac{1}{2},k+\frac{1}{2}} - g_{i,j+\frac{1}{2},k-\frac{1}{2}} + g_{i,j+\frac{1}{2},k+\frac{1}{2}}\right) +$$
$$+ \Delta\xi^2_{k+\frac{1}{2}} g_{i,j-\frac{1}{2},k-\frac{1}{2}} - \Delta\xi^2_{k-\frac{1}{2}} g_{i,j-\frac{1}{2},k+\frac{1}{2}} - \Delta\xi^2_{k+\frac{1}{2}} g_{i,j+\frac{1}{2},k-\frac{1}{2}} + \Delta\xi^2_{k-\frac{1}{2}} g_{i,j+\frac{1}{2},k+\frac{1}{2}}\Big] +$$
$$+ f_{i,j,k+1}\left[\Delta\xi^2_{k-\frac{1}{2}} g_{i,j-\frac{1}{2},k+\frac{1}{2}} - \Delta\xi^2_{k-\frac{1}{2}} g_{i,j+\frac{1}{2},k+\frac{1}{2}}\right] + f_{i,j+1,k-1}\left[-\Delta\xi^2_{k+\frac{1}{2}} g_{i,j+\frac{1}{2},k-\frac{1}{2}}\right] +$$
$$+ f_{i,j+1,k}\left[-2\Delta\xi_{k-\frac{1}{2}}\Delta\xi_{k+\frac{1}{2}}\left(g_{i,j+\frac{1}{2},k-\frac{1}{2}} - g_{i,j+\frac{1}{2},k+\frac{1}{2}}\right) + \Delta\xi^2_{k+\frac{1}{2}} g_{i,j+\frac{1}{2},k-\frac{1}{2}} - \Delta\xi^2_{k-\frac{1}{2}} g_{i,j+\frac{1}{2},k+\frac{1}{2}}\right] +$$
$$+ f_{i,j+1,k+1}\left[\Delta\xi^2_{k-\frac{1}{2}} g_{i,j+\frac{1}{2},k+\frac{1}{2}}\right]\Bigg\}$$

$$\frac{\partial}{\partial\xi}\left(g\frac{\partial f}{\partial \xi}\right)_{i,j,k} = \frac{1}{2\Delta\xi_k}\Bigg\{ f_{i,j,k-1}\left[2\frac{\Delta\xi_{k+\frac{1}{2}}}{\Delta\xi_{k-\frac{1}{2}}} g_{i,j,k-\frac{1}{2}} + \left(1 - \frac{\Delta\xi_{k-\frac{1}{2}}}{\Delta\xi_{k+\frac{1}{2}}}\right) g_{i,j,k+\frac{1}{2}}\right] +$$
$$+ f_{i,j,k}\left[-\left(1 + 2\frac{\Delta\xi_{k+\frac{1}{2}}}{\Delta\xi_{k-\frac{1}{2}}} - \frac{\Delta\xi_{k-\frac{1}{2}}}{\Delta\xi_{k+\frac{1}{2}}}\right) g_{i,j,k-\frac{1}{2}} - \left(1 + 2\frac{\Delta\xi_{k-\frac{1}{2}}}{\Delta\xi_{k+\frac{1}{2}}} - \frac{\Delta\xi_{k+\frac{1}{2}}}{\Delta\xi_{k-\frac{1}{2}}}\right) g_{i,j,k+\frac{1}{2}}\right] +$$
$$+ f_{i,j,k+1}\left[\left(1 - \frac{\Delta\xi_{k+\frac{1}{2}}}{\Delta\xi_{k-\frac{1}{2}}}\right) g_{i,j,k-\frac{1}{2}} + 2\frac{\Delta\xi_{k-\frac{1}{2}}}{\Delta\xi_{k+\frac{1}{2}}} g_{i,j,k+\frac{1}{2}}\right]\Bigg\}$$

