Knots, Links, and a Little Magic

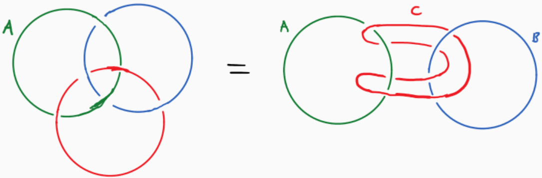
Amethyst Price

Winter 2022

Department of Mathematics
UCSC Graduate Colloquium

Let's Begin with Some Magic!

Case 1:



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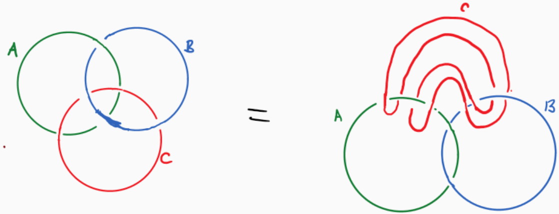

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Tools we will need

- Knots and Links
- Knot and Link Complements
- The Knot Group
- The Wirtinger Presentation
- Some Algebra

Definitions

What is a knot?

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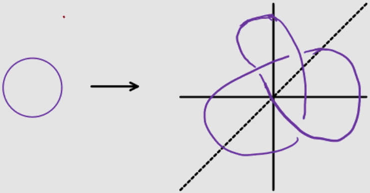
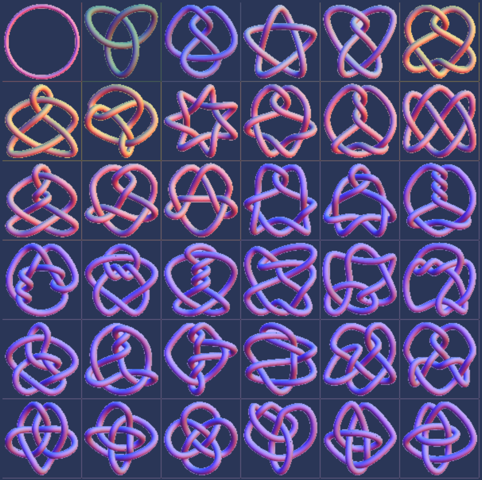
E.g: The Trefoil



Examples



What is a link?

A subset L of a 3-dimensional space X is a **link** if it is an embedding of $\mathcal{S}^0 \cup ... \cup \mathcal{S}^0 \cup \mathcal{S}^1 \cup ... \cup \mathcal{S}^1$.

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Remark: The number of components need not be finite.

Example: Antione's Necklace

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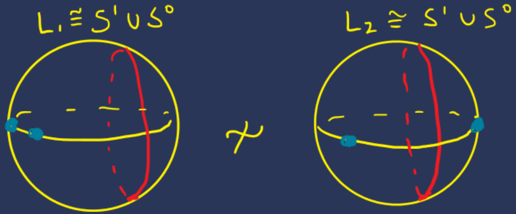
Two knots (links) K and K' are **ambient isotopic** if there exists an isotopy

$$\begin{aligned} \tilde{h} : X \times [0,1] &\to X \\ (K,0) &\mapsto K \\ (K,1) &\mapsto h(K) = K' \end{aligned}$$

We are often interested in knot types modulo ambient isotopies.

Example: Two nonequivalent links in $\mathcal{S}^2$



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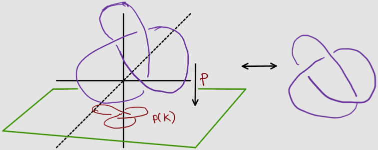
Example



Tame vs. Wild Knots

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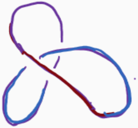

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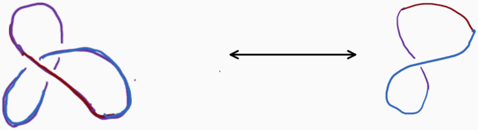

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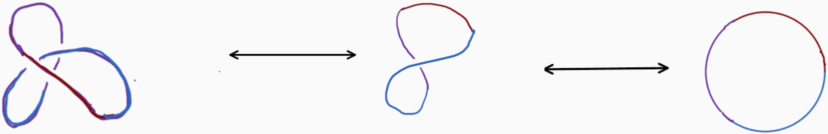

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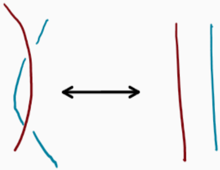

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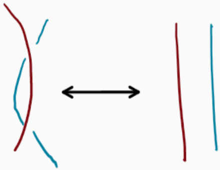

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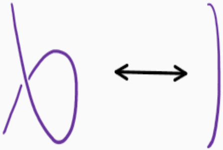

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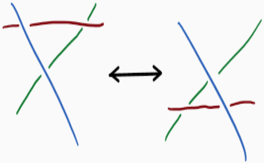

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Some natural guesses:

$[M] = [\text{Meridian}] = \langle 0, 1 \rangle$

$[L] = [\text{Longitude}] = \langle 1, 0 \rangle$

$[J] = [\text{Inessential}] = \langle 0, 0 \rangle$

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We can see that $M \not\sim J$ because J is a separating curve and M is not.

