Chapter

4

SOLUTION OF LINEAR EQUATIONS

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4.1 MATRICES AND THEIR PROPERTIES

In mathematics, a matrix is a rectangular array or table of numbers, symbols or expression arranged in rows and columns. For example, the dimension of the matrix below is 2 × 3 (read "two by three") because there are two rows and three columns.

$$\begin{bmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{bmatrix}$$

Provided that they have the same dimensions (each matrix has the same number of rows and the same number of columns as the other), two matrices can be added or subtracted element by element. The rule for matrix multiplication, however, is that two matrices can be multiplied only when the number of columns in the first equals the number of rows in the second (*i.e.*, the inner dimensions are the same, n for $(m \times n)$ – matrix times an $(n \times p)$ – matrix resulting in an $(m \times p)$ – matrix).

Definition

A system of mn numbers arranged in a rectangular array of m rows and n columns is called an $m \times n$ matrix. Such a matrix is denoted by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

Special Matrices

a) **Row and column matrices**

A matrix having a single row is called a row matrix while a matrix having a single column is called a column matrix.

b) **Square matrix**

A matrix having n rows and n columns is called a square.

c) **Non-singular matrix**

A square matrix is said to be singular if its determinant is zero otherwise it is called non-singular matrix. The elements a_{ii} in a square matrix from the leading diagonal and their sum Σa_{ii} is called the trace of the matrix.

d) **Unit matrix**

A diagonal matrix of order n which has unity for all its diagonal elements is called a unit matrix of order n and is denoted by I_m.

e) **Null matrix or zero matrix**

If all the elements of a matrix are zero, it is called a null matrix.

f) **Triangular matrix**

A square matrix all of whose elements below the leading diagonal are zero is called an upper triangular matrix. A square matrix all of whose elements above the leading diagonal are zero is called a lower triangular matrix.

g) **Symmetric and skew-symmetric matrices**

A square matrix $[a_{ij}]$ is said to be symmetric when $a_{ij} = a_{ji}$ for all i and j. If $a_{ij} = -a_{ij}$ for all i and j so that all the leading diagonal elements are zero, then the matrix is called skew-symmetric.

Examples of symmetric and skew-symmetric matrices are respectively,

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{and} \begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$$

h) **Horizontal matrix**

A matrix of order m × n is a horizontal matrix if n > m. Example,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$$

i) **Vertical matrix**

A matrix of order m × n is a vertical matrix if m > n. Example,

$$\begin{bmatrix} 2 & 5 \\ 1 & 7 \\ 4 & 6 \\ 6 & 4 \end{bmatrix}$$

j) **Diagonal matrix**

If all the elements except the principal diagonal, in a square matrix are zero, it is called a diagonal matrix. Thus a square $A = [a_{ij}]$ is a diagonal matrix, if $a_{ij} = 0$ when $i \neq j$. Example,

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

is a diagonal matrix of order 3 × 3 which can also be denoted by diagonal [2 3 4].

k) **Scalar matrix**

If all the elements in the diagonal of a diagonal matrix are equal, it is called a scalar matrix.

Thus, a square matrix $A = [a_{ij}]_{m \times n}$ is a scalar matrix if

$$a_{ij} = \begin{cases} 0 \; ; & i \neq j \\ k \; ; & i = j \end{cases}$$

where, k is a constant.

Example,

$$\begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

is a scalar matrix.

l) **Idempotent matrix**

A square matrix is idempotent, provided $A^2 = A$. For an idempotent matrix A, $A^n = A \forall n > 2, n \in N \Rightarrow A^n = A, n \geq 2$.

m) **Nilpotent matrix**

A nilpotent matrix is said to be nilpotent of index p, $(p \in N)$, if $A^p = 0$, $A^{p-1} \neq 0$, *i.e.*, p is the least positive integer for which $A^p = 0$, then A is said to be nilpotent of index p.

n) **Periodic matrix**

A square matrix which satisfies the relation $A^{k+1} = A$ for some positive integer k, then A is periodic with period k *i.e.*, if k is the least positive integer for which $A^{k+1} = A$ and A is said to be periodic with period k. If k = 1, then A is called idempotent. Example,

$$\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$$

has the period 1.

o) **Involuntary matrix**

If $A^2 = I$, the matrix is said to be an involuntary matrix. Example,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4.1.1 Determinants

The expression $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a determinant of the second order and stands for '$a_1b_2 - a_2b_1$'. It contains four numbers a_1, b_1, a_2, b_2 (called elements) which are arranged along two horizontal lines (called rows) and two vertical lines (called columns).

Similarly,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{..... (1)}$$

is called a determinant of the third order. It consists of nine elements which are arranged in three rows and three columns.

In general, a determinant of the n^{th} order is of the form,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

which is a block of n^2 elements in the form of a square along n rows and n columns. The diagonal through the left-hand top corner which contains the elements $a_{11}, a_{22}, a_{33}, \ldots\ldots, a_{nn}$ is called the leading diagonal.

Expansion of a Determinant

The cofactor of an element in a determinant is the determinant obtained by deleting the row and column which intersect at that element, with the proper sign. The sign of an element in the i^{th} row and j^{th} column is $(-1)^{i+j}$. The cofactor of an element is usually denoted by the corresponding capital letter.

For example, the cofactor of b_3 in (1) is

$$B_3 = (-1)^{3+2} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

A determinant can be expanded in terms of any row for column as follows: Multiply each element of the row (or column) in terms of which we intend expanding the determinant, by its cofactor and then add up all these products.

$\therefore$ Expanding (1) by R_1 (*i.e.*, 1^{st} row)

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1$$

$$= a_1 \begin{vmatrix} b_2 & c_3 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Similarly, expanding by C_2(*i.e.*, 2^{nd} column),

$$\Delta = b_1B_1 + b_2B_2 + b_3B_3$$

$$= -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= b_1(a_2c_3 - a_3c_2) - b_2(a_1c_3 - a_3c_1) + b_3(a_1c_2 - a_2c_1)$$

Basic Properties

i) A determinant remains unaltered by changing its rows into columns and columns into rows.

ii) If two parallel lines of a determinant are interchanged, the determinant retains its numerical value but changes in sign.

iii) A determinant vanishes if two of its parallel lines are identical.

iv) If each element of a line is multiplied by the same factor, the whole determinant is multiplied by that factor.

v) If each element of a line consists of m terms, the determinant can be expressed as the sum of m determinants.

vi) If to each element of a line, there can be added equi-multiplies of the corresponding elements of one or more parallel lines, the determinant remains unaltered.

For instance,

$$\begin{vmatrix} a_1 + pb_1 - qc_1 & b_1 & c_1 \\ a_2 + pb_2 - qc_2 & b_2 & c_2 \\ a_3 + pb_3 - qc_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + p \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} - q \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}$$

[$\because$ From (iii) property]

$$= \Delta + 0 + 0$$

$$= \Delta$$

Example 4.1

Solve the equation:

$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$$

Solution:

Operating $R_3 - (R_1 + R_2)$, we get,

$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

Operate $R_2 - R_1$ and $(R_1 + R_3)$,

$$\begin{vmatrix} x+2 & 2x+4 & 6x+12 \\ x+1 & x+1 & x+1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0.$$

or, $(x+1)(x+2)\begin{vmatrix} 0 & 2 & 6 \\ 1 & 1 & 1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$

Operate $R_1 - R_2$,

$\therefore \quad (x+1)(x+2)\begin{vmatrix} 0 & 1 & 5 \\ 1 & 1 & 1 \\ 1 & 1 & 3x+8 \end{vmatrix} = 0$

Expanding by C_1,

$\therefore \quad -(x+1)(x+2)(3x+8-5) = 0$

or, $\quad -3(x+1)(x+2)(x+1) = 0$

Hence, $x = -1, -1, -2$.

NOTE:

1. In general, $AB \neq BA$ even if both exist.
2. If A be a square matrix, then the product AA is defined as A^2. Similarly, $A \cdot A^2 = A^3$ etc.

Related Matrices

A. Transpose of a matrix

The matrix obtained from a given matrix A, by interchanging rows and columns is called the transpose of A and is denoted by A¢.

NOTE:

i) For a symmetric matrix, A¢ = A and for skew-symmetric matrix, A¢ = –A.

ii) The transpose of the product of two matrices is the product of their transposes taken in the reverse order,

i.e., $(AB)' = B'A'$

iii) Any square matrix A can be written as,

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = B + C \text{ (say)}$$

Such that;

$$B' = \frac{1}{2}(A + A')' = \frac{1}{2}(A' + A) = B$$

i.e., B is a symmetric matrix.

and, $$C' = \frac{1}{2}(A - A')' = \frac{1}{2}(A' + A) = -C$$

i.e., C is a skew-symmetric matrix.

Thus, every square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix

B. Adjoint of a square matrix A

Adjoint of a square matrix A is the transposed matrix of cofactors of A and is written as adj A. Thus the adjoint of the matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{is} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

C. Inverse of a matrix

If A is a non-singular matrix of order n, then a square matrix B of the same order such that AB = BA = I, is then called the inverse of A, I being the unit matrix.

The inverse of A is written as A^{-1} so that $AA^{-1} = A^{-1}A = I$

Also,

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

NOTE:

i) Inverse of a matrix, when it exists is unique.

ii) $(A^{-1})^{-1} = A$

iii) $(AB)^{-1} = B^{-1}A^{-1}$

Rank of a Matrix

If we select any r rows and r columns from any matrix A, deleting all other rows and columns, then the determinant formed by these $r \times r$ elements is called the minor of A of order r. Clearly, there will be a number of different minors of the same order, got by deleting different rows and columns from the same matrix.

A matrix is said to be of rank r when,

i) it has at least one non-zero minor of order r, and,

ii) every minor of order higher than r vanishes.

4.2 DIRECT METHODS OF SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS

A. Gauss Elimination Method

In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution. The method is quite general and is well adapted for computer operations.

Consider the equations,

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad \text{..... (1)}$$

Step I: To eliminate x from the second and third equations

Assuming $a_1 \neq 0$, we eliminate x from the second equation by subtracting $\left(\frac{a_2}{a_1}\right)$ times the first equation from the second equation.

Similarly, we eliminate x from the third equation by eliminating $\left(\frac{a_3}{a_1}\right)$ times the first equation from the third equation. We thus get new system.

Assuming $a_1 \neq 0$, we eliminate x from the second equation by subtracting $\left(\frac{a_2}{a_1}\right)$ times the first equation from the second equation. Similarly, we eliminmate x from the third equation by eliminating $\left(\frac{a_3}{a_1}\right)$ times the first equation from the third equation. Thus,

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ b_3'y + c_3'z &= d_3' \end{aligned}\right\} \quad \text{..... (2)}$$

Here, the first equation is called the pivotal equation and a_1 is called the first pivot.

Step II: To eliminate y from third equation in (2)

Assuming $b_2' \neq 0$, we eliminate y from the third equation of (2) by subtracting $\left(\frac{b_3'}{b_2'}\right)$ multiplied by times the second equation from the third equation. We thus, get the new system,

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ c_3''z &= d_3'' \end{aligned}\right\} \quad \text{..... (3)}$$

Here, the second equation is the pivotal equation and b_2' is the new pivot.

Step III: To evaluate the unknowns

The values of x, y, z are found from the reduced system (3) by back substitution.

NOTE:

1. On writing the given equation as,
$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$
i.e., $AX = D$
This method consists in transforming the coefficient matrix A to the upper triangular matrix by elementary row transformations only.
2. Clearly, this method will fail if any one of the pivots a_1, b_2' or c_3'' becomes zero. In such cases, we rewrite the equations in a different order so that the pivots are non-zero.
3. Partial and Complete Pivoting
In the first step, the numerically largest coefficient of x is chosen from all the equations and brought as the first pivot by interchanging the first equation with the equation having the largest coefficient of x. In the second step, the numerically largest coefficient of y is chosen from the remaining equations (leaving the first equation) and brought as the second pivot by interchanging the second equation with the equation having the largest coefficient of y. This process is continued until we arrive at the equation with the single variable. This modified procedure is called partial pivoting.

Example 4.2

Apply Gauss elimination method to solve the equations:
$x + 4y - z = -5$; $x + y - 6z = -12$; $3x - y - z = 4$

Solution:

We have,

$$x + 4y - z = -5 \quad \text{..... (1)}$$
$$x + y - 6z = -12 \quad \text{..... (2)}$$
$$3x - y - z = 4 \quad \text{..... (3)}$$

Operate (2) – (1) and (3) – 3(1) to eliminate x,

$$-3y - 5z = -7 \quad \text{..... (4)}$$
$$-13y + 2z = 19 \quad \text{..... (5)}$$

Operating (5) $-\frac{13}{3}$ (4) to eliminate y,

$$\frac{71}{3}z = \frac{148}{3} \quad \text{..... (6)}$$

By backward substitution, we get,

$$z = \frac{148}{71} = 2.0845$$

From (4),

$$y = \frac{7}{3} - \frac{5}{3}\left(\frac{148}{71}\right) = \frac{-81}{71} = -1.1408$$

From (1),

$$x = -5 - 4\left(\frac{-81}{71}\right) + \left(\frac{148}{71}\right) = \frac{117}{71} = 1.6479$$

Hence, $x = 1.6479$, $y = -1.1408$ and $z = 2.0845$

Otherwise:

We have,

$$\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$$

Operating $R_2 - R_1$ and $R_3 - 3R_1$,

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & 5 \\ 0 & -13 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 19 \end{bmatrix}$$

Operating $R_3 - \frac{13}{3}R_2$,

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71/3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148/3 \end{bmatrix}$$

Thus, we have,

$$z = \frac{148}{71} = 2.0845$$

or, $3y = 7 - 5z = 7 - 10.4225 = -3.4225$

$\therefore$ $y = -1.1408$

and, $x = -5 - 4y + z = -5 + 4(1.1408) + 2.0845 = 1.6479$

$\therefore$ $x = 1.6479, y = -1.1408$ and $z = 2.0845$

Example 4.3

Using the Gauss elimination method, solve the equations:

$$x + 2y + 3z - u = 10,$$
$$2x + 3y - 3z - u = 1,$$
$$2x - y + 2z + 3u = 7,$$
$$3x + 2y - 4z + 3u = 2.$$

Solution:

We have,

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & -3 & -1 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 7 \\ 2 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 2R_1, R_4 - 3R_1$,

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & -5 & -4 & 5 \\ 0 & -4 & -13 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} -10 \\ -19 \\ -13 \\ -28 \end{bmatrix}$$

Operate $R_3 - 5R_2, R_4 - 4R_2$,

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & 0 & 41 & 0 \\ 0 & 0 & 23 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ -19 \\ 82 \\ 48 \end{bmatrix}$$

Thus, we have,

$41z = 82$

$\therefore$ $z = 2$

or, $23z + 2u = 48$ i.e., $46 + 2u = 48$ $\therefore u = 1$

or, $-y - 9z + u = -19$ i.e., $-y - 18 + 1 = -19$ $\therefore y = 2$

or, $x + 2y + 3z - u = 10$ i.e., $x + 4 + 6 - 1 = 10$ $\therefore x = 1$

Hence, $x = 1, y = 2, z = 2$ and $u = 1$.

B. Gauss-Jordan Method

This is the modification of the Gauss elimination method. In this method, elimination of unknowns is performed not in the equation below but in the equations above also, ultimately reducing the system to a diagonal matrix form i.e., each equation involving only one unknown. From these equations, the unknowns x, y, z can be obtained readily. Thus, in this method, the labor of back-substitution for finding the unknowns is saved at the cost of additional calculations.

Example 4.4

Apply the Gauss-Jordan method to solve the equations:

$$x + y + z = 9;$$
$$2x - 3y + 4z = 13;$$
$$3x + 4y + 5z = 40$$

Solution:

Writing the equations as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 3R_1$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 13 \end{bmatrix}$$

Operate $R_3 + \frac{1}{5}R_2$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 12/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 12 \end{bmatrix}$$

Operate $-R_2 + 5R_3$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 60 \end{bmatrix}$$

Operate $R_3 + \frac{1}{6}R_3, \frac{1}{12}R_3$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \\ 5 \end{bmatrix}$$

Operate $\frac{1}{5}R_2$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 5 \end{bmatrix}$$

Operate $R_1 - R_2 - R_3$,

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

Hence, $x = 1, y = 3$ and $z = 5$.

NOTE:
Here the process of elimination of variables amounts to reducing the given coefficient matrix to a diagonal matrix by elementary row transformation only.

4.3 METHOD OF FACTORIZATION

I. Triangular Factorization Method or Dolittle Method

The coefficient matrix A of a system of linear equations can be factorized (or decomposed) into two triangular matrices L and U such that,

$$A = LU$$

where, $\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$ ..... (1)

and, $U = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & U_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & U_{nn} \end{bmatrix}$

L is known as lower triangular matrix and U is known as upper triangular matrix.

Once A is factorized into L and U, the system of equation

$$Ax = b$$

can be expressed as follows,

$$(LU)x = b$$

or, $L(Ux) = b$

Let us assume that,

$$Ux = z \quad \text{..... (2)}$$

where, z is an unknown vector-replacing equation (2) in equation (1), we get,

$$Lz = b \quad \text{..... (3)}$$

Now, we can solve the system,

$$Ax = b \quad \text{..... (4)}$$

in two stages:

1. Solve the equation $Lz = b$
   For z by forward substitution.
2. Solve the equation $Ux = z$
   For x using z (found in stage 1) by back substitution.

The elements of L and U can be determined by comparing the elements of the product of L and U with those of A. The process produces a system of n^2 equations with $n^2 + n$ unknowns (l_{ij} and m_{ij}) and, therefore, L and U are not unique. In order to produce unique factors, we should reduce the number of unknowns by n.

This is done by assuming the diagonal elements of L or U to be unity. The decomposition with L having unit diagonal values is called the Dolittle LU decomposition while the other one with U having unit diagonal elements is called the Crout LU decomposition.

Dolittle Algorithm

We can solve for the components of L and U, given A as follows:

$$A = LU$$

Implies that,

$$a_{ij} = l_{i1}u_{1j} + l_{i2}u_{2j} + \dots + l_{ii}u_{ij} \quad \text{for } i < j \quad \text{..... (5)}$$

$$a_{ij} = l_{i1}u_{1j} + l_{i2}u_{2j} + \dots + l_{ii}u_{jj} \quad \text{for } i = j \quad \text{..... (6)}$$

$$a_{ij} = l_{i1}u_{1j} + l_{i2}u_{2j} + \dots + l_{ij}u_{jj} \quad \text{for } i > j \quad \text{..... (7)}$$

where, $u_{ij} = 0$ for $i > j$ and $l_{ij} = 0$ for $i < j$

The Dolittle algorithm assumes that all the diagonal elements of L are unity.

That is,

$$l_{ij} = 1, \qquad i = 1, 2, 3, \dots, n$$

Using equations (5), (6) and (7), we can successively determine the elements of U and L as follows:

If $i \le j$,

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} U_{kj} \qquad j = 1, 2, 3, \dots, n$$

where, $u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$

Similarly,

If $i > j$,

$$l_{ij} = \frac{1}{u_{ij}} \times \left[a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \right] \qquad j = 1, 2, 3, \dots, i-1$$

where, $l_{11} = l_{22} = l_{33} = 1$

and $l_{i1} = \dfrac{a_{i1}}{u_{11}}$ for i = 2 to n

Note that, for computing any element, we need the values of elements in the previous columns as well as the values of elements in the column above that element. This suggest that we should compute the elements, column by column from left to right within each column from top to bottom.

