# Uniformly global observables for 1D maps with an indifferent fixed point

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#### Abstract

We study the property of global-local mixing for full-branched expanding maps of either the half-line or the interval, with one indifferent fixed point. Global-local mixing expresses the decorrelation of global vs local observables w.r.t. to an infinite measure  $\mu$ . Global observables are essentially bounded functions admitting an infinite-volume average, i.e., a limit for the average of the function over bigger and bigger intervals; local observables are integrable functions (both notions are relative to  $\mu$ ). Of course, the definition of global observable depends on the exact definition of infinite-volume average. The first choice for it would be to consider averages over the entire space minus a neighborhood of the indifferent fixed point (a.k.a. the "point at infinity"), in the limit where such neighborhood vanishes. This is the choice that was made in previous papers on the subject. The classes of systems for which global-local mixing was proved, with this natural choice of global observables, are ample but not really general. In this paper we consider uniformly global observables, i.e.,  $L^{\infty}$ functions whose averages over any interval V converges to a limit, uniformly as  $\mu(V) \to \infty$ . Uniformly global observables form quite an extensive subclass of all global observables. We prove global-local mixing in the sense of uniformly global observables, for two truly general classes of expanding maps with one indifferent fixed point, respectively on  $\mathbb{R}_0^+$  and on (0,1]. The technical core of the proofs is rather different from previous work.

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### 1 Introduction

In the context of infinite ergodic theory, the notion of *global-local mixing* involves two types of observable functions: local observables and global observables.

If  $(\mathcal{M}, \mathscr{A}, \mu, T)$  is a dynamical system where  $(\mathcal{M}, \mathscr{A}, \mu)$  is a  $\sigma$ -finite, infinite measure space and  $T : \mathcal{M} \longrightarrow \mathcal{M}$  is surjective, bi-measurable, and non-singular w.r.t.  $\mu$ , a local observable is a function  $f \in L^1(\mathcal{M}, \mathscr{A}, \mu)$ .

The notion of global observable requires a definition of infinite-volume average, which is given as follows. Let  $\mathscr{V}$  be an *exhaustive family*, that is, a collection of sets  $V \in \mathscr{A}$ , with  $\mu(V) < \infty$ , containing at least an increasing sequence  $(V_k)_{k \in \mathbb{N}}$  such that  $\bigcup_k V_k = \mathcal{M}$ . The *infinite-volume limit*  $\mathscr{V} \ni V \nearrow \mathcal{M}$  is defined to be the uniform limit for  $V \in \mathscr{V}$ , as  $\mu(V) \to \infty$ . In particular, a function  $F : \mathcal{M} \longrightarrow \mathbb{R}$  is said to have infinite-volume average  $\overline{\mu}_{\mathscr{V}}(F)$ , w.r.t.  $\mathscr{V}$  and  $\mu$ , if the following limit

$$\overline{\mu}_{\mathscr{V}}(F) := \lim_{\mathscr{V} \ni V \nearrow \mathcal{M}} \frac{1}{\mu(V)} \int_{V} F \, d\mu \tag{1.1}$$

exists, which means that

$$\lim_{r \to \infty} \sup_{\substack{V \in \mathcal{V} \\ \mu(V) > r}} \left| \frac{1}{\mu(V)} \int_{V} F \, d\mu - \overline{\mu}_{\mathcal{V}}(F) \right| = 0. \tag{1.2}$$

A global observable is an essentially bounded function admitting an infinite-volume average, that is, an element of

$$\mathcal{G}_{\mathcal{V}}(\mathcal{M}, \mathscr{A}, \mu) := \{ F \in L^{\infty}(\mathcal{M}, \mathscr{A}, \mu) \mid \exists \overline{\mu}_{\mathcal{V}}(F) \}.$$
 (1.3)

This space will be henceforth referred to as the *(maximal)* space of global observables, relative to  $\mathscr V$  and  $\mu$ .<sup>1</sup>

**Convention.** From now on we will abbreviate all notation such as  $\mathcal{G}_{\mathscr{V}}(\mathcal{M}, \mathscr{A}, \mu)$ ,  $L^{\infty}(\mathcal{M}, \mathscr{A}, \mu)$ , etc. to  $\mathcal{G}_{\mathscr{V}}$ ,  $L^{\infty}$ , etc. Should certain dependences need to be specified we might write  $\mathcal{G}_{\mathscr{V}}(\mu)$ ,  $L^{\infty}(\mu)$ , etc.

The definitions of infinite-volume mixing [L1, L3] concern the decorrelation properties of global and local observables. Given subspaces  $\mathcal{G} \subseteq \mathcal{G}_{\mathcal{V}}$ ,  $\mathcal{L} \subseteq L^1$ , we say that T (endowed with the measure  $\mu$ ) is global-local mixing w.r.t.  $\mathcal{V}$ ,  $\mathcal{G}$  and  $\mathcal{L}$  if

$$\forall F \in \mathcal{G}, \ \forall g \in \mathcal{L}, \quad \lim_{n \to \infty} \mu((F \circ T^n)g) = \overline{\mu}_{\mathscr{V}}(F) \,\mu(g),$$
 (1.4)

with the standard notation  $\mu(g) := \int_{\mathcal{M}} g \, d\mu$ . The reasons one may want/need to restrict the class of global and local observables, respectively, to strict subspaces of  $\mathcal{G}_{\mathscr{V}}$  and  $L^1$  has to do with the truth and feasibility of the sought result. In general, if

<sup>&</sup>lt;sup>1</sup>For a "physical" interpretation of global and local observables see [L1, L2, L3, DN].

possible, one would like to show that T is global-local mixing w.r.t.  $\mathcal{V}$ ,  $\mathcal{G}_{\mathcal{V}}$  and  $L^1$ . We refer to this property as full global-local mixing, relative to  $\mathcal{V}$ .

An equivalent way to introduce definition (1.4) is as follows:

$$\forall F \in \mathcal{G}, \forall g \in \mathcal{L}, g \ge 0, \mu(g) = 1, \quad \lim_{n \to \infty} T_*^n \mu_g(F) = \overline{\mu}_{\mathscr{V}}(F),$$
 (1.5)

where  $\mu_g$  is the measure determined by  $\frac{d\mu_g}{d\mu}=g$  and  $T_*$  is the push-forward operator for T acting on measures. This formulation makes it apparent that global-local mixing defines a sort of "convergence to equilibrium" for a class of probability measures, where the infinite-volume average plays the role of the limit measure and the global observables play the role of test functions. This seems reasonable in all cases where the dynamics tends to spread trajectories all over an infinite-measure space, and so the evolution of probability measures cannot have a limit in any standard measure-theoretic way [L1, L3, L5, BGL1, BGL2, BL, DN, GHR1].

Global observables are also used in the context of global-global mixing, another notion of 'infinite-volume mixing' that was devised in [L1] (see also [L2, L5, DN, TZ1, TZ2]), and for the question of pointwise convergence of Birkhoff averages in infinite measure [LM, DLN, BS]. In infinite ergodic theory, there exist other definitions related to mixing [S, DS], including the so-called *Hopf-Krickeberg mixing* (a short list of recent papers in this area include [MT, P, PT, GHR2], see also references therein).

Expanding maps with a sufficiently flat indifferent fixed point are paradigmatic examples of dynamical systems with an infinite measure. For the reasons set forth earlier, they are generally expected to be global-local mixing (though not global-global mixing, as their "mixing region" is essentially of finite measure in an infinite-measure space [BGL1]).

In this paper we consider full-branched expanding maps defined either on (0,1] or on  $\mathbb{R}^+_0$ , with a single indifferent fixed point respectively in 0 or at  $\infty$  (the latter case meaning that  $T(x) \sim x$ , as  $x \to +\infty$ ). Their infinite-volume mixing properties were studied in [BGL1, BL] (and partly in [BGL2]). In [BGL1] global-local mixing was proved for a class of maps of the interval with an increasing and a decreasing branch (plus certain generalizations) and for a class of maps of the half-line with finitely or infinitely many increasing branches, which preserve the Lebesgue measure (plus certain generalizations). In [BL] global-local mixing was proved for two classes of maps, of the interval and the half-line respectively, with finitely or infinitely many increasing branches: in the case of the interval, the most relevant assumption was a certain growth condition on the branches of the map; in the case of the half-line, the analogue of such condition was still assumed, though it was much more natural, and no assumption was made on the preservation of the Lebesgue measure. The maps studied in [BGL1] include the Fairy map and the ones studied in [BL] include the classical Pomeau-Manneville maps and the Liverani-Saussol-Vaienti maps.

For all the above maps, the authors proved full global-local mixing relative to the

exhaustive families

$$\mathcal{Y}_{\min} := \begin{cases} \{[a,1] \mid 0 < a < 1\}, & \text{for the case } \mathcal{M} = (0,1]; \\ \{[0,a] \mid a > 0\}, & \text{for the case } \mathcal{M} = \mathbb{R}_0^+. \end{cases}$$
(1.6)

In both cases,  $\mathcal{V}_{\min}$  is essentially the smallest exhaustive family one can take, if it is to include an increasing sequence of finite-measure sets covering  $\mathcal{M}$  (whence the notation). Therefore  $\mathcal{G}_{\mathcal{V}_{\min}}$  is essentially the largest class of global observables within the framework presented earlier. Since  $L^1$  is also the largest class of local observables within the theory, full global-local mixing w.r.t.  $\mathcal{V}_{\min}$  is the most general result of its type.

That said, the state of the art is not quite satisfactory, as the classes of maps investigated so far are not as general as one expects global-local mixing to hold for. Here we prove global-local mixing for two general classes of maps, on (0,1] and  $\mathbb{R}_0^+$ , respectively, admitting increasing and decreasing full branches, though we slightly restrict the class of global observables. We consider in fact *uniformly global* observables, that is, global observables relative to the exhaustive families

$$\mathcal{V}_{\text{unif}} := \begin{cases} \{[a,b] \mid 0 < a < b \le 1\}, & \text{for the case } \mathcal{M} = (0,1]; \\ \{[a,b] \mid 0 \le a < b\}, & \text{for the case } \mathcal{M} = \mathbb{R}_0^+. \end{cases}$$
(1.7)

In view of definitions (1.1)-(1.3), this means that averaging one of these observables over any interval with large enough measure will result in a uniformly good approximation of the infinite-volume average. Understandably,  $\mathcal{G}_{\mathcal{V}_{\text{unif}}}$  is a strict subspace of  $\mathcal{G}_{\mathcal{V}_{\text{min}}}$ . For example, in the case  $\mathcal{M} = \mathbb{R}_0^+$  and  $\mu = m$ , the Lebesgue measure, it is easy to check that

$$\lim_{r \to +\infty} \frac{1}{r} \int_0^r \cos \sqrt{x} \, dx = 0,\tag{1.8}$$

while, for all r > 0,

$$\sup_{a \in \mathbb{R}_0^+} \inf \frac{1}{r} \int_a^{a+r} \cos \sqrt{x} \, dx = \pm 1, \tag{1.9}$$

implying that  $F(x) := \cos \sqrt{x}$  belongs to  $\mathcal{G}_{\mathscr{V}_{\min}}$  but not to  $\mathcal{G}_{\mathscr{V}_{\text{unif}}}$ .

Uniformly global observables have been previously studied, together with other classes of global observables, in more general contexts than the present one, cf. [DN, DLN].

For the classes of maps we consider, cf. Sections 2.2 and 2.3, we prove full global-local mixing for uniformly global observables. The technique of the proofs is different from what was used so far: the main tools in [BGL1, BGL2, BL] were the so-called persistently monotonic local observables, that is, monotonic  $L^1$  functions which maintain their monotonicity when acted upon by the dynamics (namely, by the corresponding transfer operator). Persistently monotonic local observables form an invariant convex cone in  $L^1$ . Here we also use dynamics-invariant cones of local observables, but of a different nature, related to the logarithmic derivative of the observables.

