Stat 571B Experimental Design

Topic 17: Experiments with random factors

Montgomery: chapter -13

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Design of Engineering Experiments - Experiments with Random Factors

- Previous chapters have considered **fixed** factors
  - A specific set of factor levels is chosen for the experiment
  - Inference confined to those levels
  - Often **quantitative** factors are fixed
- When factor levels are chosen at random from a larger population of potential levels, the factor is **random**
  - Inference is about the entire population of levels
  - Industrial applications include measurement system studies

Example 1

A textile company weaves a fabric on a large number of looms. It would like the looms to be homogeneous so that it obtains a fabric of uniform strength. A process engineer suspects that, in addition to the usual variation in strength within samples of fabric from the same loom, there may also be significant variations in strength between looms. To investigate this, she selects four looms at random and makes four strength determinations on the fabric manufactured on each loom. The layout and data are given in the following.

	Observations			
looms	1	2	3	4
1	98	97	99	96
2	91	90	93	92
3	96	95	97	95
4	95	96	99	98

- Response variable is strength
- Interest focuses on determining if there is difference in strength due to the different looms
- However, the weave room contains many (100s) looms
- Solution – select a (random) **sample** of the looms, obtain fabric from each
- Consequently, “looms” is a **random factor**
- It looks like standard single-factor experiment with $a = 4$ & $n = 4$

Random Effects vs Fixed Effects

- Consider factor with numerous possible levels
- Want to draw inference **on population of levels**
- Not concerned with any specific levels
- Example of difference (1=fixed, 2=random)

  1. Compare reading ability of 10 2nd grade classes in NY

     Select $a = 10$ specific classes of interest. Randomly choose n students from each classroom. Want to compare τ_i (class-specific effects).

  2. Study the variability **among all** 2nd grade classes in NY

     Randomly choose $a = 10$ classes from large number of classes. Randomly choose n students from each classroom. Want to assess σ_τ^2 (class to class variability).

- Inference broader in random effects case
- Levels chosen randomly $\longrightarrow$ inference on population

Random Effects Model

- Similar model (as in the fixed case) with different assumptions

$$y_{ij} = \mu + \tau_i + \epsilon_{ij} \quad \begin{cases} i = 1, 2 \ldots a \\ j = 1, 2, \ldots n_i \end{cases}$$

μ - grand mean

τ_i - ith treatment effect (random)

$\epsilon_{ij} \sim \mathrm{N}(0, \sigma^2)$

- Instead of $\sum \tau_i = 0$, assume

  $\tau_i \sim \mathrm{N}(0, \sigma_\tau^2)$

  $\{\tau_i\}$ and $\{\epsilon_{ij}\}$ independent

- Var(y_{ij}) = $\sigma_\tau^2 + \sigma^2$

σ_τ^2 and σ^2 are called variance components

Relevant Hypotheses in the Random Effects (or Components of Variance) Model

- In the fixed effects model we test equality of treatment means
- This is no longer appropriate because the treatments are randomly selected
  - the individual ones we happen to have are not of specific interest
  - we are interested in the **population** of treatments
- The appropriate hypotheses are

$$H_0: \sigma_\tau^2 = 0$$
$$H_1: \sigma_\tau^2 > 0$$

Statistical Analysis

- Same ANOVA table (as before)

Source	SS	DF	MS	F_0
Between	SS_{tr}	$a-1$	MS_{tr}	$F_0 = \frac{\mathrm{MS}_{tr}}{\mathrm{MS}_E}$
Within	SS_E	$N-a$	MS_E	
Total	$\mathrm{SS}_T = \mathrm{SS}_{tr} + \mathrm{SS}_E$	$N-1$		

  - $E(MS_E)=\sigma^2$
  - $E(MS_{tr})=\sigma^2 + n\sigma_\tau^2$

See formula for $E(MS_E)$ and $E(MS_{tr})$ for fixed effect model on P72-73

- Under H_0, $F_0 \sim F_{a-1,N-a}$
- Same test as before
- Conclusions, however, pertain to entire population

Estimation

- Usually interested in estimating variances
- Use mean squares (known as ANOVA method)

$$\hat{\sigma}^2 = \mathrm{MS_E}$$

$$\hat{\sigma}_\tau^2 = (\mathrm{MS_{tr}} - \mathrm{MS_E})/n$$

If unbalanced, replace n with

$$n_0 = \frac{1}{a-1}\left(\sum_{i=1}^a n_i - \frac{\sum_{i=1}^a n_i^2}{\sum_{i=1}^a n_i}\right)$$

- Estimate of σ_τ^2 can be negative
  - Supports H_0? Use zero as estimate?
  - Validity of model? Nonlinear?
  - Other approaches (MLE, Bayesian with nonnegative prior)

Confidence intervals

- σ^2: Same as fixed case

$$\frac{(N-a)\mathrm{MS_E}}{\sigma^2} \sim \chi^2_{N-a}$$

$$\frac{(N-a)\mathrm{MS_E}}{\chi^2_{\alpha/2,N-a}} \le \sigma^2 \le \frac{(N-a)\mathrm{MS_E}}{\chi^2_{1-\alpha/2,N-a}}$$

