Department of Mathematics, UCSC

Geometry of Conformal Manifolds

Preserving Geodesics

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Outline

I. INTRODUCTION

- **A *conformal manifold*** $(M, [g])$: a smooth manifold M and an equivalence class of Riemannian metrics

$$[g] = \{\hat{g} \,|\, \hat{g} = f \cdot g \text{ where } f \in C^\infty(M) \text{ and } f > 0\}.$$

- **Model case, homogeneous space**

$$H \underset{\text{closed}}{\subset} G, \qquad \begin{array}{cc} (G, w) & \textit{Automorphism group} \\ \downarrow & \\ G/H & \textit{model} \end{array}$$

Riem. mfld $\quad G = isometries$

$$\begin{array}{c} (\mathbb{R}^n \rtimes O(n), w_{Euc}) \\ \downarrow \\ \mathbb{R}^n \end{array}$$

Conf. mlfd $\quad G = conf.\ autos.$

$$\begin{array}{c} (PO(n+1,1), w_{PO}) \\ \downarrow \\ S^n \cong PO/P_{\text{ray}} \end{array}$$

- Study of $(M, [g])$ (or (M, g)) $\quad \Leftrightarrow \quad$ Study of principal H-bundle with a $\mathfrak{g}$-valued form w, called Cartan geometry.

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$$\frac{O(n+1,1)}{\pm \mathbb{1}} = \begin{array}{c} (PO(n+1,1), w_{PO}) \\ \downarrow \\ S^n \cong PO/P_{\text{ray}} \end{array}$$

Categ. eq.

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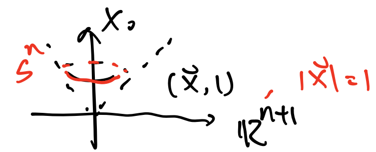
Basic Geometry on the unit sphere

0. **Embedding** $S^n \hookrightarrow \mathbb{R}^{n+1,1}$
1. **Ambient space** $\mathbb{R}^{n+1,1} \supset S^n$:
   2-dim higher Ricci flat Lorentz manifold containing S^n.
2. **Poincaré-Einstein space** $\partial B^{n+1} = S^n$:
   1-dim higher hyperbolic Einstein $H^{n+1} \overset{isom.}{=} B^{n+1}$ with boundary $\partial B^{n+1} = S^n$.
3. $(PO(n+1,1), w_{PO}) \to S^n$.
4. $(T\mathbb{R}^{n+1,1}|_{S^n}, \nabla^{\mathbb{R}^{n+1,1}}, g_{Min}, T^1_{null}) \to S^n$.

Remark:
• Motivated by 1 and 2, every $(M^{n\geq 2}, [g])$ has a unique asymptotically 2-dim higher Ricci flat Lorentz manifold containing it (and similar case for PE). [Fefferman and Graham 1985]
• The 4th case is also an associated vector bundle of Cartan bundle, called *tractor bundle*.

Basic Geometry on the unit sphere



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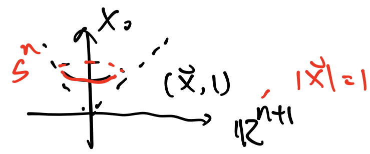

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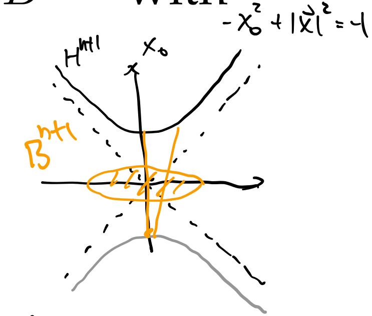

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II. NORMAL CARTAN GEOMETRY

Definition (Principal bundle & Associated bundle)

A fiber bundle (P, π, M, H) is called a *principal H-bundle* if P has a smooth H-right action that is fiber preserving and freely transitively on each fiber.

Given a smooth manifold S with a smooth H-left-action.
Define $P \times_H S = (P \times S)/\sim$ where the equivalence relation is
$[p, s] = [p \cdot h, h^{-1} \cdot s], \quad [p, s] \in P \times_H S.$
The fiber bundle $(P \times_H S, \pi_S, M, S)$ is called an *associated bundle* with fiber S.

Example

- *Frame bundle of a vector bundle* $E^r \to M$ *with rank* r
  - $\mathrm{Fr}(E^r) = \bigsqcup_{x \in M} \mathrm{Fr}(E_x) \to M$ is a principal $GL(r, \mathbb{R})$-bundle, where $\mathrm{Fr}(E_x)$ is the collection of bases of E_x;
  - $\mathrm{Fr}_{O(r)}(E^r) \to M$ is a principal $O(r)$-bundle if E has a metric. It's called *orthonormal frame bundle*.
- *Tangent bundle* $TM \cong \mathrm{Fr}(M^n) \times_{GL(n)} \mathbb{R}^n$, $\mathrm{Fr}(M) \triangleq \mathrm{Fr}(TM)$.

Given a principal H-bundle $P \xrightarrow{\pi} M$, and $\mathfrak{h}$ the Lie algebra of H

Definition

- (*Fundamental Vector Field*):

  For all $A \in \mathfrak{h}$, $\qquad \zeta_A(p) = \frac{d}{dt}\Big|_{t=0} p \cdot e^{tA} \in T_pP.$
- (*Vertical Subbundle*):
  Denote $\ker d\pi = VP$. Note that $\ker d\pi_p \cong \mathfrak{h} \quad \forall p \in P.$

Definition (Principal connection)

P has a principal connection if there is a smooth horizontal distribution HP such that

1. (*Decompostion*) $TP = VP \oplus HP$
2. (*Right invariant*) $dR_h(H_pP) = H_{ph}P$

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$p \cdot e^{tA}$

p

$\pi(p)$

$$\begin{array}{c} P \\ \downarrow \pi \\ M \end{array}$$

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$$P \xrightarrow{R_h} P$$
$$p \longmapsto p.h$$

Equivalently, there is a smooth $\mathfrak{h}$-valued 1-form γ on P satisfying the followings

1. (*Generate* $\mathfrak{h}$) $\gamma_p(\zeta_A(p)) = A$ for all $A \in \mathfrak{h}$
2. (*G-equivariance*) $R_h^* \gamma = \mathrm{Ad}(h^{-1})\gamma$ for all $h \in H$

The relation is $HP = \ker \gamma$.
Note that $V_pP = \ker(d\pi_p) \simeq \mathfrak{h}$ and $H_pP \simeq T_{\pi(p)}M$.

