Chapter

1

SOLUTION OF NON-LINEAR EQUATIONS

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1.1 INTRODUCTION

Mathematical models for a wide variety of problems in engineering can be formulated into equations of the form,

$$f(x) = 0 \qquad \text{..... (1)}$$

where, x and f(x) may be real, complex or vector quantities. The solution process often involves finding the values are called the roots of the equation. Since the function f(x) becomes zero at these values, they are also known as the zeros of the function f(x). Equation (1) may belong to one of the following types of equations:

a) Algebraic equations.

b) Polynomial equations.

c) Transcendental equations.

Any function of one variable which does not graph as a straight line in two dimensions or any function of two variables which does not graph as a plane in three dimensions, can be said to be non-linear.

Consider the function, $y = f(x)$. f(x) is a linear function, if the dependent variable y changes in direct proportion to the change in independent variable x. For example, $y = 6x + 10$ is a linear function.

On the other hand, f(x) is said to be non-linear if the response of the dependent variable y is not in direct or exact proportion to the changes in the independent variable x. For example, $y = 2x^2 + 3$ is a non-linear function.

a) Algebraic equations

An equation of type y = f(x) is said to be algebraic if it can be expressed in the form,

$$f_n y_n + f_{n-1} y_{n-1} + \cdots\cdots + f_1 y_1 + f_0 = 0 \qquad \text{..... (1)}$$

where, f_i is an i^{th} order polynomial in x. Equation (1) can be thought of as having a general form

$$f(x, y) = 0 \qquad \text{..... (2)}$$

This implies that equation (2) describes a dependence between the variables x and y.

Some examples of algebraic equations are,

i) $4x - 6y - 24 = 0$ (linear)

ii) $3x + 4xy - 30 = 0$ (non-linear)

These equations have an infinite number of pairs of values of x and y which satisfy them.

b) Polynomial Equations

Polynomial equations are a simple class of algebraic equations that are represented as follows;

$$a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots + a_1 x + a_0 = 0 \qquad \text{..... (1)}$$

This is called nth degree polynomial and has n roots. The roots may be

i) Real and different

ii) Real and repeated

iii) Complex numbers

Since complex roots appear in pairs, if n is odd, then the polynomial has atleast one real root. Some examples of polynomial equations are

i) $6x^5 - x^3 + 2x^2 = 0$

ii) $3x^2 - 4x + 8 = 0$

c) Transcendental equations

A non-algebraic equation is called a transcendental equation. These include trigonometric, exponential and logarithmic functions. Some example of transcendental equations are,

i) $3 \cos x - x = 0$

ii) $\log x - 2 = 0$

iii) $e^x \sin x - \frac{1}{4}x = 0$

A transcendental equation may have a finite or an infinite number of real roots or may not have real root at all.

1.2 ACCURACY OF NUMBERS

a) Approximate Numbers

There are two types of numbers *i.e.*, exact and approximate. Exact numbers are 1, 2, 4, 9, 13, $\frac{8}{3}$, 7.78, 14.20, ... etc. But there are numbers such as $\frac{5}{3}$ (=1.6666666......), $\sqrt{5}$ (= 2.23606......) and π (=3.141592.....) which cannot be expressed by a finite number of digits. These may be approximated by numbers 1.6666, 2.2360 and 3.1415 respectively. Such numbers which represent the given numbers to a certain degree of accuracy are called approximate numbers.

b) Significant Figure

The digits used to express a number are called significant digits (figures). Thus each of the numbers 3467, 4.689, 0.3692 contains four significant figures while the numbers 0.00468, 0.000236 contain only three significant figure since zero only help to fix the position of the decimal point. Similarly the numbers 65000 and 8400.00 have two significant figures only.

c) Rounding Off

There are numbers with large number of digits, for example: $\frac{27}{7}$ = 3.857142857. In practice, it is desirable to limit such numbers to a manageable number off digits such as 3.85 or 3.857. This process of dropping unwanted digits is called rounding off.

Rules to Round off a Number to n Significant Figures

i) Discard all digits to the right of the n^{th} digit.

ii) If this discarded number is,

  a) Less than half a unit in the n^{th} place, leave the n^{th} digit unchanged.

  b) Greater than half a unit in the n^{th} place, increase the n^{th} digit by unity.

  c) Exactly half a unit in the n^{th} place, increase the n^{th} digit by unity if it is odd otherwise leave it unchanged.

For instance, the following numbers rounded off to three significant figures are;

6.893 to 6.89
3.678 to 3.68
11.765 to 11.8
6.8254 to 6.82
84767 to 84800

Also the numbers 6.284359, 9.864651, 12.464762 rounded off to four places of decimal are 6.2844, 9.8646 and 12.4648 respectively.

> **NOTE:**
> The numbers thus rounded off to n-significant figure (or n decimal places) are said to be correct to n significant figures (or n decimal places).

1.3 ERRORS IN NUMERICAL CALCULATIONS

Approximation and errors are an integral part of human life. They are unavoidable. Errors come in a variety of forms and sizes; some are avoidable, some are not. For example, data conversion and round off errors cannot be avoided but a human error can be eliminated. Although certain errors cannot be eliminated completely, we must at least know the bounds of these errors to make use of our final solution. It is therefore essential to know how errors arise, how they grow during the numerical process and how they affect the accuracy of a solution.

In any numerical computation, we come across the following types of errors.

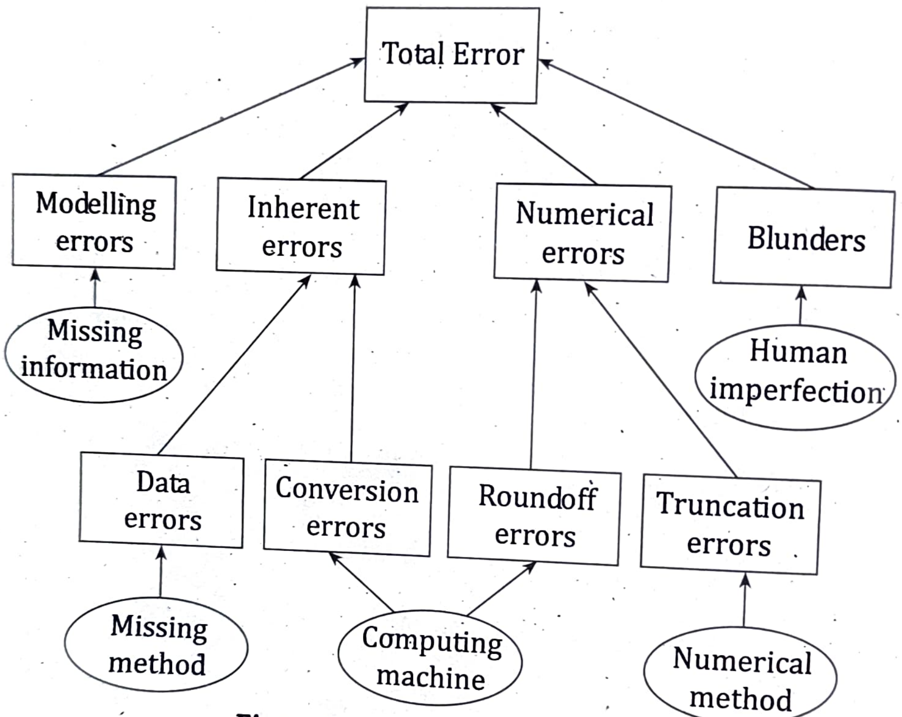


Figure 1.1: Taxonomy of errors

NOTE:

a) Inherent Errors

Errors which are already present in the statement of a problem before its solution, are called inherent errors. Such errors arise either due to the given data being approximate or due to the limitations of mathematical tables, calculators or the digital computer. Inherent errors can be minimized by taking better data or by using high precision computing aids.

b) Rounding Errors

Rounding errors arise from the process of rounding off the numbers during the computation. Such errors are unavoidable in most of the calculations due to the limitations of the computing aids. Rounding errors can, however, be reduced.

i) By changing the calculation procedure so as to avoid subtraction of nearly equal numbers or division by a small number.

OR,

ii) By retaining at least one more significant figure at each step than that given in the data and rounding off at the last step.

c) Truncation errors

Truncation errors are caused by using approximate result or on replacing an infinite process by a finite one. If we are using a decimal computer having a fixed word length of 4 digits, rounding off of 13.658 gives 13.66 whereas truncation gives 13.65. Truncation error is a type of algorithm error.

d) Absolute, relative and percentage errors

If X is the true value of a quantity and X' is its approximate value then, $|X - X'|$ *i.e.*, $|\text{Error}|$ is called the absolute error, E_a.

The relative error is defined by $E_r = \left|\frac{X - X'}{X}\right| = \frac{|\text{Error}|}{|\text{True value}|}$

and, The percentage error is, $E_p = 100\ E_r = 100 \left|\frac{X - X'}{X}\right|$

If $\bar{X}$ be such a number that $|X - X'| \leq \bar{X}$, then $\bar{X}$ is an upper limit on the magnitude of absolute error and measures the absolute accuracy.

NOTE:

1. The relative and percentage errors are independent of the units used while absolute error is expressed in terms of these units.

2. If a number is correct to n decimal places, then the error $= \frac{10^{-n}}{2}$. For example, if the number is 3.1416 correct to 4 decimal places, then the error $= \frac{10^{-4}}{2} = 0.0005$.

1. Inherent Errors

Inherent errors are those types of error that are present in the data supplied to the model. Inherent errors (also known as input efforts) contain two components, namely, data errors and conversion errors.

A. Data errors or empirical errors

Data errors arises when data for a problem are obtained by some experimental means and are, therefore, of limited accuracy and precision. This may be due to some limitations in instrumentation and reading and therefore may be unavoidable. A physical measurement, such as voltage, time period, current, distance cannot be exact. It is therefore important to remember that there is no use in performing arithmetic operations to, say, four decimal places when the original data themselves are only correct to two decimal places.

B. Conversion errors or representation errors

Conversion errors arises due to the limitations of the computer to store the data exactly. Many numbers cannot be represented exactly in a given number of decimal digits. In some cases, a decimal number 0.1 has a non terminating binary form like 0.00011001100110011........ but the computer retains only a specified number of bits. Thus, if we add 10 such numbers in a computer, the result will not be exactly 1.0 because of round off error during the conversion of 0.1 to binary form.

2. Numerical Errors

Numerical errors are introduced during the process of implementation of a numerical method. The total numerical error is the summation of round off errors and truncation errors. The total errors can be reduced by devising suitable techniques for implementing the solution.

A. Round off errors

Round off errors occurs when a fixed number of digits are used to represent exact numbers. Since the numbers are stored at every stage of computation, round off error is introduced at end of every arithmetic operation. hence, individual round off error could be very small, but cumulative effect of a series of computations can be very significant.

Rounding a number can be done in two ways. One is known as chopping and other is known as symmetric rounding.

i) **Chopping**

In chopping, the extra digits are dropped. This is called truncating the number. Suppose we are using a computer with a fixed word length of four as 32.45687 and the digits 687 will be dropped.

ii) **Symmetric error**

In the symmetric round off method, the last retained significant digit is "rounded up" by 1 if the first discarded digit is larger or equal to 5; otherwise, the retained digit is unchanged. For example, the number 32.45687 would become 32.46 and the number 33.2342 would become 33.23.

B. Truncation errors

Truncation errors arise from using an approximation in place of an exact mathematical procedure. Typically, it is the error resulting from the truncation of the numerical process. Many of the iterative procedures used in numerical computing are infinite and, hence, knowledge of this error is important. Truncation error can be reduced by using a better numerical model which usually increases the number of arithmetic operations. For example; in numerical integration, the truncation error can be reduced by increasing the number of points at which the function is integrated. But care should be exercised to see that the round off error which is bound to increase due to increase in arithmetic operations does not off-set the reduction in truncation error.

3. Modelling Errors

Mathematical models are the basis for numerical solutions. They are formulated to represent physical processes using certain parameters involved in the situations. In many situations, it is impractical or impossible to include all of the real problem and hence certain simplifying assumptions are made. Since a model is a basic input to the numerical process, no numerical method will provide adequate result if the model is erroneously conceived and formulated. We can reduce these types of errors by refining or enlarging the models by incorporating more features. But the enhancement may make the model more difficult to solve or may take more time to implement the solution process. It is also not always true that an enhanced model will provide better results. We must note that modelling, data quality and computation go hand in hand. An overly refined model with inaccurate data or an inadequate computer may not be meaningful. On the other hand, an oversimplified model may produce a result that is unacceptable. It is, therefore, necessary to strike a balance between the level of accuracy and the complexity of the model.

4. Blunders

Blunders are errors that are caused due to human imperfection. As the name indicates, such errors may cause a very serious disaster in the result. Since these errors are due to human mistakes, it should be possible to avoid them to a large extent by acquiring a sound knowledge of all aspects of the problem as well as the numerical process.

Human errors can occur at any stage of the numerical processing cycle. Some common types of errors are;

i) Lack of understanding of the problem
ii) Wrong assumptions
iii) Overlooking of some basic assumptions required for formulating the model.
iv) Errors in deriving the mathematical equation or using a model that does not describe adequately the physical system under study.
v) Selecting a wrong numerical method for solving the mathematical model.
vi) Selecting a wrong algorithm for implementing the numerical method.
vii) Making mistakes in the computer program such as testing a real number for zero and using < symbol in place of > symbol.
viii) Mistakes of data input such as misprints, giving values column-wise instead of row-wise to a matrix, forgetting a negative sign etc.
ix) Wrong guessing of initial values

All these mistakes can be avoided through a reasonable understanding of the problem and the numerical solution methods, and use of good programming techniques and tools.

Example 1.1

Round off the numbers 865250 and 37.46235 to four significant figures and compute E_a, E_r, E_p in each case.

Solution:

i) Number rounded off to four significant figures = 865200

$\therefore$ $E_a = |X - X'| = |865250 - 865200| = 50$

$$E_r = \left|\frac{X - X'}{X}\right| = \frac{50}{865250} = 6.71 \times 10^{-5}$$

$$E_p = E_r \times 100 = 6.71 \times 10^{-3}$$

ii) Number rounded off to four significant figures = 37.46

$\therefore$ $E_a = |X - X'| = |37.46235 - 37.46000| = 0.00235$

$$E_r = \left|\frac{X - X'}{X}\right| = \frac{0.00235}{37.46235} = 6.27 \times 10^{-5}$$

$$E_p = E_r \times 100 = 6.27 \times 10^{-3}$$

Example 1.2

Find the absolute error and relative error in $\sqrt{6} + \sqrt{7} + \sqrt{8}$ correct to 4 significant digits.

Solution:

Given that:

$\sqrt{6} = 2.449$

$\sqrt{7} = 2.646$

$\sqrt{8} = 2.828$

$\therefore$ $S = \sqrt{6} + \sqrt{7} + \sqrt{8} = 2.449 + 2.646 + 2.828 = 7.923$

Then the absolute error E_a in S is,

$$E_a = 0.0005 + 0.0007 + 0.0004 = 0.0016$$

This shows that S is correct to 3 significant digits only. Hence, we take S = 7.92.

Then the relative error,

$$E_r = \frac{E_a}{S} = \frac{0.0016}{7.92} = 0.0002$$

Example 1.3

The function $f(x) = \tan^{-1} x$ can be expanded as, $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \ldots\ldots$. Find n such that the series determine $\tan^{-1} x$ correct to eight significant digits at x = 1.

Solution:

If we retain n terms in the expansion of $\tan^{-1} x$, then $(n + 1)^{th}$ term,

$$= (-1)^n \frac{x^{2n+1}}{2n+1} = \frac{(-1)^n}{2n+1} \text{ for } x = 1$$

To determine $\tan^{-1}(1)$ correct to eight significant digits accuracy

$$\left|\frac{(-1)^n}{2n+1}\right| < \frac{1}{2} \times 10^{-8}$$

i.e., $2n + 1 > 2 \times 10^8$ or $n > 10^8 - \frac{1}{2}$

Hence, value of $n = 10^8 + 1$

Example 1.4

Which of the following numbers has the greatest precision.

a) **4.3201**

b) **4.32**

c) **4.320106**

Solution:

a) 4.3201 has a precision of 10^{-4}

b) 4.32 has a precision of 10^{-2}

c) 4.320106 has a precision of 10^{-6}

The last number (4.320106) has the greatest precision

Example 1.5

What is the accuracy of the following numbers?

a) **95.763**

b) **0.008472**

c) **0.0456000**

d) **36**

e) **3600**

f) **3600.00**

Solution:

a) 95.763

Ans: 95.763 has five significant digits.

b) 0.008472

Ans: 0.008472 has four significant digits. The leading or higher order zeros are only place holders.

c) 0.0456000

Ans: 0.0456000 has six significant digits.

d) 36

Ans: 36 has two significant digits.

e) 3600

Ans: Accuracy is not fixed.

f) 3600.00

Ans: 3600.00 has six significant digits. Note that the zeros were made significant by writing .00 after 3600.

Example 1.6

Find the absolute error if the number X = 0.00545828 is,

i) Truncated to three decimal digits.

ii) Rounded off to three decimal digits.

