Numerical Methods - CSC207

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System of linear equation

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\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1,\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2,\\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \quad &\\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n,
\end{aligned}
$$

where a_{jk} and b_j are given (real or complex) numbers.

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\vdots \qquad\quad \vdots \qquad\quad &\vdots\\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n,
\end{aligned}$$

  where a_{jk} and b_j are given (real or complex) numbers.
- This is *homogeneous* if all the $b_j's$ are zeros, otherwise *non-homogeneous*.
- In matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

$$\mathbf{Ax} = \mathbf{B},$$

- $\mathbf{A}\ \mathbf{x} = \mathbf{B}$, where $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.
- Here $\mathbf{A}$ is *coefficient matrix* and $\mathbf{x}, \mathbf{B}$ are *column vectors* or *matrices*.

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- Here $\mathbf{A}$ is *coefficient matrix* and $\mathbf{x}, \mathbf{B}$ are *column vectors* or *matrices*.
- Augmented matrix: The coefficient matrix $\mathbf{A}$ with column vector $\mathbf{B}$:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & b_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & : & b_n \end{bmatrix}$$

- $$[\mathbf{A} : \mathbf{B}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & b_2 \\ \vdots & \vdots & & \vdots & : & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & : & b_n \end{bmatrix}.$$
- A solution of this linear system is a tuple $(s_1, s_2, \cdots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, \cdots, s_n$ are substituted for $x_1, x_2, \cdots, x_n$, respectively.
- The set of all solutions is called *a solution of the system*.

Solution of System of linear equations

1. Analytic approach
   - Gauss-Elimination method

     (Echelon form)
   - Gauss-Jordan method

     (Normal form)
   - LU- Factorization
2. Iteration approach
   - Jacobi's iteration method
   - Gauss-Seidal method

Gauss-elimination method - complete pivoting

Ex.1: Solve $x + 6y + 2z = -1$, $3x + 5y + 2z = 8$, $6x + 2y + 8z = 26$ using Gauss elimination method with complete pivoting

Gauss-elimination method - complete pivoting

Ex.1: Solve $x + 6y + 2z = -1$, $3x + 5y + 2z = 8$, $6x + 2y + 8z = 26$ using Gauss elimination method with complete pivoting

Soln: Here, the system is

$x + 6y + 2z = -1,\ 3x + 5y + 2z = 8,\ 6x + 2y + 8z = 26$

With complete pivoting method, the system can be written as

$$8z + 2y + 6x = 26,$$

$$2z + 5y + 3x = 8,$$

$$2z + 6y + x = -1,$$

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$$8z + 2y + 6x = 26,$$

$$2z + 5y + 3x = 8,$$

$$2z + 6y + x = -1,$$

- In matrix form: $\begin{bmatrix} 8 & 2 & 6 \\ 2 & 5 & 3 \\ 2 & 6 & 1 \end{bmatrix} \begin{bmatrix} z \\ y \\ x \end{bmatrix} = \begin{bmatrix} 26 \\ 8 \\ -1 \end{bmatrix},$

$\mathbf{A}\ \mathbf{X} = \mathbf{B}.$

- In Augmented matrix: $\begin{bmatrix} 8 & 2 & 6 & : & 26 \\ 2 & 5 & 3 & : & 8 \\ 2 & 6 & 1 & : & -1 \end{bmatrix}$

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- Operating $R_2 \to R_2 - \left(\frac{2}{8}\right) R_1,\ R_3 \to R_3 - \left(\frac{2}{8}\right) R_1$, we get

$$\begin{bmatrix} 8 & 2 & 6 & : & 26 \\ 0 & 4.5 & 1.5 & : & 1.5 \\ 0 & 5.5 & -0.5 & : & -7.5 \end{bmatrix}$$

- Operating $R_2 \leftrightarrow R_3$, we get

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- Operating $R_3 \to R_3 - \left(\frac{4.5}{5.5}\right) R_2$, we get

$$\begin{bmatrix} 8 & 2 & 6 & : & 26 \\ 0 & 5.5 & -0.5 & : & -7.5 \\ 0 & 0 & \frac{21}{11} & : & \frac{84}{11} \end{bmatrix}.$$

- The system of linear equation is

$8\,z + 2y + 6x = 26, \qquad 5.5\,y - 0.5x = -7.5, \qquad \frac{21}{11}\,x = \frac{84}{11}.$

- By back substitution: $x = 4,\ y = -1,\ z = 1/2$.