Definition (Curvature)

Let γ be a principal connection on a principal H-bundle $P \xrightarrow{\pi} M$. Define the curvature form $\Omega \in \Omega^2(P, \mathfrak{h})$

$$\Omega(X, Y) = d\gamma(X, Y) + [\gamma(X), \gamma(Y)] \quad X, Y \in TP.$$

In fact, $\Omega \in \Omega^2_H(P, \mathfrak{h})$.

Principal Bundle

[Andreas Čap and Jan Slovák 2009 Cor. 1.2.7]

Given a principal H-bundle $P \xrightarrow{\pi} M$ and a representation $H \xrightarrow{\rho} GL(V)$ with $E = P \times_H V$.

Let

- $\Omega^k(M, E) = \Gamma\left(\left(\Lambda^k T^* M\right) \otimes E\right)$.
- $\Omega^k_H(P, V)$ be V-valued k-forms on P s.t. they are H-equivariant and horizontal.

There is a bijective relation $\Omega^k(M, E) \simeq \Omega^k_H(P, V)$ by

$$\alpha_{\pi(p)}(\xi_1, \ldots, \xi_k) = [p, \tilde{\alpha}_p(\tilde{\xi}_1, \ldots, \tilde{\xi}_k)],$$

where $\alpha \in \Omega^k(M, E)$, $\tilde{\alpha} \in \Omega^k_H(P, V)$ and $d\pi(\tilde{\xi}_j) = \xi_j$.

Let $P \xrightarrow{\pi} M$ be a principal H-bundle with a connection γ and $\rho\colon H \to GL(V)$ be a representation where $d\rho\colon \mathfrak{h} \to \mathfrak{gl}(V)$.

Definition (Covariant derivative on a principal bundle)

The *covariant derivative* on $\Omega^k_H(P,V)$ is

$$D\phi \triangleq (d\phi)^{hor}, \quad \forall \phi \in \Omega^k_H(P,V)$$

where $(d\phi)^{hor}(X_1,\ldots,X_k) = (d\phi)(X_1^{hor},\ldots,X_k^{hor})$ for $X_i \in TP$.

Proposition (Tu 2017 Prop. 31.16)

The covariant derivative D preserves equivariancy and horizontality,

$$D\colon \Omega^k_H(P,V) \to \Omega^{k+1}_H(P,V).$$

$(P, \gamma) \xrightarrow{\pi} M, \ \rho\colon H \to \mathrm{GL}(V), \ E = P \times_H V$

Proposition (Covariant derivative formula [Tu 2017 Thm. 31.19])

By the bijective relation $\Gamma(M, P \times_H V) \simeq \Gamma_H(P, V)$, the covariant derivative D induces a linear connection ∇ on $E = P \times_H V$, $\quad \nabla\colon \Gamma(M, E) \to \Omega(M, E)$.

Corollary

Let ξ be a section of E with $\xi(\pi(p)) = [p, \tilde{\xi}(p)]$. Then,

$$\begin{aligned}(R(X, Y)\xi)(\pi(p)) &\triangleq (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\xi(\pi(p)) \\ &= [p, \Omega(\tilde{X}^{hor}, \tilde{Y}^{hor}) \cdot \tilde{\xi}(p)], \ \text{where}\end{aligned}$$

vector fields $X, Y \in \Gamma(TM)$.

Principal Bundle

Definition (Category of principal H-bundle)

- *Objects*: Principal H-bundles (P, π, M, H)
- *Morphisms*: $(P, \pi, M) \xrightarrow{\phi} (P', \pi', M')$ where $\phi \colon P \to P'$ is fiberwise with $\phi(p \cdot h) = \phi(p) \cdot h, \quad p \in P$ and $h \in H$.

Let $H \subset G$ be a Lie subgroup of a Lie group G with the corresponding Lie algebras $\mathfrak{h} \subset \mathfrak{g}$.

Definition (Andreas Čap and Jan Slovák 2009)

Given a principal H-bundle $P_H \to M$. It is called *Cartan geometry of type* (G, H, w) where $w \in \Omega^1(P_H, \mathfrak{g})$ satisfies

- (*Generate* $\mathfrak{h}$) $w(\zeta_A(u)) = A, \quad \forall A \in \mathfrak{h}$;
- (*G-equivariant*) $R_h^* w = \mathrm{Ad}(h^{-1})\, w, \quad \forall h \in H$;
- (*Linear isomorphism*) $w_p : T_p P_H \xrightarrow{\sim} \mathfrak{g} \quad \forall p \in P_H$.

Remark:
Assume $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ and decompose w into $w = (w)_{\mathfrak{n}} + (w)_{\mathfrak{h}}$.
Then, $(w)_{\mathfrak{h}}$ is a principal connection iff $\mathfrak{n}$ is H-stable under the Adjoint action.

Given a Cartan geometry of type (G, H, w), $P_H \xrightarrow{\pi} M$.
Note that $\mathfrak{h} \cong \ker d\pi_p \Rightarrow \dim \mathfrak{g}/\mathfrak{h} = \dim T_xM$. In fact,

Proposition (TM is the associated bundle of $\mathfrak{g}/\mathfrak{h}$)

There is an isomorphism for TM by the Cartan connection w.

$$P_H \times_{\mathrm{Ad}} (\mathfrak{g}/\mathfrak{h}) \simeq TM.$$

[Andreas Čap and Jan Slovák 2009]

Definition (Curvature and torsion (G, H, w), $P_H \to M$)

- *Cartan curvature*: $K \in \Omega^2_H(P_H, \mathfrak{g})$,

$$K(\xi, \eta) = dw(\xi, \eta) + [w(\xi), w(\eta)].$$

- *Curvature function*: $\kappa \in \Gamma_H(P_H, \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g})$,

$$\kappa(X, Y) = K(w^{-1}(X), w^{-1}(Y)), \quad \forall X, Y \in \mathfrak{g}/\mathfrak{h}.$$

- *Torsion*:

$$\tau = \pi_{\mathfrak{g}/\mathfrak{h}} \circ \kappa \in \Gamma_H(P_H, \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes (\mathfrak{g}/\mathfrak{h})),$$

where $\pi_{\mathfrak{g}/\mathfrak{h}} : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical projection.
The torsion function τ defines $T \in \Omega^2(M, TM)$.
If τ vanishes, it's called *torsion free*.

