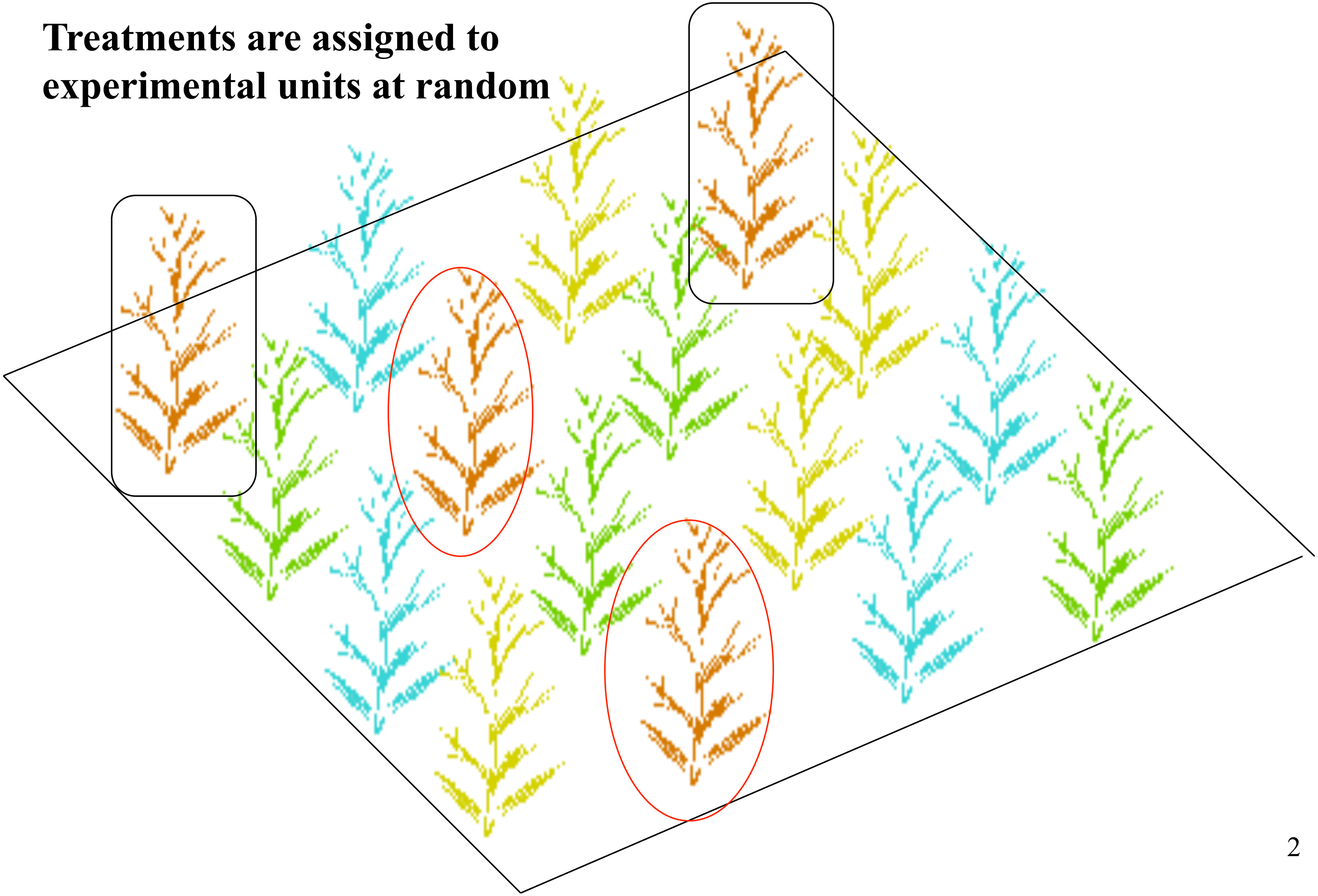
Topic 8: Latin Square Design

Montgomery: chapter 4

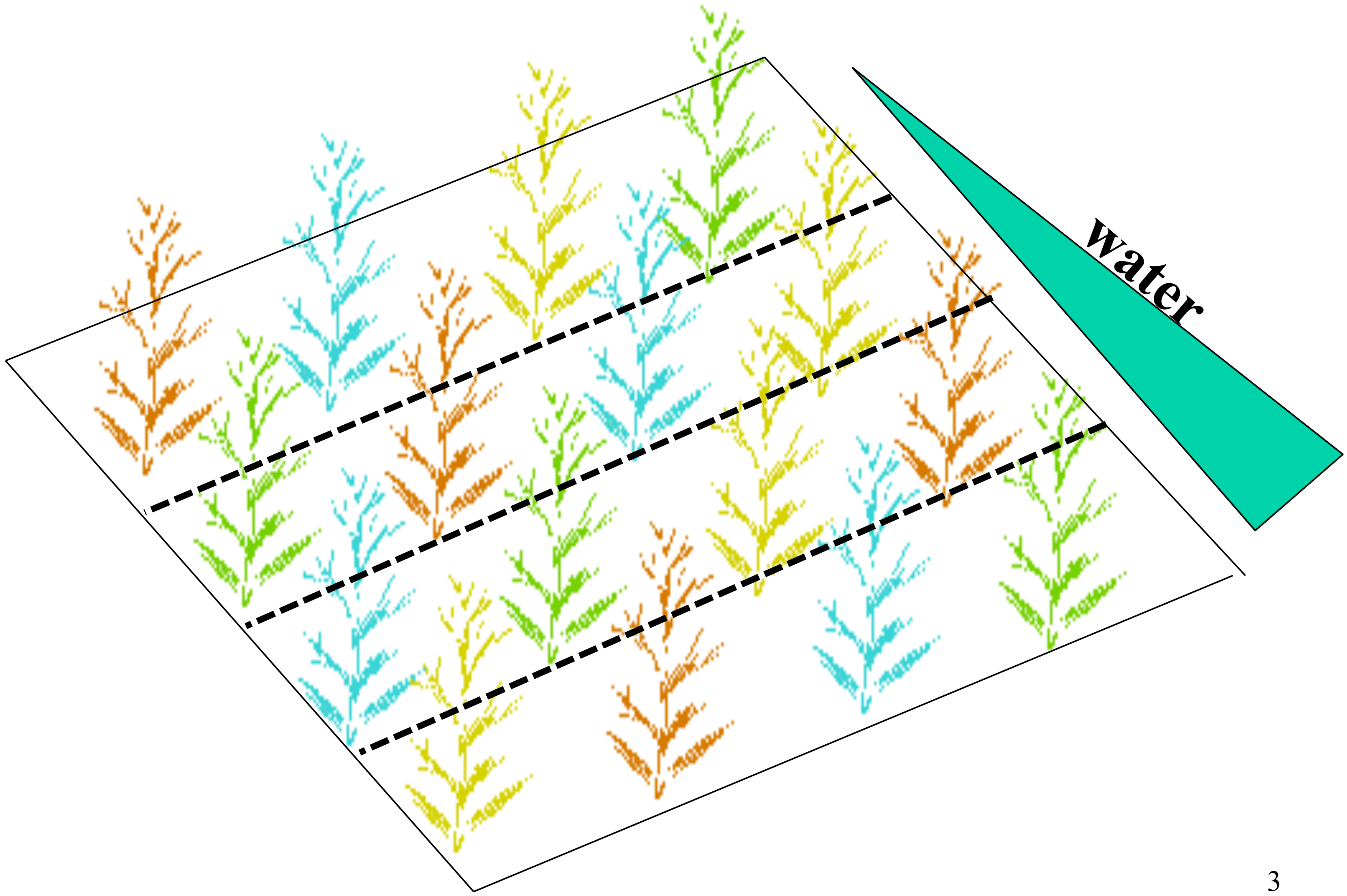