$$\left(g\frac{\partial f}{\partial \xi}\right)_{i,j,k} = \frac{1}{2\Delta\xi_k}\Bigg\{ f_{i,j,k-1}\left[-\frac{\Delta\xi_{k+\frac{1}{2}}}{\Delta\xi_{k-\frac{1}{2}}} g_{i,j,k}\right] + f_{i,j,k}\left[2\Delta\xi_k\frac{\Delta\xi_{k+\frac{1}{2}} - \Delta\xi_{k-\frac{1}{2}}}{\Delta\xi_{k+\frac{1}{2}}\Delta\xi_{k-\frac{1}{2}}} g_{i,j,k}\right] +$$
$$+ f_{i,j,k+1}\left[\frac{\Delta\xi_{k-\frac{1}{2}}}{\Delta\xi_{k+\frac{1}{2}}} g_{i,j,k}\right]\Bigg\}$$

Для удобства вывода вычислительных схем и повышения читаемости переобозначим коэффициенты при узловых значениях функции f. Для этого перечислим символьные переобоначения для коэффициентов при узлах искомой функции в указанных выше схемах.

2D:

$$\frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial x}\right) = f_{i-1,j}\left[\alpha_{xx}(-1,0)\right] + f_{i,j}\left[\alpha_{xx}(0,0)\right] + f_{i+1,j}\left[\alpha_{xx}(1,0)\right]$$

$$\frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial y}\right) = f_{i-1,j-1}\left[\alpha_{xy}(-1,-1)\right] + f_{i-1,j}\left[\alpha_{xy}(-1,0)\right] + f_{i-1,j+1}\left[\alpha_{xy}(-1,1)\right] +$$
$$+ f_{i,j-1}\left[\alpha_{xy}(0,-1)\right] + f_{i,j}\left[\alpha_{xy}(0,0)\right] + f_{i,j+1}\left[\alpha_{xy}(0,1)\right] +$$
$$+ f_{i+1,j-1}\left[\alpha_{xy}(1,-1)\right] + f_{i+1,j}\left[\alpha_{xy}(1,0)\right] + f_{i+1,j+1}\left[\alpha_{xy}(1,1)\right]$$

$$\frac{\partial}{\partial y}\left(g\frac{\partial f}{\partial x}\right) = f_{i-1,j-1}\left[\alpha_{yx}(-1,-1)\right] + f_{i-1,j}\left[\alpha_{yx}(-1,0)\right] + f_{i-1,j+1}\left[\alpha_{yx}(-1,1)\right] +$$
$$+ f_{i,j-1}\left[\alpha_{yx}(0,-1)\right] + f_{i,j}\left[\alpha_{yx}(0,0)\right] + f_{i,j+1}\left[\alpha_{yx}(0,1)\right] +$$
$$+ f_{i+1,j-1}\left[\alpha_{yx}(1,-1)\right] + f_{i+1,j}\left[\alpha_{yx}(1,0)\right] + f_{i+1,j+1}\left[\alpha_{yx}(1,1)\right]$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{i,j-1}\left[\alpha_{yy}\left(0,-1\right)\right] + f_{i,j}\left[\alpha_{yy}\left(0,0\right)\right] + f_{i,j+1}\left[\alpha_{yy}\left(0,1\right)\right]$$

Для введенных коэффициентов можно заметить некоторые свойства

$$\begin{cases} \alpha_{x,y}\left(p,q\right) = -\alpha_{y,x}\left(p,q\right), & |p| \neq |q| \\ \alpha_{x,y}\left(p,q\right) = \alpha_{y,x}\left(p,q\right), & |p| = |q| \end{cases}$$

Таким образом можно кратко записать схему для суммы.

$$\begin{aligned}\left[\frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial y}\left(g\frac{\partial f}{\partial x}\right)\right] = f_{i-1,j-1}\left[2\alpha_{x,y}\left(-1,-1\right)\right] + f_{i-1,j+1}\left[2\alpha_{x,y}\left(-1,1\right)\right] + \\ + f_{i,f}\left[2\alpha_{x,y}\left(0,0\right)\right] + f_{i+1,j-1}\left[2\alpha_{x,y}\left(1,-1\right)\right] + f_{i+1,j+1}\left[2\alpha_{x,y}\left(1,1\right)\right]\end{aligned}$$