Solution:

Given that;

$X = 0.00545828 = 0.545828 \times 10^{-2}$

i) After truncation to three decimal places, its approximate value

$X' = 0.545 \times 10^{-2}$

$\therefore$ Absolute error $= |X - X'| = 0.000828 \times 10^{-2}$

$= 0.828 \times 10^{-5}$

ii) After rounding off to three decimal places, its approximate value

$X' = 0.546 \times 10^{-2}$

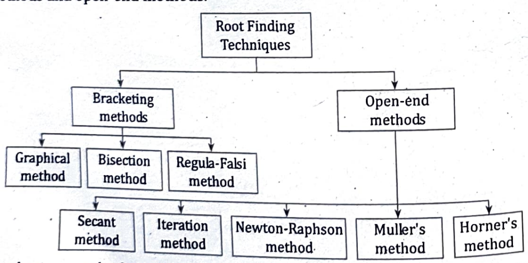
$\therefore$ Absolute error $= |X - X'|$

$= |0.545828 - 0.546| \times 10^{-2}$

$= 0.000172 \times 10^{-2} = 0.172 \times 10^{-5}$

1.4 ITERATIVE METHODS

An iterative method begins with an approximate value of the root which is generally obtained with the help of intermediate value property of the equation. This initial approximation is then successively improved iteration by iteration and this process stops when the desired level of accuracy is achieved. The various iterative methods begin their process with one or more initial approximations. Based on the number of initial approximations used, these iterative methods are divided into two categories. Bracketing methods and open-end methods.



Bracketing methods begin with two initial approximations which bracket the root. Then the width of this bracket is systematically reduced until the root is reached to desired accuracy. The commonly used methods in this category are;

1. Graphical method
2. Bisection method
3. Method of false position

Open-end methods are used on formula which require a single starting value or two starting values which do not necessarily bracket the root. Open end methods may diverge as the computation progress but when they do converge they usually do so much faster than bracketing method. The following methods fall under this category.

1. Secant method
2. Iteration method
3. Newton-Raphson method
4. Muller's method
5. Horner's method
6. Lin-Bairstow method

It may be noted that the bracketing method require to find sign changes in the function during every iteration. Open end methods do not require this.

1.4.1 Starting and Stopping an Iterative Process

A. Starting the Process

Before an iterative process is initiated, we have to determine either an approximate value of root or a "search" interval that contains a root. One simple method of guessing starting points is to plot the curve of f(x) and to identify a search interval near the root of interest. Graphical representation of a function cannot only provide us rough estimates of the roots but also help us in understanding the properties of the function, there by identifying possible problems in numerical computing. In case of polynomials, many theoretical relationships between roots and coefficients are available.

B. Largest Possible Root

For a polynomial represented by,

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots\dots\dots + a_1 x + a_0$$

The largest possible root is given by,

$$x_1^{*} = \frac{a_{n-1}}{a_n}$$

This value is taken as the initial representation when no other value is suggested by the knowledge of the problem at hand.

C. Search Bracket

Another relationship that might be useful for determining the search intervals that contain the real roots of a polynomial is,

$$|x^{*}| \le \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)}$$

where, x is the root of the polynomial. Then, the maximum absolute value of the root is,

$$|x^{*}_{max}| = \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)}$$

This means that no root exceeds x_{max} in absolute magnitude and thus, all real roots lie within the interval $(-|x^{*}_{max}|, |x^{*}_{max}|)$.

There is yet another relationship that suggests an interval for roots. All real roots x satisfy the inequality.

$$|x^{*}| \le 1 + \frac{1}{|a_n|} \max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}$$

where, the 'max' denotes the maximum of the absolute values $|a_0|, |a_1|, \ldots$ $|a_{n-1}|$.

D. Stopping Criterion

An iterative process must be terminated at some stage. When? We must have an objective criterion for deciding when to stop the process. We may use one (or combination) of the following tests, depending on the behaviour of the function, to terminate the process.

i) $|x_{i+1} - x_i| \le E_a$ (absolute error in x)

ii) $\left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \le E_r$ (Relative error in x), $x \ne 0$

iii) $|f(x_{i+1})| \le E$ (value of function at root)

iv) $|f(x_{i+1}) - f(x_i)| \le E$ (difference in function values)

v) $|f(x)| \le F_{max}$ (large function value)

vi) $|x_i| \le XL$ (large value of x)

Here, x_i represents the estimate of the root at i^{th} iteration and $f(x_i)$ is the value of the function at x_i.

There may be situations where these tests may fail when used alone. Sometimes even a combination of two tests may fail. A practical convergence test should use a combination of these tests. In cases where we do not know whether the process converges or not, we must have a limit on the number of iterations, like

Iterations $\ge$ N (limit on iterations)

1.5 BISECTION METHOD OR BINARY CHOPPING METHOD OR BOLZANO OR HALF INTERVAL OR BINARY SEARCH METHOD

The bisection method is one of the simplest and most reliable of iterative methods for the solution of non-linear equations. This method, also known as Binary chopping or half-interval method, relies on the fact that if f(x) is real and continuous in the interval $a < x < b$ and f(a) and f(b) are of opposite sings, that is $f(a)\,f(b) < 0$, then, there is at least one real root in the interval between a and b. (There may be more than one root in the interval).

Let $x_1 = a$ and $x_2 = b$. Let us also define another point x_0 to be the midpoint between a and b. That is,

$$x_0 = \frac{x_1 + x_2}{2}$$

Now, there exists the following three conditions,

a) If $f(x_0) = 0$, we have a root at x_0

b) If $f(x_0)\,f(x_1) < 0$, there is a root between x_0 and x_1

c) If $f(x_0)\,f(x_2) < 0$, there is a root between x_0 and x_2

It follows that by testing the sign of the function at midpoint, we can deduce which part of the interval contains the root. This is illustrated in figure 1.2 which shows that, sine $f(x_0)$ and $f(x_2)$ are of opposite sign, a root lies between x_0 and x_2. We can further divide this subinterval into two halves to locate a new subinterval containing the halves to locate a new subinterval containing the root. This process can be repeated until the interval containing the root is as small as we desire.

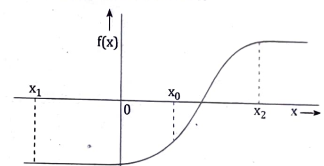


Figure 1.2: Illustration of bisection method

NOTE:
Since the new interval containing the root, is exactly half the length of the previous one, the interval width is reduced by a factor of $\frac{1}{2}$ at each step. At the end of the n^{th} step, the new interval will therefore be of length $\frac{(b-a)}{2^n}$. If on repeating this process n times, the latest interval is as small as given E, then $\frac{(b-a)}{2^n} \le E$.

or, $$n \ge \frac{[\log(b-a) - \log E}{\log 2}$$

This gives the number of iterations required for achieving an accuracy E.

In particular, the minimum number of iterations required for converging to a root in the interval (0, 1) for a E are as under:

E	:	10^{-2}	10^{-3}	10^{-4}
n	:	7	10	14

b) As the error decreases with each step by a factor of $\frac{1}{2}\left(i.e., \frac{E_{n+1}}{E_n} = \frac{1}{2}\right)$, the convergence in the bisection method is linear.

1.5.1 Algorithm for Bisection Method

1. Start.
2. Decide initial values for x_1 and x_2 and stopping criterion, E.
3. Compute $f_1 = f(x_1)$ and $f_2 = f(x_2)$.
4. If $f_1 \times f_2 > 0$, x_1 and x_2 do not bracket any root and go to step 8; otherwise continue.
5. Compute $x_0 = (x_1 + x_2)/2$ and compute $f_0 = f(x_0)$.
6. If $f_1 \times f_0 < 0$, then
   set $x_2 = x_0$
   else
   set $x_1 = x_0$
   set $f_1 = f_0$
7. If absolute value of $(x_2 - x_1)/x_2$ is less than error E_1 then,
   root $= (x_1 + x_2)/2$
   write the value of root
   go to step 8
   else
   go to step 5
8. Stop.

1.5.2 Advantages of Bisection Method

i) Convergent is guaranteed
Bisection method is bracketing method and it is always convergent.

ii) Error can be controlled
In Bisection method, increasing number of iteration always yields more accurate root.

iii) Does not involve complex calculations
Bisection method does not require any complex calculations. To perform this method, all we need is to calculate average of two numbers.

iv) Guaranteed error bound
In this method, there is guarantee error bound and it decreases with each successive iteration. The error bound decreases by $\frac{1}{2}$ with each iteration.

v) Bisection method is fast in case of multiple roots.

vii) The function does not have to be differentiable.

1.5.3 Disadvantages of Bisection Method

i) Slow rate of convergence
Although convergence of bisection method is guaranteed, it is generally slow.

ii) Choosing one guess close to root has no advantage choosing one guess close to the root may result in requiring many iterations to converge.

iii) Cannot find root of some equations. For example, $f(x) = x^2$ as there are no bracketing values.

iv) It has linear rate of convergence.

v) It fails to determine complex roots.

vi) It cannot be applied if there are discontinuities in the guess interval.

Example 1.7

Find the root of the equation $\cos x = xe^x$ using the Bisection method correct to four decimal places.

Solution:

Let, $f(x) = \cos x - xe^x$

Since, $f(0) = 1$

$f(1) = -2.18$

so, a root lies between 0 and 1.

$\therefore$ First approximation, $x_1 = \frac{1}{2}(0 + 1) = 0.5$

Now,

$f(x_1) = 0.05$ and $f(1) = -2.18$

Hence, the root lies between 1 and $x_1 = 0.5$

$\therefore$ Second approximation, $x_2 = \frac{1}{2}(0.5 + 1) = 0.75$

Now,

$f(x_2) = -0.86$ and $f(0.5) = 0.05$

Hence, the root lies between 0.5 and 0.75

$\therefore$ Third approximation, $x_3 = \frac{1}{2}(0.5 + 0.75) = 0.625$

Now,

$f(x_3) = -0.36$ and $f(0.5) = 0.05$

Hence, the root lies between 0.5 and 0.625

$\therefore$ Fourth approximation, $x_4 = \frac{1}{2}(0.5 + 0.625) = 0.5625$

Now,

$f(x_4) = -0.14$ and $f(0.5) = 0.05$

Hence, the root lies between 0.5 and 0.5625

$\therefore$ Fifth approximation, $x_5 = \frac{1}{2}(0.5 + 0.5625) = 0.5312$

Now,

$f(x_5) = -0.04$ and $f(0.5) = 0.05$

Hence, the root lies between 0.5 and 0.5312

$\therefore$ Sixth approximation, $x_6 = \frac{1}{2}(0.5 + 0.5312) = 0.5156$

Now,

$$f(fx_6) = 0.00655 \text{ and } f(1) = -2.18$$

Hence, the root lies between 1 and 0.5156

$\therefore$ Seventh approximation, $x_7 = \frac{1}{2}(0.5156 + 1) = 0.7178$

Now,

$$f(x_7) = -0.7182 \text{ and } f(0.5) = 0.05$$

Hence, the root lies between 0.5 and 0.7178

$\therefore$ Eight approximation, $x_8 = \frac{1}{2}(0.5 + 0.7178) = 0.6089$

Now,

$$f(x_8) = -0.2991 \text{ and } f(0.5) = 0.05$$

Hence, the root lies between 0.5 and 0.6089

$\therefore$ Ninth approximation, $x_9 = \frac{1}{2}(0.5 + 0.6089) = 0.5544$

Now,

$$f(x_9) = -0.1149 \text{ and } f(0.5) = 0.05$$

Hence, the root lies between 0.5 and 0.5544

$\therefore$ Tenth approximation, $x_{10} = \frac{1}{2}(0.5 + 0.5544) = 0.5272$

Now,

$$f(x_{10}) = -0.02896 \text{ and } f(0.5) = 0.05$$

Hence, the root lies between 0.5 and 0.5272

$\therefore$ 11^{th} approximation, $x_{11} = \frac{1}{2}(0.5 + 0.5272) = 0.5136$

Now,

$$f(x_{11}) = 0.0126 \text{ and } f(1) = -2.18$$

Hence, the desired approximation to the root is 0.5136

Alternative method

Let, $f(x) = xe^x - \cos x$

The initial guess be

$x_0 = 0$, $f(0) = 0e^\circ - \cos(0) = -1 < 0$

$x_1 = 1$, $f(1) = 1\,e^1 - \cos(1) = 2.177 > 0$

i.e., root lies between 0 and 1.

$\therefore$ $x_L = 0$ and $x_U = 1$

Now, first approximated root using bisection method,

$$x_N = \frac{x_L + x_U}{2} = \frac{0 + 1}{2} = 0.5$$

$\therefore$ $f(x_N) = 0.5 \times e^{0.5} - \cos(0.5) = -0.053 < 0$

so, now root lies between 0.5 and 1.

Remaining iterations are solved in Tabular form.

Iteration	x_L	$f(x_L)$	x_U	$f(x_U)$	x_N	$f(x_N)$
1	0	-1	1	2.177	0.5	-0.053
2	0.5	-0.053	1	2.177	0.75	0.8560
3	0.5	-0.053	0.75	0.8560	0.625	0.3566
4	0.5	-0.053	0.625	0.3566	0.5625	0.1412
5	0.5	-0.053	0.5625	0.1412	0.5312	0.0413
6	0.5	-0.053	0.5312	0.0413	0.5156	-0.0065
7	0.5156	-0.0065	0.5312	0.0413	0.5234	0.0172
8	0.5156	-0.0065	0.5234	0.0172	0.5195	0.0053
9	0.5156	-0.0065	0.5195	0.0053	0.5175	-0.0007
10	0.5175	-0.0007	0.5195	0.0053	0.5185	0.0022
11	0.5175	-0.0007	0.5185	0.0022	0.5180	0.0007
12	0.5175	-0.0007	0.5180	0.0007	0.5177	-0.0001
13	0.5177	-0.0001	0.5180	0.0007	0.5178	0.0001
14	0.5177	-0.0001	0.5178	0.0001	0.5177	-0.0001
15	0.5177	-0.001	0.5178	0.0001	0.5177	-0.0001

Here, the value of x_N do not change up to 4 decimal places, so required root of given function is 0.5177.

1.6 FALSE POSITION OR REGULA-FALSI OR INTERPOLATION METHOD

This is the oldest method of finding the real roots of an equation $f(x) = 0$ and closely resembles the bisection method.

Here, we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs *i.e.*, the graph of $y = f(x)$ crosses the x-axis between these points. This indicates that a root lies between x_0 and x_1 and consequently $f(x_0)\, f(x_1) < 0$.

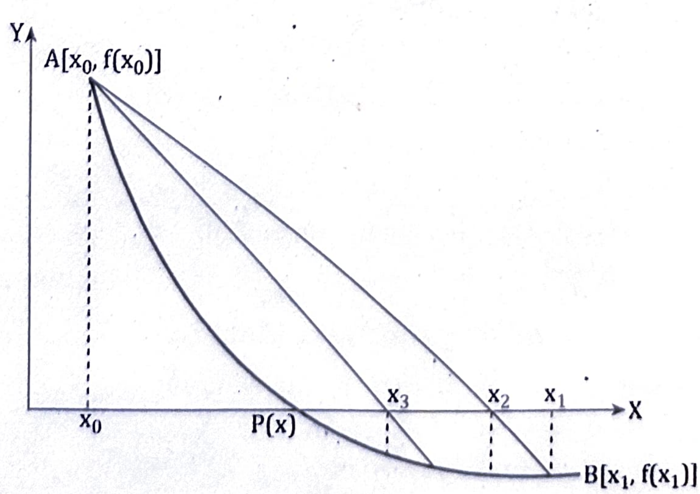


Figure 1.3

Equation of the chord joining the points A [x_0, $f(x_0)$] and B [x_1, $f(x_1)$] is,

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

This method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x-axis as an approximation to the root. So the abscissa of the point where the x-axis ($y = 0$) is given by,

$$x_2 - x_0 = \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \qquad \text{..... (1)}$$

which is an approximation to the root.

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. So replacing x_1 by x_2 in (1), we obtain the next approximation x_3. The root could as well lie between x_1 and x_2 and we would obtain x_3 accordingly. This procedure is repeated until the root is found to the desired accuracy. The iteration process based on (1) is known as the method of false position. This method has linear rate of convergence which is faster than that of the bisection method.

1.6.1 Algorithm for False Position Method

1. Start.
2. Define function $f(x)$
3. Choose initial guesses x_0 and x_1 such that $f(x_0)\,f(x_1) < 0$
4. Choose pre-specified tolerance error
5. Calculate new approximated root as
   $$x_2 = x_0 - ((x_0 - x_1) \times f(x_0))/(f(x_0) - f(x_1))$$
6. Calculate $f(x_0)\,f(x_2)$
   a) If $f(x_0)\,f(x_2) < 0$, then $x_0 = x_0$ and $x_1 = x_2$
   b) If $f(x_0)\,f(x_2) > 0$, then $x_0 = x_2$ and $x_1 = x_1$
   c) If $f(x_0)\,f(x_2) = 0$, then go to (8)
7. If $|f(x_2)| > e$, then go to (5), otherwise go to (8)
8. Display x_2 as root.
9. Stop.

A major difference between this algorithm and the bisection algorithm is the way x_2 is computed.

1.6.2 Advantages of Regula-Falsi Method

i) It does not require the derivative calculations.

ii) This method has first order rate of convergence *i.e.*, it is linearly convergent. It always converges.

iii) It is a quick method.

1.6.3 Disadvantages of Regula-Falsi Method

i) It is used to calculate only a single unknown in the equation.

ii) As it is trial and error method, in some cases it may take large time span to calculate the correct root and there by slowing down the process.

iii) It can't predict number of iterations to reach a given precision.

iv) It can be less precise than bisection method.

Example 1.8

Find the root of the equation cos x = xe^x using the regular-falsi method correct to four decimal places.

