Quasi Directed Jónsson Operations Imply Bounded Width (For fo-expansions of symmetric binary cores with free amalgamation)

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Abstract

Every $CSP(\mathbb{B})$ for a finite structure \mathbb{B} is either in P or it is NP-complete but the proofs of the finite-domain CSP dichotomy by Andrei Bulatov and Dimitryi Zhuk not only show the computational complexity separation but also confirm the algebraic tractability conjecture stating that tractability origins from a certain system of operations preserving B. The establishment of the dichotomy was in fact preceded by a number of similar results for stronger conditions of this type, i.e. for system of operations covering not necessarily all tractable finite-domain CSPs.

A similar, infinite-domain algebraic tractability conjecture is known for first-order reducts of countably infinite finitely bounded homogeneous structures and is currently wide open. In particular, with an exception of a quasi near-unanimity operation there are no known systems of operations implying tractability in this regime. This paper changes the state-of-the-art and provides a proof that a chain of quasi directed Jónsson operations imply tractability and bounded width for a large and natural class of infinite structures.

1 Introduction

Constraint Satisfaction Problems form a large class of computational problems whose complexity has been studied separately as well as within the formalism $\mathrm{CSP}(\mathbb{B})$ where \mathbb{B} is a relational structure over domain A. An instance \mathcal{I} of $\mathrm{CSP}(\mathbb{B})$ consists of a set of constraints $\mathbb{C} := ((x_1, \ldots, x_k), R)$ formed out of a tuple of variables and a k-ary relation R in \mathbb{B} . The question is whether there is a solution to \mathcal{I} , i.e., an assignment of elements from A to the variables such that for all \mathbb{C} we have $f((x_1, \ldots, x_k)) \in R$.

The formalism $CSP(\mathbb{B})$ for a finite \mathbb{B} captures a number of natural problems. Indeed, if \mathbb{B} is a graph \mathcal{H} , then we cope with an \mathcal{H} -coloring problem [18], in particular with a k-colouring problem. If the domain of \mathbb{B} has only two elements, then $CSP(\mathbb{B})$ is a Boolean satisfiability problem such as k-SAT or NAE-SAT [26]. There are also structures \mathbb{B} whose $CSP(\mathbb{B})$ comes down to solving a system of equations over a finite field [19]. The variety of natural NP-complete and polynomially solvable problems within this formalism raises a number of natural questions including:

- 1. Is there a single simple technique of showing NP-completness for CSP(B)?
- 2. Is there a single algorithm that solves all polynomially tractable $\mathrm{CSP}(\mathbb{B})$?
- 3. Are all $\mathrm{CSP}(\mathbb{B})$ either NP-complete or polynomially tractable?

The conjecture that the answer to the last question is affirmative was known as the Feder-Vardi Conjecture [17]. However, there was no reasonable progress until the questions were reformulated in the language of universal algebra [15]. The immediate consequence of that step was the conditional answer to the first question and a clear algebraic conjecture concerning the second question known as the algebraic tractability conjecture saying that a $CSP(\mathbb{B})$ is in PTIME if and only if the algebra corresponding to \mathbb{B} lies in a Taylor variety. Actually, this conjecture has a number of equivalent formulations and the one of which we care the

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most in this paper says that $CSP(\mathbb{B})$ is in PTIME iff \mathbb{B} is preserved by a Siggers operation [27]. The algebraic tractability conjecture has been confirmed independently by Bulatov [14] and Zhuk [30], and hence all the three questions for finite structures \mathbb{B} have been answered.

Although the Bulatov-Zhuk theorem is a great achievement, there are still some natural computational problems that can be expressed as CSP(B) but only when B is infinite. A perfect example is DIGRAPH-ACYCLICITY equivalent to $CSP(\mathbb{Q};<)$ where $(\mathbb{Q};<)$ is the natural linear order over the rational numbers. That structure as well as all structures with a first-order definition in (first-order reducts of) $(\mathbb{Q};<)$ a.k.a. temporal languages give rise to constraint satisfaction problems of interest in a field of artificial intelligence known as spatial and temporal reasoning, see [6] for more examples of natural computational problems expressible as infinite-domain CSPs. Furthermore, the structure $(\mathbb{Q};<)$ is a natural representative of a class of (countably infinite) finitely bounded homogeneous structures.

A τ -structure \mathbb{A} is homogeneous if every local isomorphism between finite substructures of \mathbb{A} may be extended to an automorphism of A. It is *finitely bounded* if there exists a finite set of finite τ -structures $\mathcal{F}_{\mathbb{A}}$ such that a finite structure $\mathbb D$ embeds into $\mathbb A$ if and only if there is no $\mathbb C \in \mathcal F_{\mathbb A}$ that embeds into $\mathbb D$. The following conjecture resembling the finite algebraic tractability conjecture has been formulated for cores of finite-signature first-order reducts of finitely-bounded homogeneous structures [5]. (All definitions that are omitted in the introduction are provided carefully in preliminaries.)

Conjecture 1. (Infinite-domain Algebraic Tractability Conjecture) Let $\mathbb B$ be the core of a finitesignature first-order reduct of a finitely bounded homogeneous structure. If $\mathbb B$ is preserved by a pseudo-Siggers operation, i.e., a 6-ary operation s and some unary operations α, β such that

$$\alpha s(x, y, x, z, y, z) \approx \beta s(y, x, z, x, z, y)$$

for all x, y, z in domain, then $CSP(\mathbb{A})$ is solvable in polynomial time.

The above conjecture has been confirmed in a number of special cases, in particular for temporal languages [8] or first-order reducts of homogeneous graphs [12, 9]. It is also known that if there is no pseudo-Siggers operation preserving B, then CSP(B) is NP-complete [?]. A simple strategy to attack the above conjecture in full generality would be to provide a polynomial-time algorithm for an infinite-domain CSP(B) which works only under the assumption that B is preserved by a pseudo-Siggers operation. The reality is, however, that we do not have such algorithms even for operations satisfying much stronger algebraic conditions of that type, called pseudo minor conditions or quasi minor conditions. The situation is very different for finite-domain CSPs where a number of algebraic conditions implying tractability has been identified prior to the dichotomy proofs, see [19] and [4] for the two most important results of this kind. The notable exception in an infinite-domain CSP regime is a quasi near-unanimity operation f satisfying:

$$f(y, x, ..., x) \approx \cdots \approx f(x, ..., x, y) \approx f(x, ..., x)$$

for all x, y in domain, but even in this case the proof [7] is a straightforward adaptation of the one for finite structures [17].

The next natural algebraic condition to consider for infinite structures are chains of quasi directed Jónsson operations.

Definition 1. A sequence (D_1, \ldots, D_n) of ternary operations on a set A is called a chain of quasi directed Jónsson operations if for all $x, y, z \in A$ all of the following hold:

$$D_1(x, x, y) = D_1(x, x, x), \tag{1}$$

$$D_{i}(x, y, x) = D_{i}(x, x, x),$$

$$D_{i}(x, y, x) = D_{i}(x, x, x)$$

$$D_{i}(x, y, y) = D_{i+1}(x, x, y)$$

$$for all i \in [n],$$

$$for all i \in [n-1],$$

$$(3)$$

$$D_i(x, y, y) = D_{i+1}(x, x, y)$$
 for all $i \in [n-1]$, (3)

$$D_n(x, y, y) = D_n(y, y, y). (4)$$

Every relational structure preserved by a quasi near-unanimity operation is also preserved by a chain of quasi directed Jónsson operations, see [6] for a simple proof. Another, even weaker system of operations worth considering is a chain of quasi Jónsson operations.

A finite-signature finite structure is preserved by a

- near-unanimity operation iff
- by a chain of directed Jónsson operations [20] iff
- by a chain of Jónsson operations[1].

One should however keep in mind that a direct proof of tractability for Jónsson operations [3] came prior to the equivalence of the three-conditions.

The main result of this paper, explained in details in the following subsection, states that every first-order expansion \mathbb{B} of a finitely-bounded homogeneous symmetric binary core (all relations are binary and symmetric) whose age has free amalgamation and which is preserved by a chain of directed quasi Jónsson operations has bounded width, and in consequence $CSP(\mathbb{B})$ may be solved in polynomial time by establishing local consistency.

Examples of symmetric binary structures are the countably infinite homogeneous universal graph a.k.a. the random graph \mathbb{G} or the countably infinite homogeneous universal graph omitting cliques of size k, for any $k \geq 3$, known as the k-th Henson graph \mathbb{H}_k . The complexity classifications of CSPs for first-order reducts of these structures have been obtained in [12] and respectively in [9]. The authors of the latter paper actually suggest that a reasonable intermediate step towards confirming Conjecture 1 might be to prove it for first-order reducts of structures whose age has free amalgamation. Here we make a major step towards completing that goal, which is much more challenging, even for symmetric binary cores, than the classification for homogeneous graphs presented in that paper. Homogeneous graphs have been classified in [21] — there are only few types of these. There are no such results for symmetric binary cores whose age has free amalgamation except for some partial results over multi-graphs with three kinds of edges [16].

1.1 The Main Result

In this section we present the main result of this paper.

1.1.1 Fraïssé Amalgamation and Free Amalgamation

Let \mathbb{B}_1 , \mathbb{B}_2 be two structures over the same signature τ such that all the common elements are the elements of \mathbb{A} . We say that \mathbb{C} is an amalgam of \mathbb{B}_1 and \mathbb{B}_2 over \mathbb{A} if for i=1,2 there are embeddings $f_i:\mathbb{B}_i\to\mathbb{C}$ such that $f_1(a)=f_2(a)$ for every $a\in\mathbb{A}$. An isomorphism-closed class \mathscr{C} of relational τ -structures has the amalgamation property if \mathscr{C} is closed under taking amalgams. A class of finite τ -structures that contains at most countably many non-isomorphic structures, has the amalgamation property, and is closed under taking induced substructures and isomorphisms is called an amalgamation class. It is known that there is a deep connection between amalgamation classes and homogeneous structures. The age of a homogeneous τ -structure \mathbb{A} is the class of all finite τ -structures that embed into \mathbb{A} . First of all, it is known that the age of a homogeneous structure is an amalgamation class. But what is more surprising is that out of any amalgamation class one can construct a unique, up to isomorphism, homogeneous structure.

Theorem 1. (Fraïssé) Let τ be a countable relational signature and let \mathscr{C} be an amalgamation class of τ -structures. Then there is a homogeneous and at most countable τ -structure \mathbb{A} whose age equals \mathscr{C} . The structure \mathbb{A} is unique up to isomorphism, and called the Fraïssé-limit of \mathscr{C} .

For example, we have that $(\mathbb{Q}; <)$ is the Fraïssé-limit of the class of all finite linear orders, \mathbb{G} is the Fraïssé-limit of the class of all finite graphs and that \mathbb{H}_k is the Fraïssé-limit of the class of all finite graphs omitting k-cliques. In fact, in the two last cases as an amalgam of the graphs G_1 and G_2 one can take the union $G_1 \cup G_2$ of G_1 and G_2 . We amalgamate without imposing new edges. Observe that it is in general not the case for linear orders when one usually have to add additional arcs to impose transitivity.

In general, we say that $\mathbb{B}_1 \cup \mathbb{B}_2$ is a *free amalgam* of \mathbb{B}_1 and \mathbb{B}_2 over $\mathbb{B}_1 \cap \mathbb{B}_2$. A class of structures \mathscr{C} for which taking free amalgams suffices is the class with the free amalgamation property. For the sake of simplicity we will simply say say that the class has free amalgamation.

A more general recipe of obtaining a finitely bounded homogeneous structure over a signature $\tau = \{\mathbf{A}, \mathbf{B}\}$ whose age has free-amalgamation is given in [16]. First take any finite set of finite 2-graphs $\mathcal{F}_{\mathbb{A}}$ with **A**-edges and **B**-edges: every two different vertices are connected by an **A**-edge or a **B**-edge. The class of finite structures $\mathscr{C}_{\mathbb{A}}$ over τ omitting the 2-graphs in $\mathcal{F}_{\mathbb{A}}$ has free amalgamation and the Fraïssé limit \mathbb{A} of $\mathscr{C}_{\mathbb{A}}$ is the desired τ -structures.

Although, it is not the case that the age of all homogeneous structures has free amalgamation, it is a natural restriction for structures considered in Conjecture 1.

1.2 Formulation of the Main Result

We will prove the tractability of $CSP(\mathbb{B})$ for templates under consideration by employing an algorithm establishing (k, l)-minimality for $(k \leq l)$. This algorithm is a slight variation of the better known procedure for enforcing (k, l)-consistency but better suited for measuring the level of consistency needed to solve a problem [2]. Roughly speaking, it propagates the local information about k variables in context of l variables

through the structure of a CSP instance until a fixed-point is reached. Sometimes it only narrows the search space, but in the case of 2-coloring, 2-SAT, Horn-SAT or the already mentioned DIGRAPH ACYCLICITY, the procedure simply decides whether a given instance of the problem has a solution.

We say that a structure \mathbb{B} has relational width (k, l) if establishing (k, l)-minimality on an instance of $CSP(\mathbb{B})$ decides if it has a solution. A structure \mathbb{B} has bounded (relational) width if it has relational width (k, l) for some $(k \le l)$.

Here comes the main result of this paper.

Theorem 2. Let \mathbb{A} be a finitely bounded homogeneous symmetric binary core whose age has free amalgamation and \mathbb{B} a first-order expansion of \mathbb{A} preserved by a chain of quasi directed Jónsson operations. Then $CSP(\mathbb{B})$ has relational width $(2, \mathbb{L}_{\mathbb{A}})$ where $\mathbb{L}_{\mathbb{A}}$ is the size of the largest forbidden structure in $\mathcal{F}_{\mathbb{A}}$ but not smaller than 3. In particular, $CSP(\mathbb{B})$ is in PTIME.

1.3 Bounded (Strict) Width Collapses

A structure \mathbb{B} has strict width k if every non-trivial (k, k+1)-minimal instance of $\mathrm{CSP}(\mathbb{B})$ not only has a solution but also every its partial solution may be extended to a total solution. It is known that an ω -categorical structure (all structures in this paper are ω -categorical, for a definition see [6]) has strict width k if and only if it is preserved by a (k+1)-ary quasi near-unanimity operation. Thus, any first-order expansion \mathbb{B} of a finitely bounded homogeneous symmetric binary core whose age has free amalgamation and which is preserved by a (k+1)-ary quasi near-unanimity polymorphism has relational width (k, k+1). Theorem 2 implies a stronger result.