3D:

$$\frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial x}\right) = f_{i-1,j,k}\left[\beta_{xx}\left(-1,0,0\right)\right] + f_{i,j,k}\left[\beta_{xx}\left(0,0,0\right)\right] + f_{i+1,j,k}\left[\beta_{xx}\left(1,0,0\right)\right]$$

$$\begin{aligned}\frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial y}\right) = {} & f_{i-1,j-1,k}\left[\beta_{xy}\left(-1,-1,0\right)\right] + f_{i-1,j,k}\left[\beta_{xy}\left(-1,0,0\right)\right] + f_{i-1,j+1,k}\left[\beta_{xy}\left(-1,1,0\right)\right] + \\ & + f_{i,j-1,k}\left[\beta_{xy}\left(0,-1,0\right)\right] + f_{i,j,k}\left[\beta_{xy}\left(0,0,0\right)\right] + f_{i,j+1,k}\left[\beta_{xy}\left(0,1,0\right)\right] + \\ & + f_{i+1,j-1,k}\left[\beta_{xy}\left(1,-1,0\right)\right] + f_{i+1,j,k}\left[\beta_{xy}\left(1,0,0\right)\right] + f_{i+1,j+1,k}\left[\beta_{xy}\left(1,1,0\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial \xi}\right) = {} & f_{i-1,j,k-1}\left[\beta_{x\xi}\left(-1,0,-1\right)\right] + f_{i-1,j,k}\left[\beta_{x\xi}\left(-1,0,0\right)\right] + f_{i-1,j,k+1}\left[\beta_{x\xi}\left(-1,0,1\right)\right] + \\ & + f_{i,j,k-1}\left[\beta_{x\xi}\left(0,0,-1\right)\right] + f_{i,j,k}\left[\beta_{x\xi}\left(0,0,0\right)\right] + f_{i,j,k+1}\left[\beta_{x\xi}\left(0,0,1\right)\right] + \\ & + f_{i+1,j,k-1}\left[\beta_{x\xi}\left(1,0,-1\right)\right] + f_{i+1,j,k}\left[\beta_{x\xi}\left(1,0,0\right)\right] + f_{i+1,j,k+1}\left[\beta_{x\xi}\left(1,0,1\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial y}\left(g\frac{\partial f}{\partial x}\right) = {} & f_{i-1,j-1,k}\left[\beta_{yx}\left(-1,-1,0\right)\right] + f_{i-1,j,k}\left[\beta_{yx}\left(-1,0,0\right)\right] + f_{i-1,j+1,k}\left[\beta_{yx}\left(-1,1,0\right)\right] + \\ & + f_{i,j-1,k}\left[\beta_{yx}\left(0,-1,0\right)\right] + f_{i,j,k}\left[\beta_{yx}\left(0,0,0\right)\right] + f_{i,j+1,k}\left[\beta_{yx}\left(0,1,0\right)\right] + \\ & + f_{i+1,j-1,k}\left[\beta_{yx}\left(1,-1,0\right)\right] + f_{i+1,j,k}\left[\beta_{yx}\left(1,0,0\right)\right] + f_{i+1,j+1,k}\left[\beta_{yx}\left(1,1,0\right)\right]\end{aligned}$$

$$\frac{\partial}{\partial y}\left(g\frac{\partial f}{\partial y}\right) = f_{i,j-1,k}\left[\beta_{yy}\left(0,-1,0\right)\right] + f_{i,j,k}\left[\beta_{yy}\left(0,0,0\right)\right] + f_{i,j+1,k}\left[\beta_{yy}\left(0,1,0\right)\right]$$