Solution:

Let, $f(x) = \cos x - x\,e^x - 0$

Here, $f(0) = \cos 0 - 0\,xe^0 = 1$

$f(1) = \cos 1 - e = 2.17798$

i.e., the root lies between 0 and 1

i) Taking $x_0 = 0$, $x_1 = 1$, $f(x_0) = 1$ and $f(x_2) = -2.17798$

In the regular-falsi method, we get,

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 0 + \frac{1}{3.17798} \times 1 = 0.31467$$

Now,

$f(0.31467) = 0.51987$ *i.e.,* the root lies between 0.31467 and 1

ii) Taking $x_0 = 0.31467$, $x_1 = 1$, $f(x_0) = 0.51987$ $f(x_1) = -2.17798$

$$\therefore \quad x_3 = 0.31467 + \frac{0.68533}{2.69785} \times 0.51987 = 0.44673$$

Now,

$f(0.44673) = 0.20356$ *i.e.,* the root lies between 0.44673 and 1

iii) Taking $x_0 = 0.44673$, $x_1 = 1$, $f(x_0) = 0.20356$, $f(x_1) = -2.17798$

$$\therefore \quad x_4 = 0.44673 + \frac{0.55327}{2.38154} \times 0.20356 = 0.49402$$

Repeating this process, the successive approximations are,

$$x_5 = 0.50995,\ x_6 = 0.51520,\ x_7 = 0.51692$$

$$x_8 = 0.51748,\ x_9 = 0.51767,\ x_{10} = 0.51775$$

Hence, the root is 0.5177 correct to four decimal places

Alternative method

Let, $f(x) = x\,e^x - \cos x$

The initial guess be,

$x_l = x_0 = 0, \quad f(x_0) = 0e^0 - \cos(0) = -1 < 0$

$x_U = x_1 = 1, \quad f(x_1) = 1\,e^1 - \cos(1) = 2.177 > 0$

i.e., root lies between 0 and 1.

Using false position method,

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0) = 0 - \frac{(1-0)}{2.177+1} \times (-1) = 0.3147$$

$\therefore$ $f(x_2) = -0.5197 < 0$

Now root lies between 0.3147 and 1.

Solving other iterations in tabular form as follow,

Iteration	x_L	$f(x_L)$	x_U	$f(x_U)$	$x_N = x_L - \frac{f(x_L)(x_U - x_L)}{f(x_U) - f(x_L)}$	$f(x_N)$
1	0	-1	1	2.177	0.3147	-0.5197
2	0.3147	-0.5197	1	2.177	0.4467	-0.2036
3	0.4467	-0.2036	1	2.177	0.4940	-0.0708
4	0.4940	-0.0708	1	2.177	0.5099	-0.0237
5	0.5099	-0.0237	1	2.177	0.5151	-0.0080
6	0.5151	-0.0080	1	2.177	0.5168	-0.0029
7	0.5168	-0.0029	1	2.177	0.5174	-0.0010
8	0.5174	-0.0010	1	2.177	0.5177	-0.0004
9	0.5177	-0.0004	1	2.177	0.5177	-0.0002

Here, the value of x_N does not change up to 4 decimal places. Hence, the root of given equation is 0.5177.

1.7 SECANT METHOD

This method is an important over the method of false position as it does not require the condition $f(x_0)\, f(x_1) < 0$ of that method.

Here, also the graph of the function $y = f(x)$ is approximated by a secant line but at each iteration, two most recent approximations to the root are used to find out the next approximation. Also, it is not necessary that the interval must contain the root.

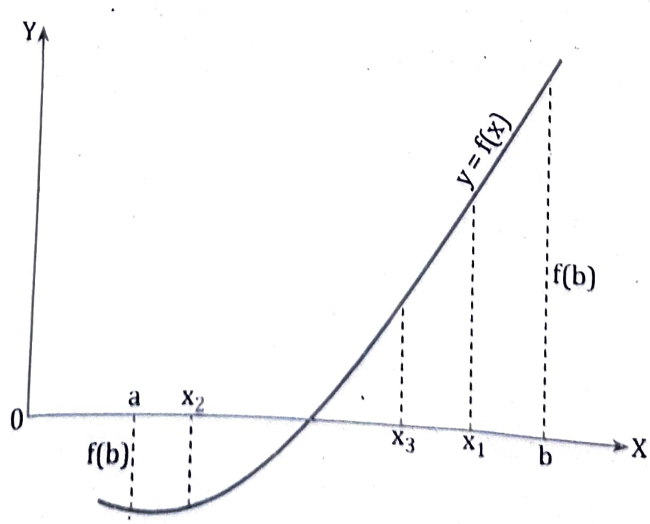


Figure: 1.4

Taking x_0, x_1 as the initial limits of the interval, we write the equation of the chord joining these as,

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1)$$

Then the abscissa of the point where it crosses the x-axis ($y = 0$) is given by,

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

which is an approximation to the root. The general formula for successive approximation is, therefore, given by,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n),\ n \geq 1$$

If at any integration $f(x_n) = f(x_{n-1})$, this method fails and shows that it does not converge necessarily. This is a drawback of secant method over false position which always converges. But if the secant method once converges, its rate of convergence is 1.6 which is faster than that of the method of false position.

1.7.1 Algorithm for Secant Method

1. Start.
2. Decide two initial points x_0 and x_1, accuracy level required, E.
3. Compute $f_0 = f(x_0)$ and $f_1 = f(x_0)$ and $f_1 = f(x_1)$
4. Compute $x_2 = \dfrac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$
5. Test for accuracy of x_2.

   If $\left|\dfrac{x_2 - x_1}{x_3}\right| > E_1$ then

   set $x_0 = x_1$ and $f_0 = f_1$

   set $x_1 = x_2$ and $f_1 = f(x_2)$

   go to step 4

   otherwise

   set root = x_2

   print results
6. Stop.

1.7.2 Advantages of Secant Method

i) It converges at faster than a linear rate, so that it is more rapidly convergent than the bisection method.

ii) It requires only one function evaluation per iteration, as compared with Newton's method which requires two.

iii) It does not require the use of derivative of the function, something that is not available in a number of applications.

1.7.3 Disadvantages of Second Method

i) It may not converge *i.e.*, may diverge.

ii) There is no guaranteed error bound for the computed iterates

Example 1.9

Find the root of the equation $xe^x = \cos x$ using secant method correct to four decimal place.

Solution:

Let $f(x) = xe^x - \cos x$

$x_0 = 0$ and $x_1 = 1$ be the initial guesses

$f(x_0) = 0e^0 - \cos(0) = -1$

$f(x_1) = 1e^1 - \cos(1) = 2.1779$

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 1 - \frac{2.1779(1-0)}{2.1779 - (-1)} = 0.3146$$

$f(x_2) = 0.3146 . e^{0.3146} - \cos(0.3146) = -0.5200$

Now, solving other iterations in tabular form as follows

Iteration	x_{n-1}	$f(x_{n-1})$	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	0	−1	1	2.1779	0.3146	−0.5200
2	1	2.1779	0.3146	−0.5200	0.4467	−0.2036
3	0.3146	−0.5200	0.4467	−0.2036	0.5317	0.0429
4	0.4467	−0.2036	0.5317	0.0429	0.5169	-2.60×10^{-3}
5	0.5317	0.0429	0.5169	-2.60×10^{-3}	0.5177	-1.74×10^{-4}
6	0.5169	-2.60×10^{-3}	0.5177	-1.74×10^{-4}	0.5177	4.47×10^{-8}

Here, the value of x_{n+1} do not change up to four decimal places.

Hence, the root of the equation is 0.5177.

1.8 NEWTON-RAPHSON METHOD

Let x_0 be an approximate root of the equation $f(x) = 0$. If $x_1 = x_0 + h$ be the exact root, then $f(x_1) = 0$.

∴ Expanding $f(x_0 + h)$ by Taylor's series

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \dots\dots\dots = (0)$$

Since h is small, neglecting h^2 and higher powers of h_1 we get,

$$f(x_0) + hf'(x_0) = 0$$

or, $$h = -\frac{f(x_0)}{f'(x_0)}$$

∴ A closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly starting with x_1, a still better approximation x_2 is given by,

$$x_2 = x_1 - \frac{f(x_0)}{f'(x)}$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (n = 0, 1, 2, \ldots\ldots\ldots)$$

which is known as the Newton Raphson formula or Newton's iteration formula.

NOTE:

1. Newton's method is useful in cases of large values of $f'(x)$ *i.e.*, when the graph of f(x) while crossing the x-axis is nearly vertical.
   If $f'(x)$ is small in the vicinity of the root, then by (1), h will be large and the computation of the root is slow or may not be possible. Thus this method is not suitable in those cases where the graph of f(x) is nearly horizontal while crossing the x-axis.
2. Newton's method is generally used to improve the result obtained by other methods. It is applicable to the solution of both algebraic and transcendental equations.

Newton's formula converges provided the initial approximation x_0 is chosen sufficiently close to the root. If it is not near the root, the procedure may lead to an endless cycle. A bad initial choice will lead one astray. Thus a proper choice of the initial guess is very important for the success of Newton's method.

We have,

$$\phi(x_n) = x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In general,

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

which gives

$$\phi'(x) = \frac{f(x)\, f''(x)}{[f'(x)]^2}$$

Since the iteration method converges if $|\phi'(x)| < 1$. So the Newton's formula will converge if, $|f(x)\, f''(x)| < |f'(x)|^2$ in the interval considered. Assuming $f(x)$, $f'(x)$ and $f''(x)$ to be continuous, we can select a small interval in the vicinity of the root α, in which the above condition is satisfied. Hence, the result.

Newton's method converges conditionally while the regular-Falsi method always converges. However when the Newton-Raphson method converges it converges faster and is preferred. The Newton-Raphson method has second order convergence.

1.8.1 Algorithm for Newton-Raphson Method

1.	Start.
2.	Assign an initial value to x_1 say x_0.
3.	Evaluate $f(x_0)$ and $f'(x_0)$.
4.	Find the improved estimate of x_0. $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.
5.	Check for accuracy of the latest estimate. Compare relative error to a predefined value E. If $\left\lvert \frac{x_1 - x_0}{x_1} \right\rvert \le E$, stop. Otherwise, continue.
6.	Replace x_0 by x_1 and repeat steps 4 and 5.

Example 1.10

Find the root of the equation $xe^x = \cos x$ using Newton Raphson method correct to four decimal places.

Solution:

Let $f(x) = xe^x - \cos x$ ..... (1)

Differentiating equation (1) with respect to x

$$f'(x) = x\,e^x + e^x + \sin x \quad ..... (2)$$

From equation (1)

Let the initial guess be

$$x_0 = 0$$

$$f(x_0) = 0e^0 - \cos(0) = -1$$

$$f'(x_0) = 0e^0 + e^0 + \sin(0) = 1$$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{1} = 1$$

$$f(x_1) = 2.1779$$

Now, continuing process in tabular form

Iteration	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	0	-1	1	2.1779
2	1	2.1779	0.6530	0.4603
3	0.6530	0.4603	0.5313	0.0416
4	0.5313	0.0416	0.5179	4.33×10^{-4}
5	0.5179	4.33×10^{-4}	0.5177	-1.74×10^{-4}
6	0.5177	-1.74×10^{-4}	0.5177	-4.90×10^{-7}

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, the desired root is 0.5177 of the equation.

1.8.2 Some Deductions from Newton-Raphson Formula

We can derive the following result from the Newton's iteration formula:

Iterative formula to find,

a) $\frac{1}{N}$ is $x_{n+1} = x_n (2 - Nx_n)$

b) $\sqrt{N}$ is $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$

c) $\frac{1}{\sqrt{N}}$ is $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{Nx_n}\right)$

d) $\sqrt[k]{N}$ is $x_{n+1} = \frac{1}{k}\left[(k-1)\,x_n + \frac{N}{x_n^{k-1}}\right]$

1.8.3 Advantages of Newton-Raphson Method

i) It converges fast if it converges, *i.e.*, in most case we get root in less number of steps.

ii) It requires only one guess.

iii) It has simple formula so it is easy to program.

iv) Formulation of this method is simple. So, it is very easy to apply.

v) Can be used to 'polish' a root found by other methods.

vi) Easy to convert to multiple dimensions.

vii) It is suitable for large size system.

viii) It is faster, reliable and the results are accurate.

1.8.4 Disadvantages of Newton-Raphson Method

i) Division by zero problem can occur.

ii) Inflection point issue might occur.

iii) In case of multiple roots, this method converges slowly.

iv) Near local maxima and local minima, due to oscillation, its convergence is slow.

v) Root jumping might take place thereby not getting intended solution.

vi) More complicated to code, particularly when implementing sparse matrix algorithms.

vii) Requires more memory.

viii) Must find the derivative

1.9 FIXED POINT ITERATION METHOD

Any function in the form of,

$$f(x) = 0 \quad \text{..... (1)}$$

can be manipulated such that x is on the left-hand side of the equation as shown below

$$x = g(x) \quad \text{..... (2)}$$

Equation (1) and (2) are equivalent and therefore, a root of equation (2) is also a root of equation (1). The root of equation (2) is given the point of intersection of the curves $y = x$ and $y = g(x)$. This intersection point is known as the fixed point of $g(x)$.

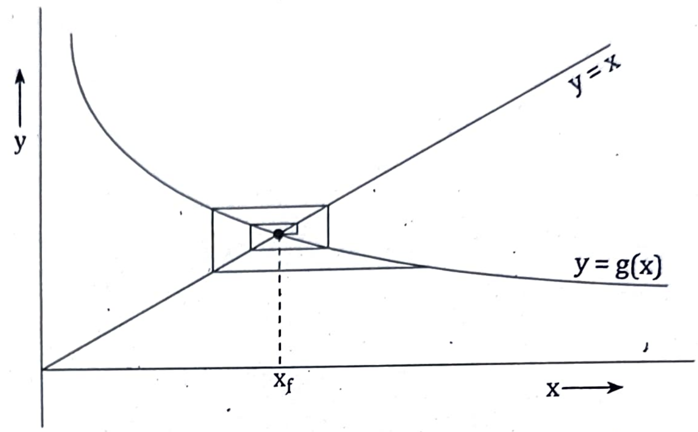


Figure 1.5: Fixed point iteration method

The above transformation can be obtained either by algebraic manipulation of the given equation or by simply adding x to both sides of the equation for example,

$$x^2 - x + 2 = 0$$

can be written as

$$x = x^2 + 2$$

or, $\quad x = x^2 + x + 2 + x = x^2 + 2x + 2$

Adding of x to both sides is normally done in situations where the original equation is not amenable to algebraic manipulations.

For example, $\tan x = 0$

Would be put into the form of equation (2) by adding x to both sides. That is, $x = \tan x + x$.

The equation $x = g(x)$ is known as the fixed point equation. If provides a convenient form for predicting the value of x as a function of x. If x_0 is the initial guess to a root, then the next approximation is given by,

$$x_1 = g(x_0)$$

Further approximation is given by,

$$x_2 = g(x_1)$$

This iteration process can be expressed in general form as,

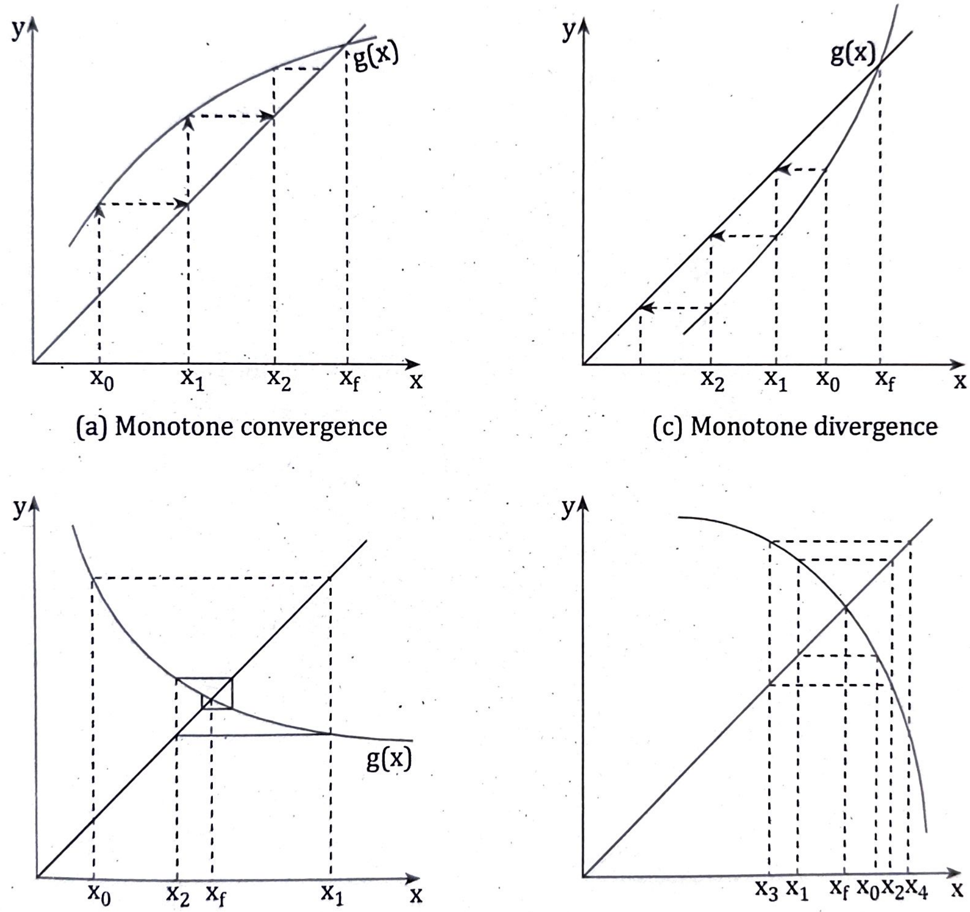
$$x_{i+1} = g(x_i),\ i = 0, 1, 2, 3, \dots\dots\dots$$

Which is called the fixed point iteration formula. This method of solution is also known as the method of successive approximation or method of direct substitution.

The algorithm is simple the iteration process would be terminated when the successive approximations agree within some specified error.

Convergence of fixed point iteration method

Convergence of the iteration process depends on the nature of $g(x)$. The process converges only when the absolute value of the slope of $y = g(x)$ curve is less than the slope of $y = x$ curve. Since the slope of $y = x$ curve is I, the necessary condition for convergence is $g'(x) < 1$.