Example 4.5

Solve the system,

$$3x_1 + 2x_2 + x_3 = 10$$
$$2x_1 + 3x_2 + 2x_3 = 14$$
$$x_1 + 2x_2 + 3x_3 = 14$$

by using Dolittle LU decomposition method.

Solution:

Factorization:

For $i = 1$, $l_{11} = 1$ and

$$u_{11} = a_{11} = 3$$
$$u_{12} = a_{12} = 2$$
$$u_{13} = a_{13} = 1$$

For i = 2,

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{2}{3} \quad \text{and} \quad l_{22} = 1$$

$$u_{22} = a_{22} - l_{21}u_{12} = 3 - \frac{2}{3} \times 2 = \frac{5}{3}$$

$$u_{23} = a_{23} - l_{21}u_{13} = 2 - \frac{2}{3} \times 1 = \frac{4}{3}$$

For i = 3,

$$l_{31} = \frac{a_{31}}{u_{11}} = \frac{1}{3}$$

$$l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}} = \frac{2 - \left(\frac{1}{3}\right) \times 2}{\left(\frac{5}{3}\right)} = \frac{4}{5}$$

$$l_{33} = 1$$

$$u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

$$= 3 \times \frac{1}{3} \times 1 - \frac{4}{5} \times \frac{4}{3} = \frac{24}{15}$$

Thus, we have,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 24/15 \end{bmatrix}$$

Forward substitution:

Solving $lz = b$ by forward substitution, we get,

$$z_1 = b_1 = 10$$

$$z_2 = b_2 - l_{21}z_1 = 14 - \frac{2}{3} \times 10 = \frac{22}{3}$$

$$z_3 = b_3 - l_{31}z_1 - l_{32}z_2 = 14 - \frac{1}{3} \times 10 - \frac{4}{5} \times \frac{22}{3} = \frac{72}{15}$$

Back substitution:

Solving $Ux = z$ by back substitution, we get,

$$x_3 = \frac{\left(\frac{27}{15}\right)}{\left(\frac{24}{15}\right)} = 3$$

$$x_2 = \frac{z_2 - u_{23}x_3}{u_{22}} = \frac{\left(\frac{22}{3}\right) - \left(\frac{4}{3}\right) \times 3}{\left(\frac{5}{3}\right)} = 2$$

$$\therefore \quad x_1 = \frac{z_1 - u_{12}x_2 - u_{13}x_3}{u_{11}} = \frac{10 - (2 \times 2) - 1 \times 3}{3} = 1$$

II. Crout Algorithm

Another approach to LU decomposition is Crout algorithm. Crout algorithm assumes unit diagonal values for U matrix and the diagonal elements of L matrix may assume any values as shown below.

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & \cdots & u_{1n} \\ 0 & 1 & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

We can use an approach that is similar to the one used in Dolittle decomopsition to evaluate the elements of L and U.

III. Cholesky Method

In case A is symmetric, the LU decomposition can be modified so that the upper factor is the transpose of the lower one or vice-versa. That is, we can factorize as,

$$A = LL^T \qquad \text{..... (1)}$$

or, $A = U^TU$

Just as for Dolittle decomposition, by multiplying the terms of equation (1) and setting them equal to each other, the following recurrence relations can be obtained.

$$\left.\begin{aligned} u_{ii} &= \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2} && (i = 1 \text{ to } n) \\ u_{ij} &= \frac{1}{u_{ii}}\left[a_{ij} - \sum_{k=1}^{i-1} u_{ki}\,u_{kj}\right] && (j > 1) \end{aligned}\right\} \qquad \text{..... (2)}$$

This decomposite is called the Cholesky's factorization or the method of square roots.

Algorithm for Cholesky's factorization

1. Given n, A
2. Set $u_{11} = \sqrt{a_{11}}$
3. Set $u_{1j} = \frac{a_{1j}}{u_{11}}$ for i = 2 to n
4. For j = 2 to n,
   - For i = 2 to j
     - Sum = aij
     - For k = 1 to i − 1
       - Sum = sum − $u_{ki}\,u_{kj}$

Repeat k

Set $u_{ij} = \frac{sum}{u_k}$ if $i < j$

Set $u_{ij} = \sqrt{sum}$ if $i = j$

Repeat i

Repeat j

5. End of factorization.

Example 4.6

Factorize the matrix using Cholesky's method

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix}$$

Solution:

We have,

$$u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2} \qquad (i = 1 \text{ to } n)$$

$$u_{ij} = \frac{1}{u_{ii}}\left[a_{ij} - \sum_{k=1}^{i-1} u_{ki}\, u_{kj}\right] \qquad (j > 1)$$

For i = 1,

$$u_{11} = \sqrt{1} = 1$$

$$u_{12} = \frac{a_{12}}{u_{11}} = \frac{2}{1} = 2$$

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{3}{1} = 3$$

For i = 2,

$$u_{22} = \sqrt{a_{22} - u_{12}^2} = \sqrt{8-4} = 2$$

$$u_{23} = \frac{a_{23} - u_{12}u_{13}}{u_{22}} = \frac{22 - 2\times 3}{2} = \frac{16}{2} = 8$$

For i = 3,

$$u_{33} = \sqrt{a_{33} - u_{13}^2 - u_{23}^2} = \sqrt{82 - 9 - 64} = 3$$

$$\text{Hence, } U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

4.4 THE INVERSE OF A MATRIX

The inverse of a matrix A is written as A^{-1} so that $AA^{-1} = A^{-1}A = I$. Thus the inverse of a matrix exists if and only if it is a non-singular square matrix. Also inverse of a matrix, when it exists is unique.

A. Gauss Elimination method

In this method, we take a unit matrix of the same order as the given matrix A and write it as AI. Now making the simultaneous row operations on AI, we try to convert A into an upper triangular matrix and then to a unit matrix. Ultimately, when A is transformed into a unit matrix, the adjacent matrix (emerged out from the transformation of I) gives the inverse of A. To increase the accuracy, the largest element in A is taken as the pivot element for performing the row operations.

B. Gauss-Jordan Method

This is similar to the guess elimination method except that instead of first converting A into upper triangular form, it is directly converted into the unit matrix.

In practice, the two matrices A and I are written side by side and the same row transformations are performed on both. As soon as A is reduced to I, the other matrix represents A^{-1}.

Example 4.7

Find the inverse of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Solution:

Here;

$$|A| = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\text{and,} \quad \text{adj } A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

Example 4.8

Using Gauss-Jordan method, find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Solution:

Writing the given matrix side by side with the unit matrix of order 3.

We have,

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

(Operate $R_2 - R_1$ and $R_3 + 2R_1$)

$$\sim\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

$\left(\text{Operate } \frac{1}{2}R_2 \text{ and } \frac{1}{2}R_3\right)$

$$\sim\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1/2 \end{array}\right]$$

(Operate $R_1 - R_2$ and $R_3 + R_2$)

$$\sim\left[\begin{array}{ccc:ccc} 1 & 0 & 6 & 3/2 & 1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & -2 & 1/2 & 1/2 & 1/2 \end{array}\right]$$

$\left(\text{Operate } R_1 + 3R_3,\ R_2 - \frac{3}{2}R_3 \text{ and } -\frac{1}{2}R_2\right)$

$$\sim\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right].$$

Hence the inverse of the given matrix is

$$\begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

Example 4.9

Using Gauss-Jordan method, find the inverse of the matrix

$$\begin{bmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$$

Solution:

Writing the given matrix side by side with the unit matrix of order 3. We have,

$$\left[\begin{array}{ccc:ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

$\left(\text{Operate } \frac{1}{2}R_1\right)$

$$\sim\left[\begin{array}{ccc:ccc} 1 & 1 & 3/2 & 1/2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

(Operate $R_2 - 2R_1$, $R_3 - R_1$)

$$\sim\left[\begin{array}{ccc:ccc} 1 & 1 & 3/2 & 1/2 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 2 & 7/2 & -1/2 & 0 & 1 \end{array}\right]$$

(Operate $R_1 + R_2$, $R_3 + 2R_2$)

$$\sim\left[\begin{array}{ccc:ccc} 1 & 0 & -1/2 & -1/2 & 1 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 0 & -1/2 & -5/2 & 2 & 1 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc:ccc} 1 & 0 & -1/2 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & -1/2 & -5/2 & 2 & 1 \end{array}\right]$$

(Operate $(-2)R_3$)

$$\sim\left[\begin{array}{ccc:ccc} 1 & 0 & -1/2 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 5 & -4 & -2 \end{array}\right]$$

$\left(\text{Operate } R_1 + \frac{1}{2}R_3,\ R_2 - 2R_3\right)$

$$\sim\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -9 & 7 & 4 \\ 0 & 0 & 1 & 5 & -4 & -2 \end{array}\right]$$

Hence the inverse of the given matrix is,

$$\begin{bmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{bmatrix}$$

C. Factorization Method

In this method, we factorize the given matrix as $A = LU$ ..... (1)

where, L is a lower triangular matrix with unit diagonal elements and U is an upper triangular matrix.

i.e., $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Now, (1) gives, ... (2)

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$$

To find L^{-1}, let $L^{-1} = X$, where, X is a lower triangular matrix.

Then, $LX = I$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying the matrices on the LHS and equating the corresponding elements, we have, ... (3)

$$x_{11} = 1,\ x_{22} = 1,\ x_{33} = 1$$

$$l_{21}x_{11} + x_{21} = 0,\ l_{31}x_{11} + l_{32}x_{21} + x_{31} = 0 \quad \text{..... (4)}$$

and, $l_{32}x_{22} + x_{32} = 0$

Equation (3) gives,

$$x_{11} = x_{22} = x_{33} = 1$$

Equation (4) gives,

$$x_{21} = -l_{21}x_{11} + x_{21},\ x_{31} = -(l_{31} + l_{32}x_{21}) \text{ and } x_{31} = -l_{32}$$

Thus, $L^{-1} = X$ is completely determined.

To find U^{-1}, let $(L^{-1} = Y$, where Y is an upper triangular matrix.

Then, $YU = I$

i.e.,
$$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & 0 & y_{33} \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

multiplying the matrices on the LHS and then equating the corresponding elements, we have,

$$\left.\begin{aligned} & y_{11}u_{11} = 1,\ y_{22}u_{22} = 1,\ y_{33}u_{33} = 1 \\ & y_{11}u_{12} + y_{12}u_{22} = 0 \\ & y_{11}u_{13} + y_{12}u_{23} + y_{13}u_{33} = 0 \end{aligned}\right\} \quad \text{.....(5)}$$

and, $y_{22}u_{13} + y_{23}u_{33} = 0$ .....(6)

From (5),
$$y_{11} = \frac{1}{u_{11}},\ y_{22} = \frac{1}{u_{22}} \text{ and } y_{33} = \frac{1}{u_{33}}$$

From (6),
$$y_{12} = -y_{11}\frac{u_{12}}{u_{22}}$$
$$y_{13} = -\frac{y_{11}u_{13} + y_{12}u_{23}}{u_{33}}$$
$$y_{23} = -\frac{y_{22}u_{23}}{u_{33}}$$

$\therefore$ We get, $U^{-1} = Y$ completely.

Hence, by (2), we get A^{-1}.

Example 4.10

Using the factorization method, find the inverse of the matrix
$$A = \begin{bmatrix} 50 & 107 & 36 \\ 27 & 54 & 20 \\ 31 & 66 & 21 \end{bmatrix}$$

Solution:

Taking $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$

and, $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Now,
$$A = LU$$
or,
$$\begin{bmatrix} 50 & 107 & 36 \\ 25 & 54 & 20 \\ 31 & 66 & 21 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$\therefore$
$$50 = u_{11}\ ;\ 107 = u_{12}\ ;\ 36 = u_{13}$$
$$25 = l_{21}u_{11}\ ;\ 54 = l_{21}u_{12}\ ;\ 20 = l_{21}u_{13} + u_{23}$$
$$31 = l_{31}u_{11}\ ;\ 66 = l_{21}u_{12} + l_{32}u_{22}\ ;\ 21 = l_{31}u_{13} + l_{32}u_{23} + u_{33}$$

or,
$$u_{11} = 50\ ;\ u_{12} = 107\ ;\ u_{13} = 36\ ;\ l_{21} = \frac{1}{2}\ ;\ u_{22} = \frac{1}{2}\ ;$$
$$u_{23} = 3\ ;\ l_{31} = \frac{31}{50}\ ;\ l_{32} = \frac{-17}{25}\ ;\ u_{23} = \frac{1}{25}$$

Thus,
$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 31/50 & 17/25 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 50 & 107 & 36 \\ 0 & 1/2 & 2 \\ 0 & 0 & 1/25 \end{bmatrix}$$

To find L^{-1}, let $L^{-1} = X$. Then $LX = I$

i.e.,
$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 31/50 & -17/25 & 1 \end{bmatrix}\begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$
$$x_{11} = 1,\ \frac{1}{2}x_{11} + x_{21} = 0$$
$$x_{22} = 1,\ \frac{31}{50}x_{11} - \frac{17}{25}x_{21} + x_{31} = 0$$
$$\frac{-17}{25}x_{22} + x_{32} = 0,\ x_{33} = 1$$

or,
$$x_{11} = x_{22} = x_{33} = 1$$
$$x_{21} = -\frac{1}{2},\ x_{31} = -\frac{24}{25},\ x_{32} = \frac{17}{25}$$

Thus,
$$L^{-1} = X = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -24/25 & 17/25 & 1 \end{bmatrix}$$

To find U^{-1}, let $U^{-1} = Y$. Then $YU = 1$

i.e.,
$$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & 0 & y_{33} \end{bmatrix}\begin{bmatrix} 50 & 107 & 36 \\ 0 & 1/2 & 2 \\ 0 & 0 & 1/25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$
$$50y_{11} = 1,\ 50y_{12} + 107y_{22} = 0,\ 50y_{13} + 107y_{23} + 36y_{33} = 0$$
$$\frac{1}{2}y_{22} = 1,\ \frac{1}{2}y_{23} + 2y_{33} = 0,\ \frac{1}{25}y_{33} = 1$$

or,
$$y_{11} = \frac{1}{50},\ y_{22} = 2,\ y_{33} = 25,\ y_{12} = \frac{-107}{25},\ y_{23} = -100,\ y_{13} = 196$$

Hence,
$$U^{-1} = \begin{bmatrix} 1/50 & -107/25 & 196 \\ 0 & 2 & -100 \\ 0 & 0 & 25 \end{bmatrix}$$

so, $$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1/50 & -107/25 & 196 \\ 0 & 2 & -100 \\ 0 & 0 & 25 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -24/25 & 17/25 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -186 & 129 & 196 \\ 95 & -66 & -100 \\ -24 & 17 & 25 \end{bmatrix}$$

4.5 ILL-CONDITIONED EQUATIONS

A linear system is said to be ill-conditioned if small changes in the coefficient of the equations result in large changes in the values of the unknowns. On the contrary, a system is well-conditioned if small changes in the coefficients of the system also produce small changes in the solution. We often come across ill-conditioning of a system is usually expected. When the determinant of the coefficient matrix is small. The coefficient matrix of an ill-conditioned system is called an ill-conditioned matrix.

While solving simultaneous equation we also come across two forms of instabilities; Inherent and induced. Inherent instability of a system is a property of the given problem and occurs due to the problem being ill conditioned. It can be avoided by reformulation of the problem suitably. Induced instability occurs because of the incorrect choice of method.

Iterative method to improve accuracy of an ill-conditioned system

Consider the system of equations,

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad \text{..... (1)}$$

Let x', y', z' be an approximate solution. Substituting these values on the left hand sides, we get new values of d_1, d_2, d_3 as d_1', d_2', d_3' so that the new system is,

$$\left.\begin{aligned} a_1x' + b_1y' + c_1z' &= d_1' \\ a_2x' + b_2y' + c_2z' &= d_2' \\ a_3x' + b_3y' + c_3z' &= d_3' \end{aligned}\right\} \quad \text{..... (2)}$$

Subtracting each equation in (2), from the corresponding equations in (1), we get,

$$\left.\begin{aligned} a_1x_e + b_1y_e + c_1z_e &= k_1 \\ a_2x_e + b_2y_e + c_2z_e &= k_2 \\ a_3x_e + b_3y_e + c_3z_e &= k_3 \end{aligned}\right\} \quad \text{..... (3)}$$

where, $x_e = x - x'$

$y_e = y - y'$

$z_e = z - z_1'$

$k_i = d_i - z_i'$

We now solve the system (3) for x_e, y_e, z_e giving $x = x' + x_e$, $y = y' + y_e$ and $z = z' + z_e$ which will be better approximation for x, y, z. We can repeat the procedure for improving the accuracy.

Example 4.11

Establish whether the system 1.01x + 2y = 2.01; x + 2y = 2 is well conditioned or not?

Solution:

It's solution is x = 1 and y = 0.5

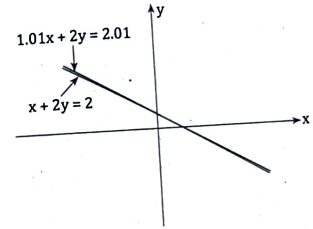
Now, consider the system,

$x + 2.01y = 2.04$

and, $x + 2y = 2$

which has the solution x = −6 and y = 4.

Hence the system is ill-conditioned.



4.6 ITERATIVE METHODS OF SOLUTION

Iterative method is that in which we start from an approximation to the true solution and obtain better and better approximation from a computation cycle repeated as often as may be necessary for achieving a desired accuracy. Thus in an iterative method, the amount of computation depends on the degree of accuracy required.

For large systems, iterative methods may be faster than the direct methods. Even the round-off errors in iterative methods are smaller. In fact, iteration is a self correcting process and any error made at any stage of computation gets automatically in the subsequent steps.

Simple iterative methods can be devised for systems in which the coefficients of the leading diagonal are large as compared to others.

4.6.1 Jacobi's Iteration Method

Consider the equation,

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad \text{..... (1)}$$

If a_1, b_2, c_3 are large as compared to other coefficients, solve the system can be written as,

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \quad \text{..... (2)}$$

Let us start with the initial approximations x_0, y_0, z_0 for the values of x, y, z respectively. Replacing these on the right sides of (2), the first approximations are given by

$$x = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

$$y = \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0)$$

$$z = \frac{1}{c_3}(d_3 - a_3x_0 - b_3y_0)$$

Replacing values of x_1, y_1, z_1 on the right sides of (2), the second approximations are given by,

$$x_2 = \frac{1}{a_1}(d_1 - b_1y_1 - c_1z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

This process is repeated until the difference between the consecutive approximations is negligible.

NOTE:
In the absence of any better estimates for x_0, y_0, z_0 these may each be taken as zero.