$$\begin{aligned}\frac{\partial}{\partial y}\left(g\frac{\partial f}{\partial \xi}\right) = {} & f_{i,j-1,k-1}\left[\beta_{y\xi}\left(0,-1,-1\right)\right] + f_{i,j-1,k}\left[\beta_{y\xi}\left(0,-1,0\right)\right] + f_{i,j-1,k+1}\left[\beta_{y\xi}\left(0,-1,1\right)\right] + \\ & + f_{i,j,k-1}\left[\beta_{y\xi}\left(0,0,-1\right)\right] + f_{i,j,k}\left[\beta_{y\xi}\left(0,0,0\right)\right] + f_{i,j,k+1}\left[\beta_{y\xi}\left(0,0,1\right)\right] + \\ & + f_{i,j+1,k-1}\left[\beta_{y\xi}\left(0,1,-1\right)\right] + f_{i,j+1,k}\left[\beta_{y\xi}\left(0,1,0\right)\right] + f_{i,j+1,k+1}\left[\beta_{y\xi}\left(0,1,1\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial \xi}\left(g\frac{\partial f}{\partial x}\right) = {} & f_{i-1,j,k-1}\left[\beta_{x\xi}\left(-1,0,-1\right)\right] + f_{i-1,j,k}\left[\gamma_{\xi x}\left(-1,0,0\right) + \beta_{x\xi}\left(-1,0,0\right)\right] + f_{i-1,j,k+1}\left[\beta_{x\xi}\left(-1,0,1\right)\right] + \\ & + f_{i,j,k-1}\left[-\beta_{x\xi}\left(0,0,-1\right)\right] + f_{i,j,k}\left[\gamma_{\xi x}\left(0,0,0\right) - \beta_{x\xi}\left(0,0,0\right)\right] + f_{i,j,k+1}\left[-\beta_{x\xi}\left(0,0,1\right)\right] + \\ & + f_{i+1,j,k-1}\left[\beta_{x\xi}\left(1,0,-1\right)\right] + f_{i+1,j,k}\left[\gamma_{\xi x}\left(1,0,0\right) + \beta_{x\xi}\left(1,0,0\right)\right] + f_{i+1,j,k+1}\left[\beta_{\xi x}\left(1,0,1\right)\right]\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial \xi}\left(g\frac{\partial f}{\partial y}\right) = {} & f_{i,j-1,k-1}\left[\beta_{y\xi}\left(0,-1,-1\right)\right] + f_{i,j-1,k}\left[\gamma_{\xi y}\left(0,-1,0\right) + \beta_{y\xi}\left(0,-1,0\right)\right] + f_{i,j-1,k+1}\left[\beta_{y\xi}\left(0,-1,1\right)\right] + \\
& + f_{i,j,k-1}\left[-\beta_{y\xi}\left(0,0,-1\right)\right] + f_{i,j,k}\left[\gamma_{\xi y}\left(0,0,0\right) - \beta_{y\xi}\left(0,0,0\right)\right] + f_{i,j,k+1}\left[-\beta_{y\xi}\left(0,0,1\right)\right] + \\
& + f_{i,j+1,k-1}\left[\beta_{y\xi}\left(0,1,-1\right)\right] + f_{i,j+1,k}\left[\gamma_{\xi y}\left(0,1,0\right) + \beta_{y\xi}\left(0,1,0\right)\right] + f_{i,j+1,k+1}\left[\beta_{y\xi}\left(0,1,1\right)\right]
\end{aligned}
$$

$$
\frac{\partial}{\partial \xi}\left(g\frac{\partial f}{\partial \xi}\right) = f_{i,j,k-1}\left[\beta_{\xi\xi}\left(0,0,-1\right)\right] + f_{i,j,k}\left[\beta_{\xi\xi}\left(0,0,0\right)\right] + f_{i,j,k+1}\left[\beta_{\xi\xi}\left(0,0,1\right)\right]
$$

Аналогично предыдущим выводам, приведем схемы для сумм операторов.

