2/14. ♥

What (the heck) is a Spectrum?

The Plan. (0) Preliminaries (topology)
(1) What is "stability"? — Motivations.
(2) Axioms we want.
(3) Definitions
(4) Examples.

Preliminaries...

Top : category of topological spaces ("badly behaves")
ob: (compactly gen'd weakly Hausdorff) spaces (technical).
mor: {continuous maps}.

CW ⊆ Top ob: {CW-complexes — built by attaching disks to spheres}.
mor: {CW-maps}

Pointed cats (Top_*, CW_*) — ob: {pointed spaces / CW-cpxes
ie. spaces w/ a distinguished point}.
mor: {cts maps / CW-maps that preserve basepts}.

$$(X, x_0) \xrightarrow{f} (Y, y_0)$$

bpt

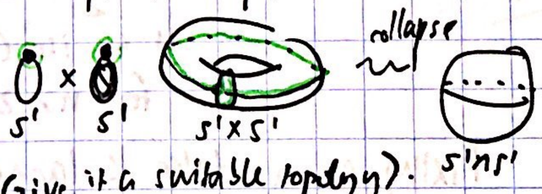
$$x_0 \longmapsto f(x_0) = y_0.$$

Given ptd spaces, we can "multiply them" via a product operation called the ~~smash product~~ ∧

$$X \wedge Y := (X \times Y)/(X \vee Y)$$

cartesian

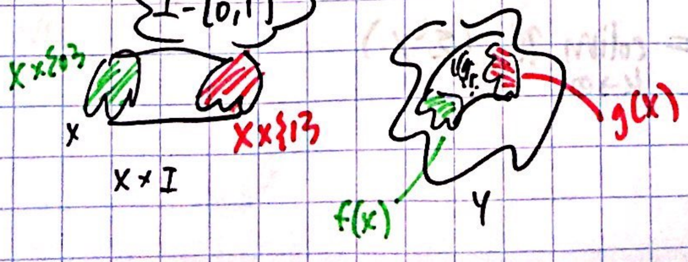
stick X & Y together along their basepoints. (give it a suitable topology).



Homotopy. Given maps $f, g : X \rightrightarrows Y$ in Top/CW, we say that a homotopy between them is a map

$$H : X \times I \longrightarrow Y \qquad \text{s.t. } H(-,0) = f, \quad H(-,1) = g$$

$I = [0,1]$



When two maps are homotopic, we write $f \sim g$.

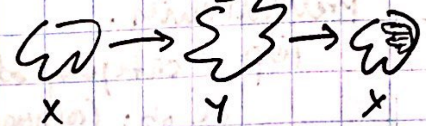
This gives us a more flexible notion of when two things are "equivalent".

In general, the notion of isomorphism in Top/CW is a homeomorphism $(X \xrightarrow{f} Y)$ s.t. $\exists f^{-1}$ w/

{"equivalence" in cat. language}

$f^{-1}f = id_X$

$ff^{-1} = id_Y$.



There's a weaker version called a homotopy equivalence.

We say f is a homotopy equivalence if $\exists f^{-1}$ s.t.

$f^{-1}f \sim id_X$

$ff^{-1} \sim id_Y$.

Given a notion of homotopy equivalence

We form a "homotopy category"

$[X]=[Y]$ iff $X \sim Y$.

eg. Top $\longrightarrow$ hoTop — ob = {equivalence classes of objects up to homotopy}. mor = {cts maps}$/\underset{htpy}{\sim}$ ← In this category, weak eq's are the isomorphisms.

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(1) Stability/motivation.

(i) Freudenthal's Suspension thm..

$\pi_i(X) = 0 \quad (i<k)$

$[S^i, X] = \mathrm{Hom}_{Top_*}(S^i, X)/\underset{htpy}{\sim}$

Thm: $X \in Top_*$ s.t. X is k-connected

then

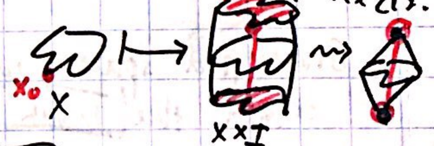
$[S^n, X]_* \xrightarrow{\Sigma} [\Sigma S^n, \Sigma X]$

$\pi_n(X) \xrightarrow{\Sigma} \pi_{n+1}(\Sigma X) = [S^{n+1}, \Sigma X]$

is an iso. (of gps) for $n \geq 2k$.