Example (Homogeneous model for Riemannian manifold)

- *Euclidean space*

$$\mathbb{R}^n \rtimes O(n) \to \mathbb{R}^n \rtimes O(n)/O(n) \simeq \mathbb{R}^n$$

is the Cartan geometry of the type $(\mathbb{R}^n \rtimes O(n), O(n), w_M)$ where w_M is the Maurer-Cartan form of the Euclidean group

$$\mathbb{R}^n \rtimes O(n) = \left\{ \begin{pmatrix} 1 & 0 \\ \vec{v} & A \end{pmatrix} \middle| \, \vec{v} \in \mathbb{R}^n \text{ and } A \in O(n) \right\}.$$

Example (Homogeneous model for conformal manifold)

- *Sphere*

$$O_+(n+1,1) \to O_+(n+1,1)/P_{ray} \simeq S^n$$

is the Cartan geometry of the type $(O_+(n+1,1), P_{ray}, w_M)$, where $O_+(n+1,1)$ is the principal P_{ray}-bundle and w_M is the Maurer-Cartan form of $O_+(n+1,1)$.

Cartan Geometry

Example (Homogeneous model for conformal manifold)

- *Sphere*

$PO(n+1,1)$

$\cong$

$$O_+(n+1,1) \to O_+(n+1,1)/P_{ray} \simeq S^n$$

time-preserving

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Definition (Category)

Let $H \subset G$ be a Lie subgroup of a Lie group G. The category of the Cartan geometry of the type (G, H) is defined by

- *Objects*: (P, π, M, w), which is a principal H-bundle with a Cartan connection w.
- *Morphisms*: $(P, \pi, M, w) \xrightarrow{\phi} (P', \pi', M', w')$ where ϕ is a principal bundle morphism with the property

$$\phi^* w' = w.$$

Given a Riemannian manifold (M, g). Motivated by the homogeneous example

$$(\mathbb{R}^n \rtimes O(n), w_M) \to \mathbb{R}^n,$$

there is actually the theorem

Theorem (Andreas Čap and Jan Slovák 2009 Thm. 1.6.1)

The category of principal $O(n)$-subbundles in $Fr(M)$ is equivalent to the category of torsion-free Cartan geometries of the type $(\mathbb{R}^n \rtimes O(n), O(n))$.

- **Construction of Cartan Geometry** $(\mathbb{R}^n \rtimes O(n), O(n), w)$

Proposition (Andreas Čap and Jan Slovák 2009)

The followings are equivalent

1. (M^n, g).
2. $P_{O(n)} \overset{i}{\hookrightarrow} \mathrm{Fr}(M)$.
3. $P_{O(n)}$ has a form $\theta \in \Omega^1_{O(n)}(P_{O(n)}, \mathbb{R}^n)$ which is strictly horizontal, called *soldering form*.

Riemannian Manifold- Curvature

Let $P_{O(n)}$ be a Cartan geometry in type $(\mathbb{R}^n \rtimes O(n), O(n), w)$.

• $\mathbb{R}^n \subset \mathbb{R}^n \oplus \mathfrak{o}(n)$ is $O(n)$-stable $\Rightarrow w = (w)_{\mathbb{R}^n} + (w)_{\mathfrak{o}(n)} = \theta + \gamma$.
• The Cartan curvature form K is decomposed to:

$$K_{\mathbb{R}^n}(\tilde{\xi}, \tilde{\eta}) = \tilde{\xi}(\theta(\tilde{\eta})) + \gamma(\tilde{\xi})\theta(\tilde{\eta}) - [\tilde{\eta}(\theta(\tilde{\xi})) + \gamma(\tilde{\eta})\theta(\tilde{\xi})] - \theta([\tilde{\xi}, \tilde{\eta}])$$
$$K_{\mathfrak{o}(n)}(\tilde{\xi}, \tilde{\eta}) = d\gamma(\tilde{\xi}, \tilde{\eta}) + [\gamma(\tilde{\xi}), \gamma(\tilde{\eta})].$$

In fact, there is the theorem

Theorem (Andreas Čap and Jan Slovák 2009 Example 1.6.1)

Cartan connections w on $Fr_{O(n)}(M)$ of the type $(R^n \rtimes O(n), O(n))$ are uniquely defined by their torsions. In particular, there is the unique torsion-free Cartan connection.

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$$K_{\mathfrak{o}(n)}(\tilde{\xi}, \tilde{\eta}) = d\gamma(\tilde{\xi}, \tilde{\eta}) + [\gamma(\tilde{\xi}), \gamma(\tilde{\eta})].$$ $\Longleftrightarrow$

$\Omega^1_{O(n)}(P, \mathfrak{o}(n)) \hat{=} \Omega^1(M, \mathrm{End}(TM))$

$R(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - \nabla_{[\xi, \eta]}$

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$T(\xi,\eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi,\eta]$

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Conformal Manifold in Cartan Geometry

For conformal manifolds

$(\mathbb{R}^n \rtimes O(n), O(n))$

$\wr$

- $(M, [g]) \Leftrightarrow (P_{G_0} \hookrightarrow \mathrm{Fr}M) \Leftrightarrow (P_{G_0}, \theta)$.
- Lots of torsion-free Cartan connections in $(\mathbb{R}^n \rtimes G_0, G_0)$, called Weyl connections.
- Need a "normal condition" to get the unique Cartan geometry for a conformal manifold.

The homogeneous model

$$(O_+(n+1,1), w_M) \to O_+(n+1,1)/P_{ray} \simeq S^n$$

tells for $(M^n, [g])$, $n \geq 3$

Theorem (Andreas Čap and Jan Slovák 2009 Thm. 1.6.7)

The category of principal G_0 subbundles in $Fr(M)$ is equivalent to the category of normal Cartan geometries of the type $(O_+(n+1,1), P_{ray})$.