Corollary 1. Let \mathbb{A} be a finitely bounded homogeneous symmetric binary core whose age has free amalgamation and \mathbb{B} a first-order expansion of \mathbb{A} with bounded strict width. Then \mathbb{B} has relational width $(2, \mathbb{L}_{\mathbb{A}})$ where $\mathbb{L}_{\mathbb{A}}$ is the size of the largest forbidden structure in $\mathcal{F}_{\mathbb{A}}$ but not smaller than 3.

Proof. The corollary follows by Theorem 2 and Proposition 6.9.10 in [6].

A similar result has been shown for first-order reducts of homogeneous graphs in [29] and first-order expansions of (not necessarily symmetric) binary cores which do not forbid any substructures of size 3, 4, 5, or 6 in [28]. That kind of results imply that in the considered cases the level of consistency needed to solve $CSP(\mathbb{B})$ for a first-order reduct (expansion) of \mathbb{A} depends rather on \mathbb{A} than on \mathbb{B} . These results are related to the original bounded width hierarchy collapse in [2] which says that a finite structure with bounded width has relational width (1,1) or (2,3). The already discussed bounded width collapses for infinite-domain CSP consider only strict width. In [22], one can find a result similar to Corollary 1 considering structures with bounded width but not necessarily bounded strict width.

1.4 Black box approach and canonical polymorphisms

An approach of converging to an algorithm for a pseudo-Siggers operation in Conjecture 1 by finding polynomial-time algorithms for weaker and weaker algebraic conditions is not the only approach that can possibly lead to resolving the conjecture. Indeed, all the algorithmic results in the classification for the first-order reducts of \mathbb{G} or \mathbb{H}_k with $k \geq 3$ may be obtained by use of a black box reduction to finite CSP from [10]. See [23] for a neat proof of the latter. That black box reduction is obtained by use of canonical polymorphisms [13] and in particular of a canonical pseudo-Siggers operation. The problem with this approach is that already getting from homogeneous graphs to homogeneous hypergraphs makes the use of the simple black box reduction impossible [25, 24].

1.5 Organisation of the Present Article

All the notions and definitions that have not been properly introduced in Section 1 are defined in Section 2. In Section 3 we treat \mathbb{B} whose relational clones are called *implicationally uniform*. We show that such \mathbb{B} have relational width $(2, \mathbb{L}_{\mathbb{A}})$. Section 4 is dedicated to \mathbb{B} whose relational clones are implicationally non-uniform — in that case \mathbb{B} is not preserved by any chains of quasi directed Jónsson operations. Armed with these results, we give a proof of Theorem 2 in Section 5.

2 Preliminaries

We use [n] for $\{1, \ldots, n\}$ and we write t[i] as a reference to the *i*-th element of an *n*-ary tuple with $i \in [n]$. A structure is usually denote by $\mathbb{A}, \mathbb{B}, \mathbb{C}$ etc. and their corresponding domains by A, B, C etc.

2.1 Structures, Relations and Formulas

All structures \mathbb{A} considered in this paper are ω -categorical, i.e., all countable models of the theory of \mathbb{A} are isomorphic. It is very well known that an automorphism group of \mathbb{A} , denoted here by $\operatorname{Aut}(\mathbb{A})$, of an ω -categorical structure \mathbb{A} is oligomorphic, i.e., there are at most finitely many orbits of k-tuples w.r.t. $\operatorname{Aut}(\mathbb{A})$ for all $k \in \mathbb{N}$. An orbit of a n-tuple t is the set $\{s \in A^n \mid \exists \alpha \in \operatorname{Aut}(\mathbb{A}) \ (s[1], \ldots, s[n]) = (\alpha(t[1]), \ldots, \alpha(t[n]))\}$. Since all orbits in this paper are w.r.t. to the automorphism group $\operatorname{Aut}(\mathbb{A})$ of a structure \mathbb{A} we will simply say orbits instead of more formally: orbits w.r.t. $\operatorname{Aut}(\mathbb{A})$. An orbit of a pair is called an *orbital*. Observe that every orbital except for = is anti-reflexive.

An ω -categorical structure \mathbb{A} is a *core*, also referred to as a *model-complete core*, if all of its endomorphisms are elementary self-embeddings, i.e., preserve all first-order formulas definable in \mathbb{A} . In this paper, we deal with finitely bounded homogeneous core structures \mathbb{A} over finite signatures. They are all ω -categorical. We additionally assume that all relations in \mathbb{A} are both binary and symmetric. Thus, we mainly cope with *finitely bounded homogeneous binary cores* \mathbb{A} and their *first-order expansions* \mathbb{B} that are structures containing all the relations in \mathbb{A} as well as some other relations with first-order definitions in \mathbb{A} .

For the sake of clarity we will use the same symbol R for a relational symbol in the signature τ of \mathbb{B} as well as for the relation $R^{\mathbb{A}}$.

Since the age of \mathbb{A} has free amalgamation, it follows that

$$\mathbf{N} = \left\{ (a_1, a_2) \in A^2 \mid \bigwedge_{R \in \tau} \neg R^{\mathbb{A}}(a_1, a_2) \right\}$$

is nonempty. It is also straightforward to see that N is an orbital w.r.t. Aut(A).

We intend to use Theorem 4.5.1 in [6] and therefore assume that all orbitals are in \mathbb{A} . Hence we need a new definition of free amalgamation known also as free amalgamation over \mathbb{N} , see e.g. [16].

Definition 2. Let \mathbb{A} be a finitely bounded homogeneous symmetric binary core over a signature containing \mathbb{N} . We say that the age of \mathbb{A} has free amalgamation (w.r.t. \mathbb{N}) if for all finite substructures \mathbb{D} , \mathbb{B}_1 , \mathbb{B}_2 such that \mathbb{D} is a common substructure of \mathbb{B}_1 and \mathbb{B}_2 of \mathbb{A} there exists an amalgam \mathbb{C} of \mathbb{B}_1 and \mathbb{B}_2 over \mathbb{D} satisfying $(a,b) \in \mathbb{N}^{\mathbb{C}}$ for all $a \in B_1 \setminus B_2$ and $b \in B_2 \setminus B_1$. We say that \mathbb{C} is a free amalgam (w.r.t. \mathbb{N}) of \mathbb{B}_1 and \mathbb{B}_2 over \mathbb{D} .

It is easy to see that the age of a finitely bounded symmetric binary core has free amalgamation according to the definition in Section 1 if and only if its expansion with all orbitals has free amalgamation w.r.t N.

Example 1. The graph $\mathbb{H}_3 = (H_3, \mathbf{E})$ pp-defines $\mathbf{N}(x_1, x_2) \equiv \exists y \ \mathbf{E}(x_1, y) \land \mathbf{E}(y, x_2)$. We have that the age of a $\{\mathbf{E}\}$ -structure \mathbb{H}_3 has free-amalgamation while the expansion of \mathbb{H}_3 seen as a $\{\mathbf{E}, \mathbf{N}\}$ -structure has free amalgamation w.r.t. \mathbf{N} .

For the sake of simplicity we will write that the age of a finitely bounded homogeneous symmetric binary core expanded with all orbitals has free amalgamation instead of more properly writing that it has free amalgamation w.r.t. **N**.

Observe now that every orbit of k-tuples w.r.t. $\operatorname{Aut}(\mathbb{A})$ where \mathbb{A} is a finitely bounded homogeneous symmetric binary core may be defined by a conjunction of atomic formulae $\mathbf{B}(x,y)$ where \mathbf{B} is an orbital w.r.t. $\operatorname{Aut}(\mathbb{A})$. It explains the following definition.

Definition 3. Let \mathbb{A} be a finitely bounded homogeneous symmetric binary core over signature τ . We say that a k-ary tuple $t = (t[1], \ldots, t[k])$ is a

$$(\mathbf{B}_{1,2},\ldots\mathbf{B}_{1,k},\mathbf{B}_{2,3},\ldots,\mathbf{B}_{2,k},\ldots,\mathbf{B}_{k-1,k})-tuple,$$

where the indices above are pairs (i < j) in $[k]^2$ ordered lexicographically, if $\mathbf{B}_{i,j}$ is an orbital w.r.t. $Aut(\mathbb{A})$ and $(t[i], t[j]) \in \mathbf{B}_{i,j}$ for all i < j.

If we do not care about other orbitals, we often simply write that t is a $(\mathbf{B}_{1,2},\ldots,\mathbf{B}_{k-1,k})$ -tuple requiring only that $(t[1],t[2]) \in \mathbf{B}_{1,2}$ and $(t[k-1],t[k]) \in \mathbf{B}_{k-1,k}$ and we do not care about other (t[i],t[j]).

Example 2. A tuple $t \in H_3^3$ where $\mathbb{H}_3 = (H_3, \mathbf{E}, \mathbf{N})$ is a $(\mathbf{E}, \mathbf{E}, \mathbf{N})$ -tuple if $(t[1], t[2]) \in \mathbf{E}$, $(t[1], t[3]) \in \mathbf{E}$ and $(t[2], t[3]) \in \mathbf{N}$. The tuple t is in particular a $(\mathbf{E}, \dots, \mathbf{N})$ -tuple.

A projection of a formula $\varphi(x_1, \ldots, x_k)$ over the set of free variables $\{x_1, \ldots, x_k\}$ to variables $\{x_{i_1}, \ldots, x_{i_l}\}$ is the relation

$$R'(y_1,\ldots,y_l) \equiv \exists x_1\cdots\exists x_k\ R(x_1,\ldots,x_k) \wedge \bigwedge_{j=1}^l y_j = x_{i_j}.$$

A projection $\Pi_{i_1,...,i_l}R$ of a k-ary relation R to coordinates $\{i_1,...,i_l\}$ is a projection of $R(x_1,...,x_k)$ to $\{x_{i_1},...,x_{i_l}\}$. We extend this notation to $\{i_1,...,i_l\}\subseteq\{-k,...,-1,1,...,k\}$ where the coordinate -l is a shorthand for k+1-l.

Definition 4. Let R be an n-ary relation with $n \geq 3$ and $A \subseteq \Pi_{1,2}(R)$. Then

$$\mathcal{A} + R = \{(c, d) \mid \exists t \in R \ (t[1], t[2]) \in \mathcal{A} \land (t[-2], t[-1]) = (c, d)\}.$$

We use the above definition mainly in the following context.

Definition 5. We say that R is a $(A \to B)$ -implication, which we denote also by $R : A \to B$ if:

- $\mathcal{A} \subsetneq \Pi_{1,2}(R), \ \mathcal{B} \subsetneq \Pi_{-2,-1}(R), \ and$
- $\mathcal{B} = \mathcal{A} + R$.

We will usually consider pairs of implications satisfying the following condition.

Definition 6. We say that two relations R_1 and R_2 agree on projections if

• $\Pi_{1,2}(R_1) = \Pi_{-2,-1}(R_2)$ and $\Pi_{1,2}(R_2) = \Pi_{-2,-1}(R_1)$.

Two implications $R_1: \mathcal{A} \to \mathcal{B}$ and $R_2: \mathcal{B} \to \mathcal{A}$ are complementary if they agree on projections.

2.2 Primitive-Positive Definability and Polymorphisms

A primitive-positive (pp)-formula is a first-order formula built up exclusively from atomic formulae, conjunction, equality and existential quantifiers. A relation R is pp-definable in a structure \mathbb{B} if it can be defined by a pp-formula.

Definition 7. A relational clone of \mathbb{B} is the set of all relations pp-definable in \mathbb{B} .

A polymorphism $f:A^m\to A$ of a structure $\mathbb B$ over domain A is a homomorphism from a power of the structure $\mathbb B$ to $\mathbb B$. A polymorphism f of a relation $R\subseteq A^n$ is a polymorphism of a structure (A,R). In this case we say that f preserves $\mathbb B$ or R.

There is a deep connection between the polymorphisms of a structure \mathbb{B} , denoted by $\text{Pol}(\mathbb{B})$, and the relations pp-definable in that structure.

Theorem 3. ([11]) Let \mathbb{B} be a countable ω -categorical structure. Then R is preserved by the polymorphisms of \mathbb{B} if and only if it has a primitive-positive definition in \mathbb{A} .

2.3 Constraints, CSP and Minimality

A constraint $\mathbb C$ over domain A is a pair $((x_1,\ldots,x_k),R)$ where (x_1,\ldots,x_k) is a tuple of pairwise different variables and $R\subseteq A^k$. A set of variables in $\mathbb C$ called also the scope of $\mathbb C$ is denoted by $\mathcal V(\mathbb C)$. A projection Π_{x_i,\ldots,x_i} $\mathbb C$ of a constraint $\mathbb C$ to variables $\{x_{i_1},\ldots,x_{i_l}\}\subseteq \mathcal V(\mathbb C)$ is a constraint $((x_{i_1},\ldots,x_{i_l}),\Pi_{i_1,\ldots,i_l}R)$.

Definition 8. An instance \mathcal{I} of $CSP(\mathbb{B})$ for a relational structure \mathbb{B} over domain A and a set of variables $\mathcal{V} = \{v_1, \ldots, v_n\}$ is a set of constraints of the form $((x_1, \ldots, x_k), R)$ such that $\{x_1, \ldots, x_k\} \subseteq \mathcal{V}$ and R is in \mathbb{B} . The question is whether there is a solution to \mathcal{I} , i.e., a mapping $f: \mathcal{V} \to A$ such that for every constraint $((x_1, \ldots, x_k), R)$ it holds that $(f(x_1), \ldots, f(x_k)) \in R$.

We say that a constraint $((x_1, \ldots, x_k), R)$ is non-trivial if R is different from \emptyset and trivial otherwise. We say that an instance \mathcal{I} is non-trivial if all constraints in it are non-trivial, and trivial otherwise.

Definition 9. An instance \mathcal{I} over variables $\mathcal{V} = \{v_1, \dots, v_n\}$ is (k, l)-minimal with $k \leq l$ if

- 1. every l-element subset of V is contained in the scope of some constraint, and
- 2. for all k-element subsets of variables $\{x_1, \ldots, x_k\}$ and all constraints $\mathbb{C}_1, \mathbb{C}_2$ whose scope contains (x_1, \ldots, x_k) we have that $\Pi_{x_1, \ldots, x_k} \mathbb{C}_1$ equals $\Pi_{x_1, \ldots, x_k} \mathbb{C}_2$.