$\Sigma : Top_* \to Top_*$

$X \longmapsto \Sigma X = X \times I / X\times\{0\} \cup X \times \{1\}$ (" $S^1 \wedge -$ ")



Fixing an n, take k large... $\Sigma^k X$ turns out to be k-connected & $(n+k) \leq 2k$

$$\pi_{n+k}(\Sigma^k X) \cong \pi_{n+k+1}(\Sigma^{k+1}X) \cong \pi_{n+k+2}(\Sigma^{k+2}X) \cong \cdots$$

ie. they are the same after some point. This is "stability"

Def. (stable homotopy groups).

$$\pi_n^{stable}(X) := \operatorname*{colim}_{k\to\infty} \pi_{n+k}(\Sigma^k X)$$

(ii) Brown Representability. "co/homological functors are representable".

(2.)

A generalized co/homology theory is a functor (in CW cpxes)

$$H^*_\bullet : CW_*^{op} \longrightarrow GrAb$$

(strings of abelian gps).

satisfying... $X \longmapsto H^*X = (H^iX)_{i\in\mathbb{Z}}$.

(i) (stability): $H^i(X) \cong H^{i+1}(\Sigma X)$

(ii) (htpy invariance): if $X \sim Y$, then $H^*(X) \cong H^*(Y)$ (as gr. abelian gps.)

(iii) (exactness): "a cofiber sequence" in CW, forms an exact seq. of gr. abelian groups.

(*) (iv) (additivity): $H(X\vee Y) \cong H^0(X)\oplus H^0(Y)$

(*) (v) (reduced/ordinary): $H^i(pt) \cong 0 \quad i>0$.

Such a functor $\bullet : CW_* \xrightarrow{H^\bullet} GrAb \xrightarrow{(i^{th}\text{ comp})} Ab$, with the composite H^i.

Brown rep: Given a co/homological functor, there's a representing space..
i.e. $\exists\ E^i_\bullet \in CW_*$ s.t. $H^i(X) \cong \mathrm{Hom}_{CW_*}(X, E^i)$

By stability $(H^i(X) = H^{i+1}(\Sigma X))$ ... $H^i(X) \cong \mathrm{Hom}_{CW_*}(X, E^i)$

$$H^{i+1}(\Sigma X) \cong \mathrm{Hom}_{CW_*}(\Sigma X, E^{i+1})$$

$$\cong \mathrm{Hom}_{CW}(X, \Omega E^{i+1})$$

$\Sigma : CW_* \rightleftarrows CW_* : \Omega$ adj.

$\Omega : X \mapsto \Omega X$ (cups.) $= [S^1, X]$. "compact open".

This holds for any $X \in CW_*$,
so by Yoneda, $E^i \cong \Omega E^{i+1} \cong \Omega\Omega E^{i+2} \cong \ldots$

So given a co/homological functor, their rep. spaces are somehow related... So we want some notion that looks like all the groups "at once". related via. Σ, Ω as above...
This is what we want spectra to be...
So what should a category of spectra involve??.
Lewis-"Is there a convenient category of Spectra?"
↳ A: NO.

(2) Axioms.

Some things we might want from Spectra..

(1) There should be an adjunction $\Sigma^\infty : Top_* \rightleftarrows (Spectra) : \Omega^\infty$

(2) Spectra should have a symm. monoidal $\otimes = \wedge$
& $\Sigma^\infty(S^0) = \mathbb{S}$ should be the unit.

(3) Should play nice w/ Σ_{Top_*} ..

(4)

⋮

Margolis: "Spectra & Steenrod Algebra".

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(3) Definitions.

Def. A $\boxed{\text{(pre)spectrum}}$ is a sequence of spaces $\{X_n\}_{n\in\mathbb{N}}$ — sometimes $\mathbb{Z}$.
$X_n \in Top_*$
$\{(X_0, X_1, X_2 \dots) \; \Sigma X_0 \to X_1 \; \Sigma X_1 \to X_2 \dots\}$

w/ maps $\{\sigma_i : \Sigma X_i \longrightarrow X_{i+1}\}_{i\in\mathbb{N}}$.