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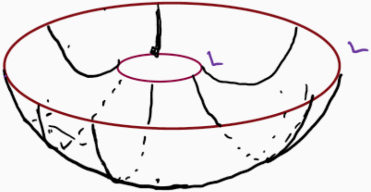

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- Any class $\langle a, b\rangle \in \pi_1(\mathbb{T}^2)$ represents and embedding $\mathcal{S}^1 \to \mathbb{T}^2 \iff \gcd(a, b) = 1$.
- There exists two self-homeomorphisms

$$h_L(e^{i\theta},^{i\phi}) = (e^{\theta+\phi}, e^{\phi}) \qquad \text{and} \quad h_M(e^{i\theta},^{i\phi}) = (e^{\theta}, e^{\phi+\theta})$$

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which extend to isomorphisms

$$h_{L*} \leftrightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } h_{M*} \leftrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(operating on the right of $\langle a, b \rangle$).

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- For any K such that $[K] \neq \langle 0, 0\rangle$, $\exists$ a homeomorphism taking K to M.
- Knots K and K' are ambient isotopic $\iff [K] = [\pm K'] \in \pi_1(\mathbb{T}^2)$.

Therefore, there are only two knot types on $\mathbb{T}^2$: the inessential, and everything else!

The Solid Torus

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- ◇ There are infinitely many ambient isotopies of longitudes.

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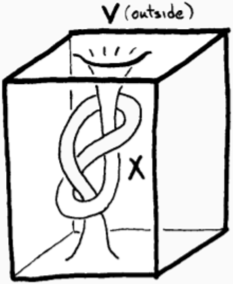
We also have that $H_n(X) = H_n(V)$, but X may or may not also be a solid torus.

The Solid Torus

If X is not a solid torus, it is sometimes called a **cube-with-knotted-hole**, and $\pi_1(X)$ may not be infinite cyclic.



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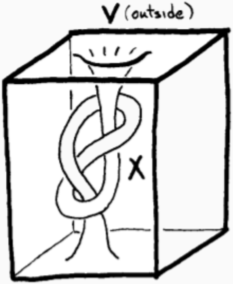

The complement X is a solid torus $\iff \pi_1(X) = \mathbb{Z}$

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Remarks:

- This is only useful for knots of codimension 2.
- The knot complement is a complete topological invariant for knots, but not for links. Though we still utilize this for many examples.

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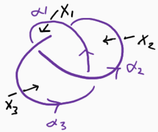
- The knot diagram consists of n arcs, separated by n over-under crossings. Label each arc $\alpha_1, \alpha_2, ..., \alpha_n$ so that α_i connects to α_{i+1} and α_n to α_1.

The Wirtinger Presentation

- At each α_i, draw an arrow labeled x_i passing under the arc using a *right-hand rule*. These arrows represent loops in the knot complement.



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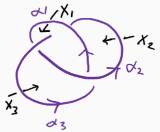
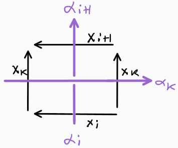


Case 1:

- At each over-under crossing, observe the relations described by the x_i's.



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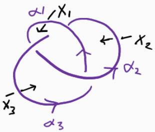

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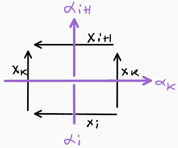

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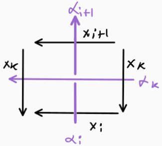

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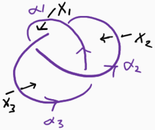

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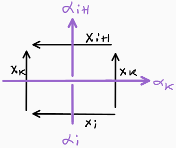

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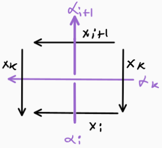
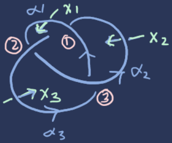
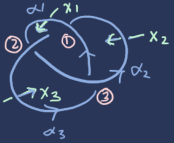


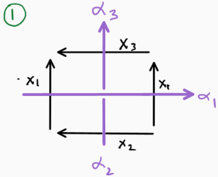
$r_i: \quad x_{i+1} x_k = x_k x_i$

This yeilds $\pi_1(X) = \langle x_1, \ldots, x_n | r_1, \ldots, r_n \rangle$.

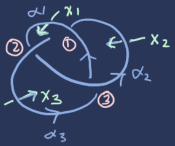


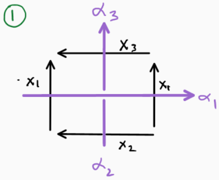
α_1
x_1
x_2
α_2
x_3
α_3
①
②
③

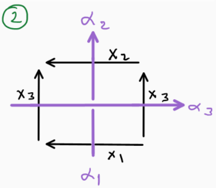




$r_1 : x_1 x_3 = x_2 x_1$







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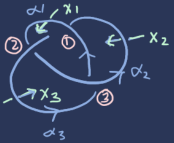