$$
\begin{aligned}
\left[\frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial y}\left(g\frac{\partial f}{\partial x}\right)\right] = {} & f_{i-1,j-1,k}\left[2\beta_{xy}\left(-1,-1,0\right)\right] + f_{i-1,j+1,k}\left[2\beta_{xy}\left(-1,1,0\right)\right] + \\
& + f_{i,f,k}\left[2\beta_{xy}\left(0,0,0\right)\right] + f_{i+1,j-1,k}\left[2\beta_{xy}\left(1,-1,0\right)\right] + f_{i+1,j+1,k}\left[2\beta_{xy}\left(1,1,0\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[\frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial \xi}\right) + \frac{\partial}{\partial \xi}\left(g\frac{\partial f}{\partial x}\right)\right] = {} & f_{i-1,j,k-1}\left[2\beta_{x\xi}\left(-1,0,-1\right)\right] + f_{i-1,j,k}\left[\gamma_{\xi x}\left(-1,0,0\right) + 2\beta_{x\xi}\left(-1,0,0\right)\right] + \\
& + f_{i-1,j,k+1}\left[2\beta_{x\xi}\left(-1,0,1\right)\right] + f_{i,f,k}\left[\gamma_{\xi x}\left(0,0,0\right)\right] + f_{i+1,j,k-1}\left[2\beta_{x\xi}\left(1,0,-1\right)\right] + \\
& + f_{i+1,j,k}\left[\gamma_{\xi x}\left(1,0,0\right) + 2\beta_{x\xi}\left(1,0,0\right)\right] + f_{i+1,j,k+1}\left[2\beta_{x\xi}\left(1,0,1\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[\frac{\partial}{\partial y}\left(g\frac{\partial f}{\partial \xi}\right) + \frac{\partial}{\partial \xi}\left(g\frac{\partial f}{\partial y}\right)\right] = {} & f_{i,j-1,k-1}\left[2\beta_{y\xi}\left(0,-1,-1\right)\right] + f_{i,j-1,k}\left[\gamma_{\xi y}\left(0,-1,0\right) + 2\beta_{y\xi}\left(0,-1,0\right)\right] + \\
& + f_{i,j-1,k+1}\left[2\beta_{y\xi}\left(0,-1,1\right)\right] + f_{i,f,k}\left[\gamma_{\xi y}\left(0,0,0\right)\right] + f_{i,j+1,k-1}\left[2\beta_{y\xi}\left(0,1,-1\right)\right] + \\
& + f_{i,j+1,k}\left[\gamma_{\xi y}\left(0,1,0\right) + 2\beta_{y\xi}\left(0,1,0\right)\right] + f_{i,j+1,k+1}\left[2\beta_{y\xi}\left(0,1,1\right)\right]
\end{aligned}
$$

Дискретизация уравнений компонентов скорости

Проведем дискретизацию уравнений для компонентов u и v скорости среды. Вывод проведем только с уравнением для компоненты u, так как уравнения идентичны с учетом замен $u \leftrightarrow v$ и $x \leftrightarrow y$, и получить второе можно заменой указанных переменных и индексов дискретизации.

