Topic 10: Balanced Incomplete Block Design (BIBD)

Montgomery: chapter 4

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Outline

- Review: RCBD
- Balanced incomplete block design

Review: Nuisance Factor

Nuisance Factor (may be present in experiment)

- Has effect on response but its effect is not of interest
- If unknown $\longrightarrow$ Protecting experiment through randomization
- If known (measurable) but uncontrollable $\longrightarrow$ Analysis of Covariance (Chapter 15 Section 3)
- If known and controllable $\longrightarrow$ Blocking

Example: Penicillin Experiment

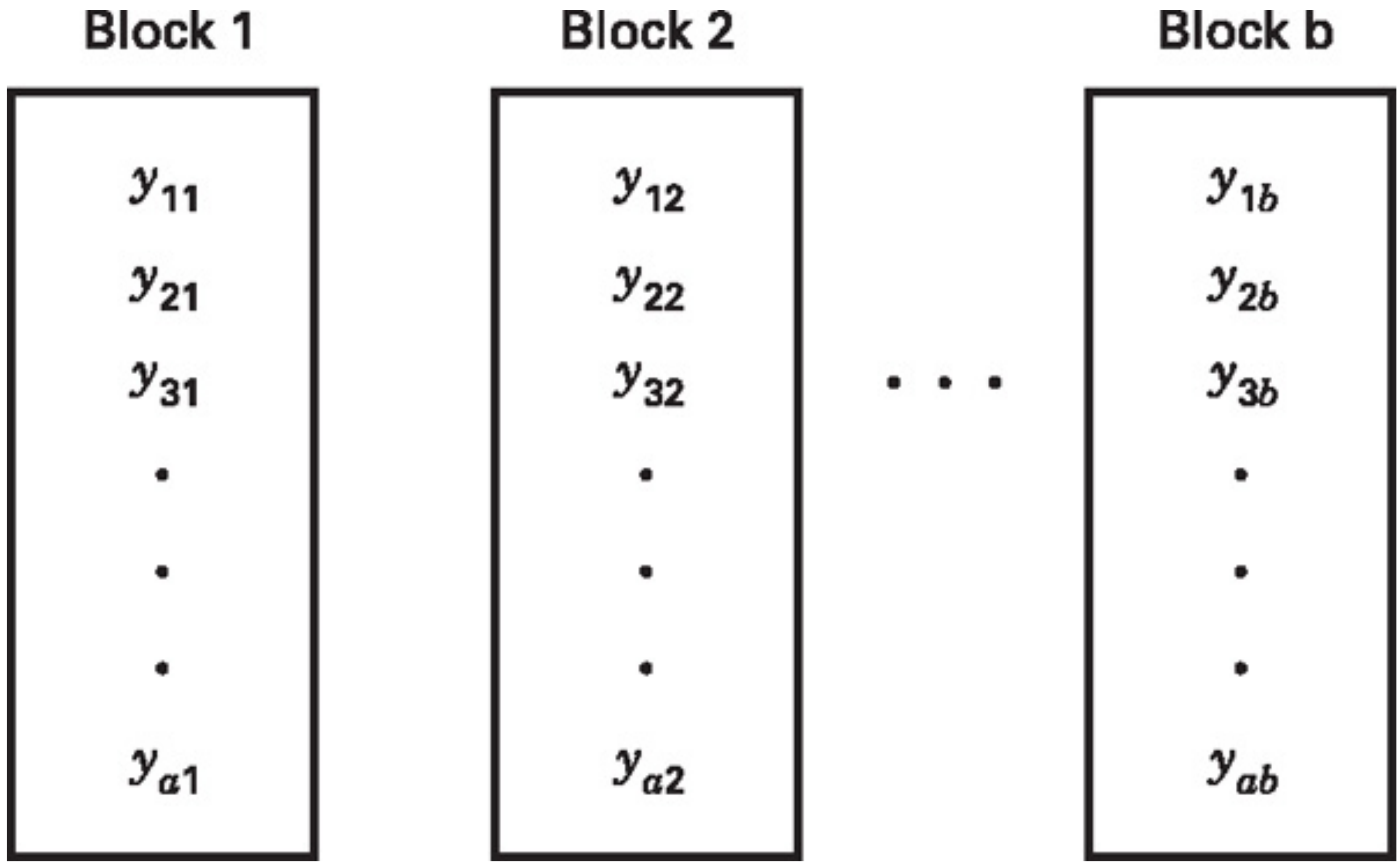
In this experiment, four penicillin manufacturing processes (A, B, C and D) were being investigated. Yield was the response. It was known that an important raw material, corn steep liquor, was quite variable. The experiment and its results were given below:

	blend 1	blend 2	blend 3	blend 4	blend 5
A	89_1	84_4	81_2	87_1	79_3
B	88_3	77_2	87_1	92_3	81_4
C	97_2	92_3	87_4	89_2	80_1
D	94_4	79_1	85_3	84_4	88_2

