Set 1 - Questions and Solutions

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1.1 Let V be a vector space, and let $f : V \to R$. Prove that f is affine if and only if $f(x) = g(x) + c$, where g is linear and $c \in R$. In the case where $f : R \to R$, the claim is that an affine function is a line that may or may not go through the origin.

If g is linear then

$f(px + (1-p)x') = g(px + (1-p)x)' + c = p(g(x) + c) + (1-p)(g(x') + c)$

which is just $pf(x) + (1-p)f(x')$, and so f is affine for any constant c.

For the other direction, we set $g(x) = f(x) - f(\mathbf{0})$ for our affine function f. We note that for any x

$$f(\mathbf{0}) = f\left(\frac{x}{2} + \frac{-x}{2}\right) = \frac{1}{2}(f(x) + f(-x))$$

from which we directly get

$$g(x) = -g(-x)$$

For any $\lambda \in (0,1)$ we have

$g(\lambda x) = f(\lambda x) - f(\mathbf{0}) = f(\lambda x + (1-\lambda)\mathbf{0}) - f(\mathbf{0}) = \lambda f(x) - \lambda(f(\mathbf{0})) = \lambda g(x)$

For $\lambda > 1$, we set $\theta = \lambda^{-1}$ and note that our previous result gives

$$\theta g((\lambda x)) = g(\theta(\lambda x))$$

for any x. Multiplying through by λ and simplifying $\theta\lambda = 1$ gives

$$g(\lambda x) = \lambda g(x)$$

Combined, these results tell us that g is homogeneous. Now, we note that

$$g(a+b) = 2g\left(\frac{a}{2} + \frac{b}{2}\right) = 2f\left(\frac{a}{2} + \frac{b}{2}\right) - 2f(\mathbf{0}) = f(a) + f(b) - 2f(\mathbf{0}) = g(a) + g(b)$$

and so g is additive as well as homogeneous. Therefore g is linear.

1.2 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and concave, and define T to be the set of all affine functions above f:

$$\boldsymbol{T = \{\phi : \phi \geq f, \phi \text{ is affine}\}}$$

Let $g(x) = \inf_{\phi \in T}\{\phi(x)\}$. Prove that $f = g$.

Intuitively, this is just the statement that a concave function lies below all its tangents. But f need not actually be differentiable, so might not have a tangent at every given point. We resolve this with the following construction:

Take an arbitrary real point x. Consider arbitrary reals a,b s.t. $a < x < b$. Then $p = \frac{b-x}{b-a}$ is s.t. $x = pa + (1-p)b$. By concavity of f

$$f(x) \geq pf(a) + (1-p)f(b) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)$$

Rearranging gives

$$\frac{f(x) - f(a)}{x-a} \geq \frac{f(b) - f(x)}{b-x}$$

a,b were arbitrary, with the only condition being $a < x < b$. Therefore the following expressions:

$$A = \inf_{a<x} \frac{f(x) - f(a)}{x-a}$$

$$B = \sup_{b>x} \frac{f(b) - f(a)}{b-x}$$

are both finite, with $B \leq A$. Now we note that the function $h(y) = A(y-x) + f(x)$ is affine via our result from question 1.1. We also note that $h(x) = f(x)$. $\forall a < x$ we have

$$h(a) = A(a-x) + f(x) \geq \frac{f(x) - f(a)}{x-a}(a-x) + f(x) = f(a)$$

(The inequality follows from the definition of A and the sign of $a - x$). A similar logic using the definition of B and the fact that $A \geq B$ shows us that $h \geq f$ everywhere. Therefore $h \in T$ and $g(x) <= h(x) = f(x)$. By definition of T, we must also have $g(x) \geq f(x)$, and so $g(x) = f(x)$. x was arbitrary, so therefore $f = g$.

1.3 Let $f : \mathbb{R} \to \mathbb{R}$ be concave that's monotone, 1-Lipschitz, and 0-increasing. Let T to be the set of all affine functions above f that are monotone, 1-Lipschitz, and 0-increasing:

$$T = \{\phi : \phi \geq f, \phi \text{ is affine, monotone, 1-Lipschitz, 0-increasing}\}$$

Let $g(x) = \inf_{\phi \in T}\{\phi(x)\}$. Prove that $f = g$.

Show also the connverse: Let $\{\phi_i\}_{i \in I}$ be a collection of affine functions $\mathbb{R}$ß$\mathbb{R}$ that are monotone, 1-Lipschitz, and 0-increasing. Let f(x) = infiIi(x). Prove also that f is continuous, concave, monotone, 1-Lipschitz, and 0-increasing.

We note that the function h