Example 4.12

Solve by Jacobi's iteration method:

$$20x + y - 2x = 17;$$
$$3x + 20y - z = -18;$$
$$2x - 3y + 20z = 25$$

Solution:

We write the given equations in the form,

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\} \quad \text{..... (2)}$$

Let, $x_0 = y_0 = z_0 = 0$,

Replacing these on the right sides of the equations (1), we get,

$$x_1 = \frac{17}{20} = 0.85, \quad y_1 = \frac{18}{20} = -0.9, \quad z_1 = \frac{25}{20} = 1.25$$

Putting these values on the right sides of the equation (1), we obtain,

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.02$$

$$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1) = -0.965$$

$$z_2 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.03$$

Replacing values on the right sides of the equations (1), we have,

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.00125$$

$$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) = 1.0015$$

$$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1.00325$$

Replacing values, we get,

$$x_4 = \frac{1}{20}(17 - y_3 + 2z_3) = 1.0004$$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) = -1.000025$$

$$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 0.9965$$

Putting these values, we have,

$$x_5 = \frac{1}{20}(-17 - y_4 + 2z_4) = 0.999966$$

$$y_5 = \frac{1}{20}(-18 - 3x_4 + z_4) = -1.000078$$

$$z_5 = \frac{1}{20}(25 - 2x_4 + 3y_4) = 0.999956$$

Again, substituting these values, we get,

$$x_6 = \frac{1}{20}(-17 - y_5 + 2z_5) = 1.0000$$

$$y_6 = \frac{1}{20}(-18 - 3x_5 + z_5) = 0.999997$$

$$z_6 = \frac{1}{20}(25 - 2x_5 + 3y_5) = 0.999992$$

The values in the fifth and sixth iterations being practically the same, we can stop. Hence the solution is,

$x = 1$, $y = -1$ and $z = 1$

4.6.2 Gauss Siedal Iteration Method

This is a modification of Jocobi's method. As before, the system of equations,

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \quad \text{..... (1)}$$

is written as,

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \quad \text{..... (2)}$$

Here, we start with the initial approximations x_0, y_0, z_0 for x, y, z respectively which may each be taken as zero. Replacing $y = y_0$, $z = z_0$ in the first of the equations (2), we get,

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

Then putting $x = x_1$, $z = z_0$ in the second of the equation (2), we have,

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_0)$$

Next substituting $x = x_1$, $y = y_1$ in the third of the equation (2), we have,

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

and so on, *i.e.*, as soon as a new approximations for an unknown is found, it is immediately used in the next step. This process of iteration is repeated until the values of x, y, z are obtained to a desired degree of accuracy.

NOTE:

1. **Jacobi and Gauss Siedal methods converge for any choice of the initial approximations if in each equation of the system, the absolute value of the largest coefficient is almost equal to or is atleast one equation greater than the sum of the absolute values of all the remaining coefficients.**
2. **The convergence in the Gauss-Siedal method is twice as fast as in Jacobi's method.**

Example 4.13

Apply the Gauss-Siedal method to solve the equations:

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Solution:

Writing the given equations as,

$$x = \frac{1}{20}(17 - y + 2z) \quad \text{..... (1)}$$

$$y = \frac{1}{20}(-18 - 3x + z) \quad \text{..... (2)}$$

$$z = \frac{1}{20}(25 - 2x + 3y) \quad \text{..... (3)}$$

First iteration by putting,

$y = y_0$, $z = z_0$ in equation (1), we get,

$$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$$

$x = x_1$, $z = z_0$ in equation (2), we get,

$$y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$$

$x = x_1$, $y = y_1$ in equation (2), we get,

$$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$$

Second iteration by putting,

$y = y_1$, $z = z_1$ in equation (1), we get,

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$$

$x = x_2$, $z = z_1$ in equation (2), we get,

$$y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$$

$x = x_2$, $y = y_2$ in equation (2), we get,

$$z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$$

Third iteration by putting,

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$$

$$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$$

$$z_3 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 1.0000$$

The values in the second and third iterations being practically the same, we can stop the iterations. Hence the solution of given equations is,

$x = 1$, $y = -1$ and $z = 1$

Example 4.14

Solve the equation

$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72$$

by the Gauss Jacobi and the Gauss Seidal method.

Solution:

Writing the given equations as,

$$x = \frac{1}{27}(85 - 6y + z) \quad \text{.....(1)}$$

$$y = \frac{1}{15}(72 - 6x - 2z) \quad \text{.....(2)}$$

$$z = \frac{1}{54}(110 - x - y) \quad \text{.....(3)}$$

a) **Gauss-Siedal's Method**

Starting from an approximation $x_0 = y_0 = z_0 = 0$.

First iteration:

$$x_1 = \frac{85}{27} = 3.148$$

$$y_1 = \frac{72}{15} = 4.8$$

$$z_1 = \frac{110}{54} = 2.037$$

Second iteration:

$$x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = 2.157$$

$$y_2 = \frac{1}{15}(72 - 6x_1 - y_1) = 3.269$$

$$z_2 = \frac{1}{54}(110 - x_1 - y_1) = 1.890$$

Third iteration:

$$x_3 = \frac{1}{27}(85 - 6y_2 + 7z_2) = 2.492$$

$$y_3 = \frac{1}{15}(72 - 6x_2 - 2z_2) = 3.685$$

$$z_3 = \frac{1}{54}(110 - x_2 - y_2) = 1.937$$

Fourth iteration:

$$x_4 = \frac{1}{27}(85 - 6y_3 + z_3) = 2.401$$

$$y_4 = \frac{1}{15}(72 - 6x_3 - 2y_3) = 3.545$$

$$z_4 = \frac{1}{54}(110 - x_3 - y_3) = 1.923$$

Fifth iteration:

$$x_5 = \frac{1}{27}(85 - 6y_4 + z_4) = 2.432$$

$$y_5 = \frac{1}{15}(72 - 6x_4 - 2y_4) = 3.583$$

$$z_5 = \frac{1}{54}(110 - x_4 - y_4) = 1.927$$

On repeating this process,

$x_6 = 2.423$,	$y_6 = 3.570$,	$z_6 = 1.926$
$x_7 = 2.426$,	$y_7 = 3.574$,	$z_7 = 1.926$
$x_8 = 2.425$,	$y_8 = 3.573$,	$z_8 = 1.926$
$x_9 = 2.426$,	$y_9 = 3.573$,	$z_9 = 1.926$

Hence, $x = 2.426$, $y = 3.573$ and $z = 1.926$.

b) **Gauss-Jacobi's Method**

First iteration by putting,

$y = y_0 = 0$, $z = z_0 = 0$ in equation (1), we get,

$$x_1 = \frac{1}{27}(85 - 6y_0 + z_0) = 3.14$$

$x = x_1$, $z = z_0$ in equation (2), we get,

$$y_1 = \frac{1}{15}(72 - 6x_1 - 2z_0) = 3.541$$

$x = x_1$, $y = y_1$ in equation (3), we get,

$$z_1 = \frac{1}{54}(110 - x_1 - y_1) = 1.913$$

Second iteration:

$$x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = 2.432$$

$$y_2 = \frac{1}{15}(72 - 6x_2 - 2z_1) = 3.572$$

$$z_2 = \frac{1}{54}(110 - x_2 - y_2) = 1.926$$

Third iteration:

$$x_3 = \frac{1}{27}(85 - 6y_2 + z_2) = 2.426$$

$$y_3 = \frac{1}{15}(72 - 6x_3 - 2z_2) = 3.573$$

$$z_3 = \frac{1}{54}(110 - x_3 - y_3) = 1.926$$

Fourth iteration:

$$x_4 = \frac{1}{27}(85 - 6y_3 + z_3) = 2.426$$

$$y_4 = \frac{1}{15}(72 - 6x_4 - 2z_3) = 3.573$$

$$z_4 = \frac{1}{54}(110 - x_4 - y_4) = 1.926$$

Hence, $x = 2.426$, $y = 3.573$ and $z = 1.926$

4.6.3 Relaxation Method

Consider the system of equations,

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

We define the residuals R_x, R_y and R_z by the relations,

$$\left.\begin{aligned} R_x &= d_1 - a_1x - b_1y - c_1z \\ R_y &= d_2 - a_2x - b_2y - c_2z \\ R_z &= d_3 - a_3x - b_3y - c_3z \end{aligned}\right\} \quad \text{....(1)}$$

To start with, we assume $x = y = z = 0$ and calculate the initial residuals. The residuals are reduced step by step, by giving increments to the variables. For this purpose, we construct the following operation table,

	δR_x	δR_y	δR_z
$\delta x = 1$	$-a_1$	$-a_2$	$-a_3$
$\delta y = 1$	$-b_1$	$-b_2$	$-b_3$
$\delta z = 1$	$-c_1$	$-c_2$	$-c_3$

We note from the equations (1) that if x is increased by (1) (Keeping y and z constant), R_x, R_y and R_z decreases by a_1, a_2, a_3 respectively. This is shown in the above table along with the effects on the residuals when y and z are given unit increments. (Table is the transpose of the coefficient matrix).

At each step, the numerically largest residual is reduced to almost zero. To reduce a particular residual, the value of the corresponding variable is changes. *e.g.*, to reduce R_x by p, x should be increased by $\frac{p}{a_1}$.

When all the residuals have been reduced to almost zero, the increments in x, y, z are added separately to give the desired solutions.

NOTE:

1. As a result, the computed values of x, y, z are substituted in (1) and the residuals are calculated. If these residuals are not all negligible, then there is some mistake and the entire process should be rechecked.
2. Relaxation method can be applied successfully only if the diagonal elements of the coefficient matrix dominate the other coefficients in the corresponding row *i.e.*, if in the equations (1),

$$|a_1| \geq |b_1| + |c_1|$$
$$|b_2| \geq |a_2| + |c_2|$$
$$|c_3| \geq |a_3| + |b_3|$$

where, > sign should be valid for atleast one row.

Example 4.15

Solve the equation by relaxation method.

$$9x - 2y + z = 50$$
$$x + 5y - 3z = 18$$
$$-2x + 2y + 7z = 19$$

Solution:

$$R_x = 50 - 9x + 2y - z$$
$$R_y = 18 - x - 5y + 3z$$
$$R_z = 19 + 2x - 2y - 7z$$

The operation table is,

	δR_x	δR_y	δR_z
$\delta x = 1$	-9	-1	2
$\delta y = 1$	2	-5	-2
$\delta z = 1$	-1	-3	-7

The relaxation table is,

	R_x	R_y	R_z	
$x = y = z = 0$	50	18	19	i
$\delta x = 5$	5	13	29	ii
$\delta z = 14$	1	25	1	iii
$\delta y = 5$	11	0	-9	iv
$\delta x = 1$	2	-1	-7	v
$\delta z = -1$	3	-4	0	vi
$\delta y = -0.8$	1.4	0	1.6	vii
$\delta y = 0.23$	1.17	0.69	-0.69	viii
$\delta y = 0.13$	0	0.56	0.17	ix
$\delta y = 0.112$	0.224	0	-0.054	x

$\Sigma\delta x = 6.13$, $\Sigma\delta y = 4.31$, $\Sigma\delta z = 3.23$

Hence, $x = 6.13$, $y = 4.31$ and $z = 3.23$.

In (i), the largest residual is 50. To reduce it, we give an increment $\delta_x = 5$ and the resulting residuals are shown in (ii) of these $R_z = 29$ is the largest and we give an increment $\delta_z = 4$ to get the results in (iii). In (vi), $R_y = -4$ is the numerically) largest value and we give and increment $\delta_y = -\frac{4}{5} = -0.8$ to obtain the results in (vii). Similarly, the other steps have been carried out.

4.7 POWER METHOD

A. Eigen Values and Eigen Vectors

If A is any square matrix of order n with elements a_{ij}, we can find a column matrix X and a constant λ such that $AX = \lambda X$ or $AX - \lambda IX = 0$ or $[A - \lambda I]X = 0$.

This matrix equation represents n homogenous linear equations,

$$\left.\begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \dots\dots\dots + a_{2n}x_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1} + a_{n2}x_2 + \dots\dots\dots + (a_{nn}-\lambda)x_n &= 0 \end{aligned}\right\} \quad \dots\dots (1)$$

which will have a non-trivial solution only if the coefficient determinant vanishes *i.e.*,

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots\dots\dots & a_{1n} \\ a_{21} & a_{12}-\lambda & \dots\dots\dots & a_{2n} \\ a_{n1} & a_{n2} & \dots\dots\dots & a_{nn}-\lambda \end{vmatrix} = 0 \quad \dots\dots (2)$$

On expansion, it gives on n^{th} degree equation in λ, called the characteristic equation of the matrix A. If roots λ_i (i = 1, 2, 3, 4, ..., n) are called the Eigen values or latent roots and corresponding to each eigen value, the equation (2) will have a non-zero solution.

$$X = [x_1, x_2, x_3, \dots\dots\dots, x_n]'$$

which is known as the eigen vector. Such an equation can ordinarily be solved easily. However, for larger systems, better methods are to be applied.

Cayley-Hamilton Theorem

Every square matrix satisfies its own characteristic equations. *i.e.,* if the characteristic equation for the nth order square matrix A is,

$$|A - \lambda I| = (-1)^n \lambda^n + k_1\lambda^{n-1} + \dots\dots\dots + k_n = 0$$

Then, $(-1)^n A^n + k_1A^{n-1} + k_n = 0$

Example 4.16

Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Solution:

The characteristic equation is $[A - \lambda I] = 0$

i.e., $\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} = 0$

or, $\lambda^2 - 7\lambda + 6 = 0$

or, $(\lambda - 6)(\lambda - 1) = 0$

$\therefore$ $\lambda = 6, 1$

Hence, the eigen values are 6 and 1.

If, x, y be the components of an eigen vector corresponding to the eigen value λ, then,

$$[A - \lambda I]\, X = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Corresponding to $\lambda = 6$, we have,

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

which gives only one independent equation $-x + 4y = 0$

$\therefore$ $\frac{x}{4} = \frac{y}{1}$ giving the eigen vector (4, 1)

Corresponding to $\lambda = 1$, we have $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

which gives only one independent equation $x + y = 0 \Rightarrow x = -y$

$\therefore$ $\frac{x}{1} = \frac{y}{-1}$ giving the eigen vector (1, –1).

Example 4.17

Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 3 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution:

The characteristic equation is,

$$|A - \lambda I| = \begin{bmatrix} 8-\lambda & -6 & 3 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

$$= \lambda^3 + 18\lambda^2 - 45\lambda = 0$$

or, $\lambda(\lambda - 3)(\lambda - 15) = 0$

$\therefore$ $\lambda = 0, 3, 15$

Thus the eigen values of A are 0, 3, 15.

If x, y, z be the components of an eigen vector corresponding to the eigen value λ, we have,

$$(A - \lambda)\, IX = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \dots\dots (1)$$

Putting $\lambda = 0$, we have, $8x - 6y + 2z = 0$, $-6x + 7y - 4z = 0$, $2x - 4y + 3z = 0$. These equations determine a single linearly independent solution which may be taken as (1, 2, 2) so that every non-zero multiple of this vector is an eigen vector corresponding to $\lambda = 0$.

Similarly, the eigen vectors corresponding to $\lambda = 3$ and $\lambda = 15$ are the arbitary non-zero multiples of the vectors (2, 1, –2) and (2, –2, 1) which are obtained from (1).

Hence the three eigen vectors may be taken as (1, 2, 2), (2, 1, –2) and (2, –2, 1).

B. Properties of Eigen Values

i) The sum of the eigen values of the matrix A is the sum of the elements of its principal diagonal.

ii) If λ is an eigen value of matrix A, then $\frac{1}{\lambda}$ is the eigen value of A^{-1}.

iii) If λ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigen value.

iv) If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigen values of matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ (m being a positive integer).

v) Any similarity transformation applied to a matrix leaves its eigen-values unchanged.

vi) If a square matrix A has a linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix whose diagonal elements are the eigen values of A.

The transformation of A by a non-singular matrix P to $P^{-1}AP$ is called a similarity transformation.

C. Power Method

If $X_1, X_2, \ldots, X_n$ are the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$, then an orbitrary column vector can written as,

$$X = k_1X_1 + k_2X_2 + \ldots + k_nX_n$$

Then, $AX = k_1AX_1 + k_2AX_2 + \ldots + k_nAX_n$

$$= k_1\lambda_1X_1 + k_2\lambda_2X_2 + \ldots + k_n\lambda_nX_n$$

Similarly,

$$A^2X = k_1\lambda_1^2X_1 + k_2\lambda_2^2X_2 + \ldots + k_n\lambda_n^2X_n$$

and, $A^rX = k_1\lambda_1^rX_1 + k_2\lambda_2^rX_2 + \ldots + k_n\lambda_n^rX_n$

If $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$, then λ_1 is the largest root and the contribution of the term $k_1\lambda_1^rX_1$ to the sum on the right increases with r and therefore, every time we multiply a column vector by A, it becomes nearer to the eigen vector X_1. Then we make the largest component of the resulting column vector unity to avoid the factor k_1.

Thus, we start with a column vector X which is as near the solution as possible and evaluate AX which is written as λ^1X^1 after normalization. This gives the first approximation λ^1 to the eigen value and X^1 to the eigen vector. Similarly, we evaluate $AX^1 = \lambda^2X^2$ which gives the second approximation. We repeat this process until $[X^r - X^{r-1}]$ becomes negligible. Then λ^r will be the largest eigen value and X^r, the corresponding eigen vector.

This iterative procedure for finding the dominant eigen value of a matrix is known as Rayleigh's power method.

NOTE:

We have, $AX = \lambda X$

as $A^{-1}AX = \lambda A^{-1}$

or, $X = \lambda A^{-1}X$

We know,

$$A^{-1}X = \frac{1}{\lambda}X$$

If we use this equation, then the above method yields the smallest eigen value.

If we use this equation, then the above method yields the smallest eigen value.

Example 4.18

Determine the largest eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}$.

Solution:

Let the initial approximations to the eigen vector corresponding to the largest eigen value of A be $X = \begin{bmatrix}1\\0\end{bmatrix}$.

Then, $AX = \begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix} = 5\begin{bmatrix}1\\0.2\end{bmatrix} = \lambda^1X^1$

So the 1st approximation to the eigen value is $\lambda^1 = 5$ and the corresponding eigen vector is $X^1 = \begin{bmatrix}1\\0.2\end{bmatrix}$.

Now,

$$AX^1 = \begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0.2\end{bmatrix} = \begin{bmatrix}5.8\\1.4\end{bmatrix} = 5.8\begin{bmatrix}1\\0.241\end{bmatrix} = \lambda^2X^2$$

Thus the second approximation to the eigen value is $\lambda^2 = 5.8$ and the corresponding eigen vector is $X^2 = \begin{bmatrix}1\\0.241\end{bmatrix}$.

Repeating the above process. We get,

$$AX^2 = \begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0.241\end{bmatrix} = 5.966\begin{bmatrix}1\\0.248\end{bmatrix} = \lambda^3X^3$$

$$AX^3 = \begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0.248\end{bmatrix} = 5.966\begin{bmatrix}1\\0.250\end{bmatrix} = \lambda^4X^4$$

$$AX^4 = \begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0.250\end{bmatrix} = 5.999\begin{bmatrix}1\\0.25\end{bmatrix} = \lambda^5X^5$$

$$AX^5 = \begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0.25\end{bmatrix} = 6\begin{bmatrix}1\\0.25\end{bmatrix} = \lambda^6X^6$$

Clearly, $\lambda^5 = \lambda^6$ and $X^5 = X^6$ upto 3 decimal places. Hence the largest eigen value is 6 and the corresponding eigen vector is $\begin{bmatrix}1\\0.25\end{bmatrix}$.