- For $\sigma_\tau^2/(\sigma^2+\sigma_\tau^2)$:

$$\frac{\mathrm{L}}{\mathrm{L}+1} \le \frac{\sigma_\tau^2}{\sigma^2+\sigma_\tau^2} \le \frac{\mathrm{U}}{\mathrm{U}+1},$$ or

$$\frac{F_0 - F_{\alpha/2,a-1,N-a}}{F_0+(n-1)F_{\alpha/2,a-1,N-a}} \le \frac{\sigma_\tau^2}{\sigma^2+\sigma_\tau^2} \le \frac{F_0 - F_{1-\alpha/2,a-1,N-a}}{F_0+(n-1)F_{1-\alpha/2,a-1,N-a}}$$

Loom Experiment (continued)

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Between	89.19	3	29.73	15.68
Within	22.75	12	1.90	
Total	111.94	15		

Highly significant result ($F_{.05,3,12} = 3.49$)

$$\hat{\sigma}_\tau^2 = (29.73 - 1.90)/4 = 6.98$$

78.6% (=6.98/(6.98+1.90)) is attributable to loom differences

Time to improve consistency of the looms

Loom Experiment - Confidence Intervals

- 95% CI for σ^2

$$\frac{\mathrm{SS_E}}{\chi^2_{.025,12}} \le \sigma^2 \le \frac{\mathrm{SS_E}}{\chi^2_{.975,12}} \quad \Rightarrow \quad (0.97, 5.17)$$

- 95% CI for $\sigma_\tau^2/(\sigma_\tau^2 + \sigma^2)$

$$\left(\frac{15.68 - 4.47}{15.68 + (4-1)4.47}, \frac{15.68 - (1/14.34)}{15.68 + (4-1)(1/14.34)}\right) = (0.385, 0.982)$$

$F_{0.025,3,12} = 4.47,\ F_{.975,3,12} = 1/14.34$

using property that

$$F_{1-\alpha/2,v_1,v_2} = 1/F_{\alpha/2,v_2,v_1}$$

```
data example;
input loom strength;
datalines;
1 98
1 97
1 99
1 96
.
.
4 98
;
```

```
/* random effect model*/
proc glm data=example;
class loom;
model strength=loom;
random loom;
output out=diag r=res
p=pred;
run;

proc plot data=diag;
plot res*pred;
run;
```

```
proc varcomp data=example method = type1;
class loom;
model strength = loom;
run;


proc mixed data=example CL covtest;
class loom;
model strength = ;
random loom;
run;
```

SAS Output (glm)

Dependent Variable: strength

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	89.1875000	29.7291667	15.68	0.0002
Error	12	22.7500000	1.8958333		
Corrected Total	15	111.9375000			

Source	DF	Type I SS	Mean Square	F Value
loom	3	89.18750000	29.72916667	15.68

Source	Type III Expected Mean Square
loom	Var(Error) + 4 Var(loom)

SAS Output (varcomp)

Variance Components Estimation Procedure

Dependent Variable: strength

Source	DF	Sum of Squares	Mean Square	Expected Mean Square
loom	3	89.187500	29.729167	Var(Error) + 4 Var(loom)
Error	12	22.750000	1.895833	Var(Error)
Corrected Total	15	111.937500	.	

Variance Component	Estimate
Var(loom)	6.95833
Var(Error)	1.89583

SAS Output (mixed)

The Mixed Procedure

Iteration History

Iteration	Evaluations	-2 Res Log Like	Criterion
0	1	75.48910190	
1	1	63.19303249	0.00000000

Convergence criteria met.

Covariance Parameter Estimates

Cov Parm	Estimate	Alpha	Lower	Upper
loom	6.9583	0.05	2.1157	129.97
Residual	1.8958	0.05	0.9749	5.1660

A Measurement Systems Capability Study

A typical gauge R&R experiment is shown below. An instrument or gauge is used to measure a critical dimension of certain part. Twenty parts have been selected from the production process, and three randomly selected operators measure each part twice with this gauge. The order in which the measurements are made is completely randomized, so this is a two-factor factorial experiment with design factors parts and operators, with two replications. Both parts and operators are random factors.

Parts	Operator 1		Operator 2		Operator 3	
1	21	20	20	20	19	21
2	24	23	24	24	23	24
3	20	21	19	21	20	22
.	.	.	.	.	.	.
19	25	26	25	24	25	25
20	19	19	18	17	19	17

Variance components equation: $\sigma_y^2 = \sigma_\tau^2 + \sigma_\beta^2 + \sigma_{\tau\beta}^2 + \sigma^2$

Total variability=Parts + Operators + Interaction + Experimental Error

=Parts + Reproducibility + Repeatability

Statistical Model with Two Random Factors

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

$$\tau_i \sim \mathrm{N}(0, \sigma_\tau^2) \quad \beta_j \sim \mathrm{N}(0, \sigma_\beta^2) \quad (\tau\beta)_{ij} \sim \mathrm{N}(0, \sigma_{\tau\beta}^2)$$

- Var(y_{ijk}) = $\sigma^2 + \sigma_\tau^2 + \sigma_\beta^2 + \sigma_{\tau\beta}^2$
- Expected MS's similar to one-factor random model