(a) Monotone convergence

(c) Monotone divergence

(b) Spiral convergence

(d) Spiral divergence

Figure 1.6: Patterns of behaviour of fixed point iteration process

We can theoretically prove this as follows;

The iteration formula is,

$$x_{i+1} = g(x_i) \quad \text{..... (3)}$$

Let, x_f be the root of the equation. Then,

$$x_f = g(x_f) \quad \text{..... (4)}$$

Subtracting equation (3) form (4) yields.

$$x_f - x_{i+1} = g(x_f) - g(x_i) \quad \text{..... (5)}$$

According to the mean value theorem, there is at least one point, say, $x = R$, in the interval x_f and x_i such that,

$$g'(R) = \frac{g(x_f) - g(x_i)}{x_f - x_i}$$

This gives,

$$g(x_f) - g(x_i) = g'(R)\,(x_f - x_i)$$

Replacing this in equation (5), we get,

$$x_f = x_{i+1} = g'(R)\,(x_f - x_i) \quad \text{...... (6)}$$

If e_i represents the error in the i^{th} iteration, then equation (6) becomes,

$$e_{i+1} = g'(R)\, e_i$$

This shows that the error will decrease with each iteration only if $g'(R) < 1$

Equation (6) implies the following,

i) Error decreases if $g'(R) < 1$

ii) Error grows if $g'(R) > 1$

iii) If $g'(R)$ is positive, the convergence is monotonic

iv) If $g'(R)$ is negative, the convergence will be oscillatory

v) The error is roughly proportional to (or less than) the error in the previous step; the fixed point method is, therefore, said to be linearly convergent.

Example 1.11

Locate root of the equation $x^2 + x - 2 = 0$ using the fixed point iteration method.

Solution:

The given equation can be expressed as,

$x = 2 - x^2$

Let us start with an initial value of $x_0 = 0$

$x_1 = 2 - 0 = 2$

$x_2 = 2 - 4 = -2$

$x_3 = 2 - 4 = -2$

Since $x_3 - x_2 = 0$, -2 is one of the roots of the equation

Let us assume that $x_0 = -1$,

Then,

$x_1 = 2 - 1 = 1$

$x_2 = 2 - 1 = 1$

Another root is 1.

1.9.1 Algorithm for Fixed Point Iteration Method for a System

1. Start.
2. Define iteration function

   $F(x, y)$ and $G(x, y)$
3. Decide starting points x_0 and y_0 and error tolerance E
4. $x_1 = F(x_0, y_0)$

   $y_1 = G(x_0, y_0)$
5. If $|x_1 - x_0| < E$ and

   $|y_1 - y_0| < E$, then

   solution obtained;

   go to step 7
6. Otherwise, set

   $x_0 = x_1$

   $y_0 = y_1$

   go to step 4
7. Write values of x_1 and y_1
8. Stop.

1.9.2 Advantages of Fixed Point Iteration Method

i) Ease of implementation

ii) Constraints satisfied

iii) Low cost per iteration

BOARD EXAMINATION SOLVED QUESTIONS

1. Find the positive root of the equation $f(x) = \cos x - 3x + 1$ correct upto 3 decimal places using Bisection method. **[2013/Fall]**

Solution:

$$f(x) = \cos x - 3x + 1$$

Let initial guess be

$x = 0$, $f(0) = \cos(0) - 3 \times 0 + 1 = 2 > 0$

$x = 1$, $f(1) = \cos(1) - 3(1) + 1 = -1.4596 < 0$

So root lies between $x = 0$ and $x = 1$

$\therefore$ $x_L = 0$ and $x_U = 1$

Now, first approximated root using bisection method

$$x_N = \frac{x_L + x_U}{2} = \frac{0 + 1}{2} = 0.5$$

$f(x_N) = 0.3775 > 0$, so now root lies between 0.5 and 1

Remaining iterations are solved in tabular form

Iteration	x_L	$f(x_L) = \cos x_L - 3x_L + 1$	x_U	$f(x_U) = \cos x_U - 3x_U + 1$	x_N	$f(x_N) = \cos x_N - 3x_N + 1$
1	0	2	1	−1.4596	0.5	0.3775
2	0.5	0.3775	1	−1.4596	0.75	−0.5183
3	0.5	0.3775	0.75	−0.5183	0.625	−0.0640
4	0.5	0.3775	0.625	−0.0640	0.5625	0.1584
5	0.5625	0.1584	0.625	−0.0640	0.5937	0.0477
6	0.5937	0.0477	0.625	−0.0640	0.6093	-7.85×10^{-3}
7	0.5937	0.0477	0.6093	-7.85×10^{-3}	0.6015	0.0199
8	0.6015	0.0199	0.6093	-7.85×10^{-3}	0.6054	6.07×10^{-3}
9	0.6054	6.07×10^{-3}	0.6093	-7.85×10^{-3}	0.6073	-7.08×10^{-4}
10	0.6054	6.07×10^{-3}	0.6073	-7.08×10^{-4}	0.6063	2.86×10^{-3}
11	0.6063	2.86×10^{-3}	0.6073	-7.08×10^{-4}	0.6068	1.07×10^{-3}

Here, the value of x_N do not change upto 3 decimal places.

Hence, the positive root of the equation is 0.6068.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = x_U$, $C = x_N$, $D = f(x_L)$, $E = f(x_U)$, $F = f(x_N)$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator as shown;

$$A : B : C = \frac{A + B}{2} : D = \cos A - 3A + 1 : E = \cos B - 3B + 1 :$$

$$F = \cos C - 3C + 1$$

Step 3: Press CALC then,

Enter the value of A? then press =

Enter the value of B? then press =

Step 4: Now press = only, again and again, to get the values for the respective row for each column.

Step 5: Update the values when A? and B? is asked again.

Step 6: Go to step 4.

2. Calculate the root of non-linear equation $3x = \cos x + 1$ using secant method. **[2013/Fall]**

Solution:

Let, $f(x) = 3x - \cos x - 1$

$x_0 = 0$ and $x_1 = 1$ be two initial guesses

$f(x_0) = -2$ and $f(x_1) = 1.4596$

Then, next approximated root by secant method is given by

$$x_2 = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)} = 1 - \frac{1.4596\,(1 - 0)}{1.4596 - (-2)} = 0.5781$$

$f(x) = -0.1032$ and now root lies between 1 and 0.5781.

Now, solving other iterations in tabular form as follows,

Iteration	x_{n-1}	$f(x_{n-1}) = 3x_{n-1} - \cos x_{n-1} - 1$	x_n	$f(x_n) = 3x_n - \cos x_n - 1$	$x_{n+1} = x_n - \frac{f(x_n)\,(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = 3x_{n+1} - \cos x_{n+1} - 1$
1	0	−2	1	1.4596	0.5781	−0.1032
2	1	1.4596	0.5781	−0.1032	0.6059	-4.28×10^{-3}
3	0.5781	−0.1032	0.6059	-4.28×10^{-3}	0.6071	-588×10^{-6}
4	0.6059	-4.28×10^{-3}	0.6071	-5.88×10^{-6}	0.6071	5.73×10^{-9}

Here, the value of x_{n+1} do not change up to 4 decimal places. Hence, the root of given non-linear equation is 0.6071.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_{n-1}$, $B = x_n$, $C = x_{n+1}$, $D = f(x_{n-1})$, $E = f(x_n)$, $F = f(x_{n+1})$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator as shown;

$$A : B : D = 3A - \cos A - 1 : E = 3B - \cos B - 1 : C = B - \frac{E(B - A)}{E - D} :$$

$$F = 3C - \cos C - 1$$

Step 3: Press CALC then,

Enter the value of A? then press =

Enter the value of B? then press =

Step 4: Now press = only, again and again to get the values for the respective row for each column.

Step 5: Update the values when A? and B? is asked again.

Step 6: Go to step 4.

3. Find a real root of the equation $x \log_{10} x = 1.2$ by using Newton Raphson (NR) method such that the root must have error less than 0.0001%. **[2013/Fall, 2018/Fall]**

Solution:

Let, $f(x) = x \log_{10} x - 1.2$ ..... (1)

Differentiating equation (1) with respect to x.

$f'(x) = 1 + \log_{10} x$ ..... (2)

From equation (1).

Let the initial guess be

$x_0 = 1$, $f(x_0) = -1.2$, $f'(x_0) = 1$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-1.2)}{1} = 2.2$$

$f(x_1) = -0.4466$

Now, continuing process in tabular form

Iteration	x_n	$f(x_n) = x_n \log_{10} x_n - 1.2$	$f'(x_n) = 1 + \log_{10} x_n$	$f(x_{n+1}) = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} \log_{10} x_{n+1} - 1.2$
1	1	-1.2	1	2.2	-0.4466
2	2.2	-0.4466	1.3424	2.5326	-0.1779
3	2.5326	-0.1779	1.4035	2.6593	-0.0704
4	2.6593	-0.0704	1.4247	2.7087	-0.0277
5	2.7087	-0.0277	1.4327	2.7280	-0.0110
6	2.7280	-0.0110	1.4358	2.7356	-4.39×10^{-3}
7	2.756	-4.39×10^{-3}	1.4370	2.7386	-1.78×10^{-3}
8	2.7386	-1.78×10^{-3}	1.4375	2.7398	-7.37×10^{-4}
9	2.7398	-7.37×10^{-4}	1.4377	2.7403	-3.01×10^{-4}
10	2.7403	-3.01×10^{-4}	1.4377	2.7405	-1.27×10^{-4}
11	2.7405	-1.27×10^{-4}	1.4378	2.7405	-5.03×10^{-5}

Here, the value of x_{n+1} do not change up to 4 decimal places and have error less than 0.0001%. Hence, required root is 2.7405.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_n$, $B = f(x_n)$, $C = f'(x_n)$, $D = x_{n-1}$, $E = f(x_{n+1})$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator as shown;

$$A : B : A \log_{10} A - 1.2 : C = 1 + \log_{10} A : D = A - \frac{B}{C} : E = D \log_{10} D - 1.2$$

Step 3: Press CALC then,

Enter the value of A? then press =

Step 4: Now press = only, again and again to get the values for the respective row for each column.

Step 5: Update the values when A? is asked again.

Step 6: Go to step 4.

4. Solve $f(x) = 3x + \sin x - e^x$ by secant method up to 5^{th} iteration.

[2013/Spring, 2017/Fall]

Solution:

$f(x) = 3x + \sin x - e^x$

Let, $x_0 = 0$ and $x_1 = 1$ be two initial guesses.

$f(x_0) = -1$ and $f(x_1) = 1.1231$

Then, next approximated root by secant method is given by

$$x_2 = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 1 - \frac{1.1231\,(1 - 0)}{1.1231 - (-1)} = 0.4710$$

$f(x_2) = 0.2651$

Now, solving up to 5^{th} iteration in tabular form as follows

Iteration	x_{n-1}	$f(x_{n-1}) = 3x_{n-1} + \sin x_{n-1} - e^{x_{n-1}}$	x_n	$f(x_n) = 3x_n + \sin x_n - e^{x_n}$	$x_{n+1} = x_n - \frac{f(x_n)\,(x_n - x_{n-1})}{f(x_n - f(x_{n-1}))}$	$f(x_{n+1}) = 3x_{n+1} + \sin x_{n+1} - e^{x_{n+1}}$
1	0	−1	1	1.1231	0.4710	0.2651
2	1	1.1231	0.4710	0.2651	0.3075	−0.1348
3	0.4710	0.2651	0.3075	−0.1348	0.3626	5.44×10^{-3}
4	0.3075	−0.1348	0.3626	5.44×10^{-3}	0.3604	-5.42×10^{-5}
5	0.3626	5.44×10^{-3}	0.3604	-5.42×10^{-5}	0.3604	-1.84×10^{-10}

Here, the value of x_{n+1} do not change up to 4 decimal places. Hence, the root of the given equation is 0.3604.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = (x_{n+1})$, $F = f(x_{n+1})$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator:

$$A : C : B = 3A + \sin A - e^A : D = 3C + \sin C - e^C : E = C - \frac{D(C - A)}{D - B} :$$

$$F = 3E + \sin E - e^E$$

Step 3: Press CALC then,

Enter the value of A? then press =

Enter the value of C? then press =

Step 4: Now press = only, again and again to get the values for the respective row for each column.

Step 5: Update the values when A? and C? is asked again.

Step 6: Go to step 4.

5. The equation $\alpha \tan\alpha = 1$ occurs in theory of vibrations. Find one of the positive real roots by using any close-end method, correct to at least three decimal places. **[2014/Spring]**

Solution:

Let, $f(\alpha) = \alpha \tan \alpha - 1$

Initial guess value be

$\alpha = 0, \quad f(0) = -1 < 0$

$\alpha = 1, \quad f(1) = 0.5574 > 0$

so, root between $\alpha = 0$ and $\alpha = 1$

$\therefore$ $X_L = 0$ and $x_U = 1$

Now, first approximated root using bisection method as closed end method

$$x_N = \frac{x_L + x_U}{2} = \frac{0+1}{2} = 0.5$$

$f(x_N) = -0.7268 < 0$

So root now lies between 0.5 and 1.

Remaining iterations are solved in tabular form.

Iteration	x_L	$f(x_L) = x_L \tan x_L - 1$	x_U	$f(x_U) = x_U \tan x_U - 1$	$x_N = \frac{x_L + x_U}{2}$	$f(x_N) = x_N \tan x_N - 1$
1	0	-1	1	0.5574	0.5	-0.7268
2	0.5	-0.7268	1	0.5574	0.75	-0.3013
3	0.75	-0.3013	1	0.5574	0.875	0.0477
4	0.75	-0.3013	0.875	0.0477	0.8125	-0.1422
5	0.8125	-0.1422	0.875	0.0477	0.8437	-0.0517
6	0.8437	-0.0517	0.875	0.0477	0.8593	-3.28×10^{-3}
7	0.8593	-3.28×10^{-3}	0.875	0.0477	0.8671	0.0217
8	0.8593	-3.28×10^{-3}	0.8671	0.0217	0.8632	9.16×10^{-3}
9	0.8593	-3.28×10^{-3}	0.8632	9.16×10^{-3}	0.8612	2.76×10^{-3}
10	0.8593	-3.28×10^{-3}	0.8612	2.76×10^{-3}	0.8602	-4.25×10^{-4}
11	0.8602	-4.25×10^{-4}	0.8612	2.76×10^{-3}	0.8607	1.16×10^{-3}

Here, the value of x_N do not change up to 3 decimal places.

Hence, the positive real root of the equation is 0.8607.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_1)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Set the following in calculator:

$$A : C : B = A \tan A - 1 : D = C \tan C - 1 : E = \frac{A + C}{2} : F = E \tan E - 1$$

CALC

6. Find the root of the equation $f(x) = x^2 - 3x + 2$ in the vicinity of $x = 0$, using Newton Raphson method. [2014/Spring]

Solution:

$$f(x) = x^2 - 3x + 2 \quad \text{..... (1)}$$

Differentiating equation (1) with respect to x

$$f'(x) = 2x - 3 \quad \text{..... (2)}$$

Let the initial guess be

$$x_0 = 0, \qquad f(x_0) = 0^2 - 3 \times (0) + 2 = 2, \qquad f'(0) = -3$$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{2}{(-3)} = 0.6667$$

$$f(x_1) = 0.4443$$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n) = x_n^2 - 3x_n + 2$	$f(x_{n+1}) = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1}^2 - 3x_{n+1} + 2$
1	0	2	0.6667	0.4443
2	0.6667	0.443	0.9332	0.0712
3	0.9332	0.0712	0.9960	4.01×10^{-3}
4	0.9960	4.01×10^{-3}	0.9999	1.00×10^{-4}
5	0.9999	1.00×10^{-4}	0.9999	1.99×10^{-8}

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, the root of the equation is 0.9999.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
> Set the following in calculator:
> $$A : B := A^2 - 3A + 2 : C = A - \frac{B}{2A - 3} : D = C^2 - 3C + 2$$
> $\boxed{\text{CALC}}$

7. Find the square root of 7 using Newton Raphson method and fixed point iteration method correct up to 4 decimal digit. [2014/Spring]

Solution:

For Newton Raphson method

Let, $x = \sqrt{N}$ or $x^2 - N = 0$

Taking $f(x) = x^2 - N$

We have,

$$f'(x) = 2x$$

Then Newton's formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

Now, taking N = 7, the above formula becomes

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{7}{x_n}\right)$$

For initial guess, taking approximate value of $\sqrt{7}$

i.e., $\sqrt{7} \approx \sqrt{9} = \sqrt{3^2} = 3$

i.e., we take $x_0 = 3$

Then, $x_1 = \frac{1}{2}\left(x_0 + \frac{7}{x_0}\right) = \frac{1}{2}\left(3 + \frac{7}{3}\right) = 2.6667$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{7}{x_1}\right) = 2.6458$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{7}{x_2}\right) = 2.6457$$

$$x_4 = \frac{1}{2}\left(x_3 + \frac{7}{x_3}\right) = 2.6457$$

Here, $x_3 = x_4$ upto 4 decimal places

Hence, the value of $\sqrt{7}$ is 2.6457

Now, for fixed point iteration method

$$x^2 = 7$$

$$f(x) = x^2 - 7$$

Differentiating with respect to x,

$$f'(x) = 2x$$

Let initial guess be $x_1 = 3$

$$f(x_1) = 3^2 - 7 = 2$$

Now, $x^2 - 7 = 0$

or, $2x^2 - x^2 = 7$

or, $x = \frac{7 + x^2}{2x}$

$$\therefore \quad x_i = \frac{\frac{7}{x} + x}{2} \qquad \text{..... (1)}$$

First iteration

$$x_1 = \frac{\frac{7}{3} + 3}{2} = 2.6666$$

Error = |2.6666 − 3| = 0.3333

Second iteration

$$x_2 = \frac{\frac{7}{2.6666} + 2.6666}{2} = 2.6458$$

Error = |2.6458 − 2.6666| = 0.0208

Third iteration

$$x_3 = \frac{\frac{7}{2.6458} + 2.6458}{2} = 2.6457$$

Error = |2.6457 – 2.6458| = 0.0001

Fourth iteration

$$x_4 = \frac{\frac{7}{2.6457} + 2.6457}{2} = 2.64575$$

Error = |2.64575 – 2.6457| = 0.00005

Here, $x_3 = x_4$ up to 4 decimal places.

