Probability Calculations

Introduction

This document presents the probability calculations of the Term Assignment[1] in the course TFE4140 Modelling and Analysis of Digital Systems, Spring 2014. What are presented are the detailed solutions of sub-problem 5 in the Term Assignment.

The Calculations

According to the Term Assignment, a microcontroller is expected to work for 6 years before failing. What is to be calculated are:

- The mathematical expression of max one error after *t* time
- The mathematical expression of max two errors after *t* time
- The mathematical expression of at least three errors after *t* time
- The Mean Time To Failure (MTTF) for the Liaison system

Chance for a failure in a controller after time t when T = 6:

$$P = P(t) = e^{-\frac{t}{T}} = e^{-\frac{t}{6}}$$

Probability for a number r failures of n controllers:

$$Q(r) = \frac{n!}{r!\,(n-r)!} * P^r * (1-P)^{n-r}$$

Since n = 4, Q(r) will be:

$$Q(r) = \frac{4!}{r!\,(4-r)!} * P^r * (1-P)^{4-r}$$

Note that: $P^r = e^{-\frac{t}{6}*r}$

Probability for max 1 error (written as P(Max (1))) after t time can be expressed as:

P(Max(1)) = Q(4)+Q(3).

$$Q(4) = \frac{4!}{4!\,(4-4)!} * P^4 * (1-P)^{4-4} = \frac{4!}{4!} * P^4 * (1-P)^0 = 1 * P^4 * 1 = P^4 = e^{-\frac{4t}{6}} = e^{-\frac{t}{2}}$$

$\underline{Q(4) = e^{-\frac{t}{2}}}$

$$Q(3) = \frac{4!}{3!\,(4-3)!} * P^3 * (1-P)^{4-3} = \frac{4!}{3! * 1!} * P^3 * (1-P)^1 = 4 * P^3 * (1-P) = 4(P^3 - P^4)$$

$$= 4\left(e^{\frac{3t}{6}} - e^{-\frac{4t}{6}}\right) = 4(e^{-\frac{t}{2}} - e^{-\frac{2t}{3}})$$

$\underline{Q(3) = 4(e^{-\frac{t}{2}} - e^{-\frac{2t}{3}})}$

$$P\big(Max(1)\big) = Q(4) + \ Q(3) = P^4 + \ 4(P^3 - P^4) = \ 4P^3 - 3P^4 = P^3 + 3(P^3 - P^4)$$
$$= e^{-\frac{3t}{6}} + 3\left(e^{-\frac{3t}{6}} - e^{-\frac{4t}{6}}\right) = \ e^{-\frac{t}{2}} + 3\left(e^{-\frac{t}{2}} - e^{-\frac{2t}{3}}\right)$$

$$\boldsymbol{P\big(Max(1)\big) = \ e^{-\frac{t}{2}} + 3\left(e^{-\frac{t}{2}} - e^{-\frac{2t}{3}}\right)}$$

Probability for max 2 errors after t time can be expressed as:

P(Max(2)) = Q(4)+Q(3) + Q(2) = P(Max(1))+Q(2)

$$Q(2) = \frac{4!}{2!\,(4-2)!} * P^2 * (1-P)^{4-2} = \frac{4!}{2! * 2!} * P^2 * (1-P)^2 = \ 6 * \ P^2 * (1-P)^2$$
$$= 6 * P^2 * (1 - 2P + P^2) = \ 6(P^2 - \ 2P^3 + P^4) = \ 6(e^{-\frac{2t}{6}} - \ 2e^{-\frac{3t}{6}} + e^{-\frac{4t}{6}})$$
$$= \ 6(e^{-\frac{t}{3}} - \ 2e^{-\frac{t}{2}} + e^{-\frac{2t}{3}})$$