Example: Penicillin Experiment -2

- Blend is a nuisance factor, treated as a block factor;
- (Complete) Blocking: all the treatments are applied within each block, and they are compared within blocks.
- Advantage: Eliminate blend-to-blend (between-block) variation from experimental error variance when comparing treatments.
- Cost: degree of freedom.

Review: Randomized Complete Block Design (RCBD)



Review: RCBD -2

- b blocks each consisting of (partitioned into) a experimental units
- a treatments are randomly assigned to the experimental units within each block
- Typically after the runs in one block have been conducted, then move to another block.
- Typical blocking factors: day, batch of raw material etc.
- Results in restriction on randomization because randomization is only within blocks.
- Data within a block are related to each other. When $a = 2$, randomized complete block design becomes paired two sample case.

What if …

Catalyst Experiment

Four catalysts are being investigated in an experiment. The experimental procedure consists of selecting a batch raw material, loading the pilot plant, applying each catalyst in a separate run and observing the reaction time. The batches of raw material are considered as blocks, however each batch is only large enough to permit three catalysts to be run.

	Block(raw material)				
Catalyst	1	2	3	4	$y_{i.}$
1	73	74	-	71	218
2	-	75	67	72	214
3	73	75	68	-	216
4	75	-	72	75	222
$y_{.j}$	221	224	207	218	870=$y_{..}$

Balanced Incomplete Block Design (BIBD)

Example 1.

	block					
treatment	1	2	3			
A	A	-	A	1	0	1
B	B	B	-	1	1	0
C	-	C	C	0	1	1

Incidence Matrix: $\mathcal{N} = (n_{ij})_{a \times b}$ where $n_{ij} = 1$, if treatment i is run in block j;
=0 otherwise.

Example 2.

	block											
treatment	1	2	3	4	5	6						
A	A	A	A	-	-	-	1	1	1	0	0	0
B	B	-	-	B	B	-	1	0	0	1	1	0
C	-	C	-	C	-	C	0	1	0	1	0	1
D	-	-	D	-	D	D	0	0	1	0	1	1

blocks must be >= # treatments

BIBD: Design Properties

- there are a treatments and b blocks.
- each block contains k (different) treatments.
- each treatment appears in r blocks.
- each pair of treatments appears together in λ blocks.

a, b, k, r, and λ are not independent

- $N = ar = bk$, where N is the total number of runs;

Balanced Incomplete Block Design (BIBD)

Example 1.

treatment	block 1	block 2	block 3			
A	A	-	A	1	0	1
B	B	B	-	1	1	0
C	-	C	C	0	1	1

Incidence Matrix: $\mathcal{N} = (n_{ij})_{a\times b}$ where $n_{ij} = 1$, if treatment i is run in block j; =0 otherwise.

$$a = 3, b = 3, k = 2, r = 2, \lambda = 1$$

Example 2.

	block											
treatment	1	2	3	4	5	6						
A	A	A	A	-	-	-	1	1	1	0	0	0
B	B	-	-	B	B	-	1	0	0	1	1	0
C	-	C	-	C	-	C	0	1	0	1	0	1
D	-	-	D	-	D	D	0	0	1	0	1	1

$$a = 4, b = 6, k = 2, r = 3, \lambda = 1$$

BIBD: Design Properties -2

- $\lambda(a-1) = r(k-1)$:

  1. for any fixed treatment i_0

  2. two different ways to count the total number of pairs including treatment i_0 in the experiment.

  I. $a-1$ possible pairs, each appears in λ blocks, so $\lambda(a-1)$;

  II. treatment i_0 appears in r blocks. Within each block, there are $k-1$ pairs including i_0, so $r(k-1)$

- $b \geq a$ (a brainteaser for math/stat students).