Gauss-Jordan Method

- Consider a system of equation

$$a_1x + b_1y + c_1z = d_1, \quad a_2x + b_2y + c_2z = d_2, \quad a_3x + b_3y + c_3z = d_3.$$

- In matrix form:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

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- Augmented matrix form :

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- Operating $R_2 \to R_2 - (\frac{a_2}{a_1})R_1, R_3 \to R_3 - (\frac{a_3}{a_1})R_1$, we get

$$\begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & b_3' & c_3' & : & d_3' \end{bmatrix}$$

Gauss Jordan Method

- Here $$\begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & b_3' & c_3' & : & d_3' \end{bmatrix}$$

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- Taking $b_2' \neq 0$. Operating $R_1 \to R_1 - (\frac{b_1'}{b_2'})R_2$, $R_3 \to R_3 - (\frac{b_3'}{b_2'})R_1$, we get
$$\begin{bmatrix} a_1 & 0 & c_1' & : & d_1' \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & 0 & c_3'' & : & d_3'' \end{bmatrix}$$

Gauss Jordan Method

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$$\begin{bmatrix} a_1 & 0 & c_1' & : & d_1' \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & 0 & c_3'' & : & d_3'' \end{bmatrix}$$

- Taking $c_3'' \neq 0$. Operating $R_1 \to R_1 - (\frac{c_1'}{c_3''})R_3$, $R_2 \to R_2 - (\frac{c_2'}{c_3''})R_3$, we get

$$\begin{bmatrix} a_1 & 0 & 0 & : & d_1'' \\ 0 & b_2' & 0 & : & d_2'' \\ 0 & 0 & c_3'' & : & d_3'' \end{bmatrix}$$

Gauss Jordan Method

- Therefore

$$\begin{bmatrix} a_1 & 0 & 0 & : & d_1'' \\ 0 & b_2' & 0 & : & d_2'' \\ 0 & 0 & c_3'' & : & d_3'' \end{bmatrix}.$$

Gauss Jordan Method

- Therefore

$$\begin{bmatrix} a_1 & 0 & 0 & : & d_1'' \\ 0 & b_2' & 0 & : & d_2'' \\ 0 & 0 & c_3'' & : & d_3'' \end{bmatrix}.$$

- Operating $R_1 \to (\frac{1}{a_1})R_1,\ R_2 \to (\frac{1}{b_2'})R_2$, and $R_3 \to (\frac{1}{c_3''})R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 & : & d_1''/a_1 \\ 0 & 1 & 0 & : & d_2''/b_2' \\ 0 & 0 & 1 & : & d_3''/c_3'' \end{bmatrix}$$

- Therefore, the solution of the system is

$$x = d_1''/a_1,\ y = d_2''/b_2',\ z = d_3''/c_3''.$$

- This method is known as Gauss-Jordan method.

Example

Ex.1 Solve the the system of the equations $x + y + z = 9$, $2x - 3y + 4z = 13$, $3x + 4y + 5z = 40$ by using Gauss-Jordan method

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Soln: Here, the system of equation is

$$x + y + z = 9,\ 2x - 3y + 4z = 13,\ 3x + 4y + 5z = 40,$$

- In matrix form : $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}.$

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- Augmented matrix : $\begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & -3 & 4 & : & 13 \\ 3 & 4 & 5 & : & 40 \end{bmatrix}.$

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- Augmented matrix : $\begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & -3 & 4 & : & 13 \\ 3 & 4 & 5 & : & 40 \end{bmatrix}$.
- Operating $R_2 \to R_2 - (2)R_1, R_3 \to R_3 - (3)R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -5 & 2 & : & -5 \\ 0 & 1 & 2 & : & 13 \end{bmatrix}.$$

Example

- Operating $R_2 \to (-1/5)R_2$, we get

$$\begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & \mathbf{1} & -2/5 & : & 1 \\ 0 & 1 & 2 & : & 13 \end{bmatrix}.$$

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- Operating $R_1 \to R_1 - R_2$, and $R_3 \to R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 0 & 7/5 & : & 8 \\ 0 & 1 & -2/5 & : & 1 \\ 0 & 0 & 12/5 & : & 12 \end{bmatrix}.$$

- Operating $R_3 \to (5/12)R_3$, we get

$$\begin{bmatrix} 1 & 0 & 7/5 & : & 8 \\ 0 & 1 & -2/5 & : & 1 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}.$$

Example

- Operating $R_2 \to (-1/5)R_2$, we get

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- Operating $R_3 \to (5/12)R_3$, we get

$$\begin{bmatrix} 1 & 0 & 7/5 & : & 8 \\ 0 & 1 & -2/5 & : & 1 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}.$$

- Operating $R_1 \to R_1 - (7/5)R_3$, and $R_2 \to R_2 + (2/5)R_3$ we get

$\begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 3 \end{bmatrix}$. Therefore $x = 1, y = 3, z = 5$.

