

# NON-SYMPLECTIC AUTOMORPHISMS OF PRIME ORDER OF O'GRADY'S TENFOLDS AND CUBIC FOURFOLDS

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**ABSTRACT.** We give a lattice-theoretic classification of non-symplectic automorphisms of prime order of irreducible holomorphic symplectic manifolds of OG10 type. We determine which automorphisms are induced by a non-symplectic automorphism of prime order of a cubic fourfold on the associated Laza–Saccà–Voisin manifolds, giving a geometric and lattice-theoretic description of the algebraic and transcendental lattices of the cubic fourfold. As an application we discuss the rationality conjecture for a general cubic fourfold with a non-symplectic automorphism of prime order.

## 1. INTRODUCTION

An irreducible holomorphic symplectic (ihs) manifold is a simply connected, compact, complex, Kähler manifold  $X$  carrying a non-degenerate holomorphic symplectic form  $\sigma_X$  which spans  $H^0(X, \Omega_X^2)$ . An automorphism of an ihs manifold is called *symplectic* if it acts trivially on the symplectic form, *non-symplectic* otherwise. A cyclic group  $G \subset \text{Aut}(X)$  is called non-symplectic if it is generated by a non-symplectic automorphism.

Cubic hypersurfaces in  $\mathbb{P}^5$  admit a Hodge decomposition of K3 type, i.e.  $H^4(Y, \mathbb{C}) = H^{3,1}(Y) \oplus H^{2,2}(Y) \oplus H^{1,3}(Y)$  and  $h^{3,1}(Y) = 1$ , hence there is a notion of symplectic and non-symplectic automorphisms. Namely, an automorphism of a cubic fourfold  $Y$  is symplectic if the induced action on  $H^4(Y, \mathbb{Z})$  acts trivially on  $H^{3,1}(Y)$ , non-symplectic otherwise.

Giovenzana–Grossi–Onorati–Veniani [18] prove that any symplectic automorphism of finite order of an ihs manifold of O'Grady's 10-dimensional deformation type is trivial.

In this paper we classify non-symplectic automorphisms of prime order of an ihs manifold  $X$  of OG10 type. If  $X$  can be realized as a compactification of the intermediate jacobian fibration of the hyperplane sections of a cubic fourfold, referring to the model due to Laza–Saccà–Voisin [31, 56], we give a lattice-theoretic criterion to determine when such an automorphism is induced by an automorphism of the cubic fourfold. To this purpose, we study the induced action on  $H^4(Y, \mathbb{Z})$  by

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a non-symplectic automorphism of prime order of a smooth cubic fourfold  $Y$ . Starting from these results we exhibit the algebraic and the transcendental lattice of a general cubic fourfold that admits a non-symplectic automorphism of prime order. Moreover, we give a geometric set of generators of the algebraic lattice in terms of planes or cubic scrolls. As a consequence, we discuss the rationality conjecture for cubic fourfolds with a non-symplectic automorphism of prime order.

The classification of the algebraic and the transcendental lattices of a cubic fourfold with a non-symplectic involution, and the discussion of the rationality conjecture for such a cubic, is the content of a recent paper by Marquand [35].

The classification of automorphisms of ihs manifolds of OG10 type is an extension of a result by Brandhorst–Cattaneo [9] that provide lattice-theoretic constraints of non-symplectic automorphisms of odd prime order of an ihs manifold in terms of isometries of unimodular lattices.

One of the advantages in studying induced automorphisms is to control the fixed locus. Automorphisms of ihs manifolds with empty fixed locus are needed to construct Enriques manifolds, the higher dimensional analogue of Enriques surfaces, see [7, 47] for more details. The authors together with Luca Giovenzana and Franco Giovenzana in an upcoming paper [3] use the results about induced automorphisms to investigate the existence of Enriques manifolds as free quotient of ihs manifolds of OG10 type.

**1.1. Automorphisms of ihs manifolds and cubic fourfolds.** Automorphisms of ihs manifolds can be classified studying the induced action on the second integral cohomology  $H^2(X, \mathbb{Z})$ , which carries a lattice structure provided by the Beauville–Bogomolov–Fujiki quadratic form. In particular, if  $X$  is an ihs manifold of OG10 type we know by [52] that  $H^2(X, \mathbb{Z})$  is isometric to the abstract lattice  $\mathbf{L} := \mathbf{U}^{\oplus 3} \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$ . A marking is an isometry  $\eta: H^2(X, \mathbb{Z}) \rightarrow \mathbf{L}$  and, whenever we fix a marked pair  $(X, \eta)$  of OG10 type, the representation map

$$(1) \quad \eta_*: \text{Aut}(X) \rightarrow \text{O}(\mathbf{L}), \quad f \mapsto \eta \circ (f^{-1})^* \circ \eta^{-1}$$

is injective by a result of Mongardi–Wandel [44]. For this reason, we study automorphisms in terms of their induced action in cohomology. More precisely, if an isometry  $\varphi \in \text{O}(\mathbf{L})$  verifies the assumptions of Hodge–theoretic Torelli theorem [34, Theorem 1.3], then there exists an automorphism  $g \in \text{Aut}(X)$  such that  $\eta_*(g) = \varphi$ .

If  $(X, \eta)$  is a marked pair of OG10 type, an isometry  $\varphi \in \text{O}(\mathbf{L})$  is *non-symplectic* if the action on the Hodge structure of  $\mathbf{L} \otimes \mathbb{C}$  (induced via the marking) is non-symplectic. A cyclic group  $G \subset \text{O}(\mathbf{L})$  is called *non-symplectic* if  $G$  is generated by a non-symplectic isometry.

Similarly if  $Y$  is a cubic fourfold, the integral cohomology  $H^4(Y, \mathbb{Z})$  with the intersection form, is an odd unimodular lattice isometric to  $[1]^{\oplus 21} \oplus [-1]^{\oplus 3}$ . Consider the square of an hyperplane section  $\eta_Y \in H^4(Y, \mathbb{Z})$ , then the primitive cohomology  $H_p^4(Y, \mathbb{Z}) = \langle \eta_Y \rangle^\perp$  with the intersection form is an even lattice isometric to  $\mathbf{F} := \mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8^{\oplus 2} \oplus \mathbf{A}_2$ . We denote by  $A(Y) := H^{2,2}(Y, \mathbb{C}) \cap H^4(Y, \mathbb{Z})$  the lattice of algebraic cycles of  $Y$ , and by  $T(Y) = A(Y)^\perp \subset H^4(Y, \mathbb{Z})$  the transcendental lattice. We denote by  $A_p(Y) := H^{2,2}(Y, \mathbb{C}) \cap H_p^4(Y, \mathbb{Z})$  the even lattice of algebraic primitive cycles. A marking is an isometry  $\gamma: H_p^4(Y, \mathbb{Z}) \rightarrow \mathbf{F}$ . Let  $(Y, \gamma)$  be a marked pair, an isometry  $\varphi \in \text{O}(\mathbf{F})$  is *non-symplectic* if the action on the Hodge structure of  $\mathbf{F} \otimes \mathbb{C}$  (induced via the marking) is non-symplectic. Automorphisms of  $Y$  are identified with Hodge isometries of  $H_p^4(Y, \mathbb{Z})$  by the Hodge–theoretic Torelli theorem [57], and then similar techniques to the ones used for ihs manifolds can be applied.

We recall the known results about automorphisms of other deformations types of ihs manifolds. Regarding prime order automorphisms of ihs manifolds of K3<sup>[2]</sup> type we refer to [1, 5, 6, 10, 13, 40,

48]. Non-symplectic automorphisms of ihs manifolds of  $K3^{[n]}$  type are treated in [11, 12], while automorphisms of ihs manifolds of  $K_n(A)$  type are classified in [43] for  $n = 2$ , and in [9] for every  $n$ . The classification of symplectic automorphisms of ihs manifolds of OG6 type is given in [21], while non-symplectic automorphisms of prime order are classified by in [20] the second named author. Symplectic birational transformations of ihs manifolds of OG10 type are studied in [36] and groups of symplectic birational transformations are studied in [37].

Automorphisms of prime order of a smooth cubic fourfold are classified in [16, 19, 59] in terms of their actions on  $\mathbb{P}^5$  and giving a general equation of  $Y$ . Groups of symplectic automorphisms of cubic fourfolds are studied in [16, 32]. The classification of algebraic and transcendental lattices of a cubic fourfold with an involution is the content of [35].

**1.2. Contents of the paper.** In §2 we introduce basic notions and results on lattice theory to study automorphisms of ihs manifolds and cubic fourfolds. Moreover, we recall Hodge-theoretic Torelli theorem for these manifolds and the construction due to Laza–Saccà–Voisin.

In §3 we classify cyclic groups  $G \subset \text{Aut}(X)$  of non-symplectic automorphisms of prime order of an ihs manifold  $X$  of OG10 type. We denote by  $\mathbf{L}^G$  the *invariant lattice* and by  $\mathbf{L}_G$  the *coinvariant lattice*. We provide a list of invariant and coinvariant sublattices of the induced action of  $G$  on  $\mathbf{L}$  up to isometry. To achieve this we consider the unique primitive embedding  $\mathbf{L} \hookrightarrow \mathbf{A} := \mathbf{U}^{\oplus 5} \oplus \mathbf{E}_8(-1)^{\oplus 2}$  and we refer to [9] for a classification of odd prime order isometries of unimodular lattices.

**Theorem 1.1** (see Theorem 3.7). *Let  $G \subset \text{O}(\mathbf{L})$  be a group of non-symplectic isometries of prime order  $p$ . Then there exists an ihs manifold  $X$  of OG10 type such that  $G \subset \text{Aut}(X)$  if and only if  $2 \leq p \leq 23$  and the pairs  $(\mathbf{L}^G, \mathbf{L}_G)$  appear in Table 6 and Table 7 for  $p = 2$ , and in Table 9 for  $p \geq 3$ .*

In §4 we give a lattice-theoretic and a geometric classification of prime order non-symplectic automorphisms of a cubic fourfold, that can be of order two or three Bb [19, 59].

Let  $G \subset \text{Aut}(Y)$  be a group of non-symplectic automorphisms of prime order of a general cubic fourfold  $Y$ , then the invariant lattice  $\mathbf{F}^G$  coincide with the primitive algebraic lattice  $A_p(Y)$ , and the coinvariant lattice  $\mathbf{F}_G$  coincides with the transcendental lattice  $T(Y)$ . In §4.1 we classify the algebraic and the transcendental lattices of a general cubic fourfold with a non-symplectic automorphism of order three.

**Theorem 1.2** (See Theorem 4.6). *Let  $G \subset \text{O}(\mathbf{F})$  be a group of non-symplectic isometries of order three. Then there exists a cubic fourfold  $Y$  such that  $G \subset \text{Aut}(Y)$  if and only if the pairs  $(\mathbf{F}^G, \mathbf{F}_G)$  appears in Table 3. For such a general  $Y$  the algebraic lattice  $A(Y)$  appear in Table 4, and the class of  $\eta_Y$  is expressed in a basis of  $A(Y)$ .*

There are many examples of rational cubic fourfolds [8, 23, 27, 55], and it is conjectured that a cubic fourfold is rational if and only if it admits an associated K3 surface (i.e the transcendental cohomology is induced from a K3 surface) [23, 26, 27]. In §4.2 we provide a geometric set of generators for the algebraic lattice of a cubic fourfold that admits a non-symplectic automorphism of order three (see Proposition 4.18, Proposition 4.11, Proposition 4.13). Moreover, knowing the algebraic and the transcendental lattices, we determine if there exists an associated K3 surface via Torelli theorem. When there is an associated K3 surface we verify the conjecture proving the rationality of the cubic, we collect information related to the rationality of the cubic fourfolds in Table 1.

TABLE 1. Cubic fourfolds with a non-symplectic automorphism of order three, with notation as in [Theorem 4.1](#).

No.	$\text{rk}(A(Y))$	Associated K3	Rational
$\phi_3^1$	1	No	?
$\phi_3^5$	7	No	?
$\phi_3^7$	9	Yes	Yes
$\phi_3^2$	13	Yes	Yes

In [§5](#) we relate an automorphism of a cubic fourfold  $Y$  to the induced automorphism of  $J(Y)$  and  $J^t(Y)$ , where  $J(Y)$  denotes the associated Laza–Saccà–Voisin manifold, and  $J^t(Y)$  denotes the associated twisted Laza–Saccà–Voisin manifold. See [§2.4](#) for precise definitions. A crucial result by Mongardi–Onorati, see [Theorem 5.1](#), relates the primitive integral cohomology  $H_p^4(Y, \mathbb{Z})$  to the integral cohomology  $H^2(J(Y), \mathbb{Z})$ , and a similar result holds true for the twisted case, see [Proposition 5.2](#).

We obtain that any non-symplectic automorphism of a general cubic fourfold  $Y$  induces a non-symplectic automorphism of the associated  $J(Y)$  and  $J^t(Y)$ . Moreover, we characterize non-symplectic automorphisms of order two and three of Laza–Saccà–Voisin manifolds that are induced by non-symplectic automorphisms of a cubic fourfold.

**Theorem 1.3** (See [Theorem 5.12](#) and [Theorem 5.13](#)). *Let  $Y$  be a general cubic fourfold and let  $G \subset \text{Aut}(Y)$  be group of non-symplectic automorphisms of prime order  $p$ . Then the automorphisms of  $G$  induce non-symplectic automorphisms of the ihs manifolds of OG10 type  $J(Y)$  and  $J^t(Y)$ , and the pairs  $(\mathbf{L}^G, \mathbf{L}_G)$  for such induced actions are classified in [Table 2](#). Viceversa, if  $X$  is a general ihs manifold of OG10 type and  $G \subset \text{Aut}(X)$  is a group of non-symplectic automorphisms of prime order such that  $(\mathbf{L}^G, \mathbf{L}_G)$  appears in [Table 2](#), then there exists a smooth cubic fourfold  $Y$  and a group of non-symplectic automorphisms  $G \subset \text{Aut}(Y)$  such that  $X$  is birational to  $J(Y)$  or  $J^t(Y)$  and the group of automorphisms is induced by automorphisms of the cubic fourfold.*

TABLE 2. Invariant and coinvariant lattices  $(\mathbf{L}^G, \mathbf{L}_G)$  of an ihs manifold of OG10 type  $X$  induced from a cubic fourfold via LSV constructions. The column  $\text{Aut}(Y)$  refers to notation of [Theorem 4.1](#). The columns  $J(Y)$  and  $J^t(Y)$  indicate whether  $X$  is birational to the models.

$\text{Aut}(Y)$	$\mathbf{L}_G$	$\mathbf{L}^G$	$\text{sgn}(\mathbf{L}^G)$	$J(Y)$	$J^t(Y)$	$p$
$\phi_2^1$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1)^{\oplus 3}$	$\mathbf{U} \oplus \mathbf{E}_6(-2)$	(1, 7)	yes	yes	2
$\phi_2^3$	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 9}$	$[2] \oplus [-2] \oplus \mathbf{E}_6(-1) \oplus \mathbf{D}_4(-1)$	(1, 11)	yes	yes	2
$\phi_3^1$	$\mathbf{U}^{\oplus} \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U}$	(1, 1)	yes	no	3
$\phi_3^1$	$\mathbf{U}^{\oplus} \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U}(3)$	(1, 1)	no	yes	3
$\phi_3^5$	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6 \oplus \mathbf{A}_2^{\oplus 3}$	$\mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	(1, 7)	yes	no	3
$\phi_3^5$	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6 \oplus \mathbf{A}_2^{\oplus 3}$	$\mathbf{U}(3) \oplus \mathbf{E}_6^*(-1)$	(1, 7)	no	yes	3
$\phi_3^7$	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 5}$	$\mathbf{U}(3) \oplus \mathbf{E}_6^*(-1) \oplus \mathbf{A}_2(-1)$	(1, 9)	yes	yes	3

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Table 2, follows from previous page

$\text{Aut}(Y)$	$\mathbf{L}_G$	$\mathbf{L}^G$	$\text{sgn}(\mathbf{L}^G)$	$J(Y)$	$J^t(Y)$	$p$
$\phi_3^2$	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	$\mathbf{U}(3) \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	$(1, 13)$	yes	yes	3

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## 2. PRELIMINARIES

**2.1. Lattices.** A *lattice*  $L$  is a free  $\mathbb{Z}$ -module of finite rank with an integral bilinear form

$$(-, -) : L \times L \rightarrow \mathbb{Z}$$

which is non-degenerate. The *signature*  $(l_+, l_-)$  of  $L$  is the signature of the real extension of  $(-, -)$ . The lattice is *positive-definite* if  $l_- = 0$  and *negative-definite* if  $l_+ = 0$ , otherwise it is called *indefinite*. A lattice is *hyperbolic* if it is indefinite and  $l_+ = 1$ . A lattice  $L$  is called *even* if  $x^2 := (x, x) \in 2\mathbb{Z}$  for any  $x \in L$ . The divisibility  $\text{div}(x, L)$  of an element  $x \in L$  is the positive generator of the ideal  $\{(x, y) | y \in L\} \subseteq \mathbb{Z}$ .

Consider the *dual lattice*

$$L^\vee = \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} | (x, l) \in \mathbb{Z} \forall l \in L\}$$

and observe that  $L \subset L^\vee$  is a finite index subgroup, the quotient  $A_L := L^\vee / L$  is called the *discriminant group* of  $L$ . The *length*  $l(A_L)$  is the minimum number of generators of  $A_L$ . There is a well-defined  $\mathbb{Q}$ -bilinear form  $b_{A_L} : A_L \times A_L \rightarrow \mathbb{Q}/\mathbb{Z}$  with associated quadratic form  $q_{A_L} : A_L \times A_L \rightarrow \mathbb{Q}/\mathbb{Z}$ .

A lattice  $L$  is called *unimodular* if the group  $A_L$  is the identity, or equivalently  $\det(L) = \pm 1$ . The lattice is called *p-elementary* for a prime number  $p$  if  $A_L \cong (\mathbb{Z}/p\mathbb{Z})^k$  for some positive integer  $k$ .

**Definition 2.1.** Let  $L$  be an even lattice, define

$$\delta(L) := \begin{cases} 0 & \text{if } q_{A_L}(x) \in \mathbb{Z}/2\mathbb{Z} \text{ for any } x \in A_L \\ 1 & \text{otherwise} \end{cases}.$$

**Definition 2.2.** We call *short root* a vector  $v \in L$  such that  $v^2 = 2$  and  $\text{div}(v, L) = 1$ . We call *long root* a vector  $v \in L$  such that  $v^2 = 6$  and  $\text{div}(v, L) = 3$ .