- Nonorthogonal design

this will be on the midterm
lamba must be an integer

BIBD: Statistical Model

- Statistical Model

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

- additive model (without interaction)
- Not all y_{ij} exist because of incompleteness
- Usual treatment and block restrictions : $\sum \tau_i = 0; \qquad \sum \beta_j = 0$
- Nonorthogonality of treatments and blocks

Use Type III Sums of Squares and lsmeans

Model Estimates

- Least squares estimates for μ, etc.

$$\hat{\mu} = \frac{y_{..}}{N}; \qquad \hat{\tau}_i = \frac{kQ_i}{\lambda a}; \qquad \hat{\beta}_j = \frac{rQ'_j}{\lambda b}$$

where

$$Q_i = y_{i.} - \frac{1}{k}\sum n_{ij} y_{.j}; \qquad Q'_j = y_{.j} - \frac{1}{r}\sum n_{ij} y_{i.}$$

$$\begin{aligned}
\mathrm{Var}(Q_i) &= \mathrm{Var}\left(y_{i.}\right) + \mathrm{Var}\left(\frac{1}{k}\sum n_{ij} y_{.j}\right) - 2\mathrm{Cov}\left(y_{i.}, \frac{1}{k}\sum n_{ij} y_{ij}\right) \\
&= r\sigma^2 + \frac{r}{k^2}k\sigma^2 - \frac{2}{k}r\sigma^2 \\
&= \frac{(k-1)r}{k}\sigma^2
\end{aligned}$$

- $\mathrm{Var}\left(\hat{\tau}_i\right) = \left(\frac{k}{\lambda a}\right)^2 \mathrm{Var}(Q_i) = \left(\frac{k}{\lambda a}\right)^2 \frac{(k-1)r}{k}\sigma^2 = \frac{k(a-1)}{\lambda a^2}\sigma^2;$

- $\mathrm{Var}\left(\hat{\tau}_i - \hat{\tau}_j\right) = \frac{2k\sigma^2}{\lambda a};$

Analysis of Variance Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Blocks	SS_{Block}	$b-1$	MS_{Block}	
Treatment	$\text{SS}_{\text{Treatment}}$	$a-1$	$\text{MS}_{\text{Treatment}}$	F_0
Error	SS_{E}	$N-a-b+1$	MS_{E}	
Total	SS_{T}	$N-1$		

- $\text{SS}_{\text{T}} = \sum\sum y_{ij}^2 - y_{..}^2/N$
- $\text{SS}_{\text{Block}} = \frac{1}{k}\sum y_{.j}^2 - y_{..}^2/N$

- $\text{SS}_{\text{Treatments}}$ needs adjustment for incompleteness

$$Q_i = y_{i.} - \frac{1}{k}\sum_{j=1}^{b} n_{ij} y_{.j} \quad \text{where} \quad n_{ij} = \begin{cases} 1 & \text{if trt } i \text{ in blk } j \\ 0 & \text{otherwise} \end{cases}$$

  - trt i's **total** minus trt i's block averages
  - $\sum Q_i = 0$

  $\text{SS}_{\text{Treatment(adjusted)}} = k \sum Q_i^2 / \lambda a = \frac{\lambda a}{k} \sum \hat{\tau}_i^2$

- SS_{E} by subtraction

- If $F_0 > F_{\alpha, a-1, N-a-b+1}$ then reject H_0

Mean Tests and Contrasts

- Must compute adjusted means (lsmeans)
- Adjusted mean is $\widehat{\mu} + \widehat{\tau}_i$
- Standard error of adjusted mean is $\sqrt{\mathrm{MS_E}(\frac{k(a-1)}{\lambda a^2} + \frac{1}{N})}$
- Contrasts based on adjusted treatment totals

  For a contrast: $\sum c_i \mu_i$

  Its estimate: $\sum c_i \hat{\tau}_i = \frac{k}{\lambda a} \sum c_i Q_i$

  Contrast sum of squares:

$$SS_C = \frac{k(\sum_{i=1}^{a} c_i Q_i)^2}{\lambda a \sum_{i=1}^{a} c_i^2}$$

Pairwise Comparison

- Pairwise comparison $\tau_i - \tau_j$:

1. Bonferroni:

$$CD = t_{\alpha/2m, ar-a-b+1} \sqrt{\mathrm{MS_E} \frac{2k}{\lambda a}}.$$

2. Tukey:

$$CD = \frac{q_\alpha(a, ar-a-b+1)}{\sqrt{2}} \sqrt{\mathrm{MS_E} \frac{2k}{\lambda a}}$$

SAS Code

```
data example;
input trt block resp @@;
datalines;
1 1 73 1 2 74 1 4 71 2 2 75 2 3 67 2 4 72
3 1 73 3 2 75 3 3 68 4 1 75 4 3 72 4 4 75
;

proc glm;
class block trt;
model resp = trt block;
lsmeans trt / tdiff pdiff adjust=bon stderr;
lsmeans trt / pdiff adjust=tukey stderr;
contrast 'a' trt 1 -1 0 0;
estimate 'b' trt 1 -1 0 0;
output out=myout r=res p=pred;
run;
```

