# UNBOUNDED VISIBILITY DOMAINS: METRIC ESTIMATES AND AN APPLICATION

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ABSTRACT. We give an explicit lower bound, in terms of the distance from the boundary, for the Kobayashi metric of a certain class of bounded pseudoconvex domains in  $\mathbb{C}^n$  with  $\mathcal{C}^2$ -smooth boundary using the regularity theory for the complex Monge–Ampère equation. Using such an estimate, we construct a family of unbounded Kobayashi hyperbolic domains in  $\mathbb{C}^n$  having a certain negative-curvature-type property with respect to the Kobayashi distance. As an application, we prove a Picard-type extension theorem for the latter domains.

# 1. Introduction and statement of results

A substantial part of the effort and most of the tools discussed in this paper are directed at the following problems that are seemingly unrelated:

- (a) Using the regularity theory for the complex Monge–Ampère equation on bounded domains  $\Omega \in \mathbb{C}^n$ ,  $n \geq 2$ , to estimate the Kobayashi pseudometric  $k_{\Omega}(z; \cdot)$  in terms of dist $(z, \partial \Omega)$ .
- (b) A Picard-type extension theorem for holomorphic mappings into a domain  $\Omega \subsetneq \mathbb{C}^n$ ,  $n \geq 2$ , where  $\Omega$  is not the complement of a divisor.

The theme that links these problems is a weak notion of negative curvature for the metric space  $(\Omega, K_{\Omega})$ , where  $K_{\Omega}$  denotes the Kobayashi pseudodistance (assumed to be a distance on domains considered in this paper). This negative-curvature-type property, called *visibility*, is that, loosely speaking, geodesic lines for  $K_{\Omega}$  joining two distinct points in  $\partial\Omega$  must bend into  $\Omega$  with some mild geometric control (reminiscent of the Poincaré disc model of the hyperbolic plane).

If the metric space  $(\Omega, K_{\Omega})$  is Cauchy-complete, then any two points in  $\Omega$  are joined by a geodesic (i.e., a path  $\sigma: I \to \Omega$ , where I is an interval, that satisfies  $K_{\Omega}(\sigma(t), \sigma(s)) = |t - s|$  for all  $s, t \in I$ ). But when  $n \geq 2$ , it is a very hard problem to tell whether, given a domain  $\Omega \subsetneq \mathbb{C}^n$ ,  $(\Omega, K_{\Omega})$  is Cauchy-complete (even when  $\Omega$  is pseudoconvex). Therefore, for the domains considered in this paper,  $(\Omega, K_{\Omega})$  will **not** be assumed to be Cauchy-complete. Thus, a formal definition of visibility (which will be provided in Section 1.2) needs to be more refined than the picture described above. This raises the question: when does a domain have the visibility property? Conditions on the quantitative behaviour of the Kobayashi pseudometric form a part of the answer to this question. This motivates our results on the Kobayashi pseudometric.

1.1. Lower bounds for the Kobayashi pseudometric. Our results on the Kobayashi pseudometric are inspired by a well-known result of Diederich-Fornæss [10, Theorem 4], which provides a lower bound for the Kobayashi pseudometric  $k_{\Omega}$  of a bounded pseudoconvex domain  $\Omega \subset \mathbb{C}^n$  with real-analytic boundary. The lower bound that they deduce for  $k_{\Omega}(z;v)$ ,  $(z,v) \in \Omega \times \mathbb{C}^n$ , is in terms of some positive power of  $(1/\text{dist}(z,\partial\Omega))$ , and has many applications. These applications are, in part, the reason for our interest in extending such an estimate to domains with just  $\mathcal{C}^2$ -smooth boundary.

Before we state our first theorem, some notation: if A and B are real quantities,  $A \gtrsim B$  will denote that there exists a constant c > 0 that is independent of all parameters determining A and

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B such that  $A \geq cB$ . In what follows,  $\mathscr{L}_{\Omega}$  will denote the Levi form of  $\partial\Omega$  determined by some fixed, but unspecified, defining function for  $\Omega$  (see Remark 1.2), and  $H_{\xi}(\partial\Omega) := T_{\xi}(\partial\Omega) \cap i T_{\xi}(\partial\Omega)$  will denote the maximal complex subspace of  $T_{\xi}(\partial\Omega)$ : the tangent space of  $\partial\Omega$  at  $\xi$ . For any  $z \in \Omega$ , we shall abbreviate  $\mathrm{dist}(z,\partial\Omega)$  to  $\delta_{\Omega}(z)$ .

**Theorem 1.1.** Let  $\Omega \subset \mathbb{C}^n$ ,  $n \geq 2$ , be a bounded pseudoconvex domain having  $\mathcal{C}^2$ -smooth boundary. Assume that there exists a  $\mathcal{C}^2$ -smooth closed submanifold of  $\partial\Omega$  such that S is totally-real and such that  $w(\partial\Omega) \subset S$ . Suppose there exists a number m > 2 such that

$$\mathscr{L}_{\Omega}(\xi; v) \gtrsim \operatorname{dist}(\xi, S)^{m-2} ||v||^2 \quad \forall v \in H_{\xi}(\partial \Omega) \text{ and } \forall \xi \in \partial \Omega \setminus S.$$
 (1.1)

Then, there exists a constant c > 0 such that

$$k_{\Omega}(z;v) \ge c \frac{\|v\|}{\delta_{\Omega}(z)^{1/m}} \quad \forall z \in \Omega \text{ and } \forall v \in \mathbb{C}^n.$$
 (1.2)

Remark 1.2. Strictly speaking,  $\mathcal{L}_{\Omega}$  requires a choice of defining function for it to be completely determined. But since, for  $\Omega$  as above, two defining functions differ by a  $\mathcal{C}^2$ -smooth factor that is non-vanishing in a neighbourhood of  $\partial\Omega$ , and given the expressions for  $H_{\xi}(\partial\Omega)$  in terms of each defining function, the inequality (1.1) makes sense even though no defining function is specified. Note also that (1.1) does not preclude the case  $w(\partial\Omega) = \emptyset$ , where  $w(\partial\Omega)$  denotes the set of points in  $\partial\Omega$  at which  $\partial\Omega$  is weakly Levi pseudoconvex.

On a first reading, the estimate (1.2) might seem unsurprising. However, to the best of our knowledge, estimates of the form (1.2) seen in the literature that are well-argued do not, for  $\Omega$  weakly pseudoconvex, provide an **explicit** exponent of  $\delta_{\Omega}$ . Moreover, there are more significant reasons for placing the estimate (1.2) on record. Namely:

- For bounded, weakly pseudoconvex, finite-type domains  $\Omega$  with  $\partial\Omega$  not real analytic, lower bounds for  $k_{\Omega}$  resembling (1.2) have been asserted on many occasions the earliest such instance being [9]. Each such claim has, eventually, relied on the difficult half of the paper [6] by Catlin. There seems to be a certain deficit in understanding the latter work nor is there any alternative exposition on the efficacy of a geometric construction called a boundary system on which the proofs of the above-mentioned assertions rely.
- We introduce a method relying on the regularity theory for the complex Monge–Ampère equation to derive lower bounds of the form (1.2). One way of deriving such an estimate is to construct plurisubharmonic peak functions satisfying certain precise estimates; see, for instance, [10, Theorem 2], [8, Proposition 4.2]. Similar peak functions, but with less restrictive requirements, are useful in proving the existence and regularity of solutions of the complex Monge–Ampère equation, as one would expect from the proof of [1, Theorem 6.2]. This underlies—in view of a result of Sibony [20]—an effective and less technical method for deriving lower bounds for the Kobayashi metric.

The latter point is substantiated by a general result whose proof (similar to that of Theorem 1.1) is the method alluded to. For its exact statement, we refer the reader to Theorem 4.2.

1.2. Visibility and a Picard-type extension theorem. One of the objectives of this work is to present a new application of visibility. This is the Picard-type extension theorem alluded to above. Both sentences call for definitions. First, we must formalise the rough idea of visibility hinted at above. We shall say that a domain  $\Omega$  is  $Kobayashi\ hyperbolic$  if  $K_{\Omega}$  is a distance.