A morphism of lattices  $L \rightarrow M$  is a linear map that preserves the bilinear forms, an injective morphism of lattices is called an *embedding*. An embedding  $L \hookrightarrow M$  is called *primitive* if its cokernel is free. If the embedding  $L \hookrightarrow M$  has finite index, then we say that  $M$  is an *overlattice* of  $L$ . Recall that by [46, Proposition 1.4.1] there is a bijective correspondence between overlattices of  $L$  and isotropic subgroups of  $A_L$ . Moreover, if  $L \hookrightarrow M$  is a primitive embedding, then  $M$  is an overlattice of  $L \oplus L^\perp$  and hence the primitive embeddings  $L \hookrightarrow M$  are determined by an anti-isometry between a subgroup of  $A_L$  and a subgroup of  $A_M$ , this subgroup is called the *gluing subgroup*, see [46, Proposition 1.15.1]. The *genus* of a lattice  $L$  is given by its signature  $\text{sgn}(L)$  and discriminant quadratic form  $q_{A_L}$ , or equivalently [46, Corollary 1.9.4] by the isometry class of  $\mathbf{U} \oplus L$

where  $\mathbf{U}$  denotes the unique even unimodular lattice of rank 2. Isometric lattices have the same genus, but lattices with the same genus might not be isometric. Typically, indefinite lattices tend to have a unique isometry class for a fixed genus and definite lattices tend to have more isometry classes for a fixed genus. If we consider a primitive embedding  $L \hookrightarrow M$  up to isometries of  $M$ , this determines uniquely the genus of the orthogonal complement  $L^\perp_M$  but it might not determine its isometry class, see e.g. [Example 2.3](#).

**Example 2.3.** Consider the following positive definite lattices

$$\mathbf{A} := \begin{pmatrix} 12 & 1 \\ 1 & 2 \end{pmatrix}, \mathbf{B} := \begin{pmatrix} 6 & 1 \\ 1 & 4 \end{pmatrix}.$$

The lattices  $\mathbf{U} \oplus \mathbf{A}$  and  $\mathbf{U} \oplus \mathbf{B}$  are isometric. On the other hand,  $\mathbf{A}$  and  $\mathbf{B}$  are not isometric but they have the same genus. This shows that  $\mathbf{U}$  admits two different primitive embeddings in  $M := \mathbf{U} \oplus \mathbf{A} \cong \mathbf{U} \oplus \mathbf{B}$  such that the respective orthogonal complements are not isometric. Note that since  $\mathbf{U}$  is unimodular, by [\[46, Proposition 1.15.1\]](#) there is a unique embedding of  $\mathbf{U}$  in  $M$  up to isometries.

There is a natural map  $O(L) \rightarrow O(A_L)$  between the group of isometries of the lattice  $L$  and the group of isometries of the discriminant group  $A_L$ . The Cartan-Dieudonné theorem [\[39, Theorem 9.10\]](#) guarantees that  $O(L \otimes_{\mathbb{Z}} \mathbb{R})$  is generated by reflections with respect to non-isotropic vectors, hence it is possible to give the following

**Definition 2.4.** The *spinor norm*  $\text{spin}: O(L) \rightarrow \{\pm 1\}$  is the group homomorphism that takes value  $+1$  on reflections for a vector  $v$  with  $v^2 < 0$ .

The kernel of the spinor norm is denoted by  $O^+(L)$  and consists of elements that preserve the orientation of a positive-definite subspace of  $L \otimes_{\mathbb{Z}} \mathbb{R}$  of maximal rank.

**Definition 2.5.** If  $G \subseteq O(L)$  is a group of isometries, we call  $L^G := \{x \in L \mid g(x) = x, \forall g \in G\}$  the *invariant lattice* and we call  $L_G := (L^G)^{\perp_L}$  the *coinvariant lattice*.

**Lemma 2.6** (see [\[4, Lemma 5.3\]](#), [\[43, Lemma 1.8\]](#)). *Let  $L$  be a lattice and  $G \subset O(L)$  the group generated by an isometry of prime order  $p$ . Then,  $m := \text{rk}(L_G)/(p-1)$  is an integer and*

$$\frac{L}{L^G \oplus L_G} \cong (\mathbb{Z}/p\mathbb{Z})^a$$

as groups, where  $a \leq m$ . Moreover, there are natural embeddings of  $\frac{L}{L^G \oplus L_G}$  in the discriminant groups  $A_{L^G}$  and  $A_{L_G}$ .

Let  $p$  be a prime number, we recall the following  $p$ -elementary lattices that will be useful in [§3](#):

$$\mathbf{K}_p := \begin{pmatrix} -(p+1)/2 & 1 \\ 1 & -2 \end{pmatrix}, \mathbf{H}_p := \begin{pmatrix} (p-1)/2 & 1 \\ 1 & -2 \end{pmatrix}.$$

We also consider the 3-elementary lattice

$$\mathbf{E}_6^*(3) := \begin{pmatrix} 4 & 2 & -1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 1 & -2 & 2 \\ -1 & 1 & 4 & 1 & -2 & -1 \\ 2 & 1 & 1 & 4 & -2 & -1 \\ -1 & -2 & -2 & -2 & 4 & -1 \\ 1 & 2 & -1 & -1 & -1 & 4 \end{pmatrix},$$

and the 17-elementary lattice

$$\mathbf{L}_{17} := \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 1 & 0 & -1 & 4 \end{pmatrix}.$$

Finally, we introduce the lattices

$$\mathbf{N}_{69} := \begin{pmatrix} 6 & 3 \\ 3 & -10 \end{pmatrix}, \mathbf{N}_{15} := \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}.$$

If  $L$  is a lattice and  $k$  is an integer, we denote by  $L(k)$  the lattice whose bilinear form is obtained by the one of  $L$  by multiplying by  $k$ . We denote by  $\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_n$  the positive-definite lattices associated to the ADE Dynkin diagrams. We recall that  $\mathbf{U}$  denotes the even unimodular lattice of rank two, and we denote by  $[k]$  the rank one lattice generated by an element of square  $k \in \mathbb{Z}$ .

**2.2. Automorphisms of ihs manifolds of OG10 type.** Let  $X$  be an ihs manifold with a marking  $\eta: \mathrm{H}^2(X, \mathbb{Z}) \rightarrow L$ , then there is a representation map

$$\mathrm{Aut}(X) \rightarrow \mathrm{O}(L),$$

which is injective if  $X$  is of OG10 type. The following theorem allows to study automorphisms in terms of Hodge isometries using the previous representation map.

**Proposition 2.7** (see, for instance, [45, §3]). *If  $(X, \eta)$  is a marked ihs manifold with marking  $\eta: \mathrm{H}^2(X, \mathbb{Z}) \rightarrow L$  and  $G \subseteq \mathrm{O}(L)$  is a group of non-symplectic isometries, then  $L^G \subseteq \mathrm{NS}(X)$  and  $\mathrm{T}(X) \subseteq L_G$ .*

We say that an ihs manifold  $X$  endowed with the action of  $G \subseteq \mathrm{Aut}(X)$  a group of non-symplectic automorphisms is *general* if one of the above inclusions is an equality, accordingly to [45, §3].

By [49] if  $X$  is an ihs manifold of OG10 type with a marking  $\eta: \mathrm{H}^2(X, \mathbb{Z}) \rightarrow \mathbf{L}$ , the monodromy group  $\mathrm{Mon}^2(X)$  coincides with the subgroup of index two of orientation preserving isometries  $\mathrm{O}^+(\mathbf{L}) \subset \mathrm{O}(\mathbf{L})$ .

**Lemma 2.8.** *If  $X$  is an ihs manifold of OG10 type and  $G \subseteq \mathrm{Aut}(X)$  is a group of non-symplectic automorphisms of prime order  $p$  then  $p \leq 23$ .*

*Proof.* Once a marking is fixed, we have  $G \subseteq \mathrm{O}^+(\mathbf{L})$  and by Lemma 2.6 we know that  $p - 1$  divides  $\mathrm{rk}(\mathbf{L}_G)$ . Since  $\mathrm{rk}(\mathbf{L}) = 24$  we conclude.  $\square$

Moreover, if  $X$  is an ihs manifold of OG10 type, *numerical wall divisors* and *numerical prime exceptional divisors* are computed in [42]:

$$\mathcal{W}_{\mathrm{OG10}}^{\mathrm{pe}x} = \{v \in \mathbf{L} \mid v^2 = -2\} \cup \{v \in \mathbf{L} \mid v^2 = -6, \mathrm{div}(v, \mathbf{L}) = 3\},$$

$$\mathcal{W}_{\mathrm{OG10}} = \mathcal{W}_{\mathrm{OG10}}^{\mathrm{pe}x} \cup \{v \in \mathbf{L} \mid v^2 = -4\} \cup \{v \in \mathbf{L} \mid v^2 = -24, \mathrm{div}(v, \mathbf{L}) = 3\}.$$

Hyperplanes orthogonal to classes in  $\mathcal{W}_{\mathrm{OG10}} \cap \mathrm{NS}(X)$  give a wall-and-chamber decomposition of the positive cone. The chamber that contains a Kähler class is the Kähler cone of  $X$ .



**2.3. Cubic fourfolds and their automorphisms.** Let  $Y \subset \mathbb{P}^5$  be a smooth cubic fourfold. The moduli space of smooth cubic fourfold  $\mathcal{M}$  is constructed in [28]. We denote by

$$\mathcal{D}/\Gamma = \{x \in \mathbb{P}(H_p^4(Y, \mathbb{Z}) \otimes \mathbb{C}) \mid x^2 = 0, x \cdot \bar{x} < 0\}^+ / \Gamma$$

the global period domain of cubic fourfolds. The period map

$$\mathcal{P} : \mathcal{M} \rightarrow \mathcal{D}/\Gamma$$

associates  $[Y] \in \mathcal{M}$  to its Hodge structure of the middle cohomology. To describe the image of this period map we define two divisors in  $\mathcal{D}/\Gamma$ . Namely the set of short roots in  $H_p^4(Y, \mathbb{Z})$  determines a  $\Gamma$ -invariant hyperplane arrangement  $H_2$  in  $\mathcal{D}$ . Let  $\mathcal{C}_2 := H_2/\Gamma \subset \mathcal{D}/\Gamma$  be the associated divisor. Similarly the set of long roots in  $H_p^4(Y, \mathbb{Z})$  determines a  $\Gamma$ -invariant hyperplane arrangement  $H_6$  in  $\mathcal{D}$ . Let  $\mathcal{C}_6 := H_6/\Gamma \subset \mathcal{D}/\Gamma$  be the associated divisor.

**Theorem 2.9** (Voisin, Hassett, Laza, Looijenga). *The period map*

$$\mathcal{P} : \mathcal{M} \rightarrow \mathcal{D}/\Gamma \setminus (\mathcal{C}_2 \cup \mathcal{C}_6)$$

*is an isomorphism.*

**Definition 2.10.** Let  $Y$  be a cubic fourfold, then a  $d$ -labeling is a rank 2 saturated sublattice  $K_d \subseteq A(Y)$  of discriminant  $d$  containing the class  $\eta_Y$ . Denote by  $\mathcal{C}_d$  the locus of the moduli space of smooth cubic fourfolds  $\mathcal{M}$  admitting a  $d$ -labeling.

Cubic fourfolds with a labeling form a divisor, as proved by Hassett [23].

**Theorem 2.11** (see [23, §3]). *The locus  $\mathcal{C}_d$  is non-empty if and only if  $d > 6$  and  $d \equiv 0, 2 \pmod{6}$ , if it is not empty then it is an irreducible divisor in the moduli space  $\mathcal{M}$ .*

Cubic fourfolds which are cohomologically related to a K3 surface are of remarkable interest.

**Definition 2.12.** We say that a polarized K3 surface  $(S, h)$  is *associated* with a cubic fourfold  $Y$  if there exists a  $d$ -labeling  $K_d$  and there is a Hodge isometry

$$H^4(Y, \mathbb{Z}) \supset K_d^\perp \cong h(-1)^\perp \subset H^2(S, \mathbb{Z})(-1).$$

In fact, the existence of an associated K3 surface only depends on the discriminant  $d$  of the  $d$ -labeling and not on the lattice  $K_d$ . Labelings for which there exists an associated K3 surface are called *admissible* and are numerically described in [23].

We have a representation map  $\text{Aut}(Y) \rightarrow \text{O}(H^4(Y, \mathbb{Z}))$  which is injective. The following theorem gives a precise description of the group of automorphisms of a smooth cubic fourfold.

**Theorem 2.13** (Hodge theoretic Torelli theorem, see [60]). *Let  $Y_1, Y_2$  be smooth cubic fourfolds. Suppose  $f : H^4(Y_2, \mathbb{Z}) \xrightarrow{\cong} H^4(Y_1, \mathbb{Z})$  is an isometry of polarized Hodge structures, then there exists a unique isomorphism  $\phi : Y_1 \xrightarrow{\cong} Y_2$  such that  $\phi^* = f$ . In particular, we have an isomorphism*

$$\text{Aut}(Y) \cong \text{O}_{Hdg}(H^4(Y, \mathbb{Z}), \eta_Y)$$

where  $\text{O}_{Hdg}(H^4(Y, \mathbb{Z}), \eta_Y)$  denotes the group of Hodge isometries of  $H^4(Y, \mathbb{Z})$  fixing the class  $\eta_Y$ .

For this reason it is natural to study isometries of  $H_p^4(Y, \mathbb{Z}) \cong \mathbf{F}$ . Similarly to Proposition 2.7, if  $G \subseteq \text{Aut}(Y)$  is a group of non-symplectic automorphisms then we have  $\mathbf{F}^G \subseteq A_p(Y)$  and  $T(Y) \subseteq \mathbf{F}_G$ . We say that a cubic fourfold  $Y$  with  $G \subseteq \text{Aut}(Y)$  a group of non-symplectic automorphisms is *general* if one of the above inclusions is an equality.



**2.4. Laza–Saccà–Voisin manifolds.** In this section we recall the construction of two geometric models of an ihs manifold of OG10 type due to Laza–Saccà–Voisin [31, 58]. Consider  $Y \subset \mathbb{P}^5$  a smooth cubic fourfold. The dual projective space  $(\mathbb{P}^5)^\vee$  parametrizes hyperplane sections  $Y_H = Y \cap H \subset Y$ , and  $U \subset (\mathbb{P}^5)^\vee$  is the open set of the smooth ones. We will often write  $\mathbb{P}^5$  instead of  $(\mathbb{P}^5)^\vee$  if it does not lead to confusion. Denote by

$$\text{Jac}(Y_H) = H^1(Y_H, \Omega_{Y_H}^2)^\vee / H_3(Y_H, \mathbb{Z})$$

the intermediate Jacobian of the hyperplane section, which is a principally polarized abelian fivefold. Over  $U$  consider the Donagi–Markman fibration

$$\pi_U : J_U(Y) \rightarrow U$$

whose fiber over the smooth hyperplane section  $Y_H$  consists of the intermediate Jacobian  $\text{Jac}(Y_H)$ . It is proved in [15] that  $J_U(Y)$  is quasi-projective and it admits a symplectic form  $\sigma_U$  for which  $\pi_U$  is a Lagrangian fibration.

Following [58], there is another Lagrangian fibration

$$\pi_U^t : J_U^t(Y) \rightarrow U$$

whose fiber over  $Y_H$  is given by the torsor  $\text{Jac}^1(Y_H)$  parametrizing degree 1 cycles.

**Theorem 2.14** (see [31, 56], and [58] for the twisted case). *Let  $Y$  be a smooth cubic fourfold. There exist smooth projective compactifications  $J(Y)$  and  $J^t(Y)$  of  $J_U(Y)$  and  $J_U^t(Y)$  respectively, with projective flat morphisms  $\pi : J(Y) \rightarrow \mathbb{P}^5$  and  $\pi^t : J^t(Y) \rightarrow \mathbb{P}^5$  extending  $\pi_U$  and  $\pi_U^t$  respectively. Moreover,  $J(Y)$  and  $J^t(Y)$  are smooth ihs manifolds of OG10 type.*

The compactification  $J(Y)$  is called the *LSV manifold* associated to  $Y$ , while  $J^t(Y)$  is called the *twisted LSV manifold* associated to  $Y$ . There is an effective relative theta divisor  $\Theta \subset J(Y)$  obtained as the closure of the union of theta divisors of the smooth fibers, it has the property that  $q_{J(Y)}(\Theta) = -2$ . There is another class  $L = \pi^* \mathcal{O}_{\mathbb{P}^5}(1)$ , that together with  $\Theta$  span a hyperbolic lattice  $\mathbf{U}_Y := \langle L, \Theta \rangle \subset \text{NS}(J(Y))$ . For a very general cubic fourfold  $Y$  one has  $\text{NS}(J(Y)) = \mathbf{U}_Y$ , in particular the family can not be locally complete since there are always two algebraic classes in the LSV manifolds. Similarly, in the twisted case there are classes  $L^t, \Theta^t \in \text{NS}(J^t(Y))$  spanning a lattice  $\mathbf{U}_Y^t := \langle L^t, \Theta^t \rangle \cong \mathbf{U}(3)$  and for a general cubic fourfold we have  $\text{NS}(J^t(Y)) = \mathbf{U}_Y^t$ .

### 3. CLASSIFICATION OF NON-SYMPLECTIC AUTOMORPHISMS OF OG10

In this section we consider the abstract lattice  $\mathbf{L} := \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 3} \oplus \mathbf{A}_2(-1)$  isometric to the second integral cohomology lattice of an ihs manifold of OG10 type. Recall that there is a unique embedding  $\mathbf{L} \hookrightarrow \mathbf{\Lambda}$  in the unimodular lattice  $\mathbf{\Lambda} := \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 5}$ , and the orthogonal complement is the rank two lattice  $\mathbf{L}^\perp \cong \mathbf{A}_2$ . The lattice  $\mathbf{L}$  is 3-elementary and the discriminant group is the cyclic group  $A_{\mathbf{L}} \cong \mathbb{Z}/3\mathbb{Z}$ . We classify non-symplectic automorphisms of prime order  $p$  by listing the possible invariant and coinvariant lattices of their induced action on the second integral cohomology  $\mathbf{L}$ . This is achieved giving the classification of isometries of the unimodular lattice  $\mathbf{\Lambda}$ .

**3.1. Isometries of  $\mathbf{\Lambda}$ .** We treat separately the case of involutions and higher order automorphisms.

**Proposition 3.1.** *Let  $G \subset \text{O}(\mathbf{\Lambda})$  be a group of isometries of order 2. If  $\text{sgn}(\mathbf{\Lambda}_G) = (2, \text{rk}(\mathbf{\Lambda}_G) - 2)$  or  $\text{sgn}(\mathbf{\Lambda}_G) = (3, \text{rk}(\mathbf{\Lambda}_G) - 3)$  then the pairs  $(\mathbf{\Lambda}^G, \mathbf{\Lambda}_G)$  appear in Table 5.*

*Proof.* Since  $G$  is cyclic of order 2 and  $\Lambda$  is unimodular, then  $\Lambda^G$  and  $\Lambda_G$  must be 2-elementary lattices and their discriminant groups are anti-isometric by Lemma 2.6, in particular they have the same length. We use [46, Theorem 3.6.2] to get all the possible isometry classes of such lattices by varying the signature, the length  $a$  and the invariant  $\delta$ . Any pair of such lattices are invariant and coinvariant lattices for the isometry that acts trivially on the invariant lattice and as  $-\text{id}$  on the coinvariant lattice. Since they are all 2-elementary and so  $-\text{id}$  acts trivially on the discriminant group.  $\square$

The following result is a direct consequence of [9, Theorem 1.1].