LU Factorization Method

- Every square matrix A can be expressed as the product of lower and upper triangular matrices if all the minors of A are non-singular,
- Consider the equation

$$a_1x + b_1y + c_1z = d_1,$$

$$a_2x + b_2y + c_2z = d_2,$$

$$a_3x + b_3y + c_3z = d_3.$$

- In matrix form

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix},$$

$$AX = B, \tag{1}$$

LU Factorization Method

- where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

- Setting

$$A = LU, \tag{2}$$

- where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

- So (1) becomes

$$LUX = B, \tag{3}$$

- Setting

$$UX = Y. \tag{4}$$

- So (3) becomes

$$LY = B,$$

or
$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix},$$

giving

$$y_1 = d_1,$$

$$l_{21}y_1 + y_2 = d_2,$$

$$l_{31}y_1 + l_{32}y_2 + y_3 = d_3.$$

Example

Solve: $2x - 3y + 10z = 3,\ -x + 4y + 2z = 20,\ 5x + 2y + z = -12$ by using LU factorization method

Soln: Here, the system of equation can be written as

$$2x - 3y + 10z = 3,$$

$$-x + 4y + 2z = 20,$$

$$5x + 2y + z = -12,$$

- and as in matrix form $\begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$,

$$AX = B, \tag{5}$$

- where $A = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix},$

Example

- Setting

$$A = LU, \tag{6}$$

- which gives $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix},$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}.$$

- Equating the corresponding elements (first columns), we get

Example

- Setting

$$A = LU, \tag{6}$$

- which gives $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$,

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}.$$

- Equating the corresponding elements (first columns), we get

$$u_{11} = 2, \qquad l_{21}u_{11} = -1, \qquad l_{31}u_{11} = 5,$$

$$\therefore \qquad l_{21} = -1/2, \qquad l_{31} = 5/2,$$

- $$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}.$$
- Equating the corresponding elements(second columns), we get

$$u_{12} = -3, \qquad l_{21}u_{12} + u_{22} = 4, \qquad l_{31}u_{12} + l_{32}u_{22} = 2,$$

- $$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}.$$
- Equating the corresponding elements(second columns), we get

$$u_{12} = -3, \qquad l_{21}u_{12} + u_{22} = 4, \qquad l_{31}u_{12} + l_{32}u_{22} = 2,$$

or $\quad u_{22} = 4 - (3/2) = 5/2, \qquad l_{32} = (2 - (5/2)(-3))/(2/5) = 19/5,$

$\therefore \qquad u_{22} = 5/2, \qquad l_{32} = 19/5,$

- Equating the corresponding elements(third columns), we get

$$u_{13} = 10, \qquad l_{21}u_{13} + u_{23} = 2, \qquad l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1,$$

- $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}.$
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- Equating the corresponding elements(third columns), we get

$$u_{13} = 10, \qquad l_{21}u_{13} + u_{23} = 2, \qquad l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1,$$

or $\quad u_{23} = 2 - (-1/2)(10) = 7, \; u_{33} = 1 - l_{31}u_{13} - l_{32}u_{23} = -253/5,$

$\therefore \quad u_{23} = 7, \quad u_{33} = -253/5.$

- $$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}.$$
- Equating the corresponding elements(second columns), we get

$$u_{12} = -3, \qquad l_{21}u_{12} + u_{22} = 4, \qquad l_{31}u_{12} + l_{32}u_{22} = 2,$$

or $\quad u_{22} = 4 - (3/2) = 5/2, \qquad l_{32} = (2 - (5/2)(-3))/(2/5) = 19/5,$

$\therefore \qquad u_{22} = 5/2, \qquad l_{32} = 19/5,$

- Equating the corresponding elements(third columns), we get

$$u_{13} = 10, \qquad l_{21}u_{13} + u_{23} = 2, \qquad l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1,$$

or $\quad u_{23} = 2 - (-1/2)(10) = 7, \; u_{33} = 1 - l_{31}u_{13} - l_{32}u_{23} = -253/5,$

$\therefore \qquad u_{23} = 7, \qquad u_{33} = -253/5.$

- Therefore $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -3 & 10 \\ 0 & \frac{5}{2} & 7 \\ 0 & 0 & -\frac{253}{5} \end{bmatrix}.$

Example

- From (5) and (6), we write $LUX = B$.
- Setting $UX = Y$. Thus $LY = B$ becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix},$$