An algorithm that transforms any instance into a (k, l)-minimal instance is straightforward and works in time $O(|\mathcal{V}|^m)$ where m is the maximum of l and the largest arity in the signature of \mathbb{B} . Indeed, we simply introduce a new constraint $((x_1, \ldots, x_l), A^l)$ for all pairwise different variables $x_1, \ldots, x_l \in \mathcal{V}$ to satisfy the first condition and then remove tuples (orbits of tuples) from the relations in constraints in the instance as long as the second condition is not satisfied. It is widely known and easy to prove that an instance \mathcal{J} of the CSP obtained by the described algorithm has the same set of solutions as \mathcal{I} : they are equivalent. In particular, if \mathcal{J} is trivial, then \mathcal{I} has no solutions. Under a natural assumption that \mathbb{B} contains all at most l-ary relations pp-definable in \mathbb{B} , we have that \mathcal{J} is an instance of $CSP(\mathbb{B})$. From now on this assumption will be in effect.

Definition 10. A relational structure \mathbb{B} has relational width (k,l) if every (k,l)-minimal instance \mathcal{I} of $CSP(\mathbb{B})$ has a solution iff it is non-trivial.

A relational structure $\mathbb B$ has bounded relational width if it has relational width (k,l) for some natural numbers $k \leq l$.

3 Implicationally Uniform Relational Clones Imply Bounded Width

In this section we deal with \mathbb{B} whose relational clones have a very particular property.

Definition 11. We say that a relational clone is implicationally uniform if for every pair of complementary implications $R_1: \mathcal{A} \to \mathcal{B}$ and $R_2: \mathcal{B} \to \mathcal{A}$ in it we have that $\mathcal{A} = \mathcal{B}$. Otherwise, we say that a relational clone is implicationally non-uniform.

In this section, we show that if a relational clone of \mathbb{B} is implicationally uniform, then $CSP(\mathbb{B})$ is solvable by establishing $(2, \mathbb{L}_{\mathbb{A}})$ -minimality where $\mathbb{L}_{\mathbb{A}}$ is the size of the largest forbidden substructure in $\mathcal{F}_{\mathbb{A}}$ but not smaller than 3. We consider instances over variables $\{v_1, \ldots, v_n\}$. Since instances \mathcal{I} of interest are $(2, \mathbb{L}_{\mathbb{A}})$ minimal, the projection of every constraint to variables v_i, v_j with $i, j \in [n]$ is the same binary relation. We denote it by $\mathcal{I}_{i,j}$. The next important notion that we are going to use is a (directed) graph of an instance.

Definition 12. Let \mathcal{I} be an instance of $CSP(\mathbb{B})$ over variables $\{v_1, \ldots, v_n\}$ and a first-order expansion \mathbb{B} of a finitely bounded homogeneous symmetric binary core \mathbb{A} . A (directed) graph $\mathcal{G}_{\mathcal{I}}$ of an instance \mathcal{I} consists of vertices of the form $((v_i, v_j), \mathcal{A})$ such that $\mathcal{A} \subsetneq \mathcal{I}_{i,j}$ has a pp-definition in \mathbb{B} .

There is an arc between $((v_i, v_j), A)$ and $((v_k, v_l), B)$ in $\mathcal{G}_{\mathcal{I}}$ if there is a relation R in the relational clone of \mathbb{B} such that

- $\Pi_{1,2}(R) = \mathcal{I}_{i,j}, \ \Pi_{-2,-1}(R) = \mathcal{I}_{k,l} \ and$
- R is a $(A \rightarrow B)$ -implication.

We are now ready to prove that the minimality algorithm solves instances of $CSP(\mathbb{B})$ such that the relational clone of \mathbb{B} is implicationally uniform clones.

Theorem 4. Let \mathbb{B} be a first-order expansion of a finitely bounded homogeneous symmetric binary core \mathbb{A} such that the relational clone of \mathbb{B} is implicationally uniform and \mathcal{I} a non-trivial $(2, \mathbb{L}_{\mathbb{A}})$ -minimal instance of $CSP(\mathbb{B})$ where $\mathbb{L}_{\mathbb{A}}$ is the maximal size of a structure in $\mathcal{F}_{\mathbb{A}}$ but not smaller than 3. Then \mathcal{I} has a solution.

Proof. We will measure the size of an instance \mathcal{I} of $CSP(\mathbb{B})$ by the sum of the number of orbits across all constraints, i.e.,

$$\operatorname{size}(\mathcal{I}) = \sum_{(\overline{x}, R) \in \mathcal{I}} \operatorname{number of orbits}(R)$$

Suppose now that the theorem does not hold and take a minimal in size instance of $\mathrm{CSP}(\mathbb{B})$ that is non-trivial $(2, \mathbb{L}_{\mathbb{A}})$ -minimal and has no solution. We start with a case where every $\mathcal{I}_{i,j}$ with $i,j \in [n]$ contains only one orbit. In this case we construct a finite structure \mathbb{C} over the domain consisting of variables in \mathcal{I} so that $(v_i, v_j) \in \mathbf{O}^{\mathbb{C}}$ if and only if $\mathcal{I}_{i,j} = \mathbf{O}$. Since $\mathbb{L}_{\mathbb{A}} \geq 3$ we have that \mathbb{C} respects the transitivity of =, i.e. we have $(v_{i_k}, v_{i_m}) \in =^{\mathbb{C}}$ whenever $(v_{i_k}, v_{i_l}) \in =^{\mathbb{C}}$ and $(v_{i_l}, v_{i_m}) \in =^{\mathbb{C}}$. In fact, $=^{\mathbb{C}}$ is an equivalence relation on $\{v_1, \ldots, v_n\}$. Thus consider a quotient structure \mathbb{D} which is $\mathbb{C}/=^{\mathbb{C}}$ and whose domain is the set of equivalence classes of $=^{\mathbb{C}}$. We have $([v_i]_{=^{\mathbb{C}}}, [v_j]_{=^{\mathbb{C}}}) \in \mathbf{O}^{\mathbb{C}}$ if and only if $\mathcal{I}_{i,j} = \mathbf{O}$. Since \mathcal{I} is $(2, \mathbb{L}_{\mathbb{A}})$ -minimal there is an embedding of \mathbb{D} into \mathbb{A} and a homomorphism of \mathbb{C} into \mathbb{A} . The latter is also a solution to \mathcal{I} which was to be proved.

The previous case represents the situation where $\mathcal{G}_{\mathcal{I}}$ is empty. Now, we move to the case where at least one $\mathcal{I}_{i,j}$ with $i \neq j$ in [n] is not an orbital. Since every orbital is pp-definable in \mathbb{A} and thereby in \mathbb{B} , and $\mathcal{I}_{i,j}$ contains at least two orbitals, from now on the graph $\mathcal{G}_{\mathcal{I}}$ is non-empty. We will look at maximal strongly connected components \mathcal{M} of $\mathcal{G}_{\mathcal{I}}$, i.e. strongly connected components such that every arc originating in \mathcal{M} ends up in \mathcal{M} . For the sake of simplicity we simply say that \mathcal{M} is a maximal component of $\mathcal{G}_{\mathcal{I}}$. We start with a simple observation. (All omitted proofs may be found in the appendix.)

Observation 1. Let \mathcal{M} be a maximal component of $\mathcal{G}_{\mathcal{I}}$, then for all pairs of vertices $((v_i, v_j), \mathcal{A})$ and $((v_k, v_l), \mathcal{B})$ in \mathcal{M} we have $\mathcal{A} = \mathcal{B}$.

Proof. Assume on the contrary that there is a cycle in \mathcal{M} containing both $((v_i, v_j), \mathcal{A})$ and $((v_k, v_l), \mathcal{B})$ with $\mathcal{A} \neq \mathcal{B}$. It implies that there is a chain of implications $R_1 : \mathcal{A}_1 \to \mathcal{A}_2, R_2 : \mathcal{A}_2 \to \mathcal{A}_3, \dots, R_a : \mathcal{A}_a \to \mathcal{B}, R_{a+1} : \mathcal{B} \to \mathcal{B}_1, R_{a+2} : \mathcal{B}_1 \to \mathcal{B}_2, \dots, R_{a+b} : \mathcal{B}_{a+b} \to \mathcal{A}$ satisfying the conditions in Definition 12. In particular all these relations are pp-definable in \mathbb{B} . By use of \circ -composition in Definition 14, we have that

 $((R_1 \circ R_2) \circ \cdots \circ R_a)$ is a $(\mathcal{A} \to \mathcal{B})$ -implication and that $((R_{a+1} \circ R_{a+2}) \circ \cdots \circ R_{a+b})$ is a $(\mathcal{B} \to \mathcal{A})$ -implication. Clearly R_1, R_2 are complementary. The fact that $\mathcal{A} \neq \mathcal{B}$ contradicts the fact that the relational clone of \mathbb{B} is implicationally uniform.

Thus, from now on we assume that $\mathcal{G}_{\mathcal{I}}$ contains a maximal strongly connected component \mathcal{M} in which all vertices are of the form $((v_i, v_j), \mathcal{A})$ with the same \mathcal{A} . We reduce the size of an \mathcal{I} so that every R in a constraint $((x_1, \ldots, x_k), R)$ in \mathcal{I} is replaced by

$$R(x_1,\ldots,x_k) \wedge \bigwedge_{((x_i,x_j),\mathcal{A})\in\mathcal{M}} \mathcal{A}(x_i,x_j).$$

The new instance will be denoted by \mathcal{J} . In order to show that it is non-trivial and $(2, \mathbb{L}_{\mathbb{A}})$ -minimal consider the following observation, which directly follows from the definition of $\mathcal{G}_{\mathcal{I}}$.

Observation 2. Let R be a k-ary relation in a relational clone of \mathbb{B} and $R(v_{i_1}, \ldots, v_{i_k})$ an atomic formula over variables of the instance \mathcal{I} such that for all $l, m \in [k]$ the projection of $R(v_{i_1}, \ldots, v_{i_k})$ to v_{i_l}, v_{i_m} is \mathcal{I}_{i_l,i_m} . Then for all c > 0 and $a_1, b_1, \ldots, a_c, b_c, l, k \in [m]$ satisfying $((v_{i_{a_j}}, v_{i_{b_j}}), \mathcal{A}) \in \mathcal{M}$ for all $j \in [c]$ we have that the projection of

$$R'(v_{i_1},\ldots,v_{i_k}) \equiv \left(R(v_{i_1},\ldots,v_{i_k}) \wedge \bigwedge_{j \in [c]} \mathcal{A}(v_{i_{a_j}},v_{i_{b_j}})\right)$$

to any v_{i_l}, v_{i_m} with $l, m \in [k]$ is either

- \mathcal{I}_{i_l,i_m} or
- \mathcal{A} and then $((v_{i_l}, v_{i_m}), \mathcal{A}) \in \mathcal{M}$.

Proof. Assume the contrary and take a minimal c such that there are v_{i_l}, v_{i_m} with $l, m \in [k]$ and such that the projection of $R'(v_{i_1}, \ldots, v_{i_k})$ to v_{i_l}, v_{i_m} is $\mathcal{B} \notin \{\mathcal{A}, \mathcal{I}_{i_l, i_m}, \emptyset\}$. Observe that since c is minimal, we can assume that $\mathcal{B} \neq \emptyset$. Then let $e \in [c]$ be such that for $I = [c] \setminus \{e\}$ we have that the projection of

$$R''(v_{i_1},\ldots,v_{i_k}) \equiv \left(R(v_{i_1},\ldots,v_{i_k}) \wedge \bigwedge_{j \in I} \mathcal{A}(v_{i_{a_j}},v_{i_{b_j}})\right)$$

to $v_{i_{a_e}}, v_{i_{b_e}}, v_{i_l}, v_{i_m}$ is a $(\mathcal{A} \to \mathcal{B})$ -implication for some $\mathcal{B} \subsetneq \mathcal{I}_{i_l, i_m}$. Since \mathcal{A} and R are pp-definable in \mathbb{B} , by the definition of $\mathcal{G}_{\mathcal{I}}$, it follows that \mathcal{M} contains $((v_{i_l}, v_{i_m}), \mathcal{B})$ which contradicts Observation 1.

The other thing that could happen is when the projection of $R'(v_{i_1}, \ldots, v_{i_k})$ to v_{i_l}, v_{i_m} is \mathcal{A} and $((v_{i_l}, v_{i_m}), \mathcal{A}) \notin \mathcal{M}$. But then again by the definition of $\mathcal{G}_{\mathcal{I}}$ the vertex $((v_{i_l}, v_{i_m}), \mathcal{A})$ should have been in \mathcal{M} . It completes the proof of the observation.

By the observation above it follows that for every constraint $((v_{i_1}, \ldots, v_{i_k}), R) \in \mathcal{I}$ a projection of

$$R(v_{i_1},\ldots,v_{i_k}) \wedge \bigwedge_{((v_{i_l},v_{i_m}),\mathcal{A})\in\mathcal{M}} \mathcal{A}(v_{i_l},v_{i_m})$$

to v_{i_l}, v_{i_m} for any $l, m \in [k]$ is either \mathcal{I}_{i_l, i_m} or \mathcal{A} and the latter case holds only when $((v_{i_l}, v_{i_m}), \mathcal{A}) \in \mathcal{M}$. It follows that \mathcal{J} is non-trivial and $(2, \mathbb{L}_{\mathbb{A}})$ -minimal. Furthermore, if \mathcal{I} does not have a solution, then \mathcal{J} does not have it either. Since \mathcal{J} is of a smaller size than \mathcal{I} we have a contradiction with the minimality of \mathcal{I} . \square

4 Implicationally Non-Uniform Clones Are Not Preserved By Chains of Quasi Directed Jónsson Operations

In this section we show that whenever a relational clone of \mathbb{B} is implicationally non-uniform, i.e. it contains complementary relations $R_1: \mathcal{A} \to \mathcal{B}$ and $R_2: \mathcal{B} \to \mathcal{A}$ with $\mathcal{A} \neq \mathcal{B}$, then \mathbb{B} is not preserved by any chain of quasi directed Jónsson operations.

We start with a definition of a directed bipartite graph \mathfrak{B}_{R_1,R_2} which reflects the structure of R_1,R_2 which agree on projections. We use it mainly for R_1,R_2 which are complementary implications.