Here is an outline of the paper: In Section 2 we first introduce the class of uniformly global observables, for both the case of the half-line and of the unit interval, discussing the dependence of their infinite-volume average on the measure defined on the system. Then we define our maps and state the main results, most notably the full global-local mixing for uniformly global observables, w.r.t. relevant equivalence classes of measures. Section 3 contains the core proofs.

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# 2 Setup

For the rest of the paper we will lighten the notation and write  $\mathscr{V} := \mathscr{V}_{\text{unif}}$  and  $\mathcal{G}_{\text{unif}}(\mu) := \mathcal{G}_{\mathscr{V}_{\text{unif}}}(\mathcal{M}, \mathscr{B}, \mu)$ , cf. (1.7) and (1.3), where  $(\mathcal{M}, \mathscr{B})$  is either  $\mathbb{R}_0^+$  or (0,1], endowed with its respective Borel  $\sigma$ -algebra, and  $\mu$  is an infinite, locally finite measure there. In other words,  $\mathcal{G}_{\text{unif}}(\mu)$  is in either case the class of *all* uniformly global observables, that is, essentially bounded functions  $F : \mathcal{M} \longrightarrow \mathbb{R}$  for which there exists  $\overline{\mu}(F) := \overline{\mu}_{\mathscr{V}_{\text{unif}}}(F)$  such that

$$\lim_{r \to \infty} \sup_{[a,b] \in \mathcal{V}_{\mu}(r)} \left| \frac{1}{r} \int_{a}^{b} F \, d\mu - \overline{\mu}(F) \right| = 0, \tag{2.1}$$

having introduced the notation

$$\mathcal{V}_{\mu}(r) := \{ [a, b] \in \mathcal{V} \mid \mu([a, b]) = r \}. \tag{2.2}$$

(Notice that (2.1)-(2.2) are equivalent to (1.2).) The Lebesgue measure, in whichever space, will always be denoted by m.

# 2.1 Dependence of the infinite-volume average on the global observable and on the measure

Let us make some simple observations on the dependence of  $\overline{\mu}(F)$  on F and  $\mu$ . Setting

$$\omega := \begin{cases} 0, & \text{for the case } \mathcal{M} = (0, 1]; \\ +\infty, & \text{for the case } \mathcal{M} = \mathbb{R}_0^+, \end{cases}$$
 (2.3)

we see that  $\omega$  is the only "point at infinity" for the measure  $\mu$ , in the sense that the complement of every neighborhood of  $\omega$  has finite measure (by the assumptions on  $\mu$ ). It is not hard to see that only the values of F around  $\omega$  count for its infinite-volume average:

**Proposition 2.1** If  $F \in \mathcal{G}_{\text{unif}}(\mu)$ ,  $G \in L^{\infty}(\mathcal{M}, \mu)$  and  $G(x) - F(x) \to 0$ , as  $x \to \omega$ , then  $G \in \mathcal{G}_{\text{unif}}(\mu)$  and  $\overline{\mu}(G) = \overline{\mu}(F)$ .

To prove this proposition it will be convenient to have a notation for the neighborhoods of  $\omega$ , which will be used in other parts of the paper as well. For  $M \in \mathcal{M}$ , let

$$U_M := \begin{cases} (0, M), & \text{for the case } \mathcal{M} = (0, 1]; \\ (M, +\infty), & \text{for the case } \mathcal{M} = \mathbb{R}_0^+. \end{cases}$$
 (2.4)

PROOF OF PROPOSITION 2.1. By possibly using  $F - \overline{\mu}(F)1$  in lieu of F, where  $1(x) \equiv 1$ , we can always assume  $\overline{\mu}(F) = 0$ . Given  $\varepsilon > 0$ , fix M such that  $|G - F| \le \varepsilon/3$  in  $U_M$ . Then, for all r > 0 and  $V \in \mathcal{V}_{\mu}(r)$ ,

$$\left| \int_{V} G \, d\mu \right| \leq \int_{V \setminus U_{M}} |G| \, d\mu + \left| \int_{V \cap U_{M}} G \, d\mu \right|$$

$$\leq \|G\|_{\infty} \, \mu(V \setminus U_{M}) + \left| \int_{V \cap U_{M}} F \, d\mu \right| + \frac{\varepsilon}{3} \, \mu(V \cap U_{M})$$

$$\leq (\|G\|_{\infty} + \|F\|_{\infty}) \, \mu(V \setminus U_{M}) + \left| \int_{V} F \, d\mu \right| + \frac{\varepsilon}{3} \, r.$$

$$(2.5)$$

Now, dividing by r and taking the sup for  $V \in \mathscr{V}_{\mu}(r)$ , we see that, for all sufficiently large r, the first term of the above r.h.s. can be made smaller or equal to  $\varepsilon/3$  (since  $\mu(V \setminus U_M) \leq \mu(\mathcal{M} \setminus U_M) < \infty$ ), and the same for the second term (since  $\overline{\mu}(F) = 0$ , see (2.1)).

Focusing instead on the dependence of the infinite-volume average on the measure, we have:

**Proposition 2.2** If  $\mu, \nu$  are two infinite, locally finite measures on  $\mathcal{M}$  with  $\nu \ll \mu$  and  $\frac{d\nu}{d\mu}(x) \to c \neq 0$ , for  $x \to \omega$ , then  $\overline{\nu} = \overline{\mu}$ , in the sense that  $\mathcal{G}_{unif}(\nu) = \mathcal{G}_{unif}(\mu)$  and  $\overline{\nu}(F) = \overline{\mu}(F)$  for all  $F \in \mathcal{G}_{unif}(\nu)$ .

PROOF. Let us first show that  $\overline{\mu}$  is a restriction of  $\overline{\nu}$ , meaning that  $\mathcal{G}_{\text{unif}}(\mu) \subseteq \mathcal{G}_{\text{unif}}(\nu)$  and, for all  $F \in \mathcal{G}_{\text{unif}}(\mu)$ ,  $\overline{\nu}(F) = \overline{\mu}(F)$ . Fix  $F \in \mathcal{G}_{\text{unif}}(\mu)$ . By Proposition 2.1, as  $V \nearrow \mathcal{M}$ ,

$$\int_{V} F \, d\nu = \mu(V) \, \frac{1}{\mu(V)} \int_{V} F \, \frac{d\nu}{d\mu} \, d\mu \sim \mu(V) c \, \overline{\mu}(F), \tag{2.6}$$

where  $\sim$  denotes *exact* asymptotics. Using 1 in place of F in the above, one also has that  $\nu(V) \sim \mu(V)c$ . Since the limit of the ratio equals the ratio of the limits for our uniform limit  $V \nearrow \mathcal{M}$  as well, our initial claim is proved.

As for the proof that  $\overline{\nu}$  is a restriction of  $\overline{\mu}$ , one uses that  $\frac{d\nu}{d\mu} > 0$  in a neighborhood  $U_M$  of  $\omega$ , where

$$\frac{d\mu}{d\nu}(x) = \left(\frac{d\nu}{d\mu}(x)\right)^{-1} \to c^{-1},\tag{2.7}$$

as  $x \to \omega$ . So one applies the first part of this proof with  $U_M, \nu|_{U_M}, \mu|_{U_M}$  in place of  $\mathcal{M}, \mu, \nu$ , respectively. The sought assertion follows from the fact that, by Proposition 2.1, the infinite-volume averages of  $F1_{U_M}$  and F, relative to either  $\mu$  or  $\nu$ , exist together and are the same (here  $1_{U_M}$  is the indicator function of  $U_M$ ). Q.E.D.

**Remark 2.3** The above assertion was already proved for general global observables, i.e., functions in  $\mathcal{G}_{\mathscr{V}_{\min}}$ , cf. [BL, Rmk. 2], but that result does not imply Proposition 2.2.

Proposition 2.2 is actually implied by a more complicated result which gives fairly general sufficient conditions for  $\overline{\mu}$  to be a restriction of  $\overline{\nu}$ :

**Proposition 2.4** Let  $\mu, \nu$  be two infinite, locally finite measures on  $\mathcal{M}$  with  $\mu \approx \nu$  (i.e.,  $\mu$  and  $\nu$  are mutually absolutely continuous). If, for all sufficiently large r > 0,

$$\lim_{M \to \omega} \sup_{\substack{V \in \mathcal{V}_{\mu}(r) \\ V \subset U_M}} \left[ \sup_{V} \frac{d\nu}{d\mu} \left( \inf_{V} \frac{d\nu}{d\mu} \right)^{-1} - 1 \right] = 0, \tag{2.8}$$

and

$$\sup_{V \in \mathcal{V}_{\mu}(r)} \nu(V) < \infty, \tag{2.9}$$

then  $\overline{\mu}$  is a restriction of  $\overline{\nu}$ .

The proof of this proposition is given in Section 3.

**Remark 2.5** In the case where  $\mu \approx m$ ,  $\nu \approx m$ , with  $\frac{d\mu}{dm}$ ,  $\frac{d\nu}{dm} \in C^1$ , the assumption (2.8) of Proposition 2.4 is implied by the following condition, that is somehow easier to check in applications:

$$\lim_{x \to \omega} \frac{\frac{d}{dx} \left(\frac{d\nu}{d\mu}\right)(x)}{\left(\frac{d\nu}{dm}\right)(x)} = 0. \tag{2.10}$$

In fact, for any given r > 0 and  $M \in \mathcal{M}$ , let us consider  $V \in \mathscr{V}_{\mu}(r)$ ,  $V \subset U_M$ , as in (2.8), and let  $x_{\text{max}}$  and  $x_{\text{min}}$  denote two points in V where  $\frac{d\nu}{d\mu}$  attains its maximum and minimum, respectively. Denoting the ordinary derivative with the usual apostrophe, we have

$$\log \sup_{V} \frac{d\nu}{d\mu} - \log \inf_{V} \frac{d\nu}{d\mu} = \int_{x_{\min}}^{x_{\max}} \left(\frac{d\nu}{d\mu}\right)' \left(\frac{d\nu}{d\mu}\right)^{-1} dm$$

$$\leq \int_{V} \left| \left(\frac{d\nu}{d\mu}\right)' \left| \left(\frac{d\nu}{d\mu} \frac{d\mu}{dm}\right)^{-1} d\mu \right|$$

$$\leq \sup_{U_{M}} \left[ \left| \left(\frac{d\nu}{d\mu}\right)' \left| \left(\frac{d\nu}{dm}\right)^{-1} \right| r, \right]$$
(2.11)

which, for fixed r, vanishes as  $M \to \omega$ , by (2.10). The uniformity in V of the above limit readily implies (2.8).

The previous propositions apply to the question of global-local mixing by means of the following simple lemma, which applies to a general dynamical system, as introduced in Section 1.

**Lemma 2.6** Suppose that  $\nu \ll \mu$  are two measures on  $\mathcal{M}$  and  $\mathcal{G}$  is a space of global observables, for the exhaustive family  $\mathcal{V}$ , where  $\overline{\nu}_{\mathcal{V}}$  and  $\overline{\mu}_{\mathcal{V}}$  coincide. If  $(T, \mu)$  is global-local mixing w.r.t.  $\mathcal{V}$ ,  $\mathcal{G}$  and  $L^1(\mu)$ , then  $(T, \nu)$  is global-local mixing w.r.t.  $\mathcal{V}$ ,  $\mathcal{G}$  and  $L^1(\nu)$ .

PROOF. Take  $F \in \mathcal{G}$  and  $g \in L^1(\nu)$ . Clearly,  $g \frac{d\nu}{d\mu} \in L^1(\mu)$ . So, for  $n \to \infty$ ,

$$\nu((F \circ T^n)g) = \mu\left((F \circ T^n)g\frac{d\nu}{d\mu}\right) \to \overline{\mu}_{\mathscr{V}}(F)\mu\left(g\frac{d\nu}{d\mu}\right) = \overline{\nu}_{\mathscr{V}}(F)\nu(g). \tag{2.12}$$
Q.E.D.