**Proposition 3.2.** *Let  $G \subset \text{O}(\Lambda)$  be a group of isometries of prime order  $3 \leq p \leq 23$ . If  $\text{sgn}(\Lambda_G) = (2, \text{rk}(\Lambda_G) - 2)$  then the pair  $(\Lambda^G, \Lambda_G)$  appears in Table 8.*

*Proof.* If  $G \subset \text{O}(\Lambda)$  has order  $p$  then by Lemma 2.6 the lattices  $\Lambda_G$  and  $\Lambda^G$  are  $p$ -elementary. The numerical criterion in [9, Theorem 1.1] determines which lattices can be invariant and coinvariant lattices of the group of isometries  $G$ , then we conclude combining [46, Corollary 1.13.5] and [54, Section 1].  $\square$

**Remark 3.3.** We point out that there is only one conjugacy class for such isometries, apart from the case  $p = 23$  where there are three conjugacy classes which are determined by the Steinitz class of  $\Lambda_G$ . The reader can refer [9] for more details.

**3.2. Invariant and coinvariant lattices of  $\mathbf{L}$ .** In this subsection we relate isometries of the lattice  $\mathbf{L}$  with isometries of the lattice  $\Lambda$ , depending on the image of the group  $G \subset \text{O}(\mathbf{L})$  via the map  $\text{O}(\mathbf{L}) \rightarrow \text{O}(A_{\mathbf{L}})$ . In this section we fix the (unique) primitive embedding  $\mathbf{L} \hookrightarrow \Lambda$ , whose orthogonal complement is  $\mathbf{L}^\perp \cong \mathbf{A}_2$ .

**Lemma 3.4.** *If  $\varphi \in \text{O}(\mathbf{L})$  is an isometry such that  $\overline{\varphi} = \text{id} \in \text{O}(A_{\mathbf{L}})$  then it extends to an element  $\tilde{\varphi} \in \text{O}(\Lambda)$  acting trivially on  $\mathbf{L}^\perp \subset \Lambda$ . If  $\varphi \in \text{O}(\mathbf{L})$  is an isometry such that  $\overline{\varphi} = -\text{id} \in \text{O}(A_{\mathbf{L}})$ , then  $\varphi$  extends to an isometry  $\tilde{\varphi} \in \text{O}(\Lambda)$  that acts on the rank 2 lattice  $\mathbf{L}^\perp \subset \Lambda$  permuting the generators.*

*Proof.* Let  $a, b$  be generators of  $\mathbf{A}_2(-1) \subset \mathbf{L}$  and consider the generator  $[\frac{a-b}{3}] = [\frac{a+2b}{3}]$  of  $\mathbf{L} \cong \mathbb{Z}/3\mathbb{Z}$ . If  $\varphi \in \text{O}(\mathbf{L})$  is such that  $\overline{\varphi} = \text{id}$  then  $\overline{\varphi}([\frac{a-b}{3}]) = [\frac{a-b}{3}]$  hence  $\varphi(a-b) = a-b+3w$  with  $w \in \mathbf{L}$ . Let  $c, d$  be generators of  $\mathbf{L}^\perp \cong \mathbf{A}_2$ , its discriminant group is also  $\mathbb{Z}/3\mathbb{Z}$  and it is generated by  $[\frac{c-d}{3}]$  with discriminant form given by  $q(\frac{c-d}{3}) = 2/3$ . Notice that  $\mathbf{L} \oplus \mathbf{A}_2$  has an overlattice isometric to  $\Lambda$  which is generated by  $\mathbf{L}$ ,  $\frac{a-b+c-d}{3}$  and  $\frac{a+2b+c+2d}{3}$ . We extend  $\varphi$  to  $\mathbf{L} \oplus \mathbf{A}_2$  by imposing  $\varphi(c) = c$  and  $\varphi(d) = d$  and we obtain an extension  $\tilde{\varphi}$  of  $\varphi$  on  $\Lambda$  as follows:

$$\tilde{\varphi}\left(\frac{a-b+c-d}{3}\right) = \frac{\varphi(a-b)+c-d}{3}$$

and

$$\tilde{\varphi}\left(\frac{a+2b+c+2d}{3}\right) = \frac{\varphi(a+2b)+c+2d}{3}.$$

If  $\varphi \in \text{O}(\mathbf{L})$  is such that  $\overline{\varphi} = -\text{id}$  then  $\overline{\varphi}([\frac{a-b}{3}]) = [\frac{b-a}{3}]$  hence we extend  $\varphi$  to  $\mathbf{L} \oplus \mathbf{A}_2$  by imposing  $\varphi(c) = d$  and  $\varphi(d) = c$  and we obtain an extension  $\tilde{\varphi}$  of  $\varphi$  on  $\Lambda$  as follows:

$$\tilde{\varphi}\left(\frac{a-b+c-d}{3}\right) = \frac{\varphi(a-b)+d-c}{3}$$

and

$$\tilde{\varphi}\left(\frac{a+2b+c+2d}{3}\right) = \frac{\varphi(a+2b)+d+2c}{3}.$$

□

**Proposition 3.5.** *Let  $G \subset O(\mathbf{L})$  be a subgroup of prime order  $p$  and consider its image  $\overline{G} \subset O(A_{\mathbf{L}})$ . Let  $c$  and  $d$  be the generators of  $\mathbf{A}_2 = \mathbf{L}^\perp \subset \mathbf{A}$ .*

- *If  $|\overline{G}| = 1$  there exists  $G' \subset O(\mathbf{A})$  a subgroup of order two such that  $G'$  restricts to  $G$  on  $\mathbf{L}$  and  $\mathbf{L}_G = \mathbf{A}_{G'}$ . In particular, we have  $\text{sgn}(\mathbf{L}_G) = \text{sgn}(\mathbf{A}_{G'})$  and  $\text{sgn}(\mathbf{L}^G) = \text{sgn}(\mathbf{A}^{G'}) - (2, 0)$ .*
- *If  $|\overline{G}| = 2$  there exists  $G' \subset O(\mathbf{A})$  a subgroup of order two such that  $G'$  restricts to  $G$  on  $\mathbf{L}$  and  $\mathbf{L}_G = (c-d)^\perp \subset \mathbf{A}_{G'}$ ,  $\mathbf{L}^G \cong (c+d)^\perp \subset \mathbf{A}^{G'}$ . In particular, we have  $\text{sgn}(\mathbf{L}_G) = \text{sgn}(\mathbf{A}_{G'}) - (1, 0)$  and  $\text{sgn}(\mathbf{L}^G) = \text{sgn}(\mathbf{A}^{G'}) - (1, 0)$ .*

*Proof.* We apply Lemma 3.4 to a generator of  $G$ . □

**Proposition 3.6.** *Let  $G \subset O(\mathbf{L})$  be a group of prime order  $p$  generated by a non-symplectic isometry, then there exists a marked pair  $(X, \eta)$  of OG10 type such that  $G \subset \text{Aut}(X)$  is a group of nonsymplectic automorphisms of prime order  $p$ .*

*Proof.* A generator  $\varphi$  of  $G$  is non-symplectic and hence one can endow  $\mathbf{L}_{\mathbb{C}} = \mathbf{L} \otimes_{\mathbb{Z}} \mathbb{C}$  with a weight-two Hodge structure such that  $\mathbf{L}^G = \mathbf{L}_{\mathbb{C}}^{1,1} \cap \mathbf{L}$ . By the surjectivity of the period map there exists a manifold  $X$  of OG10 type and a marking  $H^2(X, \mathbb{Z}) \cong \mathbf{L}$  which is an isomorphism of Hodge structures. By construction  $G$  consists of Hodge isometries, moreover since all the algebraic classes are fixed then the positive cone is fixed, and so the Kähler cone is. In this case we know that  $\text{Mon}^2(X) = O^+(\mathbf{L})$ , and we want to prove that  $G \subset O^+(\mathbf{L})$ . This is clear when  $p \neq 2$  since it is odd and  $\varphi^p = \text{id}$ . If  $p = 2$  by [20, Lemma 2.4] we have  $\text{spin}(\varphi) = +1$ , and  $\text{sign}(\mathbf{L}_G) = (2, \text{rk}(\mathbf{L}_G) - 2)$ , so that  $\varphi \in O^+(\mathbf{L})$  in any case. Since  $G \subset \text{Mon}_{\text{Hdg}}^2(X)$  and a Kähler class is preserved by  $G$  we can conclude using the Hodge-theoretic Torelli theorem [34, Theorem 1.3], as the representation map on the second cohomology is injective for manifolds of OG10 type. In particular,  $X$  is projective by Huybrecht's projectivity criterion. □

**Theorem 3.7.** *Let  $G \subset O(\mathbf{L})$  be a non-symplectic group of prime order  $p$ . Then there exists a manifold  $X$  of OG10 type such that  $G \subset \text{Aut}(X)$  if and only if  $2 \leq p \leq 23$  and the pairs  $(\mathbf{L}^G, \mathbf{L}_G)$  appear in Table 6 and Table 7 for  $p = 2$ , and in Table 9 for  $p \geq 3$ .*

*Proof.* Let  $G \subset \text{Aut}(X)$  be a subgroup of prime order  $p$ , notice that by Lemma 2.8 we have  $p \leq 23$ , and consider the induced action  $G \subset O(\mathbf{L})$ . According to Proposition 3.5, one can extend it to an action  $G' \subset O(\mathbf{A})$ . The two different extensions lead to the possible cases where  $\mathbf{L}_G = \mathbf{A}_{G'}$  or there are inclusions  $\mathbf{L}_G \subset \mathbf{A}_{G'}$  and  $\mathbf{L}^G \subset \mathbf{A}^{G'}$  with complement of rank 1.

In the first case we have  $\mathbf{L}^G = \mathbf{A}_2^{\perp \mathbf{A}^{G'}}$  since the orthogonal complement of the embedding  $\mathbf{L} \hookrightarrow \mathbf{A}$  is isometric to  $\mathbf{A}_2$ . We compute the possible embeddings  $\mathbf{A}_2 \hookrightarrow \mathbf{A}^{G'}$  and orthogonal complements for  $\mathbf{A}^{G'}$  in Table 5 and Table 8.

In the second case (it happens only for  $p = 2$ ) we consider the unique primitive embedding  $\mathbf{A}_2 \hookrightarrow \mathbf{A}$  and observe that in this case  $[2] = \langle a+b \rangle$  is  $G'$ -invariant, where  $a, b$  are generators of  $\mathbf{A}_2 = \mathbf{L}^\perp \subset \mathbf{A}$ , since  $G'$  permutes them. We consider the lattices  $\mathbf{A}^{G'}$  in Table 5 and compute the list of possible  $\mathbf{L}^G = [2]^{\perp \mathbf{A}^{G'}}$  for all the primitive embeddings  $[2] \hookrightarrow \mathbf{A}^{G'}$ . Finally, we obtain  $\mathbf{L}_G$  as the orthogonal complement of the primitive embedding  $\mathbf{L}^G \hookrightarrow \mathbf{L}$  when such an embedding exists.

Orthogonal complements of the previous embeddings are uniquely determined up to isometry because of [46, Proposition 1.14.2].

Viceversa, given a pair  $(\mathbf{L}^G, \mathbf{L}_G)$ , one can endow  $\mathbf{L}$  with a Hodge structure that makes  $\mathbf{L}^G$  and  $\mathbf{L}_G$  the invariant and coinvariant lattices of a non-symplectic automorphism, then we conclude by Proposition 3.6.  $\square$

#### 4. NON-SYMPLECTIC AUTOMORPHISMS OF ORDER THREE OF A CUBIC FOURFOLD

Finite order automorphisms of a cubic fourfold  $Y \subset \mathbb{P}^5$  are linear transformations of  $\mathbb{P}^5$  that restrict to  $Y$ .

According to [19, Theorem 2.8], there exist four families of non-symplectic automorphisms of order three of a cubic fourfold, and there are no non-symplectic automorphisms of prime order greater than three. In this section we give a lattice theoretic classification of non-symplectic automorphisms of order three of a smooth cubic fourfold.

The lattice-theoretic classification is carried on with the same techniques used for ihs manifolds of OG10 type. Definite lattices are often not uniquely determined by their genus (see Example 2.3) and enumeration of definite lattices is demanding, but the rank of lattices we look for is known thanks to the construction of moduli spaces of cubic fourfolds with a group action given in [59].

**4.1. The algebraic and transcendental lattices.** In this section we compute the algebraic and the transcendental lattice of a general cubic fourfold that admits a non-symplectic automorphism of order three (see Theorem 4.6). Here below we recall a classification result of prime order non-symplectic automorphisms of a cubic fourfold  $Y \subset \mathbb{P}^5$  in terms of the induced action of  $\mathbb{P}^5$  and giving a general equation of  $Y$ . In the following theorem we denote by  $\phi_i^j$  the  $j$ -th automorphism of order  $i$ , respecting the numbering of the list that is given in [59].

**Theorem 4.1** (see [19], and also [16, 59]). *Let  $Y = \{F = 0\} \subset \mathbb{P}^5$  be a smooth cubic fourfold with a non-symplectic automorphism  $\phi \in \text{Aut}(Y)$  of prime order  $p$ . After a linear change of coordinates that diagonalizes  $\phi$  we have  $\phi(x_0 : \dots : x_5) = (\xi^{\sigma_0} x_0 : \dots : \xi^{\sigma_5} x_5)$  and we denote by  $(\sigma_0, \dots, \sigma_5)$  such an action. If  $d$  denotes the dimension of the family of cubic fourfolds with the automorphism  $\phi$ , then we have the following possibilities:*

- $\phi_2^1$ :  $p = 2$ ,  $\sigma = (0, 0, 0, 0, 0, 1)$ ,  $d = 14$ ,

$$F = L_3(x_0, \dots, x_4) + x_5^2 L_1(x_0, \dots, x_4),$$

- $\phi_2^3$ :  $p = 2$ ,  $\sigma = (0, 0, 0, 1, 1, 1)$ ,  $d = 10$ ,

$$F = L_3(x_0, x_1, x_2) + x_0 L_2(x_3, x_4, x_5) + x_1 M_2(x_3, x_4, x_5) + x_2 N_2(x_3, x_4, x_5),$$

- $\phi_3^1$ :  $p = 3$ ,  $\sigma = (0, 0, 0, 0, 0, 1)$ ,  $d = 10$ ,

$$F = L_3(x_0, \dots, x_4) + x_5^3,$$

- $\phi_3^2$ :  $p = 3$ ,  $\sigma = (0, 0, 0, 0, 1, 1)$ ,  $d = 4$ ,

$$F = L_3(x_0, \dots, x_3) + M_3(x_4, x_5),$$

- $\phi_3^5$ :  $p = 3$ ,  $\sigma = (0, 0, 0, 1, 1, 2)$ ,  $d = 7$ ,

$$F = L_3(x_0, x_1, x_2) + M_3(x_3, x_4) + x_5^3 + x_3 x_5 L_1(x_0, x_1, x_2) + x_4 x_5 M_1(x_0, x_1, x_2),$$

- $\phi_3^7$ :  $p = 3$ ,  $\sigma = (0, 0, 1, 1, 2, 2)$ ,  $d = 6$

$$F = x_2 L_2(x_0, x_1) + x_3 M_2(x_0, x_1) + x_4^2 L_1(x_0, x_1) + x_4 x_5 M_1(x_0, x_1) + x_5^2 N_1(x_0, x_1) + x_4 N_2(x_2, x_3) + x_5 O_2(x_2, x_3)$$

where  $L_i, M_i, N_i$  and  $O_i$  are homogeneous polynomials of degree  $i$ .

From now on, non-symplectic automorphisms of order three of a smooth cubic fourfolds are denoted by  $\phi_3^1, \phi_3^2, \phi_3^5, \phi_3^7$ .

Recall that if  $Y$  is a smooth cubic fourfold and if  $G \subset \text{Aut}(Y)$  is a finite group, there is an irreducible moduli space  $\mathcal{M}_G$  of cubic fourfolds with an action of  $G$ , as constructed in [59] via GIT. Let  $Y \in \mathcal{M}_G$  and let  $\xi$  be the character of the action on  $H^{3,1}(Y)$ , denote by  $(\mathbf{F} \otimes \mathbb{C})_\xi$  the  $\xi$ -eigenspace for a fixed marking  $\gamma : H_p^4(Y, \mathbb{Z}) \rightarrow \mathbf{F}$ .

**Theorem 4.2** (see [59]). *There is an isomorphism*

$$\mathcal{P}_G : \mathcal{M}_G \xrightarrow{\cong} (\mathcal{D} \setminus \mathcal{H})/\Gamma$$

where  $\mathcal{D}$  is the period domain associated with  $(\mathbf{F} \otimes \mathbb{C})_\xi$ ,  $\mathcal{H}$  is a  $\Gamma$ -invariant hyperplane arrangement and  $\Gamma$  is an arithmetic group acting properly and discontinuously on  $\mathcal{D}$ .

In the previous statement  $\mathcal{D}$  is a symmetric domain of type IV if  $\xi = \bar{\xi}$ , and a complex hyperbolic ball otherwise.

The following result gives properties of invariant and coinvariant lattices for a prime order automorphism of a cubic fourfold.