If it is not stated otherwise we will assume that both R_1 and R_2 are quaternary. To this end, for every binary relation \mathcal{A} first-order definable in \mathbb{A} we set

- $V_L(A) := \{ O_L \mid O \text{ is an orbital contained in } A \}$, and
- $V_R(A) := \{O_R \mid O \text{ is an orbital contained in } A\}.$

Definition 13. Let R_1, R_2 be two relations that agree on projections. We define \mathfrak{B}_{R_1,R_2} to be a bipartite digraph over vertices $\mathbf{V}_L(\Pi_{1,2}(R_1)) \cup \mathbf{V}_R(\Pi_{1,2}(R_2))$ where we have two kinds of arcs:

- $(\mathbf{O}_L, \mathbf{P}_R) \in \mathbf{V}_L(\Pi_{1,2}(R_1)) \times \mathbf{V}_R(\Pi_{1,2}(R_2))$ if the relation R_1 contains a $(\mathbf{O}, \dots, \mathbf{P})$ -tuple,
- $(\mathbf{O}_R, \mathbf{P}_L) \in \mathbf{V}_R(\Pi_{1,2}(R_2)) \times \mathbf{V}_L(\Pi_{1,2}(R_1))$ if the relation R_2 contains a $(\mathbf{O}, \dots, \mathbf{P})$ -tuple.

A strongly connected component of \mathfrak{B}_{R_1,R_2} is simply called a component. A component is non-trivial if it contains more than one vertex. A component \mathcal{N} is maximal if all outgoing edges end up in \mathcal{N} and it is minimal if all ingoing edges originate in \mathcal{N} .

For the sake of simplicity we say that a subgraph of \mathfrak{B}_{R_1,R_2} induced by $(\mathbf{V}_L(\mathcal{C}) \cup \mathbf{V}_R(\mathcal{D}))$ for some binary relations $\mathcal{C} \subseteq \Pi_{1,2}(R_1)$ and $\mathcal{D} \subseteq \Pi_{1,2}(R_2)$ is a $(\mathcal{C},\mathcal{D})$ -subgraph of \mathfrak{B}_{R_1,R_2} . In particular, a component $(\mathbf{V}_L(\mathcal{C}) \cup \mathbf{V}_R(\mathcal{D}))$ of \mathfrak{B}_{R_1,R_2} induces a $(\mathcal{C},\mathcal{D})$ -subgraph and in this case we say also that that component is a $(\mathcal{C},\mathcal{D})$ -component. We say that a $(\mathcal{C},\mathcal{D})$ -component in \mathfrak{B}_{R_1,R_2} contains a tuple t if it is a $(\mathbf{C},\ldots,\mathbf{D})$ -tuple in R_1 or a $(\mathbf{D},\ldots,\mathbf{C})$ -tuple in R_2 with $\mathbf{C} \subseteq \mathcal{C}$ and $\mathbf{D} \subseteq \mathcal{D}$.

Observation 3. Let $R_1: \mathcal{A} \to \mathcal{B}$ and $R_2: \mathcal{B} \to \mathcal{A}$ be complementary. Then a $(\mathcal{A}, \mathcal{B})$ subgraph of \mathfrak{B}_{R_1, R_2} has both a minimal and a maximal component. Furthermore, a $(\Pi_{1,2}(R_1) \setminus \mathcal{A}, \Pi_{3,4} \setminus \mathcal{B})$ -subgraph of \mathfrak{B}_{R_1, R_2} has a minimal component.

Proof. Since $\mathcal{A}+R_1=\mathcal{B}$ and $\mathcal{B}+R_2=\mathcal{A}$, we have that for any vertex \mathbf{A}_L in $\mathbf{V}_L(\mathcal{A})$ there exists \mathbf{B}_R in $\mathbf{V}_B(\mathcal{A})$ such that R_1 contains an $(\mathbf{A},\ldots,\mathbf{B})$ -tuple and that for any \mathbf{B}_R in $\mathbf{V}_B(\mathcal{A})$ we have \mathbf{A}_L in $\mathbf{V}_L(\mathcal{A})$ such that a $(\mathbf{B},\ldots,\mathbf{A})$ -tuple is in R_2 . It follows that by an appropriately long walk we can reach a component from where there is no outgoing arc. It is a maximal component in the $(\mathcal{A},\mathcal{B})$ -subgraph of \mathfrak{B}_{R_1,R_2} . For the minimal component in the $(\mathcal{A},\mathcal{B})$ -subgraph of \mathfrak{B}_{R_1,R_2} notice that for any \mathbf{A}_L in $\mathbf{V}_L(\mathcal{A})$ there exists \mathbf{B}_R in $\mathbf{V}_R(\mathcal{B})$ such that a $(\mathbf{B},\ldots,\mathbf{A})$ -tuple is in R_2 and that for any \mathbf{B}_R in $\mathbf{V}_R(\mathcal{B})$ there exists \mathbf{A}_L in $\mathbf{V}_L(\mathcal{A})$ such that a $(\mathbf{A},\ldots,\mathbf{B})$ -tuple is in R_1 . Thus by walking from any vertex in $(\mathcal{A},\mathcal{B})$ -subgraph backwards but inside the subgraph we reach a minimal component in $(\mathcal{A},\mathcal{B})$. For the minimal component in a $(\Pi_{1,2}(R_1)\setminus\mathcal{A},\Pi_{3,4}\setminus\mathcal{B})$ -subgraph of \mathfrak{B}_{R_1,R_2} we do the same thing just start with any vertex in there and go against the arrows. \square

We will now define a ∘-composition of two quaternary relations, another slightly different ⋈-composition is defined later on.

Definition 14. Let R_1, R_2 be two quaternary relations such that $\pi_{-2,-1}(R_1) = \pi_{1,2}(R_2)$. We define a \circ -composition $R_1 \circ R_2$ of R_1 and R_2 to be the relation defined by the formula

$$\exists y \exists z \ R_1(x_1, x_2, y, z) \land R_2(y, z, x_3, x_4).$$

We will write $(R_1 \circ R_2)^n$ as a shorthand for the expression $((\cdots (((R_1 \circ R_2) \circ R_1) \circ R_2) \circ \cdots \circ R_1) \circ R_2)$ where both R_1 and R_2 occur n times.

Observation 4. Let $R_1 : A \to B$ and $R_2 : B \to A$ be quaternary complementary implications. Then, for all $n \geq 1$ we have both of the following:

- $S_{2n} \equiv (R_1 \circ R_2)^n$ contains an $(\mathbf{O}, \dots, \mathbf{P})$ -tuple iff there is a path in \mathfrak{B}_{R_1, R_2} of length 2n from \mathbf{O}_L to \mathbf{P}_L ;
- $S_{2n+1} \equiv (R_1 \circ R_2)^n \circ R_1$ contains an $(\mathbf{O}, \dots, \mathbf{P})$ -tuple iff there is a path in \mathfrak{B}_{R_1, R_2} of length 2n + 1 from \mathbf{O}_L to \mathbf{P}_R .

Proof. We prove the observation by induction on $n \ge 1$. Notice that if R_1 contains an $(\mathbf{O}, \dots, \mathbf{A})$ -tuple and R_2 contains an $(\mathbf{A}, \dots, \mathbf{P})$ -tuple for some orbital \mathbf{A} , then by the homogeneity of \mathbb{A} , the relation S_2 has an $(\mathbf{O}, \dots, \mathbf{P})$ -tuple. On the other hand if there are no such tuples, by the definition of S_2 , there is not an $(\mathbf{O}, \dots, \mathbf{P})$ -tuple in S_2 .

In the induction step consider $S_{2n+1} \equiv S_{2n} \circ R_1$. If there is an $(\mathbf{O}, \dots, \mathbf{P})$ -tuple in S_{2n+1} , then for some orbital \mathbf{A} , by the definition of this relation, we have an $(\mathbf{O}, \dots, \mathbf{A})$ -tuple in S_{2n} and an $(\mathbf{A}, \dots, \mathbf{P})$ -tuple in R_1 . By the induction hypothesis, there is a path of length 2n from \mathbf{O}_L to \mathbf{A}_L in \mathfrak{B}_{R_1,R_2} , and hence a path of length 2n+1 from \mathbf{O}_L to \mathbf{P}_R . On the other hand, if there is a path from \mathbf{O}_L to \mathbf{P}_R in \mathfrak{B}_{R_1,R_2} of length 2n+1, then there exists some \mathbf{A} so that there is a path of length 2n from \mathbf{O}_L to \mathbf{A}_L in \mathfrak{B}_{R_1,R_2} and a $(\mathbf{A}, \dots, \mathbf{P})$ -tuple in R_1 . By the induction hypothesis, there is an $(\mathbf{O}, \dots, \mathbf{A})$ -tuple in S_{2n} . By the homogeneity of \mathbb{A} , there is an $(\mathbf{O}, \dots, \mathbf{P})$ -tuple in S_{2n+1} . The proof for S_{2n} with n > 1 is analogous. We just replace R_1 with R_2 and S_{2n+1} with S_{2n} in the proof above. It completes the proof of the observation. \square

The next step is to extend the above observation so that it specifies what kind of a tuple we can expect, for that we need the following definition.

Definition 15. A quaternary tuple t is

- degenerated if t[1] = t[4] and t[2] = t[3],
- essentially ternary if t[2] = t[3] and $t[1] \neq t[4]$,
- essentially quaternary if $t[i] \neq t[j]$ whenever $i \in \{1, 2\}$ and $j \in \{3, 4\}$,
- partially-free if $t[1, 4] \in \mathbf{N}$.
- fully-free if $(t[i], t[j]) \in \mathbf{N}$ whenever $i \in \{1, 2\}$ and $j \in \{3, 4\}$.

We now prove observation that tells us what kind of tuples we can expect in a o-composition of two relations.

Observation 5. Let $R_1: \mathcal{A} \to \mathcal{B}$ and $R_2: \mathcal{B} \to \mathcal{C}$ be such that R_1 contains a $(\mathbf{A}, \dots, \mathbf{B})$ -tuple t_1 and R_2 contains a $(\mathbf{B}, \dots, \mathbf{C})$ -tuple t_2 . Then $R_3:=R_1\circ R_2$ is a $(\mathcal{A} \to \mathcal{C})$ -implication containing a $(\mathbf{A}, \dots, \mathbf{C})$ -tuple t_3 which is

- essentially ternary if both t_1 and t_2 are essentially ternary, A, C anti-reflexive and B is =,
- essentially quaternary if both t_1, t_2 are essentially ternary and all A, B, C are anti-reflexive,
- fully-free if both t_1, t_2 are essentially quaternary,
- partially-free if t_1 is essentially quaternary and t_2 essentially ternary and \mathbf{C} is not = or t_1 is essentially ternary and \mathbf{t}_2 essentially quaternary and \mathbf{A} is not =,
- ullet essentially quaternary (fully-free) if at least one of t_1, t_2 is essentially quaternary (fully-free)
- non-degenerated if at least one of them is non-degenerated.

Proof. It is straightforward that R_3 is a $(A \to C)$ -implication. The rest of the observation we prove case by case.

The age of \mathbb{A} has free amalgamation and therefore there are $a, b, c \in A$ such that $(a, b) \in \mathbf{A}, (b, c) \in \mathbf{B}$ and $(a, c) \in \mathbb{N}$. The first item follows.

For the second case we consider $a, b, c, d \in \mathbf{A}$ such that (a, b, b, c) is isomorphic with t_1 and (b, c, c, d) with t_2 . Since the age of \mathbb{A} has free amalgamation, we may assume that $(a, d) \in \mathbb{N}$. Observe that (a, b, c, d) is in R_3 and is essentially quaternary.

For the third case we need $a, b, c, d, e, f \in A$ such that (a, b, c, d) is isomorphic with t_1 and (c, d, e, f) with t_2 . Again using the fact that the age of \mathbb{A} has free amalgamation, we may assume that $(i, j) \in \mathbb{N}$ whenever $i \in \{a, b\}$ and $j \in \{e, f\}$. Observe that (a, b, e, f) is in R_3 and is fully free.

For the fourth case we only look at the first part. To this end we consider a, b, c, d, e such that (a, b, c, d) is isomorphic to t_1 and (c, d, e) to t_2 . Since $a \neq d$ and $d \neq e$ we may assume that $(a, e) \in \mathbf{N}$. It follows that (a, b, d, e) is a partially free and is in \mathbf{N} .

The proof of the fifth and the sixth case is straightforward and very similar to the proof of the preceding cases. \Box

We now define a self-complementary implication.

Definition 16. We say that an implication $R : A \to A$ is self-complementary if R and R are complementary and every non-trivial component of \mathfrak{B}_{R_1,R_2} is a $(\mathcal{C},\mathcal{C})$ -component.

The following will be used throughout the paper.

Observation 6. Let $R_1 : A \to B$ and $R_2 : B \to A$ be complementary implications. Then $R \equiv R_1 \circ R_2$ is a self-complementary $(A \to A)$ -implication.

Proof. The relation R is clearly a $(A \to A)$ -implication. Since every vertex in \mathfrak{B}_{R_1,R_2} has an incoming edge we have that $\Pi_{3,4}(R) = \Pi_{1,2}(R)$, and hence R, R are complementary.

Observe now that every non-trivial maximal component in considered bipartite digraphs corresponds to a maximal in the number of pairwise different vertices closed walk. From a closed walk in \mathfrak{B}_{R_1,R_2} corresponding to a maximal $(\mathcal{C},\mathcal{D})$ -component one can get a closed walk in $\mathfrak{B}_{R,R}$ corresponding to a component $(\mathcal{C},\mathcal{C})$ in $\mathfrak{B}_{R,R}$ by contracting all two arcs with a vertex from $V_R(\mathcal{D})$ in the middle into one arc. Suppose now that it is not a maximal component, i.e. there is a $(\mathcal{C}',\mathcal{C}'')$ -component in $\mathfrak{B}_{R,R}$ containing the $(\mathcal{C},\mathcal{C})$ -component. But then from a corresponding walk one can create a closed walk in \mathfrak{B}_{R_1,R_2} corresponding to a component containing vertices $(V_L(\mathcal{C}') \cup V_L(\mathcal{C}'') \cup V_R(\mathcal{D}))$. It contradicts the maximality of the $(\mathcal{C},\mathcal{D})$ -component in \mathfrak{B}_{R_1,R_2} . It follows that R is a self-complementary implication.

A non-trivial component in \mathfrak{B}_{R_1,R_2} for two R_1,R_2 which agree on projections is degenerated if all tuples in it are degenerated. Observe that a non-trivial degenerated component contains exactly two $(\mathbf{O},\mathbf{O},=,=,\mathbf{O},\mathbf{O})$ -tuples for some orbital \mathbf{O} and therefore we will call it an \mathbf{O} -degenerated component.

In the proof announced at the beginning of this section we distiguish two cases: either there is a non-trivial component in \mathfrak{B}_{R_1,R_2} which is non-degenerated or all non-trivial components in \mathfrak{B}_{R_1,R_2} are degenerated. We take care of the former case in Section 4.1 and of the latter in Section 4.2.