**Definition 1.3.** Let  $\Omega \subset \mathbb{C}^n$  be a (not necessarily bounded) Kobayashi hyperbolic domain.

(1) Let p and q be two distinct points in  $\partial\Omega$ . We say that the pair (p,q) has the visibility property with respect to  $K_{\Omega}$  if there exist neighbourhoods  $U_p$  of p and  $U_q$  of q in  $\mathbb{C}^n$  such that  $\overline{U}_p \cap \overline{U}_q = \emptyset$  and such that for each  $\lambda \geq 1$  and each  $\kappa \geq 0$ , there exists a compact set

 $K \subset \Omega$  such that the image of each  $(\lambda, \kappa)$ -almost-geodesic  $\sigma : [0, T] \longrightarrow \Omega$  with  $\sigma(0) \in U_p$  and  $\sigma(T) \in U_q$  intersects K.

(2) We say that  $\partial\Omega$  is visible if every pair of distinct points  $p, q \in \partial\Omega$  has the visibility property with respect to  $K_{\Omega}$ .

The property of  $\partial\Omega$  being visible is closely related to the notion of  $\Omega$  being a visibility domain, which was introduced by Bharali–Zimmer [2, 3]. If  $\Omega$  is bounded, then the two notions are equivalent. We shall not define visibility domains here. Instead, we will work with the properties introduced in Definition 1.3; these are closer to the terminology in Sarkar's article [19], which contributes a key idea to the proof of one of the theorems in this section. Recall the discussion on the difficulty of knowing whether  $(\Omega, K_{\Omega})$  admits geodesics:  $(\lambda, \kappa)$ -almost-geodesics (see Section 2 for their definition) serve as substitutes for geodesics. This is because if  $\Omega$  is Kobayashi hyperbolic, then (regardless of whether  $(\Omega, K_{\Omega})$  is Cauchy-complete) for any  $z, w \in \Omega$ ,  $z \neq w$ , and any  $\kappa > 0$ , there exists a  $(1, \kappa)$ -almost-geodesic joining z and w [3, Proposition 5.3].

Visibility of  $\partial\Omega$  has been used to deduce properties of holomorphic mappings into  $\Omega$ —ranging from their continuous extendability to  $\overline{\Omega}$ , to the iterative dynamics of such maps—which are too numerous to mention here. Instead, we refer readers to [2, 4, 5, 3]. Given this, it is desirable to identify families of unbounded domains  $\Omega \subsetneq \mathbb{C}^n$  such that  $\partial\Omega$  is visible. A rich collection of **planar** domains with the latter property that also satisfy other metrical conditions, and domains in  $\mathbb{C}^n$ ,  $n \geq 2$ , with the latter property and having rather wild boundaries, have been constructed in [3]. But, given an unbounded domain  $\Omega \subsetneq \mathbb{C}^n$ ,  $n \geq 2$ , such that  $\partial\Omega$  is  $\mathbb{C}^2$ -smooth, Levi pseudoconvex, but not strongly Levi pseudoconvex, are there conditions under which  $\partial\Omega$  is visible? One of our theorems addresses this natural question. Why the interest in unbounded domains, one may ask. The answer will be evident when we discuss Picard-type theorems.

We present a condition for the visibility of  $\partial\Omega$  that answers the question in italics stated above. The condition, roughly, is that with  $\Omega$  as in the above question, the set of points at which  $\partial\Omega$  is weakly Levi pseudoconvex, if non-empty, is small in a specific sense but **not** necessarily totally disconnected. The notation in the theorem below is as described prior to Theorem 1.1.

**Theorem 1.4.** Let  $\Omega \subset \mathbb{C}^n$ ,  $n \geq 2$ , be an unbounded Kobayashi hyperbolic domain that is pseudoconvex and has  $C^2$ -smooth boundary. Suppose there exists a  $C^2$ -smooth closed 1-submanifold S of  $\partial\Omega$  such that  $w(\partial\Omega) \subset S$ . Assume that for each  $p \in w(\partial\Omega)$ , there exists a neighbourhood  $U_p$  of p and  $m_p > 2$  such that

$$\mathscr{L}_{\Omega}(\xi;v)\gtrsim \operatorname{dist}(\xi,S)^{m_p-2}\|v\|^2 \quad \forall v\in H_{\xi}(\partial\Omega) \ and \ \forall \xi\in (\partial\Omega\cap U_p)\setminus S.$$

Then,  $\partial\Omega$  is visible.

Before we can present our next result, we need a general definition.

**Definition 1.5.** Let Z be a complex manifold and Y a complex submanifold of Z. We say that Y is hyperbolically imbedded in Z if for every point  $p \in \overline{Y}$  and for each neighbourhood  $U_p$  of p in Z, there exists a neighbourhood  $V_p$  of p in Z with  $V_p \subseteq U_p$  such that  $K_Y(\overline{V_p} \cap Y, Y \setminus U_p) > 0$ .

The property of being hyperbolically imbedded is relevant to a class of extension theorems that we wish to examine further. The archetypal results of this class are:

**Result 1.6** (Kiernan, [16]: paraphrased for Y, Z manifolds). Let Z be a complex manifold and let  $Y \subset Z$  be a hyperbolically imbedded relatively compact submanifold.

- (1) Then, every holomorphic map  $f: \mathbb{D}^* \to Y$  extends as a holomorphic map  $\widetilde{f}: \mathbb{D} \to Z$ .
- (2) Let X be a complex manifold, let  $k = \dim_{\mathbb{C}}(X)$ , and let  $A \subsetneq X$  be an analytic subvariety of X of dimension (k-1) having at most normal-crossing singularities. Then, any holomorphic map  $f: X \setminus A \to Y$  extends as a holomorphic map  $\widetilde{f}: X \to Z$ .

Kwack had established a result of a similar nature under an analytical hypothesis [17, Theorem 3], whose proof is repurposed in [16] to prove (1) above. Although the hypothesis of (2) is considerably more general than of (1), an inductive argument—enabled by the definition of a normal-crossing singularity—essentially reduces (2) to proving (1). Theorems such as Result 1.6 are called *Picard-type extension theorems*. The reason for this terminology is as follows: if  $Z = \mathbb{CP}^1$  and  $Y = \mathbb{C} \setminus \{0, 1\}$ , then (1) is equivalent to the Little Picard Theorem.

It is well known that the complement of (2n+1) hyperplanes in general position in  $\mathbb{CP}^n$  is hyperbolically imbedded in  $\mathbb{CP}^n$ . Beyond examples resembling the latter, it is, in general, hard to tell when Y is hyperbolically imbedded in Z: Y, Z as above. It is natural to seek geometric conditions on Y that would lead to the same conclusions as Result 1.6. A good place to seek such a theorem (i.e., with more explicit geometric conditions on Y) would be to start with  $Z = \mathbb{C}^n$  and Y a domain in  $\mathbb{C}^n$ . But observe that if  $Y \subsetneq \mathbb{C}^n$  is relatively compact, then the extension problem becomes trivial because of Riemann's removable singularities theorem. Thus, in the latter setting, we are compelled to focus on the the case when Y is an unbounded domain.

**Theorem 1.7.** Let  $\Omega \subset \mathbb{C}^n$ ,  $n \geq 2$ , be an unbounded Kobayashi hyperbolic domain with the properties stated in Theorem 1.4. Let X be a complex manifold, let  $k = \dim_{\mathbb{C}}(X)$ , and let  $A \subsetneq X$  be an analytic subvariety of dimension (k-1) having at most normal-crossing singularities. Then, any holomorphic map  $f: X \setminus A \to \Omega$  extends as a continuous map  $\tilde{f}: X \to (\mathbb{C}^n)^{\infty}$ .