We are going to state our main theorem separately for the cases  $\mathcal{M} = \mathbb{R}_0^+$  and  $\mathcal{M} = (0,1]$ . It will be morally the same result in both cases, but the two theorems will not be equivalent. Obviously, by means of a suitable conjugation  $\Psi:(0,1] \longrightarrow \mathbb{R}_0^+$  (say a smooth, decreasing function with a non-integrable singularity in 0), one can always view a dynamical system  $(\mathbb{R}_0^+, \mathscr{B}_{\mathbb{R}_0^+}, \mu, T)$  as the system  $((0,1], \mathscr{B}_{(0,1]}, \mu_o, T_o)$ , with  $\mu_o := \Psi_*^{-1}\mu$  and  $T_o := \psi^{-1} \circ T \circ \psi$ , whence a theorem for the former system can be rewritten as a theorem for the latter system, and viceversa. But the assumptions on the map in one case can become quite cumbersome when "translated" in terms of its conjugate (this occurs, for instance, for the expansivity of the map, which is not necessarily preserved by a conjugation of the above type); hence the convenience of stating a different theorem for each case.

## 2.2 Maps of the half-line

Let  $T: \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$  a Markov map w.r.t. the partition  $\{I_j\}_{j=0}^{N-1}$ , where  $I_0 = (a_1, +\infty)$  and  $I_j = (a_{j+1}, a_j)$ , for  $j = 1, \ldots, N-1$ . Here  $0 = a_N < a_{N-1} < \cdots < a_1 \in \mathbb{R}_0^+$ . Clearly,  $\{I_j\}$  is a partition only up to a Lebesgue-null set, a subtlety that we shall henceforth forget. Assume the following:

- (A1)  $T|_{I_j}$  is a bijective map onto  $\mathbb{R}^+$ , and has an extension  $\tau_j$  which is defined on  $[a_1, +\infty)$  (for j = 0), or  $[a_{j+1}, a_j]$  or  $(a_{j+1}, a_j]$  (for  $j \geq 1$ , depending on  $\tau_j$  being increasing or decreasing, respectively). The branch  $\tau_j$  is  $C^2$  (this means up to the closed endpoint of its domain).
- (A2) There exists  $\Lambda > 1$  such that  $|\tau'_j| \geq \Lambda$ , for all  $j \geq 1$ . Also,  $\tau'_0 > 1$  and  $\lim_{x \to +\infty} \tau'_0(x) = 1$ .

(A3) 
$$\lim_{x \to +\infty} \frac{\tau_0''(x)}{\tau_0'(x) - 1} = 0.$$

- (A4) For all  $j \geq 1$ , either  $\lim_{x \to a_j^-} \frac{\tau_j''(x)}{(\tau_j'(x))^2} = 0$  or  $\lim_{x \to a_{j+1}^+} \frac{\tau_j''(x)}{(\tau_j'(x))^2} = 0$ , depending on  $\tau_j$  being increasing or decreasing, respectively.
- (A5) T is exact w.r.t. m, the Lebesgue measure on  $\mathbb{R}_0^+$ .

Remark 2.7 In [BGL1, App. A] it is shown that, under the assumptions (A1)-(A4) and

(A5)' The function  $u(x) := x - \tau_0(x)$  is positive, convex and vanishing (hence decreasing), as  $x \to +\infty$ . Furthermore, u'' is decreasing (hence vanishing),

T is exact w.r.t. m. (In fact, Theorem A.1 of [BGL1] proves that, under such assumptions and the existence of an invariant measure  $\mu$ , mutually absolutely continuous with m, T is conservative and exact. But this last assumption is only needed for the conservativity of T.)

**Theorem 2.8** A map T satisfying (A1)-(A5) is fully global-local mixing, relative to  $m, w.r.t. \ \mathcal{V} = \mathcal{V}_{unif}$ .

For the convenience of the reader we recall that the above statement means that T is global-local mixing for the Lebesgue measure, relative to all uniformly global observables and all integrable local observables. An immediate consequence of Theorem 2.8 is that T is global-local mixing for a large class of other measures, though not fully, but relative to a large subclass of uniformly global observables (and all integrable local observables).

Corollary 2.9 Let  $q \in (0,1]$  and  $\nu \ll m$  on  $\mathbb{R}_0^+$  with  $\frac{d\nu}{dm}$  bounded and  $\frac{d\nu}{dm}(x) \sim x^{-q}$ , as  $x \to +\infty$ . Under the same hypotheses as Theorem 2.8,  $\overline{m}$  is a strict restriction of  $\overline{\nu}$  and  $(\mathbb{R}_0^+, \mathscr{B}, \nu, T)$  is global-local mixing w.r.t.  $\mathscr{V}$ ,  $\mathcal{G}_{\text{unif}}(m)$  and  $L^1(\nu)$ .

PROOF OF COROLLARY 2.9. Let us denote by  $\lambda_q$  the measure with  $\frac{d\lambda_q}{dm}(x)=(x+1)^{-q}$ . By Proposition 2.4 and Remark 2.5,  $\overline{m}$  is a restriction of  $\overline{\lambda_q}$ . On the other hand, by Proposition 2.2,  $\overline{\nu}=\overline{\lambda_q}$ , so  $\overline{m}$  is a restriction of  $\overline{\nu}$ . To show that it is a strict restriction, it suffices to produce a global observable in  $\mathcal{G}_{\text{unif}}(\lambda_q)\setminus\mathcal{G}_{\text{unif}}(m)$ . For  $q\in(0,1)$ , the function  $F_q(x):=\cos((x+1)^{1-q})$  does the job because, for all  $a\in\mathbb{R}_0^+$  and r>0,

$$\frac{1}{r} \int_{a}^{a+r} F_q \, d\lambda_q = \frac{\sin((a+r+1)^{1-q}) - \sin((a+1)^{1-q})}{r(1-q)},\tag{2.13}$$

implying that  $F_q \in \mathcal{G}_{\text{unif}}(\lambda_q)$  with  $\overline{\lambda_q}(F_q) = 0$ . For q = 1, one can take instead  $F_1(x) := \cos(\log(x+1))$ . For all  $a \in \mathbb{R}_0^+$  and r > 0,

$$\frac{1}{r} \int_{a}^{a+r} F_1 d\lambda_1 = \frac{\sin(\log(a+r+1)) - \sin(\log(a+1))}{r},$$
 (2.14)

implying again that  $F_1 \in \mathcal{G}_{\text{unif}}(\lambda_1)$  with  $\overline{\lambda_1}(F_1) = 0$ . On the other hand, for every  $q \in (0,1]$  and r > 0, clearly

$$\lim \sup_{a \to +\infty} / \lim \inf \frac{1}{r} \int_a^{a+r} F_q \, dm = \pm 1, \tag{2.15}$$

whence  $F_q \notin \mathcal{G}_{\text{unif}}(m)$ .

The global-local mixing of  $(\mathbb{R}_0^+, \mathcal{B}, \nu, T)$  w.r.t.  $\mathcal{V}$ ,  $\mathcal{G}_{\text{unif}}(m)$  and  $L^1(\nu)$  follows from Theorem 2.8 via Lemma 2.6, applied with  $\mathcal{G} = \mathcal{G}_{\text{unif}}(m)$ . Q.E.D.

#### 2.3 Maps of the unit interval

We now consider maps  $T:(0,1] \longrightarrow (0,1]$  which are Markov for the partition  $\{I_j\}_{j=0}^{N-1}$ , where  $I_j = (a_j, a_{j+1})$  and  $0 = a_0 < a_1 < \ldots < a_{N-1} < a_N = 1$ . Once again, we mention now and then no more that  $\{I_j\}$  is only a partition mod m, which here denotes the Lebesgue measure on (0,1]. These are the assumptions on T:

- (B1)  $T|_{I_j}$  is a bijective map onto (0,1), and has a  $C^2$  extension  $\tau_j:[a_j,a_{j+1}] \longrightarrow [0,1]$  (this means that  $\tau_j$  is  $C^2$  up to the boundary of its domain).
- (B2) There exists  $\Lambda > 1$  such that  $|\tau'_j| \geq \Lambda$ , for all  $j \geq 1$ . Also,  $\tau'_0(\xi) > 1$ , for all  $\xi \in (0, a_1]$ .
- (B3) As  $\xi \to 0^+$ ,  $T'(\xi) = 1 + \chi \xi^p + o(\xi^p)$ , for some  $p \ge 1$  and  $\chi > 0$ .
- (B4) T is exact w.r.t. m (again, the Lebesgue measure on (0,1]).

**Remark 2.10** It was proved in [T2] that, assuming (B1)-(B2),  $\tau'_0(0) = 1$  (which is part of (B3)) and

(B4)' T'' > 0 in a neighborhood of 0,

T is conservative and exact (w.r.t. m). Moreover, assuming (B1)-(B3), and (B4)', there exists a unique T-invariant measure  $\mu$  such that  $\frac{d\mu}{dm}(\xi) \sim \xi^{-p}$ , for  $\xi \to 0^+$ . This comes from [T3, §1], since (B3) implies that

$$T(\xi) = \xi + \frac{\chi}{p+1} \, \xi^{p+1} + o(\xi^{p+1}); \tag{2.16}$$

cf. also [T1] and [BL, Thm. 2.1].

The above remark shows how the measure  $\lambda_p$ , defined by

$$\frac{d\lambda_p}{dm}(\xi) = \frac{1}{\xi^p},\tag{2.17}$$

is especially relevant for maps satisfying (B3). The next theorem, which is the main result of this section, reflects this. The case p = 1 presents certain issues that lead us to consider instead (uniformly) global observables for the measure  $\lambda_{1+}$ , defined by

$$\frac{d\lambda_{1+}}{dm}(\xi) = -\frac{\log \xi}{\xi}.$$
(2.18)

**Theorem 2.11** Let T be a map satisfying (B1)-(B4). If p > 1, then T with  $\lambda_p$  is fully global-local mixing w.r.t.  $\mathscr{V} = \mathscr{V}_{\text{unif}}$ . If p = 1, then T with  $\lambda_1$  is global-local mixing w.r.t.  $\mathscr{V} = \mathscr{V}_{\text{unif}}$ ,  $\mathcal{G}_{\text{unif}}(\lambda_{1+})$  and  $L^1(\lambda_1)$ .

**Remark 2.12** The purpose of  $\lambda_{1+}$  is simply to have a measure whose singularity at 0 is slightly stronger than  $1/\xi$ . The specific form (2.18) is unimportant, except that it behaves well with the many technical arguments of the proof of Theorem 2.11 (Section 3.3). Notice that the case p = 1 was also special for the main theorem of [BL] (Thm. 2.4) about general global observables.

Our study of the restrictions of infinite-volume averages shows that, if T is fully global-local mixing for  $\lambda_p$ , p > 1, then it is global-local mixing, though not fully, for many other measures, including  $\lambda_q$ , for  $1 \le q < p$ .

Corollary 2.13 For  $1 \leq q < p$ , let  $\nu$  be an infinite, locally finite measure, absolutely continuous w.r.t. the Lebesgue measure m, with  $\frac{d\nu}{dm}(\xi) \sim \xi^{-q}$ , as  $\xi \to 0^+$ . Then  $\overline{\lambda_p}$  is a strict restriction of  $\overline{\nu}$ . Furthermore, under the same hypotheses as in Theorem 2.11,  $((0,1], \mathcal{B}, \nu, T)$  is global-local mixing w.r.t.  $\mathcal{V}$ ,  $\mathcal{G}_{\text{unif}}(\lambda_p)$  and  $L^1(\nu)$ .

