Advanced Data Structures and Algorithms

Comprehensive Assignment Solutions

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1 Matrix Decompositions

Question 4. Prove that positive-definite matrices are suitable for LU decomposition and do not require pivoting to avoid division by zero in the recursive strategy.

Detailed Solution:

Definition of Positive-Definiteness

A symmetric real matrix A is **positive-definite (PD)** if for every non-zero vector x, the scalar $x^T Ax > 0$.

Key Properties

Two fundamental properties of positive-definite matrices are crucial for this proof:

1. **Submatrix Property:** Every leading principal submatrix of a positive-definite matrix is itself positive-definite.
2. **Determinant Property:** The determinant of any positive-definite matrix is strictly positive.

Recursive Step Analysis

LU decomposition works recursively. At the first step, we choose the pivot A_{11}. Since A_{11} is a 1×1 leading principal submatrix of A, by property (1), it must be positive-definite. For a scalar, positive-definiteness implies $A_{11} > 0$. **Result:** $A_{11} \neq 0$, so division by A_{11} is safe.

After the first step of elimination, we are left with the Schur complement S in the bottom right block ($S = E - DB^{-1}C$). A powerful theorem in linear algebra states that *the Schur complement of a positive-definite matrix is also positive-definite*.

Inductive Argument

- **Base Case:** The first pivot $A_{11} > 0$, so step 1 is safe.
- **Inductive Step:** The remaining submatrix (Schur complement) to be processed is also positive-definite. Therefore, its top-left element (the next pivot) will also be strictly positive.

Illustrative Example

Consider the symmetric positive-definite matrix $A = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}$.

1. **Check Pivot:** $A_{11} = 4$. Since $4 > 0$, no pivoting is needed.
2. **Elimination:** We want to eliminate $A_{21} = 2$.

$$\text{Multiplier } l_{21} = A_{21}/A_{11} = 2/4 = 0.5$$

$$\text{New } A_{22} = 5 - (0.5 \times 2) = 4$$

3. **Result:** The remaining element is 4, which is also > 0.
4. **Decomposition:**

$$L = \begin{pmatrix} 1 & 0 \\ 0.5 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 4 & 2 \\ 0 & 4 \end{pmatrix}$$

This demonstrates how the positive-definite property guarantees non-zero pivots at every step.

Conclusion

Because every pivot encountered throughout the entire recursive decomposition process is guaranteed to be strictly positive (and thus non-zero), the algorithm will never encounter a division-by-zero error. Consequently, **pivoting** (row swapping) is not required for numerical stability or solvability when A is positive-definite.