Here,  $(\mathbb{C}^n)^{\infty}$  denotes the one-point compactification of  $\mathbb{C}^n$ . Both Theorems 1.4 and 1.7 feature the same domain  $\Omega$ . This is because, as in Result 1.6,  $\Omega$  being hyperbolically imbedded continues to be crucial to  $f: X \setminus \mathcal{A} \to \Omega$  admitting an extension with any degree of regularity. The latter condition follows if  $\partial\Omega$  is visible; see Proposition 2.5 and the remark that precedes it. Thisin view of the discussion preceding Theorem 1.7—is the reason for our interest in identifying unbounded visibility domains. Since  $\Omega$  in Theorem 1.7 is not relatively compact, it does not follow from Result 1.6. Instead, we rely on the work of Joseph-Kwack [14]; see Result 6.1. Their work does not, however, provide any geometric conditions for a domain  $\Omega$ , whether in  $\mathbb{C}^n$  or in some complex manifold, to be hyperbolically imbedded. The focus of [14] is a set of functiontheoretic characterisations of hyperbolic imbedding. It is, therefore, natural to seek geometric conditions for a domain  $\Omega \subsetneq \mathbb{C}^n$  to be hyperbolically imbedded. The hypothesis of Theorems 1.4 and 1.7 provides just such a condition. Moreover, domains satisfying the above hypothesis are abundant, as seen, for instance, from [11, Section 3.2]. We conclude our introduction with one last question: could one extend Theorem 1.7 to domains  $X \setminus \mathcal{A}$  such that  $\mathcal{A}$  has worse singularities? The condition on A stems from our use of the work cited above. Moreover, we can show that — in the notation of Theorem 1.7 — a holomorphic map  $f: X \setminus \mathcal{A} \to \Omega$  with  $\Omega$ hyperbolically imbedded does not, in general, extend continuously to X if the singularities of Aare even slightly worse than normal-crossing singularities; see Example 6.3.

# 2. Preliminaries on hyperbolic imbedding and visibility

This section is devoted to assorted observations of a technical nature that will be needed in our discussion surrounding Theorems 1.4 and 1.7. But we first explain some notation used below and in later sections (some of which has also been used without comment in Section 1).

## 2.1. Common notations.

- (1) For  $v \in \mathbb{R}^d$ , ||v|| denotes the Euclidean norm. For any  $x \in \mathbb{R}^d$  and  $A \subset \mathbb{R}^d$ , we write  $\operatorname{dist}(x,A) := \inf\{||x-a|| : a \in A\}.$
- (2) Given a point  $x \in \mathbb{R}^d$  and r > 0,  $\mathbb{B}^d(x, r)$  denotes the open Eulidean ball in  $\mathbb{R}^d$  with radius r and center x.

- (3) Given a point  $z \in \mathbb{C}^n$  and r > 0,  $B^n(z, r)$  denotes the open Eulidean ball in  $\mathbb{C}^n$  with radius r and center z. For simplicity, we write  $\mathbb{D} := B^1(0, 1)$ . Also, we write  $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ .
- (4) Given a  $\mathcal{C}^2$ -smooth function  $\phi: \Omega \to \mathbb{C}$  defined in some domain  $\Omega \subset \mathbb{C}^n$ ,  $(\mathfrak{H}_{\mathbb{C}}\phi)(z)$  denotes the complex Hessian of  $\phi$  at  $z \in \Omega$ .
- (5)  $\langle \cdot, \cdot \rangle$  denotes the standard Hermitian inner product on  $\mathbb{C}^n$ .

# 2.2. **Definitions and results.** We begin with an elementary fact:

**Lemma 2.1.** Let Z be a complex manifold. Let X and Y be domains in Z such that  $X \subsetneq Y \subsetneq Z$ . If X is hyperbolically imbedded in Z, then X is hyperbolically imbedded in Y as well.

Let p,  $U_p$  and  $V_p$  be as in Definition 1.5; by the fact that the closure of  $V_p \cap Y$  in Y equals  $(\overline{V_p \cap Y}) \cap Y$  (where  $\overline{V_p \cap Y}$  denotes the closure in Z), the above result follows immediately.

The remainder of this section focuses on definitions and facts related to the property of visibility of  $\partial\Omega$ ,  $\Omega\subset\mathbb{C}^n$  being a domain.

**Definition 2.2.** Let  $\Omega \subset \mathbb{C}^n$  be a domain and let  $I \subset \mathbb{R}$  be an interval. For  $\lambda \geq 1$  and  $\kappa \geq 0$ , a curve  $\sigma : I \to \Omega$  is called a  $(\lambda, \kappa)$ -almost-geodesic if

- $\lambda^{-1}|t-s|-\kappa \leq K_{\Omega}(\sigma(s),\sigma(t)) \leq \lambda|t-s|+\kappa$  for every  $s,t\in I$ , and
- $\sigma$  is absolutely continuous (whereby  $\sigma'(t)$  exists for almost every  $t \in I$ ) and  $k_{\Omega}(\sigma(t); \sigma'(t)) \le \lambda$  for almost every  $t \in I$ .

Next, we present a definition that formalises one of the sufficient conditions on a domain  $\Omega \subset \mathbb{C}^n$  under which  $\partial \Omega$  is visible. It is an adaptation, introduced by Bharali–Zimmer [3], to unbounded domains of a well-known property.

**Definition 2.3.** Let  $\Omega \subset \mathbb{C}^n$  be a domain. We say that  $\Omega$  satisfies a local interior-cone condition if for each R > 0 there exist constants  $r_0 > 0$ ,  $\theta \in (0, \pi)$ , and a compact subset  $K \subset \Omega$ , which depend on R, such that for each  $z \in B^n(0, R) \cap (\Omega \setminus K)$ , there is a point  $\xi_z \in \partial \Omega$  and a unit vector  $v_z$  such that

- $z = \xi_z + tv_z$  for some  $t \in (0, r_0)$ , and
- $(\xi_z + \Gamma(v_z, \theta)) \cap B^n(\xi_z, r_0) \subset \Omega$ .

Here,  $\Gamma(v_z, \theta)$  denotes the open cone

$$\Gamma(v_z, \theta) := \{ w \in \mathbb{C}^n : \operatorname{Re} \langle w, v_z \rangle > \cos(\theta/2) \|w\| \}.$$

The following result is classical in the case when  $\Omega$  is bounded. But the choice of K = K(R) in Definition 2.3 requires care when  $\Omega$  is unbounded, for which reason we provide a proof.

**Lemma 2.4.** Let  $\Omega \subset \mathbb{C}^n$  be an unbounded domain with  $C^2$ -smooth boundary. Then,  $\Omega$  satisfies a local interior-cone condition.

*Proof.* Since  $\partial\Omega$  is  $\mathcal{C}^2$ -smooth, for each  $p \in \partial\Omega$ , we can find balls  $W_p := B^n(p, R_p)$ ,  $V_p := B^n(p, r_p)$  with  $0 < r_p < R_p$  such that the following holds:

- (a) For each  $z \in V_p \cap \Omega$ , there exists unique  $\xi_z \in \partial \Omega$  such that  $\|\xi_z z\| = \delta_{\Omega}(z)$  and  $\xi_z \in W_p$ ,
- (b) If  $\xi_1 \neq \xi_2 \in \partial\Omega \cap B^n(p, R_p)$ , then  $\{\xi_1 + t\eta_{\xi_1} : t \geq 0\} \cap \{\xi_2 + t\eta_{\xi_2} : t \geq 0\} \cap B^n(p, r_p) = \emptyset$  (where  $\eta_{\xi}$  denotes the inward unit normal at  $\xi \in \partial\Omega$ ).

