Mathematics for Machine Learning - Questions and Solutions

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Note - I'm not going to write out the questions here since they are very, very inefficiently posed and no way am I going to TeX all of that.

2.1 a. In order to show that this constitutes a group, we need to show four things:

- Closure - if $a, b \in \mathbb{R}$ then clearly $ab + a + b \in \mathbb{R}$. Now, suppose that $ab + a + b = -1$. This rearranges to

$$a(b+1) = -(b+1)$$

or

$$(a+1)(b+1) = 0$$

Therefore if neither a nor b is equal to -1, $a * b$ also cannot be equal to -1, and therefore * is a valid group operation on $\mathbb{R}\backslash\{-1\}$.

- Identity - our identity is 0 since for any $a \in \mathbb{R}\backslash\{-1\}$ we have

$$a * 0 = a \cdot 0 + 0 + a = a$$

- Inverse - given a fixed $a \in \mathbb{R}\backslash\{-1\}$ we want to solve for x in the following:

$$a * x = ax + x + a = 0$$

we rearrange to get

$$x = \frac{-a}{a+1}$$

Therefore all elements in $\mathbb{R}\backslash\{-1\}$ have inverses under *

- Associativity - we consider the respective values of $(a * b) * c$ and $a * (b * c)$ for arbitrary $a, b, c \in \mathbb{R}\backslash\{-1\}$:

$$(a * b) * c = (a * b)c + a * b + c = abc + ac + bc + ab + a + b + c$$

$$(a * (b * c) = a(b * c) + b * c + a = abc + ac + bc + ab + a + b + c$$

and so we have associativity.

Now we need to show that the resulting group is Abelian, but this is clear from the definition of * being completely symmetric in its two operands.

$\square$

b. Conveniently, from our proof of associativity we know immediately that

$$3 * x * x = 3x^2 + 3x + 3x + x^2 + x + x + 3 = 4x^2 + 8x + 3$$

Therefore we need to solve $4x^2 + 8x + 3 = 15$, or rather

$$4x^2 + 8x - 12 = 4(x^2 + 2x - 3) = 4(x + 3)(x - 1) + 0$$

From this we see that the solutions are exactly $x = 1, x = -3$

2.2 a. We need to show the four group axioms:

- Closure - By definiton of $\oplus$ the result of its application is a congruence class mod n. (Well-posedness is another matter but that isn't asked for here).
- Identity - the identity is $\bar{0}$ since

$$\forall a \in \mathbb{Z}, \bar{a} \oplus \bar{0} = \overline{(a + 0)} = \bar{a}$$

- Inverses - the inverse of $\bar{a}$ for any $a \in \mathbb{Z}$ is $\overline{-a}$:

$$\forall a \in \mathbb{Z}, \bar{a} \oplus \overline{-a} = \overline{(a - a)} = \bar{0}$$

- Associativity - for any $a, b, c \in \mathbb{Z}$ we have