4.1 A non-trivial non-degenerated component

We start from a proposition which reduces the situation we take care of in this section to two simple cases.

Proposition 1. Let \mathbb{B} be a first-order expansion of a finitely bounded homogeneous symmetric binary core whose age has free amalgamation. If \mathbb{B} pp-defines complementary implications $R_1: \mathcal{A} \to \mathcal{B}, R_2: \mathcal{B} \to \mathcal{A}$ such that \mathfrak{B}_{R_1,R_2} contains a non-trivial non-degenerated component. Then R pp-defines a self-complementary $R: \mathcal{A} \to \mathcal{A}$ such that R contains

- $a(\mathbf{A}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{A})$ -tuple for some $\mathbf{A} \subseteq \mathcal{A}$ and
- a (B, N, N, N, N, B)-tuple or a (B, B, =, =, B, B)-tuple for some B $\nsubseteq A$.

Proof. Let t be a non-degenerated $(\mathbf{A}, \dots, \mathbf{C})$ -tuple in a nontrivial $(\mathcal{A}, \mathcal{A})$ -component of \mathfrak{B}_{R_1,R_2} . Without loss of generality we may assume that either t is essentially quaternary or \mathbf{A} is anti-reflexive. Indeed, if $t[2] \neq t[3]$ and t[1] = t[4] we replace R_1 with $R_1(x_2, x_1, x_4, x_3)$, if \mathbf{C} is =, then we swap R_1 with R_2 and if \mathbf{A}, \mathbf{C} are both = and t is essentially ternary, then t is degenerated which contradicts the assumption.

By Observation 4, it follows that there is k such that $R'_2 \equiv R_2 \circ (R_1 \circ R_2)^k$ contains a $(\mathbf{B}, \dots, \mathbf{A})$ -tuple and that $R \equiv R_1 \circ R'_2$ is a self-complementary implication that contains a non-degenerated $(\mathbf{A}, \dots, \mathbf{A})$ -tuple. The last tuple is non-degenerated by Observation 5. By the same observation, it follows that $R' = (R \circ R)^2$ contains an essentially quaternary $(\mathbf{A}, \dots, \mathbf{A})$ -tuple and that $S := (R' \circ R')^2$ has a fully-free $(\mathbf{A}, \dots, \mathbf{A})$ -tuple, i.e., a $(\mathbf{A}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{A})$ -tuple. The relation S is clearly a self-complementary $(\mathcal{A} \to \mathcal{A})$ -implication by Observation 6.

By Observation 3, there is a $(\mathcal{B}, \mathcal{B})$ -component in \mathfrak{B}_{R_1, R_2} with $\mathcal{A} \cap \mathcal{B} = \emptyset$. It follows that there exists l such that $S' := S^l$ contains both an $(\mathbf{A}, \mathbf{N}, \mathbf{N}, \mathbf{A})$ -tuple and a $(\mathbf{B}, \dots, \mathbf{B})$ -tuple s for some \mathbf{B} . If the latter tuple is not a $(\mathbf{B}, \mathbf{B}, =, =, \mathbf{B}, \mathbf{B})$ -tuple and \mathbf{B} is not =, then as above we show that $T = (((S' \circ S')^2)^2)$ has a $(\mathbf{B}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{B})$ -tuple. If \mathbf{B} is =, then the non-degenerated s has to be essentially quaternary. Then, again, by Observation 5, T has a $(\mathbf{B}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{B})$ -tuple. T clearly also contain a $(\mathbf{A}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{A})$ -tuple and is a self-complementary implication.

Finally, observe that $\mathfrak{B}_{T',T'}$ where

$$T'(x_1, x_2, x_3, x_4) \equiv T(x_1, x_2, x_3, x_4) \wedge T(x_4, x_3, x_2, x_1)$$

contains a $(\mathbf{A}, \dots \mathbf{B})$ -tuple only if T contains that tuple and additionally a $(\mathbf{B}, \dots \mathbf{A})$ -tuple. Since all components in $\mathfrak{B}_{T,T}$ are $(\mathcal{C}, \mathcal{C})$ -components the orbitals \mathbf{A} and \mathbf{B} cannot come from two different components. Indeed, if T contains a $(\mathbf{A}, \dots, \mathbf{B})$ -tuple where \mathbf{B}_R is in a component above the component of \mathbf{A}_L , then by the shape of the components it cannot have a $(\mathbf{B}, \dots, \mathbf{A})$ -tuple. It follows that $\mathfrak{B}_{T',T'}$ is a union of disjoint non-trivial $(\mathcal{A}, \mathcal{A})$ -components each of which is a subset of some non-trivial component in $\mathfrak{B}_{T,T}$. In particular, we have that T' is a $(\mathcal{A} \to \mathcal{A})$ -implication with $\mathbf{A} \subseteq \mathcal{A}$.

It is a trivial thing that = is transitive. We also observe that for any orbital \mathbf{O} and for all elements $a, b, c \in A$ such that $(a, b) \in \mathbf{O}$ and (b = c); or a = b and $(b, c) \in \mathbf{O}$ we have $(a, c) \in \mathbf{O}$. The latter property of = will be called *semi-transitivity*. Obviously, semi-transitivity implies transitivity.

We will now show that in both cases in the lemma above \mathbb{B} is not preserved by a chain of quasi directed Jónsson operations. Each time we will use the following definition.

Definition 17. Let \mathbb{A} be a finitely bounded homogeneous symmetric binary core whose age has free amalgamation, $U, V \subseteq [3]$ and $\mathbf{M}, \mathbf{O}, \mathbf{P} \in \{\mathbf{A}, \mathbf{B}\}$ for some orbitals \mathbf{A}, \mathbf{B} . We say that (u, v) in A^3 is a (U, V)-constant pair of triples satisfying $\mathbf{MOP}(u, v)$ if all of the following hold.

- 1. $(u[1], v[1]) \in \mathbf{M}, (u[2], v[2]) \in \mathbf{O}, (u[3], v[3]) \in \mathbf{P}.$
- 2. u[i] = u[j] iff $i, j \in U$ and v[i] = v[j] iff $i, j \in V$.
- 3. For all other $a, b \in \{u[1], u[2], u[3], v[1], v[2], v[3]\}$ for which the above conditions and the semi-transitivity of equality do not determine the orbital, we have $(a, b) \in \mathbf{N}$.

Obviously not for all $U, V \subseteq [3]$ and $\mathbf{M}, \mathbf{O}, \mathbf{P}$ there are u, v satisfying the conditions above. Therefore anytime we use that definition we make sure the appropriate u, v exist.

Lemma 1. Let \mathbb{B} be a first-order expansion of a finitely bounded homogeneous symmetric binary core \mathbb{A} whose age has free amalgamation. If \mathbb{B} pp-defines a quaternary self-complementary relation $R: \mathcal{A} \to \mathcal{A}$. which contains

- $a(\mathbf{A}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{A})$ -tuple and
- a (\mathbf{B} , \mathbf{N} , \mathbf{N} , \mathbf{N} , \mathbf{N})-tuple,

for two different orbitals $\mathbf{A} \subseteq \mathcal{A}$ and $\mathbf{B} \nsubseteq \mathcal{A}$, then R is not preserved by any chain of quasi directed Jónsson operations.

Proof. Suppose that there is a chain (D_1, \ldots, D_n) of quasi directed Jónsson operations preserving \mathbb{B} and R. We will show that this assumption contradicts the fact that $\text{Pol}(\mathbb{B})$ preserves \mathbf{B} which should be preserved since structB contains all orbitals w.r.t $\text{Aut}(\mathbb{A})$.

At various steps of the proof we simultaneously consider three different cases:

- 1. both **A** and **B** are anti-reflexive,
- 2. only **A** is anti-reflexive,
- 3. only \mathbf{B} is anti-reflexive.

In the following claim we consider two different types of (U, V)-constant pairs of triples. Inspect Figure 2 and observe that t, s is a ([3], [2])-constant pair of triples in A satisfying $\mathbf{AAB}(u, v)$ and that u, v is a ([3], {2,3})-constant pair of triples (u, v) in A satisfying $\mathbf{ABB}(u, v)$. According to Definition 17 we have that \mathbf{O} is

- 1. N if A and C are anti-reflexive,
- 2. \mathbf{B} if \mathbf{A} is = and \mathbf{B} is anti-reflexive,
- 3. **A** if **C** is = and **A** is anti-reflexive.

We are now ready to formulate and prove the claim whose proof is a large part of the proof of the lemma.

Claim 1. For all $i \in [n]$ we have both of the following.

- For all ([3], [2])-constant pairs of triples (u, v) in A satisfying $\mathbf{AAB}(u, v)$ we have that $(D_i(u), D_i(v)) \in \mathcal{A}$.
- For all ([3], {2,3})-constant pairs of triples (u, v) in A satisfying ABB(u, v) we have that $(D_i(u), D_i(v)) \in A$.

Proof. We prove the claim by the induction on $i \in [n]$.

(BASE CASE) In the base case we cope with i = 1. Let (u, v) be a ([3], [2])-constant pair of triples in A satisfying AAB(u, v). We now show that there is $u' \in A^3$ such as u with a difference that $(u'[3], v[3]) \in A$. By the fact that \mathbb{B} has free amalgamation over \mathbb{N} we have that there exists $p, r \in A^3$ satisfying the conditions from Figure 1. Since \mathbb{A} is homogeneous, there exists an automorphism α of \mathbb{A} such that $\alpha(u[i]) = \alpha(p[i])$ and $\alpha(v[i]) = \alpha(r[i])$ for $i \in [3]$. Now $\alpha^{-1}(p'[3])$ is the desired u'[3]. Since \mathbb{B} contains \mathbb{A} it follows that $(D_1(u'), D_1(v)) \in \mathbb{A}$. By (1), we have $D_1(u') = D_1(u)$, and hence $(D_1(u), D_1(v)) \in \mathbb{A}$. Since $\mathbb{A} \subseteq \mathcal{A}$, it completes the proof of the base case for the first item.

For the second bullet, let (u, v) be a $([3], \{2, 3\})$ -constant pair of triples satisfying $\mathbf{ABB}(u, v)$. Consult now Figure 2 to see that there exists a ([2], [3])-constant pair of triples $s, t \in A^3$ satisfying $\mathbf{AAC}(s, t)$ such that

- $a_1 = (s[1], t[1], u[1], v[1])$ is a $(\mathbf{A}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{A})$ -tuple,
- $a_2 = (s[2], t[2], u[2], v[2])$ is a (A, N, N, N, N, B)-tuple, and
- $a_3 = (s[3], t[3], u[3], v[3])$ is a $(\mathbf{B}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{B})$ -tuple.

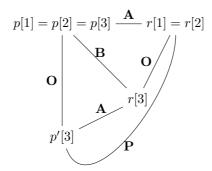


Figure 1: A structure in Age(A) needed in the proof of the base case of Claim 1. If both **A** and **B** are anti reflexive, then **O** and **P** are **N**. If **A** is =, then **O** and **P** are **B**. If **B** is =, then **O** is **A** and **P** is **N**.

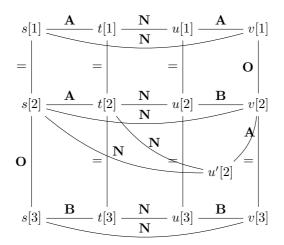


Figure 2: Since the age of \mathbb{A} has free amalgamation over \mathbb{N} , it is straightforward to show that there are four vectors $s, t, u, v \in A^3$ and a single additional element u'[2] described by the diagram above. By homogeneity of \mathbb{A} we may assume that $u, v \in A^3$ are the vectors from the proof of Claim 1. For edges not depicted in the picture the label either follows by the depicted ones and the semi-transitivity of equality or it is \mathbb{N} .

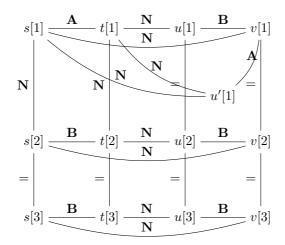


Figure 3: Triples s, t, u, v and u' used for a final step in the proof of Lemma 1. If it does not follows from the labels in the diagram and the semi-transitivity of equality, all omitted edges are labelled with N.

Let now u' be as u except for the second coordinate so that the tuple $a_2' = (s[2], t[2], u'[2], v[2])$ is a $(\mathbf{A}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{A})$ -tuple. Now all a_1, a_2', a_3 are tuples in R. Since $(D_1(s), D_1(t)) \in \mathcal{A}$ and $\mathcal{A} + R = \mathcal{A}$ we have that $(D_1(u'), D_1(v)) \in \mathcal{A}$. By (2) we have $D_1(u') = D_1(u)$, and hence $(D_1(u), D_1(v)) \in \mathcal{A}$. It completes the base case.

(INDUCTION STEP) For the induction step, consider any ([3], [2])-constant pair of triples (u, v) in A satisfying AAB(u, v). We want to show that $(D_{i+1}(u), D_{i+1}(v)) \in \mathcal{A}$. To that end consider v' such that v'[1] = v[1] and v'[2] = v'[3] = v[3]. Observe that (u, v') is a ([3], {2,3})-constant pair of triples in A satisfying ABB(u, v'). By the induction hypothesis, it follows that $(D_i(u), D_i(v')) \in \mathcal{A}$. By (3), we have $D_i(v) = D_{i+1}(v')$ and clearly $D_{i+1}(u) = D_i(u)$. It follows that $(D_{i+1}(u), D_{i+1}(v)) \in \mathcal{A}$. Now, as in the base case we show that $(D_{i+1}(u), D_{i+1}(v)) \in \mathcal{A}$ for all ([3], {2,3})-constant pair of triples (u, v) in A satisfying ABB(u, v). It completes the proof of the claim.

Finally we have to show that $(D_n(u), D_n(v)) \in \mathcal{A}$ for some ([3], [3])-constant pair of triples u, v satisfying $\mathbf{BBB}(u, v)$. Since the age of \mathbb{A} has free amalgamation and the structure is homogeneous, there exists a ({2,3}, [3])-constant pair of triples (s, t) in A satisfying $\mathbf{ABB}(s, t)$ such that

- $a_1 = (s[1], t[1], u[1], v[1])$ is a (A, N, N, N, N, B)-tuple,
- $a_2 = (s[2], t[2], u[2], v[2])$ is a $(\mathbf{B}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{B})$ -tuple, and
- $a_3 = (s[3], t[3], u[3], v[3])$ is a $(\mathbf{B}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{B})$ -tuple

(See Figure 3.) Let now u' be as u except for u'[1] which is such that $a_1' = (s[1], t[1], u'[1], v[1])$ is a $(\mathbf{A}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{A})$ -tuple. Now all a_1', a_2, a_3 are tuples in R. Since $(D_n(s), D_n(t)) \in \mathcal{A}$ and $\mathcal{A} + R = \mathcal{A}$ we have that $(D_n(u'), D_n(v)) \in \mathcal{A}$. By (4), we have $D_n(u') = D_n(u)$. It follows that $(D_n(u), D_n(v)) \in \mathcal{A}$ for (u, v) satisfying $\mathbf{BBB}(u, v)$. Since $\mathbf{B} \subsetneq \mathcal{A}$, we arrived at a contradiction. It completes the proof of the lemma.