Исходное уравнение

$$
\begin{aligned}
4\frac{\partial}{\partial x'}\left(\eta\frac{\partial u}{\partial x'}\right) + 4a_x\frac{\partial}{\partial x'}\left(\eta\frac{\partial u}{\partial \xi}\right) + \frac{\partial}{\partial y'}\left(\eta\frac{\partial u}{\partial y'}\right) + a_y\frac{\partial}{\partial y'}\left(\eta\frac{\partial u}{\partial \xi}\right) + 4a_x\frac{\partial}{\partial \xi}\left(\eta\frac{\partial u}{\partial x'}\right) + a_y\frac{\partial}{\partial \xi}\left(\eta\frac{\partial u}{\partial y'}\right) + \\
+ \left(4a_x^2 + a_y^2 + a_z^2\right)\frac{\partial}{\partial \xi}\left(\eta\frac{\partial u}{\partial \xi}\right) + (4b_x + b_y)\left(\eta\frac{\partial u}{\partial \xi}\right) = \rho g\frac{\partial s}{\partial x'} - \\
- 2\frac{\partial}{\partial x'}\left(\eta\frac{\partial v}{\partial y'}\right) - 2a_y\frac{\partial}{\partial x'}\left(\eta\frac{\partial v}{\partial \xi}\right) - \frac{\partial}{\partial y'}\left(\eta\frac{\partial v}{\partial x'}\right) - a_x\frac{\partial}{\partial y'}\left(\eta\frac{\partial v}{\partial \xi}\right) - \\
- a_y\frac{\partial}{\partial \xi}\left(\eta\frac{\partial v}{\partial x'}\right) - 2a_x\frac{\partial}{\partial \xi}\left(\eta\frac{\partial v}{\partial y'}\right) - 3a_xa_y\frac{\partial}{\partial \xi}\left(\eta\frac{\partial v}{\partial \xi}\right) - 3c_{xy}\left(\eta\frac{\partial v}{\partial \xi}\right)
\end{aligned}
$$

Дискретизация уравнения поверхности

При дискретизации уравнения поверхности можно использовать полученные выше дискретизации операторов. Например, дискретизацию для оператора $\frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial x}\right)$.

Запишем исходное уравнение и сразу перейдем к дискретной форме

$$\frac{\partial s}{\partial t} = -\frac{\partial}{\partial x}\left(D_x\frac{\partial s}{\partial x}\right) - \frac{\partial}{\partial y}\left(D_y\frac{\partial s}{\partial y}\right) + \frac{\partial b}{\partial t} + \dot{M}_s - \dot{M}_b$$

$$\begin{aligned}\frac{s_{ijl} - s_{ijl-1}}{\Delta t_l} = &-\frac{1}{\Delta x^2}\left\{s_{i-1jl}D_{x_{i-\frac{1}{2}jl}} + s_{ijl}\left[-D_{x_{i-\frac{1}{2}jl}} - D_{x_{i+\frac{1}{2}jl}}\right] + s_{i+1jl}D_{x_{i+\frac{1}{2}jl}}\right\} - \\ &-\frac{1}{\Delta y^2}\left\{s_{ij-1l}D_{y_{ij-\frac{1}{2}l}} + s_{ijl}\left[-D_{y_{ij-\frac{1}{2}l}} - D_{y_{ij+\frac{1}{2}l}}\right] + s_{ij+1l}D_{y_{ij+\frac{1}{2}l}}\right\} + \\ &+\frac{b_{ijl} - b_{ijl-1}}{\Delta t_l} + \dot{M}_{s_{ijl}} - \dot{M}_{b_{ijl}}\end{aligned}$$

$$\begin{aligned}&s_{i-1jl}\left[\frac{D_{x_{i-\frac{1}{2}jl}}}{\Delta x^2}\right] + s_{ij-1l}\left[\frac{D_{y_{ij-\frac{1}{2}l}}}{\Delta y^2}\right] + s_{ijl}\left[\frac{1}{\Delta t_l} - \frac{D_{x_{i-\frac{1}{2}jl}} + D_{x_{i+\frac{1}{2}jl}}}{\Delta x^2} - \frac{D_{y_{ij-\frac{1}{2}l}} + D_{y_{ij+\frac{1}{2}l}}}{\Delta y^2}\right] + \\ &s_{ij+1l}\left[\frac{D_{y_{ij+\frac{1}{2}l}}}{\Delta y^2}\right] + s_{i+1jl}\left[\frac{D_{x_{i+\frac{1}{2}jl}}}{\Delta x^2}\right] = s_{ijl-1}\left[\frac{1}{\Delta t_l}\right] + \frac{b_{ijl} - b_{ijl-1}}{\Delta t_l} + \dot{M}_{s_{ijl}} - \dot{M}_{b_{ijl}}\end{aligned}$$