Fix R > 0. If  $\overline{B^n(0,R)} \cap \Omega = \emptyset$  or if  $\overline{B^n(0,R)} \subset \Omega$ , then the two conditions in Definition 2.3 hold true vacuously (taking  $K = \overline{B^n(0,R)}$  in the latter case). Hence, fix R > 0 such that  $\partial \Omega \cap \overline{B^n(0,R)} \neq \emptyset$ . Write  $S := \partial \Omega \cap \overline{B^n(0,R)}$ . Let  $W := \bigcup_{p \in S} W_p$  and  $V := \bigcup_{p \in S} V_p$ . As

S is compact, we can find a finite subcover  $\{V_1, \dots, V_k\}$  of  $\{V_p : p \in S\}$  that covers S. Write  $V_j := B^n(p_j, r_j)$  and  $W_j := B^n(p_j, R_j)$ . We can choose r > 0 sufficiently small such that

$$\overline{\bigcup_{p \in S} B^n(p, r)} \subset \bigcup_{j=1}^k V_j. \tag{2.1}$$

Let K be the compact set,  $K \subset \Omega$ , defined as follows:

$$K:=\overline{B^n(0,R)}\bigcap \left(\overline{\Omega}\setminus \bigcup\nolimits_{p\in S}B^n(p,r/2)\right).$$

Let  $r_0 := r/2$ . (Note that K and  $r_0$  depend on R.)

Fix  $z \in B^n(0,R) \cap (\Omega \setminus K)$ . Then,  $z \in \bigcup_{p \in S} B^n(p,r/2)$ . Hence, by (2.1),  $z \in \bigcup_{j=1}^k V_j$ . Thus, there exists a unique point in  $\partial \Omega$ , call it  $\xi_z$ , such that  $\delta_{\Omega}(z) = ||z - \xi_z||$ . Let  $\eta_{\xi_z}$  denote the inward unit normal vector to  $\partial \Omega$  at  $\xi_z$ . Let  $z' := \xi_z + (r/2)\eta_{\xi_z}$ . If p is a point in S such that  $z \in B^n(p,r/2)$ , then it is immediate that:

- $||z-z'|| = |r/2 \delta_{\Omega}(z)|$ , whereby  $z' \in B^n(p,r)$ .
- If, for some j = 1, ..., k,  $V_j$  contains z' (owing to (2.1)), then (with  $\xi_{z'}$  having a meaning analogous to  $\xi_z$ )  $\xi_z, \xi_{z'} \in B^n(p_j, R_j) =: W_j$ .

Thus, by the property of the pair  $(V_j, W_j)$  given by (b) above,  $\xi_z = \xi_{z'}$ ; thus  $\delta_{\Omega}(z') = ||z' - \xi_z|| = r/2$ . Therefore,  $B^n(z', r/2) \subset \Omega$ .

Now, clearly,  $z = \xi_z + t\eta_{\xi_z}$ , where  $t = ||z - \xi_z|| < r/2 = r_0$ . Also, it is easy to see that there exists a uniform  $\theta \in (0, \pi)$  such that

$$(\xi_z + \Gamma(\eta_{\xi_z}, \theta)) \cap B^n(\xi_z, r_0) \subset B^n(z', r_0) \cap B^n(\xi_z, r_0) \subset \Omega.$$

Here,  $\theta$  is given by the following:

$$\cos(\theta/2) = \operatorname{Re}\langle \eta_{\xi_z}, v \rangle / \|v\|$$

$$= \operatorname{Re}\langle \eta_{\xi_z}, v \rangle / r_0, \quad \text{where } v \in \partial B^n(\xi_z, r_0) \cap \partial B^n(z', r_0).$$

In the above expression,  $\theta$  is independent of the choice of v as the inner product depends only on  $r_0$ . For this reason,  $\theta$  is also independent of  $z \in B^n(0,R) \cap (\Omega \setminus K)$ . This establishes the conditions given in Definition 2.3.

The next result is a version of a result due to Sarkar [19, Proposition 3.2-(3)]. However, unlike in [19, Proposition 3.2-(3)], we are given that  $\partial\Omega$  is visible as a part of the hypothesis of the result below. This results in a simpler proof than in [19]. Since it is so vital to proving Theorem 1.7, we shall provide a proof of the following

**Proposition 2.5.** Let  $\Omega \subset \mathbb{C}^n$  be an unbounded Kobayashi hyperbolic domain and suppose  $\partial\Omega$  is visible. Then,  $\Omega$  is hyperbolically imbedded in  $\mathbb{C}^n$ .

*Proof.* Let  $p \in \partial \Omega$ . Fix a pair of bounded  $\mathbb{C}^n$ -neighbourhoods  $U_p, V_p$  of p such that  $V_p \subseteq U_p$ . It suffices to show that  $K_{\Omega}(\overline{V_p} \cap \Omega, \Omega \setminus U_p) > 0$ .

We prove the above by contradiction. Assume that  $K_{\Omega}(\overline{V_p} \cap \Omega, \Omega \setminus U_p) = 0$ . Then, there exist a pair of sequences  $\{z_{\nu}\} \subset \overline{V_p} \cap \Omega$  and  $\{w_{\nu}\} \subset \Omega \setminus U_p$  such that  $K_{\Omega}(z_{\nu}, w_{\nu}) \to 0$  as  $\nu \to \infty$ . As  $\Omega$  is Kobayashi hyperbolic, by [3, Proposition 5.3], for each  $\nu$  there exists a  $(1, 1/\nu)$ -almost-geodesic  $\sigma_{\nu}: [a_{\nu}, b_{\nu}] \to \Omega$  joining  $z_{\nu}$  and  $w_{\nu}$ .

**Claim.** There exist a subsequence  $\{(z_{\nu_k}, w_{\nu_k})\}$  of  $\{(z_{\nu_k}, w_{\nu_k})\}$  and a compact  $K \subset \Omega$  such that  $\sigma_{\nu_k}([a_{\nu_k}, b_{\nu_k}]) \cap K \neq \emptyset$  for all k.

Proof of claim: Suppose  $\{z_{\nu} : \nu \in \mathbb{Z}_{+}\} \subseteq \Omega$ . Let  $K := \overline{\{z_{\nu} : \nu \in \mathbb{Z}_{+}\}}$ , which is contained in  $\Omega$  and is compact, since  $V_{p}$  is bounded. Clearly,  $\sigma_{\nu}([a_{\nu}, b_{\nu}]) \cap K \neq \emptyset$  for all  $\nu$ .

Now, suppose  $\overline{\{z_{\nu} : \nu \in \mathbb{Z}_{+}\}} \nsubseteq \Omega$ . Then, passing to a subsequence and relabelling, if needed, we may assume that  $z_{\nu} \to \xi$ , for some  $\xi \in \partial \Omega \cap \overline{V_p}$ . For each  $\nu$ , define

$$t_{\nu} := \inf\{t \in [a_{\nu}, b_{\nu}] : \sigma_{\nu}(t) \in \Omega \setminus U_p\}.$$

Clearly,  $t_{\nu} \in (a_{\nu}, b_{\nu})$  and  $\sigma_{\nu}(t_{\nu}) \in \partial U_{p} \cap \Omega$ . Write  $\zeta_{\nu} := \sigma_{\nu}(t_{\nu})$ . As before, if  $\{\zeta_{\nu} : \nu \in \mathbb{Z}_{+}\} \in \Omega$ , then  $K := \overline{\{\zeta_{\nu} : \nu \in \mathbb{Z}_{+}\}}$  is our desired compact set that intersects the image of  $\sigma_{\nu}$  for each  $\nu$ . If not, then we get a subsequence  $\{\zeta_{\nu_{k}}\}$  of  $\{\zeta_{\nu}\}$  and a point  $\eta \in \partial\Omega \cap \partial U_{p}$  such that  $\zeta_{\nu_{k}} \to \eta$ . Clearly,  $\eta \neq \xi = \lim_{k \to \infty} z_{\nu_{k}}$ . Observe that  $\widetilde{\sigma}_{k} := \sigma_{\nu_{k}}|_{[a_{\nu_{k}}, t_{\nu_{k}}]}$  is a (1, 1)-almost-geodesic joining  $z_{\nu_{k}}$  and  $\zeta_{\nu_{k}}$ . Thus, as  $\partial\Omega$  is visible and as  $\xi$  and  $\eta$  are distinct boundary points, there is a compact  $K \subset \Omega$  such that  $\mathrm{image}(\widetilde{\sigma}_{k}) \cap K \neq \emptyset$  for every k sufficiently large, from which the claim follows.