- $G_0 = CO(n) \subset GL(n, \mathbb{R})$

$$G_0 = \{A \in GL(n) | \, \exists \lambda > 0 \text{ s.t. } \langle Av, Aw \rangle = \lambda \langle v, w \rangle \; \forall v, w \in \mathbb{R}^n\}.$$

- Lie algebra $\mathfrak{o}(n+1,1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$

Choose a basis of $\mathbb{R}^{n+1,1}$ such that its inner product is of the form

$$\begin{pmatrix} & & 1 \\ & \mathbb{I}_n & \\ 1 & & \end{pmatrix}.$$

Then, $\mathfrak{o}(n+1,1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where

stab. of a null ray ?

$$\mathfrak{g}_{-1} \simeq \mathbb{R}^n, \; \mathfrak{g}_0 \simeq \mathfrak{co}(n) = \mathbb{R} \oplus \mathfrak{o}(n), \; \mathfrak{g}_1 \simeq \mathbb{R}^{n^*}.$$

- $P_{ray} = G_0 \ltimes e^{\mathfrak{g}_1} = \{g_0 \, e^Z | \, g_0 \in G_0, \; Z \in \mathfrak{g}_1\}$.

$\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$

Given a Cartan geometry of the type $(O_+(n+1,1), P_{ray}, w)$, $\mathcal{G} \to M^n$ with $n \geq 3$.

Definition

The Cartan connection is called *normal* on $\mathcal{G}$ if and only if

- it is torsion free;
- the $\mathfrak{g}_0$-valued curvature function $\kappa_0 : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}_0$ satisfies $(\kappa_0)_{ki}{}^{k}{}_{j} = 0$, where $\kappa_0(\hat{e}_i, \hat{e}_j) \cdot \hat{e}_k = (\kappa_0)_{ij}{}^{l}{}_{k} \hat{e}_l$ and the action $\cdot$ is the commutator between $\mathfrak{g}_0$ and $\mathfrak{g}_1$.

Weyl curv.

Construction of the normal geometry $(O_+(n+1,1), P_{ray}, w)$

• $\mathcal{G} = \mathrm{Fr}_{G_0} M \times_{G_0} P_{ray}$.

• Given a Weyl connection σ on $\mathrm{Fr}_{G_0} M$, then $\exists!$ w^σ on $\mathcal{G}$ and $\exists!$ $P^\sigma \in \Gamma_{P_{ray}}(\mathcal{G}, \mathfrak{g}_1)$ s.t.

$$w_{nor} = w^\sigma - P^\sigma.$$

Conformal Manifold- Weyl connections

- The space of Weyl connections forms an affine space over $\Gamma_{G_0}(\mathrm{Fr}_{G_0}, \mathfrak{g}_1)$ (or $\Omega^1(M)$).
- $\{\text{Weyl connection}\} \cong \Gamma_{G_0}(\mathrm{Fr}_{G_0}(M), \mathcal{G})$ $\quad \mathrm{Fr}_{G_0} M \to \mathcal{G}$
- All Levi-Civita connections w.r.t. $g \in [g]$ are Weyl connections.
- For any Weyl connection σ on $\mathrm{Fr}_{G_0}(M)$, recall the relation on $\mathcal{G}$

$$w_{nor} = w^{\sigma} - P^{\sigma}.$$

The $P^{\sigma} \in \Gamma_{P_{ray}}(\mathcal{G}, \mathfrak{g}_1)$ represents Schouten tensor w.r.t. σ.

Conformal Manifold- Curvature

Given a Levi-Civita connection σ on $\mathrm{Fr}_{G_0} M$
Recall $\mathfrak{o}(n+1,1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

- $w = w^\sigma - P$ on $\mathcal{G}$.
- $\kappa = 0 + \kappa_0 + \kappa_1$, where $\kappa_0 = \kappa_0^\sigma - \partial P$.

By $\Omega^2_{P_{ray}}(\mathcal{G}, \mathfrak{p}/\mathfrak{g}_1) \simeq \Omega^2(M, \mathrm{End}_0(TM))$,

- κ_0 represents Weyl tensor,
- κ_0^σ represents Riemann curvature tensor.

Note that κ_1 is not P_{ray}-equivariant, but it's G_0-equivariant. It represents Cotton-York tensor $C_{ijk} = 2\nabla_{[i}P_{j]k}$ by pulling back to $\mathrm{Fr}_{G_0}(M)$ along $j_\sigma \in \Gamma_{G_0}(\mathrm{Fr}_{G_0}(M), \mathcal{G})$ due to the relation $\{\text{Weyl connection}\} \cong \Gamma_{G_0}(\mathrm{Fr}_{G_0}, \mathcal{G})$.

Remark

- With respect to the decomposition
  $\mathfrak{o}(n+1,1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathfrak{g}_{-1} \oplus \mathfrak{p}$,
  the $\mathfrak{p}$-valued for Cartan connection w on $\mathcal{G}$, $(w)_{\mathfrak{p}}$, is not a principal P_{ray}-connection since $\mathfrak{g}_{-1}$ isn't P_{ray}-invariant.
- The local coordinate of $\mathcal{G}$ can be thought as

$$U \overset{(e_i)}{\to} \mathrm{Fr}_{G_0}(M) \overset{\nabla}{\to} \mathcal{G}.$$

$\{\text{Weyl}\} \simeq \{\mathrm{Fr}_{G_0} \to \mathcal{G}\}$

The local frame (e_i) is an orthonormal frame w.r.t. $g \in [g]$ and ∇ is a Weyl connection.

Conformal Manifold- Tractor Bundle

$(M^{n\geq 3}, [g])$ with the normal Cartan geometry $\mathcal{G} \to M$ of the type $(O_+(n+1,1), P_{ray}, w)$.

Pray - principal bundle

Definition

Given a finite dimensional representation

$$O_+(n+1,1) \xrightarrow{\rho} GL(V).$$

The *tractor bundle* associated to V is the associated bundle

$$\mathcal{V} = \mathcal{G} \times_{P_{ray}} V.$$

Remark: There is a general definition replacing the above representation, called $(\mathfrak{g}, P)$-module. [A. Čap and A. Rod Gover Apr. 2002]

Representation & Lie algebra

• $\mathfrak{o}(n+1,1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$

• Want the representation $O_+(n+1,1) \xrightarrow{\rho} GL(V)$ to keep the structure by $d\rho$ action.

Proposition

- $\exists! E \in \mathfrak{g}_0$ such that $[E, A_j] = jA_j \quad \forall A_j \in \mathfrak{g}_j$.
  [Čap and J. Slovák 2009 Prop. 3.1.2]
- Assume V has the eigen space decomposition for E, $V = \bigoplus_i V_i$. Then,

$$\mathfrak{g}_j \cdot V_i \subset V_{i+j}.$$

The above relation will be applied to the curvature of the linear connection on $\mathcal{V} = \mathcal{G} \times_{P_{ray}} V$ which is induced by w_{nor}.