- giving $y_1 = 3,\ y_2 = 43/2,\ y_3 = -506/5.$

- So, $UX = Y$ becomes

$$\begin{bmatrix} 2 & -3 & 10 \\ 0 & \frac{5}{2} & 7 \\ 0 & 0 & -\frac{253}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{43}{2} \\ -\frac{506}{5} \end{bmatrix}.$$

- Therefore $2\mathbf{x} - 3y + 10z = 3,\quad \frac{5}{2}\mathbf{y} + 7z = \frac{43}{2},\quad -\frac{253}{5}\mathbf{z} = -\frac{506}{5}.$
- By back substitution: $x = -4,\ y = 3$ and $z = 2.$

Inverse by Gauss-Jordan method

- Then, after augmenting by the identity, the following is obtained:

$$[A:I] = \begin{bmatrix} a_1 & a_2 & a_3 & : & 1 & 0 & 0 \\ b_1 & b_2 & b_3 & : & 0 & 1 & 0 \\ c_1 & c_2 & c_3 & : & 0 & 0 & 1 \end{bmatrix}.$$

- Performing elementary row operations on the $[A:I]$ matrix until it reaches reduced into canonical form(normal form):

$$[I:A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & : & a_1' & a_2' & a_3' \\ 0 & 1 & 0 & : & b_1' & b_2' & b_3' \\ 0 & 0 & 1 & : & c_1' & c_2' & c_3' \end{bmatrix}.$$

- The matrix augmentation can now be undone, which gives the following:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} a_1' & a_2' & a_3' \\ b_1' & b_2' & b_3' \\ c_1' & c_2' & c_3' \end{bmatrix}.$$

Ex.: Find the inverse of matrix $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ by using Gauss-Jordan method

- Augmented matrix is

$$\begin{bmatrix} 2 & 1 & 1 & :1 & 0 & 0 \\ 3 & 2 & 3 & :0 & 1 & 0 \\ 1 & 4 & 9 & :0 & 0 & 1 \end{bmatrix}$$

.

- Operating $R_1 \leftrightarrow R_3$, to get

$$\begin{bmatrix} 1 & 4 & 9 & :0 & 0 & 1 \\ 3 & 2 & 3 & :0 & 1 & 0 \\ 2 & 1 & 1 & :1 & 0 & 0 \end{bmatrix}$$

.

- Operating $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$, to get

$$\begin{bmatrix} 1 & 4 & 9 & :0 & 0 & 1 \\ 0 & -10 & -24 & :0 & 1 & -3 \\ 0 & -7 & -17 & :1 & 0 & -2 \end{bmatrix}$$

.

- Operating $R_2 \to (-\frac{1}{10})R_2$, to get

$$\begin{bmatrix} 1 & 4 & 9 & :0 & 0 & 1 \\ 0 & 1 & 12/5 & :0 & -1/10 & 3/10 \\ 0 & -7 & -17 & :1 & 0 & -2 \end{bmatrix}.$$

- Operating $R_3 \to R_3 + 7R_2,\ R_1 \to R_1 - 4R_2$, to get

$$\begin{bmatrix} 1 & 0 & -3/5 & :0 & 2/5 & -1/5 \\ 0 & 1 & 12/5 & :0 & -1/10 & 3/10 \\ 0 & 0 & -1/5 & :1 & -7/10 & 1/10 \end{bmatrix}.$$

- Operating $R_3 \to -5R_3$, to get

$$\begin{bmatrix} 1 & 0 & -3/5 & : 0 & 2/5 & -1/5 \\ 0 & 1 & 12/5 & : 0 & -1/10 & 3/10 \\ 0 & 0 & 1 & : -5 & 7/2 & -1/2 \end{bmatrix}.$$

- Operating $R_1 \to R_1 + \frac{3}{5}R_3, R_2 \to R_2 - \frac{12}{5}R_3$, to get

$$\begin{bmatrix} 1 & 0 & 0 & : -3 & 5/2 & -1/2 \\ 0 & 1 & 0 & : 12 & -17/2 & 3/2 \\ 0 & 0 & 1 & : -5 & 7/2 & -1/2 \end{bmatrix}.$$

Hence, the inverse of the given matrix is

$$A^{-1} = \begin{bmatrix} -3 & 5/2 & -1/2 \\ 12 & -17/2 & 3/2 \\ -5 & 7/2 & -1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -6 & 5 & -1 \\ 24 & -17 & 3 \\ -10 & 7 & -1 \end{bmatrix}.$$

Iterations method

- Jacobi Iteration method
- Gauss-Seidal iteration method

- Jacobi Iteration Method

Jacobi iteration method example

Ex.1: Solve $28x + 4y - z = 32$, $x + 3y + 10z = 24$, $2x + 17y + 4z = 35$ by using Jacobi iteration method accurate to three decimal places (erorr should be less than 10^{-4})