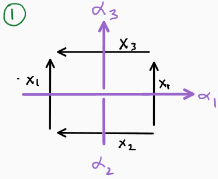

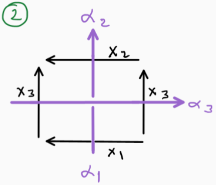

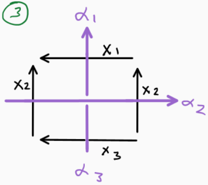

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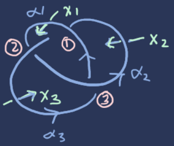

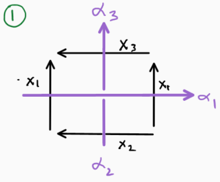

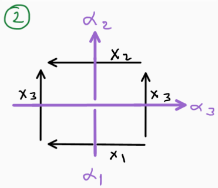

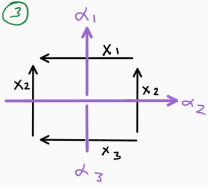

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$$\implies \pi_1(X) = \langle x_1, x_2, x_3 | x_1 x_3 = x_2 x_1 = x_3 x_2 \rangle$$
$$= \langle x_1, x_2 | x_1 x_2 x_1 = x_2 x_1 x_2 \rangle$$

The Magic Trick Revealed

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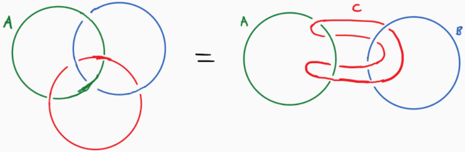
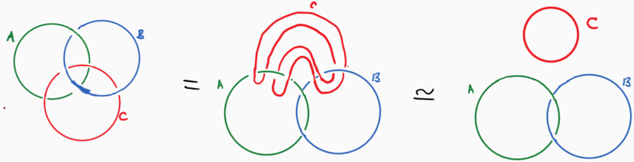
Case 1:

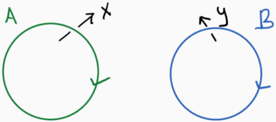


Case 2:



The Magic Trick Revealed

Let $L_1 = A \cup B$ and $X_1 = \mathbb{R}^3 \setminus L_1$.



We have $\pi_1(X_1) = \langle x, y | - \rangle$.

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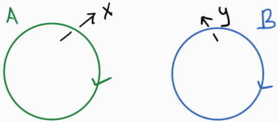
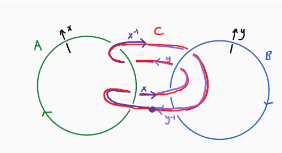


We have $\pi_1(X_1) = \langle x, y | - \rangle$.

Now consider what $C \in \pi_1(X_1)$:



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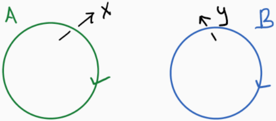

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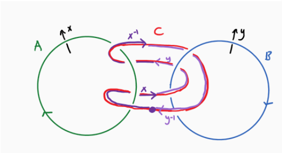

By observation $C = xyx^{-1}y^{-1} = [x, y] \in \pi_1(X_1)$, the commutator.

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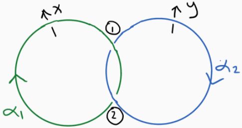
Now let $L_2 = A \cup B$ with A and B linked and $X_2 = \mathbb{R}^3 \setminus L_2$.



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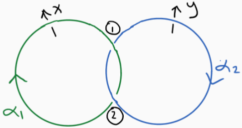
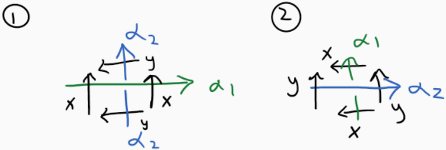


Consider the relation defined at the over-under crossing.



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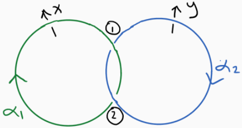

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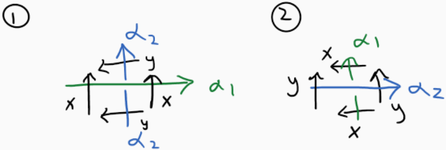

This yields $\pi_1(X_2) = \langle x, y | xy = yx \rangle$.

The Magic Trick Revealed

- We preserved the configuration of how C was linked to $A \cup B$, and therefore C still represents the commutator in $\pi_1(X_2)$.

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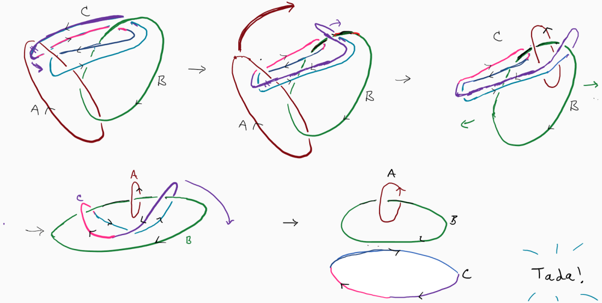

Bibliography

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[1] D. Rolfsen, *Knots and Links*. AMS Chelsea Publishing, Rhode Island, 1990, Reprinted with corrections: 2003, pp. 1–65.

Thank you!