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A Generalization of Linear Car-Following Theory

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A GENERALIZATION OF LINEAR CAR-FOLLOWING THEORY

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The linear theory of single-lane traffic flow is generalized by using an integral transform technique well-known in other branches of applied science. This technique introduces the idea of a memory function that describes the way in which a driver processes the information he receives from a lead vehicle. Several analytical examples are computed and the earlier linear model is discussed as a special case of the more general theory.

IN SEVERAL papers[1-4] a theory of car-following has been developed that depicts the way in which a driver-car unit reacts to maneuvers in the vehicle immediately in front of him. In linear car-following theory, the follower's acceleration is proportional to the relative speeds of his car and the leading car at some earlier time. The purpose of this paper is to extend this theory by introduction of a memory function that defines the way in which the following driver processes the information concerning these relative speeds.

MATHEMATICAL THEORY

THIS PAPER uses an integral transform technique that is a refinement of a method suggested in an earlier paper.[1] The transform technique makes it mathematically possible to consider a 'memory function' that defines the way in which a following driver processes his information. This kind of approach, shown schematically in Fig. 1, has been suggested by Herman in various talks on car-following.

The assumption that the relative speed is the only influencing factor on the following car's acceleration is retained. What is added here is that the response of the follower depends not on what the relative speed was at a certain earlier instant, but rather on its time history. This may be expressed mathematically by,

$$\ddot{x}_n(t)=\int_0^t M(t-t')[\dot{x}_{n-1}(t')-\dot{x}_n(t')]\,dt'. \tag{1}$$

* The work underlying this paper was performed while the author was employed by the Research Laboratories of the General Motors Corporation, Warren, Michigan.

In this equation, x_n is the displacement of the nth car in a queue of cars. The memory function M, which gives a functional representation of the way the following driver acts upon the information he has received, contains two parameters that roughly correspond to λ and τ in the old linear model.[2] These two parameters, $\bar{\lambda}$ and $\bar{\tau}$, which will be called the characteristic gain and the average retardation, are defined as

$$\bar{\lambda}=\int_0^\infty M(t)\ dt, \tag{2}$$

$$\bar{\tau}=\frac{1}{\bar{\lambda}}\int_0^\infty tM(t)\ dt. \tag{3}$$

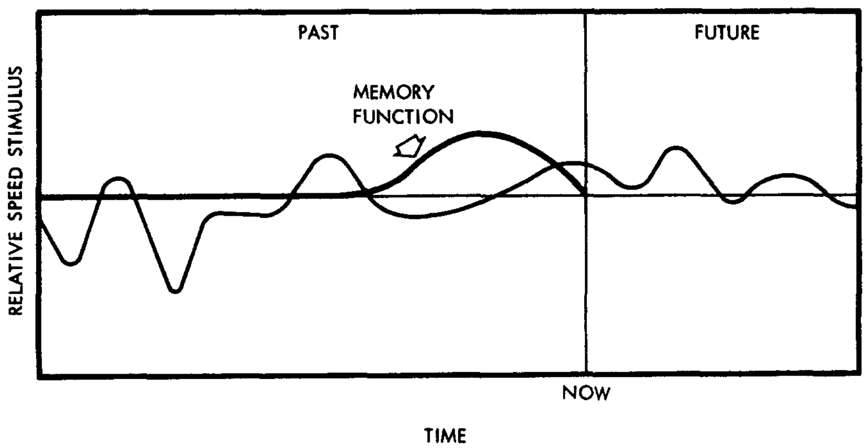

Figure 1

Equation (2) may be Laplace transformed—the right-hand side is a convolution—to give

$$\begin{aligned}s^2\bar{x}_n(s)-sx_n(0)-\dot{x}_n(0)\\ =\bar{M}(s)[s\bar{x}_{n-1}(s)-x_{n-1}(0)-s\bar{x}_n(s)+x_n(0)].\end{aligned} \tag{4}$$

