Advanced Data Structures and Algorithms

Comprehensive Assignment Solutions

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1 Graph Algorithms

Question 3. Prove that every connected component of the symmetric difference of two matchings in a graph G is either a path or an even-length cycle.

Detailed Solution:

Definitions

Let M_1 and M_2 be two valid matchings in a graph $G = (V, E)$. The **Symmetric Difference**, denoted $G' = M_1 \oplus M_2$, contains edges that appear in exactly one of the two matchings, but not both.

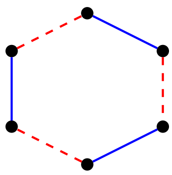
$$E(G') = (M_1 \setminus M_2) \cup (M_2 \setminus M_1)$$

Visualization of Component Types

The diagram below shows the only two possible structures for a connected component in $M_1 \oplus M_2$: an alternating path or an alternating cycle.

Case 1: Alternating Path

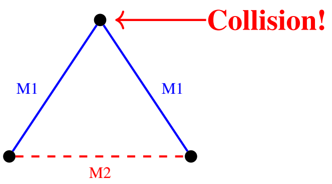
Case 2: Alternating Cycle (Even Length)



Visual Proof of Impossible Structures

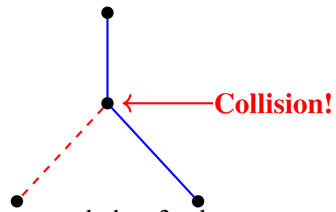
Why can't we have an odd cycle or a node with degree > 2?

Impossible Case 1: Odd Cycle



Node at top has
two M1 edges.
Violates Matching Property.

Impossible Case 2: Degree > 2



Center node has 3 edges.
Pigeonhole Principle:
At least 2 must
be M1 or M2.

Analysis of Impossible Structures

Case 1: Impossible Degree > 2
By definition, a matching is a set of edges without common vertices. For any node v:

- v is incident to at most 1 edge in M_1.
- v is incident to at most 1 edge in M_2.

The symmetric difference graph G' contains edges from both sets. Thus, the degree of v in G' is the sum of its degrees in M_1 and M_2:

$$deg_{G'}(v) \leq 1 + 1 = 2$$

Therefore, no node can have a degree of 3 or higher. This eliminates "star graphs" or any branching structures.

Case 2: Impossible Odd Cycles
If a component is a cycle, the edges must alternate between M_1 and M_2 because no two edges from the same matching can share a vertex (degree constraint). Let the edges be $e_1, e_2, \dots, e_k$.

- If $e_1 \in M_1$, then e_2 must be in M_2 (to avoid two M_1 edges at vertex v_2).
- Then e_3 must be in M_1, e_4 in M_2, and so on.

In general, edges at odd positions $(1, 3, \dots)$ are in M_1 and even positions $(2, 4, \dots)$ are in M_2. For a cycle to close, the last edge e_k connects back to the start of e_1. If the cycle length k were odd, then e_k would be an odd-numbered edge, meaning $e_k \in M_1$. But e_1 is also in M_1. This would mean the vertex v_1 is incident to two edges from M_1 (e_k and e_1), which violates the matching property. Thus, k must be even.

Conclusion

We have proven that every connected component in the symmetric difference must have a maximum degree of 2 and cannot form odd cycles. The only graph structures satisfying these constraints are isolated vertices, simple paths, and even-length cycles.