  E(MS$_\mathrm{E}$)=σ^2; E(MS$_\mathrm{A}$) = $\sigma^2 + bn\sigma_\tau^2 + n\sigma_{\tau\beta}^2$

  E(MS$_\mathrm{B}$) = $\sigma^2 + an\sigma_\beta^2 + n\sigma_{\tau\beta}^2$; E(MS$_\mathrm{AB}$) = $\sigma^2 + n\sigma_{\tau\beta}^2$
- EMS determine what MS to use in denominator

$$H_0 : \sigma_\tau^2 = 0 \longrightarrow \mathrm{MS_A}/\mathrm{MS_{AB}}$$

$$H_0 : \sigma_\beta^2 = 0 \longrightarrow \mathrm{MS_B}/\mathrm{MS_{AB}}$$

$$H_0 : \sigma_{\tau\beta}^2 = 0 \longrightarrow \mathrm{MS_{AB}}/\mathrm{MS_E}$$

Estimating Variance Components

- Using ANOVA method

$$\hat{\sigma}^2 = \mathrm{MS_E}$$

$$\hat{\sigma}^2_\tau = (\mathrm{MS_A} - \mathrm{MS_{AB}})/bn$$

$$\hat{\sigma}^2_\beta = (\mathrm{MS_B} - \mathrm{MS_{AB}})/an$$

$$\hat{\sigma}^2_{\tau\beta} = (\mathrm{MS_{AB}} - \mathrm{MS_E})/n$$

- Sometimes results in negative estimates
- Proc Varcomp and Proc Mixed compute estimates
- Can use different estimation procedures

  ANOVA method - Method = type1

  RMLE method - Method = reml

- Proc Mixed

  Variance component estimates

  Hypothesis tests and confidence intervals

Gauge Capability Example in Text 13-2

```
data randr;
input part operator resp @@;
cards;
1      1      21
1      1      20
2      1      24

……
19     3      25
20     3      19
20     3      17
;
proc glm data=randr;
class operator part;
model resp=operator|part;
random operator part operator*part/test;
run;
```

```
/* two-way random effect model by MIXED procedure and
method=type1 */

proc mixed cl maxiter=20 covtest method=type1;
class operator part;
model resp = ;
random operator part operator*part;
run;

/* two-way random effect model by MIXED procedure and
method=REML */

proc mixed cl maxiter=20 covtest method=reml;
class operator part;
model resp = ;
random operator part operator*part;
run;
```

Dependent Variable: resp

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	59	1215.091667	20.594774	20.77	<.0001
Error	60	59.500000	0.991667		
CorreTotal	119	1274.591667			

Source	DF	Type III SS	Mean Square	F Value	Pr > F
operator	2	2.616667	1.308333	1.32	0.2750
part	19	1185.425000	62.390789	62.92	<.0001
operator*part	38	27.050000	0.711842	0.72	0.8614

Source	Type III Expected Mean Square
operator	Var(Error) + 2 Var(operator*part) + 40 Var(operator)
part	Var(Error) + 2 Var(operator*part) + 6 Var(part)
operator*part	Var(Error) + 2 Var(operator*part)

Tests of Hypotheses Using the Type III
MS for operator*part as an Error Term

Source	DF	Type III SS	Mean Square	F Value	Pr > F

operator	2	2.616667	1.308333	1.84	0.1730
part	19	1185.425000	62.390789	87.65	<.0001

Tests of Hypotheses for Random Model Analysis of Variance

Dependent Variable: resp

Source	DF	Type III SS	Mean Square	F Value	Pr > F
operator	2	2.616667	1.308333	1.84	0.1730
part	19	1185.425000	62.390789	87.65	<.0001
Error	38	27.050000	0.711842		

Error: MS(operator*part)

Source	DF	Type III SS	Mean Square	F Value	Pr > F
operator*part	38	27.050000	0.711842	0.72	0.8614
Error: MS(Error)	60	59.500000	0.991667		

The Mixed Procedure

Type 1 Analysis of Variance

Source	DF	Sum of Squares	Mean Square
operator	2	2.616667	1.308333
part	19	1185.425000	62.390789
operator*part	38	27.050000	0.711842
Residual	60	59.500000	0.991667

Type 1 Analysis of Variance

Source	Expected Mean Square	Error Term	Error DF
operator	Var(Residual) + 2 Var(operator*part) + 40 Var(operator)	MS(operator*part)	38
part	Var(Residual) + 2 Var(operator*part) + 6 Var(part)	MS(operator*part)	38
operator*part	Var(Residual) + 2 Var(operator*part)	MS(Residual)	60
Residual	Var(Residual)	.	

Source	F Value	Pr > F
operator	1.84	0.1730
part	87.65	<.0001
operator*part	0.72	0.8614

Covariance Parameter Estimates

Cov Parm	Estimate	Standard Error	Z Value	Pr Z	Alpha	Lower	Upper
operator	0.0149	0.0330	0.45	0.6510	0.05	-0.0497	0.0795
part	10.2798	3.3738	3.05	0.0023	0.05	3.6673	16.8924
operator*part	-0.1399	0.1219	-1.15	0.2511	0.05	-0.3789	0.0990
Residual	0.9917	0.1811	5.48	<.0001	0.05	0.7143	1.4698

The Mixed Procedure

Estimation Method REML

Iteration History

Iteration	Evaluations	-2 Res Log Like	Criterion
0	1	624.67452320	
1	3	409.39453674	0.00003340
2	1	409.39128078	0.00000004
3	1	409.39127700	0.00000000

Convergence criteria met.