The following lemma is a version of Lemma 1 but with a slightly different condition on the second tuple and a bit more complicated proof.

Lemma 2. Let \mathbb{B} be a first-order expansion of a finitely bounded homogeneous symmetric binary core \mathbb{A} whose age has free amalgamation. If \mathbb{B} pp-defines a self-complementary implication $R: \mathcal{A} \to \mathcal{A}$ containing

- $a(\mathbf{A}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{A})$ -tuple and
- a ($\mathbf{B}, \mathbf{B}, =, = \mathbf{B}, \mathbf{B}$)-tuple,

for some orbitals $\mathbf{A} \subseteq \mathcal{A}$ and $\mathbf{B} \nsubseteq \mathcal{A}$, then \mathbb{B} is not preserved by any chain of quasi directed Jónsson operations.

Proof. The proof goes along the lines of the proof of Lemma 1. We assume on the contrary that there exists a chain (D_1, \ldots, D_n) of quasi directed Jónsson operations preserving R and show that it contradicts the fact that \mathbb{B} contains all orbitals.

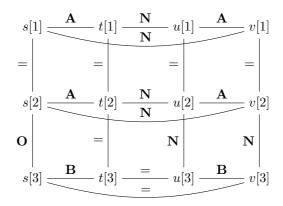


Figure 4: A substructure of A needed for the proof of the base case of induction in the proof of Claim 2.

In the formulation of the claim below we deal with ([3], [2])-constant pairs of triples (u, v) in A satisfying $\mathbf{AAB}(u, v)$ as well as ([3], {2,3})-constant pairs of triples (u, v) in A satisfying $\mathbf{ABB}(u, v)$ of which we already know they exist — see the discussion above Claim 1. For other (U, V)-constant tuples in the claim below consult Figures 4 and 6. Indeed, (u, v) in the former figure is a ([2], [2])-constant pair of triples satisfying $\mathbf{AAB}(u, v)$ while (u, v) in the latter is a ({2,3}, {2,3})-constant pair of triples in A satisfying $\mathbf{ABB}(s, t)$.

Claim 2. For all $i \in [n]$ we have both of the following.

- For all ([3], [2])-constant and all ([2], [2])-constant pairs of triples (u, v) in A satisfying $\mathbf{AAB}(u, v)$ it is the truth that $(D_i(u), D_i(v)) \in \mathbf{A}$.
- For all ([3], {2,3})-constant and all ({2,3}, {2,3})-constant pairs of triples (u, v) in A satisfying $\mathbf{ABB}(u, v)$ it is the truth that $(D_i(u), D_i(v)) \in \mathbf{A}$.

Proof. We prove the claim by the induction on $i \in [n]$.

(BASE CASE) The fact that for all ([3], [2])-constant pairs of triples (u, v) in A satisfying AAB(u, v) we have $(D_1(u), D_1(v)) \in A$ has been already proved while proving Claim 1.

Let now (u, v) be any ([2], [2])-constant pair of triples satisfying $\mathbf{AAB}(u, v)$. Since the age of $\mathbb A$ has free amalgamation (over $\mathbf N$) and the structure is homogeneous, there exists a ([2], [3])-constant pair of triples s, t in A (as in Figure 4) such that

- $a_1 = (s[1], t[1], u[1], v[1])$ is a (A, N, N, N, N, A)-tuple,
- $a_2 = (s[2], t[2], u[2], v[2])$ is a $(\mathbf{A}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{A})$ -tuple, and
- $a_3 = (s[3], t[3], u[3], v[3])$ is a $(\mathbf{B}, \mathbf{B}, =, =, \mathbf{B}, \mathbf{B})$ -tuple.

Hence all $a_1, a_2, a_3 \in R$. Since $(D_1(s), D_1(t)) \in \mathcal{A}$ and $\mathcal{A} + R = \mathcal{A}$ we have $(D_1(u), D_1(v)) \in \mathbf{A}$. It completes the proof of the base case for the first bullet.

For the second bullet, let (u, v) be now a $([3], \{2, 3\})$ -constant pair of triples in A satisfying $\mathbf{ABB}(u, v)$. Again, since the age of $\mathbb A$ has free amalgamation and the structure is homogeneous, there exists a ([2], [2])-constant pair of triples (s, t) in A (as in Figure 5) such that

- $a_1 = (s[1], t[1], u[1], v[1])$ is a $(\mathbf{A}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{A})$ -tuple,
- $a_2 = (s[2], t[2], u[2], v[2])$ is a (A, N, N, N, N, B)-tuple, and
- $a_3 = (s[3], t[3], u[3], v[3])$ is a $(\mathbf{B}, \mathbf{B}, =, =, \mathbf{B}, \mathbf{B})$ -tuple.

Let now u' be as u except for the second coordinate. We choose u'[2] so that $a'_2 = (s[2], t[2], u'[2], v[2])$ is a $(\mathbf{A}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{A})$ -tuple. Now all a_1, a'_2, a_3 are tuples in R. Since $(D_1(s), D_1(t)) \in \mathcal{A}$ and $\mathcal{A} + R = \mathcal{A}$ we have that $(D_1(u'), D_1(v)) \in \mathcal{A}$. By (2), we have $D_1(u') = D_1(u)$, and hence $(D_1(u), D_1(v)) \in \mathcal{A}$.

Finally, let (u, v) be a $(\{2, 3\}, \{2, 3\})$ -constant pair of triples in A satisfying $\mathbf{ABB}(u, v)$. Since the age of \mathbb{A} has free amalgamation and the structure is homogeneous, there is some $(\{2, 3\}, [3])$ -constant pair of triples (s, t) in A satisfying $\mathbf{ABB}(s, t)$ (as in Figure 6) such that

• $a_1 = (s[1], t[1], u[1], v[1])$ is a (A, N, N, N, N, A)-tuple,

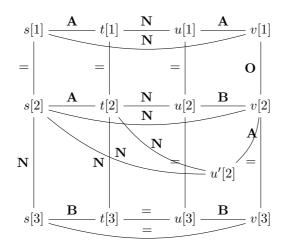


Figure 5: Triples u, v, s, t and u' which play a role in the proof of the the base case of induction in Claim 2.

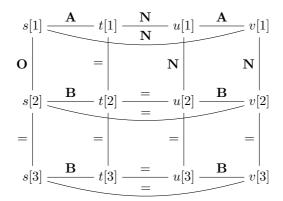


Figure 6: Triples u, v, s, t which play a role in the proof of the base case of induction in the proof of Claim 2.

- $a_2 = (s[2], t[2], u[2], v[2])$ is a $(\mathbf{B}, \mathbf{B}, =, =, \mathbf{B}, \mathbf{B})$ -tuple, and
- $a_3 = (s[3], t[3], u[3], v[3])$ is a $(\mathbf{B}, \mathbf{B}, =, =, \mathbf{B}, \mathbf{B})$ -tuple.

Observe that all $a_1, a_2, a_3 \in R$. Since $(D_1(s), D_1(t) \in A)$ and A + R = A, it follows that $(D_1(u), D_1(v) \in A)$. It completes the proof of the base case.

(INDUCTION STEP) The fact that $(D_{i+1}(u), D_{i+1}(v)) \in \mathbf{A}$. for all ([3], [2])-constant pair of triples (u, v) in A satisfying $\mathbf{AAB}(u, v)$ is shown in the same way as in the proof of Lemma 1.

The three remaining facts we need to go through the induction step may be proved in the exactly same way as for the base case but one needs to replace D_1 with D_{i+1} in all reasonings.

The final step of the proof is to show that $(D_n(u), D_n(v)) \in \mathcal{A}$ for a pair of ([3], [3])-constant pair of triples (u, v) in A satisfying $\mathbf{BBB}(u, v)$. To this end, consider such $u, v \in A^3$. Since \mathbb{A} is homogeneous and its age has free amalgamation over \mathbb{N} , there exists a ($\{2,3\}, \{2,3\}$)-constant pair of triples (s,t) in A (consult Figure 7) satisfying $\mathbf{ABB}(s,t)$ such that

- $a_1 = (s[1], t[1], u[1], v[1])$ is a (A, N, N, N, N, B)-tuple,
- $a_2 = (s[2], t[2], u[2], v[2])$ is a $(\mathbf{B}, \mathbf{B}, =, =, \mathbf{B}, \mathbf{B})$ -tuple, and
- $a_3 = (s[3], t[3], u[3], v[3])$ is a $(\mathbf{B}, \mathbf{B}, =, =, \mathbf{B}, \mathbf{B})$ -tuple.

Let now u' be as u except for the first coordinate. We choose u'[1] so that $a'_1 = (s[1], t[1], u'[1], v[1])$ is a $(\mathbf{A}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{N}, \mathbf{A})$ -tuple. Now all a'_1, a_2, a_3 are tuples in R. Since $(D_n(s), D_n(t)) \in \mathbf{A}$ and A + R = A we have that $(D_n(u'), D_n(v)) \in A$. By (4), we have $D_n(u') = D_n(u)$ and hence $(D_n(u), D_n(v)) \in A$ for u, v satisfying $\mathbf{BBB}(u, v)$. Since D_n preserves \mathbf{B} , we arrived at a contradiction. It completes the proof of the lemma.

We will now conclude Section 4.1 by a theorem which follows directly from Proposition 1 and Lemmas 1 and 2.

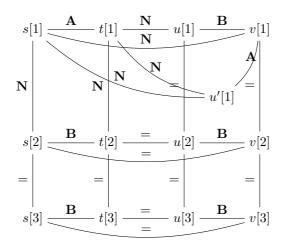


Figure 7: The figure depicts triples s, t, u, v and u' used in the final step of the proof of Lemma 2.

Theorem 5. Let \mathbb{B} be a first-order expansion of a finitely bounded homogeneous symmetric binary core which pp-defines $R_1: \mathcal{A} \to \mathcal{B}, R_2: \mathcal{B} \to \mathcal{A}$ such that \mathfrak{B}_{R_1,R_2} has a non-degenerated component. Then \mathbb{B} is not preserved by any chains of quasi directed Jónnson operations.

4.2 All non-trivial components are degenerated

By the results of the previous subsection we may assume that all nontrivial components in \mathfrak{B}_{R_1,R_2} are degenerated. We reduce this situation via Proposition 2 to three simply cases. To this end we need a number of definitions and observations.

Besides the ∘-composition defined earlier, we also define the ⋈-composition of two relations.

Definition 18. Let R_1, R_2 be two quaternary relations, then $R_1 \bowtie R_2$ is

$$R_3(x_1, x_2, x_3, x_4) \equiv \exists y_1 y_2 \ R_3(x_1, x_2, y_1, y_2) \land R_2(y_2, y_1, x_1, x_2).$$

The following observation may be proved in the exactly same way as Observation 6

Observation 7. Let $R_1: \mathcal{A} \to \mathcal{B}$ and $R_2: \mathcal{B} \to \mathcal{A}$ be complementary implications. Then $R_1 \bowtie R_2$ is a self-complementary $(\mathcal{A} \to \mathcal{A})$ -implication.

We will write $(R_1 \bowtie R_2)^n$ as a shorthand for the expression $((\cdots (((R_1 \bowtie R_2) \bowtie R_1) \bowtie R_2) \bowtie \cdots \bowtie R_1) \circ R_2)$ where both R_1 and R_2 occur n times. The next observation is analogous to Observation 4.

Observation 8. Let $R_1: A \to B$ and $R_2: B \to A$ be complementary relations. Then, for all $n \ge 1$ we have both of the following:

- $S_{2n} \equiv (R_1 \bowtie R_2)^n$ contains an $(\mathbf{O}, \dots, \mathbf{P})$ -tuple iff there is a path in \mathfrak{B}_{R_1, R_2} of length 2n from \mathbf{O}_L to \mathbf{P}_L ;
- $S_{2n+1} \equiv (R_1 \bowtie R_2)^n \circ R_1$ contains an $(\mathbf{O}, \dots, \mathbf{P})$ -tuple iff there is a path in \mathfrak{B}_{R_1, R_2} of length 2n + 1 from \mathbf{O}_L to \mathbf{P}_R .

Finally, the next observation resembles Observation 5 with a proof following strictly from a definition of the ⋈-composition.

Observation 9. Let $R_1: \mathcal{A} \to \mathcal{B}$ and $R_2: \mathcal{B} \to \mathcal{C}$ be such that R_1 contains a $(\mathbf{A}, \ldots, \mathbf{B})$ -tuple t_1 and R_2 contains a $(\mathbf{B}, \ldots, \mathbf{C})$ -tuple t_2 . Then $R_3:=R_1 \bowtie R_2$ is a $(\mathcal{A} \to \mathcal{C})$ -implication containing a $(\mathbf{A}, \ldots, \mathbf{C})$ -tuple t_3 which is essentially ternary if both t_1, t_2 are essentially ternary.

For a vertex \mathbf{O}_P with $P \in \{L, R\}$ and $S \in \{L, R\}$ we define:

FReach_S(
$$\mathbf{O}_P$$
) = { \mathbf{P}_S | there is a path in \mathfrak{B}_{R_1,R_2} from \mathbf{O}_P to \mathbf{P}_S }
BReach_S(\mathbf{O}_P) = { \mathbf{P}_S | there is a path in \mathfrak{B}_{R_1,R_2} from \mathbf{O}_P to \mathbf{P}_S }

Observation 10. Let R_1, R_2 be a pair of complementary relations pp-definable in a first-order expansion of a finitely bounded homogeneous symmetric binary core such that all non-trivial components in \mathfrak{B}_{R_1,R_2} are degenerated and \mathbf{O} an orbital such that \mathfrak{B}_{R_1,R_2} contains a non-trivial \mathbf{O} -degenerated component, then both $FReach_P(\mathbf{O}_P)$ and $BReach_P(\mathbf{O}_P)$ for $P \in \{L, R\}$ are pp-definable in \mathbb{B} .