Now, let  $o_k := \widetilde{\sigma}_k(s_k) \in \mathsf{image}(\widetilde{\sigma}_k) \cap K$ . Without loss of generality, we can assume that there is a point  $o \in K$  such that  $o_k \to o$ . Using the fact that  $\sigma_{\nu_k} : [a_{\nu_k}, b_{\nu_k}] \to \Omega$  is a  $(1, 1/\nu_k)$ -almost-geodesic, we get

$$K_{\Omega}(z_{\nu_k}, o_k) + K_{\Omega}(o_k, w_{\nu_k}) \le (s_k - a_{\nu_k}) + (b_{\nu_k} - s_k) + 2/\nu_k$$
  
=  $(b_{\nu_k} - a_{\nu_k}) + 2/\nu_k \le K_{\Omega}(z_{\nu_k}, w_{\nu_k}) + 3/\nu_k \quad \forall k.$ 

By assumption, the right-hand side of the above inequality goes to 0 as  $k \to \infty$ . Then, as  $\Omega$  is Kobayashi hyperbolic, we must have

$$\lim_{k \to \infty} z_{\nu_k} = \lim_{k \to \infty} w_{\nu_k} = \lim_{k \to \infty} o_k = o,$$

which is impossible since  $(\overline{V}_p \cap \overline{\Omega}) \cap (\overline{\Omega} \setminus U_p) = \emptyset$ . We have arrived at a contradiction. This proves the result.

## 3. Analytical preliminaries

This section is devoted to definitions and results that will be essential to the proofs of Theorems 1.1 and 1.4. Recall that the exterior derivative  $d = (\partial + \overline{\partial})$  and  $d^c := i(\partial - \overline{\partial})$ . Let  $\Omega \subset \mathbb{C}^n$ ,  $n \geq 2$ , be a bounded domain. Given two functions  $\phi \in \mathcal{C}(\partial\Omega;\mathbb{R})$  and  $h \in \mathcal{C}(\Omega;\mathbb{R})$ ,  $h \geq 0$ , the *Dirichlet problem for the complex Monge-Ampère equation* is the non-linear boundary-value problem that seeks a function  $u \in \mathcal{C}(\overline{\Omega};\mathbb{R})$  such that  $u|_{\Omega}$  is plurisubharmonic (which we shall denote as  $u \in \text{psh}(\Omega) \cap \mathcal{C}(\overline{\Omega})$  such that

$$(dd^{c}u)^{n} := \underbrace{dd^{c}u \wedge \dots \wedge dd^{c}u}_{n \text{ factors}} = h\mathcal{V}_{n},$$

$$u|_{\partial\Omega} = \phi,$$

$$(3.1)$$

where  $V_n$  is defined as

$$\mathcal{V}_n := (i/2)^n (dz_1 \wedge d\overline{z}_1) \wedge \cdots \wedge (dz_n \wedge d\overline{z}_n).$$

When  $u|_{\Omega} \notin C^2(\Omega; \mathbb{R})$ , the left-hand side of (3.1) must be interpreted as a current of bidegree (n, n). That this makes sense when  $u \in psh(\Omega) \cap C(\overline{\Omega})$  was established by Bedford-Taylor [1].

Our objective in considering the above Dirichlet problem is as follows. With  $\Omega$ , h, and  $\phi$  as above, any solution of this problem is a function  $u \in \text{psh}(\Omega) \cap \mathcal{C}(\overline{\Omega})$  that satisfies  $u|_{\partial\Omega} = \phi$ ; we would like to establish that there exist functions with the latter properties that belong to some Hölder class on  $\overline{\Omega}$ —assuming that  $\partial\Omega$  is sufficiently "nice" and  $\phi$  is sufficiently regular. The regularity theory for the complex Monge–Ampère equation provides us the means to the latter end. A regularity theorem of the type hinted at for  $\Omega$  strongly pseudoconvex was established by Bedford–Taylor [1, Theorem 9.1]. Such theorems are much harder to deduce when  $\Omega$  is weakly pseudoconvex. One such theorem is a special case of a result by Ha–Khanh [12]. Recall that  $\langle \cdot, \cdot \rangle$  denotes the standard Hermitian inner product on  $\mathbb{C}^n$ .

**Result 3.1** (special case of [12, Theorem 1.5]). Let  $\Omega \subset \mathbb{C}^n$ ,  $n \geq 2$ , be a bounded pseudoconvex domain having  $\mathcal{C}^2$ -smooth boundary, let  $\rho$  be a defining function of  $\Omega$ , and let  $m \geq 2$ . Suppose

(\*) there exists a neighbourhood U of  $\partial\Omega$ , constants c, C > 0 and, for each  $\delta > 0$  sufficiently small, there exists a plurisubharmonic function  $\varphi_{\delta}$  on U of class  $\mathcal{C}^2$  such that  $|\varphi_{\delta}| \leq 1$  and such that

$$\langle v, (\mathfrak{H}_{\mathbb{C}}\varphi_{\delta})(z)v \rangle \ge c (1/\delta)^{2/m} ||v||^2 \quad \forall v \in \mathbb{C}^n,$$
 (3.2)

$$||D\varphi_{\delta}(z)|| \le C/\delta,\tag{3.3}$$

for each  $z \in \rho^{-1}((-\delta, 0))$ .

Let  $\phi \in \mathcal{C}^{s,\alpha}(\partial\Omega)$ , s = 0,1,  $\alpha \in (0,1]$ . Then, the Dirichlet problem

$$(dd^c u)^n = 0,$$
$$u|_{\partial\Omega} = \phi,$$

has a unique plurisubharmonic solution  $u \in \mathcal{C}^{0,(s+\alpha)/m}(\overline{\Omega})$ .

The notation  $C^{j,\beta}$ ,  $j \in \mathbb{N}$ ,  $\beta \in (0,1]$ , denotes the class of all **real**-valued functions that are continuously differentiable to order j (the latter being suitably interpreted for the underlying space when  $j \geq 1$ ) and whose j-th partial derivatives satisfy a uniform Hölder condition with exponent  $\beta$ . In what follows, if j = 0 and  $\beta \in (0,1)$ , we shall denote this class simply as  $C^{\beta}$ .

The following result provides the connection between Result 3.1 and the condition on the Levi form in Theorems 1.1 and 1.4.

**Result 3.2** (special case of [15, Theorem 2.1]). Let  $\Omega \subset \mathbb{C}^n$ ,  $n \geq 2$ , be a bounded pseudoconvex domain having  $\mathbb{C}^2$ -smooth boundary. Assume there exists a  $\mathbb{C}^2$ -smooth closed submanifold S of  $\partial\Omega$  such that S is totally-real and such that  $w(\partial\Omega) \subset S$ . Suppose there exists a number m > 2 such that

$$\mathscr{L}_{\Omega}(\xi; v) \gtrsim \operatorname{dist}(\xi, S)^{m-2} ||v||^2 \quad \forall v \in H_{\xi}(\partial \Omega) \text{ and } \forall \xi \in \partial \Omega \setminus S.$$
 (3.4)

Then,  $\Omega$  satisfies the condition (\*) in Result 3.1.

Remark 3.3. Some comments about Result 3.2 are in order. Firstly, [15, Theorem 2.1] is stated for q-pseudoconvex domains satisfying a somewhat more general condition than (3.4). Result 3.2 is obtained by taking:

- q = 1, and
- $F(t) = ct^m$ , t > 0, for some c > 0,

in [15, Theorem 2.1]. (It must be noted that there is a small typo in the description of F in [15]; the asymptotic behaviour required of F is  $F(\delta)/\delta^2 \searrow 0$  as  $\delta \searrow 0$  and not what is stated on [15, p. 2769].) Secondly, the proof in [15] establishes just the estimate (3.2) (which is condition (1.5) in [15]). However, it is evident from the expression for  $\varphi_{\delta}$  given that, since m > 2, the estimate (3.3) is satisfied.