If we assume that the coordinate system is chosen so that $x_n(0)=0$, then we may rewrite equation (4) as

$$\bar{x}_n(s)=\{\dot{x}_n(0)+\bar{M}(s)[s\bar{x}_{n-1}(s)-x_{n-1}(0)]\}/s[s+\bar{M}(s)]. \tag{5}$$

The primary purpose of a car-following model is to test for instability in the motion that may lead to collisions. Two kinds of instability are defined with respect to this theory. Local instability implies that for a bounded motion in the lead car, the response of the following car becomes unbounded and results in a collision between the two. Asymptotic instability refers to that situation when a maneuver in the lead car propagates down the queue with increasing amplitude, thus causing a collision somewhere down the line.

The determination of local stability results from a perturbation analysis of the equilibrium condition in car-following—when both the lead and follower cars are moving at the same speed. An incremental change in velocity is introduced into the lead car and the resulting effect on the follower is noted. This analysis, mathematically, is carried out by investigating the nontrivial part of the system's characteristic equation, namely

$$s+\tilde{M}(s)=0. \tag{6}$$

If the roots of equation (6) have negative real parts, then the system is locally stable.

Asymptotic stability is determined by allowing the lead car to undergo a periodic maneuver about its original speed. If the amplitude of the eventual periodic maneuver in the follower is greater than that of the lead car, the system is said to be asymptotically unstable.

EXAMPLES

A. *Memory Function as Dirac-Delta Function*

If the memory function M is given by

$$M(t)=\lambda\delta(t-\tau), \tag{7}$$

then the equation of motion for the follower reduces to the linear differential-difference equation that has been analyzed extensively elsewhere.[1-3] For this impulsive memory function, the new parameters $\bar{\lambda}$ and $\bar{\tau}$ reduce to the λ and τ of the old linear model. The nontrivial part of the characteristic equation is given by

$$s+\lambda e^{-\tau s}=0. \tag{8}$$

The nature of the solutions depends on the quantity $\lambda\tau$ and is catalogued in reference 2.

There are three objections, one mathematical and two physical, that could be raised with respect to this model. Although using a delta function for the memory function does give a simple equation of motion, its characteristic equation gives rise to an infinite number of poles in the complex plane, rendering complete, closed form solutions intractable.

Physically, this kind of memory function implies that the following driver acts instantaneously to correct any relative velocity difference at a time τ ago. Intuition suggests that a more realistic memory function would be a weighted response over a finite interval of past history. Furthermore, the actual solution to equation (1), found by repeated application of the 'Shift Theorem,' shows that the follower undergoes a change in maneuver after every length of time $n\tau$, where n is an integer—even when the lead vehicle settles down to constant speed. This sort of behavior is not expected and has not been observed.

B. Memory Function as Decaying Exponential

A memory function of the type

$$M(t)=\alpha k e^{-kt}, \tag{9}$$

greatly simplifies the mathematical process because the characteristic equation is a quadratic and the transform may be inverted readily. Physically, this model is probably unrealistic because it requires a partially instantaneous reaction on the part of the following vehicle; however, the mathematics is quite instructive. A shifted exponential decay would be a similar memory function that would probably be more realistic.

For this kind of memory function, the nontrivial part of the characteristic equation is

$$s^2+ks+\alpha k=0, \tag{10}$$

since the Laplace transform of M is given by

$$\bar{M}(s)=\alpha k/(k+s). \tag{11}$$

If α and k are both assumed to be positive, the roots of equation (10) always have negative real parts. This implies that the motion is always locally stable, a fact that might be intuitively suggested by the partially instantaneous and relatively smooth reaction of the follower. It should be pointed out that local stability does not ensure that there are no collisions. Whether or not there is a collision naturally depends on the original speed and spacing in the queue and could only be determined by a detailed solution of equation (1) with appropriate initial conditions. If a system is locally *unstable*, a collision will result no matter what these conditions are.