Prof. Lingling An
University of Arizona

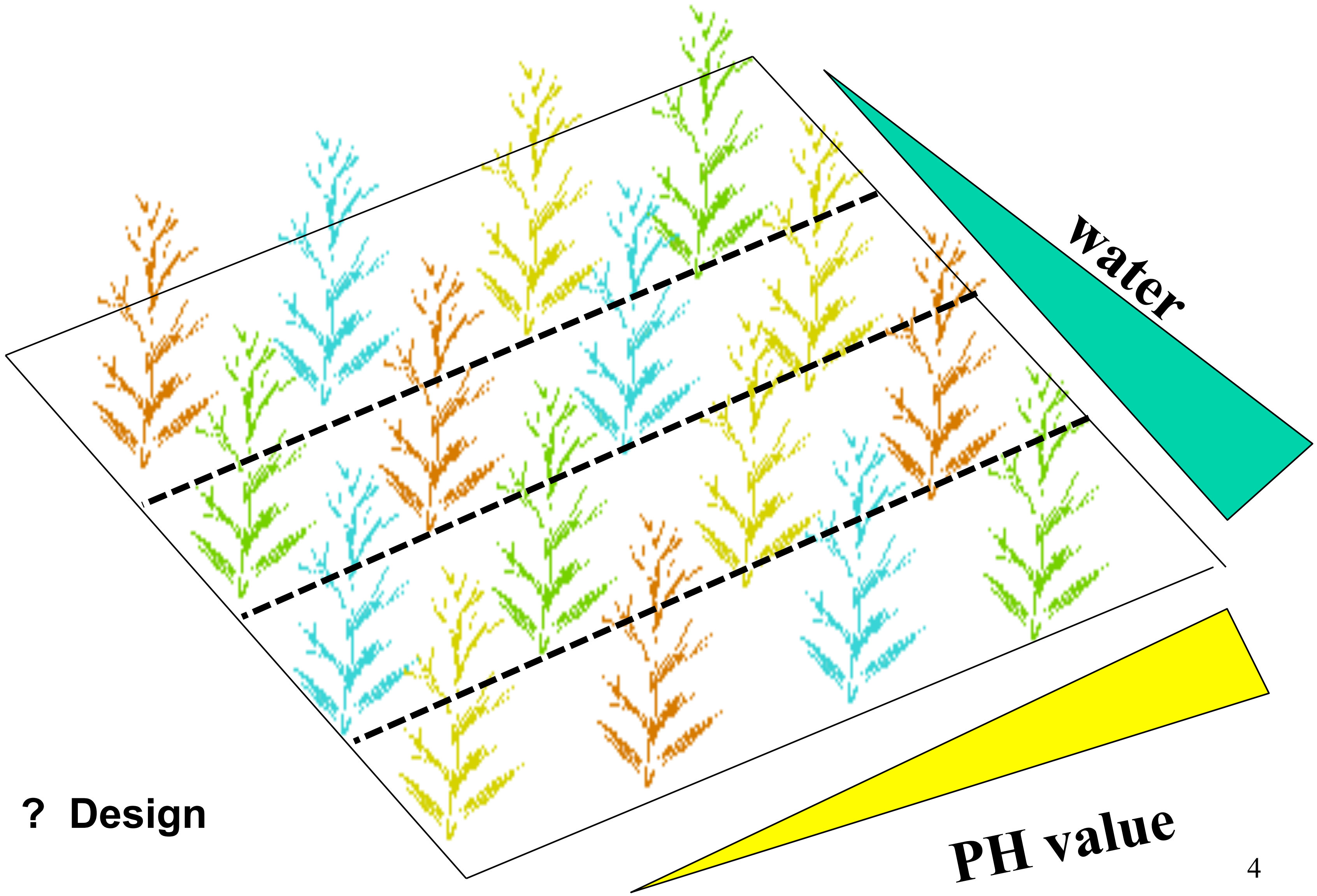
Completely randomized designs (CRD)