**Lemma 4.3.** *Let  $Y$  be a cubic fourfold and let  $G \subset \text{Aut}(Y)$  be a group of prime order  $p$ . Then*

- $A_{\mathbf{F}_G} \cong (\mathbb{Z}/p\mathbb{Z})^a$  and  $A_{\mathbf{F}^G} \cong (\mathbb{Z}/p\mathbb{Z})^a \oplus \mathbb{Z}/3\mathbb{Z}$  for some integer  $a \geq 0$ , if  $p \neq 3$ .
- $A_{\mathbf{F}_G} \cong (\mathbb{Z}/p\mathbb{Z})^{a-1}$  and  $A_{\mathbf{F}^G} \cong (\mathbb{Z}/p\mathbb{Z})^{a \pm 1}$  for some integer  $0 \leq a+1 \leq \min(\text{rk } \mathbf{F}_G, \text{rk } \mathbf{F}^G)$ , if  $p = 3$ .
- $\mathbf{F}_G$  is positive definite if  $G$  is symplectic
- $\mathbf{F}^G$  is positive definite if  $G$  is non-symplectic

*Proof.* The action of  $G$  on the unimodular lattice  $H^4(Y, \mathbb{Z})$  is trivial on  $\langle \eta_Y \rangle \cong [3]$  hence the action is trivial on the discriminant group  $A_{\langle \eta_Y \rangle} \cong A_{\mathbf{F}}$ . As a consequence, we have an isometry  $H^4(Y, \mathbb{Z})_G \cong \mathbf{F}_G$  and then  $\mathbf{F}_G$  is  $p$ -elementary by Lemma 2.6. The possible discriminant groups of  $\mathbf{F}^G$  are determined by [46, Proposition 1.5.1]. The statement about the signatures is standard.  $\square$

**Lemma 4.4.** *Let  $S$  be a lattice such that  $S \hookrightarrow \mathbf{F}$  is a primitive embedding with embedding subgroup  $A_{\mathbf{F}} \cong \mathbb{Z}/3\mathbb{Z}$ . Denote by  $\mathbf{N}$  the smallest primitive lattice containing  $\mathbf{S} \oplus \langle \eta_Y \rangle \subset H^4(Y, \mathbb{Z})$ . Then there exists  $v \in \mathbf{S}$  with  $v^2 = 6$  and divisibility  $\text{div}(v, \mathbf{F}) = 3$  if and only if there exists  $w \in \mathbf{N}$  such that  $w^2 = 1$ .*

*Proof.* The gluing subgroup of  $\mathbf{F} \oplus \langle \eta_Y \rangle \subset H^4(Y, \mathbb{Z})$  is  $A_{\langle \eta \rangle} \cong \mathbb{Z}/3\mathbb{Z}$  and, by hypothesis, also the gluing subgroup  $\mathbf{S} \oplus \langle \eta_Y \rangle \subset \mathbf{N}$  is  $A_{\langle \eta_Y \rangle} \cong \mathbb{Z}/3\mathbb{Z}$ . Suppose there exists a vector  $v \in \mathbf{S}$  with  $v^2 = 6$  and  $\text{div}(v, \mathbf{F}) = 3$ , then  $w = \frac{v - \eta_Y}{3} \in \mathbf{N}$  is such that  $w^2 = 1$ . Viceversa, if there is  $w \in \mathbf{N}$  such that  $w^2 = 1$  then  $v = 3w + \eta_Y \in \mathbf{S}$  and  $v^2 = 6$  and  $\text{div}(v, \mathbf{F}) = 3$ .  $\square$

The following lemma gives a formula to compute the rank of  $\mathbf{F}^G$  and  $\mathbf{F}_G$  knowing the dimension of  $\mathcal{M}_G$ .

**Lemma 4.5.** *Let  $Y$  be a cubic fourfold and let  $G \subset \text{Aut}(Y)$  be a finite group of prime order  $p$ . Let  $\xi$  be the associated character. Let  $\mathcal{M}_G$  be the moduli space of cubic fourfolds with an action of  $G$ . The following holds:*

- $\text{rk } \mathbf{F}^G = \dim \mathcal{M}_G + 2$  if  $G$  is symplectic,
- $\text{rk } \mathbf{F}_G = \dim \mathcal{M}_G + 2$  if  $G$  is non-symplectic and  $p = 2$ ,
- $\text{rk } \mathbf{F}_G = 2 \dim \mathcal{M}_G + 2$  if  $G$  is non-symplectic and  $p \geq 3$ .

*Proof.* By [Theorem 4.2](#) the dimension of  $\mathcal{M}_G$  equals the dimension of the associated symmetric domain, which is given by  $\dim(\mathbf{F} \otimes \mathbb{C})_\xi - 2$  if  $\xi = \bar{\xi}$  and by  $\dim(\mathbf{F} \otimes \mathbb{C})_\xi - 1$  if  $\xi \neq \bar{\xi}$ , where we denote by  $(\mathbf{F} \otimes \mathbb{C})_\xi$  the  $\xi$ -eigenspace. Suppose  $G$  is generated by a symplectic automorphism, then  $\xi = \bar{\xi} \equiv 1$  and  $\text{rk } \mathbf{F}^G = \dim(\mathbf{F} \otimes \mathbb{C})_1$ . Let now  $G$  be generated by a non-symplectic automorphism of order  $p$ . If  $p = 2$  we have  $\xi = \bar{\xi}$  and  $\text{rk } \mathbf{F}_G = \dim(\mathbf{F} \otimes \mathbb{C})_\xi$ . If  $p \geq 3$  then  $\xi \neq \bar{\xi}$  and  $\text{rk } \mathbf{F}_G = \dim(\mathbf{F} \otimes \mathbb{C})_\xi + \dim(\mathbf{F} \otimes \mathbb{C})_{\bar{\xi}} = 2 \dim(\mathbf{F} \otimes \mathbb{C})_\xi$ .  $\square$

Non-symplectic involutions on cubic fourfolds are studied by Marquand in [\[35\]](#) with a similar approach to the one that we adopt for non-symplectic automorphisms of order three. We recall that if  $Y$  is general and  $G \subset \text{Aut}(Y)$  is a group of non-symplectic automorphisms then we have  $T(Y) = \mathbf{F}_G$  and  $\mathbf{F}^G = A_p(Y)$ .

**Theorem 4.6.** *Let  $G \subset \text{O}(\mathbf{F})$  be a group of non-symplectic isometries of order three. Then there exists a cubic fourfold  $Y$  such that  $G \subset \text{Aut}(Y)$  if and only if the pairs  $(\mathbf{F}^G, \mathbf{F}_G)$  appears in [Table 3](#). For such a general  $Y$  the algebraic lattice  $A(Y)$  appear in [Table 4](#), and the class of  $\eta_Y$  has the following coordinates expressed in a basis of  $A(Y)$ :*

- $\eta_Y = (1)$  for  $\phi_3^1$ ;
- $\eta_Y = (1, 0, 0, 0, 0, 0, 0)$  for  $\phi_3^5$ ;
- $\eta_Y = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$  for  $\phi_3^7$ ;
- $\eta_Y = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$  for  $\phi_3^2$ .

*Proof.* Using [Lemma 4.5](#) we can determine the rank of  $\mathbf{F}^G$  and  $\mathbf{F}_G$ , since the dimensions of  $\mathcal{M}_G$  are available at [\[59, Theorem 6.1\]](#). Consider the composition of the two natural primitive embeddings  $\mathbf{F}_G \hookrightarrow \mathbf{F} \hookrightarrow H^4(Y, \mathbb{Z})$  and notice that  $\mathbf{F}_G \cong H^4(Y, \mathbb{Z})_G$ . Since  $H^4(Y, \mathbb{Z})$  is unimodular, we can apply [\[9, Theorem 1.1\]](#) to determine the list of the possible lengths of  $\mathbf{F}_G$ . Note that since  $\mathbf{F}_G$  is indefinite and 3-elementary, to know the length is equivalent to determine its isometry class. To determine the length of  $\mathbf{F}_G$  we can consider  $\mathbf{F}^G = A_p(Y)$ , and by [Lemma 4.3](#) determines the list of possible lengths of  $\mathbf{F}^G$ . We know by Torelli theorem for cubic fourfolds [Theorem 2.9](#) that  $A_p(Y)$  does not contain short roots and long roots. We exclude isometry classes of lattices that contain short roots via the sphere packing argument used in [\[18\]](#) or by computer algebra (we refer to OSCAR [\[50\]](#)) for the remaining cases. By [Lemma 4.4](#), to check the existence of long roots in  $\mathbf{F}^G$  is equivalent to determine the gluing isometry between  $A_{\mathbf{F}^G}$  and  $A_{\langle \eta_Y \rangle}$ . This is equivalent to determine the isometry class of  $A(Y)$ . To do that, we compute via computer algebra, the possible isometry classes for  $A(Y)$  and discard the ones containing a vector of square one. Then, for the remaining cases we compute the orthogonal complement of all the possible vectors of square three (candidates for  $\eta_Y$ ) and check if they are isometric to  $A_p(Y)$ . This happens only in one case, as we expect by [Theorem 2.13](#), this gives  $A(Y)$  and the coordinates of  $\eta_Y$  in  $A(Y)$ . The length of  $T(Y)$  equals the one of  $A(Y)$  because they are orthogonal complements in a unimodular lattice and this allows us to know the isometry class of  $T(Y)$ . The result of these computations is summarized in [Table 4](#) and [Table 3](#), where the cases are listed in an increasing order for  $\text{rk}(\mathbf{F}^G)$ .

Given such a pair  $(\mathbf{F}^G, \mathbf{F}_G)$ , the existence of a cubic fourfold with those invariant and coinvariant lattices is guaranteed using [Theorem 2.13](#) with the fact that the action of the group is trivial on the discriminant group of  $\mathbf{F}$ .  $\square$

TABLE 3. Pairs  $(\mathbf{F}^G, \mathbf{F}_G)$  for a cubic fourfold with a non-symplectic automorphism of order three.

No.	$\text{rk}(\mathbf{F}^G)$	$\mathbf{F}_G \supseteq T(Y)$	$\mathbf{F}^G \subseteq A_p(Y)$	$\text{sgn}((\mathbf{F}_G))$	$l(\mathbf{F}^G)$
$\phi_3^1$	0	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8^{\oplus 2} \oplus \mathbf{A}_2$	$\{0\}$	$(20, 2)$	0
$\phi_3^5$	6	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6 \oplus \mathbf{A}_2^{\oplus 3}$	$\begin{bmatrix} 4 & 2 & -1 & 1 & 2 & -2 \\ 2 & 4 & 1 & 2 & 1 & -1 \\ -1 & 1 & 4 & 2 & -2 & -1 \\ 1 & 2 & 2 & 4 & -1 & -2 \\ 2 & 1 & -2 & -1 & 4 & -1 \\ -2 & -1 & -1 & -2 & -1 & 4 \end{bmatrix}$	$(14, 2)$	5
$\phi_3^7$	8	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2^{\oplus 5}$	$\begin{bmatrix} 6 & 3 & 3 & 3 & 3 & 3 & -3 & 3 \\ 3 & 6 & 0 & 0 & 0 & 0 & -3 & 0 \\ 3 & 0 & 6 & 0 & 3 & 0 & 0 & 3 \\ 3 & 0 & 0 & 6 & 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 & 6 & 0 & 0 & 3 \\ 3 & 0 & 0 & 3 & 0 & 6 & -3 & 0 \\ -3 & -3 & 0 & 0 & 0 & -3 & 6 & 0 \\ 3 & 0 & 3 & 3 & 3 & 0 & 0 & 6 \end{bmatrix}$	$(12, 2)$	8
$\phi_3^2$	12	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2^{\oplus 3}$	$\begin{bmatrix} 4 & 1 & -2 & -2 & 1 & 2 & -2 & -1 & -2 & -2 & -2 & 2 \\ 1 & 4 & -2 & -1 & 1 & 0 & 0 & 1 & -2 & 1 & 1 & 2 \\ -2 & -2 & 4 & 2 & -2 & 0 & 0 & 1 & 1 & 1 & 1 & -1 \\ -2 & -1 & 2 & 4 & 0 & -2 & 1 & 2 & 0 & 2 & 0 & -2 \\ 1 & 1 & -2 & 0 & 4 & -1 & -1 & -1 & -1 & 1 & 0 & -1 \\ 2 & 0 & 0 & -2 & -1 & 4 & -2 & 0 & 0 & -1 & 0 & 2 \\ -2 & 0 & 0 & 1 & -1 & -2 & 4 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 2 & -1 & 0 & 1 & 4 & 0 & 2 & 1 & 0 \\ -2 & -2 & 1 & 0 & -1 & 0 & 0 & 0 & 4 & 0 & 0 & -2 \\ -2 & 1 & 1 & 2 & 1 & -1 & 0 & 2 & 0 & 4 & 2 & -1 \\ -2 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 4 & 0 \\ 2 & 2 & -1 & -2 & -1 & 2 & 0 & 0 & -2 & -1 & 0 & 4 \end{bmatrix}$	$(8, 2)$	6

 TABLE 4. Pairs  $(A(Y), T(Y))$  for a general cubic fourfold with a non-symplectic automorphism of order three.

No.	$\text{rk}(A(Y))$	$T(Y)$	$A(Y)$	$\text{sgn}(T(Y))$	$l(A(Y))$
$\phi_3^1$	1	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8^{\oplus 2} \oplus \mathbf{A}_2$	$[3]$	$(20, 2)$	1
$\phi_3^5$	7	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6 \oplus \mathbf{A}_2^{\oplus 3}$	$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & -1 & 1 & 2 & -2 \\ 0 & 2 & 4 & 1 & 2 & 1 & -1 \\ 0 & -1 & 1 & 4 & 2 & -2 & -1 \\ 0 & 1 & 2 & 2 & 4 & -1 & -2 \\ 0 & 2 & 1 & -2 & -1 & 4 & -1 \\ 0 & -2 & -1 & -1 & -2 & -1 & 4 \end{bmatrix}$	$(14, 2)$	6
$\phi_3^7$	9	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2^{\oplus 5}$	$\begin{bmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$	$(12, 2)$	7
$\phi_3^2$	13	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2^{\oplus 3}$	$\begin{bmatrix} 3 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 3 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 0 & 1 & -1 & 1 \\ -1 & 1 & 3 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 & 0 & 2 \\ 1 & -1 & -1 & 3 & -1 & 1 & -1 & 1 & 0 & 1 & -1 & 0 & -1 \\ -1 & 1 & -1 & -1 & 3 & -1 & 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 3 & 1 & 0 & -1 & 0 & -2 & 0 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 3 & 0 & 1 & -1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 3 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & -1 & 1 & 2 & 4 & 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 1 & 3 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 1 & -2 & -1 & 1 & 1 & 1 & 4 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 1 & 0 & 4 \end{bmatrix}$	$(8, 2)$	5



#### 4.2. Geometry of cubic fourfolds with a non-symplectic automorphism of order three.

The algebraic lattice  $A(Y)$  of a cubic fourfold  $Y$  encodes geometric information about the cubic and its conjectural rationality, since knowing the algebraic lattice allows to determine on which Hassett divisors the cubic lies. The purpose of this subsection is to describe the generators of the algebraic lattices  $A(Y)$  in terms of the geometry of the general cubic fourfold that admits a non-symplectic automorphism of order three.

**4.3. Cubic fourfolds with automorphism  $\phi_3^1$  or  $\phi_3^5$ .** A general cubic fourfold  $Y$  with the non-symplectic automorphism of order three  $\phi_3^1$  or  $\phi_3^5$  has a primitive algebraic lattice  $A_p(Y)$  computed in Table 3.

We show that such cubic fourfolds have no associated K3 surface and contain no planes.

**Lemma 4.7.** *Let  $Y$  be a general cubic fourfold with automorphism of order three  $\phi_3^1$  or  $\phi_3^5$ , then  $Y$  does not have an associated K3 surface.*

*Proof.* Suppose the cubic has an associated K3 surface, then there exists a primitive embedding of  $T(Y)(-1)$  in the K3 lattice, which is given by  $\mathbf{U}^{\oplus 3} \oplus \mathbf{E}_8(-1)^{\oplus 2}$ . It is easy to see that this is not possible for the lattices  $T(Y)$  in Table 4 corresponding to  $\phi_3^1$  and  $\phi_3^5$ .  $\square$

**Remark 4.8.** According to [24], there is also a notion of twisted K3 surface associated to a cubic fourfold. We observe that if  $Y$  is a general cubic fourfold with the automorphism  $\phi_3^5$ , then  $Y$  has an associated twisted K3 surface. Namely  $Y \in \mathcal{C}_{12}$  and 12 satisfies condition (ii) of [25, Proposition 6.23]. A general cubic fourfold with automorphisms  $\phi_3^1$  or  $\phi_3^5$  is conjecturally irrational. Moreover, a general cubic fourfold with the automorphism  $\phi_3^1$  does not lie on any Hassett divisor.

**Lemma 4.9.** *Let  $Y$  be a general cubic fourfold with an automorphism of order three of type  $\phi_3^5$ , then  $Y \notin \mathcal{C}_d$  for  $d \equiv 2 \pmod{6}$ .*

*Proof.* Any primitive lattice  $K_d$  containing  $\eta_Y$  is of the form  $\langle \eta_Y, a\eta_Y + v \rangle$  for  $0 \neq v \in A_p(Y)$  and  $a \in \mathbb{Z}$ . We can suppose  $K_d = \langle \eta_Y, v \rangle$  after applying a linear transformation, hence  $d = 3k$  in virtue of Theorem 4.6, where  $k = v^2$  and  $k$  is an even number.  $\square$

The divisor  $\mathcal{C}_8$  parametrizes cubic fourfolds containing a plane. In particular, a general cubic fourfold with automorphism  $\phi_3^5$  does not contain a plane since  $Y \notin \mathcal{C}_8$  but such a cubic belongs to  $\mathcal{C}_{12}$ , which is the closure of the locus of cubic fourfolds containing a rational cubic scroll.

**Proposition 4.10.** *Let  $Y$  be the general cubic fourfold with automorphism  $\phi_3^5$ , then the Fano variety of lines  $F(Y)$  contains 54 classes of rational curves.*

*Proof.* By the numerical description of the extremal rays of the Mori cone for hyperkähler manifolds of  $K3^{[2]}$  type in [41, Proposition 2.12], we know that divisors of square  $-2$  and divisors of square  $-10$  and divisibility 2 in  $H^2(F(Y), \mathbb{Z})$  correspond to extremal rays of the Mori cone, in particular they are classes of rational curves on  $F(Y)$ . By [2], there is an isomorphism of integral Hodge structures  $H_p^4(Y, \mathbb{Z})(-1) \cong H_p^2(F(Y), \mathbb{Z})$ . We know that there are no short roots in  $A_p(Y)$ , then the only possibility is to have vectors in  $H^{1,1}(F(Y), \mathbb{Z})$  of square  $-10$  and divisibility 2 in  $H^2(F(Y), \mathbb{Z})$ . Recall that the polarization  $H$  of  $F(Y)$  has square 6 and divisibility 2 in  $H^2(F(Y), \mathbb{Z})$ . The lattice  $A_p(Y)(-1) \cong H^{1,1}(F(Y), \mathbb{Z})$  is 3-elementary, hence vectors  $v \in H^{1,1}(F(Y), \mathbb{Z})$  of divisibility 2 in  $H^2(F(Y), \mathbb{Z})$  are of the form  $v = H + 2a$  with  $a \in H_p^{1,1}(F(Y), \mathbb{Z})$ . To get  $v^2 = -10$  we need  $4a^2 = -16$  and then  $a^2 = -4$ . By the classification in Table 3 there are exactly 54 vectors of square 4 in  $A_p(Y)$  for  $Y$  a general cubic fourfold with an action of  $\phi_3^5$ .  $\square$

**Proposition 4.11.** *Let  $Y$  be the general cubic fourfold with automorphism  $\phi_3^5$ , then  $Y$  contains 27 families of cubic scrolls  $\{T_i, T_i^\vee\}_{i=1}^{27}$  such that  $[T_i] + [T_i^\vee] = 2\eta_Y$ . Moreover, the algebraic lattice  $A(Y)$  is generated by the classes  $[T_i]$  for  $i = 1, \dots, 27$ .*

*Proof.* Every class of a rational curve on  $F(Y)$  corresponds to the class of a rational ruled surface on the cubic fourfold  $Y$ . Since, by construction, every such a surface is contained in an hyperplane section of  $Y$ , then the cubic surface is a rational cubic scroll. By Proposition 4.10, we have 54 rational cubic scrolls and by [22, Example 7.16] on a fixed cubic fourfold these scrolls are parametrized by two distinct copies of  $\mathbb{P}^2$ . Given a cubic scroll  $[T_i]$ , there is a residual scroll (the dual cubic scroll)  $[T_i^\vee]$  which is obtained by intersecting a linear hyperplane and a quadratic hypersurface containing  $[T_i]$ , as in [23] (each one of them correspond to a distinct  $(-10)$  class, as we found). One can easily check (using computer algebra [50]) that  $A_p(Y)$  contains exactly 54 vectors of square 4 that generate the entire lattice, moreover the classes  $\alpha_i := [T_i] - \eta_Y$  and  $\alpha_i^\vee := [T_i^\vee] - \eta_Y$  have square 4.  $\square$

**4.4. Cubic fourfolds with automorphism  $\phi_3^7$ .** A general cubic fourfold  $Y$  with the non-symplectic automorphism of order three  $\phi_3^7$  has a primitive algebraic lattice  $A_p(Y)$  computed in Table 3. We prove that the cubic has an associated K3 surface and it is rational. Moreover we show that the algebraic lattice of such a cubic fourfold is generated by classes of planes.