To determine asymptotic stability, we assume that the queue is initially at equilibrium and then permit the lead car to undergo a periodic maneuver about the equilibrium speed of the queue. Mathematically, we need to determine x_n from equation (1) where

$$\dot{x}_{n-1}(t)=A+B\sin\omega t. \qquad (A>B) \tag{12}$$

The initial conditions (at equilibrium) are given as

$$\dot{x}_n(0)=\dot{x}_{n-1}(0)=A;$$

$$x_n(0)=0; \qquad x_{n-1}(0)=D.$$

The displacement of the lead car is found by integrating equation (12) and applying the initial conditions,

$$x_{n-1}(t)=At-(B/\omega)\cos\omega t+D+B/\omega. \tag{13}$$

The Laplace transform of equation (13) is simply

$$\bar{x}_{n-1}(s)=(A/s^2)+[(D+B/\omega)/s]-(B/\omega)s/(s^2+\omega^2). \tag{14}$$

If this expression is substituted into equation (5), the transformed solution for the following car is given by

$$\bar{x}_n(s)=\{A+\alpha k/(s+k)[(B/\omega)+(A/s)-(B/\omega)s^2/(s^2+\omega^2)]\}\\ /s[s+\alpha k/(s+k)], \tag{15}$$

which reduces to

$$\bar{x}_n(s)=(A/s^2)+(B/\omega)(1/s)+(C_1s+C_2)/(s^2+ks+\alpha k)\\ +(C_3s+C_4)/(s^2+\omega^2), \tag{16}$$

where the C_i are values resulting from decomposition into partial fractions. Inverting this expression, the displacement of the following car is seen to be

$$x_n(t)=(B/\omega)+At+C_3\cos\omega t+(C_4/\omega)\sin\omega t\\ +L^{-1}[(C_1s+C_2)/(s^2+ks+\alpha k)]. \tag{17}$$

For α and k both positive, the last term can be neglected for considerations of asymptotic stability. Its inversion dies out in time and is therefore of importance only to the analyst who wishes to observe the way in which the follower approaches his final motion.

The speed of the following car thus eventually settles into a periodic behavior—mirroring the lead car—given by

$$\dot{x}_n(t)=A-\omega C_3\sin\omega t+C_4\cos\omega t. \tag{18}$$

The amplitude of this motion is

$$Y=(C_4^2+\omega^2C_3^2)^{1/2}. \tag{19}$$

In Appendix I, the values of C_3 and C_4 resulting from the partial fraction decomposition are carefully evaluated. For values of the product $\bar{\lambda}\bar{\tau}$ less than $\frac{1}{2}$, there are no frequencies that produce asymptotic instability. When $\bar{\lambda}\bar{\tau}$ is greater than $\frac{1}{2}$, those frequencies ω less than

$$\omega^2=(2\bar{\lambda}/\bar{\tau})-1/\bar{\tau}^2 \tag{20}$$

cause an increase in amplitude in the propagating wave, thus creating asymptotic instability. It is interesting to note that $\bar{\lambda}\bar{\tau}=\frac{1}{2}$ is also the critical value for asymptotic stability in the case where the memory function is impulsive but delayed.[2] If the motion of the lead car is a more complex periodic function than the one chosen, it may be expanded in a Fourier series and each of its frequencies analyzed.

C. Memory Function Proportional to te^{-kt}

$$M(t)=\alpha k^2 te^{-kt} \tag{21}$$

is more practical physically (initial response is zero) and does not complicate the mathematics very much. The coefficient in front was selected so that, as in the previous example, $\bar{\lambda}=\alpha$ and $\bar{\tau}=1/k$.