Treatments are assigned to experimental units at random



Randomized Complete Block Design





? Design

Outline

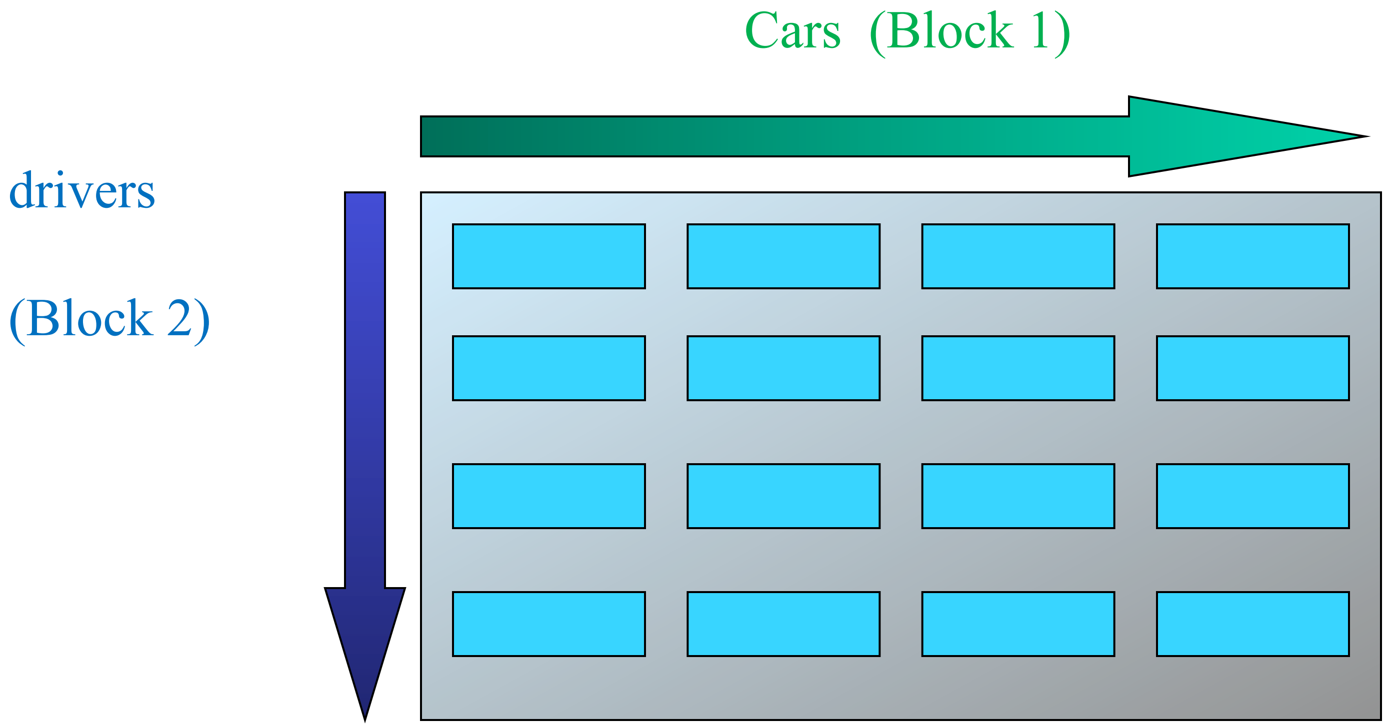
- Latin square design
- Replicated Latin square design

Automobile Emission Experiment

Four cars and **four drivers** are employed in a study of **four gasoline additives**(A, B, C, D). Even though cars can be identical models, systematic differences are likely to occur in their performance, and even though each driver may do his best to drive the car in the manner required by the test, systematic differences can occur from driver to driver. It would be desirable to eliminate both the car-to-car and driver-to-driver variations when comparing the additives.

Intro to Latin Square design

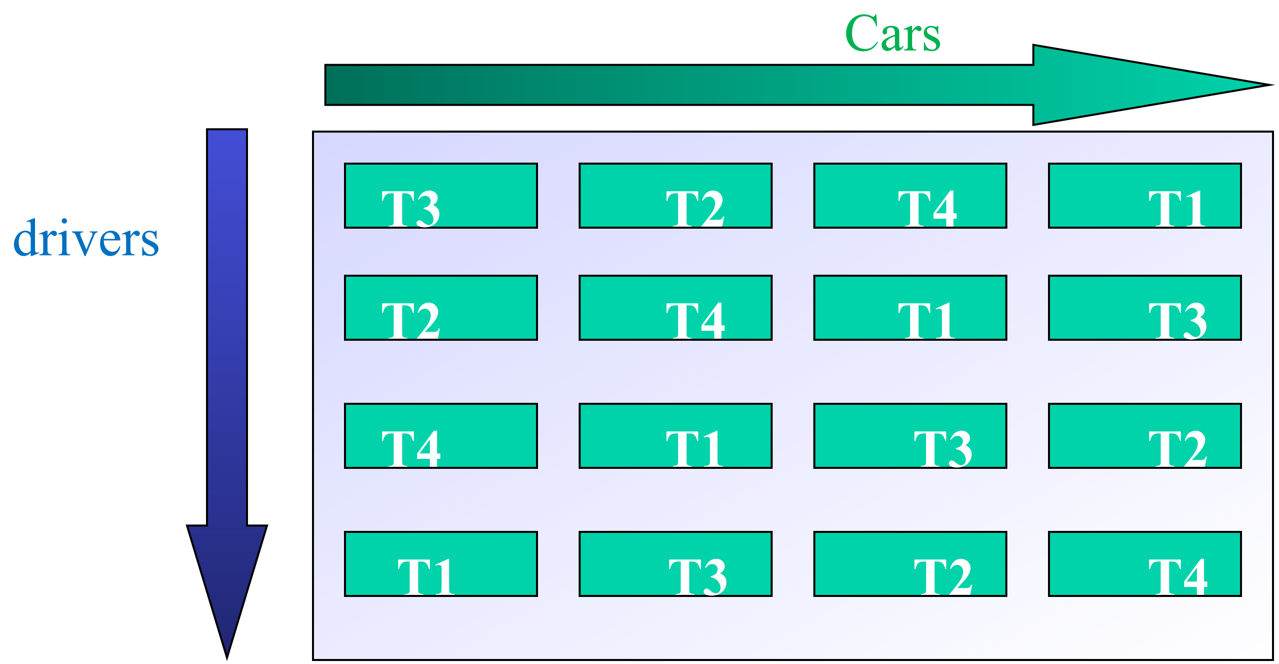
Two block factors + one treatment factor



