Advanced Data Structures and Algorithms

Comprehensive Assignment Solutions

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1   Complexity Classes

Question 3. Is the 3-SAT (3-CNF-SAT) problem NP-Hard? Justify your answer.

Detailed Solution:

Answer

Yes, the 3-SAT problem is **NP-Hard**.

Detailed Explanation

1. Introduction

The Boolean Satisfiability Problem (SAT) was the first problem proven to be **NP-Complete** (Cook-Levin Theorem). The 3-SAT problem is a specific variant of SAT where the structure of the boolean formula is restricted.

2. Definition of 3-SAT

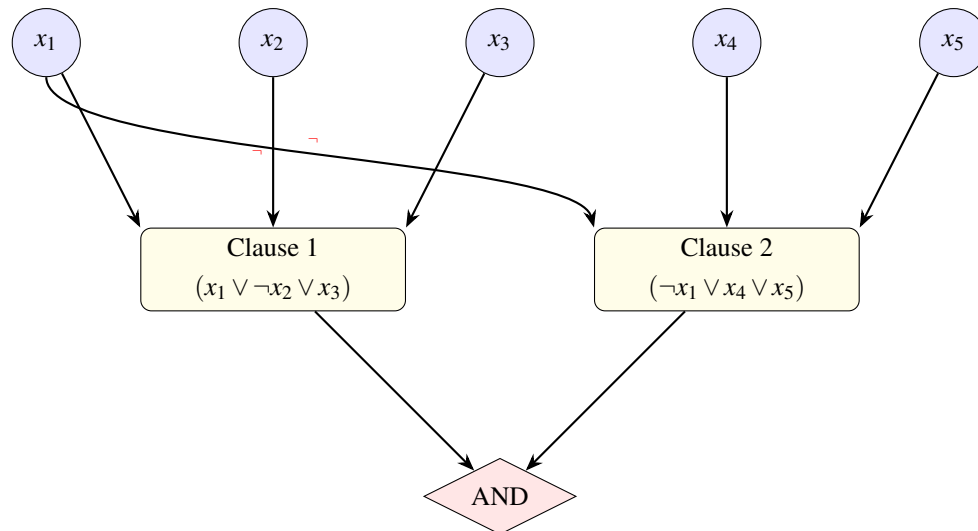For a problem to be classified as 3-SAT, the boolean formula must satisfy two conditions:

1. It must be in **Conjunctive Normal Form (CNF)**, meaning it is an AND of multiple clauses.

2. Each clause must contain **exactly three literals** (a literal is a variable or its negation).

Example Instance: Consider the following formula with 5 variables and 2 clauses:

$$\phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_4 \vee x_5)$$

3. Visual Structure

The diagram below illustrates the logic flow of the specific example above. Inputs are variables, which form clauses (OR logic), and the system is satisfied only if the global AND is true.

Result: True iff Clause 1 = True AND Clause 2 = True

4. Justification of NP-Hardness

To classify 3-SAT as NP-Hard, we use the concept of **Polynomial-Time Reduction** ($\leq_p$).

- **Premise:** We know that the general SAT problem is **NP-Complete**. This means every problem in NP can be reduced to SAT.

- **Reduction:** It has been proven that any general SAT instance (where clauses can have arbitrary length) can be transformed into an equivalent 3-SAT instance in polynomial time. This is done by splitting long clauses into chains of size-3 clauses using auxiliary variables.

- **Logic:** Since $SAT \in$ NP-Hard and $SAT \leq_p 3\text{-}SAT$, transitivity implies that 3-SAT must be at least as hard as SAT.

Conclusion

Therefore, **3-SAT is NP-Hard**. Furthermore, since a given assignment of truth values can be verified against a 3-CNF formula in polynomial time, 3-SAT is also in NP. Being both in NP and NP-Hard makes it **NP-Complete**.