FReach_L(
$$\mathbf{O}_L$$
)(x_1, x_2) $\equiv \exists y_1 y_2 \ \mathbf{O}(y_1, y_2) \land R_3(y_1, y_2, x_1, x_2)$

where

$$R_3 \equiv (R_1 \bowtie R_2)^n$$

with n being the number of all orbitals in \mathbb{A} . By Observation 8, a binary relation defined by a formula above contains all orbitals \mathbf{P} such that there is a path from \mathbf{O}_L to \mathbf{P}_L in \mathfrak{B}_{R_1,R_2} of length 2n. It is easy to see that if there is any path from \mathbf{O}_L to \mathbf{P}_L , then it is of length 2n or of even length smaller than 2n. Since there is a path of length two from \mathbf{O}_L to \mathbf{O}_R and back, the formula above defines $\mathrm{FReach}_L(\mathbf{O}_L)$. A proof for $\mathrm{FReach}_R(\mathbf{O}_R)$ is the same we just swap R_1 with R_2 .

For BReach_P(\mathbf{O}_P) we just need to invert the direction of arrows. Therefore we first set $R'_i(x_1, x_2, x_3, x_4) \equiv R_i(x_4, x_3, x_2, x_1)$ for $i \in [2]$. Now, in order to define BReach_L(\mathbf{O}_L) we use the above formula where R_1, R_2 are replaced with R'_2, R'_1 . For BReach_R(\mathbf{O}_R) we replace in the above formula R_1, R_2 with R'_1, R'_2 .

One of the cases the come up in Proposition 2 is related to the connectedness of a quaternary relation defined as follows.

Definition 19. We say that a quaternary relation R is connected if $\mathfrak{B}_{R,R'}$ is connected where $R'(x_1, x_2, x_3, x_4) \equiv R(x_4, x_3, x_2, x_1)$.

Recall that a quaternary tuple t is partially-free if $(t[1], t[4]) \in \mathbf{N}$. We are now ready for the proposition.

Proposition 2. Let \mathbb{B} be a first-order expansion of a finitely bounded homogeneous symmetric binary core whose age has free amalgamation. If \mathbb{B} pp-defines a pair of complementary implications $R_1: \mathcal{A} \to \mathcal{B}$ and $R_2: \mathcal{B} \to \mathcal{A}$ such that $\mathcal{A} \neq \mathcal{B}$ and all non-trivial components in \mathfrak{B}_{R_1,R_2} are degenerated, then \mathbb{B} pp-defines one of the following:

- 1. a relation R which is not connected and contains at least one non-degenerated tuple,
- 2. a ternary self-complementary $R: A \to A$ such that $\mathfrak{B}_{R,R}$ contains a \mathbf{A} -degenerated component, a \mathbf{B} -degenerated component for some $\mathbf{A} \subseteq A$, $\mathbf{B} \nsubseteq A$ and a $(\mathbf{B}, \mathbf{D}, \mathbf{A})$ -tuple is in R for some anti-reflexive orbital \mathbf{D} ,
- 3. a self-complementary $R: \mathcal{A} \to \mathcal{A}$ such that $\mathfrak{B}_{R,R}$ contains a \mathbf{A} -degenerated component, a \mathbf{B} -degenerated component for some anti-reflexive $\mathbf{A} \subseteq \mathcal{A}$, $\mathbf{B} \nsubseteq \mathcal{A}$ and a partially-free tuple is in R.

Proof. Since $A \neq B$ without loss of generality we can assume that there exists an orbit $\mathbf{C} \subseteq B \setminus A$. Let \mathbf{D} -degenerated component for some orbital \mathbf{D} be so that there is a path in \mathfrak{B}_{R_1,R_2} from \mathbf{C} to \mathbf{D} but there is no path from \mathbf{C} to \mathbf{D} that goes through another degenerated non-trivial component — we will say that the \mathbf{D} -degenerated component is directly above \mathbf{C} . In a similar manner, let a \mathbf{E} -degenerated component be a one directly below \mathbf{C} .

We say that π is a direct path from \mathbf{E}_L to \mathbf{D}_L in \mathfrak{B}_{R_1,R_2} if it is of the form

$$\mathbf{O}_{L}^{1}, \mathbf{O}_{R}^{2}, \dots, \mathbf{O}_{L}^{2k-1}, \mathbf{O}_{R}^{2k}, \mathbf{O}_{L}^{2k+1}$$

in \mathfrak{B}_{R_1,R_2} such that \mathbf{O}^1 is \mathbf{E} , \mathbf{O}^{2k+1} is \mathbf{D} and for all $i \in [2k]$ we have that R_1 contains a $(\mathbf{O}^i,\ldots,\mathbf{O}^{i+1})$ -tuple t_i if i is odd and that R_2 contains a $(\mathbf{O}^i,\ldots,\mathbf{O}^{i+1})$ -tuple t_i if i is even. We also require that all t_i perhaps with an exception of 1 and 2k are non-degenerated. The sentence of tuples t_i associated with π will be denoted with τ . The essential length of π is the number of non-degenerated tuples in τ . It is 2k, 2k-1 or 2k-2.

Choose now a direct path π from \mathbf{E}_L to \mathbf{D}_L in \mathfrak{B}_{R_1,R_2} of the largest essential length. Since there is one going through \mathbf{C}_R which is different than both \mathbf{E} and \mathbf{D} , the essential length of the chosen π is at least 2.

First consider the case where all non-degenerated tuples in τ are essentially ternary. By Observations 8, 7 and 9, it follows that $(R_1 \bowtie R_2)^k$ is a self-complementary $(\mathcal{A} \to \mathcal{A})$ -implication and contains an essentially ternary $(\mathbf{E}, \dots, \mathbf{D})$ -tuple. Set a new relation

$$R'(x_1, x_2, x_3) \equiv R(x_1, x_2, x_2, x_3) \land x_2 = x_3 \land C(x_1, x_2) \land D(x_2, x_3)$$

where $C = \operatorname{FReach}_L(\mathbf{E}_L) \cap \operatorname{BReach}_L(\mathbf{D}_L)$ and $\mathcal{D} = \operatorname{FReach}_R(\mathbf{E}_R) \cap \operatorname{BReach}_R(\mathbf{D}_R)$. It is easy to see that the only non-trivial components of R' are a \mathbf{D} -degenerated and an \mathbf{E} -degenerated component and thereby R' is a ternary self-complementary ($\mathbf{D} \to \mathbf{D}$)-implication that contains all tuples required in Case 2.

Observe also that the same reasoning works when either R_1 or R_2 has an essentially ternary $(\mathbf{E}, \dots, \mathbf{D})$ tuple. Thus, from now on we assume that neither of these tuples is present in any of these relations and
that τ contains at least one essentially quaternary tuple.

Consider now the case where there are at least two essentially quaternary tuples in τ . By Observations 6 and 5 we have that $R \equiv (R_1 \circ R_2)^k$ is a self-complementary $(\mathcal{A} \to \mathcal{A})$ -implication which contains a fully free $(\mathbf{E}, \dots, \mathbf{D})$ -tuple. Since $\mathbf{C} \notin \mathcal{A}$, by Observation 3, we have that \mathfrak{B}_{R_1,R_2} has at least three pairwise different \mathbf{O}_i -degenerated components and two of them are in the $(\mathcal{A}, \mathcal{A})$ -subgraph. The same holds for $\mathfrak{B}_{R,R}$. Without loss of generality assume that the $(\mathbf{O}_1, \mathbf{O}_1)$ -component is maximal in the $(\mathcal{A}, \mathcal{A})$ -subgraph, $(\mathbf{O}_2, \mathbf{O}_2)$ is minimal in the $(\mathcal{A}, \mathcal{A})$ -subgraph and $(\mathbf{O}_3, \mathbf{O}_3)$ is minimal in the $((\Pi_{1,2}(R) \setminus \mathcal{A}), (\Pi_{1,2}(R) \setminus \mathcal{A}))$ -subgraph of $\mathfrak{B}_{R,R}$. Now, if \mathbf{O}_3 is anti-reflexive, then we are in Case 3 already when we see R as a self-complementary $(\mathcal{A} \to \mathcal{A})$ -implication. Otherwise, we are in Case 3 with R seen as a self-complementary $(\mathbf{O}_1 \to \mathbf{O}_1)$ -implication.

The last case is where all but one non-degenerated tuple in τ are essentially ternary. First we assume that π is of essential length at least 3. Thus, there must be three consecutive tuples t_i, t_{i+1}, t_{i+2} of τ such that at least one of them is essentially quaternary and two remaining are essentially ternary. By Observation 5 we have what follows. If there is $j \in \{0, 1\}$ such that t_{i+j} is essentially quaternary, t_{i+j+1} is essentially ternary and \mathbf{O}_{i+j+2} is not = or there is $j \in [2]$ such that t_{i+j} is essentially quaternary, t_{i+j-1} is essentially ternary and \mathbf{O}_{i+j-1} is not =, then $R_1 \circ R_2$ in case i is odd or $R_2 \circ R_1$ in case i is even contains a partially-free tuple. Otherwise one of two cases holds:

- t_i is essentially quaternary, t_{i+1} essentially ternary and \mathbf{O}_{i+2} is =, or
- t_{i+2} is essentially quaternary, t_{i+1} essentially ternary and \mathbf{O}_{i+1} is =.

Again, by Observation 5, we have that either $R_1 \circ R_2$ or $R_2 \circ R_1$ has an essentially quaternary tuple. Clearly neither \mathbf{O}_{i+3} in the first case nor \mathbf{O}_i in the second is =. Furthermore both t_{i+2} in the first case and t_i in the second case are essentially ternary. It follows that $R_1 \circ R_2 \circ R_1$ in a case i is odd or $R_2 \circ R_1 \circ R_2$ if i is even contain a partially free tuple. By Observations 6, 4 and 5, the relation $R \equiv (R_1 \circ R_2)^k$ is a self-complementary $(\mathbf{A} \to \mathbf{A})$ -implication that contains a partially-free tuple. It is also easy to see that all non-trivial degenerated components in \mathfrak{B}_{R_1,R_2} are also in $\mathfrak{B}_{R,R}$. By Observation 3 we have that the former as well as the latter graph has at least three pairwise different \mathbf{O}_i -degenerated components so that two of them are in $(\mathcal{A}, \mathcal{A})$. Now, we complete the reasoning exactly in the same way as in the previous case.

The only case left to be considered is when τ has exactly two non-degenerated tuples. In particular we can assume that a direct path from \mathbf{E}_L to \mathbf{D}_L that goes through \mathbf{C}_R is of essential length 2. Thus we have a $(\mathbf{E}, \ldots, \mathbf{C})$ -tuple in R_1 and a $(\mathbf{C}, \ldots, \mathbf{D})$ -tuple in R_2 . One of these tuples is essentially ternary and the other is essentially quaternary. Without loss of generality assume that the former is essentially ternary. Observe now that

$$R'_1(x_1, x_2, x_3, x_4) \wedge x_2 = x_3 \wedge \mathcal{C}(x_1, x_2) \wedge \mathcal{D}(x_3, x_4)$$

where $C = \operatorname{FReach}_L(\mathbf{E}_L) \cap \operatorname{BReach}_L(\mathbf{D}_L)$ and $\mathcal{D} = \operatorname{FReach}_R(\mathbf{E}_R) \cap \operatorname{BReach}_R(\mathbf{D}_R)$ contains a non-degenerated tuple. In order to complete the proof of the proposition we will show that R'_1 is not connected, i.e.,the graph $\mathfrak{B}_{R,R'}$ where $R'(x_1,x_2,x_3,x_4) \equiv R(x_4,x_3,x_2,x_1)$ is not connected, in particular that there is no path from \mathbf{E}_L to \mathbf{D}_R . Observe that it is enough to show that there is no sequence of tuples t_1,\ldots,t_{2k+1} such that t_i is a $(\mathbf{O}_i,\ldots,\mathbf{O}_{i+1})$ -tuple in R_1 when i is odd and such that t_i is a $(\mathbf{O}_{i+1},\ldots,\mathbf{O}_i)$ -tuple in R'_1 when i is even and such that \mathbf{O}_1 is \mathbf{D} and \mathbf{O}_{2k} is \mathbf{E} . Recall that R_1 has no $(\mathbf{E},\ldots,\mathbf{D})$ -tuple and hence k>1. It follows that there is a $(\mathbf{P},\ldots,\mathbf{P}')$ -tuple in R_1 with $\mathbf{P}\subseteq \operatorname{FReach}_L(\mathbf{E}_L)\cap\operatorname{BReach}_L(\mathbf{D}_L)$ and $\mathbf{P}'\subseteq\operatorname{FReach}_L(\mathbf{E}_L)\cap\operatorname{BReach}_L(\mathbf{D}_L)$ and such that $\{\mathbf{P},\mathbf{P}'\}\cap \{\mathbf{E},\mathbf{D}\}=\emptyset$. It implies that there is a direct path from \mathbf{E}_L do \mathbf{D}_L that goes through $\mathbf{P}_L,\mathbf{P}'_R$. It has to be of essential length at least 3 which contradicts the assumption. We have that one of the three cases in the formulation of the propositions follows.

We will now show that in every case from the formulation of Proposition 2, the structure \mathbb{B} is not preserved by a chain of quasi directed Jonsson operations.

Lemma 3. Let \mathbb{B} be a first-order reduct of a finitely bounded homogenous symmetric binary core whose age has free amalgamation. If \mathbb{B} pp-defines a quaternary relation R which is not connected and which contains a non-degenerated $(\mathbf{A}, \ldots, \mathbf{B})$ -tuple, then \mathbb{B} is not preserved by any chain of quasi directed Jonsson operations.

Proof. Let $R'(x_1, x_2, x_3, x_4) \equiv R(x_4, x_3, x_2, x_1)$, and let $(\mathcal{A}, \mathcal{B})$ -component be a component of $\mathfrak{B}_{R,R'}$ such that $\mathbf{A} \subseteq \mathcal{A}$ and $\mathbf{B} \subseteq \mathcal{B}$. Since R is not connected, it is a $(\mathcal{A} \to \mathcal{B})$ -implication, R' is a $(\mathcal{B} \to \mathcal{A})$ -implication and R, R' are complementary. Since $(\mathcal{A}, \mathcal{B})$ is non-degenerated it follows by Theorem 5 that \mathbb{B} is not preserved by any chain of quasi directed Jónsson operations.

We now turn to Case 2 in the formulation of Proposition 2.