Latin Square design

- Test each additive exactly once by each car;
- Test each additive exactly once by each driver.
- For four treatments (T1, T2, T3 and T4)
- Randomization?

Sudoku puzzle ???



Latin Square design (data table)

	Column			
Row	1	2	3	4
1	A	C	D	B
2	C	B	A	D
3	B	D	C	A
4	D	A	B	C

or

	Column			
Row	1	2	3	4
1	A	B	D	C
2	D	C	A	B
3	B	D	C	A
4	C	A	B	D

. . .

	cars			
drivers	1	2	3	4
1	A=24	B=26	D=20	C=25
2	D=23	C=26	A=20	B=27
3	B=15	D=13	C=16	A=16
4	C=17	A=15	B=20	D=20

Design Matrix and Orthogonality

drivers	cars	additive
1	1	A
1	2	B
1	3	D
1	4	C
2	1	D
2	2	C
2	3	A
2	4	B
3	1	B
3	2	D
3	3	C
3	4	A
4	1	C
4	2	A
4	3	B
4	4	D

Orthogonality: for any two columns, all possible combinations appear and appear only once.

Latin Square design - 2

- Design is represented in $p \times p$ grid, rows and columns are blocks and Latin letters are treatments.
  - Every row contains all the Latin letters and every column contains all the Latin letters.

- Standard Latin Square: letters in first row and first column are in alphabetic order.

	Column			
Row	1	2	3	4
1	A	B	C	D
2	B	C	D	A
3	C	D	A	B
4	D	A	B	C

Latin Square Design

- Properties:
  - Block on two nuisance factors
  - Two restrictions on randomization
- Model and Assumptions

$$y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \epsilon_{ijk}, \quad i, j, k = 1, 2, \ldots, p$$

μ - grand mean

α_i - ith block 1 effect (ith row effect); $\sum_{i=1}^{p} \alpha_i = 0.$

τ_j - jth treatment effect; $\sum_{i=1}^{p} \tau_j = 0$

β_k - kth block 2 effect (kth column effect); $\sum_{k=1}^{p} \beta_k = 0$

$\epsilon_{ijk} \sim \mathrm{N}(0, \sigma^2)$; (Normality, Independence, Constant Variance).

Completely additive model (no interaction)

Estimation and SS partition

- Rewrite observation y_{ijk} as:

$$
\begin{array}{lllll}
=\overline{y}_{...} & +(\overline{y}_{i..}-\overline{y}_{...}) & +(\overline{y}_{.j.}-\overline{y}_{...}) & +(\overline{y}_{..k}-\overline{y}_{...}) & +(y_{ijk}-\overline{y}_{i..}-\overline{y}_{.j.}-\overline{y}_{..k}+2\overline{y}_{...}) \\
=\hat{\mu} & +\hat{\alpha}_i & +\hat{\tau}_j & +\hat{\beta}_k & +\hat{\epsilon}_{ijk}
\end{array}
$$

- Partition $\mathrm{SS_T} = \sum\sum(y_{ijk}-\bar{y}_{...})^2$ into

$$
\begin{array}{llll}
p\sum(\overline{y}_{i..}-\overline{y}_{...})^2 & +p\sum(\overline{y}_{.j.}-\overline{y}_{...})^2 & +p\sum(\overline{y}_{..k}-\overline{y}_{...})^2 & +\sum\sum\hat{\epsilon}_{ijk}^2 \\
\mathrm{SS_{Row}} & +\mathrm{SS_{Treatment}} & +\mathrm{SS_{Col}} & +\mathrm{SS_E} \\
(p-1) & (p-1) & (p-1) & (p-1)(p-2)
\end{array}
$$

Dividing SS by df gives MS: $\mathrm{MS_{Treatment}}$, $\mathrm{MS_{Row}}$, $\mathrm{MS_{Col}}$ and $\mathrm{MS_E}$

Basic Hypotheses Testing

- Basic hypotheses: $H_0 : \tau_1 = \tau_2 = \cdots = \tau_p = 0$ vs H_1: at least one is not
- Test Statistic: $F_0 = \mathrm{MS}_{\mathrm{Treatment}}/\mathrm{MS}_{\mathrm{E}} \sim F_{p-1,(p-1)(p-2)}$ under H_0.
- Caution testing row effects ($\alpha_i = 0$) and column effects ($\beta_k = 0$).