Example 4.19

Find the largest eigen value and the corresponding eigen vector of the matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

using the power method. Take $[1, 0, 0]^T$ as the initial eigen vector.

Solution:

Let the initial approximation to the required eigen vector be X[1, 0, 0]'.

Then,

$$AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^1 X^1$$

So the first approximation to the eigen value is 2 and the corresponding eigen vector

$$X(1) = [1, -0.5, 0]$$

Hence,

$$AX^1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \lambda^2 X^2$$

Repeating the above process, we get,

$$AX^2 = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^3 X^3$$

$$AX^3 = 3.43\begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \lambda^4 X^4$$

$$AX^4 = 3.41\begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \lambda^5 X^5$$

$$AX^5 = 3.41\begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \lambda^6 X^6$$

$$AX^6 = 3.41\begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = \lambda^7 X^7$$

Clearly, $\lambda^6 = \lambda^7$ and $X^6 = X^7$ approximately,

Hence, the largest eigen value is 3.41 and the corresponding eigen vector is [0.74, –1, 0.67]'.

Example 4.20

Obtain by the power method, the numerically dominant eigen value and eigen vector of the matrix A.

$$A = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$$

Solution:

Let the initial approximation to the required eigen vector be X[1, 1, 1]'.

Then,

$$AX = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ -18 \end{bmatrix} = -18\begin{bmatrix} -0.444 \\ 0.222 \\ 1 \end{bmatrix} = \lambda^1 X^1$$

So the first approximation to eigen value is –18 and the corresponding eigen vector is [–0.444, 0.222, 1]'.

Now,

$$AX^1 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} -0.444 \\ 0.222 \\ 1 \end{bmatrix} = -10.548\begin{bmatrix} 1 \\ -0.105 \\ -0.736 \end{bmatrix} = \lambda^2 X^2$$

Therefore, the second approximation to the eigen value is –10.548 and the eigen vector is [1, –0.105, –0.736]'

Repeating the process,

$$AX^2 = -18.948\begin{bmatrix} -0.930 \\ 0.361 \\ 1 \end{bmatrix} = \lambda^3 X^3$$

$$AX^3 = -18.394\begin{bmatrix} 1 \\ -0.415 \\ -0.981 \end{bmatrix} = \lambda^4 X^4$$

$$AX^4 = -19.698\begin{bmatrix} -0.995 \\ 0.462 \\ 1 \end{bmatrix} = \lambda^5 X^5$$

$$AX^5 = -19.773\begin{bmatrix} 1 \\ -480 \\ -0.999 \end{bmatrix} = \lambda^6 X^6$$

$$AX^6 = -19.922\begin{bmatrix} -0.997 \\ 0.490 \\ 1 \end{bmatrix} = \lambda^7 X^7$$

$$AX^7 = -19.956\begin{bmatrix} 1 \\ -495 \\ -0.999 \end{bmatrix} = \lambda^8 X^8$$

Since $\lambda^7 = \lambda^8$ and $X^7 = X^8$ approximately, hence the dominant eigen value and the corresponding eigen vector are given by,

$$\lambda^8 X^8 = 19.956\begin{bmatrix} 1 \\ 495 \\ 0.999 \end{bmatrix} \text{ i.e., } 20\begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix}$$

Hence, the dominant eigen value is 20 and eigen vector is [–1, 0.5, 1]'.

BOARD EXAMINATION SOLVED QUESTIONS

1. **Find the inverse of the given matrix by applying Gauss Elimination Method (GEM) with partial pivoting technique.**

$$A=\begin{bmatrix}4 & 1 & 2\\ 2 & 3 & -1\\ 1 & -2 & 2\end{bmatrix}$$

[2013/Fall]

Solution: Given that;

$$A=\begin{bmatrix}4 & 1 & 2\\ 2 & 3 & -1\\ 1 & -2 & 2\end{bmatrix}$$

Using partial pivoting technique so, arranging the matrix as

$$A=\begin{bmatrix}1 & -2 & 2\\ 4 & 1 & 2\\ 2 & 3 & -1\end{bmatrix}$$

Now, the augmented matrix is given by

$$[A:I]=\left[\begin{array}{ccc:ccc}1 & -2 & 2 & 1 & 0 & 0\\ 4 & 1 & 2 & 0 & 1 & 0\\ 2 & 3 & -1 & 0 & 0 & 1\end{array}\right]$$

Operate $R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 - 2R_1$

$$[A:I]=\left[\begin{array}{ccc:ccc}1 & -2 & 2 & 1 & 0 & 0\\ 0 & 9 & -6 & -4 & 1 & 0\\ 0 & 7 & -5 & -2 & 0 & 1\end{array}\right]$$

Operate $R_3 \to R_3 - \frac{7}{9}R_2$

$$[A:I]=\left[\begin{array}{ccc:ccc}1 & -2 & 2 & 1 & 0 & 0\\ 0 & 9 & -6 & -4 & 1 & 0\\ 0 & 0 & -1/3 & 10/9 & -7/9 & 1\end{array}\right]$$

Now, $\begin{bmatrix}1 & -2 & 2\\ 0 & 9 & -6\\ 0 & 0 & -1/3\end{bmatrix}\begin{bmatrix}x_{11}\\ x_{21}\\ x_{31}\end{bmatrix}=\begin{bmatrix}1\\ -4\\ 10/9\end{bmatrix}$

$$\begin{bmatrix}x_{11}\\ x_{21}\\ x_{31}\end{bmatrix}=\begin{bmatrix}1 & -2 & 2\\ 0 & 9 & -6\\ 0 & 0 & -1/3\end{bmatrix}\begin{bmatrix}1\\ -4\\ 10/9\end{bmatrix}$$

$\therefore \begin{bmatrix}x_{11}\\ x_{21}\\ x_{31}\end{bmatrix}=\begin{bmatrix}2.33\\ -2.66\\ -3.33\end{bmatrix}$

Also, $\begin{bmatrix}1 & -2 & 2\\ 0 & 9 & -6\\ 0 & 0 & -1/3\end{bmatrix}\begin{bmatrix}x_{12}\\ x_{22}\\ x_{32}\end{bmatrix}=\begin{bmatrix}0\\ 1\\ -7/9\end{bmatrix}$

$\therefore \begin{bmatrix}x_{12}\\ x_{22}\\ x_{32}\end{bmatrix}=\begin{bmatrix}-1.33\\ 1.66\\ 2.33\end{bmatrix}$

And, $\begin{bmatrix}1 & -2 & 2\\ 0 & 9 & -6\\ 0 & 0 & -1/3\end{bmatrix}\begin{bmatrix}x_{13}\\ x_{23}\\ x_{33}\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$

$\therefore \begin{bmatrix}x_{13}\\ x_{23}\\ x_{33}\end{bmatrix}=\begin{bmatrix}2\\ -2\\ -3\end{bmatrix}$

Hence, the inverse of matrix is

$$\begin{bmatrix}x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{bmatrix}=\begin{bmatrix}2.33 & -1.33 & 2\\ -2.66 & 1.66 & -2\\ -3.33 & 2.33 & -3\end{bmatrix}$$

2. **Solve the following system of equations by applying Gauss-Seidal iterative method. Carry out the iterations upto 6th stage**

$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

[2013/Fall]

Solution:

Arranging the equations such that magnitude of all the diagonal element is greater than the sum of magnitude of other two elements in the row i.e.,

$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35$$
$$x + 3y + 10z = 24$$

$$\begin{bmatrix}|28| > |4| + |-1|\\ |17| > |2| + |4|\\ |10| > |1| + |3|\end{bmatrix}$$

Forming the equations as

$$x=\frac{32-4y+z}{28}$$

$$y=\frac{35-2x-4z}{17}$$

$$z=\frac{24-x-3y}{10}$$

Let initial guess be 0 for x, y and z.

Solving the iterations in tabular form.

NOTE: Use the most recent values obtained to find the next one in this method.

Iteration	$x=\frac{32-4y+z}{28}$	$y=\frac{35-2x-4z}{17}$	$z=\frac{24-x-3y}{10}$
Guess	0	0	0
1	$\frac{32-4(0)+0}{28}$ = 1.142	$\frac{35-2(1.142)-4(0)}{17}$ = 1.924	$\frac{24-1.142-3(1.924)}{10}$ = 1.708
2	0.929	1.547	1.843
3	1.130	1.492	1.839
4	0.995	1.509	1.847
5	0.993	1.507	1.848
6	1.136	1.490	1.839

NOTE:
Procedure to iterate in programmable calculator
Let, A = x, B = y, C = z
Step 1: Set the following in calculator

$$A = \frac{31 - 4B + C}{28} : B = \frac{35 - 2A - 4C}{17} : C = \frac{24 - A - 3B}{10}$$

Step 2: Press CALC then
enter the value of B? then press =
enter the value of C? then press =
Step 3: Now press = only, again and again to get the values for respective row for each column.
Step 4: The values are updated automatically so continue pressing = till the required number of iterations.

3. Solve the following system of equations using Gauss elimination method.

$$10x_1 - 7x_2 + 3x_3 + 5x_4 = 6$$
$$-6x_1 + 8x_2 - x_3 + 4x_4 = 5$$
$$3x_1 + x_2 + 4x_3 + 11x_4 = 2$$
$$5x_1 - 9x_2 - 2x_3 + 4x_4 = 7$$

[2013/Spring]

Solution:

Writing the given system of equations in matrix form,

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ -6 & 8 & -1 & -4 \\ 3 & 1 & 4 & 11 \\ 5 & -9 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

Operate $R_2 \to R_2 - \left(-\frac{6}{10}\right) R_1,\ R_3 \to R_3 - \frac{3}{10} R_1,\ R_4 \to R_4 - \frac{5}{10} R_1$

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 3.1 & 3.1 & 9.5 \\ 0 & -5.5 & -3.5 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ 0.2 \\ 4 \end{bmatrix}$$

Operate $R_3 \to R_3 - \frac{3.1}{3.8} R_2,\ R_4 \to R_4 - \frac{-5.5}{3.8} R_2$

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44 & 10.31 \\ 0 & 0 & -2.34 & 0.05 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ -6.81 \\ 16.44 \end{bmatrix}$$

Operate $R_4 \to R_4 - \left(-\frac{2.34}{2.44}\right) R_3$

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44 & 10.31 \\ 0 & 0 & 0 & 0.993 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ -6.81 \\ 9.90 \end{bmatrix}$$

Now, performing back substitution,

$9.93x_4 = 9.90$

$\therefore\ x_4 = 0.99 \approx 1$

$2.44x_3 + 10.31x_4 = -6.81$

or, $2.44x_3 = -6.81 - 10.31 \times -1$

$\therefore\ x_3 = -7.01 \approx -7$

$3.8x_2 + 0.8x_3 - x_4 = 8.6$

or, $3.8x_2 + 0.8(-7) + 1 = 8.6$

$\therefore\ x_2 = 3.47 \approx 3.5$

$10x_1 - 7x_2 + 3x_3 + 5x_4 = 6$

or, $10x_1 - 7(3.5) + 3(-7) + 5(1) = 6$

$\therefore\ x_1 = 4.65$

4. Determine the highest even value and its corresponding eigen vector for the following matrix using power method.

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$$

[2013/Spring]

Solution:

Let the vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then, the iterations are carried out as,

$$AX_0 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 13 \end{bmatrix}$$

The highest value in AX_0 is 13 so dividing each element by 13.

$$AX_0 = 13 \begin{bmatrix} 0.2307 \\ 0.6923 \\ 1 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.2307 \\ 0.6923 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.3076 \\ 6.0767 \\ 12.5385 \end{bmatrix} = 12.5385 \begin{bmatrix} 0.1042 \\ 0.4864 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.1042 \\ 0.4864 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5634 \\ 5.2854 \\ 11.8414 \end{bmatrix} = 11.8414 \begin{bmatrix} 0.0475 \\ 0.4463 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.0475 \\ 0.4463 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3864 \\ 5.0351 \\ 11.7377 \end{bmatrix} = 11.7377 \begin{bmatrix} 0.0329 \\ 0.4289 \\ 1 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.0329 \\ 0.4289 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3196 \\ 4.9565 \\ 11.6827 \end{bmatrix} = 11.6827\begin{bmatrix} 0.0273 \\ 0.4242 \\ 1 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.0273 \\ 0.4242 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2999 \\ 4.9303 \\ 11.6695 \end{bmatrix} = 11.6695\begin{bmatrix} 0.0256 \\ 0.4224 \\ 1 \end{bmatrix}$$

Hence, the required eigen value $11.6695 \approx 12$.

And, required eigen vector $= \begin{bmatrix} 0.0256 \\ 0.4224 \\ 1 \end{bmatrix}$.

NOTE:

Procedure to solve in programmable calculator

Step 1: Press MODE then select MATRIX by pressing 6.

Step 2: Select MatA by pressing 1 and select 3 × 3 by pressing 1.

Step 3: Initialize the given matrix from the question.

Step 4: Press SHIFT then 4(MATRIX) and select Dim by pressing 1.

Step 5: Select MatB by pressing 2 and select 3 × 1 by pressing 3.

Step 6: Initialize the initial vector value and press AC.

Step 7: Press SHIFT then 4(MATRIX) and select MatA by pressing 3 and then press Multiply (×).

Step 8: Press SHIFT then 4(MATRIX) and select MatB by pressing 4 and then press =

Step 9: Now find the largest value in matrix and then press Divide (÷) and enter the largest value and then press =

Step 10: Now for next iteration press AC

Step 11: Press SHIFT then 4(MATRIX) and select MatA by pressing 3 then Multiply(×).

Step 12: Press SHIFT then 4(MATRIX) and select MatAns by pressing 6 and then press =

Step 13: Go to step 9.

5. **Using Factorization method, solve the following system of linear equations:**

$$3x + 2y + 7z = 4$$
$$2x + 3y + z = 5$$
$$3x + 4y + z = 7$$

[2013/Spring]

Solution:

In matrix form

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} \quad i.e., AX = B$$

In factorization method, we represent A as

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Solving for unknown values

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{12}u_{11} & u_{12}l_{21} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

$u_{11} = 3$	$u_{12} = 2$	$u_{13} = 7$
$l_{21} = \frac{2}{3} = 0.667$	$2 \times 0.667 + u_{22} = 3$ $\therefore u_{22} = 1.666$	$0.667 \times 7 + u_{23} = 1$ $\therefore u_{23} = -3.669$
$l_{31} = \frac{3}{3} = 1$	$l \times 2 + l_{32}(1.666) = 4$ $\therefore l_{32} = 1.2$	$1 \times 7 + 1.2(-3.669) + u_{33} = 1$ $\therefore u_{33} = -1.597$

Now, substituting obtained coefficients, we have overall system of

LUX = B

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.667 & 1 & 0 \\ 1 & 1.2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 7 \\ 0 & 1.666 & -3.669 \\ 0 & 0 & -1.597 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

Let UX = V then

LV = B

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.667 & 1 & 0 \\ 1 & 1.2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

Using forward substitution

$\therefore \quad v_1 = 4$

or, $\quad 0.667v_1 + v_2 = 5$

$\therefore \quad v_2 = 2.332$

or, $\quad 1v_1 + 1.2v_2 + v_3 = 7$

$\therefore \quad v_3 = 0.201$

Using the obtained values at UX = V

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 1.666 & -3.669 \\ 0 & 0 & -1.597 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2.332 \\ 0.201 \end{bmatrix}$$

Using backward substitution

$\therefore \quad z = \frac{0.201}{-1.597} = -0.125$

or, $\quad 1.666y - 3.669z = 2.332$

$\therefore \quad y = 1.174$

or, $\quad 3x + 2y + 7z = 4$

$\therefore \quad x = 0.842$

6. **Solve the following system of equations by applying Gauss Elimination Method (GEM) with partial pivoting technique. And also determine the determinant value.**

$$2x + 2y + z = 6$$
$$4x + 2y + 3z = 4$$
$$x - y + z = 0$$

Solution: [2014/Fall]

By partial pivoting technique, the system of linear equation can be arranged as,

$$4x + 2y + 3z = 4$$
$$2x + 2y + z = 6$$
$$x - y + z = 0$$

The augmented matrix can be written as

$$[A : B] = \begin{bmatrix} 4 & 2 & 3 & : & 4 \\ 2 & 2 & 1 & : & 6 \\ 1 & -1 & 1 & : & 0 \end{bmatrix}$$

Operate $R_1 \to R_1 - 3R_3$, $R_2 \to R_2 - 2R_3$

$$= \begin{bmatrix} 1 & 5 & 0 & : & 4 \\ 0 & 4 & -1 & : & 6 \\ 1 & -1 & 1 & : & 0 \end{bmatrix}$$

Operate $R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 5 & 0 & : & 4 \\ 0 & 4 & -1 & : & 6 \\ 0 & -6 & 1 & : & -4 \end{bmatrix}$$

Operate $R_2 \to \frac{R_2}{4}$

$$= \begin{bmatrix} 1 & 5 & 0 & : & 4 \\ 0 & 1 & -1/4 & : & 3/2 \\ 0 & -6 & 1 & : & -4 \end{bmatrix}$$

Operate $R_3 \to R_3 + 6R_2$

$$[A : B] = \begin{bmatrix} 1 & 5 & 0 & : & 4 \\ 0 & 1 & -1/4 & : & 3/2 \\ 0 & 0 & -1/2 & : & 5 \end{bmatrix}$$

Performing back substitution,

$$-\frac{1}{2}z = 5$$

$\therefore$ $z = -10$

Then, $y - \frac{1}{4}z = \frac{3}{2}$

or, $y + \frac{10}{4} = \frac{3}{2}$

$\therefore$ $y = -1$

and, $x + 5y + 0 = 4$

or, $x + 5(-1) = 4$

$\therefore$ $x = 9$

Also, determinant value

$$\begin{vmatrix} 4 & 2 & 3 \\ 2 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= 4(2 + 1) - 2(2 - 1) + 3(-2 - 2)$$

$$= -2$$

7. **Find the largest eigen value and the corresponding eigen vector correct upto 3 decimal places using power method for the matrix**

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$ **[2014/Fall, 2017/Fall, 2019/Spring]**

Solution:

Let initial eigen vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Then the iterations are carried out as

$$AX_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} = 4\begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -3.5 \\ 2.5 \end{bmatrix} = 3.5\begin{bmatrix} 0.7142 \\ -1 \\ 0.7142 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.7142 \\ -1 \\ 0.7142 \end{bmatrix} = \begin{bmatrix} 2.4284 \\ -3.4284 \\ 2.4284 \end{bmatrix} = 3.4284\begin{bmatrix} 0.7083 \\ -1 \\ 0.7083 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.7083 \\ -1 \\ 0.7083 \end{bmatrix} = \begin{bmatrix} 2.4166 \\ -3.4166 \\ 2.4166 \end{bmatrix} = 3.4166\begin{bmatrix} 0.7073 \\ -1 \\ 0.7073 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.7073 \\ -1 \\ 0.7073 \end{bmatrix} = \begin{bmatrix} 2.4146 \\ -3.4146 \\ 2.4146 \end{bmatrix} = 3.4146\begin{bmatrix} 0.707 \\ -1 \\ 0.707 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.707 \\ -1 \\ 0.707 \end{bmatrix} = \begin{bmatrix} 2.414 \\ -3.414 \\ 2.414 \end{bmatrix} = 3.414\begin{bmatrix} 0.707 \\ -1 \\ 0.707 \end{bmatrix}$$

Hence the required eigen value is 3.414 correct upto 3 decimal places.

$$\text{And required eigen vector} = \begin{bmatrix} 0.707 \\ -1 \\ 0.707 \end{bmatrix}$$

8. Solve the following system of by using Gauss Seidal method.

$$10x - 5y - 2z = 3$$
$$x + 6y - 10z = -3$$
$$4x - 10y + 3z = -3$$

[2014/Fall]

Solution:

$$10x - 5y - 2z = 3$$
$$x + 6y - 10z = -3$$
$$4x - 10y + 3z = -3$$

Arranging the equations such that magnitude of all the diagonal element is greater than the sum of magnitude of other two elements in the row.

$$10x - 5y - 2z = 3$$
$$4x - 10y + 3z = -3$$
$$x + 6y - 10z = -3$$

Now, forming the equations as,

$$x = \frac{3 + 5y + 2z}{10}$$
$$y = \frac{-3 - 3z - 4x}{-10} = \frac{3 + 3z + 4x}{10}$$
$$z = \frac{-3 - x - 6y}{-10} = \frac{3 + x + 6y}{10}$$

Let initial guess be 0 for x, y and z.