We require one last result for proving Theorems 1.1 and 1.4.

**Result 3.4** (paraphrasing [20, Proposition 6]). Let  $\Omega \subset \mathbb{C}^n$  be a domain and  $z \in \Omega$ . If there exists a negative plurisubharmonic function u on  $\Omega$  that is of class  $C^2$  in a neighbourhood of z and satisfies

$$\langle v, (\mathfrak{H}_{\mathbb{C}}u)(z)v \rangle \ge c||v||^2 \quad \forall v \in \mathbb{C}^n,$$

for some c > 0, then

$$k_{\Omega}(z;v) \ge \left(\frac{c}{\alpha}\right)^{1/2} \frac{\|v\|}{|u(z)|^{1/2}} \quad \forall v \in \mathbb{C}^n,$$

where  $\alpha > 0$  is a universal constant.

#### 4. Lower bounds for the Kobayashi metric

We begin by stating and proving the general result relying on the complex Monge–Ampére equation to estimate the Kobayashi metric that was hinted at in Section 1.1. Before we state it, we need a definition.

**Definition 4.1.** A function  $\omega : [0, \infty) \to [0, \infty)$  is called a *modulus of continuity* if it is concave, monotone increasing, and such that  $\lim_{x\to 0^+} \omega(x) = \omega(0) = 0$ .

**Theorem 4.2.** Let  $\Omega \subset \mathbb{C}^n$ ,  $n \geq 2$ , be a bounded domain. Suppose there exists a modulus of continuity  $\omega : ([0,\infty),0) \to ([0,\infty),0)$  and that, for each Lipschitz function  $\phi : \partial\Omega \to \mathbb{R}$ , there exists a function  $u_{\phi} : \overline{\Omega} \to \mathbb{R}$  such that  $u_{\phi}|_{\Omega}$  solves the complex Monge-Ampère equation

$$(dd^c u)^n = 0,$$
$$u|_{\partial\Omega} = \phi,$$

and satisfies

$$|u_{\phi}(z_1) - u_{\phi}(z_2)| \le C_{\phi} \,\omega(\|z_1 - z_2\|) \quad \forall z_1, z_2 \in \overline{\Omega},$$
 (4.1)

for some constant  $C_{\phi} > 0$ . Then there exists a constant c > 0 such that

$$k_{\Omega}(z;v) \ge c \frac{\|v\|}{\omega(\delta_{\Omega}(z))^{1/2}} \quad \forall z \in \Omega \text{ and } v \in \mathbb{C}^n.$$
 (4.2)

Before proving Theorem 4.2 we state the following elementary lemma.

**Lemma 4.3.** Let  $\omega : [0, \infty) \to [0, \infty)$  be a concave, monotone increasing function such that  $\omega(0) = 0$ . Then, for all  $\lambda, x \geq 0$ ,  $\omega(\lambda x) \leq (\lambda + 1)\omega(x)$ .

The proof of Theorem 4.2. Define  $\phi: \partial\Omega \to (-\infty, 0]$  by  $\phi(z) := -2||z||^2$ . As this function is Lipschitz, there exists a function  $u_{\phi}: \overline{\Omega} \to \mathbb{R}$  with the properties stated in the hypothesis of Theorem 4.2. Let us define

$$\Phi(z) := u_{\phi}(z) + ||z||^2 \quad \forall z \in \overline{\Omega}.$$

For  $\nu \in \mathbb{N}$ , write  $\Omega_{\nu} := \{z \in \Omega : \delta_{\Omega}(z) > 1/2^{\nu}\}$ . Let  $\nu_0 \in \mathbb{Z}_+$  and be so large that  $\Omega_{\nu}$  is connected for every  $\nu \geq \nu_0$ . It follows from [18, Satz 4.2] by Richberg that there exists a plurisubharmonic function  $\Psi$  on  $\Omega$  of class  $\mathcal{C}^{\infty}$  such that for all  $\nu \geq \nu_0$ 

$$0 \le \Psi(z) - \Phi(z) \le \omega(2^{-\nu}) \quad \forall z \in \Omega \setminus \Omega_{\nu}. \tag{4.3}$$

Clearly,  $\Psi$  extends continuously to  $\overline{\Omega}$  (we shall refer to this extension as  $\Psi$  as well) and

$$\Psi(z) = -\|z\|^2 \quad \forall z \in \partial\Omega. \tag{4.4}$$

Now, let us write  $U(z) := \Psi(z) + ||z||^2$  for each  $z \in \overline{\Omega}$ . Since  $\Psi$  is plurisubharmonic,

$$\langle v, (\mathfrak{H}_{\mathbb{C}}U)(z)v \rangle \ge ||v||^2 \quad \forall z \in \Omega \text{ and } v \in \mathbb{C}^n.$$
 (4.5)

Fix a z such that  $\delta_{\Omega}(z) \leq 1/2^{\nu_0}$ . As  $\partial\Omega$  is compact, there exists a point  $\xi_z \in \partial\Omega$  such that

$$\delta_{\Omega}(z) = \|z - \xi_z\|.$$

There exists an integer  $\nu_z \geq \nu_0$  such that

$$1/2^{(\nu_z+1)} < \delta_{\Omega}(z) \le 1/2^{\nu_z}.$$

It follows from (4.3) that

$$|U(z)| \le |\Psi(z) - \Phi(z)| + |\Phi(z) + ||z||^{2}|$$

$$\le \omega(2^{-\nu_{z}}) + |(\Phi(z) + ||z||^{2}) - (\Phi(\xi_{z}) + ||\xi_{z}||^{2})|. \tag{4.6}$$

Now, owing to our hypothesis on  $u_{\phi}$ , there exists a constant  $C_1 > 0$  such that

$$|(\Phi(z) + ||z||^2) - (\Phi(\xi_z) + ||\xi_z||^2)| \le C_\phi \,\omega(\delta_\Omega(z)) + C_1 \delta_\Omega(z).$$

Here, we have used the condition (4.1) and the fact that  $||z - \xi_z|| = \delta_{\Omega}(z)$ . Combining the last estimate with (4.6), we get, in view of Lemma 4.3:

$$|U(z)| \leq \left(\frac{2^{-\nu_z}}{\delta_{\Omega}(z)} + 1\right) \omega(\delta_{\Omega}(z)) + C_{\phi} \,\omega(\delta_{\Omega}(z)) + C_1 \delta_{\Omega}(z)$$
  
$$\leq (3 + C_{\phi}) \omega(\delta_{\Omega}(z)) + C_1 \delta_{\Omega}(z).$$

From the latter estimate, the fact that  $\omega$  is concave, and that z—apart from the constraint  $\delta_{\Omega}(z) \leq 1/2^{-\nu_0} \leq 1/2$ —was chosen arbitrarily, we have

$$|U(z)| \le C\omega(\delta_{\Omega}(z)) \quad \forall z \in \Omega \text{ such that } \delta_{\Omega}(z) \le 1/2^{\nu_0}$$

for some constant C > 0. Since the set  $\{z \in \Omega : \delta_{\Omega}(z) \ge 1/2^{\nu_0}\}$  is compact, raising the value of C > 0 if needed, we get

$$|U(z)| \le C\omega(\delta_{\Omega}(z)) \quad \forall z \in \Omega.$$
 (4.7)

By (4.4), we get  $U|_{\partial\Omega} = 0$ . Thus, by the Maximum Principle, U is a smooth negative plurisubharmonic function. Thus, from (4.5), (4.7), and Result 3.4, we conclude that

$$k_{\Omega}(z;v) \ge \left(\frac{1}{C\alpha}\right)^{1/2} \frac{\|v\|}{\omega(\delta_{\Omega}(z))^{1/2}} \quad \forall z \in \Omega \text{ and } v \in \mathbb{C}^n,$$

which is the desired lower bound.