Covariance Parameter Estimates

Cov Parm	Estimate	Standard Error	Z Value	Pr Z	Alpha	Lower	Upper
operator	0.0106	0.03286	0.32	0.3732	0.05	0.001103	3.7E12
part	10.2513	3.3738	3.04	0.0012	0.05	5.8888	22.1549
operator*part	0	.	.	.	.	.	.
Residual	0.8832	0.1262	7.00	<.0001	0.05	0.6800	1.1938

Confidence Intervals for Variance Components

- Can use asymptotic variance estimates to form CI
- Known as Wald's approximate CI
- Mixed: option CL=WALD or METHOD=TYPE1

  Use standard normal $\longrightarrow$ 95% CI uses 1.96

  $$\hat{\sigma}^2_\beta \pm 1.96(.0330) = (-0.05, 0.08)$$

  $$\hat{\sigma}^2_\tau \pm 1.96(3.3738) = (3.67, 16.89)$$

Two-Factor Mixed Effects Model

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$$

- Assume A fixed and B random
  1 $\sum \tau_i = 0$ and $\beta \sim \mathrm{N}(0, \sigma^2_\beta)$ usual assumptions
  2 $(\tau\beta)_{ij} \sim \mathrm{N}(0, (a-1)\sigma^2_{\tau\beta}/a)$ $(a-1)/a$ simplifies EMS
  3 $\sum (\tau\beta)_{ij} = 0$ for β level j added restriction
- Due to added restriction
  - Not all $(\tau\beta)_{ij}$ indep, Cov$((\tau\beta)_{ij}, (\tau\beta)_{i'j}) = -\frac{1}{a}\sigma^2_{\tau\beta}$
  - Cov$(y_{ijk}, y_{i'jk'}) = \sigma^2_\beta - \frac{1}{a}\sigma^2_{\tau\beta}, i \neq i'$.
- Known as **restricted** mixed effects model
- This model coincides with EMS algorithm

  E(MS$_\mathrm{E}$)=σ^2

  E(MS$_\mathrm{A}$) = $\sigma^2 + bn\sum\tau_i^2/(a-1) + n\sigma^2_{\tau\beta}$

  E(MS$_\mathrm{B}$) = $\sigma^2 + an\sigma^2_\beta$

  E(MS$_\mathrm{AB}$) = $\sigma^2 + n\sigma^2_{\tau\beta}$

Hypotheses Testing and Diagnostics

- Testing hypotheses:

$$H_0 : \tau_1 = \tau_2 = \ldots = 0 \longrightarrow \mathrm{MS_A}/\mathrm{MS_{AB}}$$

$$H_0 : \sigma^2_\beta = 0 \longrightarrow \mathrm{MS_B}/\mathrm{MS_E}$$

$$H_0 : \sigma^2_{\tau\beta} = 0 \longrightarrow \mathrm{MS_{AB}}/\mathrm{MS_E}$$

- Variance Estimates (Using ANOVA method)

$$\hat{\sigma}^2 = \mathrm{MS_E}$$

$$\hat{\sigma}^2_\beta = (\mathrm{MS_B} - \mathrm{MS_E})/an$$

$$\hat{\sigma}^2_{\tau\beta} = (\mathrm{MS_{AB}} - \mathrm{MS_E})/n$$

- Diagnostics
  - Histogram or QQplot

    Normality or Unusual Observations
  - Residual Plots

    Constant variance or Unusual Observations

Gauge Capability Example in Text 12-3

```
options nocenter ls=75;

data randr;
 input part operator resp @@;
 cards;
1 1 21 1 1 20 1 2 20 1 2 20 1 3 19 1 3 21
2 1 24 2 1 23 2 2 24 2 2 24 2 3 23 2 3 24
3 1 20 3 1 21 3 2 19 3 2 21 3 3 20 3 3 22
4 1 27 4 1 27 4 2 28 4 2 26 4 3 27 4 3 28
.........................................
20 1 19 20 1 19 20 2 18 20 2 17 20 3 19 20 3 17
;
proc glm;
 class operator part;
 model resp=operator|part;
run;
==================================================
```

Usual output

Dependent Variable: resp

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	59	1215.091667	20.594774	20.77	<.0001
Error	60	59.500000	0.991667		
CorrTotal	119	1274.591667			

Source	DF	Type I SS	Mean Square	F Value	Pr > F
operator	2	2.616667	1.308333	1.32	0.2750
part	19	1185.425000	62.390789	62.92	<.0001
operator*part	38	27.050000	0.711842	0.72	0.8614

Need use correct error term for calculation

- $H_0 : \tau_1 = \tau_2 = \tau_3 = 0$:

$$F_0 = \frac{MS_A}{MS_{AB}} = \frac{1.308}{0.712} = 1.84$$

P-value based on $F_{2,38}$: 0.173.