Maps between Spectra $X \longrightarrow Y$
$X = (X_i)_i$, $Y = (Y_i)_i$

are a bunch of maps $\{f_i : X_i \longrightarrow Y_i\}_{i\in\mathbb{N}}$ (of top. spaces)

s.t. the f_i play nice w/ σ_i^X, σ_i^Y

ie.
$$\begin{tikzcd} \Sigma X_i \arrow[r, "\Sigma f_i"] \arrow[d, "\sigma^X"'] \arrow[dr, phantom, "="] & \Sigma Y_i \arrow[d, "\sigma^Y"] \\ X_{i+1} \arrow[r, "f_{i+1}"'] & Y_{i+1} \end{tikzcd}$$
commute.

Note that maps $(\sigma_i : \Sigma X_i \longrightarrow X_{i+1})$ give rise (via adjunction $\Sigma \dashv \Omega$) to maps $(\tilde{\sigma}_i : X_i \longrightarrow \Omega X_{i+1})$

If $\tilde{\sigma}_i^X$'s are weak equivalences, then we call X a $\boxed{\Omega\text{-(pre)spectrum}}$

These are nice versions... eg. $\pi_n(X_i) = [S^n, X_i]$
$$\xrightarrow{\sim} [S^n, \Omega X_{i+1}]$$
$$\cong [\Sigma S^n, X_{i+1}]$$
$$= [S^{n+1}, X_{i+1}]$$
$$\cong \pi_{n+1}(X_{i+1})$$

& $X_0 \simeq \Omega X_1 \simeq \Omega^2 X_2 \simeq \dots$

ie X_0 is an $\boxed{\infty\text{-loop-space}}$

Def. If all the X_i are CW-cpxes we call X a $\boxed{\text{CW-spectrum}}$. — Adams

Def. A $\boxed{\text{spectrum}}$ (May) is a prespectrum s.t. the maps $\sigma_i : \Sigma X_i \xrightarrow{\approx} X_{i+1}$ are homeo's (!)

(4) Examples.

e.g. For any top. space, $X \in Top_*$, there's a "suspension spectrum"

$$\Sigma^\infty X = ((\Sigma^\infty X)_i := \Sigma^i X)$$

$$\therefore \Sigma(\Sigma^\infty X)_i = \Sigma(\Sigma^i X) = \Sigma^{i+1} X \xrightarrow{id} \Sigma^{i+1} X = (\Sigma^\infty X)_{i+1}.$$

e.g. $\Sigma^\infty S^0 = \mathbb{S}$, the sphere spectrum.

$\Sigma^\infty : Top \rightleftarrows Spectra : \Omega^\infty$

$X_0 \leftarrow (X_i)_i$

e.g. (Eilenberg-Maclane Spectra)
For any abelian gp $A \in Ab$,

$K(A,n)$

E-M space
$\pi_i K(A,n) = \begin{cases} A & i = n \\ 0 & i \neq n \end{cases}$

$HA := (HA_n = K(A,n))_{n \in \mathbb{N}}$

rep. ordinary coho/homology

e.g. $A := \mathbb{Z}$.

e.g. Thom spectra.. There's a whole world.

$$MO = (MO(n) = T(\gamma^n))$$

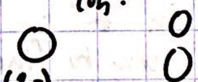
universal bundle $\gamma^n \to BO(n)$

"Thom constr."

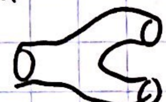
represents cobordism -

Two n-mflds N, N' are cobordant if $\exists$ (n+1)-mfld M s.t. $\partial M = N \cup N'$

e.g. $S^1 \underset{coh.}{\sim} S^1 \cup S^1$



by the (2-)manifold



$\Omega^O_n = \{n\text{-mflds}\} / \text{cobordisms}.$

$\pi^{st.}_n MO = \Omega^O_n.$

MU - complex version.

MG - any top. gp. G.

e.g. K-theory. KU, KO, ...

e.g. TMF. Represents $TMF^* : \{p\} \to Sets$
$p \mapsto TMF(p)$

"I love you, Patricia ♥"

Further reading... (lots of examples.)

- Strickland - An Introduction to the Category of Spectra"
- Beaudry, Campbell - A Guide for computing stable homotopy groups. (good concise intuitive intro.)
- Greenlees - Spectra for Commutative Algebraists.. (for algebraists)
- Malkiewich - "The Stable Homotopy Category"
- Gregoric - "Spectra Are Your Friends." (beware, uses ∞-categories!)