A substantial part of the proof of Theorem 1.1 is the same as that of the previous theorem. However, Theorem 1.1 is not a special case of Theorem 4.2; the assumption on  $\partial\Omega$  gives us better boundary behaviour of the solutions of the same Dirichlet problem considered in the proof above. With these words, we give:

The proof of Theorem 1.1. Given our assumptions on  $\partial\Omega$ , Result 3.2 implies that  $\Omega$  satisfies the condition (\*) in Result 3.1. As in the proof of Theorem 4.2, define  $\phi:\partial\Omega\to(-\infty,0]$  by  $\phi(z):=-2\|z\|^2$ . As  $\phi\in\mathcal{C}^{1,1}(\partial\Omega)$ , taking the values s=1 and  $\alpha=1$  in the conclusion of Result 3.1, we see that the Dirichlet problem stated in Result 3.1, with  $\phi$  as above, has a unique solution of class  $\mathcal{C}^{2/m}(\overline{\Omega})$ . Let us denote this solution by  $u_{\phi}$ . At this stage, exactly the same argument as in the proof of Theorem 4.2 with

$$\omega(r) := r^{2/m}, \quad r \in [0, \infty),$$

gives us a function U defined on  $\overline{\Omega}$  such that  $U|_{\Omega}$  is a smooth negative plurisubharmonic function that satisfies the conditions

$$|U(z)| \le C\delta_{\Omega}(z)^{2/m},\tag{4.8}$$

$$\langle v, (\mathfrak{H}_{\mathbb{C}}U)(z)v \rangle \ge ||v||^2$$
 (4.9)

(for some constant C > 0) for every  $z \in \Omega$  and  $v \in \mathbb{C}^n$ . From these inequalities and Result 3.4, we conclude that

$$k_{\Omega}(z;v) \ge \left(\frac{1}{C\alpha}\right)^{1/2} \frac{\|v\|}{\delta_{\Omega}(z)^{1/m}} \quad \forall z \in \Omega \text{ and } v \in \mathbb{C}^n,$$

which is the desired lower bound.

#### 5. The proof of Theorem 1.4

Before we can give the proof of Theorem 1.4, we give a definition that will be useful in the latter proof.

**Definition 5.1** (Bharali–Zimmer, [3]). Let  $\Omega \subset \mathbb{C}^n$  be a Kobayashi hyperbolic domain. Given a subset  $A \subset \overline{\Omega}$ , we define the function  $r \mapsto M_{\Omega, A}(r)$ , r > 0, as

$$M_{\Omega,A}(r) := \sup \left\{ \frac{1}{k_{\Omega}(z;v)} : z \in A \cap \Omega, \, \delta_{\Omega}(z) \le r, \, ||v|| = 1 \right\}.$$

The function  $M_{\Omega, A}$  is involved in one of the two conditions that a point  $p \in \partial \Omega$ , for  $\Omega$  as in the above definition, must satisfy to be what is called a "local Goldilocks point" by Bharali–Zimmer in [3]; see [3, Definition 1.3]. The connection between local Goldilocks points and the visibility property is given by the following

**Result 5.2** (paraphrasing [3, Theorem 1.4]). Let  $\Omega \subset \mathbb{C}^n$  be a Kobayashi hyperbolic domain. If the set of points in  $\partial\Omega$  that are not local Goldilocks points is a totally disconnected set, then  $\partial\Omega$  is visible.

We are now in a position to give

The proof of Theorem 1.4. The proof of Theorem 1.4 will be carried out in two steps.

Step 1. For  $p \in \partial \Omega$ , constructing a bounded subdomain  $D_p$  such that  $\partial \Omega \cap \partial D_p \ni p$  and is large  $\mathbf{Fix}\ p \in \partial \Omega$ . Consider a unitary change of coordinate  $Z = (Z_1, \ldots, Z_n)$  centered at p (i.e., Z(p) = 0) with respect to which  $T_p(\partial \Omega) = \{(Z_1, \ldots, Z_n) \in \mathbb{C}^n : \operatorname{Im}(Z_n) = 0\}$ , the outward unit normal to  $\partial \Omega$  at p = 0 is  $(0, \ldots, 0, -i)$ , and such that there exist a neighbourhood  $U_p^2 := \mathbb{B}^{2n-1}(0, r_2) \times (-r_2, r_2)$  and a function  $\varphi_p : (\mathbb{B}^{2n-1}(0, r_2), 0) \to (\mathbb{R}, 0)$  such that  $Z(\partial \Omega) \cap U_p^2$  is connected and

$$Z(\Omega) \cap U_p^2 \subset \{(Z', Z_n) \in \mathbb{B}^{2n-1}(0, r_2) \times \mathbb{R} : \operatorname{Im}(Z_n) > \varphi_p(Z', \operatorname{Re}(Z_n))\}$$

(here,  $r_2$  depends on p but, for simplicity of notation, we will omit suffixes and understand that this dependence is implied). Shrinking  $r_2$  if necessary, we will assume, additionally, that:

• 
$$\varphi_p(Z', x_n) \in (-r_2/3, r_2/3)$$
 for every  $(Z', x_n) \in \mathbb{B}^{2n-1}(0, r_2)$  and

$$\{Z\in\mathbb{B}^{2n-1}(0,r_2)\times\mathbb{R}:\varphi_p(Z',\operatorname{Re}(Z_n))<\operatorname{Im}(Z_n)<\varphi_p(Z',\operatorname{Re}(Z_n))+r_2/3\}\subset Z(\Omega),$$

• 
$$(Z(\partial\Omega)\cap U_p^2)\cap Z(w(\partial\Omega))=\emptyset$$
 if  $p\notin w(\partial\Omega)$ , and  $U_p^2\Subset U_p$  if  $p\in w(\partial\Omega)$ 

(where  $U_p$  is as in the statement of Theorem 1.4). Fix  $r_1 \in (0, r_2/2)$ . Let  $\psi_p : \mathbb{R}^{2n-1} \to [0, \infty)$  be a smooth, non-negative, radial convex function such that

$$\psi_p|_{\mathbb{B}^{2n-1}(0,r_1)} \equiv 0$$
, and  $\psi_p|_{\mathbb{R}^{2n-1}\setminus\mathbb{B}^{2n-1}(0,r_1)}$  is strongly convex. (5.1)

Clearly,

$$\operatorname{\mathsf{Graph}}\Bigl( (\varphi_p + \psi_p)|_{\mathbb{B}^{2n-1}(0,r_2) \setminus \overline{\mathbb{B}^{2n-1}(0,r_1)}} \Bigr)$$
 is a strongly Levi pseudoconvex hypersurface. (5.2)

Also, we can find a  $\psi_p$  that satisfies all the above conditions and such that  $0 \le \psi_p < r_2/3$  on  $\mathbb{B}^{2n-1}(0,r_2)$ , due to which

$$\mathcal{S}_p := \mathsf{Graph}\Big(\left.(\varphi_p + \psi_p)\right|_{\mathbb{B}^{2n-1}(0,r_2)}\Big) \Subset Z(\Omega) \cup \mathsf{Graph}\left.\left(\varphi_p\right|_{\overline{\mathbb{B}^{2n-1}(0,r_1)}}\right).$$

Owing to this and to (5.2), we can construct a bounded domain  $\widetilde{D}_p$  such that

- (a)  $\widetilde{D}_p \subsetneq Z(\Omega)$ ,
- (b)  $\mathcal{S}_p \subset \partial \widetilde{D}_p$ ,
- $(c) \ \ S_p := \operatorname{Graph} \left( \varphi_p|_{\overline{\mathbb{B}^{2n-1}(0,r_1)}} \right) = Z(\partial\Omega) \cap \partial \widetilde{D}_p,$
- (d)  $\partial \widetilde{D}_p$  is strongly Levi pseudoconvex at each  $\xi \in \partial \widetilde{D}_p \setminus S_p$  whenever  $p \in w(\partial \Omega)$ , and  $\widetilde{D}_p$  is a strongly Levi pseudoconvex domain when  $p \notin w(\partial \Omega)$ .