ANOVA table for Latin Square design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Rows	SS_{Row}	$p-1$	MS_{Row}	
Treatment	$SS_{\text{Treatment}}$	$p-1$	$MS_{\text{Treatment}}$	F_0
Column	SS_{Column}	$p-1$	MS_{Column}	
Error	SS_{E}	$(p-2)(p-1)$	MS_{E}	
Total	SS_{T}	p^2-1		

$SS_{\text{T}} = \sum\sum\sum y_{ijk}^2 - y_{...}^2/p^2$; $SS_{\text{Row}} = \frac{1}{p}\sum y_{i..}^2 - y_{...}^2/p^2$

$SS_{\text{Treatment}} = \frac{1}{p}\sum y_{.j.}^2 - y_{...}^2/p^2$ $SS_{\text{Column}} = \frac{1}{p}\sum y_{..k}^2 - y_{...}^2/p^2$

$SS_{\text{Error}} = SS_{\text{T}} - SS_{\text{Row}} - SS_{\text{Treatment}} - SS_{\text{Col}}$

- Decision Rule:

If $F_0 > F_{\alpha,p-1,(p-2)(p-1)}$ then reject H_0.

Another example

Consider an experiment to investigate the effect of 4 diets on milk production. There are 4 cows. Each lactation period the cows receive a different diet. Assume there is a washout period so previous diet does not affect future results. Will block on lactation period and cow. A 4 by 4 Latin square is used.

	Cows			
Periods	1	2	3	4
1	A=38	B=39	C=45	D=41
2	B=32	C=37	D=38	A=30
3	C=35	D=36	A=37	B=32
4	D=33	A=30	B=35	C=33

SAS code: Latin square design

```
options nocenter ls=75;
data new;
input cow period trt resp @@;
datalines;
1 1 1 38 1 2 2 32 1 3 3 35 1 4 4 33
2 1 2 39 2 2 3 37 2 3 4 36 2 4 1 30
3 1 3 45 3 2 4 38 3 3 1 37 3 4 2 35
4 1 4 41 4 2 1 30 4 3 2 32 4 4 3 33
;
proc glm;
class cow trt period;
model resp=trt period cow;
means trt/ lines tukey;
means period cow;
output out=new1 r=res p=pred;
run;
```

SAS output

Overall ANOVA table

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	9	242.5625000	26.9513889	33.17	0.0002
Error	6	4.8750000	0.8125000		
Corrected Total	15	247.4375000			

Type III model ANOVA

Source	DF	Type III SS	Mean Square	F Value	Pr > F
trt	3	40.6875000	13.5625000	16.69	0.0026
period	3	147.1875000	49.0625000	60.38	<.0001
cow	3	54.6875000	18.2291667	22.44	0.0012

Tukey's Studentized Range (HSD) Test for resp

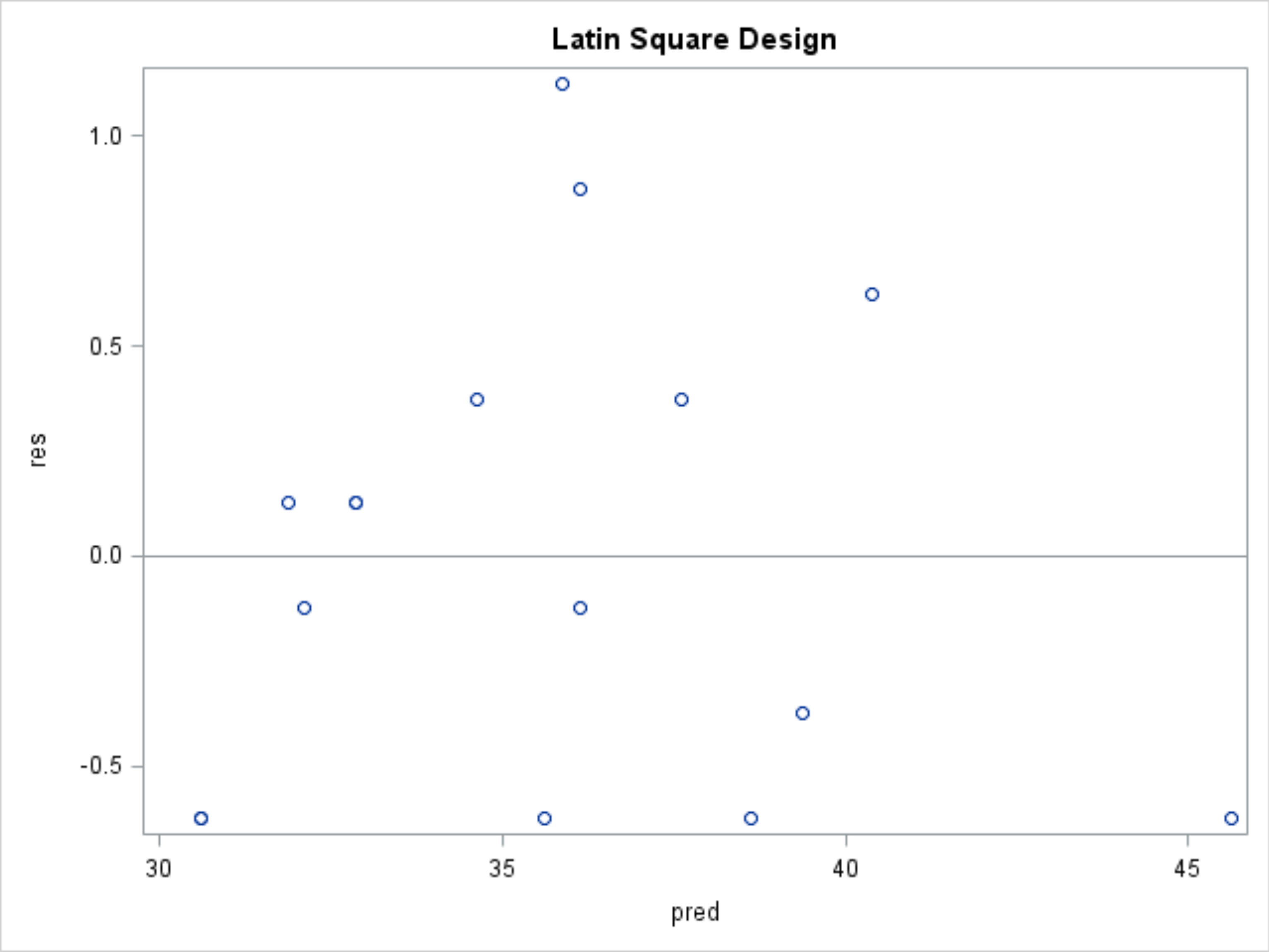
Alpha	0.05
Error Degrees of Freedom	6
Error Mean Square	0.8125
Critical Value of Studentized Range	4.89559
Minimum Significant Difference	2.2064