SAS output

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	6	77.75000000	12.95833333	19.94	0.0024
Error	5	3.25000000	0.65000000		
Corrected Total	11	81.00000000			

Source	DF	**Type I SS**	Mean Square	F Value	Pr > F
trt	3	11.66666667	3.88888889	5.98	0.0415
block	3	66.08333333	22.02777778	33.89	0.0010

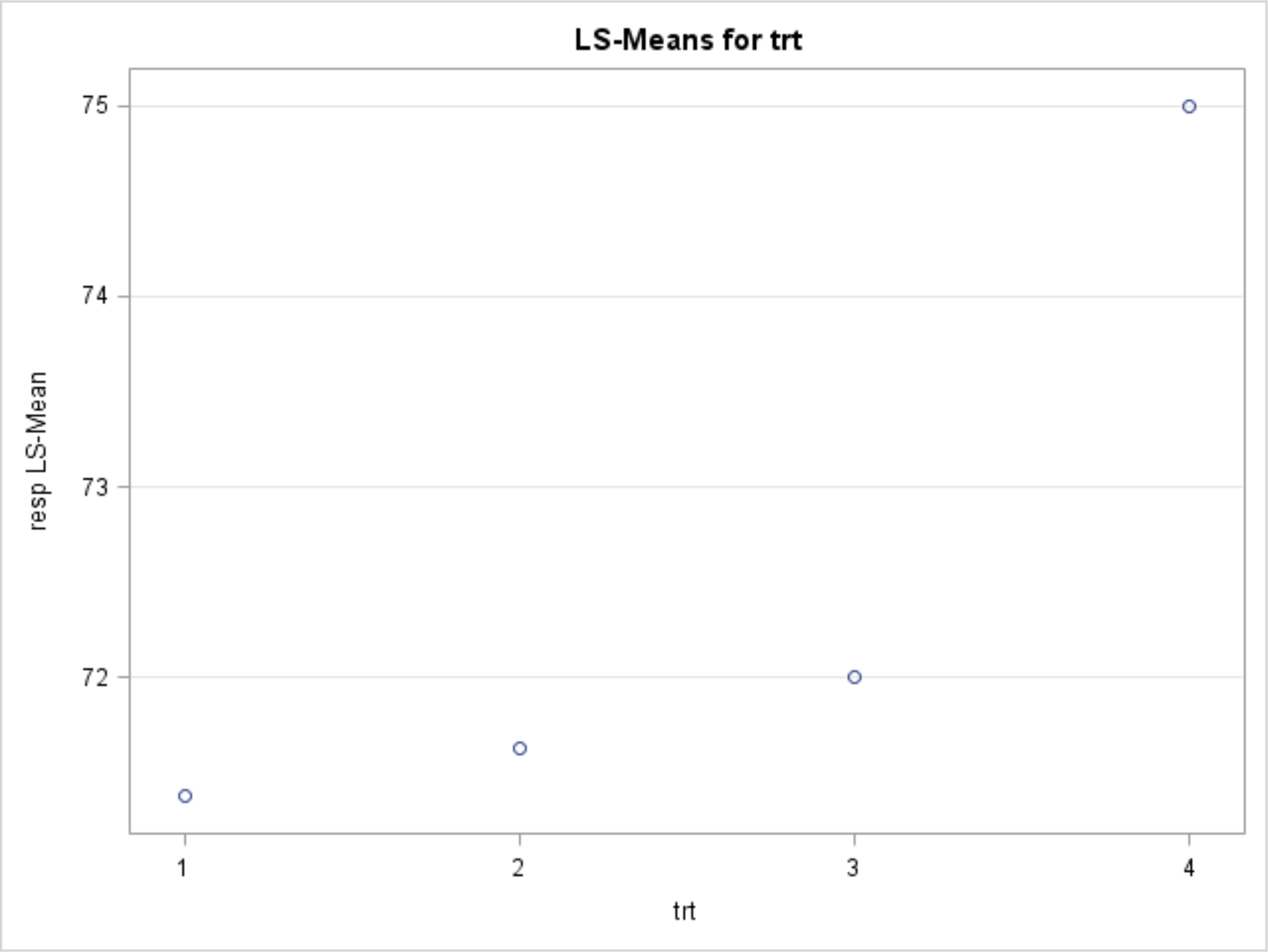
Source	DF	**Type III SS**	Mean Square	F Value	Pr > F
trt	3	22.75000000	7.58333333	11.67	**0.0107**
block	3	66.08333333	22.02777778	33.89	0.0010