Soln: Here, the elements in coefficients matrix are diagonally dominant as

$$28x + 4y - z = 32,\ 2x + 17y + 4z = 35,\ x + 3y + 10z = 24,$$

- The system of equation can be written as

$$x = \frac{1}{28}(32 - 4y + z), \tag{7}$$

$$y = \frac{1}{17}(35 - 2x - 4z), \tag{8}$$

$$z = \frac{1}{10}(24 - x - 3y). \tag{9}$$

- Putting $x_0 = y_0 = z_0 = 0$ in (7) - (9), the *first approximation* is

$$x_1 = \frac{1}{28}(32 - 0 + 0) = \frac{32}{28} = 1.14285,$$

$$y_1 = \frac{1}{17}(35 - 0 - 0) = \frac{35}{17} = 2.0588,$$

$$z_1 = \frac{1}{10}(24 - 0 - 0) = \frac{24}{10} = 2.4.$$

Iterations	x	y	z
1	1.14285	2.0588	2.4
2	0.9345	1.3597	1.6681
3	1.0082	1.5564	1.8986
4	0.9883	1.4935	1.8323
5	0.9949	1.5114	1.8531
6	0.9931	1.5058	1.8471
7	0.9937	1.5074	1.8489
8	0.9935	1.5069	1.8489
9	0.9936	1.5070	1.8486
10	0.9936	1.5070	1.8485

- the values of x, y and z in the last two iterations are almost same with very sufficient small error.
- the roots are $x = 0.993$, $y = 1.507$, $z = 1.848$.

- Gauss-Seidal Iteration Method

Example

Ex.1: Solve $10x + y - z = 11.19$, $x + 10y + z = 28.08$, $-x + y + 10z = 35.61$, Gauss-Seidal iteration method correct to two decimal places

Soln: Here, the system of equations can be rearranged as

$$10x + y - z = 11.19,$$

$$x + 10y + z = 28.08,$$

$$-x + y + 10z = 35.61,$$

- the given system of equation can be written as

$$x = \frac{1}{10}(11.19 - y + z), \tag{10}$$

$$y = \frac{1}{10}(28.08 - x - z), \tag{11}$$

$$z = \frac{1}{10}(35.61 + x - y). \tag{12}$$

Example(cont...)

- Taking the initial approximations $y_0 = z_0 = 0$.
- *First Iteration* Substituting $y_0 = z_0 = 0$ in (10), we get

$$x_1 = \frac{1}{10}(11.19 - 0 + 0) = \frac{11.19}{10} = 1.119.$$

Substituting $x = x_1, z = z_0$ in (11), we get

$$y_1 = \frac{1}{10}(28.08 - 1.119 - 0) = \frac{26.9610}{10} = 2.6961.$$

Substituting $x = x_1, y = y_1$ in (12), we get

$$z_1 = \frac{1}{10}(35.61 + 1.119 - 2.6961) = \frac{34.0329}{10} = 3.40329.$$

Examp...

- The other iterations are the following:

Iterations	x	y	z
1	1.119	2.6961	3.40329
2	1.1897	2.3487	3.4451
3	1.2286	2.3406	3.4498
4	1.2299	2.3400	3.4499

.

- The values of x, y and z in the last two iterations are same correct to two decimal places.
- Hence, the roots are $x = 1.22,\ y = 2.34,\ z = 3.45$.

Example

Ex. 2: Solve $-x_1 - x_2 - 2x_3 + 10x_4 = -9$, $10x_1 - 2x_2 - x_3 - x_4 = 3$, $-2x_1 + 10x_2 - x_3 - x_4 = 15$, $-x_1 - x_2 + 10x_3 - 2x_4 = 27$ using Gauss-Seidal iteration method

Soln: Here, the system of equations can be rearranged as

$10x_1 - 2x_2 - x_3 - x_4 = 3$, $-2x_1 + 10x_2 - x_3 - x_4 = 15$,
$-x_1 - x_2 + 10x_3 - 2x_4 = 27$, $-x_1 - x_2 - 2x_3 + 10x_4 = -9$.