PROOF. Using Proposition 2.4 and Remark 2.5, it is easy to see that  $\overline{\lambda_p}$  is a restriction of  $\overline{\lambda_q}$ , and by Proposition 2.2 the same holds for  $\overline{\nu}$ . To show that the restriction is strict, it suffices to produce a global observable in  $\mathcal{G}_{\text{unif}}(\lambda_q) \setminus \mathcal{G}_{\text{unif}}(\lambda_p)$ . For q > 1, we may take the function  $F_q(\xi) := \cos(\xi^{-q+1})$ , because, for all  $[a, b] \subset (0, 1]$ ,

$$\frac{1}{\lambda_q([a,b])} \int_a^b F_q \, d\lambda_q = \frac{\sin(a^{-q}) - \sin(b^{-q})}{\lambda_q([a,b]) \, q},\tag{2.19}$$

implying that  $F_q \in \mathcal{G}_{\text{unif}}(\lambda_q)$ , with  $\overline{\lambda_q}(F_q) = 0$ . On the other hand, let r > 0 and  $V_b = [b-c,b] \subset (0,1]$ , such that  $\lambda_p(V_b) = r$ . A simple calculation shows that  $c = b - (b^{1-p} + r(p-1))^{1/(1-p)}$ . In particular,  $c \sim rb^p$ , for  $b \to 0^+$ . Therefore, since q < p, the function  $F_q$  is approximately constant on  $V_b$ , when b is very close to 0, giving

$$\lim \sup_{b \to 0^+} / \lim \inf \frac{1}{r} \int_{V_b} F_q \, d\lambda_p = \pm 1. \tag{2.20}$$

For q = 1, a similar argument holds for the global observable  $F_1(\xi) := \cos(\log(\xi^{-1}))$ , yielding that  $\mathcal{G}_{\text{unif}}(\lambda_1) \setminus \mathcal{G}_{\text{unif}}(\lambda_p)$  is non-empty for all p > 1.

The global-local mixing of  $((0,1], \mathcal{B}, \nu, T)$  as in the statement of the corollary follows once again from Lemma 2.6 (with  $\mathcal{G} = \mathcal{G}_{\text{unif}}(\lambda_p)$ ). Q.E.D.

# 3 Proofs

In this section we prove our main results and Proposition 2.4, that was left behind. We start with the latter.

#### 3.1 Proof of Proposition 2.4

Let  $F \in \mathcal{G}_{\text{unif}}(\mu)$  and, without loss of generality, assume that  $\overline{\mu}(F) = 0$ . We want to prove that  $\overline{\nu}(F)$  exists and equals 0.

Let us start by noting that, for any  $V \in \mathcal{V}$ ,

$$\left| \int_{V} F \frac{d\nu}{d\mu} d\mu - \frac{\nu(V)}{\mu(V)} \int_{V} F d\mu \right| = \left| \int_{V} F \left( \frac{d\nu}{d\mu} - \frac{\nu(V)}{\mu(V)} \right) d\mu \right|$$

$$\leq \|F\|_{\infty} \left( \sup_{V} \frac{d\nu}{d\mu} - \inf_{V} \frac{d\nu}{d\mu} \right) \mu(V)$$

$$= \|F\|_{\infty} \left[ \sup_{V} \frac{d\nu}{d\mu} \left( \inf_{V} \frac{d\nu}{d\mu} \right)^{-1} - 1 \right] \mu(V) \inf_{V} \frac{d\nu}{d\mu},$$
(3.1)

where we have used that, for any  $x \in V$ ,

$$\left| \frac{d\nu}{d\mu}(x) - \frac{\nu(V)}{\mu(V)} \right| = \left| \frac{d\nu}{d\mu}(x) - \frac{1}{\mu(V)} \int_{V} \frac{d\nu}{d\mu} d\mu \right| \le \sup_{V} \frac{d\nu}{d\mu} - \inf_{V} \frac{d\nu}{d\mu}. \tag{3.2}$$

Dividing both sides by  $\nu(V)$  and using that  $\mu(V)\inf_{V}\frac{d\nu}{d\mu} \leq \nu(V)$ , we have the following inequality that we will use later,

$$\left| \frac{1}{\nu(V)} \int_{V} F \, d\nu - \frac{1}{\mu(V)} \int_{V} F \, d\mu \right| \le \|F\|_{\infty} \left[ \sup_{V} \frac{d\nu}{d\mu} \left( \inf_{V} \frac{d\nu}{d\mu} \right)^{-1} - 1 \right]. \tag{3.3}$$

Fix  $\varepsilon > 0$ . Since  $\overline{\mu}(F) = 0$ , there exists  $r = r(\varepsilon) > 0$  so large that

$$\sup_{V \in \mathscr{V}_{\mu}(r)} \left| \frac{1}{r} \int_{V} F \, d\mu \right| \le \frac{\varepsilon}{4} \tag{3.4}$$

and (2.8)-(2.9) hold. So, by (2.8), there exists  $M \in \mathcal{M}$  such that, for all  $V \in \mathcal{V}_{\mu}(r)$ ,  $V \subset U_M$ ,

$$\left[\sup_{V} \frac{d\nu}{d\mu} \left(\inf_{V} \frac{d\nu}{d\mu}\right)^{-1} - 1\right] \le \frac{\varepsilon}{4\|F\|_{\infty}}.$$
 (3.5)

For R > 0, which one should think of as much larger than r, let us consider  $W \in \mathscr{V}_{\nu}(R)$ . Partition W into adjacent intervals  $V_0, V_1, \ldots, V_l$ , from left to right in the case  $\mathcal{M} = \mathbb{R}_0^+$ , and from right to left in the case  $\mathcal{M} = (0, 1]$ , such that  $\mu(V_i) = r$ , for  $i = 1, \ldots, l$ , and  $\mu(V_0) \leq r$ ; see Fig. 1. Let l' be the smallest integer between 1 and l such that  $V_{l'} \subset U_M$ . Of course,  $V_i \subset U_M$  for all  $i \geq l'$ .

Now write

$$\int_{W} F \, d\nu = \sum_{i=0}^{l'-1} \int_{V_i} F \, d\nu + \sum_{i=l'}^{l} \int_{V_i} F \, d\nu. \tag{3.6}$$

$$V_0$$
  $V_1$   $\cdots$   $V_{l'-1}$   $V_{l'}$   $\cdots$   $V_l$ 

FIGURE 1: Partition of W for the proof of Proposition 2.4 (case  $\mathcal{M} = \mathbb{R}_0^+$ ).

For  $i \geq l'$ , since  $V_i \subset U_M$ ,

$$\left| \frac{1}{\nu(V_i)} \int_{V_i} F \, d\nu - \frac{1}{\mu(V_i)} \int_{V_i} F \, d\mu \right| \le \frac{\varepsilon}{4},\tag{3.7}$$

where we have used (3.3) and (3.5). The estimate (3.7), together with (3.4) and the fact that  $V_i \in \mathcal{V}_{\mu}(r)$ , implies that, for all  $i \geq l'$ ,

$$\left| \int_{V_i} F \, d\nu \right| \le \nu(V_i) \, \frac{\varepsilon}{2}. \tag{3.8}$$

As for the other terms in the r.h.s. of (3.6), observe that

$$\sum_{i=0}^{l'-1} \int_{V_i} F \, d\nu \le ||F||_{\infty} \left( \nu(\mathcal{M} \setminus U_M) + \sup_{V \in \mathscr{V}_{\mu}(r)} \nu(V) \right), \tag{3.9}$$

having estimated  $|\int_{V_{l'-1}} F d\nu| \le ||F||_{\infty} \sup_{V \in \mathscr{V}_{\mu}(r)} \nu(V)$ . Notice that this estimate, and thus (3.9), also holds in the case l' = 1, because  $\mu(V_0) \le r$ . Setting

$$\overline{R} = \frac{2}{\varepsilon} \|F\|_{\infty} \left( \nu(\mathcal{M} \setminus U_M) + \sup_{V \in \mathscr{V}_{\mu}(r)} \nu(V) \right)$$
(3.10)

one has that for any  $R \geq \overline{R}$ , the l.h.s. of (3.9) does not exceed  $R\varepsilon/2$ . This fact, (3.6) and (3.9) imply that, for all  $R \geq \overline{R}$  and  $W \in \mathcal{V}_{\nu}(R)$ ,

$$\left| \int_{W} F \, d\nu \right| \le \frac{R\varepsilon}{2} + \frac{\varepsilon}{2} \sum_{i=l'}^{l} \nu(V_i) \le R\varepsilon, \tag{3.11}$$

which is what was to be proved.

Q.E.D.

#### 3.2 Proof of Theorem 2.8

Here and for the rest of the paper we denote by  $\phi_j := \tau_j^{-1}$  the *inverse branches* of T, for both cases  $\mathcal{M} = \mathbb{R}_0^+$  and  $\mathcal{M} = (0, 1]$ .

We begin the proof of Theorem 2.8 with a number of simple lemmas on the properties of the branches and inverse branches of T.

**Lemma 3.1** For  $j \geq 1$ , either  $\lim_{x \to a_j^-} \tau'_j(x) = +\infty$  or  $\lim_{x \to a_{j+1}^+} \tau'_j(x) = -\infty$ , depending on  $\tau_j$  being increasing or decreasing, respectively.

PROOF. Let us denote with  $x \to \bullet$  either limit  $x \to a_j^-$  or  $x \to a_{j+1}^+$ , depending on the two cases  $\tau_j' > 0$  or  $\tau_j' < 0$ , as in the statement of the lemma. By (A4),

$$\lim_{x \to \bullet} \left( \frac{1}{\tau_i'(x)} \right)' = 0. \tag{3.12}$$

Therefore  $1/\tau'_j(x)$  has a limit, for  $x \to \bullet$ , in  $\mathbb{R}$ . Then, so does  $\tau'_j(x)$ , in  $\mathbb{R} \cup \{\pm \infty\}$  (because  $\tau'_j$  never changes sign). The fact that  $\tau_j$  is monotonic and surjective onto  $\mathbb{R}^+_0$  forces the conclusion of the lemma. Q.E.D.

Lemma 3.2 For all j,

$$\sup_{\mathbb{R}_{0}^{+}} \frac{|\phi_{j}''|}{|\phi_{j}'|(1-|\phi_{j}'|)} < \infty. \tag{3.13}$$

Furthermore,

$$\lim_{x \to +\infty} \frac{\phi_0''(x)}{\phi_0'(x)(1 - \phi_0'(x))} = 0 \tag{3.14}$$

and, for all  $j \geq 1$ ,

$$\lim_{x \to +\infty} \phi_j'(x) = \lim_{x \to +\infty} \frac{\phi_j''(x)}{\phi_j'(x)} = 0. \tag{3.15}$$

PROOF. Since  $\phi'_j = \frac{1}{\tau'_j} \circ \phi_j$  and  $\phi''_j = -\frac{\tau''_j}{(\tau'_j)^3} \circ \phi_j$ , one gets

$$\frac{|\phi_j''|}{|\phi_j'| (1 - |\phi_j'|)} = \frac{|\tau_j''|}{|\tau_j'| (|\tau_j'| - 1)} \circ \phi_j.$$
(3.16)

Let us prove (3.13) for j = 0. From (A1)-(A3), the function  $\frac{|\tau_0''|}{\tau_0'(\tau_0'-1)}$  is continuous on  $[a_0, +\infty)$  (observe that  $\tau_0' > 1$  up to and including  $a_0$ ) and vanishes at  $+\infty$ , implying that (3.16) is bounded for j = 0. In addition, (A3) gives (3.14).

For the case  $j \geq 1$ , let us rewrite the r.h.s. of (3.16) as

$$\frac{|\tau_j''|}{|\tau_j'|^2 \left(1 - \frac{1}{|\tau_j'|}\right)} \circ \phi_j. \tag{3.17}$$

Since  $|\tau'_j| \ge \Lambda > 1$ , a sufficient condition for (3.13) is an upper bound for  $|\tau''_j|/|\tau'_j|^2$ , which we readily get from (A4) and the continuity of  $\tau''_j$ . Finally, since

$$\frac{\phi_j''}{\phi_j'} = -\frac{\tau_j''}{(\tau_j')^2} \circ \phi_j, \tag{3.18}$$

(3.15) follows from Lemma 3.1 and (A4). Q.E.D.

