Stat 571B Experimental Design

Topic 9: Graeco-Latin Square Design

Montgomery: chapter 4

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Outline

- Graeco-Latin Square
  - Example
  - Model and SAS code

Graeco-Latin Square: An Example

An experiment is conducted to compare four gasoline additives by testing them on four cars with four drivers over four days. Only four runs can be conducted in each day. The response is the amount of automobile emission.

Treatment factor: gasoline additive, denoted by A, B, C and D.

Block factor 1: driver, denoted by 1, 2, 3, 4.

Block factor 2: day, denoted by 1, 2, 3, 4.

Block factor 3: car, denoted by α, β, γ, δ.

	days			
drivers	1	2	3	4
1	A	B	C	D
2	B	A	D	C
3	C	D	A	B
4	D	C	B	A

	days			
drivers	1	2	3	4
1	α	β	γ	δ
2	δ	γ	β	α
3	β	α	δ	γ
4	γ	δ	α	β

drivers	days 1	2	3	4
1	$A\alpha = 32$	$B\beta = 25$	$C\gamma = 31$	$D\delta = 27$
2	$B\delta = 24$	$A\gamma = 36$	$D\beta = 20$	$C\alpha = 25$
3	$C\beta = 28$	$D\alpha = 30$	$A\delta = 23$	$B\gamma = 31$
4	$D\gamma = 34$	$C\delta = 35$	$B\alpha = 29$	$A\beta = 33$

Graeco-Latin Square

- Consider a p x p Latin square, and superpose on it a second p x p Latin square in which the treatments are denoted by Greek/Latin letters.
- If the two squares when superimposed have the property that each Greek letter appears once and only once with each Latin letter, the two Latin squares are said to be orthogonal.
- the superimposed square is called Graeco-Latin square.
- Graceo-Latin squares exist for all $p \geq 3$, but 6 x 6 Graeco-Latin square does not exist.

Graeco-Latin Square Design Matrix:

driver	day	additive	car
1	1	A	α
1	2	B	β
1	3	C	γ
1	4	D	δ
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	1	D	γ
4	2	C	δ
4	3	B	α
4	4	A	β

Model and Assumptions

$$y_{ijkl} = \mu + \alpha_i + \tau_j + \beta_k + \zeta_l + \epsilon_{ijkl} \quad \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \\ l = 1, 2, \ldots, p \end{cases}$$

μ - grand mean

α_i - ith block 1 effect (row effect) $\quad \sum \alpha_i = 0$

τ_j - jth treatment effect $\quad \sum \tau_j = 0$

β_k - kth block 2 effect (column effect) $\quad \sum \beta_k = 0$

ζ_l - lth block 3 effect (Greek letter effect) $\quad \sum \zeta_l = 0$

$\epsilon_{ijk} \sim \mathrm{N}(0, \sigma^2)$ (independent)

- Completely additive model (no interaction)

Estimation and ANOVA

- Rewrite observation as:

  $y_{ijkl} = \overline{y}_{....} + (\overline{y}_{i...} - \overline{y}_{....}) + (\overline{y}_{.j..} - \overline{y}_{....}) + (\overline{y}_{..k.} - \overline{y}_{....}) +$
  $(\overline{y}_{...l} - \overline{y}_{....}) + (y_{ijkl} - \overline{y}_{i...} - \overline{y}_{.j..} - \overline{y}_{..k.} - \overline{y}_{...l} + 3\overline{y}_{....})$
  $= \hat{\mu} + \hat{\alpha}_i + \hat{\tau}_j + \hat{\beta}_k + \hat{\zeta}_l + \hat{\epsilon}_{ijkl}$

- Partition $\mathrm{SS_T}$ into:

  $p\sum(\overline{y}_{i...} - \overline{y}_{....})^2 + p\sum(\overline{y}_{.j..} - \overline{y}_{....})^2 + p\sum(\overline{y}_{..k.} - \overline{y}_{....})^2 +$
  $p\sum(\overline{y}_{...l} - \overline{y}_{....})^2 + \sum\sum\hat{\epsilon}_{ijkl}^2$
  $= \mathrm{SS_{Row}} + \mathrm{SS_{Treatment}} + \mathrm{SS_{Col}} + \mathrm{SS_{Greek}} + \mathrm{SS_E}$

  with degree of freedom $p-1, p-1, p-1, p-1$ and $(p-3)(p-1)$, respectively.

Analysis of Variance Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Rows	SS_{Row}	$p-1$	MS_{Row}	
Treatment	$\text{SS}_{\text{Treatment}}$	$p-1$	$\text{MS}_{\text{Treatment}}$	F_0
Column	$\text{SS}_{\text{Column}}$	$p-1$	$\text{MS}_{\text{Column}}$	
Greek	SS_{Greek}	$p-1$	MS_{Greek}	
Error	SS_{E}	$(p-3)(p-1)$	MS_{E}	
Total	SS_{T}	p^2-1		

$\text{SS}_{\text{T}} = \sum\sum\sum y_{ijkl}^2 - y_{....}^2/p^2;$ $\text{SS}_{\text{Row}} = \frac{1}{p}\sum y_{i...}^2 - y_{....}^2/p^2;$

$\text{SS}_{\text{Treatment}} = \frac{1}{p}\sum y_{.j..}^2 - y_{....}^2/p^2$ $\text{SS}_{\text{Column}} = \frac{1}{p}\sum y_{..k.}^2 - y_{....}^2/p^2;$

$\text{SS}_{\text{Greek}} = \frac{1}{p}\sum y_{...l}^2 - y_{....}^2/p^2;$ $\text{SS}_{\text{Error}} =$ Use subtraction;

Decision Rule: If $F_0 > F_{\alpha,p-1,(p-3)(p-1)}$ then reject H_0

Sas Code and Output

```
data new;
input row col trt greek resp @@;
datalines;
1 1 1 1 32 1 2 2 2 25
1 3 3 3 31 1 4 4 4 27
2 1 2 4 24 2 2 1 3 36
2 3 4 2 20 2 4 3 1 25
3 1 3 2 28 3 2 4 1 30
3 3 1 4 23 3 4 2 3 31
4 1 4 3 34 4 2 3 4 35
4 3 2 1 29 4 4 1 2 33
;
proc glm data=new;
class row col trt greek;
model resp=row col trt greek;
run;
```

Overall ANOVA table

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	12	296.7500000	24.7291667	2.83	0.2122
Error	3	26.1875000	8.7291667		
Corrected Total	15	322.9375000			

Type III model ANOVA

Source	DF	Type III SS	Mean Square	F Value	Pr > F
row	3	90.6875000	30.2291667	3.46	0.1674
col	3	68.1875000	22.7291667	2.60	0.2263
trt	3	36.6875000	12.2291667	1.40	0.3942
greek	3	101.1875000	33.7291667	3.86	0.1481

- Model adequacy checking is as same as previous models

- Multiple comparison can be carried out using similar methods.

Last slide

- Read Section: 4.3