- $H_0 : \sigma^2_\beta = 0$:

$$F_0 = \frac{MS_B}{MS_E} = \frac{62.391}{0.992} = 62.89$$

P-value based on $F_{19,60}$: 0.000

- H_0: $\sigma^2_{\tau\beta} = 0$:

$$F_0 = \frac{MS_{AB}}{MS_E} = \frac{0.712}{0.992} = 0.72$$

P-value based on $F_{38,60}$: 0.86

- Variance components estimates:

$$\hat{\sigma}_{\beta}^{2} = \frac{62.39 - 0.99}{(3)(2)} = 10.23$$

$$\hat{\sigma}_{\tau\beta}^{2} = \frac{0.71 - 0.99}{2} = -.14(\approx 0)$$

$$\hat{\sigma}^{2} = 0.99$$

- Pairwise comparison for τ_1, τ_2 and τ_3.

Unrestricted Mixed Model

- SAS uses **unrestricted mixed model** in analysis
- $y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$

  $\sum \tau_i = 0$ and $\beta_j \sim \mathrm{N}(0, \sigma^2_\beta)$

  $(\tau\beta)_{ij} \sim \mathrm{N}(0, \sigma^2_{\tau\beta})$
- Expected mean squares:

  E(MS$_{\mathrm{E}}$)=σ^2

  E(MS$_{\mathrm{A}}$) = $\sigma^2 + bn \sum \tau_i^2/(a-1) + n\sigma^2_{\tau\beta}$

  E(MS$_{\mathrm{B}}$) = $\sigma^2 + an\sigma^2_\beta + n\sigma^2_{\tau\beta}$

  E(MS$_{\mathrm{AB}}$) = $\sigma^2 + n\sigma^2_{\tau\beta}$
- random statement in SAS also gives these results

- Differences
  - E($\mathrm{MS_B}$)
  - Test $H_0 : \sigma_\beta^2 = 0$ using $\mathrm{MS_{AB}}$ in denominator
  - Cov($y_{ijk}, y_{i'jk'}$)=σ_β^2, $i \neq i'$.

- Connection

$$(\bar{\tau\beta})_{.j} = (\sum_i (\tau\beta)_{ij})/a$$

$$y_{ijk} = \mu + \tau + (\beta_j + (\bar{\tau\beta})_{.j}) + ((\tau\beta)_{ij} - (\bar{\tau\beta})_{.j}) + \epsilon_{ijk}$$

  Check the model above satisfies the conditions of restricted mixed model

- Restricted model is slightly more general.

SAS code: Gauge Capability Example (Unrestricted Model)

```
options nocenter ls=75;

data randr;
 input part operator resp @@;
 cards;
1 1 21 1 1 20 1 2 20 1 2 20 1 3 19 1 3 21
2 1 24 2 1 23 2 2 24 2 2 24 2 3 23 2 3 24
3 1 20 3 1 21 3 2 19 3 2 21 3 3 20 3 3 22
4 1 27 4 1 27 4 2 28 4 2 26 4 3 27 4 3 28
.
.
;

proc glm;
 class operator part;
 model resp=operator|part;
 random part operator*part / test;
 means operator / tukey lines E=operator*part;
 lsmeans operator / adjust=tukey E=operator*part tdiff stderr;
```

```
proc mixed alpha=.05 cl covtest;
 class operator part;
 model resp=operator / ddfm=kr;
 random part operator*part;
 lsmeans operator / alpha=.05 cl diff adjust=tukey;
run;
```

Covariance Parameter Estimates							
Cov Parm	Estimate	Standard Error	Z Value	Pr > Z	Alpha	Lower	Upper
part	10.2513	3.3738	3.04	0.0012	0.05	5.8888	22.1549
operator*part	0	.	.	.	.	.	.
Residual	0.8832	0.1262	7.00	<.0001	0.05	0.6800	1.1938

Least Squares Means									
Effect	operator	Estimate	Standard Error	DF	t Value	Pr > |t|	Alpha	Lower	Upper
operator	1	22.3000	0.7312	20.1	30.50	<.0001	0.05	20.7752	23.8248
operator	2	22.2750	0.7312	20.1	30.46	<.0001	0.05	20.7502	23.7998
operator	3	22.6000	0.7312	20.1	30.91	<.0001	0.05	21.0752	24.1248

Differences of Least Squares Means

Effect	operator	_operator	Estimate	Standard Error	DF	t Value	Pr > |t|	Adjustment	Adj P	Alpha	Lower	Upper	Adj Lower	Adj Upper
operator	1	2	0.02500	0.2101	98	0.12	0.9055	Tukey-Kramer	0.9922	0.05	-0.3920	0.4420	-0.4751	0.5251
operator	1	3	-0.3000	0.2101	98	-1.43	0.1566	Tukey-Kramer	0.3308	0.05	-0.7170	0.1170	-0.8001	0.2001
operator	2	3	-0.3250	0.2101	98	-1.55	0.1252	Tukey-Kramer	0.2739	0.05	-0.7420	0.09201	-0.8251	0.1751