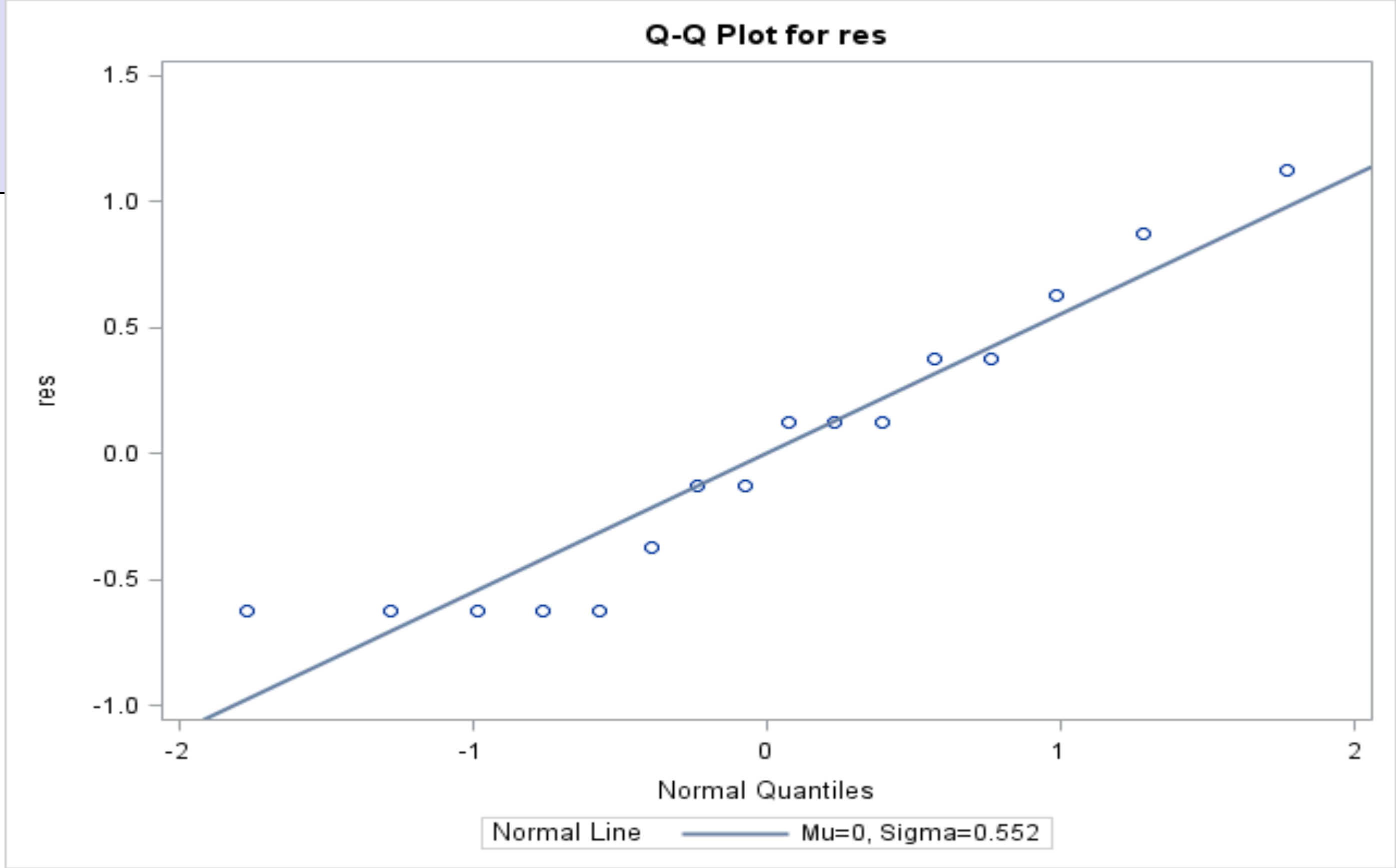
Means with the same letter are not significantly different.			
Tukey Grouping	Mean	N	trt
A	37.5000	4	3
A			
A	37.0000	4	4
B	34.5000	4	2
B			
B	33.7500	4	1

SAS code: model adequacy checking

```
title 'Latin Square Design';
proc sgplot;
scatter x=pred y=res;
refline 0;
run;

proc univariate data=new1 normal;
var res;
qqplot res/normal (L=1 mu=0 sigma=est);
run;
```


Latin Square Design
1.0
0.5
0.0
-0.5
res
30
35
40
45
pred



Tests for Normality				
Test	Statistic		p Value	
Shapiro-Wilk	W	0.90641	Pr < W	0.1019
Kolmogorov-Smirnov	D	0.176031	Pr > D	>0.1500
Cramer-von Mises	W-Sq	0.068568	Pr > W-Sq	>0.2500
Anderson-Darling	A-Sq	0.511559	Pr > A-Sq	0.1730

Replicating Latin Squares

- Latin Squares result in small degree of freedom for $\mathrm{SS_E}$:
  $\mathrm{df} = (p-1)(p-2)$.
  - If 3 treatments: $\mathrm{df_E} = 2$
  - If 4 treatments $\mathrm{df_E} = 6$
  - If 5 treatments $\mathrm{df_E} = 12$

  Use replication to increase $\mathrm{df_E}$

- Different ways for replicating Latin squares:
  1. Same rows and same columns
  2. New rows and same columns
  3. Same rows and new columns
  4. New rows and new columns

  Degree of freedom for $\mathrm{SS_E}$ depends on which method is used;

  Often need to include an additional blocking factor for "replicate" effect.