**Lemma 4.12.** *Let  $Y$  be a general cubic fourfold with automorphism  $\phi_3^7$ , then  $Y$  has an associated K3 surface and  $Y$  is rational.*

*Proof.* Consider a general cubic fourfold  $Y$  with automorphism  $\phi_3^7$ , then it is easy to see that the lattice  $T(Y)(-1)$  in Table 4 corresponding to  $\phi_3^7$  admits a primitive embedding in a K3 lattice, hence  $Y$  has an associated K3 surface. Moreover if  $Y$  admits an automorphism  $\phi_3^7$  then  $Y \in \mathcal{C}_{14}$ , as  $K_{14} = \langle \eta_Y, m_1 + m_2 \rangle$  gives a labeling if  $(\eta_Y, m_1, \dots, m_8)$  is a basis for the matrix of  $A(Y)$  in Table 4. It is well known that any cubic fourfold on  $\mathcal{C}_{14}$  is rational [2].  $\square$

**Proposition 4.13.** *Let  $Y$  be a general cubic fourfold with automorphism  $\phi_3^7$ . Then the cubic fourfold contains exactly nine disjoint planes  $F_1, \dots, F_9$  and a basis of  $A(Y)$  is given by  $\{\eta_Y, [F_1], \dots, [F_8]\}$ .*

*Proof.* If a plane is contained in  $Y$  it has to be invariant for the action of  $\phi_3^7$  on  $\mathbb{P}^5$ . If a plane is invariant for the action of  $\phi_3^7$  it has equation

$$F_{\{a,b,c,d,e,f\}} := \{ax_0 = bx_1, cx_2 = dx_3, ex_4 = fx_5\}.$$

We want to study the intersection  $Y \cap F_{\{a,b,c,d,e,f\}}$ . The equation of a cubic fourfold  $Y$  with an action of  $\phi_3^7$  is given in Theorem 4.1 and it is straightforward to see that  $Y \cap F_{\{a,b,c,d,e,f\}}$  coincides with

$$\{x_0^2 x_2 P_{\{2,1,0\}}(a, b, c, d, e, f) + x_4^2 x_0 P_{\{1,0,2\}}(a, b, c, d, e, f) + x_2^2 x_4 P_{\{0,2,1\}}(a, b, c, d, e, f) = 0\},$$

where  $P_{\{i,j,k\}}$  are polynomials of multi-degrees  $(i, j, k)$  in  $\mathbb{P}_{[a:b]}^1 \times \mathbb{P}_{[c:d]}^1 \times \mathbb{P}_{[e:f]}^1$ . From this description, we deduce that the planes contained in  $Y$  correspond to the set

$$S := \{([a:b], [c:d], [e:f]) \text{ such that } P_{\{2,1,0\}} = P_{\{1,0,2\}} = P_{\{0,2,1\}} = 0\} \subset \mathbb{P}_{[a:b]}^1 \times \mathbb{P}_{[c:d]}^1 \times \mathbb{P}_{[e:f]}^1.$$

Consider the projections  $p_{a,b}, p_{c,d}, p_{e,f}$  from  $\mathbb{P}_{[a:b]}^1 \times \mathbb{P}_{[c:d]}^1 \times \mathbb{P}_{[e:f]}^1$  to its factors, and denote by  $f_1, f_2, f_3$  the fibers of these three projections. The set  $S$  coincides with the intersection of the following three

divisors:

$$(2) \quad \begin{aligned} &2f_1 + f_2, \\ &f_1 + 2f_3, \\ &2f_2 + f_3, \end{aligned}$$

then  $S$  is a finite set of degree  $(2f_1 + f_2)(f_1 + 2f_3)(2f_2 + f_3) = 9$ . This means that the intersection of these three divisors gives exactly none points in  $\mathbb{P}_{[a:b]}^1 \times \mathbb{P}_{[c:d]}^1 \times \mathbb{P}_{[e:f]}^1$ , that correspond to the nine planes that we denote by  $F_1, \dots, F_9$ . Observe that the lattice generated by the classes  $\{\frac{1}{3} \sum_{i=1}^9 [F_i], [F_1], \dots, [F_8]\}$  has intersection matrix as in Table 4 and it is a saturated sublattice of  $A(Y)$ , it follows that  $\eta_Y = \frac{1}{3} \sum_{i=1}^9 [F_i]$  and  $A(Y) = \langle \eta_Y, [F_1], \dots, [F_8] \rangle$ . In particular, by construction the planes are disjoint as claimed.  $\square$

**4.5. Cubic fourfolds with automorphism  $\phi_3^2$ .** A cubic fourfold  $Y$  with the non-symplectic automorphism of order three  $\phi_3^2$  has a primitive algebraic lattice  $A_p(Y)$  computed in Table 3.

In this section we explore the geometry of a cubic fourfold with such an action making a couple of complementary remarks, referring to [30, §2]. In particular, we prove that a cubic fourfold  $Y$  with such an action contains eightyone invariant planes that are related to the existence of three Eckardt points on  $Y$ .

**Lemma 4.14.** *Let  $Y$  be a general cubic fourfold with automorphism  $\phi_3^2$ , then  $Y$  has an associated K3 surface and  $Y$  is rational.*

*Proof.* Same proof as Lemma 4.12, where the labeling is given by  $K_{14} = \langle \eta_Y, -m_6 + m_7 \rangle$  where  $(\eta_Y, m_1, \dots, m_{12})$  is a basis for the matrix of  $A(Y)$  in Table 4.  $\square$

Here we recall the definition of an Eckardt point for a cubic fourfold given in [30, Definition 1.5], where we only consider smooth cubic fourfolds.

**Definition 4.15.** Let  $Y$  be a smooth cubic fourfold. We say that  $p \in Y$  is an *Eckardt point* if  $p$  has multiplicity 3 in  $T_p Y \cap Y$ , equivalently if  $T_p Y \cap Y$  is a cone with vertex  $p$  over a cubic surface.

We prove that a cubic fourfold with the automorphism  $\phi_3^2$  contains three Eckardt points.

**Proposition 4.16.** *Let  $Y$  be a general cubic fourfold with an order three non-symplectic automorphism  $\phi_3^2$ . Then  $Y$  contains exactly eightyone (invariant) planes, associated to three fixed Eckardt points  $P_1, P_2, P_3 \in Y$ .*

*Proof.* Recall that a general cubic fourfold  $Y$  with an action of automorphism  $\phi_3^2$  is described as

$$L_3(x_0, \dots, x_3) + M_3(x_4, x_5) = 0,$$

where  $L_3$  and  $M_3$  are two homogeneous polynomials of degree 3 in  $x_0, \dots, x_3$  and  $x_4, x_5$  respectively. If  $P = [0 : 0 : 0 : 0 : p : q]$  is a zero of the polynomial  $M_3(x_4, x_5)$ , then we can write

$$M_3(x_4, x_5) = (qx_4 - px_5)M_2(x_4, x_5).$$

Note that the equation of the tangent plane  $T_P Y$  is  $\{qx_4 - px_5 = 0\}$  since  $\frac{\partial F}{\partial x_4}|_P = q$  and  $\frac{\partial F}{\partial x_5}|_P = p$ . Then the intersection  $T_P Y \cap Y$  is a cone with vertex  $P$  over the invariant cubic surface  $S = Y \cap \{x_4 = x_5 = 0\}$ , hence  $P$  is an Eckardt point. There are three such Eckardt points  $P_1, P_2, P_3$  corresponding to the points  $M_3 = 0$ . By the argument above each one of them gives an invariant Eckardt point  $P_i = [0 : 0 : 0 : 0 : a_i : b_i] \in Y$  for  $i = 1, 2, 3$ . The cone over the invariant surface with vertex any of the Eckardt points is still invariant and contained in  $Y$ . The cubic surface  $S$  contains

exactly 27 lines, then any Eckardt point determines 27 invariant planes passing through the lines of the surface. If  $F \subset Y$  is a plane, then by [57], its cohomology class is such that  $[F]^2 = 3$  and  $[F] \cdot \eta_Y = 1$ , moreover any class with these numerical properties is represented by a unique plane. Using computer algebra, one can check that there are exactly eightyone such classes in  $A(Y)$ .  $\square$

**Corollary 4.17.** *Let  $Y$  be a cubic with an automorphism  $\phi_3^2$ , then  $\mathbb{Z}/3\mathbb{Z} \times D_4 \subset \text{Aut}(Y)$ , where  $D_4$  denotes the dihedral group.*

*Proof.* The automorphism of order three is given by  $\phi_3^2$  and, according to [30] for any Eckardt point there is an associated involution given by a hyperplane reflection. From the matrix description of the automorphisms, choosing an appropriate set of coordinates, the reflections generate the dihedral group and commute with the automorphism  $\phi_3^2$ .  $\square$

**Proposition 4.18.** *Let  $Y$  be a general cubic fourfold with automorphism  $\phi_3^2$ . In the notation of the proof of Proposition 4.16, consider the 6 disjoint lines on the cubic surface  $S$  contained in  $Y$ , and consider the classes of the planes  $[F_{i,j}]$  for  $i \in \{1, 2, 3\}$  and for  $j \in \{1, \dots, 6\}$  that are the unique planes passing through the Eckardt point  $P_i$  and one of the disjoint lines in  $S$ . Then the algebraic lattice  $A(Y)$  has a basis given by the classes of the following planes:*

- $[F_{1,j}]$  for all  $j \in \{1, \dots, 6\}$
- $[F_{2,k}]$  for  $k \in \{1, \dots, 5\}$
- $[F_{1,0}]$  and  $[F_{2,0}]$

where  $F_{1,0}$  and  $F_{2,0}$  are classes of the cones of the pullback of a general line via  $S = \text{Bl}_6 \mathbb{P}^2 \rightarrow \mathbb{P}^2$  with vertex, respectively, the Eckardt points  $P_1$  and  $P_2$ .

*Proof.* The intersection numbers of the classes corresponding to planes in the cone with vertex an Eckardt point are described in [30, Lemma 2.4], the cases where the planes pass through different Eckardt points can be easily deduced. The classes  $[F_{1,0}]$  and  $[F_{2,0}]$  correspond to a union of planes  $P_0 + P'_0 + P''_0$  and  $Q_0 + Q'_0 + Q''_0$  respectively, where  $P_0$  shares a line with  $P'_0$  and  $P''_0$  and the intersection  $P'_0 \cap P''_0$  is a point (and the same behaviour for  $Q_0, Q'_0, Q''_0$ ) (see [53]). The rank 13 sublattice of  $A(Y)$  generated by  $\{[F_{1,j}], [F_{2,k}], [F_{1,0}], [F_{2,0}]\}_{j,k}$  coincides with  $A(Y)$  since the intersection matrix has the same determinant as the one of  $A(Y)$  in Table 4.  $\square$

**Remark 4.19.** According to [14, Proposition 3.3] for a hypersurface of degree  $d = 3$  in  $\mathbb{P}^5$  the Eckardt points of the hypersurface have to be at most  $d = 3$  points on a line  $l$  not contained in  $Y$ . This configuration is actually verified for a general cubic fourfold with an order three non-symplectic automorphism  $\phi_3^2$ .

Note that in [29, Theorem 1.8] and in [30] the authors study the minimal algebraic lattice that a cubic fourfold  $Y$  needs to have in order to admit an Eckardt point, and they find that a cubic fourfold  $Y$  with a non-symplectic involution  $\phi_2^1$  has an Eckardt point. In this situation they prove that  $A_p(Y) \cong \mathbf{E}_6(2)$  and the cubic fourfold has no associated K3 surface by [35, Theorem 1.2]. In the case of a cubic fourfold with an automorphism of order three  $\phi_3^2$  the cubic fourfold has more algebraic classes, it admits also the involution  $\phi_2^1$  as we showed in Corollary 4.17, but it is rational and has an associated K3 surface, as we prove in Lemma 4.14.

**Remark 4.20.** Note that by lattice-theoretic considerations we can detect that a cubic fourfold with automorphism  $\phi_3^2$  contains an Eckardt point. Namely let  $Y$  be a general cubic fourfold with an order three non-symplectic automorphism  $\phi_3^2$ , then it is easy to see that there exists a primitive embedding  $\mathbf{E}_6(2) \hookrightarrow A_p(Y)$ , and this is enough to conclude that  $Y$  contains an Eckardt point by [30, Proposition 2.8].

## 5. INDUCED ACTION ON LAZA-SACCA-VOISIN MANIFOLDS

In this section we study non-symplectic automorphisms of ihs manifolds of OG10 type that are constructed as Laza–Saccà–Voisin manifolds and we investigate when these automorphisms are induced by non-symplectic automorphisms of the underlying cubic fourfold.

The main result that we recall here is the following theorem due to Mongardi–Onorati.

**Theorem 5.1** (see [18, Addendum, Theorem 3.4]). *Let  $Y$  be a cubic fourfold, and let  $J(Y)$  be the associated LSV manifold, then there is a Hodge isometry*

$$H_p^4(Y, \mathbb{Z})(-1) \xrightarrow{\cong} \mathbf{U}_Y^\perp \subset H^2(J(Y), \mathbb{Z}).$$

**Proposition 5.2.** *Let  $Y$  be a cubic fourfold, and let  $J^t(Y)$  be the associated twisted LSV manifold, then there is a Hodge isometry*

$$H_p^4(Y, \mathbb{Z})(-1) \xrightarrow{\cong} (\mathbf{U}_Y^t)^\perp \subset H^2(J^t(Y), \mathbb{Z}).$$

*Proof.* The twisted LSV manifold is birational to the Li–Pertusi–Zhao manifold  $\tilde{M}_\sigma(2(\lambda_1 + \lambda_2), \mathcal{A}_Y)$  by [33, Theorem 1.3]. Consider the symplectic resolution  $\tilde{M}_\sigma(2(\lambda_1 + \lambda_2), \mathcal{A}_Y)$  of the moduli space of Bridgeland  $\sigma$ -semi-stable objects on the Kuznetsov component  $\mathcal{A}_Y$  of the cubic fourfold  $Y$ . We know by [17, Example 2.13] that there is a Hodge embedding  $H_p^4(Y, \mathbb{Z})(-1) \hookrightarrow H^2(\tilde{M}_\sigma(2(\lambda_1 + \lambda_2), \mathcal{A}_Y), \mathbb{Z})$  with orthogonal complement of  $(1, 1)$  type and isometric to  $\mathbf{U}(3)$ . The moduli space  $\tilde{M}_\sigma(2(\lambda_1 + \lambda_2), \mathcal{A}_Y)$  is birational to  $J^t(Y)$  by [33, Theorem 1.3]. For a general cubic fourfold  $Y$  the algebraic lattice of  $\tilde{M}_\sigma(2(\lambda_1 + \lambda_2), \mathcal{A}_Y)$  is isometric to  $\mathbf{U}(3)$  and the algebraic lattice of  $J^t(Y)$  is isometric to  $\mathbf{U}_Y^t$ , hence composing the Hodge isometries we get the following Hodge isometry

$$H_p^4(Y, \mathbb{Z})(-1) \xrightarrow{\cong} (\mathbf{U}_Y^t)^\perp \subset H^2(J^t(Y), \mathbb{Z}).$$

□

We give a numerical criterion for an ihs manifold of OG10 type to be birational to a LSV manifold  $J(Y)$  or a twisted LSV manifold  $J^t(Y)$ .

**Proposition 5.3** (see [37, Proposition 7.5] and [51]). *Let  $X$  be an ihs manifold of OG10 type. There exists a smooth cubic fourfold  $Y$  such that  $X$  is birational to  $J(Y)$  if and only if  $\mathbf{U} \subset \text{NS}(X)$  and  $\mathbf{U}^{\perp_{\text{NS}(X)}} \cap \mathcal{W}_{\text{pex}}(X) = \emptyset$ .*

**Proposition 5.4.** *Let  $X$  be an ihs manifold of OG10 type. There exists a smooth cubic fourfold  $Y$  such that  $X$  is birational to  $J^t(Y)$  if and only if there is a primitive embedding  $\mathbf{U}(3) \subset \text{NS}(X)$  and  $\mathbf{U}(3)^{\perp_{\text{NS}(X)}} \cap \mathcal{W}_{\text{pex}}(X) = \emptyset$ .*

*Proof.* If  $X$  and  $J^t(Y)$  are birational, then there is an Hodge isometry  $H^2(J^t(Y), \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ , so the embedding  $\mathbf{U}_Y^t \subset \text{NS}(J^t(Y))$  induces the embedding of  $\mathbf{U}(3)$  in  $\text{NS}(X)$ . We know by Proposition 5.2 that the lattice  $(\mathbf{U}_Y^t)^\perp \subset H^2(J(Y), \mathbb{Z})$  is Hodge-isometric to  $H_p^4(Y, \mathbb{Z})(-1)$ . The description of the image of the period map of cubic fourfold Theorem 2.9 ensures that there are no long or short roots in  $H_p^4(Y, \mathbb{Z})$ , which coincide with classes of prime exceptional divisors of ihs manifolds of OG10 type up to a sign (see §2.2).

