Advanced Data Structures and Algorithms

Comprehensive Assignment Solutions

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1 Matrix Decompositions

Question 1. Explain the LU decomposition of a matrix using Gaussian Elimination. Clearly describe each step involved in the process.

Detailed Solution:

Introduction to LU Decomposition

LU decomposition is a factorization of a square matrix A into two triangular matrices:

$$A = L \cdot U$$

where L is a **Lower Triangular Matrix** (typically with 1s on the diagonal) and U is an **Upper Triangular Matrix**. This factorization effectively records the steps of Gaussian Elimination.

The Gaussian Elimination Process

We transform A into U via a sequence of row operations. The multipliers used to zero out elements below the diagonal are stored in L.

Initialization:

- Let $U^{(0)} = A$.
- Let L be the identity matrix I.

Step-by-Step Procedure: For each column k from 1 to $n-1$:

1. **Identify Pivot:** Let the pivot element be U_{kk}. (Assume $U_{kk} \neq 0$; otherwise, row swapping/pivoting is needed).
2. **Eliminate Entries Below Pivot:** For each row i from $k+1$ to n:
   (a) Compute the **multiplier** λ:
   $$\lambda = \frac{U_{ik}}{U_{kk}}$$
   (b) This multiplier λ represents the factor needed to cancel out the element U_{ik} using the pivot row. We store this multiplier in the lower triangular matrix:
   $$L_{ik} = \lambda$$

(c) **Update Row i:** Subtract λ times row k from row i to produce the new row i in U.

$$U_{ij} \leftarrow U_{ij} - \lambda \cdot U_{kj} \quad \text{for } j = k, \ldots, n$$

Outcome

Upon completion of the loops:

- The matrix U contains the **Row Echelon Form** of A. All elements below the main diagonal are zero.
- The matrix L contains the unit diagonal (1s) and the multipliers used during elimination in the strict lower triangular part.

Example Structure

If $A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$:

1. Pivot $A_{11} = 2$. To eliminate $A_{21} = 4$, multiplier is $4/2 = 2$.
2. $L_{21} = 2$.
3. Row 2 becomes: $[4,3] - 2 \times [2,1] = [0,1]$.

Result:

$$L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

Complexity Analysis

The complexity is dominated by the nested loops in the elimination step.

- The outer loop runs n times for each pivot k.
- For each pivot, we iterate through the remaining $n-k$ rows (i loop).
- For each row, we perform subtraction across the remaining $n-k$ columns (j loop).

Total operations roughly:

$$\sum_{k=1}^{n-1} (n-k)^2 \approx \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{n^3}{3}$$

Thus, the time complexity of LU decomposition is $O(n^3)$.