Lemma 4. Let \mathbb{B} be a first-order expansion of a finitely bounded homogeneous symmetric binary core whose age has free amalgamation. If \mathbb{B} pp-defines a ternary self-complementary relation $R: \mathcal{A} \to \mathcal{A}$ which contains:

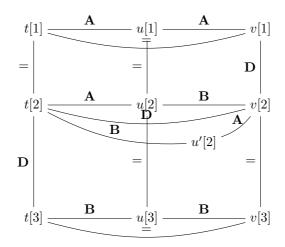


Figure 8: The figure needed for a proof of the second item in Claim 3.

- a **A**-degenerated component, a **B**-degenerated component for some orbitals $\mathbf{A} \subseteq \mathcal{A}$, $\mathbf{B} \nsubseteq \mathcal{A}$, and
- a (**B**, **D**, **A**)-tuple for some anti-reflexive orbital **D**.

then \mathbb{B} is not preserved by any chains of quasi directed Jónsson operations.

Proof. The proof goes along the lines of similar proofs in the previous subsection. Except for a trivial case of (3,3)-constant pairs of triples, we use Definition 17 only for ([3],[2])-constant and $([3],\{2,3\})$ -constant pairs u,v of triples in A satisfying $\mathbf{A}\mathbf{A}\mathbf{B}$ and $\mathbf{A}\mathbf{B}\mathbf{B}$ respectively with a difference to the original definition such that $(v[2],v[3]) \in \mathbf{D}$ in the first case and $(v[1],v[2]) \in \mathbf{D}$ in the second case. There is a $(\mathbf{B},\mathbf{D},\mathbf{A})$ -tuple t in R, and hence both kinds of (U,V)-constant pairs of triples mentioned above exist in A.

We assume on the contrary that there is a chain (D_1, \ldots, D_n) of quasi directed Jónsson operations preserving \mathbb{B} and R. We show that it contradicts the fact that $\operatorname{Pol}(\mathbb{B})$ preserves \mathcal{A} which is pp-definable.

Claim 3. For all $i \in [n]$ we have both of the following.

- For all ([3], [2])-constant pairs of triples (u, v) in A satisfying $\mathbf{AAB}(u, v)$ we have that $(D_i(u), D_i(v)) \in \mathcal{A}$.
- For all ([3], {2,3})-constant pairs of triples (u, v) in A satisfying $\mathbf{ABB}(u, v)$ we have that $(D_i(u), D_i(v)) \in \mathcal{A}$.

Proof. We prove the claim by the induction on $i \in [n]$.

(BASE CASE) For the first bullet we start with a ([3], [2])-constant pair of triples (u, v) in A satisfying AAB(u, v). If we replace v with v' which is identical to v with a difference that v'[3] = v[2]. It is clear that (u, v') is a ([3], [3])-constant pair of triples in A satisfying AAA, and hence $(D_1(u), D_1(v')) \in A$. By (1) we have $D_1(v') = D_1(v)$. It follows $(D_1(u), D_1(v)) \in A$.

For the second bullet in the formulation of the claim, we start with a ([3], $\{2,3\}$)-constant pair of triples (u,v) satisfying $\mathbf{ABB}(u,v)$ and extend it with $t \in A^3$ as pictured in Figure 8. Observe that no new elements are needed, and hence the extension is trivial. Indeed, we have that t[1] = t[2] = v[2] and t[3] = v[3]. There also exists u' identical with u with a difference that $u'[2] \neq u[2]$ so that

- $a_1 = (t[1], u[1], v[1])$ is a $(\mathbf{A}, =, \mathbf{A})$ -tuple in R,
- $a_2 = (t[2], u'[2], v[2])$ is a $(\mathbf{B}, \mathbf{D}, \mathbf{A})$ -tuple in R, and
- $a_3 = (t[3], u[3], v[3])$ is a $(\mathbf{B}, =, \mathbf{B})$ -tuple in R.

Indeed, since the age of \mathbb{A} has free amalgamation we have that there are $a, b, c, d \in A$ such that (a, b, c) is a $(\mathbf{A}, \mathbf{D}, \mathbf{B})$ -tuple, (a, d, c) is a $(\mathbf{B}, \mathbf{D}, \mathbf{A})$ -tuple and $(b, d) \in \mathbf{N}$. If α is an automorphism of \mathbb{A} sending (t[2], u[2], v[2]) to (a, b, c), then we may take $\alpha^{-1}(d)$ for u'[2].

By the first bullet we have that $(D_1(t), D_1(u)) \in \mathcal{A}$, by (2) that $(D_1(t), D_1(u')) \in \mathcal{A}$. Since R is a $(\mathcal{A} \to \mathcal{A})$ -implication, it follows that $(D_1(u',v)) \in \mathcal{A}$ and $(D_1(u,v)) \in \mathcal{A}$. It completes the proof of the base case.

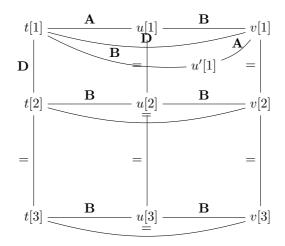


Figure 9: A figure which we use in the final step of the proof of Lemma 4.

(INDUCTION STEP) For the induction step, consider any ([3], [2])-constant pair of triples (u, v) in A satisfying $\mathbf{AAB}(u, v)$. We want to show that $(D_{i+1}(u), D_{i+1}(v)) \in \mathcal{A}$. To that end consider v' such that v'[1] = v[1] and v'[2] = v'[3] = v[3]. Observe that (u, v') is a ([3], $\{2, 3\}$)-constant pair of triples in A satisfying $\mathbf{ABB}(u, v')$. By the induction hypothesis, it follows that $(D_i(u), D_i(v')) \in \mathcal{A}$. By (3), we have $D_i(v) = D_{i+1}(v')$ and clearly $D_{i+1}(u) = D_i(u)$. It follows that $(D_{i+1}(u), D_{i+1}(v)) \in \mathcal{A}$. Now, as in the base case we show that $(D_{i+1}(u), D_{i+1}(v)) \in \mathbf{A}$ for all ([3], $\{2, 3\}$)-constant pair of triples (u, v) in \mathbf{A} satisfying $\mathbf{ABB}(u, v)$. It completes the proof of the claim.

For the final step of the proof of the lemma consider a ([3], [3])-constant pair of triples u, v satisfying $\mathbf{BBB}(u, v)$ and $t \in A^3$ so that u, t forms a ({2,3}, [3])-constant pair of triples satisfying $\mathbf{ABB}(u, v)$ as in Figure 9. By the assumption we have that there is a ($\mathbf{B}, \mathbf{D}, \mathbf{A}$)-tuple (c, a, b) in A. Set α to be an automorphism of \mathbb{A} sending (a, b) to (u[1], v[1]). Observe that we may take $(\alpha(c), \alpha(b), \alpha(b))$ for t.

Let now u' be as u with a difference that u'[1] satisfying

- $a_1 = (t[1], u'[1], v[1])$ is a $(\mathbf{B}, \mathbf{D}, \mathbf{A})$ -tuple,
- $a_2 = (t[2], u[2], v[2])$ is a $(\mathbf{B}, =, \mathbf{B})$ -tuple, and
- $a_3 = (t[3], u[3], v[3])$ is a $(\mathbf{B}, =, \mathbf{B})$ -tuple.

as well as $(u'[1], u[i]) \in \mathbf{N}$ for $i \in \{2, 3\}$. Again, we use the fact that the age of \mathbb{A} has free amalgamation in order to see that there are $a, b, c, d \in A$ such that (a, b, c) is a $(\mathbf{A}, \mathbf{D}, \mathbf{B})$ -tuple, (a, d, c) is a $(\mathbf{B}, \mathbf{D}, \mathbf{A})$ -tuple and $(b, d) \in \mathbf{N}$. Sending (t[1], u[1], v[1]) by an automorphism α to (a, b, c) we get that u'[1] is $\alpha^{-1}(d)$. Thus, it exists.

By (4) we have that $(D_n(t), D_n(u')) \in \mathcal{A}$. Since all a_i with $i \in [3]$ are in R and R is a $(\mathcal{A} \to \mathcal{A})$ -implication, it follows that $(D_n(u'), D_n(v)) \in \mathcal{A}$. Again, by (4), we have $(D_n(u), D_n(v)) \in \mathcal{A}$. It contradicts the assumption that D_n preserves \mathbf{B} and completes the proof of the lemma.

We finally turn to the third case in Proposition 2.

Lemma 5. Let \mathbb{B} be a first-order expansion of a finitely bounded homogeneous symmetric binary core \mathbb{A} whose age has free amalgamation. If \mathbb{B} pp-defines a self-complementary implication $R: \mathcal{A} \to \mathcal{A}$ such that $\mathfrak{B}_{R,R}$ contains a \mathbf{A} -degenerated component and a \mathbf{B} -degenerated component where $\mathbf{A} \subseteq \mathcal{A}$ and $\mathbf{B} \nsubseteq \mathcal{A}$ are both anti-reflexive and a partially-free $(\mathbf{O}_{11}, \mathbf{O}_{12}, \mathbf{O}_{13}, \mathbf{N}, \mathbf{O}_{23}, \mathbf{O}_{24}, \mathbf{O}_{34})$ -tuple is in R, then \mathbb{B} is not preserved by any chain of quasi directed Jónsson operations.

Proof. Again, we assume on the contrary that there is a chain (D_1, \ldots, D_n) of quasi directed Jónsson operations preserving \mathbb{B} and R and as usual we show that it contradicts the fact that $Pol(\mathbb{B})$ preserves all orbitals.

In contrast to the proof of Lemma 4, we use the original version of Definition 17. We provide a claim as usual.

Claim 4. For all $i \in [n]$ we have both of the following.

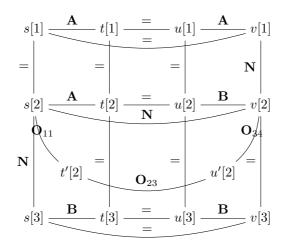


Figure 10: A substructure of \mathbb{A} used in the proof of Claim 4.

- For all ([3], [2])-constant pairs of triples (u, v) in A satisfying $\mathbf{AAB}(u, v)$ we have that $(D_i(u), D_i(v)) \in \mathcal{A}$.
- For all ([3], {2,3})-constant pairs of triples (u, v) in A satisfying ABB(u, v) we have that $(D_i(u), D_i(v)) \in A$.

Proof. We prove the claim by the induction on $i \in [n]$.

(BASE CASE) The first bullet is a special case of what we proved for Claim 1. For the second bullet consider a ([3], $\{2,3\}$)-constant pair of triples u,v in A satisfying ABB(u,v) and a ([3], [2])-constant pair of triples s,t satisfying AAB(t,s) as in Figure 10. Observe that all values in vectors s,t are already in u,v and hence s,t exist.

We will now show that there exist t', u' identical to t, u but with a difference that t'[2], u'[2] are new and such that

- $a_1 = (s[1], t[1], u[1], v[1])$ is a $(\mathbf{A}, \mathbf{A}, =, =, \mathbf{A}, \mathbf{A})$ -tuple,
- $a_2 = (s[2], t'[2], u'[2], v[2])$ is a $(\mathbf{O}_{12}, \mathbf{O}_{13}, \mathbf{N}, \mathbf{O}_{23}, \mathbf{O}_{24}, \mathbf{O}_{34})$ -tuple, and
- $a_3 = (s[3], t[3], u[3], v[3])$ is a $(\mathbf{B}, \mathbf{B}, =, = \mathbf{B}, \mathbf{B})$ -tuple.

Indeed, since the age of \mathbb{A} has free amalgamation there are elements $a, b, c, d, e \in A$ such that (a, b, c) is a $(\mathbf{A}, \mathbf{N}, \mathbf{B})$ -tuple, (a, d, e, c) is a $(\mathbf{O}_{12}, \mathbf{O}_{13}, \mathbf{N}, \mathbf{O}_{23}, \mathbf{O}_{24}, \mathbf{O}_{34})$ -tuple and all other orbitals between the elements a, b, c, d, e either follow from semi-transitivity of equality or are \mathbf{N} . If α is an automorphism of \mathbb{A} sending (a, b, c) to (s[2], t[2], v[2]), then we take $\alpha^{-1}(d)$ for t'[2] and $\alpha^{-1}(e)$ for u'[2].

By the first item we have that $(D_1(s), D_1(t)) \in \mathcal{A}$. By (2), it follows that $(D_1(s), D_1(t')) \in \mathcal{A}$ Since $a_i \in R$ for $i \in [3]$ we have $(D_1(u'), D_1(v)) \in \mathcal{A}$. By (2), we have that $(D_1(u), D_1(v)) \in \mathcal{A}$, which was to be proved. It completes the proof for the base case.

(INDUCTION STEP) The first bullet we prove exactly like in the proof of Claim 1, and the second bullet as in the base case.

In the final step, as usual, we show that under the working assumptions there is a (3,3)-constant part of triples (u,v) satisfying **BBB** such that $(D_n(u),D_n(v)) \in \mathcal{A}$. We provide the reasoning analogous to reasoning in the proof of the claim with a difference that we use Figure 11 instead of Figure 10.

We will now conclude Section 4.2 by a theorem which follows directly from Proposition 2 and Lemmas 3, 4 and 5.

Theorem 6. Let \mathbb{B} be a first-order expansion of a finitely bounded homogeneous symmetric binary core which pp-defines complementary implications $R_1: \mathcal{A} \to \mathcal{B}$ and $R_2: \mathcal{B} \to \mathcal{A}$ with $\mathcal{A} \neq \mathcal{B}$ and such that every non-trivial component of \mathfrak{B}_{R_1,R_2} is degenerated. Then \mathbb{B} is not preserved by any chains of quasi directed Jónnson operations.

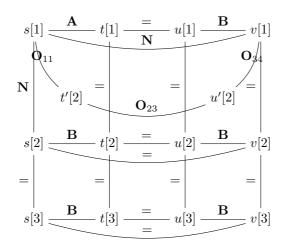


Figure 11: A substructure of A used in the final step of the proof of Lemma 5

5 Main Result

We are ready to prove the main theorem of the paper.

Proof of Theorem 2 The relational clone of \mathbb{B} is clearly either implicationally uniform or not. In the former case the theorem holds by Theorem 4 while in the latter \mathbb{B} pp-defines complementary implications $R_1: \mathcal{A} \to \mathcal{B}, R_2: \mathcal{B} \to \mathcal{A}$ such that $\mathcal{A} \neq \mathcal{B}$. Now, either \mathfrak{B}_{R_1,R_2} contains a non-trivial non-degenerated component or all non-trivial components of this graph are degenerated. In either case, the structure \mathbb{B} is not preserved by any chain of quasi directed Jónsson operations by Theorems 5 or 6. \square

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