**Lemma 3.3** For  $x \in I_0$ ,  $\tau_0(x) < x$ .

PROOF. If the claim is false, there must exist  $a \in I_0$  such that  $\tau_0(a) = a$ . Since  $\tau'_0 > 1$ , it follows that  $\tau_0(x) > x$ , for all x > a. Thus, chosen some  $x_0 > a$  and denoted  $x_1 := T(x_0)$ , we have that  $x_1 > x_0$  and  $\{T^n[x_0, x_1)\}_{n \in \mathbb{N}}$  is a partition of  $[x_0, +\infty)$ . This contradicts the exactness of T via the so-called Miernowski-Nogueira criterion [MN]. (See Appendix A.1 of [L4] for a generalization of said criterion. In the language of that reference,  $[x_0, x_1)$  is not asymptotically intersecting and therefore T cannot be exact.) Q.E.D.

Let us introduce the main objects that will be used in the proof of Theorem 2.8:

Cones of local observables: For  $M, D, \varepsilon > 0$ , set

$$\mathcal{C}_{M,D,\varepsilon} := \left\{ g \in L^1(\mathbb{R}_0^+, m) \cap C^1(\mathbb{R}_0^+) \mid g > 0, \frac{|g'|}{g} \le D \text{ and } \frac{|g'(x)|}{g(x)} \le \varepsilon, \forall x \ge M \right\}. \tag{3.19}$$

This is a *cone*, in the sense that  $g \in \mathcal{C}_{M,D,\varepsilon}$  and c > 0 implies  $cg \in \mathcal{C}_{M,D,\varepsilon}$ .

**Transfer operator:** We define  $\widehat{T}: L^1(m) \longrightarrow L^1(m)$  via the identity

$$\forall F \in L^{\infty}(m), g \in L^{1}(m), \qquad \int_{\mathbb{R}_{0}^{+}} (F \circ T)g \, dm = \int_{\mathbb{R}_{0}^{+}} F \, \widehat{T}g \, dm. \tag{3.20}$$

In this case, where the reference measure is the Lebesgue measure,  $\widehat{T}$  is also referred to as the  $Perron-Frobenius\ operator$  and its explicit formula is notoriously:

$$\widehat{T}g = \sum_{j=0}^{N-1} |\phi'_j| (g \circ \phi_j) = \sum_{j=0}^{N-1} \sigma_j \phi'_j (g \circ \phi_j),$$
(3.21)

where  $\sigma_j := \operatorname{sgn}(\phi'_j)$  (recall that  $\phi_j$  is monotonic).

Now set

$$\underline{D} := \max_{0 \le j \le N-1} \sup_{\mathbb{R}_0^+} \frac{|\phi_j''|}{|\phi_j'| (1 - |\phi_j'|)}, \tag{3.22}$$

which is finite by Lemma 3.2. The following result is a crucial statement that says that the dynamics preserves densities that are "almost flat at infinity".

**Lemma 3.4** Let  $D \geq \underline{D}$ . For all  $\varepsilon > 0$ , there exists M > 0 such that  $\widehat{T}\mathcal{C}_{M,D,\varepsilon} \subseteq \mathcal{C}_{M,D,\varepsilon}$ .

PROOF. Let  $g \in C^1$ , g > 0. We first establish the inequality

$$|(\widehat{T}g)'| \le \max_{j} \left( \left| \frac{\phi_{j}''}{\phi_{j}'} \right| + \frac{|(g' \circ \phi_{j})\phi_{j}'|}{g \circ \phi_{j}} \right) \widehat{T}g, \tag{3.23}$$

that will be used repeatedly in the present proof. A simple computation out of (3.21) yields

$$(\widehat{T}g)' = \sum_{j=0}^{N-1} \left( \sigma_j \phi_j'' (g \circ \phi_j) + \sigma_j (\phi_j')^2 (g' \circ \phi_j) \right). \tag{3.24}$$

If in each of the above summands we divide and multiply the left term by  $\phi'_j$  and the right term by  $g \circ \phi_j$ , we obtain (3.23) after a triangular inequality.

Now let  $\varepsilon > 0$ . By (3.14) and (3.15) there exists M > 0 such that, for all  $x \geq M$ ,

$$\frac{|\phi_0''(x)|}{\phi_0'(x)(1 - \phi_0'(x))} \le \varepsilon; \tag{3.25}$$

$$\max_{j \ge 1} \left( \left| \frac{\phi_j''}{\phi_j'} \right| + D|\phi_j'| \right) \le \varepsilon. \tag{3.26}$$

Let us show that  $C_{D,M,\varepsilon}$  is  $\widehat{T}$ -invariant, as in the claim of Lemma 3.4. Take  $g \in C_{M,D,\varepsilon}$ . Since  $\phi_j \in C^2$ ,  $\widehat{T}g \in C^1$ . Also, the surjectivity of T and g > 0 imply  $\widehat{T}g > 0$ . Moreover, by (3.23),  $|g'|/g \leq D$  and  $D \geq \underline{D}$ , cf. (3.22), we obtain

$$|(\widehat{T}g)'| \le \max_{j} \left( \left| \frac{\phi_{j}''}{\phi_{j}'} \right| + D|\phi_{j}'| \right) \widehat{T}g \le D \widehat{T}g.$$
(3.27)

At this point, take  $x \geq M$ . By Lemma 3.3,  $\phi_0(x) > x \geq M$ . The properties of  $g \in \mathcal{C}_{D,M,\varepsilon}$  imply that

$$\begin{aligned} |(\widehat{T}g)'(x)| &\leq \max_{j} \left( \left| \frac{\phi_{j}''(x)}{\phi_{j}'(x)} \right| + \left| \frac{g'(\phi_{j}(x)) \phi_{j}'(x)}{g(\phi_{j}(x))} \right| \right) \widehat{T}g(x) \\ &\leq \max_{j} \left\{ \left( \frac{|\phi_{0}''(x)|}{\phi_{0}'(x)} + \varepsilon \phi_{0}'(x) \right), \max_{j \geq 1} \left( \left| \frac{\phi_{j}''(x)}{\phi_{j}'(x)} \right| + D|\phi_{j}'(x)| \right) \right\} \widehat{T}g(x) \\ &\leq \varepsilon \widehat{T}g(x), \end{aligned}$$

$$(3.28)$$

where the last inequality comes from (3.25)-(3.26). This concludes the proof of Lemma 3.4.

The next lemma formalizes the intuitive fact that if a local observable is almost flat in a neighborhood of infinity, its integral against a uniformly global observable in that neighborhood results approximately in the infinite-volume average of the global observable.

**Lemma 3.5** Let  $F \in \mathcal{G}_{\mathrm{unif}}(m)$ ,  $\overline{m}(F) = 0$ . For each  $\delta > 0$ , there exists  $\varepsilon > 0$  such that for, every  $g \in C^1(\mathbb{R}^+_0)$  and  $M \geq 0$  with

- (i) g > 0;
- (ii) m(q) = 1;

(iii) 
$$\frac{|g'(x)|}{g(x)} \le \varepsilon$$
, for all  $x \ge M$ ,

one has

$$\left| \int_{M}^{\infty} Fg \, dm \right| \le \delta.$$

PROOF. Let us assume that F is not almost everywhere 0, otherwise the statement is trivial. By the hypothesis  $\overline{m}(F) = 0$ , given  $\delta > 0$  there exists r > 0 such that

$$\sup_{V \in \mathscr{V}_m(r)} \left| \frac{1}{r} \int_V F \, dm \right| \le \frac{\delta}{2}. \tag{3.29}$$

Set  $\varepsilon := \frac{\delta}{2r\|F\|_{\infty}}$ . For any g and M verifying (i)-(iii), we decompose

$$\int_{M}^{\infty} Fg \, dm = \sum_{k=0}^{\infty} \int_{M+kr}^{M+(k+1)r} Fg \, dm.$$
 (3.30)

For  $k \in \mathbb{N}$ , call  $x_{\min,k}$  a minimum point of g in [M+kr, M+(k+1)r] (recall that g is continuous). By (3.29), (i), (iii) and the definition of  $\varepsilon$ ,

$$\left| \int_{M+kr}^{M+(k+1)r} Fg \, dm \right| = \left| \int_{M+kr}^{M+(k+1)r} F(x) \left( g(x_{\min,k}) + \int_{x_{\min,k}}^{x} g'(t) \, dt \right) dx \right|$$

$$\leq \frac{\delta}{2} r g(x_{\min,k}) + r \|F\|_{\infty} \varepsilon \left( \int_{M+kr}^{M+(k+1)r} g(t) \, dt \right)$$

$$\leq \left( \frac{\delta}{2} + \frac{\delta}{2} \right) \left( \int_{M+rK}^{M+(k+1)r} g(t) \, dt \right). \tag{3.31}$$

We obtain the assertion of Lemma 3.5 by (3.30)-(3.31), using (ii). Q.E.D.

We are now ready for the final arguments of the proof of Theorem 2.8. It suffices to establish that

$$\lim_{n \to \infty} m((F \circ T^n)g) = 0, \tag{3.32}$$

for all  $F \in \mathcal{G}_{\text{unif}}(m)$  with  $\overline{m}(F) = 0$  and all  $g \in L^1(m)$ . If m(g) = 0, the above follows from the weak form of global-local mixing called (GLM1), which is a simple consequence of the exactness of T [L3, Thm. 3.5].<sup>2</sup> So we turn to the case  $m(g) \neq 0$ . By the linearity of (3.32) in q, we may assume m(q) = 1. Since T is exact, it is enough

<sup>&</sup>lt;sup>2</sup>This theorem and Lemma 3.6 in the same reference, which we use momentarily, were stated for measure-preserving systems, with another assumptions that ensured that the infinite-volume average was dynamics-invariant. None of these assumptions are needed for the proofs of the statements invoked here. See also Appendix A.1 of [BL].

to show that, for all  $\delta > 0$ , there exists  $g_{\delta} \in L^{1}(m)$ , which may depend on F, such that  $m(g_{\delta}) = 1$  and

$$\limsup_{n \to \infty} |m((F \circ T^n)g_{\delta})| \le 2\delta. \tag{3.33}$$

In fact, [L3, Lem. 3.6] states that if the above conditions holds for  $g_{\delta}$ , then it holds for all g with m(g) = 1. The arbitrariness of  $\delta$  then implies (3.32) for all such g.

It remains to prove (3.33). Fix  $\delta > 0$  and let  $\varepsilon > 0$  be given by Lemma 3.5 for the chosen value of  $\delta$ . Then let M > 0 be given by Lemma 3.4 for the selected value of  $\varepsilon$  and any fixed value of  $D \geq \underline{D}$ . Let  $g_{\delta}$  be any element of  $\mathcal{C}_{M,D,\varepsilon}$  with  $m(g_{\delta}) = \|g_{\delta}\|_1 = 1$  (such an element evidently exists). Then, for all  $n \in \mathbb{N}$ ,

$$|m((F \circ T^{n})g_{\delta})| \leq |m((1_{[0,M]}F) \circ T^{n})g_{\delta})| + |m((1_{(M,\infty)}F) \circ T^{n})g_{\delta})|$$

$$= |m((1_{[0,M]}F) \circ T^{n})g_{\delta})| + \left| \int_{M}^{\infty} F(\widehat{T}^{n}g_{\delta}) dm \right|.$$
(3.34)

We conclude the proof of Theorem 2.8 if we are able to bound each the above terms with  $\delta$ , for n large. The first bound comes from the exactness of T, which implies full local-local mixing (LLM) w.r.t. m, namely,

$$\forall f \in L^{\infty}(m) \cap L^{1}(m), \ \forall g \in L^{1}(m), \quad \lim_{n \to \infty} m((f \circ T^{n})g) = 0; \tag{3.35}$$

cf. [L3, Thm. 3.5]. The second bound holds for all n and comes directly from Lemma 3.5, since  $\widehat{T}^n g_{\delta} \in \mathcal{C}_{M,D,\varepsilon}$  (by Lemma 3.4) and  $m(\widehat{T}^n g_{\delta}) = \|\widehat{T}^n g_{\delta}\|_1 = \|g_{\delta}\|_1 = 1$ . Q.E.D.

#### 3.3 Proof of Theorem 2.11

Once again, we start with a few calculus lemmas.