- The equation can be written as

$$x_1 = \frac{1}{10}(3 + 2x_2 + x_3 + x_4), \tag{13}$$

$$x_2 = \frac{1}{10}(15 + 2x_1 + x_3 + x_4), \tag{14}$$

$$x_3 = \frac{1}{10}(27 + x_1 + x_2 + 2x_4), \tag{15}$$

$$x_4 = \frac{1}{10}(-9 + x_1 + x_2 + 2x_3), \tag{16}$$

Example

- Taking initial approximations $x_2 = x_3 = x_4 = 0$

$$x_1 = \frac{1}{10}(3 + 0 + 0 + 0) = 0.3,$$

$$x_2 = \frac{1}{10}(15 + 2 \times 0.3 + 0 + 0) = 1.56,$$

$$x_3 = \frac{1}{10}(27 + 0.3 + 1.56 + 0) = 2.886,$$

$$x_4 = \frac{1}{10}(-9 + 0.3 + 1.56 + 2 \times 2.886) = -0.1368,$$

Example

Iterations	x_1	x_2	x_3	x_4
1	0.3	1.56	2.886	-0.1368
2	0.8869	1.9523	2.9565	-0.0247
3	0.9836	1.9899	2.9924	-0.0004
4	0.9968	1.9981	2.9986	-0.0007
5	0.9994	1.9996	2.9997	-0.0001
6	0.9998	1.9999	2.9999	-0.0000
7	0.9999	1.9999	2.9999	0.0000
8	0.9999	1.9999	2.9999	0.0000

  .

- The values of x_1, x_2, x_3 and x_4 in the last two iterations are almost same with very sufficient small error.
- Hence, the roots are $x_1 = 0.9999, x_2 = 1.9999, x_3 = 2.9999, x_4 = 0.0000$.

Eigenvalue and Eigenvector by Power Method

- Let A be a square matrix of size $n \times n$, and let $\mathbf{X}$ be a $n \times 1$ vector. If

$$AX = \lambda X,$$

then λ is called Eigenvalue and X is called Eigenvector of A.

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- The Power Method is an iterative technique used to approximate the eigenvalue of a square matrix A that has the largest absolute value (also known as the dominant Eigenvalue), along with its associated Eigenvector.

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- The Power Method is an iterative technique used to approximate the eigenvalue of a square matrix A that has the largest absolute value (also known as the dominant Eigenvalue), along with its associated Eigenvector.

Definition

*(**Dominant Eigenvalue and Dominant Eigenvector**): Let A be a matrix of size $n \times n$ and $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ be the Eigenvalues of the matrix A satisfying $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$, then λ_1 is called dominant Eigenvalue and the any Eigenvector corresponding to λ_1 is called dominant Eigenvector.*

Largest Eigenvalue and Eigenvector

Ex.1: Find the largest eigenvalue and eigenvector of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

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Soln: Here, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Initial guess of Eigenvector $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- *First Iteration:*

$$AX = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 0.3333 \\ 1 \end{bmatrix} = \lambda^{(1)} X^{(1)}.$$

Largest Eigenvalue and Eigenvector

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Soln: Here, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Initial guess of Eigenvector $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- *First Iteration:*

$$AX = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 0.3333 \\ 1 \end{bmatrix} = \lambda^{(1)} X^{(1)}.$$

- Eigenvalue is $\lambda^{(1)} = 3$, Eigenvector is $X^{(1)} = \begin{bmatrix} 0.3333 \\ 1 \end{bmatrix}$
- *Second Iteration:*

$$AX^{(1)} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.3333 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 4.9999 \end{bmatrix} = 4.9999 \begin{bmatrix} 0.4667 \\ 1 \end{bmatrix} = \lambda^{(2)} X^{(2)}.$$

- Eigenvalue is $\lambda^{(2)} = 4.9999$, Eigenvector is $X^{(2)} = \begin{bmatrix} 0.4667 \\ 1 \end{bmatrix}$

Largest Eigenvalue and Eigenvector

- *Third Iteration:*

$$AX^{(2)} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4667 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4667 \\ 5.4001 \end{bmatrix} = 5.4001 \begin{bmatrix} 0.4568 \\ 1 \end{bmatrix} = \lambda^{(3)} X^{(3)}.$$

Largest Eigenvalue and Eigenvector

- *Third Iteration:*

$$AX^{(2)} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4667 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4667 \\ 5.4001 \end{bmatrix} = 5.4001 \begin{bmatrix} 0.4568 \\ 1 \end{bmatrix} = \lambda^{(3)} X^{(3)}.$$

- Eigenvalue is $\lambda^{(3)} = 5.4001$, Eigenvector is $X^{(3)} = \begin{bmatrix} 0.4568 \\ 1 \end{bmatrix}$

Largest Eigenvalue and Eigenvector

- *Third Iteration:*

$$AX^{(2)} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4667 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4667 \\ 5.4001 \end{bmatrix} = 5.4001 \begin{bmatrix} 0.4568 \\ 1 \end{bmatrix} = \lambda^{(3)} X^{(3)}.$$