Solving the iterations in tabular form.

Iteration	$x = \frac{3 + 5y + 2z}{10}$	$y = \frac{3 + 3z + 4x}{10}$	$z = \frac{3 + x + 6y}{10}$
Guess	0	0	0
1	0.3	0.42	0.582
2	0.6264	0.7251	0.7977
3	0.8220	0.8681	0.9030
4	0.9146	0.9367	0.9534
5	0.9590	0.9696	0.9776
6	0.9803	0.9854	0.9892
7	0.9905	0.9929	0.9947
8	0.9953	0.9965	0.9974
9	0.9977	0.9983	0.9987
10	0.9988	0.9991	0.9993
11	0.9994	0.9995	0.9996

Hence the approximated values of x, y, and z is $0.999 \approx 1$.

> ***NOTE:***
> Procedure to iterate in programmable calculator:
> Let $A = x$, $B = y$, $C = z$
> Set the following in calculator:
> $A = \frac{3 + 5B + 2C}{10}$: $B = \frac{3 + 3C + 4A}{10}$: $C = \frac{3 + A + 6B}{10}$
> Now press CALC and enter the initial value of B and C and continue pressing = only for the required no. of iterations.

9. Use Gauss Elimination Method to solve the equation. Use partial pivoting method where necessary.

$$4x_1 + 5x_2 - 6x_3 = 28$$
$$2x_1 - 7x_3 = 29$$
$$-5x_1 - 8x_2 = -64$$

[2014/Spring]

Solution:

Writing the given system of equation in matrix form,

$$\begin{bmatrix} 4 & 5 & -6 \\ 2 & 0 & -7 \\ -5 & -8 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 28 \\ 29 \\ -64 \end{bmatrix}$$

Operate $R_2 \to R_2 - \left(\frac{2}{4}\right)R_1$ and $R_3 \to R_3 - \frac{(-5)}{4}R_1$

$$\begin{bmatrix} 4 & 5 & -6 \\ 0 & -2.5 & -4 \\ 0 & -1.75 & -7.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 28 \\ 15 \\ -29 \end{bmatrix}$$

Operate $R_3 \to R_3 - \frac{-1.75}{-2.5}R_2$

$$\begin{bmatrix} 4 & 5 & -6 \\ 0 & -2.5 & -4 \\ 0 & 0 & -4.7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 28 \\ 15 \\ -39.5 \end{bmatrix}$$

Now, performing back substitution

$$-4.7x_3 = -39.5$$

$\therefore$ $x_3 = 8.404$

$$-2.5x_2 - 4x_3 = 15$$

or, $-2.5x_2 - 4(8.404) = 15$

$\therefore$ $x_2 = -19.446$

$$4x_1 + 5x_2 - 6x_3 = 28$$

or, $4x_1 + 5(-19.446) - 6(8.404) = 28$

$\therefore$ $x_1 = 43.913$

10. Find the largest eigen value λ and the corresponding eigen vector X of the matrix

[2014/Spring]

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution:

Let initial eigen vector be $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

Then the iterations are carried out as

$$AX_0 = \begin{bmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\3\\-1\end{bmatrix} = 3\begin{bmatrix}1\\1\\-0.333\end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\1\\-0.333\end{bmatrix} = \begin{bmatrix}3\\3\\0.333\end{bmatrix} = 3\begin{bmatrix}1\\1\\0.111\end{bmatrix}$$

$$AX_2 = \begin{bmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\1\\0.111\end{bmatrix} = \begin{bmatrix}3\\3\\-0.111\end{bmatrix} = 3\begin{bmatrix}1\\1\\-0.037\end{bmatrix}$$

$$AX_3 = \begin{bmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\1\\-0.037\end{bmatrix} = \begin{bmatrix}3\\3\\0.037\end{bmatrix} = 3\begin{bmatrix}1\\1\\0.012\end{bmatrix}$$

$$AX_4 = \begin{bmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\1\\0.012\end{bmatrix} = \begin{bmatrix}3\\3\\-0.012\end{bmatrix} = 3\begin{bmatrix}1\\1\\-0.004\end{bmatrix}$$

Hence the largest eigen value λ is 3 and largest eigen vector is $\begin{bmatrix}1\\1\\0\end{bmatrix}$.

11. **Solve the following by Gauss-Siedal Method**

$$b + 3c + 2d = 19$$
$$3b + 2c + 2d = 20$$
$$a + 4b + 2d = 17$$
$$-2a + 2b + c + d = 9$$

Solution: [2014/Spring]

Here, the provided system is not diagonally dominant as the magnitude of all the diagonal element is not greater than the sum of magnitude of other elements in the row.

i.e., |coefficient of a| $\ngtr$ |sum coefficient of b, c and d|.

Hence we cannot solve for the convergence from this method.

If it is to be solved from other methods the acquired values a, b, c and d are;

a = 1 b = 2

c = 3 d = 4

12. **Solve the following set of equation using LU factorization method.**

$$3x + 2y + z = 10$$
$$2x + 3y + 2z = 14$$
$$x + 2y + 3z = 14$$

[2015/Fall, 2017/Fall, 2019/Spring]

Solution:

Writing the equation in matrix from AX = B

$$\begin{bmatrix}3 & 2 & 1\\2 & 3 & 2\\1 & 2 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}10\\14\\14\end{bmatrix}$$

Here, we represent A as

$$\begin{bmatrix}1 & 0 & 0\\l_{21} & 1 & 0\\l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\0 & u_{22} & u_{23}\\0 & 0 & u_{33}\end{bmatrix} = \begin{bmatrix}3 & 2 & 1\\2 & 3 & 2\\1 & 2 & 3\end{bmatrix}$$

Solving for unknown values,

$$\begin{bmatrix}u_{11} & u_{12} & u_{13}\\l_{21}u_{11} & u_{12}l_{21} + u_{22} & l_{21}u_{13} + u_{23}\\l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33}\end{bmatrix} = \begin{bmatrix}3 & 2 & 1\\2 & 3 & 2\\1 & 2 & 3\end{bmatrix}$$

$u_{11} = 3$	$u_{12} = 2$	$u_{13} = 1$
$l_{21} = \frac{2}{3} = 0.667$	$u_{12}l_{21} + u_{22} = 3$ $\therefore u_{22} = 1.666$	$0.667 \times 1 + u_{23} = 2$ $\therefore u_{23} = 1.333$
$l_{31} = \frac{1}{3} = 0.333$	$0.333\times 2 + l_{32}(1.666) = 2$ $\therefore l_{32} = 0.8$	$0.333\times1+0.8\times1.333+u_{33} = 1$ $\therefore u_{33} = 1.6$

Substituting the values,

$$\underset{L}{\begin{bmatrix}1 & 0 & 0\\0.667 & 1 & 0\\0.333 & 0.8 & 1\end{bmatrix}}\underset{U}{\begin{bmatrix}3 & 2 & 1\\0 & 1.666 & 1.333\\0 & 0 & 1.6\end{bmatrix}}\underset{X}{\begin{bmatrix}x\\y\\z\end{bmatrix}} = \underset{B}{\begin{bmatrix}10\\14\\14\end{bmatrix}}$$

Let LUX = B

$\Rightarrow$ UX = V

so, LV = B

$$\begin{bmatrix}1 & 0 & 0\\0.667 & 1 & 0\\0.333 & 0.8 & 1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}10\\14\\14\end{bmatrix}$$

Now, performing forward substitution,

$\therefore$ $v_1 = 10$

$\Rightarrow$ $0.667v_1 + v_2 = 14$

$\therefore$ $v_2 = 7.33$

$\Rightarrow$ $0.333v_1 + 0.8v_2 + v_3 = 14$

$\therefore$ $v_3 = 4.80$

Then, UX = V becomes

$$\begin{bmatrix}3 & 2 & 1\\0 & 1.666 & 1.333\\0 & 0 & 1.6\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}10\\7.33\\4.80\end{bmatrix}$$

Performing backward substitution,

$$z = \frac{4.8}{1.6} = 3$$

$\Rightarrow$ $1.666y - 1.333z = 7.33$

$\therefore$ $y = 1.99 \approx 2$

$\Rightarrow$ $3x + 2y + z = 10$

$\therefore$ $x = \frac{3.02}{3} = 1.02 \approx 1$

Hence, $x = 1$;

$y = 2$;

and, $z = 3$.

13. Use Gauss-Seidal Iterative method to solve given equations.

$$40x - 20y - 10z = 390$$
$$10x - 60y + 20z = -280$$
$$10x - 30y + 120z = -860$$

[2015/Fall]

Solution:

Here the equations have the dominance of diagonal element so forming the equations as

$$x = \frac{390 + 20y + 10z}{40}$$

$$y = \frac{-280 - 10x - 20z}{-60}$$

$$z = \frac{-860 - 10x - 30y}{120}$$

Let the initial guess be 0 for x, y and z.

Now, solving the iteration in tabular form

Iteration	$x = \frac{390 + 20y + 10z}{40}$	$y = \frac{-280 - 10x - 20z}{-60}$	$z = \frac{-860 - 10x - 30y}{120}$
Guess	0	0	0
1	9.75	6.291	-9.551
2	10.507	3.234	-8.850
3	9.154	3.242	-8.74
4	9.186	3.284	-8.753
5	9.203	3.282	-8.754
6	9.202	.3282	-8.754
7	9.202	3.282	-8.754

Here, the values of x, y and z are correct upto 3 decimal places.

So the approximate values of $x = 9.202$, $y = 3.282$ and $z = -8.754$

> ***NOTE:***
> *Procedure to iterate in programmable calculator*
> *Let* $A = x$, $B = y$, $C = z$
> *Set the following in calculator*
> $A = \frac{390 + 20B + 10C}{40}$: $B = \frac{280 + 10A + 20C}{60}$: $C = \frac{-860 - 10A - 30B}{120}$
> *Now press CALC and enter the initial value of B and C and continue pressing = only for the required no. of iterations.*

14. Find the eigen vlaue and corresponding eigen vector of given matrix

$$\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

[2015/Fall]

Solution:

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then the iterations are carried out as

$$AX_0 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ -0.333 \\ 1 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -0.333 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0.666 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0.111 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.111 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -0.222 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ -0.037 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -0.037 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0.074 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0.012 \\ 1 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.012 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -0.024 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ -0.004 \\ 1 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -0.004 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0.008 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0.001 \\ 1 \end{bmatrix}$$

Hence the required eigen value = 6.

And the required eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

15. Find the largest eigen value and corresponding eigen vector of the following square matrix using power method.

[2015/Spring]

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & 3 \\ 4 & 3 & 5 \end{bmatrix}$$

Solution:

Let the initial vector be $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

Then the iterations are carried out as

$$AX_0 = \begin{bmatrix}3 & 1 & 4\\1 & 2 & 3\\4 & 3 & 5\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}8\\6\\12\end{bmatrix} = 12\begin{bmatrix}0.667\\0.5\\1\end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix}3 & 1 & 4\\1 & 2 & 3\\4 & 3 & 5\end{bmatrix}\begin{bmatrix}0.667\\0.5\\1\end{bmatrix} = \begin{bmatrix}6.501\\4.667\\9.168\end{bmatrix} = 9.168\begin{bmatrix}0.709\\0.509\\1\end{bmatrix}$$

$$AX_2 = \begin{bmatrix}3 & 1 & 4\\1 & 2 & 3\\4 & 3 & 5\end{bmatrix}\begin{bmatrix}0.709\\0.509\\1\end{bmatrix} = \begin{bmatrix}6.636\\4.727\\9.363\end{bmatrix} = 9.363\begin{bmatrix}0.708\\0.504\\1\end{bmatrix}$$

$$AX_3 = \begin{bmatrix}3 & 1 & 4\\1 & 2 & 3\\4 & 3 & 5\end{bmatrix}\begin{bmatrix}0.708\\0.504\\1\end{bmatrix} = \begin{bmatrix}6.628\\4.716\\9.344\end{bmatrix} = 9.344\begin{bmatrix}0.709\\0.504\\1\end{bmatrix}$$

$$AX_4 = \begin{bmatrix}3 & 1 & 4\\1 & 2 & 3\\4 & 3 & 5\end{bmatrix}\begin{bmatrix}0.709\\0.504\\1\end{bmatrix} = \begin{bmatrix}6.631\\4.717\\9.348\end{bmatrix} = 9.348\begin{bmatrix}0.709\\0.504\\1\end{bmatrix}$$

$$AX_5 = \begin{bmatrix}3 & 1 & 4\\1 & 2 & 3\\4 & 3 & 5\end{bmatrix}\begin{bmatrix}0.709\\0.504\\1\end{bmatrix} = \begin{bmatrix}6.631\\4.717\\9.348\end{bmatrix} = 9.348\begin{bmatrix}0.709\\0.504\\1\end{bmatrix}$$

Hence the required eigen value = 9.348.

And the required eigen vector is $\begin{bmatrix}0.709\\0.504\\1\end{bmatrix}$.

16. Solve the following system of equation by the process of Gauss elimination. (Use partial pivoting if necessary)

$$3x + 2y + z = 10$$
$$2x + 3y + 2z = 14$$
$$x + 2y + 3z = 14$$

[2015/Spring]

Solution:

Writing given equations in matrix form

$$\begin{bmatrix}3 & 2 & 1\\2 & 3 & 2\\1 & 2 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}10\\14\\14\end{bmatrix}$$

Operate $R_2 \to R_2 - \frac{2}{3}R_1$ and $R_3 \to R_3 - \frac{1}{3}R_1$

$$\begin{bmatrix}3 & 2 & 1\\0 & 5/3 & 4/3\\0 & 4/3 & 8/3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}10\\22/3\\32/3\end{bmatrix}$$

Operate $R_3 \to R_3 - \frac{4}{5}R_2$

$$\begin{bmatrix}3 & 2 & 1\\0 & 5/3 & 4/3\\0 & 0 & 8/5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}10\\22/3\\24/5\end{bmatrix}$$

Now, performing backward substitution,

or, $\frac{8}{5}z = \frac{24}{5}$

$\therefore$ $z = 3$

or, $\frac{5}{3}y + \frac{4}{3}z = \frac{22}{3}$

$\therefore$ $y = 2$

or, $3x + 2y + z = 10$

$x = \frac{10 - 2y - z}{3}$

$\therefore$ $x = 1$

17. Use Gauss Seidal iteration method to solve

$$2x + y + z = 5$$
$$3x + 5y + 2z = 15$$
$$2x + y + 4z = 8$$

[2015/Spring]

Solution:

Here the equations are in diagonally dominant form.

Forming the equations as

$$x = \frac{5 - y - z}{2}$$

$$y = \frac{15 - 3x - 2z}{5}$$

$$x = \frac{8 - 2x - y}{4}$$

Let initial guess be 0 for x, y and z.

Solving the iterations in tabular form.

Iteration	$x = \frac{5-y-z}{2}$	$y = \frac{15-3x-2z}{5}$	$z = \frac{8-2x-y}{4}$
Guess	0	0	0
1	2.5	1.5	0.375
2	1.562	1.912	0.741
3	1.173	1.999	0.931
4	1.044	2.008	0.976
5	1.008	2.004	0.995
6	1.0005	2.0017	0.9993
7	0.999	2.0008	1.0003
8	0.999	2.0008	1.0003

Hence the required valued of x, y and z are 1, 2 and 1 respectively.

NOTE:

Procedure to iterate in programmable calculator

Let A = x, B = y, C = z

Set the following in calculator

$$A = \frac{5-B-C}{2} : B = \frac{15-3A-2C}{5} : C = \frac{8-2A-B}{4}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required no. of iterations.

18. Solve the following system of equations by using Gauss elimination method with partial pivoting technique.

$$x + y + z + w = 2$$
$$x + y + 3z - 2w = -6$$
$$2x + 3y - z + 2w = 7$$
$$x + 2y + z - w = -2$$

Solution: [2016/Fall]

Writing the system of equations in matrix form,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & -2 \\ 2 & 3 & -1 & 2 \\ 1 & 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 7 \\ -2 \end{bmatrix}$$

Interchanging R_1 and R_3 but not variable x and z as partial pivoting

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 1 & 1 & 3 & -2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \\ 2 \\ -2 \end{bmatrix}$$

Operate $R_2 \to R_2 - \frac{1}{2}R_1$, $R_3 \to R_3 - \frac{1}{2}R_1$, $R_4 \to R_4 - \frac{1}{2}R_1$

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 0 & -0.5 & 3.5 & -3 \\ 0 & -0.5 & 1.5 & 0 \\ 0 & 0.5 & 1.5 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7 \\ -9.5 \\ -1.5 \\ -5.5 \end{bmatrix}$$

Operate $R_3 \to R_3 - R_2$, $R_4 \to R_4 + R_2$

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 0 & -0.5 & 3.5 & -3 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7 \\ -9.5 \\ 8 \\ -15 \end{bmatrix}$$

Interchanging R_3 and R_4 but not the variable z and w as partial pivoting.