Why only use Type III? marginal effect

Output: Bonferroni method

Least Squares Means
Adjustment for Multiple Comparisons: Bonferroni

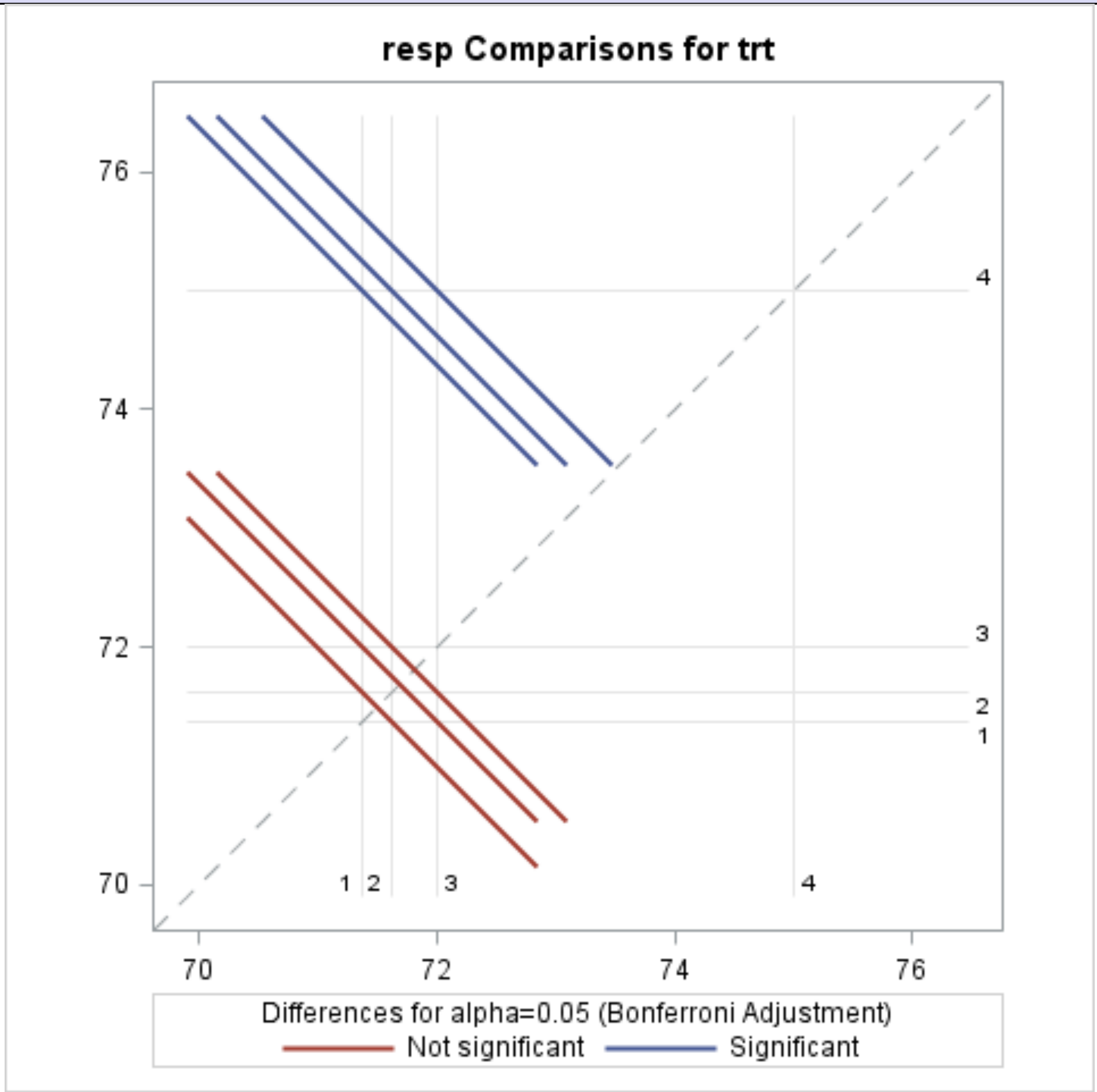
trt	resp LSMEAN	Standard Error	Pr > |t|	LSMEAN Number
1	71.3750000	0.4868051	<.0001	1
2	71.6250000	0.4868051	<.0001	2
3	72.0000000	0.4868051	<.0001	3
4	75.0000000	0.4868051	<.0001	4