Viceversa, if there are no short or long roots in  $\mathbf{U}(3)^\perp \subset \text{NS}(X)(-1)$  then by Theorem 2.9 we know that  $\mathbf{U}(3)^\perp \subset H^2(X, \mathbb{Z})(-1)$  is Hodge isometric to  $H_p^4(Y, \mathbb{Z})$  for a cubic fourfold  $Y$ , which is also Hodge isometric to  $(\mathbf{U}_Y^t)^\perp \subset H^2(J^t(Y), \mathbb{Z})(-1)$ . Extending the Hodge isometry to the entire lattice  $H^2(X, \mathbb{Z}) \cong H^2(J^t(Y), \mathbb{Z})$  via [46, Corollary 1.5.2], we conclude that  $X$  and  $J^t(Y)$  are birational. □

**Remark 5.5.** Combining [33, Theorem 1.3] with [17, Theorem 4.3] we obtain that  $J(Y)$  is birational to  $J^t(Y)$  if and only if the cubic fourfold  $Y$  admits a  $d$ -labeling with  $d \equiv 2(6)$ . As an application of the results in §4, we obtain that  $J(Y)$  and  $J^t(Y)$  are birational if  $Y$  is general with an automorphism  $\phi_3^7$  or  $\phi_3^2$  but they are not birational if  $Y$  is general with an automorphism  $\phi_3^1$  or  $\phi_3^5$ . We also point out that by [35] it follows that  $J(Y)$  and  $J^t(Y)$  are birational if  $Y$  is general with one of the non-symplectic involutions  $\phi_2^1$  or  $\phi_2^3$ . We collect this information in Table 2.

We study the induced action on the second integral cohomology of birational transformations of LSV manifolds induced by automorphisms of cubic fourfolds. Recall that an automorphism of a cubic fourfold  $Y$  induces a birational transformation of the LSV manifold  $J(Y)$ .

**Remark 5.6.** An automorphism of the cubic fourfold  $Y$  induces a birational transformation of the twisted LSV manifold  $J^t(Y)$ . In fact, let  $u : \mathcal{V}_U \rightarrow U \subset \mathbb{P}^5$  be the family of smooth hyperplane sections of  $Y$ , recall that Deligne-Belinson cohomology gives an exact sequence of sheaves of groups

$$0 \rightarrow J_U(Y) \rightarrow H_{\mathcal{D}}^4(\mathcal{V}_U, \mathbb{Z}(2)) \xrightarrow{c} R^4 u_* \mathbb{Z} \rightarrow 0$$

where the sheaf  $R^4 u_* \mathbb{Z}$  is canonically isomorphic to  $\mathbb{Z}$ . By definition  $J_U^t = c^{-1}(1)$ , functoriality gives an automorphism of  $J_U^t$ .

Let  $\phi \in \text{Aut}(Y)$ , we denote by  $\tilde{\phi} \in \text{Bir}(J(Y))$  and by  $\tilde{\phi}^t \in \text{Bir}(J^t(Y))$  the induced birational transformations.

**Lemma 5.7** (see [42, Lemma 7.1] or [56, Lemma 3.2]). *Let  $Y$  be a cubic fourfold,  $\phi \in \text{Aut}(Y)$  is symplectic if and only if  $\tilde{\phi} \in \text{Bir}(J(Y))$  and  $\tilde{\phi}^t \in \text{Bir}(J^t(Y))$  are symplectic.*

**Proposition 5.8.** *Let  $Y$  be a general cubic fourfold with a non-symplectic automorphism  $\phi \in \text{Aut}(Y)$  of finite order and let  $\tilde{\phi} \in \text{Bir}(J(Y))$  be the induced birational transformation on the LSV manifold  $J(Y)$ . Then there is an isometry*

$$(H_p^4(Y, \mathbb{Z}))^{\phi}(-1) \cong (\mathbf{U}_Y^{\perp})^{\tilde{\phi}} \subset H^2(J(Y), \mathbb{Z})^{\tilde{\phi}}.$$

*The same statement holds for  $J^t(Y)$ , replacing  $\mathbf{U}_Y$  with  $\mathbf{U}_Y^t \cong \mathbf{U}(3)$  and  $\tilde{\phi}$  with  $\tilde{\phi}^t$ .*

*Proof.* We prove the statement for the untwisted case, the twisted case is analogous.

By [56, Section 3.1] every birational transformation  $\tilde{\phi} \in \text{Bir}(J(Y))$  of a LSV manifold which is induced by an automorphism  $\phi \in \text{Aut}(Y)$  of the cubic fourfold  $Y$  fixes the two generators of  $\mathbf{U}_Y$ . We consider the isometry  $\tilde{\phi}$  restricted on  $\mathbf{U}_Y^{\perp}$ , which is still a Hodge isometry since the two generators in  $\mathbf{U}_Y$  are of  $(1, 1)$  type. According to [42, Lemma 7.1] there is an isogeny of Hodge structures

$$\alpha : H_p^4(Y, \mathbb{Z})(-1) \rightarrow \mathbf{U}_Y^{\perp} \subset H^2(J(Y), \mathbb{Z}),$$

where  $\alpha$  is the restriction of the morphism  $[Z]_* \circ q^* : H^4(Y, \mathbb{Z}) \rightarrow H^2(J(Y), \mathbb{Z})$ . Here  $q : \mathcal{U}_Y \rightarrow Y$  is the inclusion of linear sections and  $[Z]_*(x) = \pi_{1*}(\pi_2^* x \cdot Z)$  where  $Z \in \text{CH}^2(J(Y) \times_{\mathbb{P}^5} \mathcal{U}_Y)_{\mathbb{Q}}$  is the closure of a distinguished cycle  $Z_U \in \text{CH}^2(J_U(Y) \times_{\mathbb{P}^5} \mathcal{U}_U)_{\mathbb{Q}}$ , and  $\pi_1, \pi_2$  are the respective projections. For an integer  $k$ , it is well defined the cycle  $\phi^k(Z_U) \in \text{CH}^2(J_U(Y) \times_{\mathbb{P}^5} \mathcal{U}_U)_{\mathbb{Q}}$ , consider its closure  $\overline{\phi^k(Z_U)} \in \text{CH}^2(J(Y) \times_{\mathbb{P}^5} \mathcal{U}_Y)_{\mathbb{Q}}$ . Replacing  $Z$  with

$$\tilde{Z} = \frac{1}{\sqrt{\text{ord}(\phi)}} \sum_{k=0}^{\text{ord}(\phi)-1} \overline{\phi^k(Z_U)}$$



in the above definition, since  $\phi(\tilde{Z}) = \tilde{Z}$ , one gets a  $(\phi, \tilde{\phi})$ -equivariant isogeny of Hodge structures

$$\tilde{\alpha}: H_p^4(Y, \mathbb{Z})(-1) \rightarrow \mathbf{U}_Y^\perp \subset H^2(J(Y), \mathbb{Z}).$$

There is an isometry  $A_p(Y)(-1) \cong (\mathbf{U}_Y^\perp)^{1,1}$  by [Theorem 5.1](#), and by the equivariance of  $\tilde{\alpha}$  it follows that  $H_p^4(Y, \mathbb{Z})^\phi(-N) \subseteq (\mathbf{U}_Y^\perp)^{\tilde{\phi}}$  for some integer  $N > 0$ . The cubic fourfold is general then we have  $H_p^4(Y, \mathbb{Z})^\phi \cong A_p(Y)$  and there are finite index embeddings

$$A_p(Y)(-N) \cong H_p^4(Y, \mathbb{Z})^\phi(-N) \subseteq (\mathbf{U}_Y^\perp)^{\tilde{\phi}} \subseteq (\mathbf{U}_Y^\perp)^{1,1},$$

we conclude by observing that the last embedding is primitive.  $\square$

We can recover a cubic fourfold with an automorphism from a certain automorphism of a manifold of OG10 type.

**Proposition 5.9.** *Let  $X$  be a manifold of OG10 type with a marking  $H^2(X, \mathbb{Z}) \cong \mathbf{L}$  and let  $f \in \text{Bir}(X)$  be a general non-symplectic birational transformation of prime order. Then  $f$  is induced by an automorphism of a cubic fourfold if and only if it acts trivially on the discriminant group  $A_{\mathbf{L}}$ , and one of the following holds:*

- i) there is a primitive embedding  $\mathbf{U} \hookrightarrow \text{NS}(X)$  such that  $\mathbf{U}^{\perp_{\text{NS}(X)}} \cap \mathcal{W}_{P_{ex}} = \emptyset$ ,*
- ii) there is a primitive embedding  $\mathbf{U}(3) \hookrightarrow \text{NS}(X)$ , such that  $\mathbf{U}(3)^{\perp_{\text{NS}(X)}} \cap \mathcal{W}_{P_{ex}} = \emptyset$ .*

*Proof.* By generality assumption we have  $\mathbf{L}^f = \text{NS}(X)$ . If  $f$  is natural the statement follows from [Proposition 5.8](#) and [Theorem 2.13](#).

Viceversa, assume  $f \in \text{Bir}(X)$  is a non-symplectic birational transformation acting trivially on  $A_{\mathbf{L}}$  and such that *i)* holds. Consider the lattice  $\mathbf{N} := \mathbf{U}^{\perp_{\mathbf{L}(-1)}}$ . Since  $\mathbf{U}^{\perp_{\text{NS}(X)}} \cap \mathcal{W}_{P_{ex}} = \emptyset$  and prime exceptional divisors of an OG10 type manifold correspond to short and long roots up to the sign, by [Theorem 2.9](#) there exists a smooth cubic fourfold  $Y$  such that  $H_p^4(Y, \mathbb{Z}) \cong \mathbf{N}$  is a Hodge isometry. The restriction of the isometry  $f$  to  $\mathbf{N}$  extends to an isometry of  $H^4(Y, \mathbb{Z})$  that fixes the class  $\langle \eta_Y \rangle = H_p^4(Y, \mathbb{Z})^\perp \subset H^4(Y, \mathbb{Z})$  if and only if  $f$  acts trivially on the discriminant group  $A_{\mathbf{L}(-1)} \cong A_{\mathbf{N}}$ , as in our assumption. We conclude by [Proposition 5.8](#) and [Theorem 2.13](#). The same proof holds true assuming *ii)*.  $\square$

**Lemma 5.10.** *Let  $Y$  be a general cubic fourfold with a non-symplectic automorphism  $\phi \in \text{Aut}(Y)$  of finite order. Then, the induced birational transformations  $\tilde{\phi} \in \text{Bir}(J(Y))$  and  $\tilde{\phi}^t \in \text{Bir}(J^t(Y))$  are automorphisms.*

*Proof.* From [Proposition 5.8](#) we know that from generality assumption  $A_p(Y) = H_p^4(Y, \mathbb{Z})^\phi$  then  $\text{NS}(J(Y)) = H^2(J(Y), \mathbb{Z})^{\tilde{\phi}}$  and  $\text{NS}(J^t(Y)) = H^2(J^t(Y), \mathbb{Z})^{\tilde{\phi}^t}$ . By the Hodge theoretic Torelli theorem for ihs manifolds [[34](#), Theorem 1.3] we conclude that the transformations are regular.  $\square$

**Remark 5.11.** This shows that the converse of [[56](#), Proposition 3.11] does not hold. Namely, some birational transformations induced by automorphisms of a cubic fourfold extend to regular automorphisms even if the fibers of the Lagrangian fibration are reducible. Reducible fibers arise for cubic fourfolds containing planes or cubic scrolls, see [[38](#)].

**Theorem 5.12.** *Let  $Y$  be a general cubic fourfold with a non-symplectic automorphism of prime order. Then, the invariant and coinvariant lattices of the induced actions on  $J(Y)$  and  $J^t(Y)$  are described in [Table 2](#).*



*Proof.* We have a classification of possible invariant and coinvariant sublattices of  $H_p^4(Y, \mathbb{Z})$  for non-symplectic automorphisms of a general  $Y$ . This is the content of [35, Theorem 1.1] for the case of involutions, and of Theorem 4.6 for the case of automorphisms of order three. Using Proposition 5.8 and Table 6 we conclude the statement.  $\square$

**Theorem 5.13.** *Let  $X$  be an ihs manifold of OG10 type and let  $f \in \text{Aut}(X)$  be a general non-symplectic automorphism of prime order, and let  $G = \langle f \rangle$ . If the pair  $(\mathbf{L}^G, \mathbf{L}_G)$  appears in Table 2, then there exists a cubic fourfold  $Y$  with an automorphism  $\phi \in \text{Aut}(Y)$  such that  $X$  is birational to  $J(Y)$  or  $J^t(Y)$  and  $f$  is compatible with  $\tilde{\phi} \in \text{Aut}(J(Y))$  or  $\tilde{\phi}^t \in \text{Aut}(J^t(Y))$  respectively.*

*Proof.* Suppose  $G \subset \text{Aut}(X)$  is as in Table 2, then the hypothesis of Proposition 5.3 or Proposition 5.4 are satisfied so that  $X$  is birational to an LSV manifold. Finally, Proposition 5.9 concludes the proof.  $\square$

#### APPENDIX A. TABLES OF INVARIANT AND COINVARIANT LATTICES OF NON-SYMPLECTIC AUTOMORPHISMS OF OG10

Here we collect tables of (isometry classes) of invariant and coinvariant lattices for the action on  $\Lambda$  and  $\mathbf{L}$ , i.e. pairs  $(\mathbf{L}^G, \mathbf{L}_G)$  and  $(\Lambda^G, \Lambda_G)$  for a group  $G$  of prime order isometries, with given signatures.

##### A.1. $p = 2$ .

TABLE 5. Pairs  $(\Lambda^G, \Lambda_G)$  for  $G \subset \text{O}(\Lambda)$  of prime order  $p = 2$  and  $\text{sgn}(\Lambda_G) = (2, \text{rk}(\Lambda_G) - 2)$ , i.e. trivial action on the discriminant group.

No.	$\text{rk}(\Lambda^G)$	$\Lambda_G$	$\Lambda^G$	$\text{sgn}(\Lambda_G)$	$a$	$\delta$
1	4	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{U} \oplus [2]^{\oplus 2}$	(2, 20)	2	1
2	4	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 3}$	$[2]^{\oplus 3} \oplus [-2]$	(2, 20)	4	1
3	5	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [-2]$	$\mathbf{U}^{\oplus 2} \oplus [2]$	(2, 19)	1	1
4	5	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]$	(2, 19)	3	1
5	5	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 2}$	(2, 19)	5	1
6	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2}$	$\mathbf{U}^{\oplus 3}$	(2, 18)	0	0
7	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus \mathbf{U}(2)$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(2)$	(2, 18)	2	0
8	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2] \oplus [-2]$	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]$	(2, 18)	2	1
9	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}(2)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2}$	(2, 18)	4	0
10	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [2]^{\oplus 2}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	(2, 18)	4	1
11	6	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus \mathbf{U}(2)$	$\mathbf{U}(2)^{\oplus 3}$	(2, 18)	6	0
12	6	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [2] \oplus [-2]$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 3}$	(2, 18)	6	1
13	7	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2]$	$\mathbf{U}^{\oplus 3} \oplus [-2]$	(2, 17)	1	1
14	7	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 2}$	(2, 17)	3	1
15	7	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 3}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	(2, 17)	5	1
16	7	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 4}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 4}$	(2, 17)	7	1
17	8	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2}$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 2}$	(2, 16)	2	1
18	8	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 2}$	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 3}$	(2, 16)	4	1
19	8	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [2]^{\oplus 2}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 4}$	(2, 16)	6	1
20	8	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 7}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 5}$	(2, 16)	8	1
21	9	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 3}$	(2, 15)	3	1
22	9	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [2]$	(2, 15)	5	1
23	9	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	(2, 15)	7	1
24	9	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 7}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 6}$	(2, 15)	9	1
25	10	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)$	(2, 14)	4	0
26	10	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 4}$	(2, 14)	4	1
27	10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1)^{\oplus 3}$	$\mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2} \oplus \mathbf{D}_4(-1)$	(2, 14)	6	0
28	10	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 5}$	(2, 14)	6	1
29	10	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)^{\oplus 3}$	$\mathbf{U}(2)^{\oplus 3} \oplus \mathbf{D}_4(-1)$	(2, 14)	8	0
30	10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]^{\oplus 4}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 6}$	(2, 14)	8	1

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Table 5, follows from previous page

No.	rk( $\Lambda^G$ )	$\Lambda_G$	$\Lambda^G$	sgn( $\Lambda_G$ )	$a$	$\delta$
31	10	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 5}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 7}$	(2, 14)	10	1
32	11	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 3}$	(2, 13)	3	1
33	11	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 5}$	(2, 13)	5	1
34	11	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 6}$	(2, 13)	7	1
35	11	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 7}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 7}$	(2, 13)	9	1
36	11	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 8}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 8}$	(2, 13)	11	1
37	12	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]^{\oplus 2}$	(2, 12)	2	1
38	12	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 3} \oplus [-2]$	(2, 12)	4	1
39	12	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 4}$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 6}$	(2, 12)	6	1
40	12	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 6}$	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 7}$	(2, 12)	8	1
41	12	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 7}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 8}$	(2, 12)	10	1
42	12	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 11}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 9}$	(2, 12)	12	1
43	13	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [2]$	(2, 11)	1	1
44	13	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]$	(2, 11)	3	1
45	13	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 2}$	(2, 11)	5	1
46	13	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 5}$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 7}$	(2, 11)	7	1
47	13	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 6}$	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 8}$	(2, 11)	9	1
48	13	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 7}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 9}$	(2, 11)	11	1
49	13	$[2]^{\oplus 2} \oplus [-2]^{\oplus 11}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 10}$	(2, 11)	13	1
50	14	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 3}$	(2, 10)	0	0
51	14	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{U}(2)$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{U}(2)$	(2, 10)	2	0
52	14	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]$	(2, 10)	2	1
53	14	$\mathbf{E}_8(-1) \oplus \mathbf{U}(2)^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2}$	(2, 10)	4	0
54	14	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	(2, 10)	4	1
55	14	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U}(2)^{\oplus 3}$	(2, 10)	6	0
56	14	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 4}$	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 3}$	(2, 10)	6	1
57	14	$\mathbf{U}(2)^{\oplus 2} \oplus \mathbf{D}_4(-1)^{\oplus 2}$	$\mathbf{D}_4(-1)^{\oplus 2} \oplus \mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2}$	(2, 10)	8	0
58	14	$\mathbf{D}_4(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 8}$	(2, 10)	8	1
59	14	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{E}_8(-2)$	$\mathbf{D}_4(-1)^{\oplus 2} \oplus \mathbf{U}(2)^{\oplus 3}$	(2, 10)	10	0
60	14	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 9}$	$\mathbf{D}_4(-1)^{\oplus 2} \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 3}$	(2, 10)	10	1
61	14	$\mathbf{E}_8(-2) \oplus \mathbf{U}(2)^{\oplus 2}$	$\mathbf{E}_8(-2) \oplus \mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2}$	(2, 10)	12	0
62	14	$[2]^{\oplus 2} \oplus [-2]^{\oplus 10}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 10}$	(2, 10)	12	1
63	15	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 3} \oplus [-2]$	(2, 9)	1	1
64	15	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 2}$	(2, 9)	3	1
65	15	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	(2, 9)	5	1
66	15	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 4}$	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 4}$	(2, 9)	7	1
67	15	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 9}$	(2, 9)	9	1
68	15	$[2]^{\oplus 2} \oplus [-2]^{\oplus 9}$	$\mathbf{U}^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 9}$	(2, 9)	11	1
69	16	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 2}$	(2, 8)	2	1
70	16	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus \mathbf{U} \oplus [2]^{\oplus 2}$	(2, 8)	4	1
71	16	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 6}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 4}$	(2, 8)	6	1
72	16	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 7}$	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 5}$	(2, 8)	8	1
73	16	$[2]^{\oplus 2} \oplus [-2]^{\oplus 8}$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 10}$	(2, 8)	10	1
74	17	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 3}$	(2, 7)	3	1
75	17	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 4}$	(2, 7)	5	1
76	17	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 6}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	(2, 7)	7	1
77	17	$[2]^{\oplus 2} \oplus [-2]^{\oplus 7}$	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 6}$	(2, 7)	9	1
78	18	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 3} \oplus \mathbf{D}_4(-1)$	(2, 6)	2	0
79	18	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)$	(2, 6)	4	0
80	18	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]$	(2, 6)	4	1
81	18	$\mathbf{U}(2)^{\oplus 2} \oplus \mathbf{D}_4(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2} \oplus \mathbf{D}_4(-1)$	(2, 6)	6	0
82	18	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 5}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	(2, 6)	6	1
83	18	$[2]^{\oplus 2} \oplus [-2]^{\oplus 6}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 3}$	(2, 6)	8	1
84	19	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 3}$	(2, 5)	3	1
85	19	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 4}$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	(2, 5)	5	1
86	19	$[2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	(2, 5)	7	1
87	20	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2]^{\oplus 2}$	(2, 4)	2	1
88	20	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 3} \oplus [-2]$	(2, 4)	4	1
89	20	$[2]^{\oplus 2} \oplus [-2]^{\oplus 4}$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 3}$	(2, 4)	6	1
90	21	$\mathbf{U}^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [2]$	(2, 3)	1	1
91	21	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]$	(2, 3)	3	1
92	21	$[2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 2}$	(2, 3)	5	1
93	22	$\mathbf{U}^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 3}$	(2, 2)	0	0
94	22	$\mathbf{U} \oplus \mathbf{U}(2)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{U}(2)$	(2, 2)	2	0
95	22	$\mathbf{U} \oplus [2] \oplus [-2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]$	(2, 2)	2	1
96	22	$\mathbf{U}(2)^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2}$	(2, 2)	4	0

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Table 5, follows from previous page

No.	$\text{rk}(\Lambda^G)$	$\Lambda_G$	$\Lambda^G$	$\text{sgn}(\Lambda_G)$	$a$	$\delta$
97	22	$[2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	(2, 2)	4	1
98	23	$\mathbf{U} \oplus [2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 3} \oplus [-2]$	(2, 1)	1	1
99	23	$[2]^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 2}$	(2, 1)	3	1
100	24	$[2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 2}$	(2, 0)	2	1

TABLE 6. Pairs  $(\mathbf{L}^G, \mathbf{L}_G)$  for  $G \subset \text{O}(\mathbf{L})$  of prime order  $p = 2$  and  $\text{sgn}(\mathbf{L}_G) = \text{sgn}(\Lambda_G)$ , i.e. trivial action on the discriminant group.