- Eigenvalue is $\lambda^{(3)} = 5.4001$, Eigenvector is $X^{(3)} = \begin{bmatrix} 0.4568 \\ 1 \end{bmatrix}$
- *Fourth Iteration:*

Largest Eigenvalue and Eigenvector

- *Third Iteration:*

$$AX^{(2)} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4667 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4667 \\ 5.4001 \end{bmatrix} = 5.4001 \begin{bmatrix} 0.4568 \\ 1 \end{bmatrix} = \lambda^{(3)} X^{(3)}.$$

- Eigenvalue is $\lambda^{(3)} = 5.4001$, Eigenvector is $X^{(3)} = \begin{bmatrix} 0.4568 \\ 1 \end{bmatrix}$
- *Fourth Iteration:*

$$AX^{(3)} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4568 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4568 \\ 5.3704 \end{bmatrix} = 5.3704 \begin{bmatrix} 0.4575 \\ 1 \end{bmatrix} = \lambda^{(4)} X^{(4)}.$$

- Eigenvalue is $\lambda^{(4)} = 5.3704$, and Eigenvector is $X^{(4)} = \begin{bmatrix} 0.4575 \\ 1 \end{bmatrix}$.

Largest Eigenvalue and Eigenvector

- *Fifth Iteration:*

Largest Eigenvalue and Eigenvector

- *Fifth Iteration:*

$$AX^{(4)} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4575 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4575 \\ 5.3725 \end{bmatrix} = 5.3725 \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix} = \lambda^{(5)} X^{(5)}.$$

- Eigenvalue is $\lambda^{(5)} = 5.3725$, and Eigenvector is $X^{(5)} = \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix}$
- *Sixth Iteration:*

Largest Eigenvalue and Eigenvector

- *Fifth Iteration:*

$$AX^{(4)} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4575 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4575 \\ 5.3725 \end{bmatrix} = 5.3725 \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix} = \lambda^{(5)} X^{(5)}.$$

- Eigenvalue is $\lambda^{(5)} = 5.3725$, and Eigenvector is $X^{(5)} = \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix}$
- *Sixth Iteration:*

$$AX^{(5)} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4574 \\ 5.3722 \end{bmatrix} = 5.3722 \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix} = \lambda^{(6)} X^{(6)}.$$

Largest Eigenvalue and Eigenvector

- *Fifth Iteration:*

$$AX^{(4)} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4575 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4575 \\ 5.3725 \end{bmatrix} = 5.3725 \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix} = \lambda^{(5)} X^{(5)}.$$

- Eigenvalue is $\lambda^{(5)} = 5.3725$, and Eigenvector is $X^{(5)} = \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix}$
- *Sixth Iteration:*

$$AX^{(5)} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4574 \\ 5.3722 \end{bmatrix} = 5.3722 \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix} = \lambda^{(6)} X^{(6)}.$$

- Eigenvalue is $\lambda^{(6)} = 5.3725$, Eigenvector is $X^{(6)} = \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix}$.
- Largest Eigenvalue, and Eigenvector in fifth and sixth iterations are almost same. Hence, $\lambda = 5.372$, and $X = \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix}$.

Largest Eigenvalue and Eigenvector

Ex.1 **Find the largest Eigenvalue and Eigenvector of** $\begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$ **using Power method**

Largest Eigenvalue and Eigenvector

Ex.1 **Find the largest Eigenvalue and Eigenvector of** $\begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$ **using Power method**

Soln: Here, $A = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$. Initial guess, $X = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}'$.

- *First Iteration:*

$$AX = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ -20 \end{bmatrix} = 20 \begin{bmatrix} 0.75 \\ -0.5 \\ -1 \end{bmatrix} = \lambda^{(1)} X^{(1)}.$$

Largest Eigenvalue and Eigenvector

Ex.1 **Find the largest Eigenvalue and Eigenvector of** $\begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$ **using Power method**

Soln: Here, $A = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$. Initial guess, $X = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}'$.