$\underline{Q(2) = \ 6(A^{2t} - \ 2A^{3t} + A^{4t})}$

$$P\big(Max(2)\big) = P\big(Max(1)\big) + \ Q(2) = P^3 + 3(P^3 - P^4) + \ 6(P^2 - \ 2P^3 + P^4)$$
$$= 4P^3 - 3P^4 + 6P^2 - 12P^3 + 6P^4 = 3P^4 - 8P^3 + 6P^2 = 3e^{-\frac{4t}{6}} - 8e^{-\frac{3t}{6}} + 6e^{-\frac{2t}{6}}$$
$$= 3e^{-\frac{2t}{3}} - 8e^{-\frac{t}{2}} + 6e^{-\frac{t}{3}}$$

$$\underline{\boldsymbol{P\big(Max(2)\big) = 3e^{-\frac{2t}{3}} - 8e^{-\frac{t}{2}} + 6e^{-\frac{t}{3}}}}$$

Probability for error in at least 3 (written as P(Least(3))controllers can be expressed as:

$$P\big(Least(3)\big) = \ 1 - P(\mathrm{Max}(2)) = 1 - \left(3e^{-\frac{2t}{3}} - 8e^{-\frac{t}{2}} + 6e^{-\frac{t}{3}}\right) = 1 - 3e^{-\frac{2t}{3}} + 8e^{-\frac{t}{2}} - 6e^{-\frac{t}{3}}$$

$$\underline{\boldsymbol{P\big(Least(3)\big) = 1 - 3e^{-\frac{2t}{3}} + 8e^{-\frac{t}{2}} - 6e^{-\frac{t}{3}}}}$$

The reason this works is that at least 3 errors are the same as 3 OR 4 errors.

That would be Q(3) + Q(4). Since Q(0) + … + Q(4) = 1, subtracting Q(0), Q(1) and Q(2) from 1 should provide the answer. The total probability of Q(0) + Q(1) + Q(2) is already expressed in P(Max(2)). Therefore, P(Least(3)) = 1 – P(Max(2)).

Mean time to failure:

The assumption is that when 3 microcontrollers have failed, the entire system has failed. As long as 2 or more controllers are working, the system works. R(t) must then be P(Max(2))

$\underline{R(t) = 3e^{-\frac{2t}{3}} - 8e^{-\frac{t}{2}} + 6e^{-\frac{t}{3}}}$

$$MTTF = \int_0^\infty R(t)dt = \int_0^\infty 3e^{-\frac{2t}{3}} - 8e^{-\frac{t}{2}} + 6e^{-\frac{t}{3}}dt = \left[\frac{3}{-\frac{2}{3}} e^{-\frac{2t}{3}} - \frac{8}{-\frac{1}{2}}e^{-\frac{t}{2}} + \frac{6}{-\frac{1}{3}}e^{-\frac{t}{3}}\right]_0^\infty$$

$$= \left[-\frac{9}{2} e^{-\frac{2t}{3}} + 16\, e^{-\frac{t}{2}} - 18e^{-\frac{t}{3}}\right]_0^\infty$$

$$= \left(-\frac{9}{2} * 0 + 16 * 0 - 18 * 0\right) - \left(-\frac{9}{2} * 1 + 16 * 1 - 18 * 1\right) = \frac{9}{2} - 16 + 18$$

$$= 4{,}5 + 2 = 6{,}5$$

$\boldsymbol{MTTF = 6,5}$

Estimated mean time to failure is 6,5 years.

Summary

$P(Max(1)) = e^{-\frac{t}{2}} + 3\left(e^{-\frac{t}{2}} - e^{-\frac{2t}{3}}\right)$

$P(Max(2)) = 3e^{-\frac{2t}{3}} - 8e^{-\frac{t}{2}} + 6e^{-\frac{t}{3}}$

$P(Least(3)) = 1 - 3e^{-\frac{2t}{3}} + 8e^{-\frac{t}{2}} - 6e^{-\frac{t}{3}}$

$MTTF = 6{,}5$

Sources:

[1] TFE4140 Modelling and Analysis of Digital Systems Term Assignment 2014