For this memory function, the nontrivial part of the characteristic equation is given by

$$s^3+2ks^2+k^2s+\alpha k^2=0, \tag{22}$$

since

$$\bar{M}(s)=\alpha k^2/(s+k)^2. \tag{23}$$

As would be expected since the initial response is zero, for this memory function there exist values of $\bar{\lambda}\bar{\tau}$ that produce local instability. Using standard techniques it can be shown that when

$$\bar{\lambda}\bar{\tau}>2, \tag{24}$$

there is a root of equation (22) with a positive real part, thus causing local instability.

Asymptotic stability will be determined in a manner similar to the preceding example. Consider a queue of cars initially at some equilibrium speed A. Allow a lead car to undergo a maneuver,

$$\dot{x}_{n-1}(t)=A+B\sin\omega t, \qquad (A>B) \tag{12}$$

where the initial conditions are the same as they were when the memory function was a decaying exponential. The displacement of the lead car is still given by equation (13) and its transform by equation (14); however, because of the different memory function, the transformed solution for the following car now becomes

$$\bar{x}_n(s)=\{A+[\alpha k^2/(s+k)^2][(B/\omega)+(A/s)-(B/\omega)s^2/s^2+\omega^2]\}/ s[s+\alpha k^2/(s+k)^2]. \tag{25}$$

This expression can be reduced to

$$\bar{x}_n(s)=(A/s^2)+(B/\omega)(1/s)+(C_1s+C_2)/(s^2+\omega^2) +(C_3s^2+C_4s+C_5)/(s^3+2ks^2+k^2s+\alpha k^2), \tag{26}$$

where, as before, the C_i are values resulting from decomposition into partial fractions. If equation (26) is inverted, the result gives the displacement of the follower car as

$$x_n(t)=(B/\omega)+At+C_1\cos\omega t+(C_2/\omega)\sin\omega t$$
$$+L^{-1}[(C_3s^2+C_4s+C_5)/(s^3+2ks^2+k^2s+\alpha k^2)]. \quad (27)$$

Just as in the previous example, the last term can be neglected for considerations of asymptotic stability. For $\bar{\lambda}\bar{\tau}<2$, its contribution to the motion dies out in time. If $\bar{\lambda}\bar{\tau}>2$, then the system is certainly asymptotically unstable since it is locally unstable.

When $\bar{\lambda}\bar{\tau}<2$, the speed of the following car eventually becomes periodic, governed by the equation

$$\dot{x}_n(t)=A-\omega C_1\sin\omega t+C_2\cos\omega t. \quad (28)$$

Its amplitude is

$$Y=(C_2^2+\omega^2C_1^2)^{1/2}. \quad (29)$$

In Appendix II, the values of C_1 and C_2 are calculated from the partial fraction decomposition. For values of the product $\bar{\lambda}\bar{\tau}$ less than ¼, there are no frequencies that give rise to asymptotic instability. When $\bar{\lambda}\bar{\tau}$ does have a value greater than ¼, asymptotic instability results when the frequency of the lead car is less than

$$\omega^2=(4\bar{\lambda}/\bar{\tau}^3)^{1/2}-1/\bar{\tau}^2. \quad (30)$$

This equation is derived in Appendix II.

D. *Memory Function as Unitary Square Wave*

Another possible memory function would be

$$M(t)=\lambda\{[S_{\tau-p}(t)-S_{\tau+p}(t)]/2p\}, \qquad (\tau>p) \quad (31)$$

where S is the Heaviside step function. This memory function has the nice mathematical property that $\bar{\lambda}=\lambda$ and $\bar{\tau}=\tau$ for *all* values of p. Since the limit of this function as p tends to zero is the Dirac-delta function, an analysis of this square wave shows the effect of 'smearing' the memory function while keeping constant $\bar{\lambda}$ and $\bar{\tau}$. Physically, this memory function implies that the following driver's response is spread over a finite interval. For varying interval lengths, the product of the length and the intensity of the response remains a constant.