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 0 & -0.5 & 3.5 & -3 \\ 0 & 0 & 5 & -5 \\ 0 & 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7 \\ -9.5 \\ -15 \\ 8 \end{bmatrix}$$

Operate $R_4 \to R_4 - \frac{(-2)}{5}R_3$

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 0 & -0.5 & 3.5 & -3 \\ 0 & 0 & 5 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7 \\ -9.5 \\ -15 \\ 2 \end{bmatrix}$$

Performing backward substitution

or, $1w = 2$

$\therefore$ $w = 2$

or, $5z - 5w = -15$

$\therefore$ $z = -1$

or, $-0.5y + 3.5yz - 3w = -9.5$

$\therefore$ $y = 0$

or, $2x + 3y - z + 2w = 7$

$\therefore$ $x = 1$

19. Solve the following system of equations by using Crout's algorithm.

$$2x - 3y + 10z = 3$$
$$-x + 4y + 2z = 20$$
$$5x + 2y + z = -12$$

[2016/Fall]

Solution:

Writing the system of equations in matrix form,

$$\underset{A}{\begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}}\underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}}$$

Now, using Crout's algorithm, we represent A as

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{22} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

Solving for unknown values,

$l_{11} = 2$	$l_{11}u_{12} = -3$ $\therefore u_{12} = -1.5$	$l_{11}u_{13} = 10$ $\therefore u_{13} = 5$
$l_{21} = -1$	$l_{21}u_{12} + l_{22} = 4$ $\therefore l_{22} = 2.5$	$l_{21}u_{13} + l_{22}u_{23} = 2$ $\therefore u_{23} = 2.8$
$l_{31} = 5$	$l_{31}u_{12} + l_{32} = 2$ $\therefore l_{32} = 9.5$	$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 1$ $\therefore l_{33} = -50.6$

Now, substituting obtained coefficients as LUX = B

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 2.5 & 0 \\ 5 & 9.5 & -50.6 \end{bmatrix}\begin{bmatrix} 1 & -1.5 & 5 \\ 0 & 1 & 2.8 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

Let UX = V, so LV = B then,

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 2.5 & 0 \\ 5 & 9.5 & -50.6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

Using forward substitution,

$\therefore \quad v_1 = 1.5$

or $\quad -1v_1 + 2.5v_2 = 20$

$\therefore \quad v_2 = 8.6$

or, $\quad 5v_1 + 9.5v_2 - 50.6v_3 = -12$

$\therefore \quad v_3 = 2$

Then, UX = V

$$\begin{bmatrix} 1 & -1.5 & 5 \\ 0 & 1 & 2.8 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1.5 \\ 8.6 \\ 2 \end{bmatrix}$$

Performing backward substitution,

$\therefore \quad z = 2$

or, $\quad y + 2.8z = 8.6$

$\therefore \quad y = 3$

or, $\quad x - 1.5y + 5z = 1.5$

$\therefore \quad x = -4$

20. Find the largest eigen value and corresponding eigen vector of given matrix using power method.

$$\begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}$$

[2016/Fall]

Solution:

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then performing the iterations as follows,

$$AX_0 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 10 \\ 8 \\ 5 \end{bmatrix}=10\begin{bmatrix} 1 \\ 0.8 \\ 0.5 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.8 \\ 0.5 \end{bmatrix}=\begin{bmatrix} 8.8 \\ 5.5 \\ 3.5 \end{bmatrix}=8.8\begin{bmatrix} 1 \\ 0.625 \\ 0.397 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.625 \\ 0.397 \end{bmatrix}=\begin{bmatrix} 7.75 \\ 4.316 \\ 3.191 \end{bmatrix}=7.75\begin{bmatrix} 1 \\ 0.556 \\ 0.411 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.556 \\ 0.411 \end{bmatrix}=\begin{bmatrix} 7.336 \\ 4.013 \\ 3.233 \end{bmatrix}=7.336\begin{bmatrix} 1 \\ 0.547 \\ 0.440 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.547 \\ 0.440 \end{bmatrix}=\begin{bmatrix} 7.282 \\ 4.055 \\ 3.320 \end{bmatrix}=7.282\begin{bmatrix} 1 \\ 0.556 \\ 0.455 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.556 \\ 0.455 \end{bmatrix}=\begin{bmatrix} 7.336 \\ 4.145 \\ 3.365 \end{bmatrix}=7.336\begin{bmatrix} 1 \\ 0.565 \\ 0.458 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.565 \\ 0.458 \end{bmatrix}=\begin{bmatrix} 7.390 \\ 4.199 \\ 3.374 \end{bmatrix}=7.390\begin{bmatrix} 1 \\ 0.568 \\ 0.456 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.568 \\ 0.456 \end{bmatrix}=\begin{bmatrix} 7.408 \\ 4.208 \\ 3.368 \end{bmatrix}=7.408\begin{bmatrix} 1 \\ 0.568 \\ 0.454 \end{bmatrix}$$

$$AX_8 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.568 \\ 0.454 \end{bmatrix}=\begin{bmatrix} 7.408 \\ 4.202 \\ 3.362 \end{bmatrix}=7.408\begin{bmatrix} 1 \\ 0.567 \\ 0.453 \end{bmatrix}$$

Hence the required eigen vector is $\begin{bmatrix} 1 \\ 0.567 \\ 0.453 \end{bmatrix}$.

And the required eigen value 7.408.

21. Using Gauss Seidal method solve the following system of liner equations.

$$10x_1 + 6x_2 - 5x_3 = 27$$
$$3x_1 + 8x_2 + 10x_3 = 27$$
$$4x_1 + 10x_2 + 3x_3 = 27$$

[2016/Spring]

Solution:

Arranging the system of liner equations in diagonally dominant forms,

$$10x_1 + 6x_2 - 5x_3 = 27$$
$$4x_1 + 10x_2 + 3x_3 = 27$$
$$3x_1 + 8x_2 + 10x_3 = 27$$

Forming the equations as,

$$x_1 = \frac{27 - 6x_2 + 5x_3}{10}$$

$$x_2 = \frac{27 - 4x_1 - 3x_3}{10}$$

$$x_3 = \frac{27 - 3x_1 - 8x_2}{10}$$

Let the initial guess be 0 for x_1, x_2 and x_3

Solving the iterations in tabular form.

Iteration	$x_1 = \dfrac{27 - 6x_2 + 5x_3}{10}$	$x_2 = \dfrac{27 - 4x_1 - 3x_3}{10}$	$x_3 = \dfrac{27 - 3x_1 - 8x_2}{10}$
Guess	0	0	0
1	2.7	1.62	0.594
2	2.025	1.711	0.723
3	2.034	1.669	0.754
4	2.075	1.643	0.763
5	2.095	1.633	0.765
6	2.102	1.629	0.766
7	2.105	1.628	0.766
8	2.106	1.627	0.766
9	2.106	1.627	0.766

Hence the required valued of x_1, x_2 and x_3 are 2.106, 1.627 and 0.766 respectively which are correct upto 3 decimal places.

NOTE:

Procedure to iterate in programmable calculator

Let $A = x_1, B = x_2, C = x_3$

Set the following in calculator

$$A = \frac{27 - 6B + 5C}{10};\ B = \frac{27 - 4A - 3C}{10};\ C = \frac{27 - 3A - 8B}{10}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required no. of iterations.

22. Find the largest eigen value and corresponding eigen vector of the matrix

$$\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$$

[2016/Spring, 2018/Spring]

Solution:

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then performing the iterations as

$$AX_0 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 13 \end{bmatrix} = 13\begin{bmatrix} 0.230 \\ 0.692 \\ 1 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.230 \\ 0.692 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.306 \\ 6.074 \\ 12.538 \end{bmatrix} = 12.538\begin{bmatrix} 0.104 \\ 0.484 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.104 \\ 0.484 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.556 \\ 5.28 \\ 11.832 \end{bmatrix} = 11.832\begin{bmatrix} 0.046 \\ 0.446 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.046 \\ 0.446 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.384 \\ 5.03 \\ 11.738 \end{bmatrix} = 11.738\begin{bmatrix} 0.032 \\ 0.428 \\ 1 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.032 \\ 0.428 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.316 \\ 4.952 \\ 11.68 \end{bmatrix} = 11.68\begin{bmatrix} 0.027 \\ 0.423 \\ 1 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.027 \\ 0.423 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.296 \\ 4.927 \\ 11.665 \end{bmatrix} = 11.668\begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.291 \\ 4.919 \\ 11.663 \end{bmatrix} = 11.663\begin{bmatrix} 0.024 \\ 0.421 \\ 1 \end{bmatrix}$$

Hence the required eigen value 11.663.

And the required eigen vector is $\begin{bmatrix} 0.024 \\ 0.421 \\ 1 \end{bmatrix}$.

23. Find the inverse of the matrix by using Gauss Jordan Method. **[2017/Fall]**

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

Solution:

The augmented matrix can be written as

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

Operate $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - R_1$

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -5 & -3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

Operate $R_2 \to R_2 - 3R_3$

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & 1 & -2 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

Operate $R_3 \to R_3 - R_2$

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & 1 & -2 \\ 0 & 0 & 5 & 0 & -1 & 3 \end{array}\right]$$

Operate $R_3 \rightarrow \frac{1}{5} R_3$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & 1 & -2 \\ 0 & 0 & 1 & 0 & -0.2 & 0.6 \end{array}\right]$$

Operate $R_2 \rightarrow R_2 + 5R_3$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -0.2 & 0.6 \end{array}\right]$$

Operate $R_1 \rightarrow R_1 + R_2$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -0.2 & 0.6 \end{array}\right]$$

Operate $R_1 \rightarrow R_1 - 2R_3$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0.4 & -0.2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -0.2 & 0.6 \end{array}\right]$$

Now, inverse of matrix,

$$[A : I] = [I : A]$$

$$[A = A^{-1}]$$

Hence, $A^{-1} = \begin{bmatrix} 0 & 0.4 & -0.2 \\ -1 & 0 & 1 \\ 1 & -0.2 & 0.6 \end{bmatrix}$

24. Solve the following set of equation using LU factorization method.

$$5x - 2y + z = 4$$
$$7x + y - 5z = 8$$
$$3x + 7y + 4z = 10$$

[2017/Spring]

Solution:

Writing the system of equations in matrix form.

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

In LU factorization method, we represent A as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

Solving for unknown values,

$u_{11} = 5$	$u_{12} = -2$	$u_{13} = 1$
$l_{21}u_{11} = 7$ $\therefore l_{21} = 1.4$	$l_{21}u_{12} + u_{22} = 1$ $\therefore u_{22} = 3.8$	$l_{21}u_{13} + u_{23} = -5$ $\therefore u_{23} = -6.4$
$l_{31}u_{11} = 3$ $\therefore l_{31} = 0.6$	$l_{31}u_{12} + l_{32}u_{22} = 7$ $\therefore l_{32} = 2.15$	$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4$ $\therefore u_{33} = 17.16$

Now, substituting obtained coefficient and we have overall system of

$$\underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1.4 & 1 & 0 \\ 0.6 & 2.15 & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} 5 & -2 & 1 \\ 0 & 3.8 & -6.4 \\ 0 & 0 & 17.16 \end{bmatrix}}\underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}}$$

Let, $LUX = B$

$UX = V$

$LV = B$, then,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1.4 & 1 & 0 \\ 0.6 & 2.15 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

Now, performing forward substitution,

$\therefore \quad v_1 = 4$

or, $\quad 1.4v_1 + v_2 = 8$

$\therefore \quad v_2 = 2.4$

or, $\quad 0.6v_1 + 2.15v_2 + v_3 = 10$

$\therefore \quad v_3 = 2.44$

Now,

$UX = V$

$$\begin{bmatrix} 5 & -2 & 1 \\ 0 & 3.8 & -6.4 \\ 0 & 0 & 17.16 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2.4 \\ 2.44 \end{bmatrix}$$

Performing backward substitution,

or, $\quad 17.16z = 2.44$

$\therefore \quad z = 0.142$

or, $\quad 3.8y - 6.4z = 2.4$

$\therefore \quad y = 0.870$

or, $\quad 5x - 2y + z = 4$

$\therefore \quad x = \frac{5.598}{5} = 1.119$

25. Solve the equation by Gauss-Jacobi method.

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

[2017/Spring]

Solution:

Given that;

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

The given equations are in diagonally dominant form.

Now, forming the equations as,

$$x = \frac{1}{20}[17 - y + 2z]$$

$$y = \frac{1}{20}[-18 + z - 3x]$$

$$z = \frac{1}{20}[25 + 3y - 2x]$$

Let $x_0 = 0, y_0 = 0$ and $z_0 = 0$ be initial guesses.

And solving the iterations in tabular form

Iteration	$x = \frac{1}{20}[17 - y + 2z]$	$y = \frac{1}{20}[-18 + z - 3x]$	$z = \frac{1}{20}[25 + 3y - 2x]$
Guess	0	0	0
1	0.85	−0.9	1.25
2	1.02	−0.965	1.03
3	1.00125	−1.0015	1.00325
4	1.0004	−1.000025	0.99965
5	0.99996	−1.00007	0.99995

Hence the required values of x, y and z are 1, −1 and 1 respectively.

NOTE:

Procedure to iterate in programmable calculator

Let, A = x, B = y, C = z

Step 1: Set the following in calculator

$$A : B : C : D = \frac{17 - B + 2C}{20} : E = \frac{-18 + C - 3A}{20} : F = \frac{25 + 3B - 2A}{20}$$

Step 2: Press CALC then

enter the value of A? then press =

enter the value of B? then press =

enter the value of C? then press =

Step 3: Now press = only, again and again to get the values for respective row for each column.

Step 4: Update the values of A?, B? and C? when asked again.

Step 5: Got to step 3.

26. **Determine the largest eigen value and the corresponding eigen vector of the matrix using power method.**

$$A = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$$

[2017/Spring, 2018/Fall]

Solution:

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Now using power method, the iterations are carried out as

$$AX_0 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ -18 \end{bmatrix} = -18\begin{bmatrix} -0.444 \\ 0.222 \\ 1 \end{bmatrix}$$

NOTE: Here $|-18| > 8$ and $|-4|$

Again,

$$AX_1 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} -0.444 \\ 0.222 \\ 1 \end{bmatrix} = \begin{bmatrix} -10.548 \\ 1.104 \\ 7.768 \end{bmatrix} = -10.548\begin{bmatrix} 1 \\ -0.105 \\ -0.736 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.105 \\ -0.736 \end{bmatrix} = -18.948\begin{bmatrix} -0.930 \\ 0.361 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} -0.930 \\ 0.361 \\ 1 \end{bmatrix} = -18.394\begin{bmatrix} 1 \\ -0.415 \\ -0.981 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.415 \\ -0.981 \end{bmatrix} = -19.698\begin{bmatrix} -0.995 \\ 0.462 \\ 1 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} -0.995 \\ 0.462 \\ 1 \end{bmatrix} = -19.773\begin{bmatrix} 1 \\ -0.480 \\ -0.999 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.480 \\ -0.999 \end{bmatrix} = -19.922\begin{bmatrix} -0.997 \\ 0.490 \\ 1 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} -0.997 \\ 0.490 \\ 1 \end{bmatrix} = -19.956\begin{bmatrix} 1 \\ -0.495 \\ -0.999 \end{bmatrix}$$

$$AX_8 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.495 \\ 0.999 \end{bmatrix} = 19.956\begin{bmatrix} 1 \\ -0.495 \\ -0.999 \end{bmatrix} \approx 20\begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix}$$

Hence the dominant eigen value is 20 and eigen vector is $\begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix}$.

27. Find the inverse of matrix using Gauss Jordan method [2018/Fall]

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Solution:

The augmented matrix can be written as,

$$[A:I] = \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 3 & 3 & -3 & : & 0 & 1 & 0 \\ -2 & -4 & -4 & : & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 + 2R_1$,

$$[A:I] = \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 0 & -12 & : & -3 & 1 & 0 \\ 0 & -2 & 2 & : & 2 & 0 & 1 \end{bmatrix}$$

Interchanging R_2 and R_3,

$$[A:I] = \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & -2 & 2 & : & 2 & 0 & 1 \\ 0 & 0 & -12 & : & -3 & 1 & 0 \end{bmatrix}$$

Operate $R_2 \to \frac{R_2}{-2}$ and $R_3 \to \frac{R_3}{-12}$,

$$[A:I] = \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & -1 & : & -1 & 0 & -0.5 \\ 0 & 0 & 1 & : & 0.25 & -0.083 & 0 \end{bmatrix}$$

Operate $R_2 \to R_2 + R_3$,

$$[A:I] = \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & -0.75 & -0.083 & -0.5 \\ 0 & 0 & 1 & : & 0.25 & -0.083 & 0 \end{bmatrix}$$

Operate $R_1 \to R_1 - R_2$,

$$[A:I] = \begin{bmatrix} 1 & 0 & 3 & : & 1.75 & 0.083 & 0.5 \\ 0 & 1 & 0 & : & -0.75 & -0.083 & -0.5 \\ 0 & 0 & 1 & : & 0.25 & -0.083 & 0 \end{bmatrix}$$

Operate $R_1 \to R_1 - 3R_3$,

$$[A:I] = \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0.332 & 0.5 \\ 0 & 1 & 0 & : & -0.075 & -0.083 & -0.5 \\ 0 & 0 & 1 & : & 0.25 & -0.083 & 0 \end{bmatrix}$$

For inverse of matrix,

$[A:I] = [I:A]$

$[A = A^{-1}]$

Hence,

$$A^{-1} = \begin{bmatrix} 1 & 0.332 & 0.5 \\ -0.75 & -0.083 & -0.5 \\ 0.25 & -0.083 & 0 \end{bmatrix}$$

28. Solve the following system of equation

$$6x_1 - 2x_2 + x_3 = 4$$
$$-2x_1 + 7x_2 + 2x_3 = 5$$
$$x_1 + 2x_2 - 5x_3 = -1$$

Using Gauss factorization method. [2018/Fall]

Solution:

Writing the given system of equation in matrix from AX = B

$$\begin{bmatrix} 6 & -2 & 1 \\ -2 & 7 & 2 \\ 1 & 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$$

In Gauss factorization method, we decompose matrix A in the following form,

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{12} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 6 & -2 & 1 \\ -2 & 7 & 2 \\ 1 & 2 & -5 \end{bmatrix}$$

Here, A = LU

Solving for unknown values,

$u_{11} = 6$	$u_{12} = -2$	$u_{13} = 1$
$l_{21}u_{11} = -2$ $\therefore l_{21} = -0.333$	$u_{12}l_{21} + u_{22} = 7$ $\therefore u_{22} = 6.334$	$l_{21}u_{13} + u_{23} = 2$ $\therefore u_{23} = 2.333$
$l_{31}u_{11} = 1$ $\therefore l_{31} = 0.167$	$l_{31}u_{12} + l_{32}u_{22} = 2$ $\therefore l_{32} = 0.368$	$l_{31}u_{13} + l_{32}u_{23} + u_{33} = -5$ $\therefore u_{33} = -6.025$

Now, substituting obtained coefficients, we have overall system as,

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -0.333 & 1 & 0 \\ 0.167 & 0.368 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 6 & -2 & 1 \\ 0 & 6.334 & 2.333 \\ 0 & 0 & -6.025 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}}_{B}$$

Let UX = V,

so, LV = B then

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.333 & 1 & 0 \\ 0.167 & 0.368 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$$

Using forward substitution

$\therefore$ $v_1 = 4$

or, $-0.333v_1 + v_2 = 5$

$\therefore$ $v_2 = 6.332$

or, $0.167v_1 + 0.368v_2 + v_3 = -1$

$\therefore$ $v_3 = -3.998$

Now,

UX = V

$$\begin{bmatrix} 6 & -2 & 1 \\ 0 & 6.334 & 2.333 \\ 0 & 0 & -6.025 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6.332 \\ -3.998 \end{bmatrix}$$