Hence, the value of $\sqrt{7}$ is 2.64575

8. The flux equation of an iron core electric circuit is given by $f(\phi) = 10 - 2.1\,\phi - 0.01\,\phi^3$. The steady state value of flux is obtained by solving the equation $f(\phi) = 0$. By using any close end method, estimate the steady state value of "ϕ" correct to 3 decimal places. [2014/Fall]

Solution:

$$f(\phi) = 10 - 2.1\,\phi - 0.01\,\phi^3$$

Let initial guess be

$x = \phi = 4$, $f(4) = 10 - 2.1 \times 4 - 0.01\,(4)^3 = 0.96 > 0$

$x = \phi = 5$, $f(5) = 10 - 2.1 \times 5 - 0.01 \times 5^3 = -1.75 < 0$

So root lies between x = 4 and x = 5

∴ $x_L = 4$ and $x_U = 5$

Now, first approximated root using bisection method as close end method,

$$x_N = \frac{x_L + x_U}{2} = \frac{4+5}{2} = 4.5$$

$f(x_N) = -0.3612 < 0$ so now root lies between 4 and 4.5

Remaining iterations are solved in tabular form.

Iteration	x_L	$f(x_L) = 10 - 2.1x_L - 0.01x_L^3$	x_U	$f(x_U) = 10 - 2.1x_U - 0.01x_U^3$	$x_N = \frac{x_L + x_U}{2}$	$f(x_N) = 10 - 2.1x_N - 0.01x_N^3$
1	4	0.96	5	–1.75	4.5	–0.3612
2	4	0.96	4.5	–0.3612	4.25	0.3073
3	4.25	0.3073	4.5	–0.3612	4.375	–0.0249
4	4.25	0.3073	4.375	–0.0249	4.3125	0.1417
5	4.3125	0.1417	4.375	–0.0249	4.3437	0.0586
6	4.3437	0.0586	4.375	–0.0249	4.3593	0.0170
7	4.3593	0.0170	4.375	–0.0249	4.3671	-3.78×10^{-3}
8	4.3593	0.1070	4.3671	-3.78×10^{-3}	4.3632	6.63×10^{-3}

9	4.3632	6.63×10^{-3}	4.3671	-3.78×10^{-3}	4.3651	1.55×10^{-3}
10	4.3651	1.55×10^{-3}	4.3671	-3.78×10^{-3}	4.3661	-1.11×10^{-3}
11	4.3651	1.55×10^{-3}	4.3661	-1.11×10^{-3}	4.3656	2.23×10^{-4}
12.	4.3656	2.23×10^{-4}	4.3661	-1.11×10^{-3}	4.3658	-3.10×10^{-4}

Here, the value of x_N do not change up to 3 decimal places.

Hence, the steady state value of ϕ is 4.3658

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Set the following in calculator:

$$A : C : B = 10 - 2.1\,A - 0.01\,A^3 : D = 10 - 2.1\,C - 0.01\,C^3 :$$

$$E = \frac{A + C}{2} : F = 10 - 2.1\,E - 0.01\,E^3$$

$\boxed{\text{CALC}}$

9. Evaluate one of the real roots of the given equation $xe^x - \cos x = 0$ by NR method accurate to at least 4 decimal places. [2014/Fall]

Solution:

Let $f(x) = xe^x - \cos x$ ..... (1)

Differentiating equation (1) with respect to x.

$$f'(x) = x\,e^x + e^x + \sin x \quad \text{..... (2)}$$

From equation (1)

Let the initial guess be

$$x_0 = 0$$

$$f(x_0) = 0e^0 - \cos(0) = -1$$

$$f'(x_0) = 0e^0 + e^0 + \sin(0) = 1$$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{1} = 1$$

$$f(x_1) = 2.1779$$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n) = x_n e^{x_n} - \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} e^{x_{n+1}} - \cos x_{n+1}$
1	0	-1	1	2.1779
2	1	2.1779	0.6530	0.4603
3	0.6530	0.4603	0.5313	0.0416
4	0.5313	0.0416	0.5179	4.33×10^{-4}
5	0.5179	4.33×10^{-4}	0.5177	-1.74×10^{-4}
6	0.5177	-1.74×10^{-4}	0.5177	-4.90×10^{-7}

Here, the value of x_{n+1} do not change up to 4 decimal places.

9	4.3632	6.63×10^{-3}	4.3671	-3.78×10^{-3}	4.3651	1.55×10^{-3}
10	4.3651	1.55×10^{-3}	4.3671	-3.78×10^{-3}	4.3661	-1.11×10^{-3}
11	4.3651	1.55×10^{-3}	4.3661	-1.11×10^{-3}	4.3656	2.23×10^{-4}
12	4.3656	2.23×10^{-4}	4.3661	-1.11×10^{-3}	4.3658	-3.10×10^{-4}

Here, the value of x_N do not change up to 3 decimal places.

Hence, the steady state value of ϕ is 4.3658

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Set the following in calculator:

$$A : C : B = 10 - 2.1\,A - 0.01\,A^3 : D = 10 - 2.1\,C - 0.01\,C^3 :$$

$$E = \frac{A + C}{2} : F = 10 - 2.1\,E - 0.01\,E^3$$

CALC

9. Evaluate one of the real roots of the given equation $xe^x - \cos x = 0$ by NR method accurate to at least 4 decimal places. **[2014/Fall]**

Solution:

Let $f(x) = xe^x - \cos x$ ..... (1)

Differentiating equation (1) with respect to x.

$f'(x) = x\,e^x + e^x + \sin x$ ..... (2)

From equation (1)

Let the initial guess be

$x_0 = 0$

$f(x_0) = 0e^0 - \cos(0) = -1$

$f'(x_0) = 0e^0 + e^0 + \sin(0) = 1$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{1} = 1$$

$f(x_1) = 2.1779$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n) = x_n e^{x_n} - \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} e^{x_{n+1}} - \cos x_{n+1}$
1	0	-1	1	2.1779
2	1	2.1779	0.6530	0.4603
3	0.6530	0.4603	0.5313	0.0416
4	0.5313	0.0416	0.5179	4.33×10^{-4}
5	0.5179	4.33×10^{-4}	0.5177	-1.74×10^{-4}
6	0.5177	-1.74×10^{-4}	0.5177	-4.90×10^{-7}

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, the desired root is 0.5177 of the equation.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_N$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
> Set the following in calculator:
>
> $$A : B = Ae^A - \cos A : C = A - \frac{B}{Ae^A + e^A + \sin A} : D = Ce^C - \cos C$$
>
> CALC

10. Determine the root of $e^x = x^3 + \cos 25\,x$ using secant method correct to four decimal place. **[2015/Fall]**

Solution:

Let $f(x) = e^x - x^3 - \cos 25\,x$

Let $x_0 = 4$ and $x_1 = 5$ be two initial guesses

$f(x_0) = -10.2641$ and $f(x_1) = 22.6254$

Then, next approximated root by secant method is given by

$$x_2 = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 5 - \frac{22.6254\,(5-4)}{22.2654 - (-10.2641)} = 4.3210$$

$f(x_2) = -6.1371$

Now, solving other iterations in tabular form as follows

Itn.	x_{n-1}	$f(x_{n-1}) = e^{x_{n-1}} - x_{n-1}^3 - \cos 25\,x_{n-1}$	x_n	$f(x_n) = e^{x_n} - x_n^3 - \cos 25\,x_n$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = e^{x_{n+1}} - x_{n+1}^3 - \cos 25 x_{n+1}$
1	4	−10.2641	5	22.6254	4.3120	−6.1371
2	5	22.6254	4.3120	−6.1371	4.4587	−2.2048
3	4.3120	−6.1371	4.4587	−2.2048	4.5409	−0.7681
4	4.4587	−2.2048	4.5409	−0.7681	4.5848	1.5611
5	4.5409	−0.7681	4.5848	1.5611	4.5553	−0.0979
6	4.5848	1.5611	4.5553	−0.0979	4.5570	−0.0112
7	4.5553	−0.0979	4.5570	−0.0112	4.5572	-9.43×10^{-4}
8	4.5570	−0.0112	4.5572	-9.43×10^{-4}	4.5572	3.39×10^{-6}

Here, the value of x_{n+1} do not change up to four decimal place.

Hence, the root of the equation is 4.5572.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = x_n$, $E = x_{n+1}$, $F = f(x_{n+1})$
> Set the following in calculator:
>
> $$A : C : B = e^A - A^3 - \cos 25\,A : D = e^C - C^3 - \cos 25\,C : E = C - \frac{D(C-A)}{D-B} :$$
>
> $$F = e^E - E^3 - \cos 25\,E$$
>
> CALC

11. The current i in an electric circuit is given by $i = 10\, e^{-x} \sin 2\pi x$ where x is in seconds. Using NR method, find the value of x correct up to 3 decimal places for i = 2 ampere. **[2015/Fall]**

Solution:

Given that;

$$i = 10e^{-x} \sin 2\pi x$$

At i = 2 Ampere

$$2 = 10e^{-x} \sin 2\pi x$$

Let, $f(x) = 10e^{-x} \sin 2\pi x - i$ ..... (1)

or, $f(x) = (10e^{-x} \sin 2\pi x) - 2$ for i = 2 amp

Differentiating equation (1) with respect to x,

$$f'(x) = 10\,(e^{-x}\, 2\pi \cos 2\pi x - \sin 2\pi x \cdot e^{-x})$$

$$= 10\, e^{-x}\,(2\pi \cos 2\pi x - \sin 2\pi x) \quad .....(2)$$

From equation (1),

Let the initial guess be,

$x_0 = 0$, $\quad f(x_0) = 10e^0 \sin 0 - 2 = -2$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-2)}{62.8318} = 0.0318$$

$f(x_1) = -0.0773$

Now, continuing process in tabular form

Iteration	x_n	$f(x_n) = 10e^{-x_n} \sin 2\pi x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = 10e^{-x_{n+1}} \sin 2\pi\, x_{n+1}$
1	0	-2	0.0318	-0.0773
2	0.0318	-0.0773	0.0331	-2.45×10^{-3}
3	0.0331	-2.45×10^{-3}	0.0331	-1.20×10^{-6}

Here, the value of x_{n+1} do not change up to 3 decimal places.

Hence, the value of x is 0.0331 seconds.

12. Solve the equation $\log x - \cos x = 0$ correct to three significant digit after decimal using bracketing method. **[2015/Fall]**

Solution:

Let $f(x) = \log x - \cos x$

Let initial guess be

$x = 1$, $\quad f(1) = \log (1) - \cos (1) = -0.5403 < 0$

$x = 2$, $\quad f(2) = \log (2) - \cos (2) = 0.7171 > 0$

so, root lies between x = 1 and x = 2

$\therefore \quad x_L = 1$ and $x_U = 2$

Now, first approximated root using bisection method.

$$x_N = \frac{x_L + x_U}{2} = \frac{1+2}{2} = 1.5$$

$f(x_N) = 0.1053 > 0$ so now root lies between 1 and 1.5.

Remaining iterations are carried out in tabular form

Itn.	x_L	$f(x_L) = \log x_L - \cos x_L$	x_U	$f(x_U) = \log x_U - \cos x_U$	$x_N = \frac{x_L+x_U}{2}$	$f(x_N) = \log x_N - \cos x_N$
1	1	-0.5403	2	0.7171	1.5	0.1053
2	1	-0.5403	1.5	0.1053	1.25	-0.2184
3	1.25	-0.2184	1.5	0.1053	1.375	-0.0562
4	1.375	-0.0562	1.5	0.1053	1.4375	0.0247
5	1.375	-0.0562	1.4375	0.0247	1.4062	-0.0157
6	1.4062	-0.0157	1.4375	0.0247	1.4218	4.39×10^{-3}
7	1.4062	-0.0157	1.4218	4.39×10^{-3}	1.4140	-5.70×10^{-3}
8	1.4140	5.70×10^{-3}	1.4218	4.39×10^{-3}	1.4179	-6.55×10^{-4}
9	1.4179	-6.55×10^{-4}	1.4218	4.39×10^{-3}	1.4198	1.80×10^{-3}
10	1.4179	-6.55×10^{-4}	1.4198	1.8×10^{-3}	1.4188	5.09×10^{-4}
11	1.4179	-6.55×10^{-4}	1.4188	5.09×10^{-4}	1.4183	-1.37×10^{-4}

Here, the value of x_N do not change up to three significant digits after decimal. Hence, the root of the equation is 1.4183.

> **NOTE:**
>
> Procedure to iterate in programmable calculator:
>
> Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$
>
> Set the following in calculator:
>
> $$A : C : B = \log A - \cos A : D = \log C - \cos C : E = \frac{A+C}{2} :$$
>
> F log E – cos E
>
> $\boxed{\text{CALC}}$

13. **Find the root of the equation $x - 1.5 \sin x - 2.5 = 0$ using Newton Raphson method so that relative error is less than 0.01%. [2015/Spring]**

Solution:

Let $f(x) = x - 1.5 \sin x - 2.5$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = 1 - 1.5 \cos x$ ..... (2)

From equation (1),

Let the initial guess be

$x_0 = 3$

$f(x_0) = 3 - 1.5 \sin(3) - 2.5 = 0.2883$

$f'(x_0) = 1 - 1.5 \cos(3) = 2.4849$

Now, using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{0.2883}{2.4849} = 2.8839$$

$$f(x_1) = 1.624 \times 10^{-3}$$

Now continuing process in tabular form.

Iteration	x_n	$f(x_n)$ $x_n - 1.5 \sin x_n - 2.5$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} - 1.5 \sin x_{n+1} - 2.5$
1	3	0.2883	2.8839	1.624×10^{-3}
2	2.8839	1.624×10^{-3}	2.8832	-9.034×10^{-5}
3	2.8832	-9.034×10^{-5}	2.8832	-5.250×10^{-9}

Here, the value of x_{n+1} do not change up to 4 decimal places and relative error is also less than 0.01%. Hence, the root of the equation is 2.8832.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
Set the following in calculator:

$$A : B = A - 1.5 \sin A - 2.5 : C = A - \frac{B}{1 - 1.5 \cos A} :$$

$$D = C - 1.5 \sin C - 2.5$$

$\boxed{\text{CALC}}$

14. Find the root of the equation $xe^x = \cos x$ using secant method correct to four decimal place. [2015/Spring]

Solution:

Let, $f(x) = xe^x - \cos x$

$x_0 = 0$ and $x_1 = 1$ be the initial guesses

$f(x_0) = 0e^0 - \cos(0) = -1$

$f(x_1) = 1 \times e^1 - \cos(1) = 2.1779$

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 1 - \frac{2.1779(1-0)}{2.1779 - (-1)} = 0.3146$$

$$f(x_2) = -0.5200$$

Now, solving other iterations in tabular form as follows

Itn.	x_{n-1}	$f(x_{n-1}) = x_{n-1} e^{x_{n-1}} - \cos x_{n-1}$	x_n	$f(x_n) = x_n e^{x_n} - \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = x_{n+1} e^{x_{n+1}} - \cos x_{n+1}$
1	0	-1	1	2.1779	0.3146	-0.5200
2	1	2.1779	0.3146	-0.5200	0.4467	-0.2036
3	0.3146	-0.5200	0.4467	-0.2036	0.5317	0.0429
4	0.4467	-0.2036	0.5317	0.0429	0.5169	-2.60×10^{-3}
5	0.5317	0.0429	0.5169	-2.60×10^{-3}	0.5177	-1.74×10^{-4}
6	0.5169	-2.60×10^{-3}	0.5177	-1.74×10^{-4}	0.5177	4.47×10^{-8}

Here, the value of x_{n+1} do not change up to four decimal places.

Hence, the root of the equation is 0.5177.

> **NOTE:**
>
> Procedure to iterate in programmable calculator:
>
> Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$
>
> Set the following in calculator:
>
> $$A : C : B = Ae^A - \cos A : D = Ce^C - \cos C : E = C - \frac{D(C-A)}{D-B} :$$
>
> $$F = Ee^E - \cos E$$
>
> CALC

15. Using the bisection method, find the approximate root of the equation $\sin x = \frac{1}{x}$ that lies between x = 1 and x = 1.5 (in radian's). Carry out up to 7th stage. [2013/Spring, 2015/Spring, 2017/Fall]

Solution:

$$\text{Let } f(x) = \sin x - \frac{1}{x}$$

The initial guess be,

$$x = 1, \qquad f(1) = \sin 1 - \frac{1}{1} = -0.1585 < 0$$

$$x = 1.5, \qquad f(1.5) = \sin(1.5) - \frac{1}{1.5} = 0.3308 > 0$$

As root lies between x = 1 and x = 1.5,

$\therefore \quad x_L = 1$ and $x_U = 1.5$

Now, first approximated root using bisection method,

$$x_N = \frac{x_L + x_U}{2} = \frac{1 + 1.5}{2} = 1.25$$

$f(x_N) = 0.1489 > 0$ so now root lies between 1 and 1.25.

Performing the iterations up to 7th stage in tabular form.