The characteristic equation for such a memory function is a transcendental equation with an infinite number of roots in the complex plane. If the half-width p of the wave is written

$$p=k\tau, \qquad (0\leqq k\leqq 1) \quad (32)$$

it can be shown, after some algebraic manipulations, that the critical value for $\bar{\lambda}\bar{\tau}$ forming the upper bound of local stability is given by

$$\bar{\lambda}\bar{\tau}(k)=k\pi^2/4\sin(k\pi/2). \qquad (0\leqq k\leqq 1) \quad (33)$$

As k approaches zero, the critical value for $\bar{\lambda}\bar{\tau}$ approaches $\pi/2$. This is consistent with the results derived in the earlier literature.[2]

From the point of view of local stability, the effect of 'smearing' is very evident. As the width of the memory function increases, the region of local

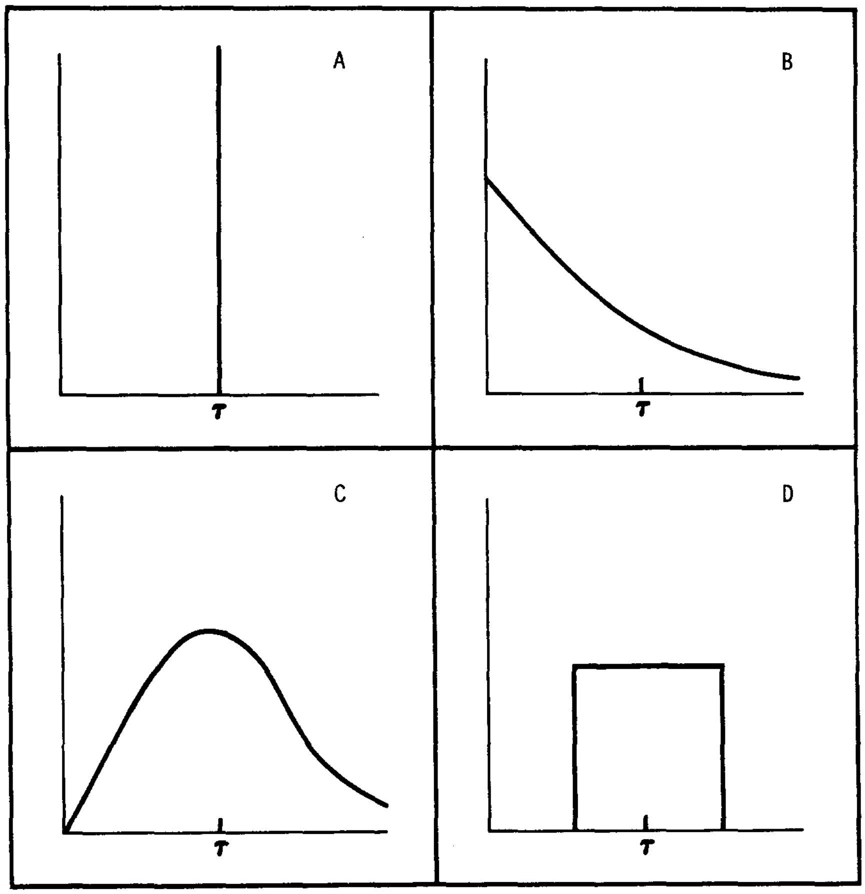

Figure 2

stability becomes larger; at $k=1$, the system is locally stable for $\lambda\tau<\pi^2/4$.

The investigation of the asymptotic stability for this kind of memory function would be computationally difficult. It appears plausible, however, in view of the results of the preceding examples, that the asymptotic stability will depend on the product $\bar{\lambda}\bar{\tau}$.

CONCLUSION

An attempt has been made in this paper to generalize the theory of single-lane traffic flow. This generalization uses the concept of a 'memory function' to explain the way in which a following driver reacts to a given maneuver in the lead car. Several examples of possible memory functions (Fig. 2) have been worked out and the earlier linear model discussed as a special case of the more general theory.