Example

- $V = \mathfrak{o}(n+1,1)$, $V_i = \mathfrak{g}_i$.
- $V = \mathbb{R}^{n+1,1}$. Let (e_+, e_i, e_-) be a basis which makes the inner product of the form

$$\begin{pmatrix} & & 1 \\ & \mathbb{I} & \\ 1 & & \end{pmatrix}.$$

Then, $V_1 = \operatorname{span} e_+$, $V_0 = \operatorname{span}_i(e_i)$, $V_{-1} = \operatorname{span} e_-$.

• There is an inner product on $\mathcal{G} \times_{P_{ray}} \mathbb{R}^{n+1,1}$ induced from the Minkowski inner product on $\mathbb{R}^{n+1,1}$.

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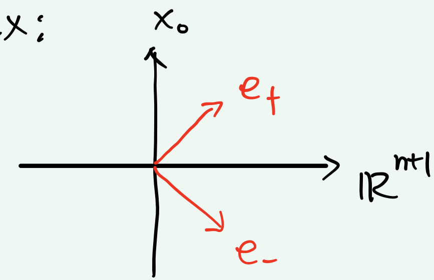

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• There is an inner product on $\mathcal{G} \times_{P_{ray}} \mathbb{R}^{n+1,1}$ induced from the Minkowski inner product on $\mathbb{R}^{n+1,1}$.

Tractor connection and coordinate description

- $\hat{\mathcal{G}} = \mathcal{G} \times_{P_{ray}} O_+(n+1,1)$, $\quad \mathcal{G} \xrightarrow{j} \hat{\mathcal{G}}, u \mapsto [u, id]$
- $V = \mathbb{R}^{n+1,1}$ and $\mathcal{V} = \mathcal{G} \times_{P_{ray}} V \cong \hat{\mathcal{G}} \times_{O_+(n+1,1)} V$.

Proposition (Type $(O_+(n+1,1), P_{ray}, w)$)

- A (Normal) Cartan connection w on $\mathcal{G}$ induces a unique principal connection γ^w on $\hat{\mathcal{G}}$ s.t. $j^*\gamma^w = w$.
- γ^w induces $\nabla^{\mathcal{V}}$ on $\mathcal{V}$, called *tractor connection*.

[Andreas Čap and Jan Slovák 2009 Thm. 1.5.6]

Recall

- $\{\text{Weyl connection}\} \cong \Gamma_{G_0}(\mathrm{Fr}_{G_0} M, \mathcal{G})$ $\qquad \mathrm{Fr}_{G_0} M \to \mathcal{G}$
- $\mathbb{R}^{n+1,1} \cong \overset{e_+}{V_1} \oplus \overset{e_i}{V_0} \oplus \overset{e_-}{V_{-1}}$.
- $\mathcal{G} \times_{P_{ray}} V \overset{\sigma}{\cong} \mathrm{Fr}_{G_0} M \times_{G_0} V \cong \overset{e_+}{\mathcal{E}[-1]} \oplus \overset{e_i}{TM[-1]} \oplus \overset{e_-}{\mathcal{E}[1]}$.

For a given Levi-Civita connection σ w.r.t. a metric $g \in [g]$. The covariant derivative on $t = [t_1, t_0, t_{-1}] \in \Gamma(M, \mathcal{G} \times_{P_{ray}} \mathbb{R}^{n+1,1})$ is

$$\nabla_i^{\mathcal{V}} t(x) = \begin{pmatrix} \nabla_i^\sigma t_1 - P^\sigma_{ij}\, t_0^j \\ \nabla_i^\sigma t_0 + (t_1 \delta_i^k + t_{-1} P^{\sigma}{}_i{}^k) e_k \\ \nabla_i^\sigma t_{-1} - t_0^i \end{pmatrix},$$

where (e_i) is an orthonormal frame of g and $\nabla_i^\sigma t_{\pm 1} = e_i(t_{\pm 1})$.

$$\mathcal{V} = \mathcal{G} \times_{P_{ray}} \mathbb{R}^{n+1,1}$$

$K = (K)_{-1} + (K)_0 + \underbrace{(K)_1}$

$= 0 + \text{Weyl curv} + \text{Cotto York}$

For a Levi-Civita connection ∇, the curvature of the tractor connection is

$$\left(\nabla_i^{\mathcal{V}}\nabla_j^{\mathcal{V}} - \nabla_j^{\mathcal{V}}\nabla_i^{\mathcal{V}} - \nabla_{[i,j]}^{\mathcal{V}}\right)\begin{pmatrix}\alpha\\ \mu^k\\ \rho\end{pmatrix} = \begin{pmatrix}0 & -C_{ijl} & 0\\ 0 & W_{ij}{}^k{}_l & C_{ij}{}^k\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \mu^l\\ \rho\end{pmatrix}$$

where $C_{ijk} = 2\nabla_{[a}P_{b]c}$ is the Cotton-York tensor.

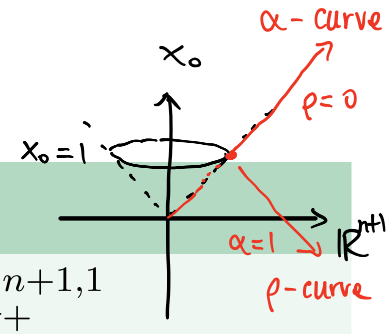
Example (Sphere $S^n \hookrightarrow \mathbb{R}^{n+1,1}$)

Use the coordinate transformation (α, Ω, ρ) on $\mathbb{R}^{n+1,1}_+$

$$\alpha = \frac{1}{2}(|\vec{x}| + x_0),\ \alpha\rho = (|\vec{x}| - x_0),\ \Omega \in S^n.$$

- The $\nabla^{\mathbb{R}^{n+1,1}}$ on $T\mathbb{R}^{n+1,1}|_{S^n}$ w.r.t. the above coordinate is same as tractor connection. The coordinate is called normal coordinate in [Fefferman and Graham Dec. 2011].
- The coordinate order (α, Ω, ρ) represents (V_1, V_0, V_{-1}).

Conformal Manifold- Tractor Bundle



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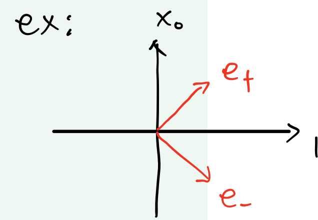

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- The coordinate order (α, Ω, ρ) represents (V_1, V_0, V_{-1}).