Using backward substitution

or, $-6.025x_3 = -3.998$

∴ $x_3 = 0.663$

or, $6.33x_2 + 2.333x_3 = 6.332$

∴ $x_2 = 0.755$

or, $6x_1 - 2x_2 + x_3 = 4$

∴ $x_1 = 0.807$

29. Solve the following system of equations using factorization method.

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

[2018/Spring]

Solution:

Writing the system of equations in matrix form,

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

In factorization method, we decompose matrix in the following form $A = LU$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Solving for unknown values

$u_{11} = 2$	$u_{12} = 3$	$u_{13} = 1$
$l_{21}u_{11} = 1$ ∴ $l_{21} = 0.5$	$l_{21}u_{12} + u_{22} = 2$ ∴ $u_{22} = 0.5$	$l_{21}u_{13} + u_{23} = 3$ ∴ $u_{23} = 2.5$
$l_{31}u_{11} = 3$ ∴ $l_{31} = 1.5$	$l_{31}u_{12} + l_{32}u_{22} = 1$ ∴ $l_{32} = -7$	$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 5$ ∴ $u_{33} = 21$

Now, substituting obtained coefficient, we have overall system of $LUX = B$ as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & 0.5 & 2.5 \\ 0 & 0 & 21 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$UX = V$

Then, $LV = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & -7 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Now, performing forward substitution,

∴ $v_1 = 9$

→ $0.5v_1 + v_2 = 6$

$v_2 = 1.5$

∴ $1.5v_1 - 7v_2 + v_3 = 8$

→ $v_3 = 5$

∴ Then, $UX = V$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 0.5 & 2.5 \\ 0 & 0 & 21 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1.5 \\ 5 \end{bmatrix}$$

Now, performing backward substitution

or, $12z = 5$

∴ $z = 0.238$

or, $05y + 2.5z = 1.5$

∴ $y = 1.81$

or, $2x + 3y + 1z = 9$

∴ $x = 1.66$

30. Find inverse of the matrix, using Gauss Jordan method

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

[2019/Fall]

Solution:

The augumented matrix can be written as

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

Operate $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 2R_1$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

Operate $R_2 \to \frac{R_2}{2}$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1/2 & -1/2 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

Operate $R_3 \to R_3 + 2R_2$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$$

Operate $R_3 \to \frac{R_3}{-4}$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$$

Operate $R_1 \to R_1 - R_2$

$$[A : I] = \begin{bmatrix} 1 & 0 & 6 & : & 1.5 & -0.5 & 0 \\ 1 & 1 & -3 & : & -0.5 & +0.5 & 0 \\ 0 & 0 & 1 & : & -0.25 & -0.25 & -0.25 \end{bmatrix}$$

Operate $R_1 \to R_1 - 6R_3$

$$[A : I] = \begin{bmatrix} 1 & 0 & 0 & : & 3 & 1 & 1.5 \\ 0 & 1 & -3 & : & -0.5 & +0.5 & 0 \\ 0 & 0 & 1 & : & -0.25 & -0.25 & -0.25 \end{bmatrix}$$

Operate $R_2 \to R_2 + 3R_3$

$$[A : I] = \begin{bmatrix} 1 & 0 & 0 & : & 3 & 1 & 1.5 \\ 0 & 1 & 0 & : & -1.25 & -0.25 & -0.75 \\ 0 & 0 & 1 & : & -0.25 & -0.25 & -0.25 \end{bmatrix}$$

Now, for inversion of matrix

$$[A : I] = [I : A]$$

$$[A = A^{-1}]$$

Hence,

$$A^{-1} = \begin{bmatrix} 3 & 1 & 1.5 \\ -1.25 & -0.25 & -0.75 \\ -0.25 & -0.25 & -0.25 \end{bmatrix}$$

31. Determine the largest eigen value and the corresponding eigen vector of the matrix using power method.

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$$ **[2019/Fall]**

Solution:

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$AX_0 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 14 \end{bmatrix} = 14\begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ 6.5 \end{bmatrix} = 6.5\begin{bmatrix} 0.076 \\ 0.153 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.076 \\ 0.153 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.617 \\ -0.084 \\ 5.915 \end{bmatrix} = 5.915\begin{bmatrix} 0.273 \\ -0.014 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.273 \\ -0.014 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.429 \\ -0.116 \\ 6.482 \end{bmatrix} = 6.482\begin{bmatrix} 0.374 \\ -0.017 \\ 1 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.374 \\ -0.017 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.425 \\ 0.428 \\ 7.193 \end{bmatrix} = 7.193\begin{bmatrix} 0.337 \\ 0.059 \\ 1 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.337 \\ 0.059 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.160 \\ 0.584 \\ 7.199 \end{bmatrix} = 7.199\begin{bmatrix} 0.300 \\ 0.081 \\ 1 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.300 \\ 0.081 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.057 \\ 0.524 \\ 7.043 \end{bmatrix} = 7.043\begin{bmatrix} 0.292 \\ 0.074 \\ 1 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.292 \\ 0.074 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.07 \\ 0.464 \\ 6.974 \end{bmatrix} = 6.974\begin{bmatrix} 0.296 \\ 0.066 \\ 1 \end{bmatrix}$$

$$AX_8 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.296 \\ 0.066 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.098 \\ 0.066 \\ 1 \end{bmatrix} = 6.974\begin{bmatrix} 0.3 \\ 0.064 \\ 1 \end{bmatrix}$$

Hence the required largest eigen value is $6.974 \approx 7$

And corresponding eigen vector is $\begin{bmatrix} 0.3 \\ 0.064 \\ 1 \end{bmatrix}$.

32. Use relaxation method to solve the given systems of equations.

$$20x + y - 2z = 17$$
$$3x + 20y - z = 18$$
$$2x - 3y + 20z = 25$$

[2019/Fall]

Solution:

The diagonal elements of the coefficient matrix dominate the other coefficients in the corresponding row.

i.e., $|20| \geq |1| + |-2|$

$|20| \geq |3| + |-1|$

$|20| \geq |2| + |-3|$

Now, using relaxation method.

The residuals are given by

$$R_x = 17 - 20x - y + 2z$$
$$R_y = 18 - 3x - 20y + z$$
$$R_z = 25 - 2x + 3y - 20z$$

The operation table is

	δR_x	δR_y	δR_z
$\delta x = 1$	−20	−3	−2
$\delta y = 1$	−1	−20	3
$\delta z = 1$	2	1	−20

Now, the relaxation table is shown below.

Taking $z = y = z = 0$ as initial assumption

	R_x	R_y	R_z	
$x = y = z = 0$	17	18	25	
$\delta z = 1$	$17+(1\times2)=19$	$18+(1\times1)=19$	$25-(20\times1)=5$	
$\delta x = 0.5$	$19-(20\times0.5)=9$	$19+(-3\times0.5)=17.5$	$5-(5\times0.5)=4$	i
$\delta y = 0.5$	$9+(-1\times0.5)=8.5$	$17.5-(20\times0.5)=7.5$	$4+3(0.5)=5.5$	ii
$\delta x = 0.5$	$8.5+(-20\times0.5)=-1.5$	$7.5-(3\times0.5)=6$	$5.5-2\times0.5=4.5$	iii
$\delta y = 0.33$	−1.83	−0.6	5.49	
$\delta z = 0.28$	−1.27	−0.32	−0.11	
$\delta x = -0.06$	−0.07	−0.14	0.010	
$\delta y = -0.007$	−0.063	0.00	−0.010	
$\delta x = -0.003$	−0.003	0.009	0.006	

Now,

$$\Sigma\delta x = 0.5 + 0.5 - 0.06 - 0.003 = 0.937$$
$$\Sigma\delta y = 0.5 + 0.33 - 0.007 = 0.823$$
$$\Sigma\delta z = 1 + 0.28 = 1.28$$

Thus, $x = 0.937$, $y = 0.823$ and $z = 1.28$

> **NOTE:**
> In (i) in the table, the largest residual is 25 so to reduce it, we give an increment in δz at $\delta z = 1$ and the resulting residuals are shown in (ii).
> *i.e.*, larger residuals are reduced by assuming suitable increment values. Similarly the steps are carried out.
> Also when increment is done in either δx or δy or δz, use the operation table respectively.

33. Solve the equation by relaxation method

$$9x - y + 2z = 9$$
$$x + 2y - 2z = 15$$
$$2x - 2y - 13z = -17$$

[2020/Fall]

Solution:

$$9x - y + 2z = 9$$
$$x + 2y - 2z = 15$$
$$2x - 2y - 13z = -17$$

Using relaxation method,

The residuals are given by,

$$R_x = 9 - 9x + y - 2z$$
$$R_y = 15 - x - 2y + 2z$$
$$R_z = -17 - 2x + 2y + 13z$$

The operation table is

	δR_x	δR_y	δR_z
$\delta x = 1$	−9	−1	−2
$\delta y = 1$	1	−2	2
$\delta z = 1$	−2	2	13

Taking initial guess of $x = y = z = 0$.

Now, the relaxation table is,

	R_x	R_y	R_z
0	9	15	−17
$\delta z = 1$	7	17	−4
$\delta y = 8$	15	1	12
$\delta z = 2$	−3	−1	8
$\delta z = -0.615$	−1.77	−2.23	0.005
$\delta y = -1.115$	−2.885	0	−0.225
$\delta x = -0.32$	−0.005	0.32	0.415
$\delta z = -0.031$	0.057	0.25	0.012
$\delta y = 0.125$	0.182	0	0.262

Now,

$$\Sigma\delta x = 2 - 0.32 = 1.68$$
$$\Sigma\delta y = 8 - 1.115 + 0.125 = 7.01$$
$$\Sigma\delta z = 1 - 0.615 - 0.031 = 0.354$$

Thus, $x = 1.68$, $y = 7.01$ and $z = 0.354$

34. Determine the largest eigen value and the corresponding eigen vector of the matrix using the power method.

$$A = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}$$

[2020/Fall]

Solution:

$$A = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}$$

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then using power method, performing the iterations as,

$$AX_0 = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 13 \end{bmatrix} = 13\begin{bmatrix} 0.692 \\ 1 \\ 1 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0.692 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.692 \\ 11.768 \\ 11.768 \end{bmatrix} = 11.768\begin{bmatrix} 0.738 \\ 1 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0.738 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.738 \\ 11.952 \\ 11.952 \end{bmatrix} = 11.952\begin{bmatrix} 0.731 \\ 1 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0.731 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.731 \\ 11.924 \\ 11.924 \end{bmatrix} = 11.924\begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.732 \\ 11.928 \\ 11.928 \end{bmatrix} = 11.928\begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.732 \\ 11.928 \\ 11.928 \end{bmatrix} = 11.928\begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix}$$

Hence the required eigen value is 11.928.

and the eigen vector is $\begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix}$

35. Solve the following set of equations by using LU decomposition method.

$$3x + 2y + 7z = 32$$
$$2x + 3y + z = 40$$
$$3x + 4y + z = 56$$

[2020/Fall]

Solution:

Writing the system of equations in matrix form AX = B

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 32 \\ 40 \\ 56 \end{bmatrix}$$

In LU factorization method, we represent A as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

Solving for unknown values

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & u_{12}l_{21} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

Solving for unknown values,

$u_{11} = 3$	$u_{12} = 2$	$u_{13} = 7$
$l_{21}u_{11} = 2$ $\therefore l_{21} = 0.667$	$l_{21}u_{12} + u_{22} = 3$ $\therefore u_{22} = 1.666$	$l_{21}u_{13} + u_{23} = 1$ $\therefore u_{23} = -3.669$
$l_{31}u_{11} = 3$ $\therefore l_{31} = 1$	$l_{31}u_{12} + l_{32}u_{22} = 4$ $\therefore l_{32} = 1.2$	$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$ $\therefore u_{33} = -1.597$

Substituting the values

$$\underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0.667 & 1 & 0 \\ 1 & 1.2 & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} 3 & 2 & 7 \\ 0 & 1.666 & -3.669 \\ 0 & 0 & -1.597 \end{bmatrix}}\underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 32 \\ 40 \\ 56 \end{bmatrix}}$$

Here, $LUX = B$

Let $UX = V$

so, $LV = B$ then,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.667 & 1 & 0 \\ 1 & 1.2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 32 \\ 40 \\ 56 \end{bmatrix}$$

Performing forward substitution

$\therefore$ $v_1 = 32$

or, $0.667\,v_1 + v_2 = 40$

$\therefore$ $v_2 = 18.656$

or, $v_1 + 1.2\,v_2 + v_3 = 56$

$\therefore$ $v_3 = 1.612$

Now, $UX = V$

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 1.666 & -3.669 \\ 0 & 0 & -1.597 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 32 \\ 18.656 \\ 1.612 \end{bmatrix}$$

Again, performing backward substitution

or, $-1.597z = 1.612$

$\therefore$ $z = -1.009 \approx -1$

or, $1.666y - 3.669z = 18.656$

$\therefore$ $y = \dfrac{14.953}{1.666} = 8.975 \approx 9$

or, $3x + 2y + 7z = 32$

$\therefore$ $x = 7.037 \approx 7$

36. Write short notes on: Relaxation method. **[2014/Fall]**

Solution: See the topic 4.6.3.

37. Write short notes on Ill conditioned system. **[2014/Spring, 2016/Spring, 2019/Spring]**

Solution: See the topic 4.5.

38. Write short notes on: Gauss Seidel method of iteration. **[2017/Fall]**

Solution: See the topic 4.6.2.

39. Write a program in any high level language C or C++ to solve a system of linear equation, using gauss elimination method. **[2016/Spring]**

Solution: See the "Appendix", program number 11.

40. Write a program to solve a system of linear equations by Gauss Seidal method. **[2018/Spring]**

Solution: See the "Appendix", program number 16.

ADDITIONAL QUESTION SOLUTION

1. **Solve the following system of equation using LU factorization method.**

$5x_1 + 2x_2 + 3x_3 = 31$

$3x_1 + 3x_2 + 2x_3 = 25$

$x_1 + 2x_2 + 4x_3 = 25$

Solution:

Writing the system of equations in matrix form AX = B.

$$\begin{bmatrix} 5 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 31 \\ 25 \\ 25 \end{bmatrix}$$

In LU factorization method, we represent A as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 5 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & u_{12}l_{21} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 5 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

Solving for unknown values,

$u_{11} = 5$	$u_{12} = 2$	$u_{13} = 3$
$l_{21}u_{11} = 3$ $\therefore l_{21} = 0.6$	$l_{21}u_{12} + u_{22} = 3$ $\therefore u_{22} = 1.8$	$l_{21}u_{13} + u_{23} = 2$ $\therefore u_{23} = 0.2$
$l_{31}u_{11} = 1$ $\therefore l_{31} = 0.2$	$l_{31}u_{12} + l_{32}u_{22} = 2$ $\therefore l_{32} = 0.88$	$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4$ $\therefore u_{33} = 3.224$

Now substituting obtained coefficient and we have overall system of

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 1 & 0 \\ 0.2 & 0.88 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 5 & 2 & 3 \\ 0 & 1.8 & 0.2 \\ 0 & 0 & 3.224 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 31 \\ 25 \\ 25 \end{bmatrix}}_{B}$$

Here, LUX = B

Let UX = V

so, LV = B then,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 1 & 0 \\ 0.2 & 0.88 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 31 \\ 25 \\ 25 \end{bmatrix}$$

Now, performing forward substitution

$\therefore$ $v_1 = 31$

or, $0.6v_1 + v_2 = 25$

$\therefore$ $v_2 = 6.4$

or, $0.2v_1 + 0.88v_2 + v_3 = 25$

$\therefore$ $v_3 = 13.168$

Then, UX = V becomes

$$\begin{bmatrix} 5 & 2 & 3 \\ 0 & 1.8 & 0.2 \\ 0 & 0 & 3.224 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 31 \\ 6.4 \\ 13.168 \end{bmatrix}$$

Performing backward substitution,

$3.224x_3 = 13.168$

or, $x_3 = 4.084$

$\therefore$ $1.8x_2 + 0.2x_3 = 6.4$

or, $x_2 = 3.101$

$\therefore$ $5x_1 + 2x_2 + 3x_3 = 31$

or, $x_1 = 2.509$

$\therefore$

2. **Apply Gauss Seidal Iterative method to solve the linear equations correct to 2 decimal places.**

$10x + y - z = 11.19$

$x + 10y + z = 28.08$

$-x + y + 10z = 35.61$

Solution:

Here, the provided equations are in diagonally dominant form, so forming the equations as,

$$x = \frac{11.19 - y + z}{10}$$

$$y = \frac{28.08 - x - z}{10}$$

$$z = \frac{35.61 + x - y}{10}$$

Let the initial guess be 0 for x, y, and z.

Solving the iterations in tabular form,

Iteration	$x = \frac{11.19 - y + z}{10}$	$y = \frac{28.08 - x - z}{10}$	$z = \frac{35.61 + x - y}{10}$
Guess	0	0	0
1	1.119	2.6961	3.4032
2	1.1897	2.3487	3.4451
3	1.2286	2.3406	3.4498
4	1.2299	2.3400	3.4499

Here, the values of x, y and z are correct upto 2 decimal places.