The next natural development of this theory, to try to determine the form of the memory function from experimental data already taken, is now in progress.[5]

APPENDIX I

Derivation of Amplitude in Equation (19)

Rewrite equation (15) as

$$\bar{x}_n(s) = A(s+k) + (B\alpha k/\omega) + (A\alpha k/s) - (B\alpha k/\omega)s^2/(s^2+\omega^2) / s(s^2+ks+\alpha k). \tag{34}$$

If we multiply the numerator and denominator by $\omega s(s^2+\omega^2)$, the result is

$$\bar{x}_n(s) = [A(s^2+ks+\alpha k)(s^2+\omega^2) + B\alpha k s\omega]/s^2(s^2+ks+\alpha k)(s^2+\omega^2), \tag{35}$$

which may be written as

$$\bar{x}_n(s) = (A/s^2) + (D/s) + (C_1 s + C_2)/(s^2+ks+\alpha k) + (C_3 s + C_4)/(s^2+\omega^2). \tag{36}$$

The values D and C_i result from a decomposition of the second term of (36) into partial fractions and are the solutions to the following system of equations,

$$\begin{aligned} D+C_1+C_3&=0,\\ kD+C_2+kC_3+C_4&=0,\\ (\alpha k+\omega^2)D+\omega^2 C_1+\alpha k C_3+kC_4&=0,\\ k\omega^2 D+\omega^2 C_2+\alpha k C_4&=0,\\ \alpha k\omega^2 D&=B\alpha k\omega. \end{aligned} \tag{37}$$

Clearly the value for D is given by

$$D = B/\omega. \tag{38}$$

To determine the amplitude of the periodic motion in the following car, we need to obtain the values C_3 and C_4 from (37). We can eliminate C_1 and C_2 from (37) by

$$\begin{aligned} C_1 &= -(B/\omega) - C_3,\\ C_2 &= (-C_4\alpha k - Bk\omega)/\omega^2, \end{aligned} \tag{39}$$

to yield two simultaneous equations in C_3 and C_4,

$$\begin{aligned} kC_3+C_4[1-(\alpha k/\omega^2)]&=0,\\ (\alpha k-\omega^2)C_3+kC_4&=-B\alpha k/\omega. \end{aligned} \tag{40}$$

These equations may be solved to give

$$C_4 = -B\alpha k^2/M\omega, \tag{41}$$

$$C_3 = -B\alpha k^3[(\alpha k/\omega^2)-1]/Mk\omega, \tag{42}$$

where

$$M = k^2 + [\omega - (\alpha k/\omega)]^2. \tag{43}$$

The amplitude of the follower car's periodic motion is given by equation (19):

$$Y = (C_4^2 + \omega^2 C_3^2)^{1/2}.$$

Asymptotic instability results when this value Y is greater than B.

Substituting (41) and (42) into (19), we obtain

$$Y = B(\alpha k/\omega)M^{-1/2}. \tag{44}$$

From equation (44) it is obvious that asymptotic instability results when

$$(\alpha k/\omega)M^{-1/2} > 1. \tag{45}$$

The critical value (when the amplitude remains the same) is found by making (45) an equality. Thus

$$\omega^4 + \omega^2(k^2 - 2\alpha k) = 0. \tag{46}$$

This expression may be factored to yield

$$\omega^2(\omega^2 + k^2 - 2\alpha k) = 0. \tag{47}$$

For real frequencies, the critical value occurs when

$$\omega^2 = 2\alpha k - k^2. \tag{48}$$

In terms of $\bar{\lambda}$ and $\bar{\tau}$ [*see* equations (3) and (4)], this may be written

$$\omega^2 = (2\bar{\lambda}/\bar{\tau}) - 1/\bar{\tau}^2. \tag{20}$$

APPENDIX II

Derivation of Amplitude in Equation (29)