No.	$\text{rk}(\Lambda^G)$	$\mathbf{L}_G$	$\mathbf{L}^G$	$\text{sgn}(\mathbf{L}_G)$	$a(\mathbf{L}_G)$	$\delta(\mathbf{L}_G)$
1	4	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2}$	$[2] \oplus [-6]$	(2, 20)	2	1
2	5	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [-2]$	$\mathbf{A}_2(-1) \oplus [2]$	(2, 19)	1	1
3	5	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2}$	$[2] \oplus [-2] \oplus [-6]$	(2, 19)	3	1
4	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{A}_2(-1)$	(2, 18)	0	0
5	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus \mathbf{U}(2)$	$\mathbf{U}(2) \oplus \mathbf{A}_2(-1)$	(2, 18)	2	0
6	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2] \oplus [-2]$	$\mathbf{A}_2(-1) \oplus [2] \oplus [-2]$	(2, 18)	2	1
7	6	$\mathbf{E}_8(-1) \oplus [-2]^{\oplus 2} \oplus [2]^{\oplus 2}$	$[2] \oplus [-2]^{\oplus 2} \oplus [-6]$	(2, 18)	4	1
8	7	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2]$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]$	(2, 17)	1	1
9	7	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	(2, 17)	3	1
10	7	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 3}$	$[2] \oplus [-2]^{\oplus 3} \oplus [-6]$	(2, 17)	5	1
11	8	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 2}$	(2, 16)	2	1
12	8	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 2}$	$\mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 3}$	(2, 16)	4	1
13	8	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [2]^{\oplus 2}$	$[2] \oplus [-2]^{\oplus 4} \oplus [-6]$	(2, 16)	6	1
14	9	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 3}$	(2, 15)	3	1
15	9	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{A}_2(-1) \oplus [-2]^{\oplus 4} \oplus [2]$	(2, 15)	5	1
16	9	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$[2] \oplus [-2]^{\oplus 5} \oplus [-6]$	(2, 15)	7	1
17	10	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)$	$\mathbf{U}(2) \oplus \mathbf{A}_2(-1) \oplus \mathbf{D}_4(-1)$	(2, 14)	4	0
18	10	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 4}$	(2, 14)	4	1
19	10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1)^{\oplus 3}$	$\mathbf{U} \oplus \mathbf{E}_6(-2)$	(2, 14)	6	0
20	10	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 5}$	(2, 14)	6	1
21	10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]^{\oplus 4}$	$[2] \oplus [-2]^{\oplus 6} \oplus [-6]$	(2, 14)	8	1
22	11	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2]$	(2, 13)	3	1
23	11	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 5}$	(2, 13)	5	1
24	11	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 6}$	(2, 13)	7	1
25	11	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 7}$	$[2] \oplus [-2]^{\oplus 7} \oplus [-6]$	(2, 13)	9	1
26	12	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-6]$	(2, 12)	2	1
27	12	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 3}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]$	(2, 12)	4	1
28	12	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 4}$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 6}$	(2, 12)	6	1
29	12	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 6}$	$\mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 7}$	(2, 12)	8	1
30	12	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 7}$	$[2] \oplus [-2]^{\oplus 8} \oplus [-6]$	(2, 12)	10	1
31	13	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus [2]$	(2, 11)	1	1
32	13	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2] \oplus [-6]$	(2, 11)	3	1
33	13	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	(2, 11)	5	1
34	13	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 5}$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 7}$	(2, 11)	7	1
35	13	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 6}$	$\mathbf{U} \oplus [-2]^{\oplus 8} \oplus [-6]$	(2, 11)	9	1
36	13	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 7}$	$[2] \oplus [-2]^{\oplus 9} \oplus [-6]$	(2, 11)	11	1
37	14	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{A}_2(-1)$	(2, 10)	0	0
38	14	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{U}(2)$	$\mathbf{E}_8(-1) \oplus \mathbf{U}(2) \oplus \mathbf{A}_2(-1)$	(2, 10)	2	0
39	14	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]$	(2, 10)	2	1
40	14	$\mathbf{E}_8(-1) \oplus \mathbf{U}(2)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	(2, 10)	4	0
41	14	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 2} \oplus [-6]$	(2, 10)	4	1
42	14	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 4}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 3}$	(2, 10)	6	1
43	14	$\mathbf{U}(2)^{\oplus 2} \oplus \mathbf{D}_4(-1)^{\oplus 2}$	$\mathbf{D}_4(-1) \oplus \mathbf{U} \oplus \mathbf{E}_6(-2)$	(2, 10)	8	0
44	14	$\mathbf{D}_4(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 8}$	(2, 10)	8	1
45	14	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{E}_8(-2)$	$\mathbf{D}_4(-1) \oplus \mathbf{U}(2) \oplus \mathbf{E}_6(-2)$	(2, 10)	10	0
46	14	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 9}$	$[2] \oplus [-2] \oplus \mathbf{E}_6(-2) \oplus \mathbf{D}_4(-1) = \mathbf{U} \oplus \mathbf{M}$	(2, 10)	10	1
47	14	$\mathbf{U}(2)^{\oplus 2} \oplus [-2]^{\oplus 8}$	$\mathbf{U}(2) \oplus [-2]^{\oplus 9} \oplus [-6]$	(2, 10)	12	1
48	15	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]$	(2, 9)	1	1
49	15	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	(2, 9)	3	1
50	15	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 3} \oplus [-6]$	(2, 9)	5	1
51	15	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 4}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 4}$	(2, 9)	7	1

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Table 6, follows from previous page

No.	rk( $\Lambda^G$ )	$\mathbf{L}_G$	$\mathbf{L}^G$	$\text{sgn}(\mathbf{L}_G)$	$a$	$\delta$
52	15	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 9}$	(2, 9)	9	1
53	15	$[2]^{\oplus 2} \oplus [-2]^{\oplus 9}$	$\mathbf{A}_2(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 9}$	(2, 9)	11	1
54	16	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-6]$	(2, 8)	4	1
55	16	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 6}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus \mathbf{U} \oplus [-2]^{\oplus 4}$	(2, 8)	6	1
56	16	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 7}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 5}$	(2, 8)	8	1
57	16	$[2]^{\oplus 2} \oplus [-2]^{\oplus 8}$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 10}$	(2, 8)	10	1
58	17	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 3}$	(2, 7)	3	1
59	17	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 4}$	(2, 7)	5	1
60	17	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 5} \oplus [-6]$	(2, 7)	7	1
61	17	$[2]^{\oplus 2} \oplus [-2]^{\oplus 7}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 6}$	(2, 7)	9	1
62	18	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{A}_2(-1) \oplus \mathbf{D}_4(-1)$	(2, 6)	2	0
63	18	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)$	(2, 6)	4	0
64	18	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]$	(2, 6)	4	1
65	18	$\mathbf{U}(2)^{\oplus 2} \oplus \mathbf{D}_4(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{E}_6(-2)$	(2, 6)	6	0
66	18	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 2} \oplus [-6]$	(2, 6)	6	1
67	18	$[2]^{\oplus 2} \oplus [-2]^{\oplus 6}$	$\mathbf{D}_4(-1) \oplus \mathbf{D}_6(-1) \oplus [2] \oplus [-2]^{\oplus 3}$	(2, 6)	8	1
68	19	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2]$	(2, 5)	3	1
69	19	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 4}$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	(2, 5)	5	1
70	19	$[2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 3} \oplus [-6]$	(2, 5)	7	1
71	20	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2] \oplus [-6]$	(2, 4)	2	1
72	20	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]$	(2, 4)	4	1
73	20	$[2]^{\oplus 2} \oplus [-2]^{\oplus 4}$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 3}$	(2, 4)	6	1
74	21	$\mathbf{U}^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1) \oplus [2]$	(2, 3)	1	1
75	21	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2] \oplus [-2] \oplus [-6]$	(2, 3)	3	1
76	21	$[2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	(2, 3)	5	1
77	22	$\mathbf{U}^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus \mathbf{A}_2(-1)$	(2, 2)	0	0
78	22	$\mathbf{U} \oplus \mathbf{U}(2)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1) \oplus \mathbf{U}(2)$	(2, 2)	2	0
79	22	$\mathbf{U} \oplus [2] \oplus [-2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]$	(2, 2)	2	1
80	22	$\mathbf{U}(2)^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	(2, 2)	4	0
81	22	$[2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 2} \oplus [-6]$	(2, 2)	4	1
82	23	$\mathbf{U} \oplus [2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]$	(2, 1)	1	1
83	23	$[2]^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	(2, 1)	3	1
84	24	$[2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 2}$	(2, 0)	2	1

TABLE 7. Pairs  $(\mathbf{L}^G, \mathbf{L}_G)$  for  $G \subset \text{O}(\mathbf{L})$  of prime order  $p = 2$  and  $\text{sgn}(\mathbf{L}_G) = \text{sgn}(\Lambda_G) - (1, 0)$ , i.e. non-trivial action on the discriminant group.

No.	rk( $\mathbf{L}_G$ )	$\mathbf{L}_G = (\mathbf{L}^G)^{\perp \mathbf{L}}$	$\mathbf{L}^G = [2]^{\perp \mathbf{L}^G}$	$\text{sgn}(\mathbf{L}^G)$	$a(\mathbf{L}^G)$	$\delta(\mathbf{L}^G)$
1	3	$[2]^{\oplus 2} \oplus [-6]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [-2]^{\oplus 3}$	(1, 20)	3	1
2	4	$[2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [-2]^{\oplus 2}$	(1, 19)	2	1
3	4	$[2]^{\oplus 2} \oplus [-2] \oplus [-6]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 3}$	(1, 19)	3	1
4	5	$\mathbf{U} \oplus [2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [-2]$	(1, 18)	1	1
6	5	$[2]^{\oplus 2} \oplus [-2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 2}$	(1, 18)	3	1
7	5	$[2]^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-6]$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]$	(1, 18)	5	1
8	6	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}$	(1, 17)	0	0
9	6	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}(2)$	(1, 17)	2	0
11	6	$\mathbf{U} \oplus [2] \oplus [-2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2] \oplus [-2]$	(1, 17)	2	1
12	6	$\mathbf{U}(2)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2}$	(1, 17)	4	0
13	6	$[2]^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_6(-1) \oplus [-2]^{\oplus 2}$	(1, 17)	4	1
14	6	$[2]^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-6]$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 4}$	(1, 17)	6	1
15	7	$\mathbf{U}^{\oplus 2} \oplus [-2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]$	(1, 16)	1	1
16	7	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_6(-1) \oplus [-2]$	(1, 16)	3	1
17	7	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-6]$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 3}$	(1, 16)	5	1
18	7	$[2]^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [-6]$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [-2]^{\oplus 7}$	(1, 16)	7	1
19	8	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_6(-1)$	(1, 15)	2	0
20	8	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 2}$	(1, 15)	4	1
21	8	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2] \oplus [-6]$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 3}$	(1, 15)	6	1
22	8	$[2]^{\oplus 2} \oplus [-2]^{\oplus 5} \oplus [-6]$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 7}$	(1, 15)	8	1
23	9	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]$	(1, 14)	3	1
24	9	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	(1, 14)	5	1

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Table 7, follows from previous page

No.	rk( $L_G$ )	$L_G = (L^G)^{\perp L}$	$L^G = [2]^{\perp \Lambda^G}$	$\text{sgn}(L^G)$	$a(L^G)$	$\delta(L^G)$
25	9	$[2]^{\oplus 2} \oplus [-2]^{\oplus 5} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 6}$	(1, 14)	7	1
26	9	$[2]^{\oplus 2} \oplus [-2]^{\oplus 6} \oplus [-6]$	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]^{\oplus 5}$	(1, 14)	9	1
27	10	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1) \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus \mathbf{U}(2)$	(1, 13)	4	0
28	10	$\mathbf{U} \oplus [2] \oplus [-2] \oplus \mathbf{D}_4(-1) \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]$	(1, 13)	4	1
29	10	$\mathbf{U}(2)^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 3}$	(1, 13)	6	0
30	10	$\mathbf{D}_6(-1) \oplus [2]^{\oplus 3} \oplus [-6]$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 5}$	(1, 13)	6	1
31	10	$[2]^{\oplus 2} \oplus [-2]^{\oplus 6} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]^{\oplus 4}$	(1, 13)	8	1
32	10	$\mathbf{U}(2)^{\oplus 2} \oplus \mathbf{E}_6(-2)$	$\mathbf{U} \oplus \mathbf{E}_8(-2) \oplus \mathbf{D}_4(-1)$	(1, 13)	10	0
33	10	$[2]^{\oplus 2} \oplus [-2]^{\oplus 7} \oplus [-6]$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 8}$	(1, 13)	10	1
34	11	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-6]$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [-2]^{\oplus 3}$	(1, 12)	3	1
35	11	$\mathbf{U} \oplus 2 \oplus [-2]^{\oplus 6} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 4}$	(1, 12)	5	1
36	11	$\mathbf{D}_4(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-6]$	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]^{\oplus 3}$	(1, 12)	7	1
37	12	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [-6]$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 7}$	(2, 12)	9	1
38	11	$[2]^{\oplus 2} \oplus [-2]^{\oplus 8} \oplus [-6]$	$\mathbf{U} \oplus [-2]^{\oplus 11}$	(1, 12)	11	1
39	12	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [-2]^{\oplus 2}$	(1, 11)	2	1
40	12	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [2] \oplus [-2]$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 3}$	(1, 11)	4	1
41	12	$\mathbf{D}_4(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]^{\oplus 2}$	(1, 11)	6	1
42	12	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 6}$	(1, 11)	8	1
43	12	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5} \oplus [-6]$	$\mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 7}$	(1, 11)	10	1
44	12	$[2]^{\oplus 2} \oplus [-2]^{\oplus 9} \oplus [-6]$	$[2] \oplus [-2]^{\oplus 11}$	(1, 11)	12	1
45	13	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [-2]$	(1, 10)	1	1
46	13	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	(1, 10)	3	1
47	13	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [2] \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]$	(1, 10)	5	1
48	13	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 4} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 5}$	(1, 10)	7	1
49	13	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 8} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus [-2]^{\oplus 9}$	(1, 10)	9	1
50	13	$[2]^{\oplus 2} \oplus [-2]^{\oplus 9} \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]^{\oplus 10}$	(1, 10)	11	1
51	14	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U}$	(1, 9)	0	0
52	14	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U}(2)$	(1, 9)	2	0
53	14	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]$	(1, 9)	2	1
54	14	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2} \oplus [-6]$	$\mathbf{U} \oplus \mathbf{D}_6(-1) \oplus [-2]^{\oplus 2}$	(1, 9)	4	1
55	14	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U}(2) \oplus \mathbf{D}_4(-1)^{\oplus 2}$	(2, 9)	6	0
56	14	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 4} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 4}$	(1, 9)	6	1
57	14	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-2) \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{E}_8(-2)$	(1, 9)	8	0
58	14	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 5} \oplus \mathbf{A}_2(-1)$	$\mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 5}$	(1, 9)	8	1
59	14	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{E}_8(-2) \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-2) \oplus \mathbf{U}(2)$	(1, 9)	10	0
60	14	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 9} \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]^{\oplus 9}$	(1, 9)	10	1
61	15	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus [2]$	(1, 8)	1	1
62	15	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 3}$	(1, 8)	5	1
63	15	$\mathbf{U} \oplus \mathbf{D}_6(-1) \oplus [2] \oplus [-2]^{\oplus 4} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus [-2]^{\oplus 7}$	(1, 8)	7	1
64	15	$\mathbf{D}_6(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5} \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]^{\oplus 8}$	(1, 8)	9	1
65	16	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 2}$	(1, 7)	4	1
66	16	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus [-2]^{\oplus 6}$	(1, 7)	6	1
67	16	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5} \oplus [-6]$	$[2] \oplus [-2]^{\oplus 7}$	(1, 7)	8	1
68	17	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]$	(1, 6)	3	1
69	17	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 4} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus [-2]^{\oplus 5}$	(1, 6)	5	1
70	17	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5} \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]^{\oplus 6}$	(1, 6)	7	1
71	18	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus [-2]^{\oplus 4}$	(1, 5)	4	1
72	18	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 5} \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]^{\oplus 5}$	(1, 5)	6	1
73	19	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-6]$	$\mathbf{U} \oplus [-2]^{\oplus 3}$	(1, 4)	3	1
74	19	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 5} \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]^{\oplus 4}$	(1, 4)	5	1
75	20	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus [-2]^{\oplus 2}$	(1, 3)	2	1
76	20	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2] \oplus [-6]$	$[2] \oplus [-2]^{\oplus 3}$	(1, 3)	4	1
77	21	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2] \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus [-2]$	(1, 2)	1	1
78	21	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2] \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]^{\oplus 2}$	(1, 2)	3	1
79	22	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U}$	(1, 1)	1	1
80	22	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2] \oplus [-2] \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]$	(1, 1)	2	1
81	23	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [2] \oplus \mathbf{A}_2(-1)$	$[2]$	(1, 0)	1	1

A.2.  $p \geq 3$ .

