Problem Set Template

Your name
Course number: Course name

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Problem 1.1. True or false? The set $V = (0, \infty)$ (positive reals) is a vector space with addition and scalar multiplication given by $x + y = xy$ and $\alpha \cdot x = x^\alpha$ for all $x, y \in (0, \infty), \alpha \in \mathbb{R}$

Solution. This is a vector space - we can go through all 8 of the definitional statements of a vector space and see that they hold:

- $\forall x_1, x_2, x_3 \in V, (x_1 + x_2) + x_3 = x_1x_2x_3 = x_1 + (x_2 + x_3)$
- $1 \in V$ is our additive identity
- The additive inverse of $x \in V$ is $1/x$
- $\forall x_1, x_2 \in V, x_1 + x_2 = x_2 + x_1$ since basic multiplication is commutative
- $\forall x \in V, 1 \cdot x = x^1 = x$
- $\forall \alpha, \beta \in \mathbb{R}, x \in V, (\alpha\beta) \cdot x = x^{\alpha\beta} = (x^\beta)^\alpha = \alpha \cdot (\beta \cdot x)$
- $\forall \alpha, \beta \in \mathbb{R}, x \in V, (\alpha + \beta) \cdot x = x^{\alpha+\beta} = x^\alpha x^\beta = (\alpha \cdot x) + (\beta \cdot x)$
- $\forall x, y \in V, \alpha \in \mathbb{R}, \alpha \cdot (x + y) = (xy)^\alpha = x^\alpha y^\alpha = \alpha \cdot x + \alpha \cdot y$

$\square$

Problem 1.2. Show that $C[0, 1]$, 1s with the usual pointwise operations is not a finite dimensional vector space.

Solution. Suppose that $C[0, 1]$ has finite dimension d. It is easy to concieve of a family of continuous functions $\{f_i | i \in (\mathbb{N} \cap [1, d+1])\}$ s.t. the support of f_i is contained in $((i-1)(d+1), i(d+1))$ and that no function in this family is everywhere 0. These are clearly lineary independent members of $C[0, 1]$, which is a contradiction since now we have $d + 1$ linearly independent members of a vector space of dimension d. $\square$

Problem 1.3. Let $S := \{\mathbf{x} = C^1[a, b] : \mathbf{x}(a) = y_a, \mathbf{x}(b) = y_b\}$, where $y_a, y_b \in \mathbb{R}$. Prove that S is a subspace of $C[a, b]$ iff $y_a = y_b = 0$.

Solution. Suppose that $y_a \neq 0$. Let $f_1, f_2 \in S$. Since S is a subspace, $f_1 + f_2 \in S$. Then $(f_1 + f_2)(a) = 2y_a \neq y_a$, and so by definition of S, $f_1 + f_2 \notin S$, and thus we have a contradiction. A similar argument holds when $y_b \neq 0$.

For the other direction we need to show that S is a vector space if $y_a = y_b = 0$. Most of the required properties are inherited directly from $C^1[a, b]$ - we only need to show additive and multiplicative closure, and that the additive identity lives in S. The additive identity in this case is the zero function which trivially lives in S. Note also that for any $f_1, f_2 \in S, \alpha \in \mathbb{R}$, we have

- $(f_1 + f_2)(a) = f_1(a) + f_2(a) = 0 + 0 = 0$
- $(f_1 + f_2)(b) = f_1(b) + f_2(b) = 0 + 0 = 0$

- $(\alpha \cdot f_1)(a) = \alpha f_1(a) = \alpha \cdot 0 = 0$
- $(\alpha \cdot f_1)(b) = \alpha f_1(b) = \alpha \cdot 0 = 0$

Therefore we have additive and multiplicative closure, and thus S is a subspace of $C^1[a, b]$. $\square$

Problem 1.4. In $C[0, 1]$ equipped with the $\|.\|_\infty$-norm, calculate the norms of t, $-t$, t^n and $\sin(2\pi nt)$, where $n \in \mathbb{N}$.

Solution. We have

- $\|t\|_\infty = \sup_{t\in(0,1)} |t| = 1$
- $\| - t\|_\infty = \sup_{t\in(0,1)} | - t| = 1$
- $\|t^n\|_\infty = \sup_{t\in(0,1)} |t^n| = 1$
- $\|t\|_\infty = \sup_{t\in(0,1)} |\sin(2\pi nt)| = 1$

$\square$

Problem 1.5. Let $(X, \| \cdot \|)$ be a normed space. Prove that $\forall x, y \in X, |\|x\| - \|y\|| \leq \|x - y\|$

Solution. By definition of norm, we have that $\|x\| = \|(x-y)+y\| \leq \|x-y\|+\|y\|$, and so $\|x\|-\|y\| \leq \|x-y\|$. By symmetry, we also have $\|y\| - \|x\| \leq \|y - x\|$. $(y - x) = -1 \cdot (x - y)$ and so by definition of norm, $\|y - x\| = | - 1| \cdot \|x - y\| = \|x - y\|$ Therefore

$$max(\|x\| - \|y\|, \|y\| - \|x\|) = |\|x\| - \|y\|| \leq \|x - y\|$$

$\square$

Problem 1.6. If $x \in \mathbb{R}$, then let $\|x\| = |x|^2$. Is $\|.\|$ a norm on $\mathbb{R}$?

Solution. $\|.\|$ is not a norm. We have $\|2x\| = |(2x)^2| = 4|x^2| = 4\|x\| \neq |2|\|x\|$(for $x \neq 0$). $\square$

Problem 1.7. Let X be a normed space with norm $\| \cdot \|_X$, and Y be a subspace of X. Prove that Y is also a normed space with the norm $\| \cdot \|_Y$ defined simply as the restriction of $\| \cdot \|_X$ to Y.

$\|.\|$ is not a norm. We have $\|2x\| = |(2x)^2| = 4|x^2| = 4\|x\| \neq |2|\|x\|$(for $x \neq 0$).

Solution. Additive and multiplicative closure of Y along with the norm properties of $\| \cdot \|_X$ has this drop out pretty trivially. $\square$

Problem 1.8. Let $1 < p < \inf$ and q s.t. $\frac{1}{p} + \frac{1}{q} = 1$.
Then if $x_1, ..., x_d, y_1, ..., y_d \in \mathbb{C}$, prove Hölder's inequality:

$$\sum_{n=1}^{d} |x_n y_n| \leq \left(\sum_{n=1}^{d} |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{d} |y_n|^q\right)^{\frac{1}{q}}$$

Solution. $\square$