Rules For Expected Mean Squares

- In models so far, EMS fairly straightforward
- Could calculate EMS using brute force method
- For mixed models, good to have formal procedure
- Montgomery describes procedure for **restricted** model

0 Write the error term in the model as $\epsilon_{(ij..)m}$, where m represents the replication subscript

1 Write each variable term in model as a row heading in a two-way table

2 Write the subscripts in the model as column headings. Over each subscript write F if factor fixed and R if random. Over this, write down the levels of each subscript

3 For each row, copy the number of observations under each subscript, providing the subscript does not appear in the row variable term

4 For any bracketed subscripts in the model, place a 1 under those subscripts that are inside the brackets

5 Fill in remaining cells with a 0 (if subscript represents a fixed factor) or a 1 (if random factor).

6 To find the expected mean square of any term (row), cover the entries in the columns that contain non-bracketed subscript letters in this term in the model. For those rows with at least the same subscripts, multiply the remaining numbers to get coefficient for corresponding term in the model.

This will be on the final

Two-Factor Fixed Model:

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$$

Sigma + levels in EMS == Fixed

	F	F	R	
	a	b	n	
term	i	j	k	EMS
τ_i	0	b	n	$\sigma^2 + \frac{bn\Sigma\tau_i^2}{a-1}$
β_j	a	0	n	$\sigma^2 + \frac{an\Sigma\beta_j^2}{b-1}$
$(\tau\beta)_{ij}$	0	0	n	$\sigma^2 + \frac{n\Sigma\Sigma(\tau\beta)_{ij}^2}{(a-1)(b-1)}$
$\epsilon_{(ij)k}$	1	1	1	σ^2

Two-Factor Random Model:

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$$

	R	R	R	
	a	b	n	
term	i	j	k	EMS
τ_i	1	b	n	$\sigma^2 + n\sigma^2_{\tau\beta} + bn\sigma^2_\tau$
β_j	a	1	n	$\sigma^2 + n\sigma^2_{\tau\beta} + an\sigma^2_\beta$
$(\tau\beta)_{ij}$	1	1	n	$\sigma^2 + n\sigma^2_{\tau\beta}$
$\epsilon_{(ij)k}$	1	1	1	σ^2

variances in EMS == random

Two-Factor Mixed Model (A Fixed):

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$$

tau has levels and sigma = Fixed
beta has variances, no sigma == Random

	F	R	R	
	a	b	n	
term	i	j	k	EMS
τ_i	0	b	n	$\sigma^2 + n\sigma^2_{\tau\beta} + \frac{bn\Sigma\tau_i^2}{a-1}$
β_j	a	1	n	$\sigma^2 + an\sigma^2_\beta$
$(\tau\beta)_{ij}$	0	1	n	$\sigma^2 + n\sigma^2_{\tau\beta}$
$\epsilon_{(ij)k}$	1	1	1	σ^2

this will not be on the exam, it's motivation for the next slide

Three-Factor Mixed Model (A Fixed):

$$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \epsilon_{ijkl}$$

	F	R	R	R	
	a	b	c	n	
term	i	j	k	l	EMS
τ_i	0	b	c	n	$\sigma^2 + cn\sigma^2_{\tau\beta} + bn\sigma^2_{\tau\gamma} + n\sigma^2_{\tau\beta\gamma} + \frac{bcn\Sigma\tau_i^2}{a-1}$
β_j	a	1	c	n	$\sigma^2 + an\sigma^2_{\beta\gamma} + acn\sigma^2_{\beta}$
γ_k	a	b	1	n	$\sigma^2 + an\sigma^2_{\beta\gamma} + abn\sigma^2_{\gamma}$
$(\tau\beta)_{ij}$	0	1	c	n	$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + cn\sigma^2_{\tau\beta}$
$(\tau\gamma)_{ik}$	0	b	1	n	$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + bn\sigma^2_{\tau\gamma}$
$(\beta\gamma)_{jk}$	a	1	1	n	$\sigma^2 + an\sigma^2_{\beta\gamma}$
$(\tau\beta\gamma)_{ijk}$	0	1	1	n	$\sigma^2 + n\sigma^2_{\tau\beta\gamma}$
ϵ_{ijkl}	1	1	1	1	σ^2

Approximate F Tests

- For some models, no exact F-test exists
- Recall 3 Factor Mixed Model (A - fixed)
- No exact test for A based on EMS

Assume $a = 3, b = 2, c = 3, n = 2$ and following MS were obtained

Source	DF	MS	EMS	F	P
A	2	0.7866	$12\phi_A + 6\sigma^2_{AB} + 4\sigma^2_{AC}$ $+2\sigma^2_{ABC} + \sigma^2$	?	?
B	1	0.0010	$18\sigma^2_B + 6\sigma^2_{BC} + \sigma^2$	0.33	.622
AB	2	0.0056	$6\sigma^2_{AB} + 2\sigma^2_{ABC} + \sigma^2$	2.24	.222
C	2	0.0560	$12\sigma^2_C + 6\sigma^2_{BC} + \sigma^2$	18.87	.051
AC	4	0.0107	$4\sigma^2_{AC} + 2\sigma^2_{ABC} + \sigma^2$	4.28	.094
BC	2	0.0030	$6\sigma^2_{BC} + \sigma^2$	10.00	.001
ABC	4	0.0025	$2\sigma^2_{ABC} + \sigma^2$	8.33	.001
Error	18	0.0003	σ^2		

- Possible approaches:
  - Could assume some variances are negligible, not recommended without "conclusive" evidence
  - Pool (insignificant) means squares with error, also risky, not recommended when df for error is already big.