Hence, the required values are;

$x = 1.2299 \approx 1.23$, $y = 2.34$, $z = 3.4499 \approx 3.45$

3. **Find inverse of the matrix, using Gauss Jordan method**

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

Solution:

The augumented matrix can be written as

$$[A:I]=\left[\begin{array}{ccc:ccc} 3 & 1 & 2 & 1 & 0 & 0\\ 1 & 2 & 3 & 0 & 1 & 0\\ 2 & 3 & 5 & 0 & 0 & 1\end{array}\right]$$

Operate R_1 and R_2

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 0 & 1 & 0\\ 3 & 1 & 2 & 1 & 0 & 0\\ 2 & 3 & 5 & 0 & 0 & 1\end{array}\right]$$

Operate $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 0 & 1 & 0\\ 0 & -5 & -7 & 1 & -3 & 0\\ 0 & -1 & -1 & 0 & -2 & 1\end{array}\right]$$

Operate $R_3 \to R_3 - \frac{1}{5}R_2$

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 0 & 1 & 0\\ 0 & -5 & -7 & 1 & -3 & 0\\ 0 & 0 & 2/5 & -1/5 & -7/5 & 1\end{array}\right]$$

Operate $R_3 \to \frac{5}{2}R_3$

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 0 & 1 & 0\\ 0 & -5 & -7 & 1 & -3 & 0\\ 0 & 0 & 1 & -1/2 & -7/2 & 5/2\end{array}\right]$$

Operate $R_2 \to \frac{R_2}{-5}$

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 0 & 1 & 0\\ 0 & 1 & 7/5 & -1/5 & 3/5 & 0\\ 0 & 0 & 1 & -1/2 & -7/2 & 5/2\end{array}\right]$$

Operate $R_2 \to R_2 - \frac{7}{5}R_3$

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 0 & 1 & 0\\ 0 & 1 & 0 & 1/2 & 11/2 & -7/2\\ 0 & 0 & 1 & -1/2 & -7/2 & 5/2\end{array}\right]$$

Operate $R_1 \to R_1 - 2R_2$

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 0 & 3 & -1 & -10 & 7\\ 0 & 1 & 0 & 1/2 & 11/2 & -7/2\\ 0 & 0 & 1 & -1/2 & -7/2 & 5/2\end{array}\right]$$

Operate $R_1 \to R_1 - 3R_3$

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2\\ 0 & 1 & 0 & 1/2 & 11/2 & -7/2\\ 0 & 0 & 1 & -1/2 & -7/2 & 5/2\end{array}\right]$$

For inversion of matrix

$[A:I]=[I:A]$

$[A=A^{-1}]$

Hence,

$$A^{-1}=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{11}{2} & \frac{-7}{2}\\ \frac{-1}{2} & \frac{-7}{2} & \frac{5}{2}\end{bmatrix}$$

4. **Find the largest eigen value and the corresponding eigen vector of the following matrix using the power method with an accuracy of 2 decimal points,**

$$\begin{bmatrix} 1 & 2 & 1\\ 2 & 1 & 2\\ 1 & 2 & -1\end{bmatrix}.$$

Solution:

Let the initial vector be $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

Now, using power method, performing the iterations as

$$AX_0=\begin{bmatrix} 1 & 2 & 1\\ 2 & 1 & 2\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}4\\5\\2\end{bmatrix}=5\begin{bmatrix}0.8\\1\\0.4\end{bmatrix}$$

Again,

$$AX_1=\begin{bmatrix} 1 & 2 & 1\\ 2 & 1 & 2\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}0.8\\1\\0.4\end{bmatrix}=\begin{bmatrix}3.2\\3.4\\2.4\end{bmatrix}=3.4\begin{bmatrix}0.9412\\1\\0.7059\end{bmatrix}$$

$$AX_2=\begin{bmatrix} 1 & 2 & 1\\ 2 & 1 & 2\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}0.9412\\1\\0.7059\end{bmatrix}=\begin{bmatrix}3.6471\\4.2942\\2.2353\end{bmatrix}=4.2942\begin{bmatrix}0.8493\\1\\0.5205\end{bmatrix}$$

$$AX_3=\begin{bmatrix} 1 & 2 & 1\\ 2 & 1 & 2\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}0.8493\\1\\0.5205\end{bmatrix}=\begin{bmatrix}3.3698\\3.7396\\2.3288\end{bmatrix}=3.7396\begin{bmatrix}0.9011\\1\\0.6227\end{bmatrix}$$

$$AX_4=\begin{bmatrix} 1 & 2 & 1\\ 2 & 1 & 2\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}0.9011\\1\\0.6227\end{bmatrix}=\begin{bmatrix}3.5238\\4.0476\\2.2784\end{bmatrix}=4.0476\begin{bmatrix}0.8706\\1\\0.5629\end{bmatrix}$$

$$AX_5=\begin{bmatrix} 1 & 2 & 1\\ 2 & 1 & 2\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}0.8706\\1\\0.5629\end{bmatrix}=\begin{bmatrix}3.4335\\3.8670\\2.3077\end{bmatrix}=3.8670\begin{bmatrix}0.8879\\1\\0.5968\end{bmatrix}$$

$$AX_6=\begin{bmatrix} 1 & 2 & 1\\ 2 & 1 & 2\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}0.8879\\1\\0.5968\end{bmatrix}=\begin{bmatrix}3.4847\\3.9694\\2.2911\end{bmatrix}=3.9694\begin{bmatrix}0.8779\\1\\0.5772\end{bmatrix}$$

$$AX_7=\begin{bmatrix} 1 & 2 & 1\\ 2 & 1 & 2\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}0.8779\\1\\0.5772\end{bmatrix}=\begin{bmatrix}3.4551\\3.9102\\2.3007\end{bmatrix}=3.9102\begin{bmatrix}0.8836\\1\\0.5884\end{bmatrix}$$

$$AX_8 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8836 \\ 1 \\ 0.5884 \end{bmatrix} = \begin{bmatrix} 3.4720 \\ 3.9440 \\ 2.2952 \end{bmatrix} = 3.9440\begin{bmatrix} 0.8803 \\ 1 \\ 0.5819 \end{bmatrix}$$

$$AX_9 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8803 \\ 1 \\ 0.5819 \end{bmatrix} = \begin{bmatrix} 3.4622 \\ 3.9244 \\ 2.2984 \end{bmatrix} = 3.9244\begin{bmatrix} 0.8822 \\ 1 \\ 0.5857 \end{bmatrix}$$

$$AX_{10} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8822 \\ 1 \\ 0.5857 \end{bmatrix} = \begin{bmatrix} 3.4679 \\ 3.9358 \\ 2.2965 \end{bmatrix} = 3.9358\begin{bmatrix} 0.8811 \\ 1 \\ 0.5835 \end{bmatrix}$$

$$AX_{11} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8811 \\ 1 \\ 0.5835 \end{bmatrix} = \begin{bmatrix} 3.4646 \\ 3.9292 \\ 2.2976 \end{bmatrix} = 3.9292\begin{bmatrix} 0.8818 \\ 1 \\ 0.5848 \end{bmatrix}$$

$$AX_{12} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8818 \\ 1 \\ 0.5848 \end{bmatrix} = \begin{bmatrix} 3.4666 \\ 3.9332 \\ 2.2970 \end{bmatrix} = 3.9332\begin{bmatrix} 0.8814 \\ 1 \\ 0.5840 \end{bmatrix}$$

$$AX_{13} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8814 \\ 1 \\ 0.5840 \end{bmatrix} = \begin{bmatrix} 3.4654 \\ 3.9308 \\ 2.2974 \end{bmatrix} = 3.9308\begin{bmatrix} 0.8816 \\ 1 \\ 0.5845 \end{bmatrix}$$

Here the values are correct upto 2 decimal places.

Hence the required eigen values is 3.9308.

And the required eigen vector is $\begin{bmatrix} 0.8816 \\ 1 \\ 0.5845 \end{bmatrix}$.

5. Solve the following linear equations using Guass elimination method using partial pivoting.

$$2x + 3y + 2z = 2$$
$$10x + 3y + 4z = 16$$
$$3x + 6y + z = 6$$

Solution:

Writing the given system of equations in matrix form

$$\begin{bmatrix} 2 & 3 & 2 \\ 10 & 3 & 4 \\ 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 16 \\ 6 \end{bmatrix}$$

Interchanging R_1 and R_2 but not x and y as partial pivoting.

$$\begin{bmatrix} 10 & 3 & 4 \\ 2 & 3 & 2 \\ 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \\ 6 \end{bmatrix}$$

Operate $R_2 \to R_2 - \frac{2}{10}R_1$ and $R_3 \to R_3 - \frac{3}{10}R_1$

$$\begin{bmatrix} 10 & 3 & 4 \\ 0 & 2.4 & 1.2 \\ 0 & 5.1 & -0.2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -1.2 \\ 1.2 \end{bmatrix}$$

Interchanging R_2 and R_3 but not y and z variable

$$\begin{bmatrix} 10 & 3 & 4 \\ 0 & 5.1 & -0.2 \\ 0 & 2.4 & 1.2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 1.2 \\ -1.2 \end{bmatrix}$$

Operate $R_3 \to R_3 - \frac{2.4}{5.1}R_2$

$$\begin{bmatrix} 10 & 3 & 4 \\ 0 & 5.1 & -0.2 \\ 0 & 0 & 1.2941 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 1.2 \\ -1.7647 \end{bmatrix}$$

Now performing backward substitution,

or, $1.2941z = -1.7647$

$\therefore$ $z = -1.3637$

or, $5.1y - 0.2z = 1.2$

$\therefore$ $y = 0.1818$

or, $10x + 3y + 4z = 16$

$\therefore$ $x = 2.0909$

6. Solve the following system of linear algebraic equations using the Gauss elimination method.

$$2x_1 + 3x_2 + 2x_3 + 5x_4 = 11$$
$$4x_1 + 2x_2 + 2x_3 + 4x_4 = 11$$
$$4x_1 + x_2 + 4x_3 + 5x_4 = 11$$
$$5x_1 - 5x_2 + 3x_3 + x_4 = 11$$

Solution:

Writing the given system of equations in matrix form

$$\begin{bmatrix} 2 & 3 & 2 & 5 \\ 4 & 2 & 2 & 4 \\ 4 & 1 & 4 & 5 \\ 5 & -5 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11 \\ 11 \\ 11 \\ 11 \end{bmatrix}$$

Operate $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - \frac{5}{2}R_1$

$$\begin{bmatrix} 2 & 3 & 2 & 5 \\ 0 & -4 & -2 & -6 \\ 0 & -5 & 0 & -5 \\ 0 & -12.5 & -2 & -11.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11 \\ -11 \\ -11 \\ -16.5 \end{bmatrix}$$

Operate $R_3 \to R_3 - \frac{5}{4}R_2$ and $R_4 \to R_4 - \frac{12.5}{4}R_2$

$$\begin{bmatrix} 2 & 3 & 2 & 5 \\ 0 & -4 & -2 & -6 \\ 0 & 0 & 2.5 & 2.5 \\ 0 & 0 & 4.25 & 7.25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11 \\ -11 \\ 2.75 \\ 17.875 \end{bmatrix}$$

Operate $R_4 \to R_4 - \dfrac{4.25}{2.5} R_3$

$$\begin{bmatrix} 2 & 3 & 2 & 5 \\ 0 & -4 & -2 & -6 \\ 0 & 0 & 2.5 & 2.5 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11 \\ -11 \\ 2.75 \\ 13.2 \end{bmatrix}$$

Now, performing backward substitution,

or, $3x_4 = 13.2$

$\therefore$ $x_4 = 4.4$

or, $2.5x_3 + 2.5x_4 = 2.75$

$\therefore$ $x_3 = -3.3$

or, $-4x_2 - 2x_3 - 6x_4 = -11$

$\therefore$ $x_2 = -2.2$

or, $2x_1 + 3x_2 + 2x_3 + 5x_4 = 11$

$\therefore$ $x_1 = 1.1$

Hence, the required values of the equation are;

$x_1 = 1.1, x_2 = -2.2, x_3 = -3.3, x_4 = 4.4$

7. Solve the following system of linear equations using the Gauss Seidal iteration method.

$$x_1 + 3x_2 - x_3 + 7x_4 = 19$$
$$2x_1 + 8x_2 + x_3 - x_4 = 17$$
$$3x_1 + x_2 + 9x_3 - x_4 = 15$$
$$9x_1 - x_2 - x_3 + 2x_4 = 13$$

Solution:

Arranging the given linear equations in diagonally dominant form

$$9x_1 - x_2 - x_3 + 2x_4 = 13$$
$$2x_1 + 8x_2 + x_3 - x_4 = 17$$
$$3x_1 + x_2 + 9x_3 - x_4 = 15$$
$$x_1 + 3x_2 - x_3 + 7x_4 = 19$$

Now, forming the equations as

$$x_1 = \frac{13 + x_2 + x_3 - 2x_4}{9}$$

$$x_2 = \frac{17 - 2x_1 - x_3 + x_4}{8}$$

$$x_3 = \frac{15 - 3x_1 - x_2 + x_4}{9}$$

$$x_4 = \frac{19 - x_1 - 3x_2 + x_3}{7}$$

Let the initial guess be 0 for x_1, x_2, x_3 and x_4.

Solving the iterations in tabulator form

Iteration	$x_1 = \frac{13+x_2+x_3-2x_4}{9}$	$x_2 = \frac{17-2x_1-x_3+x_4}{8}$	$x_3 = \frac{15-3x_1-x_2+x_4}{9}$	$x_4 = \frac{19-x_1-3x_2+x_3}{7}$
Guess	0	0	0	0
1	1.4444	1.7639	0.9892	1.8933
2	1.3296	1.9056	1.2221	1.8822
3	1.3737	1.8641	1.2108	1.8921
4	1.3656	1.8688	1.2141	1.8917
5	1.3666	1.8681	1.2138	1.8918
6	1.3665	1.8681	1.2138	1.8919

Here, the values of x_1, x_2, x_3 and x_4 are correct upto 3 decimal places.

So, the approximated values of $x_1 = 1.3665$, $x_2 = 1.8681$, $x_3 = 1.2138$ and $x_4 = 1.8919$.

8. Find the largest eigen value and the corresponding vector of the following matrix using the power method.

$$\begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}$$

Solution:

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Now, using power method, performing the iterations as

$$AX_0 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 14 \end{bmatrix} = 14\begin{bmatrix} 0.5714 \\ 0.4286 \\ 1 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.5714 \\ 0.4286 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.2858 \\ 4.9998 \\ 11.8572 \end{bmatrix} = 11.8572\begin{bmatrix} 0.3615 \\ 0.4217 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.3615 \\ 0.4217 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.8315 \\ 3.9641 \\ 11.6266 \end{bmatrix} = 11.6266\begin{bmatrix} 0.3295 \\ 0.3410 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.3295 \\ 0.3410 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.3640 \\ 3.9655 \\ 11.3525 \end{bmatrix} = 11.3525\begin{bmatrix} 0.2963 \\ 0.3493 \\ 1 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2963 \\ 0.3493 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.3391 \\ 3.7829 \\ 11.3442 \end{bmatrix} = 11.3442\begin{bmatrix} 0.2943 \\ 0.3335 \\ 1 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2943 \\ 0.3335 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2561 \\ 3.8045 \\ 11.2948 \end{bmatrix} = 11.2948\begin{bmatrix} 0.2883 \\ 0.3368 \\ 1 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2883 \\ 0.3368 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2606 \\ 3.7679 \\ 11.2987 \end{bmatrix} = 11.2987\begin{bmatrix} 0.2886 \\ 0.3335 \\ 1 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2886 \\ 0.3335 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2447 \\ 3.7760 \\ 11.2891 \end{bmatrix} = 11.2891\begin{bmatrix} 0.2874 \\ 0.3345 \\ 1 \end{bmatrix}$$

$$AX_8 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2874 \\ 0.3345 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2473 \\ 3.7680 \\ 11.2909 \end{bmatrix} = 11.2909\begin{bmatrix} 0.2876 \\ 0.3337 \\ 1 \end{bmatrix}$$

$$AX_9 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2876 \\ 0.3337 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2437 \\ 3.7706 \\ 11.2887 \end{bmatrix} = 11.2887\begin{bmatrix} 0.2873 \\ 0.3340 \\ 1 \end{bmatrix}$$

$$AX_{10} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2873 \\ 0.3340 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2446 \\ 3.7685 \\ 11.2893 \end{bmatrix} = 11.2893\begin{bmatrix} 0.2874 \\ 0.3338 \\ 1 \end{bmatrix}$$

$$AX_{11} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2874 \\ 0.3338 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2438 \\ 3.7694 \\ 11.2888 \end{bmatrix} = 11.2888\begin{bmatrix} 0.2873 \\ 0.3339 \\ 1 \end{bmatrix}$$

Hence, the required eivgen value is $11.2888 \approx 11.29$.

And the corresponding vector is $\begin{bmatrix} 0.2873 \\ 0.3339 \\ 1 \end{bmatrix}$.

9. Solve the following set of linear equations using LU factorization method.

$$x - 3y + 10z = 3$$
$$-x + 4y + 2z = 20$$
$$5x + 2y + z = -12$$

Solution:

Writing the given set of equations in matrix form AX = B

$$\begin{bmatrix} 1 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

In LU factorization method, we represent A as,

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

Solving for unknown values,

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & u_{12}l_{21} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

$u_{11} = 1$	$u_{12} = -3$	$u_{13} = 10$
$l_{21}u_{11} = -1$ $\therefore l_{21} = -1$	$l_{21}u_{12} + u_{22} = 4$ $\therefore u_{22} = 1$	$l_{21}u_{13} + u_{23} = 2$ $\therefore u_{23} = 12$
$l_{31}u_{11} = 5$ $\therefore l_{31} = 5$	$l_{31}u_{12} + l_{32}u_{22} = 2$ $\therefore l_{32} = 17$	$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$ $\therefore u_{33} = -253$

Substituting the values

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 17 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -3 & 10 \\ 0 & 1 & 12 \\ 0 & 0 & -253 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}}_{B}$$

Here, $LUX = B$

Let, $UX = V$

so, $LV = B$ then,

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 17 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

Performing forward substitution,

$\therefore \quad v_1 = 3.$

or, $\quad -v_1 + v_2 = 20$

$\therefore \quad v_2 = 23$

or, $\quad 5v_1 + 17v_2 + v_3 = -12$

$\therefore \quad V_3 = -418$

Then, $UX = V$ becomes

$$\begin{bmatrix} 1 & -3 & 10 \\ 0 & 1 & 12 \\ 0 & 0 & -253 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 23 \\ -418 \end{bmatrix}$$

Performing backward substitution,

or, $\quad -253z = -418$

$\therefore \quad z = 1.6522$

or, $\quad y + 12z = 23$

$\therefore \quad y = 3.1736$

or, $\quad x - 3y + 10z = 3$

$\therefore \quad x = -4.0012$

10. Find the inverse of the matrix, using Gauss Jordan elimination method

$$A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & 5 & 3 \end{bmatrix}$$

Solution:

The augumented matrix can be written as

$$[A : I] = \left[\begin{array}{ccc|ccc} 4 & 3 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

Interchanging R_1 and R_2

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 4 & 3 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

Operate $R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 - 3R_1$

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -5 & 1 & -4 & 0 \\ 0 & 2 & 0 & 0 & -3 & 1 \end{array}\right]$$

Operate $R_3 \to R_3 + 2R_2$

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -5 & 1 & -4 & 0 \\ 0 & 0 & -10 & 2 & -11 & 1 \end{array}\right]$$

Operate $R_3 \to \frac{R_3}{-10}$

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -5 & 1 & -4 & 0 \\ 0 & 0 & -1 & -1/5 & 11/10 & -1/10 \end{array}\right]$$

Operate $R_2 \to \frac{R_2}{-1}$

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 5 & -1 & 4 & 0 \\ 0 & 0 & 1 & -1/5 & 11/10 & -1/10 \end{array}\right]$$

Operate $R_1 \to R_1 - R_2$

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 0 & -4 & 1 & -3 & 0 \\ 0 & 1 & 5 & -1 & 4 & 0 \\ 0 & 0 & 1 & -1/5 & 11/10 & -1/10 \end{array}\right]$$

Operate $R_2 \to R_2 - 5R_3$

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 0 & -4 & 1 & -3 & 0 \\ 0 & 1 & 0 & 0 & -1.5 & 0.5 \\ 0 & 0 & 1 & -1/5 & 1.1 & -0.1 \end{array}\right]$$

Operate $R_1 \to R_1 + 4R_3$

$$[A:I]=\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0.2 & 1.4 & -0.4 \\ 0 & 1 & 0 & 0 & -1.5 & 0.5 \\ 0 & 0 & 1 & -0.2 & 1.1 & -0.1 \end{array}\right]$$

For inversion of matrix

$$[A:I]=[I:A]$$

$$[A=A^{-1}]$$

Hence,

$$A^{-1}=\begin{bmatrix} 0.2 & 1.4 & -0.4 \\ 0 & -1.5 & 0.5 \\ -0.2 & 1.1 & -0.1 \end{bmatrix}$$