LS-Means for trt
75
74
73
72
resp LS-Mean
1
2
3
4
trt

Difference matrix: Bonferroni

Least Squares Means for Effect trt
t for H0: LSMean(i)=LSMean(j) / Pr > |t|
Dependent Variable: resp

i/j	1	2	3	4
1		-0.35806	-0.89514	-5.19183
		1.0000	1.0000	0.0209
2	0.358057		-0.53709	-4.83378
	1.0000		1.0000	0.0284
3	0.895144	0.537086		-4.29669
	1.0000	1.0000		0.0464
4	5.191833	4.833775	4.296689	
	0.0209	0.0284	0.0464	


resp Comparisons for trt
76
74
72
70
4
3
2
1
1 2 3 4
70 72 74 76
Differences for alpha=0.05 (Bonferroni Adjustment)
Not significant
Significant

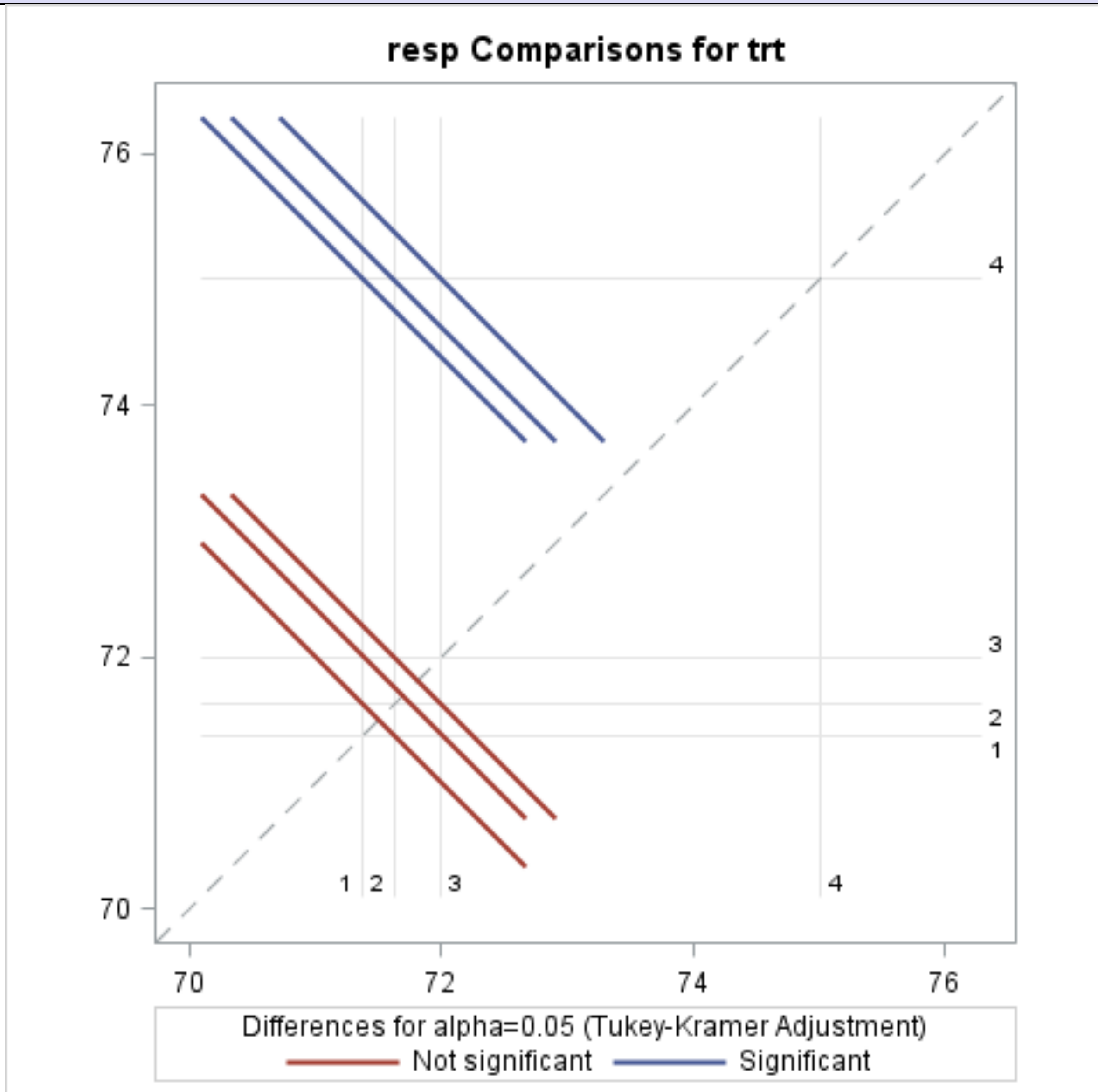
Output: Tukey's method

Least Squares Means
Adjustment for Multiple Comparisons: Tukey-Kramer

trt	resp LSMEAN	Standard Error	Pr > |t|	LSMEAN Number
1	71.3750000	0.4868051	<.0001	1
2	71.6250000	0.4868051	<.0001	2
3	72.0000000	0.4868051	<.0001	3
4	75.0000000	0.4868051	<.0001	4