Satterthwaite's Approximate F-test

- H_0 : effect $= 0$, e.g., $H_0 : \tau_1 = \cdots = \tau_a = 0$ or equivalently $H_0 : \sum \tau_i^2 = 0$.
  No exact test exists.

- Get two linear combinations of mean squares

  $$MS' = \mathrm{MS}_r \pm \ldots \pm \mathrm{MS}_s$$

  $$MS'' = \mathrm{MS}_u \pm \ldots \pm \mathrm{MS}_v$$

  such that 1) MS' and MS'' do not share common mean squares; 2) $E(MS') - E(MS'')$ is a multiple of the effect.

  – approximate test statistic F: $F = \frac{MS'}{MS''} = \frac{\mathrm{MS}_r \pm \ldots \pm \mathrm{MS}_s}{\mathrm{MS}_u \pm \ldots \pm \mathrm{MS}_v} \approx F_{p,q}$

    where $p = \frac{(\mathrm{MS}_r \pm \ldots \pm \mathrm{MS}_s)^2}{\mathrm{MS}_r^2/f_r + \ldots + \mathrm{MS}_s^2/f_s}$ and $q = \frac{(\mathrm{MS}_u \pm \ldots \pm \mathrm{MS}_v)^2}{\mathrm{MS}_u^2/f_u + \ldots + \mathrm{MS}_v^2/f_v}$

  – f_i is the degrees of freedom associated with MS_i

  – p and q may not be integers, interpolation is needed. SAS can handle noninteger dfs.

  – Caution when subtraction is used

Example: 3-Factor Mixed Model (A Fixed)

$H_0 : \tau_1 = \tau_2 = \tau_3 = 0$

$MS' = \mathrm{MS}_A$

$MS'' = \mathrm{MS}_{AB} + \mathrm{MS}_{AC} - \mathrm{MS}_{ABC}$

$E(MS' - MS'') = 12\phi_A = 12\frac{\Sigma\tau_i^2}{3-1}$

$$F = \frac{\mathrm{MS}_A}{\mathrm{MS}_{AB} + \mathrm{MS}_{AC} - \mathrm{MS}_{ABC}} = \frac{.7866}{.0107 + .0056 - .0025} = 57.0$$

$$p = 2 \qquad q = \frac{.0138^2}{.0107^2/4 + .0056^2/2 + .0025^2/4} = 4.15$$

- Interpolation needed

$$\mathrm{P}(F_{2,4} > 57) = .0011 \qquad \mathrm{P}(F_{2,5} > 57) = .0004$$

$$\mathrm{P} = .85(.0011) + .15(.0004) = .001$$

- SAS can be used to compute P-values and quantile values for F and χ^2 values with noninteger degrees of freedom.

  Upper Tail Probability: probf(x,df1,df2) and probchi(x,df)

  Quantiles : finv(p,df1,df2) and cinv(p,df)

```
data one;
  p=1-probf(57,2.0,4.15);
  f=finv(.95,2.0,4.15);
  c1=cinv(.025,18.57);
  c2=cinv(.975,18.57);
proc print data=one;

OBS         P           F          C1          C2
 1       .00096      6.7156     8.61485     32.2833
```

Another Approach to Testing $H_0 : \tau_1 = \tau_2 = \tau_3 = 0$

$MS' = \mathrm{MS}_A + \mathrm{MS}_{ABC}$

$MS'' = \mathrm{MS}_{AB} + \mathrm{MS}_{AC}$

$E(MS' - MS'') = ?$

$$F = \frac{\mathrm{MS}_A + \mathrm{MS}_{ABC}}{\mathrm{MS}_{AB} + \mathrm{MS}_{AC}} = \frac{.7866 + .0025}{.0107 + .0056} = 48.41$$

$$p = \frac{.7891^2}{.7866^2/2 + .0025^2/4} = 2.01 \qquad q = \frac{.0163^2}{.0107^2/4 + .0056^2/2} = 6.00$$

P-value= $P(F > 48.41) = 0.002$

- This is again found significant
- Avoid subtraction,summation should be preferred.

Approximate Confidence Intervals

Suppose we are interested in σ_x^2.