Conformal Manifold- Tractor Bundle

[A. Čap and A. Gover 2003]

Conversely, to recover a conformal structure from a rank $n+2$ vector bundle $\mathcal{V}$ in bijective relation, one needs $(\mathcal{V}, m, \mathcal{V}^1, \nabla)$, where

- m is a Minkowski metric on $\mathcal{V}$.
- $\mathcal{V}^1$ is a simple null line bundle in $\mathcal{V}$.
- ∇ is a metric connection on $\mathcal{V}$ w.r.t. m and satisfies
  1. (*Nondegenerate*) $\forall x \in M, \forall \xi \in TM, \exists \sigma \in \Gamma(\mathcal{V}^1)$ s.t. $\nabla_\xi \sigma \notin \mathcal{V}^1$.
  2. (*Torsion-free*) $R(\xi, \eta)\mathcal{V}^1 \subset \mathcal{V}^1$.
  3. (*Ricci-trace-free*) The Ricci-type contraction of $W \in \Gamma(\Lambda^2 T^*M \otimes L(TM, TM))$ vanishes, where W is induced from R.

III. GEODESIC

Given a (M^n, g) with the Levi-Civita connection ∇.

Definition

A curve γ in M is called a *Riemannian geodesic* if $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. The geodesic equation w.r.t. a local frame (e_i) is

$$\ddot{\gamma}^k + \dot{\gamma}^i\dot{\gamma}^j\Gamma^k_{ij} = 0 \quad , \nabla_{e_i}e_j = \Gamma^k_{ij}e_k.$$

Riemannian Geodesic

$\left(\mathrm{Fr}_{O(n)}(M^n) \xrightarrow{\pi} M\right)$ the torsion-free Cartan geometry of type $(\mathbb{R}^n \rtimes O(n), O(n), w)$, recall $w_p \colon T_p\mathrm{Fr}_{O(n)}(M) \to \mathbb{R}^n \oplus \mathfrak{o}(n)$.

Proposition

Consider a vector field $w^{-1}(\mathcal{X})$ on $\mathrm{Fr}_{O(n)}(M)$, where $\mathcal{X} = \mathcal{X}^i \hat{e}_i$ is constant in $\mathbb{R}^n \subset \mathbb{R}^n \oplus \mathfrak{o}(n)$, and any its integral curve $\tilde{\gamma} \subset \mathrm{Fr}_{O(n)}(M)$. Then, the projection $\gamma = \pi(\tilde{\gamma})$ is a Riemannian geodesic in M.

Conversely, any Riemannian geodesic $\gamma \colon I \to M$ is locally the projections of integral curves $\tilde{\gamma} \subset \mathrm{Fr}_{O(n)}(M)$ for some constant vector fields $w^{-1}(\mathcal{X})$.

Riemannian Geodesic

Integral curve equation for $w^{-1}(\mathcal{X})$ **on** $\mathrm{Fr}_{O(n)}(M)$
A curve $\tilde{\gamma} \subset \mathrm{Fr}_{O(n)}(M)$ locally is $(x(t), C(t))$, where $x(t) \in M$ and $C(t) \in O(n)$, then the integral curve equation is

$$\begin{cases} \dot{x}^i & = \mathcal{X}^j C^i{}_j \\ \dot{C}^i{}_j & = -\mathcal{X}^k C^l{}_k C^a{}_j \Gamma^i_{la} \end{cases}.$$

Therefore,

$$\begin{cases} \dot{x}^i & = \mathcal{X}^j C^i{}_j \\ \mathcal{X}^j \dot{C}^i{}_j & = \ddot{x}^i = -\mathcal{X}^k C^l{}_k \mathcal{X}^j C^a{}_j \Gamma^i_{la} = -\dot{x}^l \dot{x}^a \Gamma^i_{la}. \end{cases}$$

Given $(M^{n\geq 3}, [g])$, $\mathcal{G} = \mathrm{Fr}_{G_0}(M) \times_{G_0} P_{ray} \xrightarrow{\pi} M$

- $\mathcal{G}$ is of the normal type $(O_+(n+1,1), P_{ray}, w)$.
- $w_p \colon T_p\mathcal{G} \xrightarrow{\simeq} \mathfrak{g} = \mathfrak{g}_{-1} \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\text{vertical}}$,
  where $\mathfrak{g}_{-1} \simeq \mathbb{R}^n$, $\mathfrak{g}_0 \simeq \mathbb{R} \oplus \mathfrak{o}(n)$, $\mathfrak{g}_1 \simeq (\mathbb{R}^n)^*$.
- $P_{ray} = G_0 \rtimes e^{\mathfrak{g}_1}$

Consider integral curves $\tilde{\gamma}$ in $\mathcal{G}$ for vector fields $w^{-1}(\mathcal{X})$, $\mathcal{X} = \mathcal{X}^i \hat{e}_i \in \mathfrak{g}_{-1}$.

- Local section $U \xrightarrow{(e_i)} \mathrm{Fr}_{G_0}(M) \xrightarrow{\nabla} \mathcal{G}$,
  where (e_i) is an orthonormal frame of $g \in [g]$ and ∇ is g's Levi-Civita connection.
- A curve $\tilde{\gamma} \subset \mathcal{G}$ locally is $(x(t), C(t), b(t))$ where $C(t) \in G_0$ and $b(t)$ is considered as a one-form along $x(t)$.

Integral curve equation for $w^{-1}(\mathcal{X})$ on $\mathcal{G}$

[Friedrich and Schmidt Nov. 9, 1987]

$$\begin{cases} \dot{x}^i & = C^i{}_j \mathcal{X}^j; \\ \dot{C}^i{}_j & = -(\Gamma^i_{kl} + b^i_{kl}) C^k{}_n \mathcal{X}^n C^l{}_j, \quad b^j_{ki} = \delta^j_k b_i + \delta^j_i b_k - g_{ki} g^{jl} b_l; \\ \dot{b}_k & = (b_j \Gamma^j_{ik} + \frac{1}{2} b_j b^j_{ik} + P_{ik}) C^i{}_n \mathcal{X}^n, \end{cases}$$

The above equations can be expressed

$$\begin{cases} (\nabla_{\dot{x}} \dot{x})^i & = -b^i_{jk} \dot{x}^j \dot{x}^k; \\ (\nabla_{\dot{x}} c_j)^i & = -b^i_{lk} \dot{x}^l C^k{}_j, \quad c_j = e_k C^k{}_j; \\ (\nabla_{\dot{x}} b)_i & = (\frac{1}{2} b_j b^j_{ki} + P_{ki}) \dot{x}^k, \end{cases}$$