**Lemma 3.6** For p > 1,  $\phi_0''(\xi) \to 0$  as  $\xi \to 0^+$ .

PROOF. Since  $\phi_0'' = -\frac{\tau_0''}{(\tau_0')^3} \circ \phi_0$ , and  $\lim_{\xi \to 0^+} \tau_0'(\xi) = 1$ , it is enough to prove that  $\lim_{\xi \to 0^+} \tau_0''(\xi) = 0$ . By (B1),  $\tau_0''(\xi) \to c \in \mathbb{R}$ , as  $\xi \to 0^+$ . If  $c \neq 0$ , the Taylor expansion of  $\tau_0'$  at 0 would be  $\tau_0' = 1 + c\xi + o(\xi)$ , contradicting (B3). Q.E.D.

Let  $\ell_p, \ell_{1+}: (0,1] \longrightarrow \mathbb{R}^+$  denote the (infinite) densities of the measures  $\lambda_p, \lambda_{1+}$ , respectively:

$$\ell_p(\xi) := \frac{1}{\xi^p};\tag{3.36}$$

$$\ell_{1+}(\xi) := -\frac{\log \xi}{\xi}.\tag{3.37}$$

As already mentioned in Section 2.3, we will work with the function  $\ell_p$ , for p > 1, and  $\ell_{1+}$ , for p = 1. Since we prove the same statements for all these functions, it is convenient to introduce the common notation<sup>3</sup>

$$p \oplus = \begin{cases} p, & \text{if } p > 1; \\ 1+, & \text{if } p = 1. \end{cases}$$
 (3.38)

The analogue of Lemma 3.2 for interval maps is the combination of the upcoming Lemmas 3.10 and 3.11, which require a few preparatory results.

#### **Lemma 3.7** Under the above assumptions,

$$\lim_{\xi \to 0^{+}} \frac{1 - \phi_{0}'(\xi) \frac{\ell_{p \oplus}(\xi) \ell_{p \oplus}'(\phi_{0}(\xi))}{\ell_{p \oplus}'(\xi) \ell_{p \oplus}(\phi_{0}(\xi))}}{1 - \phi_{0}'(\xi) \frac{\ell_{p \oplus}(\phi_{0}(\xi))}{\ell_{p \oplus}(\xi)}} = p.$$

PROOF. Set  $\kappa := \frac{\chi}{p+1}$ . By (B3) and (2.16), we have, as  $\xi \to 0^+$ ,

$$\phi_0'(\xi) = 1 - \chi \xi^p + o(\xi^p); \tag{3.39}$$

$$\phi_0(\xi) = \xi - \kappa \xi^{p+1} + o(\xi^{p+1}). \tag{3.40}$$

Let us first consider the case p > 1. The above expansions give

$$\frac{1 - \phi_0'(\xi) \frac{\ell_p(\xi) \ell_p'(\phi_0(\xi))}{\ell_p'(\xi) \ell_p(\phi_0(\xi))}}{1 - \phi_0'(\xi) \frac{\ell_p(\phi_0(\xi))}{\ell_p(\xi)}} = \frac{1 - (1 - \chi \xi^p + o(\xi^p)) \frac{\xi}{\phi_0(\xi)}}{1 - (1 - \chi \xi^p + o(\xi^p)) \frac{\xi^p}{(\phi_0(\xi))^p}}$$

$$= \frac{1 - (1 - \chi \xi^p + o(\xi^p)) \frac{\xi^p}{(\phi_0(\xi))^p}}{1 - (1 - \chi \xi^p + o(\xi^p)) \frac{1 - \kappa \xi^p + o(\xi^p)^{-1}}{1 - (1 - \chi \xi^p + o(\xi^p)) (1 - \kappa \xi^p + o(\xi^p))^{-p}}$$

$$= \frac{\chi - \kappa + o(1)}{\chi - p\kappa + o(1)}$$

$$= p + o(1), \tag{3.41}$$

as claimed. In the case p=1 we have instead

$$\frac{\ell_{1+}(\xi)\,\ell'_{1+}(\phi_0(\xi))}{\ell'_{1+}(\xi)\,\ell_{1+}(\phi_0(\xi))} = \frac{\xi}{\phi_0(\xi)} \left[ \frac{\log \xi}{\log \phi_0(\xi)} \frac{\log \phi_0(\xi) - 1}{\log \xi - 1} \right]. \tag{3.42}$$

The reader might think that a better notation would be to define  $\ell_p$  as in (3.36) and  $\ell_1(\xi) := -\log \xi/\xi$ , but unfortunately we also need the function  $\ell_1(\xi) := 1/\xi$ .

Therefore, since  $\chi - \kappa \neq 0$ ,

$$\frac{1 - \phi_0'(\xi) \frac{\ell_{1+}(\xi) \ell_{1+}(\phi_0(\xi))}{\ell_{1+}'(\xi) \ell_{1+}(\phi_0(\xi))}}{1 - \phi_0'(\xi) \frac{\ell_{1+}(\phi_0(\xi))}{\ell_{1+}(\xi)}} = \frac{1 - (1 - \chi \xi + o(\xi)) \frac{\xi}{\phi_0(\xi)} \left[ \frac{\log \xi}{\log \phi_0(\xi)} \frac{\log \phi_0(\xi) - 1}{\log \xi - 1} \right]}{1 - (1 - \chi \xi + o(\xi)) \frac{\xi}{\phi_0(\xi)} \frac{\log \phi_0(\xi)}{\log \xi}} = \frac{1 - (1 - \chi \xi + o(\xi)) (1 + \kappa \xi + o(\xi)) \left[ \frac{\log \xi}{\log \phi_0(\xi)} \frac{\log \phi_0(\xi) - 1}{\log \xi - 1} \right]}{1 - (1 - \chi \xi + o(\xi)) (1 + \kappa \xi + o(\xi)) \frac{\log \phi_0(\xi)}{\log \xi}}.$$
(3.43)

Using (3.40) with p = 1, we obtain

$$\frac{\log \phi_0(\xi)}{\log \xi} = 1 + o(\xi); \tag{3.44}$$

Q.E.D.

$$\frac{\log \phi_0(\xi) - 1}{\log \xi - 1} = 1 + o(\xi),\tag{3.45}$$

so that (3.43) is equal to 1 + o(1), proving Lemma 3.7.

The next lemma is obtained through (some of) the same estimates as presented above, so we omit its proof.

**Lemma 3.8** Under the above assumptions,

$$\lim_{\xi \to 0^+} \left[ \ell_{p \oplus}(\xi) \left( 1 - \phi_0'(\xi) \frac{\ell_{p \oplus}(\phi_0(\xi))}{\ell_{p \oplus}(\xi)} \right) \right] = \begin{cases} \frac{\chi}{p+1}, & \text{if } p > 1; \\ +\infty, & \text{if } p = 1. \end{cases}$$

It will be convenient below to use modifications of the densities  $\ell_{p\oplus}$  defined as follows:

$$z_{p\oplus}(\xi) := \begin{cases} \ell_{p\oplus}(\xi), & \xi \in (0, \eta/2]; \\ \vartheta(\xi), & \xi \in (\eta/2, \eta); \\ c & \xi \in [\eta, 1], \end{cases}$$

$$(3.46)$$

where  $\eta \in (0, a_1)$  will be fixed in Lemma 3.9,  $c \in (\ell_{p\oplus}(\eta), \ell_{p\oplus}(\eta/2))$ , and  $\vartheta_{p\oplus} \in$  $C^1((\eta/2,\eta))$  is a decreasing function with

$$\lim_{\xi \to (\eta/2)^+} \vartheta(\xi) = \ell_{p\oplus}(\eta/2), \qquad \lim_{\xi \to \eta^-} \vartheta(\xi) = c; \tag{3.47}$$

$$\lim_{\xi \to (\eta/2)^{+}} \vartheta(\xi) = \ell_{p\oplus}(\eta/2), \qquad \lim_{\xi \to \eta^{-}} \vartheta(\xi) = c; \qquad (3.47)$$

$$\lim_{\xi \to (\eta/2)^{+}} \vartheta'(\xi) = \ell'_{p\oplus}(\eta/2), \qquad \lim_{\xi \to \eta^{-}} \vartheta'(\xi) = 0; \qquad (3.48)$$

$$|\vartheta'(\xi)| \le |\ell'_{p\oplus}(\xi)|, \quad \forall \xi \in (\eta/2, \eta);$$
 (3.49)

see Fig. 2. Here  $\eta, c, \vartheta$  depend on  $p \oplus$ . Observe that, by construction,

$$\ell_{p\oplus} \le z_{p\oplus}, \qquad |z'_{p\oplus}| \le |\ell'_{p\oplus}|.$$

$$(3.50)$$

We also denote by  $\zeta_{p\oplus}$  the (infinite) measure on (0,1] with density  $z_{p\oplus}$ . The raison d'être of  $z_{p\oplus}$  is the following property.

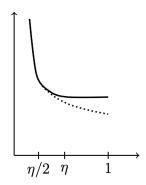


FIGURE 2: Graph of  $z_{p\oplus}$  (solid line) vs graph of  $\ell_{p\oplus}$  (dashed line)

**Lemma 3.9** For  $\eta$  small enough (depending on  $p \oplus$ ) and  $j \in \{0, 1, ..., N-1\}$ ,

$$|\phi_j'| \frac{z_{p\oplus} \circ \phi_j}{z_{p\oplus}} < 1.$$

Furthermore, for  $j \in \{1, ..., N-1\}$ , the above l.h.s. is less than or equal to  $\Lambda^{-1}$ .

PROOF. First, observe that, for all  $j \geq 1$  and  $\xi \in (0,1]$ ,  $\phi_j(\xi) \geq a_1$ . By the monotonicity of  $z_{p\oplus}$  and  $\eta < a_1$ ,

$$z_{p\oplus}(\phi_j(\xi)) \le z_{p\oplus}(a_1) = c = \min_{(0,1]} z_{p\oplus},$$
 (3.51)

which, together with  $|\phi'_j| \leq \Lambda^{-1}$ , proves the second statement of the lemma (and thus the first, limited to  $j \geq 1$ ).

For the case j = 0, notice that, by Lemma 3.8, there exists  $\delta > 0$  such that, for all  $\xi \in (0, \delta)$ ,

$$1 - \phi_0'(\xi) \frac{\ell_{p\oplus}(\phi_0(\xi))}{\ell_{p\oplus}(\xi)} > 0, \quad \text{i.e.,} \quad \phi_0'(\xi) \frac{\ell_{p\oplus}(\phi_0(\xi))}{\ell_{p\oplus}(\xi)} < 1.$$
 (3.52)

Now choose any  $\eta \leq \phi_0(\delta)$ . For  $\xi \in [\delta, 1]$ , using the fact that  $\phi_0$  is increasing and  $z_{p\oplus}$  is decreasing, we have

$$\frac{z_{p\oplus}(\phi_0(\xi))}{z_{p\oplus}(\xi)} \le \frac{z_{p\oplus}(\eta)}{\min z_{p\oplus}} = 1. \tag{3.53}$$

By (B2) and since  $\tau_0(0) = 0$ , for  $\xi \in (0, 1]$ ,

$$\phi_0(\xi) < \xi, \tag{3.54}$$

for all  $\xi \in (0,1]$ . Let us now establish some useful equivalences:

$$\frac{z_{p\oplus}(\phi_0(\xi))}{z_{p\oplus}(\xi)} \le \frac{\ell_{p\oplus}(\phi_0(\xi))}{\ell_{p\oplus}(\xi)} \tag{3.55}$$

$$\iff \frac{z_{p\oplus}(\xi) + \int_{\xi}^{\phi_0(\xi)} z'_{p\oplus}(s) \, ds}{z_{p\oplus}(\xi)} \le \frac{\ell_{p\oplus}(\xi) + \int_{\xi}^{\phi_0(\xi)} \ell'_{p\oplus}(s) \, ds}{\ell_{p\oplus}(\xi)} \tag{3.56}$$

$$\iff \ell_{p\oplus}(\xi) \int_{\phi_0(\xi)}^{\xi} |z'_{p\oplus}(s)| \, ds \le z_{p\oplus}(\xi) \int_{\phi_0(\xi)}^{\xi} |\ell'_{p\oplus}(s)| \, ds. \tag{3.57}$$

So, (3.55) holds true by virtue of (3.57), (3.54) and (3.50). Therefore, using (3.52),

$$\phi_0'(\xi) \, \frac{z_{p\oplus}(\phi_0(\xi))}{z_{p\oplus}(\xi)} \le \phi_0'(\xi) \, \frac{\ell_{p\oplus}(\phi_0(\xi))}{\ell_{p\oplus}(\xi)} < 1, \tag{3.58}$$

for all  $\xi \in (0, \delta)$ . It remains to prove that the leftmost term above is less than 1 also for  $\xi \in [\delta, 1]$ . But this is an easy consequence of (3.53) and  $\phi'_0 < 1$ . Q.E.D.