Itn.	x_L	$f(x_L) = \sin x_L - \frac{1}{x_L}$	x_U	$f(x_U) = \sin x_U - \frac{1}{x_U}$	$x_N = \frac{x_L+x_U}{2}$	$f(x_N) = \sin x_N - \frac{1}{x_N}$
1	1	-0.1585	1.5	0.3308	1.25	0.1489
2	1	-0.1585	1.25	0.1489	1.125	0.0133
3	1	-0.1585	1.125	0.0133	1.0625	-0.0676
4	1.0625	-0.0676	1.125	0.0133	1.09375	-0.0259
5	1.09375	-0.0259	1.125	0.0133	1.109375	-5.98×10^{-3}
6	1.109375	-5.98×10^{-3}	1.125	0.0133	1.1171875	3.76×10^{-3}
7	1.109375	-5.98×10^{-3}	1.1171875	3.76×10^{-3}	1.11328125	-1.09×10^{-3}

Thus, the desired approximation to the root carried out up to 7th stage is 1.11328125.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Set the following in calculator:

$$A : C : B = \sin A - \frac{1}{A} : D = \sin C - \frac{1}{C} : E = \frac{A + C}{2} : F = \sin E - \frac{1}{E}$$

CALC

16. Find a real root of the equation $xe^x = 3$ by using any bracketing method correct to three decimal places (Take $x_1 = 1$ and $x_2 = 1.5$). [2016/Fall]

Solution:

Let $f(x) = x\,e^x - 3$

And, initial guess be the provided value

i.e., $x = 1$, $f(1) = 1e^1 - 3 = -0.2817 < 0$

$x = 1.5$, $f(1.5) = 1.5e^{1.5} - 3 = 3.7225 > 0$

Root lies between $x = 1$ and $x = 1.5$,

$\therefore$ $x_L = 1$ and $x_U = 1.5$

Now, first approximated root using bisection method as bracketing method

$$x_N = \frac{x_L + x_U}{2} = \frac{1 + 1.5}{2} = 1.25$$

$f(x_N) = 1.3629 > 0$ so now root lies between 1 and 1.25.

Remaining iterations are carried out in tabular form.

Itn.	x_L	$f(x_L) = x_L e^{x_L} - 3$	x_U	$f(x_U) = x_U e^{x_U} - 3$	$x_N = \frac{x_L + x_U}{2}$	$f(x_N) = x_N e^{x_N} - 3$
1	1	−0.2817	1.5	3.7225	1.25	1.3629
2	1	−0.2817	1.25	1.3629	1.125	0.4652
3	1	−0.2817	1.125	0.4652	1.0625	0.0744
4	1	−0.2817	1.0625	0.0744	1.0312	−0.1077
5	1.0312	−0.1077	1.0625	0.0744	1.0468	−0.0181
6	1.0468	−0.0181	1.0625	0.0744	1.0546	0.0275
7	1.0468	−0.0181	1.0546	0.0275	1.0507	4.63×10^{-3}
8	1.0468	−0.0181	1.0507	4.63×10^{-3}	1.0487	-7.07×10^{-3}
9	1.0487	-7.07×10^{-3}	1.0507	4.63×10^{-3}	1.0497	-1.22×10^{-3}
10	1.0497	-1.22×10^{-3}	1.0507	4.63×10^{-3}	1.0502	1.70×10^{-3}
11	1.0497	-1.22×10^{-3}	1.0502	1.70×10^{-3}	1.0499	-5.21×10^{-5}
12	1.0499	-5.21×10^{-5}	1.0502	1.70×10^{-3}	1.0500	5.33×10^{-4}
13	1.0499	-5.21×10^{-5}	1.0500	5.33×10^{-4}	1.0499	-5.21×10^{-5}
14	1.0499	-5.21×10^{-5}	1.0500	5.33×10^{-4}	1.0499	-5.21×10^{-5}

Here, the value of x_N do not change up to 3 decimal places.

Hence, the real root of the equation is 1.0499.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Set the following in calculator:

$$A : C : B = Ae^A - 3 : D = Ce^C - 3 : E = \frac{A + C}{2} : F = Ee^E - 3$$

$\boxed{\text{CALC}}$

17. Obtain a real root of the equation sin x + 1 = 2x by using secant method such that the real root must have relative error less than 0.0001. **[2016/Fall]**

Solution:

Let $f(x) = \sin x + 1 - 2x$

Let $x_0 = 0$ and $x_1 = 1$ be two initial guesses.

$f(x_0) = 1$ and $f(x_1) = -0.1585$

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)} = 1 - \frac{-0.1585(1 - 0)}{-0.1585 - 1} = 0.8631$$

$f(x_2) = 0.0336$

Now, solving other iterations in tabular form as follows

Itn.	x_{n-1}	$f(x_{n-1}) = \sin x_{n-1} + 1 - 2x_{n-1}$	x_n	$f(x_n) = \sin x_n + 1 - 2x_n$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = \sin x_{n+1} + 1 - 2x_{n+1}$
1	0	1	1	-0.1585	0.8631	0.0336
2	1	-0.1585	0.8631	0.0336	0.8870	1.18×10^{-3}
3	0.8631	0.0336	0.8870	1.18×10^{-3}	0.8878	8.51×10^{-5}
4	0.8870	1.18×10^{-3}	0.8878	8.51×10^{-5}	0.8878	4.43×10^{-8}

Here, the value of x_{n+1} do not change up to 4 decimal places and have relative error less than 0.0001.

Hence, the real root of the equation is 0.8878

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$

Set the following in calculator:

$$A : C : B = \sin A + 1 - 2A : D = \sin C + 1 - 2C : E = C - \frac{D(C - A)}{D - B} :$$

$$F = \sin E + 1 - 2E$$

$\boxed{\text{CALC}}$

18. Find the root of the equation x sin x + cos x = 0 using Newton Raphson's method so that relative error is less than 0.1. **[2016/Fall]**

Solution:

Let, $f(x) = x \sin x + \cos x$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = x \cos x$ ..... (2)

From equation (1)

Let the initial guess be

$x_0 = 2$, $f(x_0) = 1.4024$, $f'(x_0) = -0.8322$

Using NR method, next approximated root is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{1.4024}{-0.8322} = 3.6851$$

$f(x_1) = -2.7616$

Now, continuing the process in tabular form.

Itn.	x_n	$f(x_n) = x_n \sin x_n + \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} \sin x_{n+1} + \cos x_{n+1}$
1	2	1.4024	3.6851	−2.7616
2	3.6851	−2.7616	2.8095	−0.0294
3	2.8095	−0.0294	2.7984	-0.03×10^{-3}
4	2.7984	-0.03×10^{-3}	2.7983	2.26×10^{-4}
5	2.7983	2.26×10^{-4}	2.7983	7.32×10-7

Here, the value of x_{n+1} do not change up to 4 decimal places. And, relative error is also less than 0.1.

$$\text{Relative error} = \left(\frac{|x_{n+1} - x_n|}{x_{n+1}}\right)$$

$$= \left(\frac{|2.7983 - 2.7984|}{2.7983}\right)$$

$$= 0.003574$$

Hence, the desired root of the equation is 2.7983.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
> Set the following in calculator:
>
> $$A : B : A \sin A + \cos A : C = A - \frac{B}{A \cos A} : D = C \sin C + \cos C$$
>
> $\boxed{\text{CALC}}$

19. Using Newton-Raphson method find a root of the equation $xe^x = 2$.

Solution: **[2016/Spring]**

Let, $f(x) = xe^x - 2$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = e^x + x\,e^x$ ..... (2)

From equation (1),

Let the initial guess be,

$x_0 = 0, \qquad f(x_0) = 0e^0 - 2 = -2, \qquad f'(x_0) = e^0 + 0e^0 = 1$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-2}{1} = 2$$

$$f(x_1) = 12.7781$$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n) = x_n e^{x_n} - 2$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} e^{x_{n+1}} - 2$
1	0	-2	2	12.7781
2	2	12.7781	1.4235	3.9098
3	1.4235	3.9098	1.0349	0.9130
4	1.0349	0.9130	0.8755	0.1012
5	0.8755	0.1012	0.8530	1.71×10^{-3}
6	0.8530	1.71×10^{-3}	0.8526	-2.39×10^{-5}
7	0.8526	-2.39×10^{-5}	0.8526	-1.01×10^{-8}

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, a root of the equation is 0.8526.

> **NOTE:**
>
> Procedure to iterate in programmable calculator:
>
> Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
>
> Set the following in calculator:
>
> $$A : B = Ae^A - 2 : C = A - \frac{B}{e^A + Ae^A} : D = Ce^C - e$$
>
> $\boxed{\text{CALC}}$

20. Find a real root of the cos x = 3x – 1, correct to three decimal places, using fixed point method. **[2016/Spring]**

Solution:

Let, $f(x) = \cos x - 3x + 1 = 0$ ..... (1)

or, $\cos x + 1 = 3x$

or, $x = \dfrac{1 + \cos x}{3}$

i.e., $g(x) = \dfrac{1 + \cos x}{3}$ ..... (2)

Let initial guess be $x_0 = 1$ then,

$$|g'(x_0)| = \left|\frac{1}{3}(-\sin x)\right| = \left|\frac{1}{3}(-\sin 1)\right| = 0.2804$$

Here, $|0.2804| < 1$

Then next approximated root by fixed point method is given by,

$$g(x_0) = x_1 = \frac{1 + \cos(1)}{3} = 0.5134$$

Now, continuing the process in tabular form.

Itn.	x_n	$f(x_n) = \cos x_n - 3x_{n+1}$	$x_{n+1} = g(x_n) = \frac{1 + \cos x_n}{3}$	$f(x_{n+1}) = \cos x_{n+1} - 3x_{n+1} + 1$
1	1	−1.45	0.5134	0.3308
2	0.5134	0.3308	0.6236	−0.0590
3	0.6236	−0.0590	0.6039	0.0114
4	0.6039	0.0114	0.6077	-2.13×10^{-3}
5	0.6077	-2.13×10^{-3}	0.6069	7.19×10^{-4}
6	0.6069	7.19×10^{-4}	0.6071	5.88×10^{-6}
7	0.6071	5.88×10^{-6}	0.6071	-1.11×10^{-6}

Here, the value of $g(x_n)$ do not change up to 4 decimal places.

Hence, the real real of the equation is 0.6071.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1} = g(x_n)$, $D = f(x_{n+1})$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator:

$$A : B = \cos A - 3A + 1 : C = \frac{1 + \cos A}{3} : D = \cos C - 3C + 1$$

Step 3: Press CALC then,

Enter the value of A? then press =

Step 4: Now press = only, again and again to get the values for the respective row for each column.

Step 5: Update the values when A? is asked again.

Step 6: Go to step 4.

21. Find a real root of $e^{\cos x} - \sin x - 1 = 0$ correct to 4 decimal places using false position method. **[2017/Spring]**

Solution:

Let, $f(x) = e^{\cos x} - \sin x - 1$

The initial guess be,

$$x_L = x_0 = 0, \quad f(x_0) = e^{\cos(0)} - \sin(0) - 1 = 1.71828 > 0$$

$$x_U = x_1 = 1, \quad f(x_1) = e^{\cos(1)} - \sin(1) - 1 = -0.12494 < 0$$

i.e., Root lies between 0 and 1.

Now, using false position method,

$$x_2 = x_0 - \frac{(x_1 - x_0)\, f(x_0)}{f(x_1) - f(x_0)} = 0 - \frac{(1-0) \times 1.71828}{(-0.12494 - 1.71828)} = 0.93221$$

$\therefore \quad f(x_2) = 0.01201$

Since the value of $f(x_2)$ is positive, now root lies between 0.9322 and 1.

Solving other iterations in tabular form as follows,

Itn.	x_L	$f(x_L) = e^{\cos x_L} - \sin x_L - 1$	x_U	$f(x_U) = e^{\cos x_U} - \sin x_U - 1$	$x_N = x_L - \frac{f(x_L)(x_U - x_L)}{f(x_U) - f(x_L)}$	$(x_N) = e^{\cos x_N} - \sin x_N - 1$
1	0	1.71828	1	–0.12494	0.93221	0.01201
2	0.93221	0.01201	1	–0.12494	0.93815	-1.64×10^{-4}
3	0.93221	0.01201	0.93815	-1.64×10^{-4}	0.93806	1.95×10^{-5}
4	0.93806	1.95×10^{-5}	0.93815	-1.64×10^{-4}	0.93806	1.95×10^{-5}

Here, the value of x_N do not change up to 4 decimal places.

Hence, the root of the equation is 0.93806.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator:

$$A : C : B = e^{\cos A} - \sin A - 1 : D = e^{\cos C} - \sin C - 1 : E = A - \frac{(C - A)B}{D - B} :$$

$$F = e^{\cos E} - \sin E - 1$$

Step 3: Press CALC then,

Enter the value of A? then press =

Enter the value of C? then press =

Step 4: Now press = only, again and again to get the values for the respective row for each column.

Step 5: Update the values when A? and C? is asked again.

Step 6: Go to step 4.

22. Find the root of the equation $3x = \cos x + 1$ using NR method with the tolerance is 10E – 5. **[2017/Spring]**

Solution:

Let, $f(x) = 3x - \cos x - 1$ ..... (1)

Differentiating equation (1) with respect to x,

$$f'(x) = 3 + \sin x \quad \text{..... (2)}$$

From equation (1),

Let the initial guess be,

$x_0 = 0,\quad f(x_0) = 3 \times 0 - \cos 0 - 1 = -2 < 0$

$x_1 = 1,\quad f(x_1) = 3 \times 1 - \cos(1) - 1 = 1.4596 > 0$

so, a root lies between 0 and 1.

Using Newton Raphson method, next approximated root is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-2)}{3} = 0.6667$$

$$f(x_1) = 0.2142$$

Now, continuing process in tabular form.

Itn.	x_n	$f(x_n) = 3x_n - \cos x_n - 1$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = 3x_{n+1} - \cos x_{n+1} - 1$
1	0	-2	0.6667	0.2142
2	0.6667	0.2142	0.6075	1.422×10^{-3}
3	0.6075	1.422×10^{-3}	0.6071	-5.88×10^{-6}
4	0.6071	-5.88×10^{-6}	0.6071	-4.53×10^{-9}

Here, the value of x_{n+1} do not change up to 4 decimal places.
Hence, the desired root is 0.6071 of the equation.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
Set the following in calculator:

$$A : B = 3A - \cos A - 1 : C = A - \frac{B}{3 + \sin A} : D = 3C - \cos C - 1$$

$\boxed{\text{CALC}}$

23. Find the root of $e^x \tan x = 1$ by creating iterative formula of Newton-Raphson method. **[2018/Spring]**

Solution:

Let $f(x) = e^x \tan x - 1$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = e^x (\tan x + \sec^2 x)$ ..... (2)

From equation (1),
Let the initial guess be,

$x_0 = 0$

$f(x_0) = e^0 \tan 0 - 1 = -1$

$f'(x_0) = e^0(\tan 0 + \sec^2 0) = 1$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{1} = 1$$

$f(x_1) = 3.2334$

Now, continuing process in tabular form.

Itn.	x_n	$f(x_n) = e^{x_n} \tan x_n - 1$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = e^{x_{n+1}} \tan x_{n+1} - 1$
1	0	-1	1	3.2334
2	1	3.2334	0.7612	1.0396
3	0.7612	1.0396	0.5914	0.2132
4	0.5914	0.2132	0.5357	0.0142
5	0.5357	0.0142	0.5314	3.007×10^{-5}
6	0.5314	3.007×10^{-5}	0.5313	-2.988×10^{-4}
7	0.5313	-2.988×10^{-4}	0.5313	3.311×10^{-8}

Now, continuing process in tabular form.

Itn.	x_n	$f(x_n) = 3x_n - \cos x_n - 1$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = 3x_{n+1} - \cos x_{n+1} - 1$
1	0	−2	0.6667	0.2142
2	0.6667	0.2142	0.6075	1.422×10^{-3}
3	0.6075	1.422×10^{-3}	0.6071	-5.88×10^{-6}
4	0.6071	-5.88×10^{-6}	0.6071	-4.53×10^{-9}

Here, the value of x_{n+1} do not change up to 4 decimal places.
Hence, the desired root is 0.6071 of the equation.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
Set the following in calculator:

$$A : B = 3A - \cos A - 1 : C = A - \frac{B}{3 + \sin A} : D = 3C - \cos C - 1$$

$\boxed{\text{CALC}}$

23. Find the root of $e^x \tan x = 1$ by creating iterative formula of Newton-Raphson method. **[2018/Spring]**

Solution:

Let $f(x) = e^x \tan x - 1$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = e^x (\tan x + \sec^2 x)$ ..... (2)

From equation (1),

Let the initial guess be,

$x_0 = 0$

$f(x_0) = e^0 \tan 0 - 1 = -1$

$f'(x_0) = e^0(\tan 0 + \sec^2 0) = 1$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{1} = 1$$

$f(x_1) = 3.2334$

Now, continuing process in tabular form.