Since the 1st and the 3rd equations can determine the 2nd one, then define

Definition

Given $g \in [g]$ with its Levi-Civita connection. A *conformal geodesic* (or called *conformal circle*) is a curve $x(\tau)$ in M with a 1-form $b(\tau)$ along it such that

$$\begin{cases} (\nabla_{\dot{x}}\dot{x})^i & = -b^i_{jk}\dot{x}^j\dot{x}^k; \\ (\nabla_{\dot{x}}b)_i & = (\frac{1}{2}b_j b^j_{ki} + P_{ki})\dot{x}^k. \end{cases}$$

Equivalent definitions for conformal geodesics

[Bailey, Eastwood, and A. R. Gover 1994, Bailey and Eastwood Jan. 1990]

- $(\nabla_{\dot{x}}\dot{x})^i = -b^i_{jk}\dot{x}^j\dot{x}^k$, $(\nabla_{\dot{x}}b)_i = (\frac{1}{2}b_j b^j_{ki} + P_{ki})\dot{x}^k$ for $g \in [g]$.
- (x, b) is locally the projections of integral curves $\tilde{\gamma} \in \mathcal{G}$ for some constant vector fields $w^{-1}(\mathcal{X})$.
- There is a particular null tractor field X^I along a curve x s.t. its acceleration A^I is null and parallel along x.
- Let $V^i = \dot{x}^i$, $A^i = V^j \nabla_j V^i$ and $b_i = \frac{A_i}{V^2} - 2\frac{V\cdot A}{V^4}V_i$ w.r.t. the metric g. Then,

$$V^i \nabla_i A^j = \frac{3V\cdot A}{V^2}A^j - \frac{3A^2}{2V^2}V^j + V^2 V^i P_i{}^j - 2P_{ik}V^i V^k V^j.$$

The above equation is conformally invariant.

Conformal Geodesic / Conformal Circle

Example (Conformal geodesics in $\mathbb{R}^n$, Tod 2012)

In the unit length parametrization of a conformal geodesic $\vec{x} \subset \mathbb{R}^n$, its acceleration $\vec{a}$ is of constant length. So, conformal geodesics in $\mathbb{R}^n$ are classified to

- (*straight line*, $||\vec{a}|| = 0$): $\quad \vec{x}(s) = \vec{u}_0\, s + \vec{x}_0.$
- (*planar circle*, $||\vec{a}|| \neq 0$):

$$\vec{x}(s) - \vec{x}_0 - \frac{\vec{a}_0}{a^2} = \frac{\vec{u}_0}{||\vec{a}||} \sin(||\vec{a}||\, s) - \frac{\vec{a}_0}{||\vec{a}||^2} \cos(||\vec{a}||\, s).$$

It's the circle with radius $\frac{1}{||\vec{a}||}$ sitting on the plane spanned by $\vec{u}_0$ and $\vec{a}_0$.

- **Homogeneous model**[Andreas Čap and Jan Slovák 2009 1.1.6]

The homogeneous model of strictly pseudoconvex partially integrable almost CR-manifolds is the CR-sphere $S^{2n+1} \cong PSU(n+1,1)/P$ where $P \subset PSU(n+1,1)$ is the stabalizer of a null line. Similar to the real case, the corresponding Lie algebra is

$$\mathfrak{su}(n+1,1) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where $\mathfrak{g}_{-2} \cong \mathfrak{g}_2 \cong \mathbb{R}, \quad \mathfrak{g}_{-1} \cong \mathbb{C}^n$, $\mathfrak{g}_1 \cong (\mathbb{C}^n)^*$, $\mathfrak{g}_0 \cong \mathfrak{su}(n) \oplus \mathbb{C}$.

(The relation between CR manifold and Cartan geometry can be seen in [Andreas Čap and Jan Slovák 2009 4.2.4].)

- **Chern-Moser Chain**

By [A. Čap, J. Slovák, and Žádník 2004], those curves on a base manifold M corresponding to local projections of constant-$\mathfrak{g}_{-2}$ vector fields are called *Chern-Moser chains*.

IV. DIFFEOMORPHISM PRESERVING GEODESICS

Preserving Geodesics

Given a diffeomorphism $f\colon M \to M$

Q: If f preserves geodesics*, is f an automorphism?
$*$ In Riemmanian case (M, g), it means $\forall$ geodesics $\gamma \subset M$, f is defined to satisfy $\nabla_{df(\dot\gamma)} df(\dot\gamma) = 0 \quad \& \quad \nabla_{df^{-1}(\dot\gamma)} df^{-1}(\dot\gamma) = 0$.

Fact:

- f preserves Riemannian geodesics $\Leftrightarrow$ f preserves Levi-Civita connection: $df(\nabla_X Y) = \nabla_{df(X)} df(Y)$.
  [Corollary from the result in Spivak 1999 Chap. 6 Addendum 1]
- In nondegenerate CR case, if f preserves Chern-Moser chains, then f is either a CR isomorphism or a conjugate CR isomorphism. [Cheng 1988]
- $f\colon \mathbb{R}^n \to \mathbb{R}^n,\ \vec{v} \mapsto A\vec{v} + \vec{d}, \quad A \in \mathrm{GL}(n),\ \vec{d} \in \mathbb{R}^n,$
  is geodesic-preserving but not an isometry in general.

Riemannian Geodesics

Theorem (Kobayashi Oct. 1955)

Given a connected complete Riemannian manifold (M, g) with a connection-preserving diffeomorphism $(M, \nabla^g) \xrightarrow{f} (M, \nabla^g)$. Assume the holonomy group of ∇^g is irreducible, then f is an isometry except when $M = \mathbb{R}$.

[Kobayashi and Nomizu Feb. 22, 1996]
Given a principal G-bundle with a principal connection $(P, \gamma) \to M$.
Define $p \sim q$ if $p, q \in P$ are joined by a piecewise smooth horizontal curve.

Definition

The holonomy group $Hol_p(\gamma)$ at $p \in P$ is

$$Hol_p(\gamma) = \{g \in G | \, p \sim p \cdot g\}.$$

- If M is simply connected, then $Hol_p(\gamma)$ is connected.
- If M and P are connected, then for all two points $p, q \in P$, $\exists! g \in G$ s.t. $q \sim p \cdot g$ and so $Hol_q(\gamma) = g^{-1} Hol_p(\gamma) g$.