#### Lemma 3.10 For all j,

$$\sup_{(0,1]} \left| \frac{\frac{z'_{p\oplus}}{z_{p\oplus}^2} - \phi'_j \frac{z'_{p\oplus} \circ \phi_j}{z_{p\oplus}(z_{p\oplus} \circ \phi_j)}}{1 - |\phi'_j| \frac{z_{p\oplus} \circ \phi_j}{z_{p\oplus}}} \right| < \infty.$$

PROOF. Let us first consider the case j=0. Since  $z_{p\oplus}=\ell_{p\oplus}$  in  $(0,\eta/2)$ , by Lemma 3.7 there exists  $\eta_o\in(0,\eta/2)$  such that

$$\sup_{(0,\eta_{o})} \left| \frac{z'_{p\oplus}}{z^{2}_{p\oplus}} - \phi'_{0} \frac{z'_{p\oplus} \circ \phi_{0}}{z_{p\oplus}(z_{p\oplus} \circ \phi_{0})} \right| = \sup_{(0,\eta_{o})} \left| \frac{\ell'_{p\oplus}}{\ell^{2}_{p\oplus}} - \phi'_{0} \frac{\ell'_{p\oplus} \circ \phi_{0}}{\ell_{p\oplus}(\ell_{p\oplus} \circ \phi_{0})} \right| \\
= \sup_{(0,\eta_{o})} \left| \frac{\ell'_{p\oplus}}{\ell^{2}_{p\oplus}} - \phi'_{0} \frac{\ell'_{p\oplus} \circ \phi_{0}}{\ell_{p\oplus}(\ell_{p\oplus} \circ \phi_{0})} \right| \\
\leq \sup_{(0,\eta_{o})} \frac{|\ell'_{p\oplus}|}{\ell^{2}_{p\oplus}} \cdot \sup_{(0,\eta_{o})} \frac{1 - \phi'_{0} \frac{\ell_{p\oplus}(\ell'_{p\oplus} \circ \phi_{0})}{\ell'_{p\oplus}(\ell_{p\oplus} \circ \phi_{0})}}{1 - \phi'_{0} \frac{\ell_{p\oplus} \circ \phi_{0}}{\ell_{p\oplus}}} < \infty \tag{3.59}$$

On the other hand, by Lemma 3.9, using continuity,

$$\inf_{[\eta_o,1]} \left( 1 - \phi_0' \, \frac{z_{p\oplus} \circ \phi_0}{z_{p\oplus}} \right) > 0. \tag{3.60}$$

Since the numerator of the leftmost term in (3.59) is bounded outside a neighborhood of the origin (recall that  $z_{p\oplus} \geq c > 0$ ), we have proved the claim of the Lemma when j = 0.

Let us now consider the case  $j \geq 1$ . Lemma 3.9 implies that

$$1 - |\phi_j'| \frac{z_{p\oplus} \circ \phi_j}{z_{p\oplus}} \ge 1 - \Lambda^{-1} > 0.$$
 (3.61)

Also,  $z'_{p\oplus} \circ \phi_j = 0$ , because  $\phi_j \ge a_1 > \eta$ , Thus,

$$\sup_{(0,1]} \left| \frac{z'_{p\oplus}}{z^2_{p\oplus}} - \phi'_j \frac{z'_{p\oplus} \circ \phi_j}{z_{p\oplus}(z_{p\oplus} \circ \phi_j)} \right| = \sup_{(0,1]} \frac{|z'_{p\oplus}|}{z^2_{p\oplus}} < \infty, \tag{3.62}$$

because the argument of the sup is bounded in a right neighborhood of 0 and identically null in a left neighborhood of 1. The previous two estimates conclude the proof of Lemma 3.10.

Q.E.D.

#### Lemma 3.11 For all j,

$$\sup_{(0,1]} \frac{\left| \frac{\phi_j''}{\phi_j'} \right|}{z_{p\oplus} \left( 1 - |\phi_j'| \frac{z_{p\oplus} \circ \phi_j}{z_{p\oplus}} \right)} < \infty.$$

PROOF. We proceed as in the previous proof. For j = 0, by Lemma 3.8 there exists  $\eta_o \in (0, \eta/2)$  such that

$$\sup_{(0,\eta_o)} \frac{\frac{|\phi_0''|}{\phi_0'}}{z_{p\oplus} \left(1 - \phi_0' \frac{z_{p\oplus} \circ \phi_0}{z_{p\oplus}}\right)} = \sup_{(0,\eta_o)} \frac{\frac{|\phi_0''|}{\phi_0'}}{\ell_{p\oplus} \left(1 - \phi_0' \frac{\ell_{p\oplus} \circ \phi_0}{\ell_{p\oplus}}\right)} \\
\leq \frac{\sup_{(0,\eta_o)} \frac{|\phi_0''|}{\phi_0'}}{\inf_{(0,\eta_o)} \ell_{p\oplus} \left(1 - \phi_0' \frac{\ell_{p\oplus} \circ \phi_0}{\ell_{p\oplus}}\right)} < \infty$$
(3.63)

(recall that is  $\phi_0''$  continuous on [0,1]). The bound on  $[\eta_o,1]$  follows from (3.60) and the definition of  $z_{p\oplus}$ .

In the case  $j \geq 1$ , the assertion follows directly form the second statement of Lemma 3.9, the definition of  $z_{p\oplus}$  and the fact that  $|\phi_i''|/|\phi_i'|$  is bounded. Q.E.D.

As in the proof of Theorem 2.8, we need cones of local observables and a suitable transfer operator to act on them.

Cones of local observables: For  $\delta, D, \varepsilon > 0$ , set

$$C_{\delta,D,\varepsilon} := \left\{ g \in L^1(\zeta_{p\oplus}) \cap C^1 \mid g > 0, \frac{|g'|}{g} \le Dz_{p\oplus} \text{ and } \frac{|g'(\xi)|}{g(\xi)} \le \varepsilon z_{p\oplus}(\xi), \forall \xi \le \delta \right\},$$
(3.64)

where we have dropped from the indication that all functions are defined in (0,1]. These cones are analogous to the ones defined in (3.19) for the half-line case, except that they are relative to the distance

$$dist(\xi_1, \xi_2) := \left| \int_{\xi_1}^{\xi_2} z_{p\oplus}(\xi) \, d\xi \right| \tag{3.65}$$

in (0,1] (notice that the diameter of (0,1] is infinity, w.r.t. dist).

**Transfer operator:** We define  $\widehat{T}_{p\oplus}: L^1(\zeta_{p\oplus}) \longrightarrow L^1(\zeta_{p\oplus})$  via the identity

$$\forall F \in L^{\infty}(\zeta_{p\oplus}), g \in L^{1}(\zeta_{p\oplus}), \qquad \int_{(0,1]} (F \circ T) g \, d\zeta_{p\oplus} = \int_{(0,1]} F \, \widehat{T}_{p\oplus} g \, d\zeta_{p\oplus}. \tag{3.66}$$

This operator describes the evolution of densities w.r.t. the measure  $\zeta_{p\oplus}$ . Since  $\frac{d\zeta_{p\oplus}}{dm} = z_{p\oplus}$ , a standard computation gives

$$\widehat{T}_{p\oplus} g = \frac{\widehat{T}(gz_{p\oplus})}{z_{p\oplus}} = \frac{1}{z_{p\oplus}} \sum_{i=0}^{N-1} |\phi_j'| (gz_{p\oplus}) \circ \phi_j, \tag{3.67}$$

where  $\widehat{T}$  is the Perron-Frobenius operator for T on (0,1]. Set

$$\underline{D} := \max_{0 \le j \le N-1} \sup_{(0,1]} \left( \frac{\frac{|\phi_j''|}{|\phi_j'|}}{z_{p\oplus} \left(1 - |\phi_j'| \frac{z_{p\oplus} \circ \phi_j}{z_{p\oplus}}\right)} + \frac{\left|\frac{z_{p\oplus}'}{z_{p\oplus}} - \phi_j' \frac{z_{p\oplus}' \circ \phi_j}{z_{p\oplus}(z_{p\oplus} \circ \phi_j)}\right|}{\left(1 - |\phi_j'| \frac{z_{p\oplus} \circ \phi_j}{z_{p\oplus}}\right)} \right). \quad (3.68)$$

By Lemmas 3.11 and 3.10,  $\underline{D} < \infty$ .

**Lemma 3.12** Let  $D \geq \underline{D}$ . For all  $\varepsilon > 0$ , there exists  $\delta \in (0,1)$  such that  $\widehat{T}_{p\oplus} \mathcal{C}_{\delta,D,\varepsilon} \subseteq \mathcal{C}_{\delta,D,\varepsilon}$ .

Proof. A somewhat lengthy computation gives

$$(\widehat{T}_{p\oplus} g)' = \sum_{j=0}^{N-1} |\phi'_j| [(gz_{p\oplus}) \circ \phi_j] \mathcal{D}_j(g),$$
(3.69)

where

$$\mathcal{D}_{j}(g) := \frac{1}{z_{p\oplus}} \left( \frac{\phi_{j}''}{\phi_{j}'} + \phi_{j}' \frac{g' \circ \phi_{j}}{g \circ \phi_{j}} + \phi_{j}' \frac{z_{p\oplus}' \circ \phi_{j}}{z_{p\oplus} \circ \phi_{j}} - \frac{z_{p\oplus}'}{z_{p\oplus}} \right), \tag{3.70}$$

intended as a function  $(0,1] \longrightarrow \mathbb{R}$ . For all g > 0 with  $\frac{|g'|}{g} \leq Dz_{p\oplus}$ , we have

$$\max_{j} \|\mathcal{D}_{j}(g)\|_{\infty} \leq \max_{j} \sup_{(0,1]} \left[ \frac{1}{z_{p\oplus}} \left( \left| \frac{\phi_{j}''}{\phi_{j}'} \right| + D(z_{p\oplus} \circ \phi_{j}) |\phi_{j}'| + \left| \phi_{j}' \frac{z_{p\oplus}' \circ \phi_{j}}{z_{p\oplus}} \circ \phi_{j} - \frac{z_{p\oplus}'}{z_{p\oplus}} \right| \right) \right]. \tag{3.71}$$

Using that  $D \geq \underline{D}$ , one has that (3.71) is less than or equal to D. Therefore

$$\left| (\widehat{T}_{p\oplus} g)' \right| \le \max_{j} \|\mathcal{D}_{j}(g)\|_{\infty} \, \widehat{T}(gz_{p\oplus}) \le D \, \widehat{T}(gz_{p\oplus}) = Dz_{p\oplus} \, \widehat{T}_{p\oplus} \, g, \tag{3.72}$$

cf. (3.67).