Output: difference matrix for Tukey's method

Least Squares Means for effect trt Pr > |t| for H0: LSMean(i)=LSMean(j) Dependent Variable: resp				
i/j	1	2	3	4
1		0.9825	0.8085	0.0130
2	0.9825		0.9462	0.0175
3	0.8085	0.9462		0.0281
4	0.0130	0.0175	0.0281	



resp Comparisons for trt
76
74
72
70
4
3
2
1
1 2 3 4
70 72 74 76
Differences for alpha=0.05 (Tukey-Kramer Adjustment)
Not significant
Significant

Dependent Variable: resp

Contrast	DF	Contrast SS	Mean Square	F Value	Pr > F
a	1	0.08333333	0.08333333	0.13	0.7349

Parameter	Estimate	Standard Error	t Value	Pr > |t|
b	-0.25000000	0.69821200	-0.36	0.7349

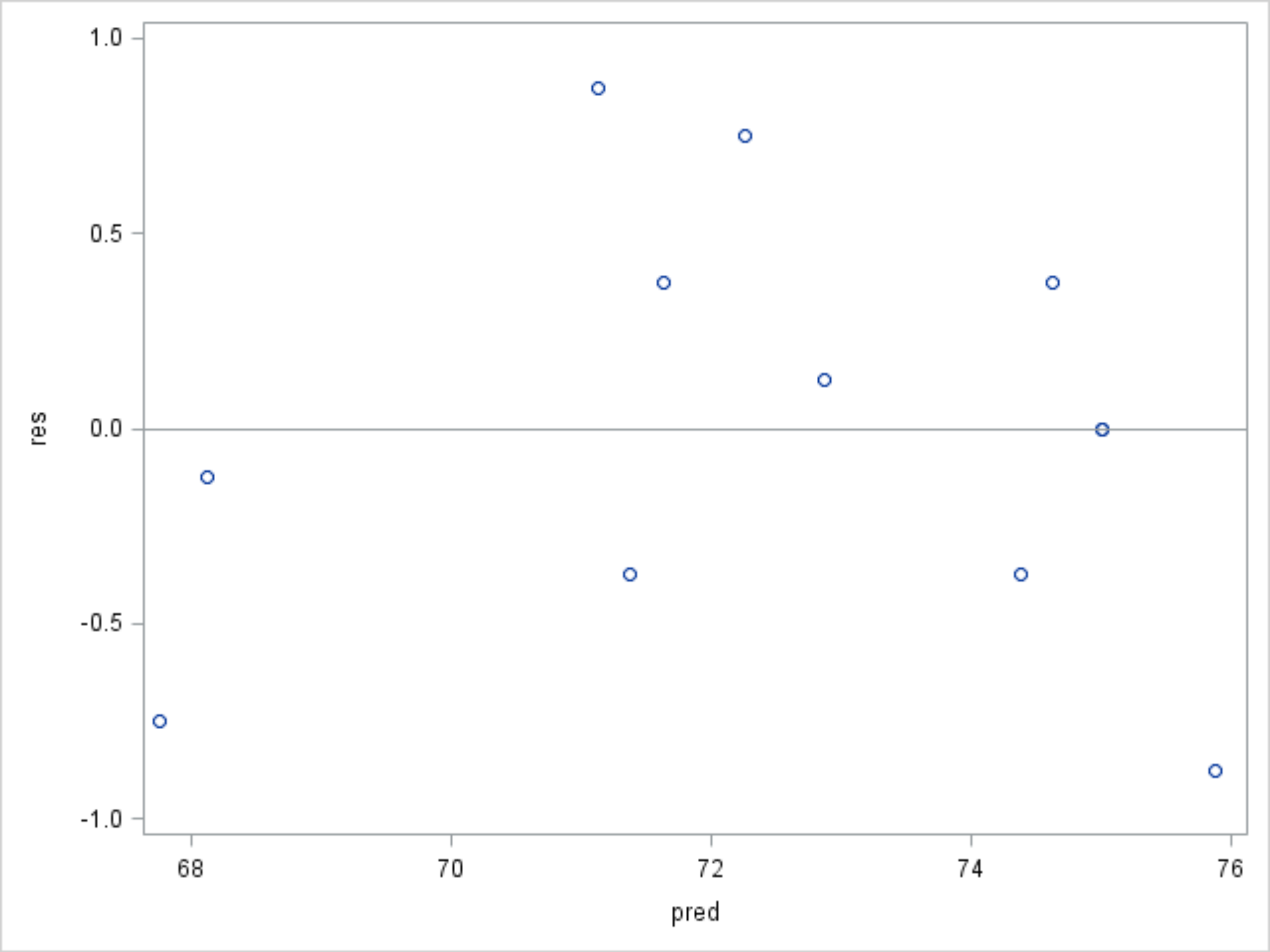
What's the difference between a and b?