Itn.	x_n	$f(x_n) = e^{x_n} \tan x_n - 1$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = e^{x_{n+1}} \tan x_{n+1} - 1$
1	0	−1	1	3.2334
2	1	3.2334	0.7612	1.0396
3	0.7612	1.0396	0.5914	0.2132
4	0.5914	0.2132	0.5357	0.0142
5	0.5357	0.0142	0.5314	3.007×10^{-5}
6	0.5314	3.007×10^{-5}	0.5313	-2.988×10^{-4}
7	0.5313	-2.988×10^{-4}	0.5313	3.311×10^{-8}

Here, the value of x_{n+1} do not change up to 4 decimal places.
Hence, the desired root is 0.5313 of the equation.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
Set the following in calculator:

$$A : B = e^A \tan A - 1 : C = A - \frac{B}{e^A(\tan A + \sec^2 A)} : D = e^C \tan C - 1$$

CALC

24. Solve $f(x) = xe^x - 1$ by secant method for tolerance value 0.0001. **[2018/Spring]**

Solution:

$$f(x) = x\,e^x - 1$$

Let $x_0 = 0$ and $x_1 = 1$ be two initial guesses.
Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 1 - \frac{(1.7182)\,(1-0)}{1.7182 - (-1)} = 0.3678$$

$$f(x_2) = -0.4686$$

Now, solving other iterations in tabular form as follows,

Itn.	x_{n-1}	$f(x_{n-1}) = x_{n-1} e^{x_{n-1}} - 1$	x_n	$f(x_n) = x_n e^{x_n} - 1$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = x_{n+1} e^{x_{n+1}} - 1$
1	0	–1	1	1.7182	0.3678	–0.4686
2	1	1.7182	0.3678	–0.4686	0.5032	–0.1677
3	0.3678	–0.4686	0.5032	–0.1677	0.5786	0.0319
4	0.5032	–0.1677	0.5786	0.0319	0.5665	–0.0017
5	0.5786	0.0319	0.5665	–0.0017	0.5671	–0.0001
6	0.5665	–0.0017	0.5671	–0.0001	0.5671	–0.0001

Here, the value of x_{n+1} do not change up to 4 decimal places with the tolerance value of 0.0001.

Hence, the root of the equation is 0.5671.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$
Set the following in calculator:

$$A : C : B = Ae^A - 1 : D = Ce^C - 1 : E = C - \frac{D(C - A)}{D - B} : F = Ee^E - 1$$

CALC

25. Using secant method, find a root of the equation $e^x \sin x - x^2 = 0$ correct up to three decimal places. **[2018/Fall]**

Solution:

Let, $f(x) = e^x \sin x - x^2$

and, $x_0 = 2$ and $x_1 = 3$ be two initial guesses.

$f(x_0) = e^2 \sin(2) - 2^2 = 2.7188$ and $f(x_1) = -6.1655$

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 3 - \frac{-6.1655(3-2)}{-6.1655 - 2.7188}$$

$$= 2.3060$$

$f(x_2) = 2.1246$

Now, solving other iterations in tabular form as follows,

Itn.	x_{n-1}	$f(x_{n-1}) = e^{x_{n-1}} \sin x_{n-1} - x_{n-1}^2$	x_n	$f(x_n) = e^{x_n} \sin x_n - x_n^2$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = e^{x_{n+1}} \sin x_{n+1} - x_{n+1}^2$
1	2	2.7188	3	-6.1655	2.3060	2.1246
2	3	-6.1655	2.3060	2.1246	2.4838	1.1590
3	2.3060	2.1246	2.4838	1.1590	2.6972	-0.8958
4	2.4838	1.1590	2.6972	-0.8958	2.6041	0.1401
5	2.6972	-0.8958	2.6041	0.1401	2.6166	0.0144
6	2.6041	0.1401	2.6166	0.0144	2.6180	1.43×10^{-4}
7	2.6166	0.0144	2.6180	1.43×10^{-4}	2.6180	-8.70×10^{-7}

Here, the value of x_{n+1} do not change up to three decimal places.

Hence, the root of given equation is 2.6180.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$

Set the following in calculator:

$$A : C : B = e^A \sin A - A^2 : D = e^C \sin C - C^2 : E = C - \frac{D(C-A)}{D-B} :$$
$$F = e^E \sin E - E^2$$

$\boxed{\text{CALC}}$

26. Find where the graph of $y = x - 3$ and $y = ln(x)$ intersect using bisection method. Get the intersection value correct to four decimal places. **[2019/Fall]**

Solution:

$y = x - 3$ and $y = ln(x)$

$f(x_1) = x - 3,$ $f(x_2) = ln\,(x)$

In order to intersect,

$$f(x_1) - f(x_2) = 0$$

i.e., $f(x) = x - 3 - \ln(x) = 0$

Let initial guess be,

$x = 1,\quad f(1) = 1 - 3 - \ln(1) = -2$

$x = 2,\quad f(2) = -1.6991 < 0$

$x = 3,\quad f(3) = -1.0986 < 0$

$x = 4,\quad f(4) = -0.3862 < 0$

$x = 5,\quad f(5) = 0.3905 > 0$

so, root lies between x = 4 and x = 5.

$\therefore \quad x_L = 4$ and $x_U = 5$

Now, first approximated root using bisection method,

$$x_N = \frac{x_L + x_U}{2} = \frac{4+5}{2} = 4.5$$

$f(x_N) = -0.0040 < 0$ so now root lies between 4.5 and 5.

Remaining iterations are solved in tabular form.

Itn.	x_L	$f(x_L) = x_L - 3 - \ln(x_L)$	x_U	$f(x_U) = x_U - 3 - \ln(x_U)$	x_N	$f(x_N) = x_N - 3 - \ln(x_N)$
1	4	−0.3862	5	0.3905	4.5	−0.0040
2	4.5	−0.0040	5	0.3905	4.75	0.1918
3	4.5	−0.0040	4.75	0.1918	4.625	0.0935
4	4.5	−0.0040	4.625	0.0935	4.5625	0.0446
5	4.5	−0.0040	4.5625	0.0446	4.53125	0.0202
6	4.5	−0.0040	4.53125	0.0202	4.515625	0.0080
7	4.5	−0.0040	4.515625	0.0080	4.5078125	0.0020
8	4.5	−0.0040	4.5078125	0.0020	4.50390625	−0.0010
9	4.50390625	−0.0010	4.5078125	0.0020	4.505859	0.0004
10	4.503906	−0.0010	4.505859	0.0004	4.504882	−0.0002
11	4.504882	−0.0002	4.505859	0.0004	4.505370	−0.0001
12	4.504882	−0.0002	4.505370	0.0001	4.505126	-8.985×10^{-5}
13	4.505126	-8.985×10^{-5}	4.505370	0.0001	4.505248	5.060×10^{-6}
14	4.505126	-8.985×10^{-5}	4.505248	5.060×10^{-6}	4.505187	-4.239×10^{-5}
15	4.505187	-4.239×10^{-5}	4.505248	5.060×10^{-6}	4.5052175	-1.866×10^{-5}
16	4.5052175	-1.866×10^{-5}	4.505248	5.060×10^{-6}	4.505232	-6.804×10^{-6}

Here, the value of x_N do not change up to 4 decimal places.

Hence, the graph of $y = x - 3$ and $y = ln(x)$ intersects at $x = 4.505232$.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$
> Set the following in calculator:
>
> $$A : C : B = A - 3 - ln(A) : D = C - 3 - ln(C) : E = \frac{A + C}{2} : F = E - 3 - ln(E)$$
>
> $\boxed{\text{CALC}}$

27. **Find value of $\sqrt{18}$ using Newton Raphson method.** **[2019/Fall]**

Solution:

Let $x = \sqrt{N}$ or $x^2 - N = 0$

Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$

Then Newton's formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

Now, taking $N = 18$, the above formula becomes

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{18}{x_n}\right)$$

For initial guess, taking approximate value of $\sqrt{18}$

i.e., $\sqrt{18} = \sqrt{4^2} = \sqrt{16} = 4$

i.e., we take $x_0 = 4$

Then,

$$x_1 = \frac{1}{2}\left(x_n + \frac{18}{x_0}\right) = \frac{1}{2}\left(4 + \frac{18}{4}\right) = 4.25$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{18}{x_1}\right) = \frac{1}{2}\left(4.25 + \frac{18}{4.25}\right) = 4.2426$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{18}{x_2}\right) = \frac{1}{2}\left(4.2426 + \frac{18}{4.2426}\right) = 4.2426$$

Here, $x_2 = x_3$ up to 4 decimal places.

Hence, the value of $\sqrt{18}$ is 4.2426.

28. **Using secant method, find the zero of function $f(x) = 2x - \log_{10} x - 7$ correct up to three decimal places.** **[2019/Spring]**

Solution:

$$f(x) = 2x - \log_{10} x - 7$$

Let, $x_0 = 1$ and $x_1 = 2$ be two initial guesses.

> **NOTE:**
> 0 is not taken as initial guess because it gives the undetermined value of $f(x)$ at $x = 0$.

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 2 - \frac{(-3.3010)(2-1)}{-3.3010-(-5)}$$

$$= 3.9429$$

$$f(x_2) = 0.2899$$

Now, solving other iterations in tabular form as follows

Itn.	x_{n-1}	$f(x_{n-1}) = 2x_{n-1} - \log_{10}x_{n-1} - 7$	x_n	$f(x_n) = 2x_n - \log_{10} x_n - 7$	$x_{n+1} = x_n - \frac{f(x_n)(x_n-x_{n-1})}{f(x_n)-f(x_{n-1})}$	$f(x_{n+1}) = 2x_{n+1} - \log_{10}x_{n+1} - 7$
1	1	-5	2	-3.3010	3.9429	0.2899
2	2	-3.3010	3.9429	0.2899	3.7860	-6.180×10^{-3}
3	3.9429	0.2899	3.7860	-6.180×10^{-3}	3.7892	-1.475×10^{-3}
4	3.7860	-6.180×10^{-3}	3.7892	-1.475×10^{-3}	3.7902	-0.1508
5	3.7892	-1.475×10^{-3}	3.7902	-0.1508	3.7891	-3.360×10^{-4}
6	3.7902	-0.1508	3.7891	-3.360×10^{-4}	3.7890	-5.246×10^{-4}
7	3.7891	-3.360×10^{-4}	3.7890	-5.246×10^{-4}	3.7892	-1.475×10^{-4}

Here, the value of x_{n+1} do not change up to 3 decimal places.

Hence, the zero of function f(x) is 3.7892.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$

Set the following in calculator:

$$A : C : B = 2A - \log_{10} A - 7 : D = 2C - \log_{10} C - 7 : E = C - \frac{D(C-A)}{D-B} :$$

$$F = 2E - \log_{10} E - 7$$

$\boxed{\text{CALC}}$

29. Find the root of the equation log x – cos x = 0 correct up to three decimal placed by using N-R method. [2019/Spring]

Solution:

Let, $f(x) = \log x - \cos x$ ..... (1)

Differentiating equation (1) with respect to x,

$$f'(x) = \frac{1}{x} + \sin x \quad \text{..... (2)}$$

From equation (1),

Let the initial guess be,

$$x_0 = 1, f(x_0) = -0.5403, f'(x_0) = 1.8414$$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-0.5403}{1.8414} = 1.2934$$

$$f(x_1) = -0.1621$$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n) = \log x_n - \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = \log x_{n+1} - \cos x_{n+1}$
1	1	−0.5403	1.2934	−0.1621
2	1.2934	−0.1621	1.3868	−0.0409
3	1.3868	−0.0409	1.4107	-9.97×10^{-3}
4	1.4107	-9.97×10^{-3}	1.4165	-2.46×10^{-3}
5	1.4165	-2.46×10^{-3}	1.4179	-6.55×10^{-4}
6	1.4179	-6.55×10^{-4}	1.4182	-2.67×10^{-4}
7	1.4182	-2.67×10^{-4}	1.4183	-1.37×10^{-4}

Here, the value of x_{n+1} do not change up to 3 decimal places.

Hence, the desired root is 1.4183 of the equation.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
> Set the following in calculator:
>
> $$A : B = \log A - \cos A : C = A - \frac{B}{\sin A + \frac{1}{A}} : D = \log C \cos C$$
>
> CALC

30. Find the positive real root of the equation $\cos x + e^x + x^2 = 3$. Using false position method, correct to 3 decimal places **[2020/Fall]**

Solution:

Let, $(x) = \cos x + e^x + x^2 - 3$

The initial guess be,

$x_0 = 0$, $\quad f(x_0) = \cos 0 + e^0 + 0^2 - 3 = -1 < 0$

$x_1 = 1$, $\quad f(x_1) = \cos(1) + e^1 + 1^2 - 3 = 1.2585 > 0$

so, root lies between 0 and 1.

Now, using false position method,

$$x_2 = x_0 - \frac{(x_1 - x_0)\, f(x_0)}{f(x_1) - f(x_0)}$$

$$= 0 - \frac{(1-0)(-1)}{1.2585 - (-1)} = 0.4427$$

$\therefore \quad f(x_2) = -0.3435$

Since the value of $f(x_2)$ is negative, now root lies between 0.4427 and 1.

Solving other iterations in tabular form as follows,

Itn.	x_L	$f(x_L) = \cos x_L + e^{x_L} + x_L^2 - 3$	x_U	$f(x_U) = \cos x_U + e^{x_U} + x_U^2 - 3$	$x_N = x_L - \dfrac{f(x_L)(x_U - x_L)}{f(x_U) - f(x_L)}$	$f(x_N) = \cos x_N + e^{x_N} + x_N^2 - 3$
1	0	–1	1	1.2585	0.4427	–0.3435
2	0.4427	–0.3435	1	1.2585	0.5621	–0.0835
3	0.5621	–0.0835	1	1.2585	0.5893	–0.0186
4	0.5893	–0.0186	1	1.2585	0.5952	-4.30×10^{-3}
5	0.5952	-4.30×10^{-3}	1	1.2585	0.5965	-1.12×10^{-3}
6	0.5965	-1.12×10^{-3}	1	1.2585	0.5968	-3.94×10^{-4}

Here, the value of x_N do not change up to three decimal places.

Hence, the positive real root of the equation is 0.5968.

NOTE:

Procedure to iterate in programmable calculator:

Let, A = x_L, B = $f(x_L)$, C = x_U, D = $f(x_U)$, E = x_N, F = $f(x_N)$

Set the following in calculator:

$$A : C : B = \cos A + e^A + A^2 - 3 : D = \cos C + e^C + C^2 - 3 :$$

$$E = A - \frac{(C - A)B}{D - B} : F = \cos E + e^E + E^2 - 3$$

CALC

31. Find the real root of the equation x sin x – cos x = 0 using Newton-Raphson method, correct to 3 decimal places. [2020/Fall]

Solution:

Let, $f(x) = x \sin x - \cos x$ ..... (1)

Differentiating equation (1) with respect to x,

$$f'(x) = x \cos x + \sin x + \sin x \quad \text{..... (2)}$$

$$= x \cos x + 2 \sin x$$

From equation (1),

Let the initial guess be,

$x_0 = 1$, $f(x_0) = 0.3011$, $f'(x_0) = 2.2232$

Using NR method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{0.3011}{2.2232} = 0.8645$$

$$f(x_1) = 8.66 \times 10^{-3}$$

Now, continuing process in tabular form.

Itn.	x_n	$f(x_n) = x_n \sin x_n - \cos x_n$	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} \sin x_{n+1} - \cos x_{n+1}$
1	1	0.3011	0.8645	8.66×10^{-3}
2	0.8645	8.66×10^{-3}	0.8603	-6.97×10^{-5}
3	0.8603	-6.97×10^{-5}	0.8603	-7.02×10^{-8}

Here, the value of x_{n+1} do not change up to 3 decimal places.

Hence, the desired root of the equation is 0.8603.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$

Set the following in calculator:

$$A : B = A \sin A - \cos A : C = A - \frac{B}{A \cos A + 2 \sin A} :$$

$$D = C \sin C - \cos C$$

CALC

32. Find the real root of the equation $x \log_{10} x - 1.2 = 0$ correct to four places of decimal using bracketing method. [2014/Fall]

Solution:

Let, $f(x) = x \log_{10} x - 1.2$

Let initial guess be,

$x = 2.5$, $f(2.5) = -0.2051 < 0$

$x = 3$, $f(3) = 0.2313 > 0$

so, root lies between $x = 2.5$ and $x = 3$

$\therefore$ $x_L = 2.5$ and $x_U = 3$

Now, first approximated root using bisection method,

$$x_N = \frac{x_L + x_U}{2} = \frac{2.5 + 3}{2} = 2.75$$

$f(x_N) = 8.16 \times 10^{-3} > 0$ so now root lies between 2.5 and 2.75

Remaining iterations are solved in tabular form.

Itn.	x_L	$f(x_L) = x_L \log_{10} x_L - 1.2$	x_U	$f(x_U) = x_U \log_{10} x_U - 1.2$	$x_N = \frac{x_L + x_U}{2}$	$f(x_N) = x_N \log_{10} x_N - 1.2$
1	2.5	-0.2051	3	0.2313	2.75	8.16×10^{-3}
2	2.5	-0.2051	2.75	8.160×10^{-3}	2.625	-0.0997
3	2.625	-0.0997	2.75	8.16×10^{-3}	2.6875	-0.0461
4	2.6875	-0.0461	2.75	8.16×10^{-3}	2.7187	-0.0191
5	2.7187	-0.0191	2.75	8.16×10^{-3}	2.7343	-5.53×10^{-3}
6	2.7343	-5.53×10^{-3}	2.75	8.16×10^{-3}	2.7421	1.26×10^{-3}
7	2.7343	-5.53×10^{-3}	2.7421	1.26×10^{-3}	2.7382	-2.13×10^{-3}
8	2.7382	-2.13×10^{-3}	2.7421	1.26×10^{-3}	2.7401	-4.76×10^{-4}
9	2.7401	-4.76×10^{-4}	2.7421	1.26×10^{-3}	2.7411	3.95×10^{-4}
10	2.7401	-4.76×10^{-4}	2.7411	3.95×10^{-4}	2.7406	-4.02×10^{-5}
11	2.7406	-4.02×10^{-5}	2.7411	3.95×10^{-4}	2.7408	1.34×10^{-4}
12	2.7406	-4.02×10^{-5}	2.7408	1.34×10^{-4}	2.7407	4.70×10^{-5}
13	2.7406	-4.02×10^{-5}	2.7407	4.70×10^{-5}	2.7406	-4.02×10^{-5}
14	2.7406	-4.02×10^{-5}	2.7407	4.70×10^{-5}	2.7406	-4.022×10^{-5}

Here, the value of x_N do not change up to 4 decimal places.