- Case 1: there exists a mean square MS_x with df$_x$ such that $E(MS_x) = \sigma_x^2$. Then $\hat{\sigma}_x^2 = MS_x$, and

  $$\frac{\text{df}_x MS_x}{\sigma_x^2} \sim \chi^2(\text{df}_x)$$

  Exact 100(1-α)% CI: $\frac{\mathrm{df}_x MS_x}{\chi^2_{\alpha/2,\mathrm{df}_x}} \le \sigma_x^2 \le \frac{\mathrm{df}_x MS_x}{\chi^2_{1-\alpha/2,\mathrm{df}_x}}$

- Case 2: there exist

  $MS' = \mathrm{MS}_r + \ldots + \mathrm{MS}_s$ and, $MS'' = \mathrm{MS}_u + \ldots + \mathrm{MS}_v$

  such that $E(MS' - MS'') = k\sigma_x^2$. Then

$$\hat{\sigma}_x^2 = \frac{MS' - MS''}{k}, \text{ and } \frac{\mathrm{df}_x \hat{\sigma}_x^2}{\sigma_x^2} \approx \chi^2(\mathrm{df}_x)$$

where

$$\mathrm{df}_x = \frac{(\hat{\sigma}_x^2)^2}{\sum \frac{MS_i}{k^2 f_i}} = \frac{(\mathrm{MS}_r + \cdot + \mathrm{MS}_s - \mathrm{MS}_u - \cdot - \mathrm{MS}_v)^2}{\mathrm{MS}_r^2/f_r + \cdot + \mathrm{MS}_s^2/f_s + \mathrm{MS}_u^2/f_u + \cdot + \mathrm{MS}_v^2/f_v}$$

Approximate 100(1-α)% CI:

$$\frac{\mathrm{df}_x \hat{\sigma}_x^2}{\chi^2_{\alpha/2,\mathrm{df}_x}} \le \sigma_x^2 \le \frac{\mathrm{df}_x \hat{\sigma}_x^2}{\chi^2_{1-\alpha/2,\mathrm{df}_x}}$$

Gauge Capability Example (Both Factors are Random)

Dependent Variable: RESP

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	59	1215.09166667	20.594774	20.77	0.0001
Error	60	59.50000000	0.991667		
CorrecTotal	119	1274.59166667			

Source	DF	Type III SS	Mean Square	F Value	Pr > F
OPERATOR	2	2.61666667	1.308333	1.32	0.2750
PART	19	1185.42500000	62.390789	62.92	0.0001
OPERATOR*PART	38	27.05000000	0.711842	0.72	0.8614

$$\hat{\sigma}_\tau^2 = \frac{MS_A - MS_{AB}}{bn} = (62.39 - 0.71)/6 = 10.28$$

$$\text{df} = \frac{(62.39 - 0.71)^2}{62.39^2/19 + 0.71^2/38} = 18.57$$

CI: $(18.57(10.28)/32.28, 18.57(10.28)/8.61) = (5.91, 22.17)$

$$\hat{\sigma}_\beta^2 = \frac{MS_B - MS_{AB}}{an} = (1.31 - 0.71)/40 = 0.015$$

$$\text{df} = \frac{(1.31 - 0.71)^2}{1.31^2/2 + 0.71^2/38} = .413$$

CI: $(.413(.015)/3.079, .413(.015)/2.29 \times 10^{-8}) = (.002, 270781)$

PROC GLM and PROC MIXED

- PROC GLM:
  - Uses the method of least squares to fit general linear models. No other parameter estimation method can be specified.
- PROC MIXED
  - Fits a variety of mixed linear models to data and allows specification of the parameter estimation method to be used.

Comparison

PROC GLM	PROC MIXED
(1) Designed for "fixed effects" models with allowance for some adjustments when random effects are present.	(1) Designed for "random" and "mixed effects" models
(2) Estimation of fixed effects: Estimates are based on Ordinary Least Squares.	(2) Estimation of fixed effects: Based on Generalized Least Squares in a normal error model.(estimates are maximum likelihood estimates under normality).
(3) Estimation of variance components via the RANDOM statement: Uses method of moments estimation of solving expected mean squares for variance components.	(3) Estimation of variance components: Uses maximum likelihood or restricted maximum likelihood estimation methods for variance components (unless type 3 is requested).
(4) Both fixed and random effects are listed in the model statement.	(4) Only the fixed effects are listed in the model statement. The random effects are listed in the random statement.
(5) The LSMEANS statement interprets all effects as fixed, even the random effects. While the least square means are correct, their standard errors are not necessarily correct. This is true even if a RANDOM statement is used.	(5) Effects in the MODEL statement are assumed fixed. The standard error estimates for least square means account for the random effects.

(6) The RANDOM statement under PROC GLM invokes the calculation of expected mean squares for the listed effects and the appropriate test using the "test" option. The randomness of the effect is not incorporated into the tests of main effects, lsmeans, contrasts etc.	(6) Signals incorporation of the listed effects in all aspects of inference. Various correlation structures, describing the dependencies of multiple random effects can be selected using different options.
(7) The TEST statement is an important statement to use correct error terms in testing model effects (e.g. Subsampling designs, split-plot design).	(7) No TEST statement under PROC MIXED
(8) The REPEATED statement in PROC GLM is used to specify various transformations with which to conduct the traditional univariate or multivariate tests.	(8) The REPEATED statement in PROC MIXED is used to specify covariance structures for repeated measurements on subjects. The approach used in PROC MIXED is more flexible and more widely applicable than either the univariate or multivariate approaches.
(9) PROC GLM is not efficient in handling missing values.	(9) PROC MIXED has a better mechanism for handling missing values.

Last slide

- Read Sections: 13.1-13.7