For a given  $\varepsilon > 0$ , we now describe how to choose  $\delta$  so that  $\mathcal{C}_{\delta,D,\varepsilon}$  is invariant. There exists  $\delta_1 \in (0, \eta/2)$ , cf. (3.46), such that the following functional (in)equalities are true, when restricted to  $(0, \delta_1]$ :

$$\left(\frac{|\phi_0''|}{z_{p\oplus}\phi_0'} + \left|\frac{z_{p\oplus}'}{z_{p\oplus}^2} - \phi_0' \frac{(z_{p\oplus}' \circ \phi_0)}{(z_{p\oplus} \circ \phi_0)z_{p\oplus}}\right|\right) \left(1 - \phi_0' \frac{z_{p\oplus} \circ \phi_0}{z_{p\oplus}}\right)^{-1}$$

$$= \frac{|\phi_0''|}{\phi_0' \ell_{p\oplus} \left(1 - \phi_0' \frac{\ell_{p\oplus} \circ \phi_0}{\ell_{p\oplus}}\right)} + \frac{|\ell_{p\oplus}'|}{\ell_{p\oplus}^2} \frac{\left|1 - \phi_0' \frac{\ell_{p\oplus}(\ell_{p\oplus}' \circ \phi_0)}{\ell_{p\oplus}(\ell_{p\oplus} \circ \phi_0)}\right|}{1 - \phi_0' \frac{\ell_{p\oplus} \circ \phi_0}{\ell_{p\oplus}}}$$

$$< \varepsilon. \tag{3.73}$$

In fact, calling  $\xi$  the argument of all the above functions, when  $\xi \to 0^+$  the second term of the second line of (3.73) vanishes by Lemma 3.7 and the fact that  $\frac{|\ell'_{p\oplus}(\xi)|}{\ell^2_{p\oplus}(\xi)} \to 0$ . The same is true for the first term, as a consequence of Lemmas 3.6 and 3.8. Moreover, there exists  $\delta_2 > 0$  such that, in  $(0, \delta_2]$ ,

$$\frac{|\phi_j''|}{z_{p\oplus}|\phi_j'|} + \frac{D(z_{p\oplus} \circ \phi_j)|\phi_j'|}{z_{p\oplus}} + \frac{|z_{p\oplus}'|}{z_{p\oplus}^2} \le \varepsilon, \tag{3.74}$$

whenever  $j \geq 1$ . In fact, for the first two terms, both  $D|\phi'_j|(z_{p\oplus} \circ \phi_j)$  and  $|\phi''_j/\phi'_j|$  are bounded, for  $j \geq 1$ . As for the third term,

$$\lim_{\xi \to 0^+} \frac{|z'_{p\oplus}(\xi)|}{z^2_{p\oplus}(\xi)} = \lim_{\xi \to 0^+} \frac{|\ell'_{p\oplus}(\xi)|}{\ell^2_{p\oplus}(\xi)} = 0.$$
 (3.75)

Set  $\delta := \min\{\delta_1, \delta_2\}$  and let  $g \in \mathcal{C}_{\delta, D, \varepsilon}$ . We show that  $\widehat{T}_{p \oplus} g \in \mathcal{C}_{\delta, D, \varepsilon}$ . Evidently,  $\widehat{T}_{p \oplus} g > 0$  and  $\widehat{T}_{p \oplus} g \in C^1$ . By (3.72),  $\frac{|\widehat{T}_{p \oplus} g'|}{\widehat{T}_{p \oplus} g} \leq Dz_{p \oplus}$ . For  $\xi \leq \delta$ , by (3.54),  $\phi_0(\xi) \leq \delta$ , whence  $\frac{|g' \circ \phi_0(\xi)|}{g \circ \phi_0(\xi)} \leq \varepsilon z_{p \oplus} \circ \phi_0(\xi)$ .

Therefore, looking at (3.69)-(3.70) and using that  $z'_{p\oplus} \circ \phi_j \equiv 0$ , for all  $j \geq 1$ , we can write that, on  $(0, \delta]$ ,

$$\left| (\widehat{T}_{p\oplus} g)' \right| \leq \max_{j \geq 0} \left| \mathcal{D}_{j}(g) \right| \widehat{T}(gz_{p\oplus})$$

$$\leq \max \left\{ \frac{1}{z_{p\oplus}} \left( \frac{|\phi_{0}''|}{\phi_{0}'} + \phi_{0}' \varepsilon(z_{p\oplus} \circ \phi_{0}) + \left| \frac{z_{p\oplus}'}{z_{p\oplus}} - \frac{(z_{p\oplus}' \circ \phi_{0})\phi_{0}'}{z_{p\oplus} \circ \phi_{0}} \right| \right), \qquad (3.76)$$

$$\max_{j \geq 1} \left[ \frac{1}{z_{p\oplus}} \left( \frac{|\phi_{0}''|}{|\phi_{0}'|} + |\phi_{0}'| D(z_{p\oplus} \circ \phi_{0}) + \frac{|z_{p\oplus}'|}{z_{p\oplus}} \right) \right] \right\} \widehat{T}(gz_{p\oplus}).$$

Once again, the above inequality holds when all the functions are restricted to  $(0, \delta]$ . Using (3.73) and (3.74), one shows that, with the same restriction on the arguments, both terms inside the above braces are  $\leq \varepsilon$ , concluding that, for  $0 < \xi \leq \delta$ ,

$$\left| (\widehat{T}_{p\oplus} g)'(\xi) \right| \le \varepsilon \widehat{T}(g z_{p\oplus})(\xi) = \varepsilon z_{p\oplus}(\xi)(\widehat{T}_{p\oplus} g)(\xi). \tag{3.77}$$

This ends the proof of Lemma 3.12.

Q.E.D.

The analogue of Lemma 3.5 is the following lemma, whose proof is included for the reader's convenience.

**Lemma 3.13** Let  $F \in \mathcal{G}_{\text{unif}}(\zeta_{p\oplus})$ ,  $\overline{\zeta_{p\oplus}}(F) = 0$ . For each  $\rho > 0$ , there exists  $\varepsilon > 0$  such that for, every  $g \in C^1((0,1])$  and  $\delta \geq 0$  with

- (i) g > 0;
- (ii)  $\zeta_{p\oplus}(g) = 1$ ;
- (iii)  $\frac{|g'(\xi)|}{g(\xi)} \le \varepsilon z_{p\oplus}(\xi)$ , for all  $\xi \le \delta$ ,

one has

$$\left| \int_0^\delta Fg \, d\zeta_{p\oplus} \right| \le \rho.$$

PROOF. We assume  $||F||_{\infty} > 0$ , otherwise the statement is trivial. By the hypothesis  $\overline{\zeta_{p\oplus}}(F) = 0$ , for any  $\rho > 0$ , there exists r > 0 such that

$$\sup_{V \in \mathcal{V}_{\zeta_{p\oplus}}(r)} \left| \frac{1}{r} \int_{V} F \, d\zeta_{p\oplus} \right| \le \frac{\rho}{2}. \tag{3.78}$$

Set  $\varepsilon := \frac{\rho}{2r||F||_{\infty}}$ . We partition  $(0, \delta]$  into a countable family of adjacent intervals  $V_k \in \mathscr{V}_{\zeta_{p\oplus}}(r)$ . For all g and  $\delta$  verifying (i)-(iii),

$$\int_0^\delta Fg \, d\zeta_{p\oplus} = \sum_{k=0}^\infty \int_{V_k} Fg \, d\zeta_{p\oplus}. \tag{3.79}$$

For  $k \in \mathbb{N}$ , denote by  $\xi_{\min,k}$  a minimum point of g in  $V_k$ . By (3.78), (i), (iii) and the definition of  $\varepsilon$ ,

$$\left| \int_{V_k} Fg \, d\zeta_{p\oplus} \right| = \left| \int_{V_k} F(\xi) \left( g(\xi_{\min,k}) + \int_{\xi_{\min,k}}^{\xi} g'(t) \, dt \right) d\zeta_{p\oplus}(\xi) \right|$$

$$\leq \frac{\rho}{2} rg(\xi_{\min,k}) + r \|F\|_{\infty} \varepsilon \left( \int_{V_k} g(t) \, z_{p\oplus}(t) \, dt \right)$$

$$\leq \left( \frac{\rho}{2} + \frac{\rho}{2} \right) \left( \int_{V_k} g(t) \, d\zeta_{p\oplus}(t) \right).$$
(3.80)

and we conclude the proof of the lemma by (3.79), (3.80) and (ii). Q.E.D.

As a penultimate step towards the proof of Theorem 2.11, we show that T is fully global-local mixing w.r.t.  $\mathscr V$  and the measure  $\zeta_{p\oplus}$ . To this end, in complete analogy with the case of maps of  $\mathbb R_0^+$ —see the discussion before (3.33)—it suffices to prove that, for all  $F \in \mathcal G_{\mathrm{unif}}(\zeta_{p\oplus})$  with  $\overline{\zeta_{p\oplus}}(F) = 0$  and  $\rho > 0$ , there exists  $g_\rho \in L^1(\zeta_{p\oplus})$  with  $\zeta_{p\oplus}(g_\rho) = 1$  such that

$$\limsup_{n \to \infty} |\zeta_{p \oplus}((F \circ T^n)g_{\rho})| \le 2\rho. \tag{3.81}$$

To this end, given  $\rho > 0$ , we consider  $\varepsilon$  as given by Lemma 3.13 and  $\delta$  as given by Lemma 3.12 for such value of  $\varepsilon$ . Let  $D \geq \underline{D}$  and  $g_{\rho}$  be any element of  $\mathcal{C}_{\delta,D,\varepsilon}$  with  $\zeta_{p\oplus}(g_{\rho}) = 1$ . For all  $n \in \mathbb{N}$ ,

$$|\zeta_{p\oplus}((F \circ T^n)g_{\rho})| \leq |\zeta_{p\oplus}((1_{[\delta,1]}F) \circ T^n)g_{\rho})| + |\zeta_{p\oplus}((1_{(0,\delta)}F) \circ T^n)g_{\rho})|$$

$$= |\zeta_{p\oplus}((1_{[\delta,1]}F) \circ T^n)g_{\rho})| + \left| \int_0^{\delta} F(\widehat{T}_{p\oplus}^n g_{\rho}) d\zeta_{p\oplus} \right|. \tag{3.82}$$

For  $n \to \infty$ , the first term vanishes by local-local mixing, for T is exact (w.r.t  $z_{p\oplus}$  as well as m). The second term is bounded above by  $\rho$  by Lemma 3.13, since  $\widehat{T}^n_{p\oplus} g_{\rho} \in \mathcal{C}_{\delta,D,\varepsilon}$  (Lemma 3.12) and  $\zeta_{p\oplus}(\widehat{T}^n_{p\oplus} g_{\rho}) = \|\widehat{T}^n_{p\oplus} g_{\rho}\|_{L^1(\zeta_{p\oplus})} = \|g_{\rho}\|_{L^1(\zeta_{p\oplus})} = 1$ . Finally, full global-local mixing w.r.t.  $\zeta_{p\oplus}$  and w.r.t.  $\lambda_{p\oplus}$  are equivalent properties

Finally, full global-local mixing w.r.t.  $\zeta_{p\oplus}$  and w.r.t.  $\lambda_{p\oplus}$  are equivalent properties by Proposition 2.2 and Lemma 2.6. This proves the statement of Theorem 2.11 for the case p > 1. As for the case p = 1, one sees by Proposition 2.4 and Remark 2.5 that  $\overline{\zeta_{1+}}$ is a restriction of  $\overline{\lambda_1}$ . But  $\overline{\lambda_{1+}} = \overline{\zeta_{1+}}$  (Proposition 2.2), so  $\overline{\lambda_{1+}}$  is a restriction of  $\overline{\lambda_1}$ . Then one applies Lemma 2.6 to conclude that  $((0,1], \mathcal{B}, \lambda_1, T)$  is global-local mixing w.r.t.  $\mathcal{V}$ ,  $\mathcal{G}_{\text{unif}}(\lambda_{1+})$  and  $L^1(\lambda_1)$ . Q.E.D.

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<sup>&</sup>lt;sup>4</sup>A strict restriction, in fact, as one can see that  $F_1(\xi) = \cos(\log(\xi^{-1}))$  belongs to  $\mathcal{G}_{\text{unif}}(\lambda_1)$  but not to  $\mathcal{G}_{\text{unif}}(\lambda_{1+})$ ; cf. proof of Corollary 2.13 in Section 2.3.

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