Method 1: same rows and same columns in additional squares

```
Example:
  1  2  3         data          replication
1 A  B  C      7.0 8.0 9.0
2 B  C  A      4.0 5.0 6.0            1
3 C  A  B      6.0 3.0 4.0

  1  2  3          data
1 C  B  A       8.0 4.0 7.0
2 B  A  C       6.0 3.0 6.0           2
3 A  C  B       5.0 8.0 7.0

  1  2  3          data
1 B  A  C        9.0 6.0 8.0
2 A  C  B        5.0 7.0 6.0          3
3 C  B  A        9.0 3.0 7.0
```

Model and ANOVA Table for Method 1

Usually includes replicate (e.g., time) effects $(\delta_1, \ldots, \delta_n)$

$$y_{ijkl} = \mu + \alpha_i + \tau_j + \beta_k + \delta_l + \epsilon_{ijkl} \quad \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \\ l = 1, 2, \ldots, n \end{cases}$$

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Rows	SS_{Row}	$p-1$		
Columns	$\text{SS}_{\text{Column}}$	$p-1$		
Replicate	$\text{SS}_{\text{Replicate}}$	$n-1$		
Treatment	$\text{SS}_{\text{Treatment}}$	$p-1$	$\text{MS}_{\text{Treatment}}$	F_0
Error	SS_{E}	$(p-1)(n(p+1)-3)$	MS_{E}	
Total	SS_{T}	np^2-1		

data input and SAS code for Method 1

```
data new;
input rep row col trt resp;
datalines;
1 1 1 1 7.0
1 1 2 2 8.0
1 1 3 3 9.0
1 2 1 2 4.0
1 2 2 3 5.0
1 2 3 1 4.0
1 3 1 3 6.0
1 3 2 1 3.0
1 3 3 2 4.0
2 1 1 3 8.0
2 1 2 2 4.0
2 1 3 1 7.0
2 2 1 2 6.0
2 2 2 1 3.0
2 2 3 3 6.0
2 3 1 1 5.0
2 3 2 3 8.0
2 3 3 2 7.0
```

```
3 1 1 2 9.0
3 1 2 1 6.0
3 1 3 3 8.0
3 2 1 1 5.0
3 2 2 3 7.0
3 2 3 2 6.0
3 3 1 3 9.0
3 3 2 2 3.0
3 3 3 1 7.0
;
```

```
proc glm data=new;
class rep row col trt;
model resp=rep row col trt;
run;
```

SAS Output for Method 1

Overall ANOVA table

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	8	61.18518519	7.64814815	4.21	0.0054
Error	18	32.66666667	1.81481481		
Corrected Total	26	93.85185185			

Type III model ANOVA

Source	DF	Type III SS	Mean Square	F Value	Pr > F
rep	2	5.62962963	2.81481481	1.55	0.2391
row	2	23.40740741	11.70370370	6.45	0.0077
col	2	9.85185185	4.92592593	2.71	0.0933
trt	2	22.29629630	11.14814815	6.14	0.0093

Method 2 for replicating Latin Square Design

Method 2: New (different) rows and same columns

```
Example:
  1  2  3          data            replication
1 A  B  C      7.0 8.0 9.0
2 B  C  A      4.0 5.0 6.0              1
3 C  A  B      6.0 3.0 4.0

  1  2  3           data
4 C  B  A      8.0 4.0 7.0
5 B  A  C      6.0 3.0 6.0              2
6 A  C  B      5.0 8.0 7.0

  1  2  3           data
7 B  A  C      9.0 6.0 8.0
8 A  C  B      5.0 7.0 6.0              3
9 C  B  A      9.0 3.0 7.0
```

Model and ANOVA Table for Method 2

- Row effects are **nested** within square
- α_i can be different for different squares, so they are denoted $\alpha_{i(l)}$ for $i = 1, \ldots, p$ and $l = 1, \ldots, n$, and satisfy by $\sum_{i=1}^{p} \alpha_{i(l)} = 0$ for any fixed l.

$$y_{ijkl} = \mu + \alpha_{i(l)} + \tau_j + \beta_k + \delta_l + \epsilon_{ijkl} \quad \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \\ l = 1, 2, \ldots, n \end{cases}$$

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Rows	SS_{Row}	$n(p-1)$		
Columns	SS_{Column}	$p-1$		
Replicate	$SS_{\text{Replicate}}$	$n-1$		
Treatment	$SS_{\text{Treatment}}$	$p-1$	$MS_{\text{Treatment}}$	F_0
Error	SS_{E}	$(p-1)(np-2)$	MS_{E}	
Total	SS_{T}	np^2-1		

data input and SAS code for Method 2

```
data new;
input rep row col trt resp;
datalines;
1 1 1 1 7.0
1 1 2 2 8.0
1 1 3 3 9.0
1 2 1 2 4.0
1 2 2 3 5.0
1 2 3 1 4.0
1 3 1 3 6.0
1 3 2 1 3.0
1 3 3 2 4.0
2 4 1 3 8.0
2 4 2 2 4.0
2 4 3 1 7.0
2 5 1 2 6.0
2 5 2 1 3.0
2 5 3 3 6.0
2 6 1 1 5.0
2 6 2 3 8.0
2 6 3 2 7.0
```

```
3 7 1 2 9.0
3 7 2 1 6.0
3 7 3 3 8.0
3 8 1 1 5.0
3 8 2 3 7.0
3 8 3 2 6.0
3 9 1 3 9.0
3 9 2 2 3.0
3 9 3 1 7.0
;
proc glm data=new;
class rep row col trt;
model resp=rep row(rep) col trt;
run;
```