We can rewrite equation (25) as

$$\bar{x}_n(s) = A(s+k)^2 + \alpha k^2[(B/\omega) + (A/s) - (B/\omega)s^2/(s^2+\omega^2)]/ s(s^3 + 2ks^2 + k^2 s + \alpha k^2). \tag{49}$$

Multiplying the numerator and denominator by the factor $\omega s(s^2+\omega^2)$, equation (49) becomes

$$\bar{x}_n(s) = (A/s^2) + \alpha k^2 B\omega / s(s^2+\omega^2)(s^3 + 2ks^2 + k^2 s + \alpha k^2). \tag{50}$$

This may be written as

$$\bar{x}_n(s) = (A/s^2) + (D/s) + (C_1 s + C_2)/(s^2+\omega^2) + (C_3 s^2 + C_4 s + C_5)/ (s^3 + 2ks^2 + k^2 s + \alpha k^2), \tag{51}$$

where the values D and C_i are solutions to a system of equations given by

$$
\begin{aligned}
D+C_1+C_3&=0,\\
2kD+2kC_1+C_2+C_4&=0,\\
(\omega^2+k^2)D+k^2C_1+2kC_2+\omega^2C_3+C_5&=0,\\
(2k\omega^2+\alpha k^2)D+\alpha k^2C_1+k^2C_2+\omega^2C_4&=0,\\
k^2\omega^2D+\alpha k^2C_2+\omega^2C_5&=0,\\
D\omega^2\alpha k^2&=B\alpha k^2\omega.
\end{aligned}
\tag{52}
$$

It is obvious, as in the previous example, that

$$D=B/\omega. \tag{53}$$

We need to determine C_1 and C_2 to calculate the amplitude of the following car's periodic maneuver. Eliminate C_3, C_4, and C_5 from (52) by substituting

$$
\begin{aligned}
C_3&=-C_1-(B/\omega),\\
C_4&=(-2Bk/\omega)-2kC_1-C_2,\\
C_5&=(-\alpha k^2C_2-Bk^2\omega)/\omega^2.
\end{aligned}
\tag{54}
$$

This gives us two simultaneous equations in C_1 and C_2,

$$
\begin{aligned}
C_1(k^2-\omega^2)+C_2[2k-(\alpha k^2/\omega^2)]&=0,\\
C_1(\alpha k^2-2k\omega^2)+C_2(k^2-\omega^2)&=-B\alpha k^2/\omega,
\end{aligned}
\tag{55}
$$

which have the solution

$$C_1=[(2B\alpha k^3/\omega)-(B\alpha^2k^4/\omega^3)]/N, \tag{56}$$

and

$$C_2=[B\alpha k^2\omega-(B\alpha k^4/\omega)]/N, \tag{57}$$

where

$$N=(k^2-\omega^2)^2+(\alpha^2k^4/\omega^2)+4k^2\omega^2-4\alpha k^3. \tag{58}$$

The amplitude Y of the follower is given by equation (29) as

$$Y=(C_2^2+\omega^2C_1^2)^{1/2}.$$

If we substitute our values for C_1 and C_2 into this equation, the result is

$$Y=B(\alpha k^2/\omega)N^{-1/2}. \tag{59}$$

Asymptotic instability occurs when

$$(\alpha k^2/\omega)N^{-1/2}>1. \tag{60}$$

The amplitude remains the same when

$$\alpha^2k^4/\omega^2N=1, \tag{61}$$

or

$$\omega^2(\omega^4+2k^2\omega^2+k^2-4\alpha k^3)=0. \tag{62}$$

For real frequencies, this critical value may be written

$$\omega^2=2(\alpha k^3)^{1/2}-k^2. \tag{63}$$

In terms of the quantities $\bar{\lambda}$ and $\bar{\tau}$, the critical frequency value is

$$\omega^2=(4\bar{\lambda}/\bar{\tau}^3)^{1/2}-1/\bar{\tau}^2. \tag{64}$$

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