TABLE 8. Pairs  $(\Lambda^G, \Lambda_G)$  for  $G \subset O(\Lambda)$  of prime order  $p \geq 3$  and  $\text{sgn}(\Lambda_G) = (2, \text{rk}(\Lambda_G) - 2)$ .

No.	$\text{rk}(\Lambda^G)$	$\Lambda_G$	$\Lambda^G$	$\text{sgn}(\Lambda_G)$	$a$	$p$
1	24	$\mathbf{A}_2$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	(2, 0)	1	3
2	22	$\mathbf{U}^{\oplus 2}$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{E}_8(-1)^{\oplus 2}$	(2, 2)	0	3
3	22	$\mathbf{U} \oplus \mathbf{U}(3)$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1)^{\oplus 2}$	(2, 2)	2	3
4	20	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{E}_8(-1) \oplus \mathbf{E}_6(-1)$	(2, 4)	1	3
5	20	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	(2, 4)	3	3
6	18	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)^{\oplus 2}$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{E}_6(-1)^{\oplus 2}$	(2, 6)	2	3
7	18	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 2}$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6(-1)^{\oplus 2}$	(2, 6)	4	3
8	16	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_6(-1)$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	(2, 8)	1	3
9	16	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6(-1)$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)^{\oplus 2}$	(2, 8)	3	3
10	16	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)^{\oplus 2}$	(2, 8)	5	3
11	14	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1)$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{E}_8(-1)$	(2, 10)	0	3
12	14	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1)$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1)$	(2, 10)	2	3
13	14	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)^{\oplus 4}$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{A}_2(-1)^{\oplus 4}$	(2, 10)	4	3
14	14	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 4}$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 4}$	(2, 10)	6	3
15	12	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{E}_6(-1)$	(2, 12)	1	3
16	12	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6(-1)$	(2, 12)	3	3
17	12	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)^{\oplus 5}$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	(2, 12)	5	3
18	12	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 5}$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6^*(-1)$	(2, 12)	7	3
19	10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_6(-1)^{\oplus 2}$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{A}_2(-1)^{\oplus 2}$	(2, 14)	2	3
20	10	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6(-1)^{\oplus 2}$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 2}$	(2, 14)	4	3
21	10	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	$\mathbf{U} \oplus \mathbf{U}(3)^{\oplus 2} \oplus \mathbf{A}_2(-1)^{\oplus 2}$	(2, 14)	6	3
22	10	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 6}$	$\mathbf{U}(3)^{\oplus 3} \oplus \mathbf{A}_2(-1)^{\oplus 2}$	(2, 14)	8	3
23	8	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{E}_6(-1)$	(2, 16)	1	3
24	8	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1) \oplus \mathbf{E}_6(-1)$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)$	(2, 16)	3	3
25	8	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)^{\oplus 4}$	$\mathbf{U} \oplus \mathbf{U}(3)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	(2, 16)	5	3
25*	8	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)^{\oplus 4}$	$\mathbf{U}(3)^{\oplus 3} \oplus \mathbf{A}_2(-1)$	(2, 16)	7	3
26	6	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1)^{\oplus 2}$	$\mathbf{U}^{\oplus 3}$	(2, 18)	0	3
27	6	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1)^{\oplus 2}$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(3)$	(2, 18)	2	3
28	6	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)^{\oplus 4}$	$\mathbf{U} \oplus \mathbf{U}(3)^{\oplus 2}$	(2, 18)	4	3
29*	6	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)^{\oplus 5}$	$\mathbf{U}(3)^{\oplus 3}$	(2, 18)	6	3
30	4	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{A}_2$	(2, 20)	1	3
31	4	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U}(3) \oplus \mathbf{A}_2$	(2, 20)	3	3
32	22	$\mathbf{U} \oplus \mathbf{H}_5$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{H}_5 \oplus \mathbf{E}_8(-1)^{\oplus 2}$	(2, 2)	1	5
33	18	$\mathbf{U} \oplus \mathbf{H}_5 \oplus \mathbf{A}_4(-1)$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{H}_5 \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_4(-1)$	(2, 6)	2	5
34	14	$\mathbf{U} \oplus \mathbf{H}_5 \oplus \mathbf{E}_8(-1)$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{H}_5 \oplus \mathbf{E}_8(-1)$	(2, 10)	1	5
35	14	$\mathbf{U} \oplus \mathbf{H}_5 \oplus \mathbf{A}_4(-1)^{\oplus 2}$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{H}_5 \oplus \mathbf{A}_4(-1)^{\oplus 2}$	(2, 10)	3	5
36	10	$\mathbf{U} \oplus \mathbf{H}_5 \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_4(-1)$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{H}_5 \oplus \mathbf{A}_4(-1)$	(2, 14)	2	5
37	10	$\mathbf{U} \oplus \mathbf{H}_5 \oplus \mathbf{A}_4(-1)^{\oplus 3}$	$\mathbf{U} \oplus \mathbf{U}(5) \oplus \mathbf{H}_5 \oplus \mathbf{A}_4(-1)$	(2, 14)	4	5
38	6	$\mathbf{U} \oplus \mathbf{H}_5 \oplus \mathbf{E}_8(-1)^{\oplus 2}$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{H}_5$	(2, 18)	1	5
39	6	$\mathbf{U} \oplus \mathbf{H}_5 \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_4(-1)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{U}(5) \oplus \mathbf{H}_5$	(2, 18)	3	5
40*	6	$\mathbf{U}(5)^{\oplus 2} \oplus \mathbf{A}_4(-1)$	$\mathbf{U}(5)^{\oplus 2} \oplus \mathbf{H}_5$	(2, 18)	5	5
41	20	$\mathbf{U}^{\oplus 2} \oplus \mathbf{K}_7(-1)$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_6(-1)$	(2, 4)	1	7
42	14	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1)$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{E}_8(-1)$	(2, 10)	0	7
43	14	$\mathbf{U} \oplus \mathbf{U}(7) \oplus \mathbf{E}_8(-1)$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(7) \oplus \mathbf{E}_8(-1)$	(2, 10)	2	7
44	8	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_6(-1)$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{K}_7$	(2, 16)	1	7
45	8	$\mathbf{U} \oplus \mathbf{U}(7) \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_6(-1)$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(7) \oplus \mathbf{K}_7$	(2, 16)	3	7
46	16	$\mathbf{K}_{11} \oplus \mathbf{E}_8(-1)$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{A}_{10}(-1)$	(2, 8)	1	11
47	6	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1)^{\oplus 2}$	$\mathbf{U}^{\oplus 3}$	(2, 18)	0	11
48	6	$\mathbf{U} \oplus \mathbf{U}(11) \oplus \mathbf{E}_8(-1)^{\oplus 2}$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(11)$	(2, 18)	2	11
49	14	$\mathbf{U} \oplus \mathbf{H}_{13} \oplus \mathbf{E}_8(-1)$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{H}_{13} \oplus \mathbf{E}_8(-1)$	(2, 10)	1	13
50	10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1) \oplus \mathbf{L}_{17}(-1)$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{L}_{17}(-1)$	(2, 14)	1	17
51	8	$\mathbf{K}_{19} \oplus \mathbf{E}_8(-1)^{\oplus 2}$	$\mathbf{U}^{\oplus 3} \oplus \mathbf{K}_{19}(-1)$	(2, 16)	1	19
52	4	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{K}_{23}(-1)$	$\mathbf{U} \oplus \mathbf{K}_{23}$	(2, 20)	1	23

TABLE 9. Pairs  $(\mathbf{L}^G, \mathbf{L}_G)$  for  $G \subset \mathrm{O}(\mathbf{L})$  of prime order  $p \geq 3$  and  $\mathrm{sgn}(\mathbf{L}_G) = \mathrm{sgn}(\mathbf{A}_G)$ .

No.	$\mathrm{rk}(\mathbf{L}^G)$	$\mathbf{L}_G$	$\mathbf{L}^G$	$\mathrm{sgn}(\mathbf{L}_G)$	$H$	$p$
1	22	$\mathbf{A}_2$	$\mathbf{U} \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)^{\oplus 2}$	(2, 0)	id	3
2	20	$\mathbf{U}^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	(2, 2)	id	3
3	20	$\mathbf{U} \oplus \mathbf{U}(3)$	$\mathbf{U}(3) \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	(2, 2)	id	3
4	20	$\mathbf{U} \oplus \mathbf{U}(3)$	$\mathbf{U} \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	(2, 2)	$\mathbb{Z}/3\mathbb{Z}$	3
5	18	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{E}_8(-1) \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)$	(2, 4)	id	3
6	18	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{E}_8(-1)^{\oplus 2}$	(2, 4)	$\mathbb{Z}/3\mathbb{Z}$	3
7	18	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)^{\oplus 4}$	(2, 4)	id	3
8	18	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{E}_8(-1) \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)$	(2, 4)	$\mathbb{Z}/3\mathbb{Z}$	3
9	16	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{E}_6(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	(2, 6)	id	3
10	16	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{E}_8(-1) \oplus \mathbf{E}_6(-1)$	(2, 6)	$\mathbb{Z}/3\mathbb{Z}$	3
11	16	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 2}$	$\mathbf{U}(3) \oplus \mathbf{E}_6(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	(2, 6)	id	3
12	16	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{E}_6(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	(2, 6)	$\mathbb{Z}/3\mathbb{Z}$	3
13	14	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_6(-1)$	$\mathbf{U} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)^{\oplus 2}$	(2, 8)	id	3
14	14	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)^{\oplus 3}$	$\mathbf{U} \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	(2, 8)	id	3
15	14	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)^{\oplus 3}$	$\mathbf{U} \oplus \mathbf{E}_6(-1)^{\oplus 2}$	(2, 8)	$\mathbb{Z}/3\mathbb{Z}$	3
16	14	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	$\mathbf{U}(3) \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	(2, 8)	id	3
17	14	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	$\mathbf{U} \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	(2, 8)	$\mathbb{Z}/3\mathbb{Z}$	3
18	12	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1)$	$\mathbf{U} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	(2, 10)	id	3
19	12	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1)$	$\mathbf{U}(3) \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	(2, 10)	id	3
20	12	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1)$	$\mathbf{U} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	(2, 10)	$\mathbb{Z}/3\mathbb{Z}$	3
21	12	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)^{\oplus 4}$	$\mathbf{U} \oplus \mathbf{A}_2(-1)^{\oplus 5}$	(2, 10)	id	3
22	12	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)^{\oplus 4}$	$\mathbf{U}(3) \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	(2, 10)	$\mathbb{Z}/3\mathbb{Z}$	3
23	12	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 4}$	$\mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 5}$	(2, 10)	id	3
24	12	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 4}$	$\mathbf{U} \oplus \mathbf{A}_2(-1)^{\oplus 5}$	(2, 10)	$\mathbb{Z}/3\mathbb{Z}$	3
25	10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)$	(2, 12)	id	3
26	10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{E}_8(-1)$	(2, 12)	$\mathbb{Z}/3\mathbb{Z}$	3
27	10	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	$\mathbf{U}(3) \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)$	(2, 12)	id	3
28	10	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)$	(2, 12)	$\mathbb{Z}/3\mathbb{Z}$	3
29	10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)^{\oplus 5}$	$\mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 4}$	(2, 12)	id	3
30	10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)^{\oplus 5}$	$\mathbf{U}(3) \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)$	(2, 12)	$\mathbb{Z}/3\mathbb{Z}$	3
31	10	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 5}$	$\mathbf{U}(3) \oplus \mathbf{E}_6^*(-1) \oplus \mathbf{A}_2(-1)$	(2, 12)	id	3
32	10	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 5}$	$\mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 4}$	(2, 12)	id	3
33	8	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_6(-1)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{A}_2(-1)^{\oplus 3}$	(2, 14)	id	3
34	8	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_6(-1)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{E}_6(-1)$	(2, 14)	$\mathbb{Z}/3\mathbb{Z}$	3
35	8	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6(-1)^{\oplus 2}$	$\mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	(2, 14)	id	3
36	8	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6(-1)^{\oplus 2}$	$\mathbf{U}(3) \oplus \mathbf{E}_6(-1)$	(2, 14)	$\mathbb{Z}/3\mathbb{Z}$	3
37	8	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	$\mathbf{U}(3) \oplus \mathbf{E}_6^*(-1)$	(2, 14)	id	3
38	8	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	$\mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 3}$	(2, 14)	$\mathbb{Z}/3\mathbb{Z}$	3
39	8	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 6}$	$\mathbf{U}(3) \oplus \mathbf{E}_6^*(-1)$	(2, 14)	$\mathbb{Z}/3\mathbb{Z}$	3
40	6	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{E}_6(-1)^{\oplus 2}$	(2, 16)	id	3
41	6	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1) \oplus \mathbf{E}_6(-1)$	$\mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 2}$	(2, 16)	id	3
42	6	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1) \oplus \mathbf{E}_6(-1)$	$\mathbf{U} \oplus \mathbf{A}_2(-1)^{\oplus 2}$	(2, 16)	$\mathbb{Z}/3\mathbb{Z}$	3
43	6	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)^{\oplus 4}$	$\mathbf{U}(3) \oplus \mathbf{A}_2(-1)^{\oplus 2}$	(2, 16)	$\mathbb{Z}/3\mathbb{Z}$	3
44	4	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{A}_2(-1)$	(2, 18)	id	3
45	4	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1)^{\oplus 2}$	$\mathbf{U}(3) \oplus \mathbf{A}_2(-1)$	(2, 18)	id	3
46	4	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{A}_2(-1)$	(2, 18)	$\mathbb{Z}/3\mathbb{Z}$	3
47	4	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)^{\oplus 4}$	$\mathbf{U}(3) \oplus \mathbf{A}_2(-1)$	(2, 18)	$\mathbb{Z}/3\mathbb{Z}$	3
48	2	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U}(3)$	(2, 20)	id	3
49	2	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U}$	(2, 20)	$\mathbb{Z}/3\mathbb{Z}$	3
50	4	$\mathbf{U} \oplus \mathbf{U}(3) \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U}(3)$	(2, 20)	$\mathbb{Z}/3\mathbb{Z}$	3
51	20	$\mathbf{U} \oplus \mathbf{H}_5$	$\mathbf{H}_5 \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	(2, 2)	id	5
52	16	$\mathbf{U} \oplus \mathbf{H}_5 \oplus \mathbf{A}_4(-1)$	$\mathbf{H}_5 \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_4(-1) \oplus \mathbf{A}_2(-1)$	(2, 6)	id	5
53	12	$\mathbf{U} \oplus \mathbf{H}_5 \oplus \mathbf{E}_8(-1)$	$\mathbf{H}_5 \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	(2, 10)	id	5
54	12	$\mathbf{U} \oplus \mathbf{H}_5 \oplus \mathbf{A}_4(-1)^{\oplus 2}$	$\mathbf{H}_5 \oplus \mathbf{A}_4(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	(2, 10)	id	5
55	8	$\mathbf{U} \oplus \mathbf{H}_5 \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_4(-1)$	$\mathbf{H}_5 \oplus \mathbf{A}_4(-1) \oplus \mathbf{A}_2(-1)$	(2, 14)	id	5
56	8	$\mathbf{U} \oplus \mathbf{H}_5 \oplus \mathbf{A}_4(-1)^{\oplus 3}$	$\mathbf{U}(5) \oplus \mathbf{A}_4(-1) \oplus \mathbf{N}_{15}(-1)$	(2, 14)	id	5
57	4	$\mathbf{U} \oplus \mathbf{H}_5 \oplus \mathbf{E}_8(-1)^{\oplus 2}$	$\mathbf{H}_5 \oplus \mathbf{A}_2(-1)$	(2, 18)	id	5
58	4	$\mathbf{U} \oplus \mathbf{H}_5 \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_4(-1)^{\oplus 2}$	$\mathbf{U}(5) \oplus \mathbf{N}_{15}(-1)$	(2, 18)	id	5
59	18	$\mathbf{U}^{\oplus 2} \oplus \mathbf{H}_7$	$\mathbf{U} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_6(-1) \oplus \mathbf{A}_2(-1)$	(2, 4)	id	7
60	12	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1)$	$\mathbf{U} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	(2, 10)	id	7
61	12	$\mathbf{U} \oplus \mathbf{U}(7) \oplus \mathbf{E}_8(-1)$	$\mathbf{U}(7) \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	(2, 10)	id	7
62	6	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_6(-1)$	$\mathbf{U} \oplus \mathbf{K}_7(-1) \oplus \mathbf{A}_2(-1)$	(2, 16)	id	7
63	6	$\mathbf{U} \oplus \mathbf{U}(7) \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_6(-1)$	$\mathbf{U}(7) \oplus \mathbf{K}_7(-1) \oplus \mathbf{A}_2(-1)$	(2, 16)	id	7
64	14	$\mathbf{U} \oplus \mathbf{H}_{11} \oplus \mathbf{E}_8(-1)$	$\mathbf{H}_{11} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	(2, 8)	id	11

Continues on next page



Table 9, follows from previous page

No.	$\text{rk}(\mathbf{L}^G)$	$\mathbf{L}_G$	$\mathbf{L}^G$	$\text{sgn}(\mathbf{L}_G)$	$H$	$p$
65	4	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{A}_2(-1)$	(2, 18)	id	11
66	4	$\mathbf{U} \oplus \mathbf{U}(11) \oplus \mathbf{E}_8(-1)^{\oplus 2}$	$\mathbf{U}(11) \oplus \mathbf{A}_2(-1)$	(2, 18)	id	11
67	12	$\mathbf{U} \oplus \mathbf{H}_{13} \oplus \mathbf{E}_8(-1)$	$\mathbf{H}_{13} \oplus \mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1)$	(2, 10)	id	13
68	6	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1) \oplus \mathbf{L}_{17}$	$\mathbf{U} \oplus \mathbf{L}_{17} \oplus \mathbf{A}_2(-1)$	(2, 14)	id	17
69	6	$\mathbf{K}_{19} \oplus \mathbf{E}_8(-1)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus \mathbf{K}_{19}(-1)$	(2, 16)	1	19
70	2	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{K}_{23}$	$\mathbf{N}_{69}(-1)$	(2, 20)	id	23

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