- *First Iteration:*

$$AX = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ -20 \end{bmatrix} = 20 \begin{bmatrix} 0.75 \\ -0.5 \\ -1 \end{bmatrix} = \lambda^{(1)} X^{(1)}.$$

Second Iteration:

$$AX^{(1)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.75 \\ -0.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 16.25 \\ -7.5 \\ -15 \end{bmatrix} = 16.25 \begin{bmatrix} 1 \\ -0.4615 \\ -0.9230 \end{bmatrix} = \lambda^{(2)} X^{(2)}.$$

Largest Eigenvalue and Eigenvector

- *Third Iteration:*

$$AX^{(2)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.4615 \\ -0.9231 \end{bmatrix} = \begin{bmatrix} 19.5541 \\ -9.8158 \\ -19.9386 \end{bmatrix} = 19.9386 \begin{bmatrix} 0.9807 \\ -0.4923 \\ -1 \end{bmatrix} = \lambda^{(3)} X^{(3}$$

Largest Eigenvalue and Eigenvector

- *Third Iteration:*

$$AX^{(2)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.4615 \\ -0.9231 \end{bmatrix} = \begin{bmatrix} 19.5541 \\ -9.8158 \\ -19.9386 \end{bmatrix} = 19.9386 \begin{bmatrix} 0.9807 \\ -0.4923 \\ -1 \end{bmatrix} = \lambda^{(3)} X^{(3}$$

- *Fourth Iteration:*

$$AX^{(3)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.9807 \\ -0.4923 \\ -1 \end{bmatrix} = \begin{bmatrix} 19.6797 \\ -9.7146 \\ -19.5832 \end{bmatrix} = 19.6797 \begin{bmatrix} 1 \\ -0.4936 \\ -0.9951 \end{bmatrix} = \lambda^{4)} X^{(4)}$$

Largest Eigenvalue and Eigenvector

- *Third Iteration:*

$$AX^{(2)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.4615 \\ -0.9231 \end{bmatrix} = \begin{bmatrix} 19.5541 \\ -9.8158 \\ -19.9386 \end{bmatrix} = 19.9386 \begin{bmatrix} 0.9807 \\ -0.4923 \\ -1 \end{bmatrix} = \lambda^{(3)} X^{(3}$$

- *Fourth Iteration:*

$$AX^{(3)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.9807 \\ -0.4923 \\ -1 \end{bmatrix} = \begin{bmatrix} 19.6797 \\ -9.7146 \\ -19.5832 \end{bmatrix} = 19.6797 \begin{bmatrix} 1 \\ -0.4936 \\ -0.9951 \end{bmatrix} = \lambda^{4)} X^{(4)}$$

- *Fifth Iteration:*

$$AX^{(4)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.4936 \\ -0.9951 \end{bmatrix} = \begin{bmatrix} 19.9597 \\ -9.9526 \\ -19.9842 \end{bmatrix} = 19.9842 \begin{bmatrix} 0.9988 \\ -0.4980 \\ -1 \end{bmatrix} = \lambda^{(5)} X$$

Largest Eigenvalue and Eigenvector

- *Sixth Iteration:*

$$AX^{(5)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.9988 \\ -0.4980 \\ -1 \end{bmatrix} = \begin{bmatrix} 19.9740 \\ -9.9640 \\ -19.968 \end{bmatrix} = 19.974 \begin{bmatrix} 1 \\ -0.4988 \\ -0.9997 \end{bmatrix} = \lambda^{(6)} X^{(6)}.$$

Largest Eigenvalue and Eigenvector

- *Sixth Iteration:*

$$AX^{(5)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.9988 \\ -0.4980 \\ -1 \end{bmatrix} = \begin{bmatrix} 19.9740 \\ -9.9640 \\ -19.968 \end{bmatrix} = 19.974 \begin{bmatrix} 1 \\ -0.4988 \\ -0.9997 \end{bmatrix} = \lambda^{(6)} X^{(6)}.$$

- *Seventh Iteration:*

$$AX^{(6)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.4988 \\ -0.9997 \end{bmatrix} = \begin{bmatrix} 19.9943 \\ -9.9874 \\ -19.9958 \end{bmatrix} = 19.9958 \begin{bmatrix} 0.9999 \\ -0.4995 \\ -1 \end{bmatrix} = \lambda^{(7)} X^{(7$$

Largest Eigenvalue and Eigenvector

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- *Eighth Iteration:*

$$AX^{(7)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.9999 \\ -0.4995 \\ -1 \end{bmatrix} = \begin{bmatrix} 19.9965 \\ -9.9930 \\ -19.9960 \end{bmatrix} = 19.9965 \begin{bmatrix} 1 \\ -0.4997 \\ -1 \end{bmatrix} = \lambda^{(8)} X^{(8)}$$

Largest Eigenvalue and Eigenvector

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Largest Eigenvalue and Eigenvector

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Largest Eigenvalue and Eigenvector

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- The largest eigenvalue corresponding to the vectors of seventh and eighth iterations are almost same.

Hence, Eigenvalue, $\lambda = 19.99$, Eigenvector, $X = \begin{bmatrix} 1.000 \\ -0.499 \\ -1 \end{bmatrix}$