Review Holonomy group

[Kobayashi and Nomizu Feb. 22, 1996]

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Define $p \sim q$ if $p, q \in P$ are joined by a piecewise smooth horizontal curve.

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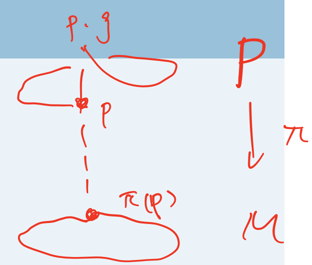

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- If M and P are connected, then for all two points $p, q \in P$, $\exists! g \in G$ s.t. $q \sim p \cdot g$ and so $Hol_q(\gamma) = g^{-1} Hol_p(\gamma) g$.

Assume $(P, \gamma) \to M$ with M connected.

Theorem

Let $p \in P$. The Lie algebra $\mathfrak{hol}_p(\gamma)$ of $Hol_p(\gamma)$ is

$$\mathfrak{hol}_p(\gamma) = span\{\Omega_q(\tilde{X}^{hor}, \tilde{Y}^{hor}) |\ q \sim p\}.$$

[Kobayashi and Nomizu Feb. 22, 1996 Chap.II Thm 8.1]

Vector Bundle with a Linear Connection $(E, \nabla) \to M$

Definition

$$Hol_x(\nabla) = \{g \in \mathrm{GL}(E_x) |\ V(1) = g \cdot V(0) \text{ where } V \subset E \text{ along a closed curve } c \subset M \text{ centered at } x \text{ with } \nabla_{\dot{c}} V = 0 \text{ and } V(0), V(1) \in E\}$$

Associated Vector Bundle $(P_G, \gamma) \xrightarrow{\pi} M$ and $G \xrightarrow{\rho} \mathrm{GL}(V)$

If $E = P_G \times_G V$ and ∇ is induced from γ, then

$$Hol_p(\gamma) \stackrel{\rho}{=} Hol_{\pi(p)}(\nabla).$$

[Kobayashi and Nomizu Feb. 22, 1996, Chap IV theorem 5.4]

Let (M, g) be a Riemannian manifold, then T_xM has the orthogonal decomposition

$$T_xM = \bigoplus_{i=0}^{k} T_x^{(i)}M, \quad \text{where}$$

$T_x^{(0)}M$ is the subspace s.t. $\mathrm{Hol}(x, \nabla)$ is a trivial representation on it and $T_x^{(i)}M$ are irreducible representations of $\mathrm{Hol}(x, \nabla)$ for $i = 1 \sim k$.
By parallel translation, these subspaces introduce involutive distributions and orthogonal to each other. Then,
For any point $y \in M$, $\exists$ its open neighborhood $V \subset M$ s.t.

$$V = \sum_{i=0}^{k} V_i, \quad V_i \underset{open}{\subset} M_i \text{ integral manifolds of } T^{(i)}M \text{ through } y$$

$$\text{and} \quad g|_V = \bigoplus_{i=0}^{k} g|_{V_i}, \quad g|_{V_0} \text{ is isometric to Euclidean metric.}$$

Given a $(M^{n\geq 3}, [g])$ with the standard tractor bundle $(\mathcal{V}, m^{[g]}, \mathcal{V}^1, \nabla^{\mathcal{V}})$ and a vector bundle isomorphism

$$\begin{tikzcd} \mathcal{V} \arrow[r, "F"] \arrow[d] & \mathcal{V} \arrow[d] \\ M \arrow[r, "f"] & M \end{tikzcd}.$$

Q:

1. f preserves conformal geodescis $\overset{?}{\Leftrightarrow}$ F preserves $\nabla^{\mathcal{V}}$
2. If F preserves $\nabla^{\mathcal{V}}$, is $[f^*g] = [g]$?
3. If 1 and 2 are true, then f preserves conformal geodesics $\Rightarrow [f^*g] = [g]$.

Discussion

- From Riemannian manifold perspective, I may need to know irreducible representation of $Hol_x(\nabla^{\mathcal{V}}) \to \mathrm{GL}(\mathcal{V}_x)$.
- The de Rham decomposition leads to sum of irred. reps. of $Hol_x(\nabla^g)$, maybe I need to know the de Rham decomposition in conformal version.

Facts

- For the standard tractor bundle $(\mathcal{V}, m^{[g]}, \mathcal{V}^1, \nabla^{\mathcal{V}})$ of a conformal manifold $(M^{n\geq 3}, [g])$, $Hol_x(\nabla^{\mathcal{V}}) \subset O(n+1, 1)$.
- For $(S^{n\geq 3}, [g_{S^n}])$, $Hol_x(\nabla^{\mathcal{V}}) = id$.

Irreducible representation $Hol_x(\nabla^{\mathcal{V}}) \to \mathrm{GL}(\mathcal{V}_x)$

Theorem (Scala and Olmos 2001)

Let G be a connected Lie subgroup of $SO(N,1)$ and assume $G \hookrightarrow GL(\mathbb{R}^{N+1})$ is irreducible, then $G = SO_+(N,1)$.

By the above result, if $(M^{n\geq 3}, [g])$ is simply connected and $Hol_x(\nabla^{\mathcal{V}}) \to \mathrm{GL}(\mathcal{V}_x)$ is irred., then $Hol_x(\nabla^{\mathcal{V}}) \cong SO_+(n+1,1)$.

Conformal de Rham local decomposition

Theorem (Armstrong 2005)

Assume $\mathcal{V}$ has a holonomy-preserved subbundle of rank $2 \leq k \leq n$, then locally $\exists\ g \in [g]$ s.t. $U^n = U_1^{k-1} \times U_2^{n-(k-1)}$ where $g|_{U_1}$ and $g|_{U_2}$ are Einstein.

Indecomposable
$(M^{n\geq 3}, [g])$ is called *indecomposable* if M is conformal Einstein and M cannot be decomposed as above.

Theorem (Armstrong 2005)

If $(M^{n\geq 3}, [g])$ is conformal Einstein, then there is a parallel section s of $\mathcal{V}$. In particular, the line bundle produced by $s \in \Gamma(\mathcal{V})$ is holonomy preserved.

By the above result, any indecomposable representation $Hol_x(\nabla^{\mathcal{V}}) \to \mathcal{V}_x$ is reducible.

Reference I

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