Advanced Data Structures and Algorithms

Comprehensive Assignment Solutions

Arya Raag
Roll No: A125002

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1 Matrix Decompositions

Question 3. Prove that if matrix A is non-singular, then its Schur complement is also non-singular.

Detailed Solution:

Definitions

Let A be a block partitioned matrix:

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$$

Assume B is non-singular (invertible). The **Schur Complement** of B in A, denoted S, is defined as:

$$S = E - DB^{-1}C$$

LDU Decomposition of Block Matrix

We can decompose the block matrix A using block Gaussian elimination logic. Specifically, A can be factored as:

$$A = \begin{pmatrix} I & 0 \\ DB^{-1} & I \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I & B^{-1}C \\ 0 & I \end{pmatrix}$$

Let's verify the center and right matrices product:

$$\begin{pmatrix} B & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I & B^{-1}C \\ 0 & I \end{pmatrix} = \begin{pmatrix} B & C \\ 0 & S \end{pmatrix}$$

Now multiply by the left matrix:

$$\begin{pmatrix} I & 0 \\ DB^{-1} & I \end{pmatrix} \begin{pmatrix} B & C \\ 0 & S \end{pmatrix} = \begin{pmatrix} B & C \\ DB^{-1}B & DB^{-1}C + S \end{pmatrix} = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$$

(Since $S = E - DB^{-1}C$, so $DB^{-1}C + S = E$).

Determinant Analysis

Using the property that $\det(XY) = \det(X)\det(Y)$, we take the determinant of the decomposition:

$$\det(A) = \det(L) \cdot \det(\text{BlockDiag}) \cdot \det(U)$$

The matrices L and U are block triangular with Identity matrices on the diagonals, so their determinants are exactly 1. Thus:

$$\det(A) = 1 \cdot \det \begin{pmatrix} B & 0 \\ 0 & S \end{pmatrix} \cdot 1$$

For a block diagonal matrix, the determinant is the product of the determinants of the blocks:

$$\det(A) = \det(B) \cdot \det(S)$$

Conclusion of Proof

We are given that A is non-singular, which means $\det(A) \neq 0$. From the equation $\det(A) = \det(B) \cdot \det(S)$, it implies that the product of the two scalars on the right side is non-zero. Therefore, it is necessary that $\det(S) \neq 0$.

Since the determinant of S is non-zero, the Schur complement S is **non-singular**.