Overall ANOVA table

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	12	74.00000000	6.16666667	4.35	0.0054
Error	14	19.85185185	1.41798942		
Corrected Total	26	93.85185185			

Type III model ANOVA

Source	DF	Type III SS	Mean Square	F Value	Pr > F
rep	2	5.62962963	2.81481481	1.99	0.1742
row(rep)	6	36.22222222	6.03703704	4.26	0.0120
col	2	9.85185185	4.92592593	3.47	0.0596
trt	2	22.29629630	11.14814815	7.86	0.0051

Latin Rectangle

- If there do not exist "replicate effects", the distinction between the squares can be neglected, so the replicated Latin squares form a Latin Rectangle with np separate rows.
- Row effects are $\alpha_1, \alpha_2, \ldots, \alpha_{np}$ satisfying $\sum_{i=1}^{np} \alpha_i = 0$.
- Model:

$$y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, \ldots, np \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \end{cases}$$

- ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Rows	SS_{Row}	$np-1$		
Columns	SS_{Column}	$p-1$		
Treatment	$SS_{\text{Treatment}}$	$p-1$	$MS_{\text{Treatment}}$	F_0
Error	SS_{E}	$(p-1)(np-2)$	MS_{E}	
Total	SS_{T}	np^2-1		

Method 3: Same rows and new (different) columns, is similar to Method 2. Details are omitted

Latin Rectangle

- If there do not exist ”replicate effects”, the distinction between the squares can be neglected, so the replicated Latin squares form a Latin Rectangle with np separate columns

Crossover (Changeover) Design

- Consider an experiment for investigating the effects of 3 diets (A, B, C) on milk production. Suppose the experiment involves 3 lactation periods. Cows take different diets in different periods, that is, a cow will not take the same diet more than once (crossover).

- Case 1: 3 cows are used.

	cow 1	cow 2	cow 3
period 1	A	B	C
period 2	B	C	A
period 3	C	A	B

- Case 2: 6 cows are employed:

	cow 1	cow 2	cow 3	cow 4	cow 5	cow 6
pd 1	A	B	C	A	B	C
pd 2	B	C	A	C	A	B
pd 3	C	A	B	B	C	A

Crossover (Changeover) Design -2

- In general, there are p treatments, np experimental units, and p periods. n Latin squares ($p \times p$) are needed to form a rectangle ($p \times np$). So that

  I. Each unit has each treatment for one period

  II. In each period, each treatment is used on n units.

Source of Variation	DF
units	$np - 1$
periods	$p - 1$
treatment	$p - 1$
error	$np^2 - (n+2)p + 2$
total	$np^2 - 1$

Method 4: New (different) rows and new columns in additional squares

Example:

	1	2	3	data			replication
1	A	B	C	7.0	8.0	9.0	
2	B	C	A	4.0	5.0	6.0	1
3	C	A	B	6.0	3.0	4.0	

	4	5	6	data			
4	C	B	A	8.0	4.0	7.0	
5	B	A	C	6.0	3.0	6.0	2
6	A	C	B	5.0	8.0	7.0	

	7	8	9	data			
7	B	A	C	9.0	6.0	8.0	
8	A	C	B	5.0	7.0	6.0	3
9	C	B	A	9.0	3.0	7.0	

Model and ANOVA Table for Method 4

- Usually both row effects ($\alpha_{i(l)}$) and column effects ($\beta_{k(l)}$) are **nested** with squares.

$$y_{ijkl} = \mu + \alpha_{i(l)} + \tau_j + \beta_{k(l)} + \delta_l + \epsilon_{ijkl} \quad \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \\ l = 1, 2, \ldots, n \end{cases}$$

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Rows	SS_{Row}	$n(p-1)$		
Columns	$\text{SS}_{\text{Column}}$	$n(p-1)$		
Replicate	$\text{SS}_{\text{Replicate}}$	$n-1$		
Treatment	$\text{SS}_{\text{Treatment}}$	$p-1$	$\text{MS}_{\text{Treatment}}$	F_0
Error	SS_{E}	$(p-1)(n(p-1)-1)$	MS_{E}	
Total	SS_{T}	np^2-1		

data input and SAS code for Method 4

```
data new;
input rep row col trt resp;
datalines;
1 1 1 1 7.0
1 1 2 2 8.0
1 1 3 3 9.0
1 2 1 2 4.0
1 2 2 3 5.0
1 2 3 1 4.0
1 3 1 3 6.0
1 3 2 1 3.0
1 3 3 2 4.0
2 4 4 3 8.0
2 4 5 2 4.0
2 4 6 1 7.0
2 5 4 2 6.0
2 5 5 1 3.0
2 5 6 3 6.0
2 6 4 1 5.0
2 6 5 3 8.0
2 6 6 2 7.0
```

```
3 7 7 2 9.0
3 7 8 1 6.0
3 7 9 3 8.0
3 8 7 1 5.0
3 8 8 3 7.0
3 8 9 2 6.0
3 9 7 3 9.0
3 9 8 2 3.0
3 9 9 1 7.0
;
proc glm data=new;
class rep row col trt;
model resp=rep row(rep) col(rep) trt;
run;
```

SAS output for method 4

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	16	77.70370370	4.85648148	3.01	0.0411
Error	10	16.14814815	1.61481481		
Corrected Total	26	93.85185185			

Source	DF	Type III SS	Mean Square	F Value	Pr > F
rep	2	5.62962963	2.81481481	1.74	0.2242
row(rep)	6	36.22222222	6.03703704	3.74	0.0324
col(rep)	6	13.55555556	2.25925926	1.40	0.3042
trt	2	22.29629630	11.14814815	6.90	0.0131

Last slide

- Read Sections: 4. 2