Write  $D_p := Z^{-1}(\widetilde{D}_p)$ . Finally, we can extend  $S \cap Z^{-1}(S_p)$ , whenever the latter is non-empty, to a  $\mathcal{C}^2$ -smooth closed 1-submanifold of  $\partial D_p$ .

**Step 2.** Showing that each  $p \in \partial \Omega$  is a local Goldilocks point.

It is well-known that p is a local Goldilocks point if  $p \notin w(\partial\Omega)$ . For a  $p \in w(\partial\Omega)$ , it follows from the discussion in the second paragraph of Step 1 and from the properties (a)–(d) that  $D_p$  satisfies all the conditions of Theorem 1.1 with  $m=m_p$ . Let  $\mathcal{U}_p:\overline{D}_p\to (-\infty,0]$  denote the function constructed in the proof of Theorem 1.1 that is plurisubharmonic on  $D_p$  and satisfies the conditions (4.8) and (4.9) with  $m=m_p$ . Let  $W_p$  be a neighbourhood of p having the following properties:

- $\overline{W_p \cap D_p} \cap \partial D_p \subsetneq \partial \Omega \cap \partial D_p$ . •  $\delta_{\Omega}(z) = \delta_{D_p}(z)$  for all  $z \in W_p$ .
- By a standard construction [11, Section 2], we can find an upper semicontinous plurisubharmonic function  $\widetilde{\mathcal{U}}_p: \Omega \to (-\infty, 0)$  such that

$$\mathcal{U}_p(z) = \widetilde{\mathcal{U}}_p(z) \quad \forall z \in W_p \cap D_p = W_p \cap \Omega. \tag{5.3}$$

Thus,  $\widetilde{\mathcal{U}}_p$  is a negative plurisubharmonic function on  $\Omega$  that is of class  $\mathcal{C}^2$  in a neighbourhood of z for each  $z \in W_p \cap D_p = W_p \cap \Omega$ . Hence, by Result 3.4, the conditions (4.8) and (4.9) applied to  $\mathcal{U}_p$ , and by (5.3), there exists a constant  $C_p > 0$  such that

$$k_{\Omega}(z;v) \ge \left(\frac{1}{C_p \alpha}\right)^{1/2} \frac{\|v\|}{\delta_{D_p}(z)^{1/m_p}}$$

$$= \left(\frac{1}{C_p \alpha}\right)^{1/2} \frac{\|v\|}{\delta_{\Omega}(z)^{1/m_p}} \quad \forall z \in W_p \cap \Omega \text{ and } v \in \mathbb{C}^n.$$

The equality above is due to the fact that  $\delta_{\Omega}(z) = \delta_{D_p}(z)$  for all  $z \in W_p$ . Write  $A^p := W_p \cap \overline{\Omega}$ . It follows from the above estimate that the quantity  $M_{\Omega, A^p}$  satisfies the estimate

$$M_{\Omega, A^p}(r) \le c_p r^{1/m_p}$$
 for all  $r > 0$  sufficiently small, (5.4)

for some  $c_p > 0$ .

By Lemma 2.4,  $\Omega$  satisfies a local interior cone condition. Thus, by [3, Lemma 2.2] and (5.4) it follows that p satisfies the conditions for being a local Goldilocks point. Since  $p \in w(\partial\Omega)$  was chosen arbitrarily, it follows that every boundary point is a local Goldilocks point. Thus, by Result 5.2,  $\partial\Omega$  is visible.

# 6. The proof of Theorem 1.7

In this section, we give the (short) proof of Theorem 1.7. This is a Picard-type extension theorem, as discussed in Section 1. Such a theorem relies upon  $\Omega \subset \mathbb{C}^n$ , although not relatively compact, being hyperbolically imbedded, just as in Result 1.6. We state the result by Joseph–Kwack, alluded to in Section 1, that formalises the latter statement. Here,  $Z^{\infty}$  denotes the one-point compactification of Z.

**Result 6.1** (Joseph–Kwack, [14, Corollary 7]: paraphrased for Y, Z manifolds). Let Z be a complex manifold and Y be a complex submanifold of Z such that Y is hyperbolically imbedded in Z. Let X be a complex manifold, let  $k = \dim_{\mathbb{C}}(X)$ , and let  $A \subseteq X$  be an analytic subvariety of dimension (k-1) having at most normal-crossing singularities. Then, any holomorphic map  $f: X \setminus A \to Y$  extends as a continuous map  $\widetilde{f}: X \to Z^{\infty}$ .

Remark 6.2. A comment on certain terminology and definitions used in Joseph–Kwack [14] are in order. With Y, Z as above, they defined a notion of when a point in  $\overline{Y} \subset Z$  is a hyperbolic point for Y (see [14, p. 363] for the definition). Many of the foundational results in [14] give certain necessary and sufficient conditions for a point in  $\overline{Y}$  to be a hyperbolic point for Y. A careful reading of the proofs in [14] indicates that the proof of [14, Corollary 7] relies on the latter

results, and its conclusion holds true under the assumption that each point in  $\overline{Y}$  is a hyperbolic point for Y. It turns out that Y is hyperbolically imbedded in Z (in the sense of Definition 1.5) if and only if each point in  $\overline{Y}$  is a hyperbolic point for Y; see [13], [21, Corollary 2].

Proof of Theorem 1.7. Since  $\Omega$  satisfies the hypothesis of Theorem 1.4,  $\partial\Omega$  is visible. So, owing to Proposition 2.5,  $\Omega$  is hyperbolically imbedded in  $\mathbb{C}^n$ . Thus, by Result 6.1, the proof follows immediately.

6.1. **An example.** We conclude this discussion with a basic example which shows that, for X,  $\mathcal{A}$  as in Section 1, a holomorphic map  $f: X \setminus \mathcal{A} \to \Omega$ , where  $\Omega$  is a hyperbolically imbedded domain, does not, in general, extend continuously to X if the singularities of  $\mathcal{A}$  are even slightly worse than normal-crossing singularities.

**Example 6.3.** An example of an unbounded planar domain  $\Omega$  that is hyperbolically imbedded in  $\mathbb{C}$  and a holomorphic function  $f:(\mathbb{D}^2 \setminus \mathcal{A}) \to \Omega$ , where  $\mathcal{A}$  is a closed analytic set in  $\mathbb{D}^2$  of codimension 1 containing singular points, but not normal-crossing singularities, such that f does **not** extend:

- either to a holomorphic function on  $\mathbb{D}^2$ ,
- or to a continuous map from  $\mathbb{D}^2$  to  $\mathbb{C}^{\infty}$ .

Let  $\Omega := \mathbb{C} \setminus \{0,1\}$ . It is a long-established fact that  $\mathbb{CP}^1 \setminus \{[0:1], [1:0], [1:1]\}$  is hyperbolically imbedded in  $\mathbb{CP}^1$ . So, it follows by Lemma 2.1 that  $\Omega$  is hyperbolically imbedded in  $\mathbb{C}$ . Define

$$A := \{(z, w) \in \mathbb{D}^2 : z(w - z)w = 0\}.$$

If we define  $f:(\mathbb{D}^2\setminus\mathcal{A})\to\mathbb{C}$  by

$$f(z, w) := z/w \quad \forall (z, w) \in (\mathbb{D}^2 \setminus \mathcal{A}),$$

then it is elementary to see that, by construction, f is holomorphic and that  $\mathsf{range}(f) \subseteq \Omega$ . For each fixed  $\lambda \in \mathbb{C} \setminus \{0,1\}$ ,  $(\lambda \zeta, \zeta) \in \mathbb{D}^2 \setminus \mathcal{A}$  for every  $\zeta \in \mathbb{C}^*$  with sufficiently small  $|\zeta|$ . We have

$$\lim_{\zeta \to 0} f(\lambda \zeta, \zeta) = \lambda.$$

Since  $\lambda \in \mathbb{C} \setminus \{0,1\}$  was arbitrary, the above shows that (0,0) is a point of indeterminacy of f. Hence, the extension of f to  $\mathbb{D}^2$  in either of the two above-mentioned ways is impossible.

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