Hence, the real of the equation is 2.7406.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Set the following in calculator:

$$A : C : B = A \log_{10} A - 1.2 : D = C \log_{10} C - 1.2 : E = \frac{A + C}{2} :$$

$$F = E \log_{10} E - 1.2$$

$\boxed{\text{CALC}}$

33. Write short notes on error in numerical calculations.

[2013/Spring, 2014/Spring, 2016/Spring, 2019/Fall]

Solution: See the topic 1.3.

34. Write short notes on monotonic and oscillatory divergence in fixed point iteration method. **[2014/Fall]**

Solution:

If $g:[a, b] \longrightarrow [a, b]$ is continuous, then $g(x)$ has a fixed point, x^*, such that $g(x^*) = x^*$. Furthermore, if $|g'(x)| < 1$ for all $x \in [a, b]$, then this fixed point is unique on $[a, b]$ and the fixed point iteration $x_{n+1} = g(x_n)$ will coverage to x^* for all choices of $x_0 \in [a, b]$.

We have,

$$|g(x) - (y)| \leq |g'(z)|\,|x - y| < |x - y|$$

where, $Z \in [x, y] \subset [a, b]$. Thus the mapping contracts and by the contraction mapping theorem $x_n \longrightarrow x^*$, the unique fixed point.

The plane R^2 may be divided into four regions for some $g(x)$ with respect to one of its fixed point x^*, depending on the behaviour $g(x)$ takes while in a given region. Chiefly, one is concerned with when $|g(x) - x^*| < |x - x^*|$, so that $g(x)$ is said to converge.

The region where $g(x)$ converges is bounded by points satisfying $|g(x) - x^*| < |x - x^*|$. For such a boundary, either $g(x) - x^* = x - x^*$ or $g(x) - x^* = x^* - x$. The former indicates that the boundary consists of additional fixed points, $g(x) = x$. The latter gives the boundary $g(x) = 2x^* - x$. One may summarize that if $g(x)$ lies between the lines x and $2x^* - x$, then $g(x)$ will be closer to the fixed point x^* than x.

The second division of behaviour is whether sign $[g(x - x^*)] = \text{sign}\,(x - x^*)$. If this is true, then $g(x)$ converges or diverges monotonically towards or away from x^* in this region. If this is false then $g(x)$ oscillates around x^*. The boundary between these two regions is where sign $[g(x) - x^*) = 0]$ or where $g(x) = x^*$.

We are now prepared to describe the four regions of a fixed point function is 1D, based on these behaviours (convergence/divergence, monotonic/ oscillation).

Region 1

If $g(x) < x < x^*$ or $g(x) > x > x^*$, then $g(x)$ diverges monotonically from x^*

Region 2

If $x < g(x) < x^*$ or $x > g(x) > x^*$, then $g(x)$ converges monotonically towards x^*

Region 3

If $x < x^* < g(x) < 2x^* - x$ or, $x > x^* > g(x) > 2x^* - x$, then $g(x)$ converges with oscillations towards x^*.

Region 4

If $x < x^* < 2x^* - x < g(x)$ or $x > x^* > 2x^* - x > g(x)$, then $g(x)$ diverges with oscillations from x^*.

35. Write short notes on an algorithm for NR-method. **[2014/Spring]**

Solution: See the topic 1.8.1.

36. Write short notes on convergence of Newton-Raphson methods. **[2015/Fall]**

Solution: See the topic 1.8.

37. Write short notes on importance of numerical methods in Engineering. **[2018/Fall]**

Solution:

Numerical simulation is a powerful tool to solve scientific and engineering problem. it plays an important role in many aspects of fundamental research and engineering applications. For example mechanism of turbulent flow with/without visco-elastic additives, optimization of processes, prediction of oil/gas production and online control of manufacturing. The soul of numerical simulation is numerical method which is driven by the above demands and in return pushes science and technology by the successful applications of advanced numerical methods. With the development of mathematical theory and computer hardware, various numerical methods are proposed. The new numerical methods or their new applications lead to important progress in the related fields. For example, parallel computing largely promote the precision of direct numerical simulations of turbulent flow to capture undiscovered flow structures. Proper orthogonal decomposition method greatly reduces the simulation time of oil pipelining transportation. Thus, numerical methods become more and more important and their modern developments are worth exploring.

A numerical method is a complete and definite set of procedures for the solution of a problem, together with computable error estimates. The study and implementation of such methods is the province of numerical analysis.

Numerical methods may be regard as a new 'philosophy' in the development of the computer based scientific methods. Even the computer based approaches are deterministic or randomness based *i.e.*, semi-numerical methods. The major advantage of numerical methods is that a numerical value can be obtained even when the problem has no analytical solution.

In many aspects of our life, a huge amount of different materials are used. Glass, wood, metals, concrete, which are directly used almost every minute in our everyday life. Thus, the modification of materials and prediction of their properties are very important objectives for the manufactures. In order to produce high quality materials, the engineers in industry, among other problems, are very much interested in the elastic behaviour or loading capacity of the material. While it is known that the bonding forces between the atoms of the material are responsible for their physical and chemical properties. So to manufacture a new product with higher quality, a detailed investigation of the material on the atomic level is not required in most cases. A mathematical model is needed for the quantitative description of the change of material properties under external influences. The concept of differential equations come to help us as an excellent tool for the development of such a model.

38. Write short notes on convergence of fixed point iteration method. **[2018/Spring]**

Solution: See the topic 1.9.

39. Write short notes on: algorithm of Bisection method. **[2019/Spring]**

Solution: See the topic 1.6.1.

40. Write an algorithm to find a real of a non-linear equation using secant method. **[2016/Spring]**

Solution: See the topic 1.7.1.

ADDITIONAL QUESTION SOLUTION

1 **Round off the number 75462 to four significant digits and then calculate the absolute error and percentage error.**

Solution:

Given that;

$$x = 75462$$

Now, rounding off the number up to four significant digits

$$x_1 = 75460$$

Then,

$$\text{Absolute error } (E_a) = |x - x_1|$$

$$= |75462 - 75460| = 2$$

and, $\text{Percentage error } (E_p) = E_r \times 100$

$$= \left|\frac{x - x_1}{x}\right| \times 100$$

$$= \left|\frac{75462 - 75460}{75462}\right| \times 100$$

$$= 0.0027$$

2. **If 0.333 is the approximate value of $\frac{1}{3}$, find the absolute and relative errors.**

Solution:

We have,

$$\text{Exact value } (x) = \frac{1}{3}$$

$$\text{Approximate value } (x_1) = 0.333$$

Then,

$$\text{Absolute error, } E_a = |x - x_1| = \left|\frac{1}{3} - 0.333\right| = 0.0003$$

$$\text{Relative error, } E_r = \left|\frac{x - x_1}{x}\right| = \left|\frac{\frac{1}{3} - 0.333}{\frac{1}{3}}\right| = 0.0010$$

3. **The height of an observation tower was estimated to be 47 m, whereas its actual height was 45 m. Calculate the percentage relative error in the measurement.**

Solution:

We have,

$$\text{Actual height of tower } (x) = 45 \text{ m}$$

$$\text{Estimated height of tower } (x_1) = 47 \text{ m}$$

Then,

Percentage relative error, $E_p = E_r \times 100$

$$= \left|\frac{x - x_1}{x}\right| \times 100 = \left|\frac{45 - 47}{45}\right| \times 100$$

$$= 4.4444$$

4. Find a real root of the following equation, correct to six decimals, using the fixed point iteration method.

$$\sin x + 3x - 2 = 0$$

Solution:

Let, $f(x) = \sin x + 3x - 2 = 0$

or, $3x = 2 - \sin x$ ..... (1)

or, $x = \dfrac{2 - \sin x}{3}$

i.e., $g(x) = \dfrac{2 - \sin x}{3}$ ..... (2)

Differentiating equation (2) with respect to x

$$g'(x) = \frac{1}{3}(0 - \cos x) = \frac{-\cos x}{3}$$

Let initial guess be $x_0 = 1$ then

$$|g'(x_0)| = \left|\frac{-\cos(1)}{3}\right| = 0.180101$$

Here; $0.180101 < 1$

Then next approximated root by fixed point method is given by,

$$g(x_0) = x_1 = \frac{2 - \sin(1)}{3} = 0.386176$$

Now continuing the process in tabular form

Iteration	x_L	$f(x_n)$	$x_{n+1} = g(x_n)$	$f(x_{n+1})$
1	1	1.841471	0.386176	−0.464823
2	0.386176	−0.464823	0.541117	0.138445
3	0.541117	0.138445	0.494969	−0.040089
4	0.494969	−0.040089	0.508332	0.011717
5	0.508332	0.011717	0.514426	−0.003417
6	0.504426	−0.003417	0.505565	0.000997
7	0.505565	0,000997	0.505233	−0.000290
8	0.505233	−0.000290	0.505330	0.000086
9	0.505330	0.000086	0.505301	−0.000026
10	0.505301	−0.000026	0.505310	0.000009
11	0.505310	0.000009	0.505307	−0.000003
12	0.505307	−0.000003	0.505308	0.000001
13	0.505308	0.000001	0.505308	0.000001

Here, the value of $g(x_n)$ do not change up to 6 decimal places.

Hence, the real root of the equation is 0.505308.

5. **Find a real root of the equation $\sin x = e^{-x}$ correct up to four decimal places using N.R method.**

Solution:

Let, $f(x) = \sin x - e^{-x}$ ..... (1)

Differentiating equation (1) with respect to x

$$f'(x) = \cos x + e^{-x} \quad \text{..... (2)}$$

From equation (1)

Let the initial guess be

$$x_0 = 0, \qquad f(x_0) = -1, \qquad f'(x_0) = 2$$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{2} = 0.5$$

$$f(x_1) = -0.12711$$

Now continuing process in tabular form.

Iteration	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	0	−1	0.5	−0.12711
2	0.5	−0.12711	0.58565	−0.00400
3	0.58565	−0.00400	0.58853	-3.80×10^{-6}
4	0.58853	-3.80×10^{-6}	0.58853	-6×10^{-9}

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, the desired real root of the equation is 0.58853.

6. **Find a real root of $\cos x + e^x - 5 = 0$ accurate to 4 decimal places using the secant method.**

Solution:

Let, $f(x) = \cos x + e^x - 5$

Let, $x_0 = 1$ and $x_1 = 2$ be two initial guesses.

Then,

$$f(x_0) = -1.74142 \text{ and } f(x_1) = 1.97291$$

Next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 2 - \frac{1.97291\,(2-1)}{1.97291 + 1.74142}$$

$$= 1.46884$$

$$f(x_2) = -0.55403$$

Now solving other iterations in tabular form as

Iteration	x_{n-1}	$f(x_{n-1})$	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	1	-1.74142	2	1.97291	1.46884	-0.55403
2	2	1.97291	1.46884	-0.55403	1.58530	-0.13375
3	1.46884	-0.55403	1.58530	-0.13375	1.62236	0.01349
4	1.58530	-0.13375	1.62236	0.01349	1.61896	-0.00031
5	1.62236	0.01349	1.61896	-0.00031	1.61904	0.00002
6	1.61896	-0.00031	1.61904	0.00002	1.61904	-2.91×10^{-6}

Here, the value of x_{n+1} do not change up to 4 decimal places.
Hence, the root of the equation is 1.61904.

7. Using the Bisection method, find a real root of the equation $f(x) = 3x - \sqrt{1 + \sin x}$ correct up to three decimal points.

Solution:

Given that;

$$f(x) = 3x - \sqrt{1 + \sin x}$$

Let initial guess be

$x = 0,\quad f(0) = -1 < 0$

$x = 1,\quad f(1) = 1.64299 > 0$

so, root lies between $x = 0$ and $x = 1$

$\therefore\quad x_L = 0$ and $x_U = 1$

Now first approximated root using bisection method

$$x_N = \frac{x_L + x_U}{2} = \frac{0+1}{2} = 0.5$$

$$f(x_N) = 0.2837 > 0$$

so, now root lies between 0 and 0.5

Remaining iterations are solved in tabular form

Iteration	x_L	$f(x_L)$	x_U	$f(x_U)$	x_N	$f(x_N)$
1	0	-1	1	1.64299	0.5	0.2837
2	0	-1	0.5	0.2837	0.25	-0.3669
3	0.25	-0.3669	0.5	0.2837	0.375	-0.0439
4	0.375	-0.0439	0.5	0.2837	0.4375	0.1193
5	0.375	-0.0439	0.4375	0.1193	0.4063	0.0377
6	0.375	-0.0439	0.4063	0.0377	0.3907	-0.0030
7	0.3907	-0.0030	0.4063	0.0377	0.3985	0.0174
8	0.3907	-0.0030	0.3985	0.0174	0.3946	0.0072
9	0.3907	-0.0030	0.3946	0.0072	0.3927	0.0022
10	0.3907	-0.0030	0.3927	0.0022	0.3917	-0.0004
11	0.3917	-0.0004	0.3927	0.0022	0.3922	0.0009
12	0.3917	-0.0004	0.3922	0.0009	0.3920	0.0004

Here, the value of x_N do not change up to three decimal places.

Hence, the real root of the equation is 0.3920.

8. **Find a root of $e^x = 3x$ using bisection method and Newton Raphson method correct up to 3 decimal places.**

Solution:

Let, $f(x) = e^x - 3x$

i) Using bisection method

Let initial guess be

$x = 0.5,\quad f(0.5) = 0.1487 > 0$

$x = 1,\quad f(1) = -0.2817 < 0$

so, root lies between x = 0.5 and x = 1

$\therefore\quad x_L = 0.5$ and $x_U = 1$

Now, first approximated root using bisection method

$$x_N = \frac{x_L + x_U}{2} = \frac{0.5 + 1}{2} = 0.75$$

$f(x_N) = -0.1330 < 0$ so root lies between 0.5 and 0.75

Remaining iterations are solved in tabular form

Iteration	x_L	$f(x_L)$	x_U	$f(x_U)$	x_N	$f(x_N)$
1	0.5	0.1487	1	−0.2817	0.75	−0.1330
2	0.5	0.1487	0.75	−0.1330	0.625	−0.0068
3	0.5	0.1487	0.625	−0.0068	0.5625	0.0676
4	0.5625	0.0676	0.625	−0.0068	0.5938	0.0295
5	0.5938	0.0295	0.625	−0.0068	0.6094	0.0111
6	0.6094	0.0111	0.625	−0.0068	0.6172	0.0021
7	0.6172	0.0021	0.625	−0.0068	0.6211	−0.0023
8	0.6172	0.0021	0.6211	−0.0023	0.6192	−0.0002
9	0.6172	0.0021	0.6192	−0.0002	0.6182	0.0010
10	0.6182	0.0010	0.6192	−0.0002	0.6187	0.0004

Here, the value of x_N do not change up to three decimal places.

Hence, the root of the equation is 0.6187.

ii) Using NR method

$f(x) = e^x - 3x$ ..... (1)

Differentiating equation (1) with respect to x

$f'(x) = e^x - 3$

From equation (1)

Let the initial guess be

$x_0 = 0,\ f(x_0) = 1,\ f'(x_0) = -2$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{(-2)} = 0.5$$

$$f(x_1) = 0.1487$$

Now, continuing the iterations in tabular form.

Iteration	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	0	1	0.5	0.1487
2	0.5	0.1487	0.6100	0.0104
3	0.61	0.0104	0.6190	0.0001
4	0.619	0.0001	0.6191	-0.00004

Here, the value of x_{n+1} do not change up to 3 decimal places.

Hence, the desired root of the equation is 0.6191.

9. Find a real root of $x^5 - 3x^3 - 1 = 0$ correct up to four decimal places using the secant method.

Solution:

Let, $f(x) = x^5 - 3x^3 - 1$

and, $x_0 = 1$ and $x_1 = 2$ be two initial guesses

$f(x_0) = -3$ and $f(x_1) = 7$

Then next approximated root by secant method is given by

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 2 - \frac{7(2-1)}{7+3} = 1.3$$

$$f(x_2) = -3.8781$$

Now, solving other iterations in tabular form as,

Iteration	x_{n-1}	$f(x_{n-1})$	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	1	-3	2	7	1.3	-3.8781
2	2	7	1.3	-.38781	1.5496	-3.2279
3	1.3	-3.8781	1.5496	-3.2279	2.7887	102.5969
4	1.5496	-3.2279	2.7887	102.5969	1.5874	-2.9206
5	2.7887	102.5969	1.5874	-2.9206	1.6207	-2.5893
6	1.5874	-2.9206	1.6207	-2.5893	1.8810	2.5816
7	1.6207	-2.5893	1.8810	2.5816	1.7510	-0.6457
8	1.8810	2.5816	1.7510	-0.6457	1.7770	-0.1149
9	1.7510	-0.6457	1.7770	-0.1149	1.7826	0.0064
10	1.7770	-0.1149	1.7826	0.0064	1.7823	-0.0002
11	1.7826	0.0064	1.7823	-0.0002	1.7823	-0.0002

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, a real root of the equation is 1.7823.

10. Evaluate the real root of $f(x) = 4 \sin x - e^x$ using Newton Raphson method. The absolute error of root in consecutive iteration should be less than 0.01.

Solution:

Let, $f(x) = 4 \sin x - e^x$ ..... (1)

Differentiating equation (1) with respect to x

$$f'(x) = 4 \cos x - e^x$$

From equation (1)

Let the initial guess be

$$x_0 = 0, \qquad f(x_0) = -1, \qquad f'(x_0) = 3$$

Using NR method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{3} = 0.3333$$

$$f(x_1) = -0.0869$$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	0	−1	0.3333	−0.0869
2	0.3333	−0.0869	0.3697	−0.0020
3	0.3697	−0.0020	0.3706	0.0001
4	0.3706	0.0001	0.3706	0.0001

Here, the value of x_{n+1} do not change up to 4 decimal places and have error less than 0.01. Hence, required root is 0.3706.