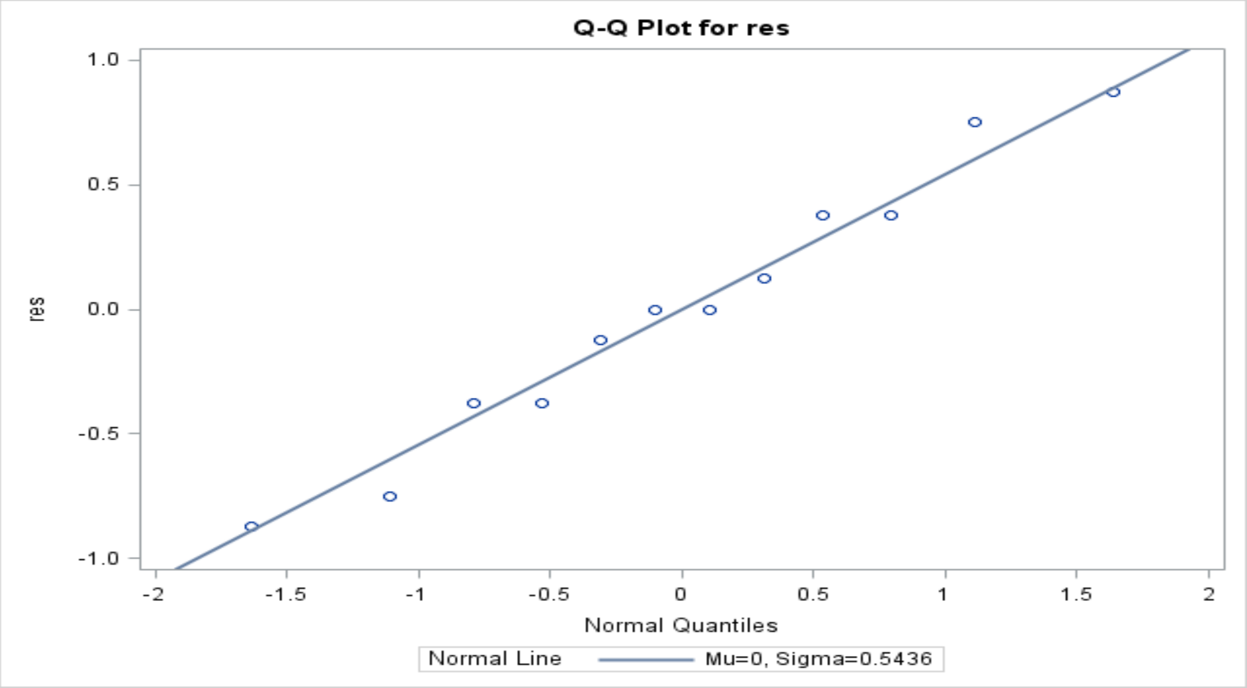
t2 == F

Model adequacy checking

```
proc sgplot data=myout;
scatter y=res x=pred;
refline 0;
run;

proc univariate data=myout
normal;
var  res;
qqplot res/normal(mu=est
sigma=est
color=red L=1);
run;
```


1.0
0.5
0.0
-0.5
-1.0
res
68
70
72
74
76
pred



Tests for Normality				
Test	Statistic		p Value	
Shapiro-Wilk	W	0.969455	Pr < W	0.9050
Kolmogorov-Smirnov	D	0.088205	Pr > D	>0.1500
Cramer-von Mises	W-Sq	0.023458	Pr > W-Sq	>0.2500
Anderson-Darling	A-Sq	0.17265	Pr > A-Sq	>0.2500

Last slide

- Read Section: 4.4