Chapter

6

SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

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6.1 INTRODUCTION

Partial differential equations arise in the study of many branches of applied mathematics. For example; in fluid dynamics, heat transfer, boundary layer flow, elasticity, quantum mechanics and electro-magnetic theory. Only a few of these equations can be solved by analytical methods which are also complicated by requiring use of advanced mathematical techniques. In most of the cases, it is easier to develop approximate solutions by numerical methods. Of all the numerical methods available for the solution of partial differential equations, the method of finite differences is most commonly used. In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite difference equations. Then the given equation is changed to a system of linear equations which are solved by iterative procedures. This process is slow but produces good results in many boundary value problems. An added advantage of this method is that the computation can be carried by electronic computers. To accelerate the solution, sometimes the method of relaxation proves quite effective.

6.2 CLASSIFICATION OF SECOND ORDER EQUATIONS

The general linear partial differential equation of the second order in two independent variables is of the form.

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} + \left(x, y, u\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad \text{..... (1)}$$

Such a partial differential equation is said to be

a) Elliptic if $B^2 - 4AC < 0$

b) Parabolic if $B^2 - 4AC = 0$

c) Hyperbolic if $B^2 - 4AC > 0$

A partial equation is classified according to the region in which it is desired to be solved. For instance, the partial differential equation $f_{xx} + f_{yy} = 0$ is elliptic if $y > 0$, parabolic if $y = 0$ and hyperbola if $y < 0$.

A. Finite Difference Approximations to Partial Derivatives

Consider a rectangular region R in the x, y plane. Divide this region into a rectangular network of sides $\Delta x = h$ and $\Delta y = k$ as shown in figure 6.1. The points of intersection of the dividing lines are called mesh points, nodal points or grid points.

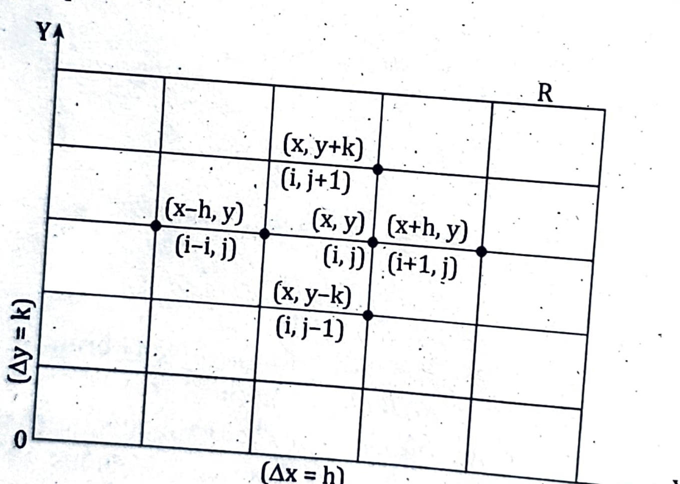


Figure 6.1

Then we have the finite difference approximations for the partial derivatives in x-direction.

$$\frac{\partial u}{\partial x} = \frac{u(x+h, y) - u(x, y)}{h} + O(h)$$

$$= \frac{u(x, y) - u(x-h, y)}{h} + O(h)$$

$$= \frac{u(x+h, y) - u(x-h, y)}{2h} + O(h^2)$$

and, $$\frac{\partial^2 u}{\partial x^2} = \frac{(x-h, y) - 2u(x, y) + u(x+h, y)}{h^2} + O(h^2)$$

Writing u(x, y) = u(ih, jk) as simply $u_{i,j}$ the above approximations become,

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h) \quad \text{..... (1)}$$

$$= \frac{u_{i,j} - u_{i-1,j}}{h} + O(h) \quad \text{..... (2)}$$

$$= \frac{u_{i+1} - u_{i-1,j}}{2h} + O(h^2) \quad \text{..... (3)}$$

$$u_{xx} = \frac{u_{i-1} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2) \quad \text{..... (4)}$$

Similarly, we have approximations for the derivatives with respect to y,

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k) \quad \text{..... (5)}$$

$$= \frac{u_{i,j} - u_{i,j-1}}{k} + O(k) \quad \text{..... (6)}$$

$$= \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2) \quad \text{..... (7)}$$

$$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + O(k^2) \quad \text{..... (8)}$$

Replacing the derivatives in any partial differential equation by their corresponding difference approximations (1) to (8), we obtain the finite-difference analogues of the given equation.

B. Elliptic Equations

The Laplace equation,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and the Poisson's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

are examples of elliptic partial differential equations. The Laplace equation arises in steady-state flow and potential problem. Poisson's equation arises in fluid mechanics, electricity and magnetism and torsion problem.

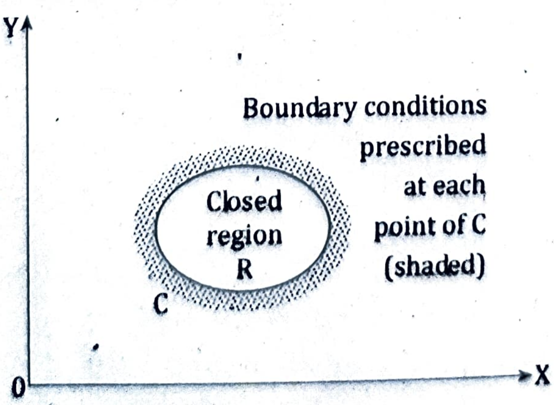


Figure 6.2

The solution of these equations is a function u(x, y) which is satisfied at every point of region R subject to certain boundary conditions specified on the closed curve.

In general, problem concerning steady viscous flow, equilibrium stress in elastic structures etc lead to elliptic type of equations.

C. Solutions of Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{..... (1)}$$

Consider a rectangular region R for which u(x, y) is known at the boundary. Divide this region into a network of square mesh of side h as shown in figure 6.3. (Assuming that an exact sub-division of R is possible). Replacing the derivatives in (1) by their difference approximations, we have,

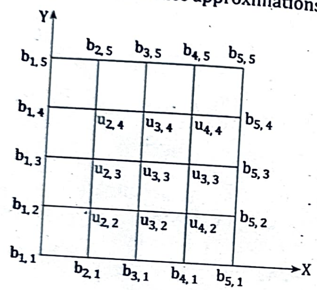


Figure 6.3

$$\frac{1}{h^2}[u_{i-1} - 2u_{i+1,j}] + \frac{1}{h^2}[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}] = 0$$

or, $$u_{i,j} = \frac{1}{4}[u_{i-1} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}] \quad \text{..... (2)}$$

This shows that the value of u at any interior mesh point is the average of its values at four neighboring points to the left, right, above and below. Equation (2) is called the standard 5-point formula which is exhibited in figure 6.4.

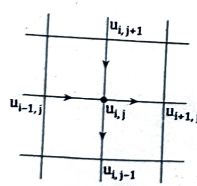


Figure 6.4

Sometimes a formula similar to equation (2) is used which is given by,

$$u_{i,j} = \frac{1}{4}(u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}) \quad \text{..... (3)}$$

This shows that the value of $u_{i,j}$ is the average of its values at the four neighboring diagonal mesh points. Equation (3) is called the diagonal five-point formula which is represented in figure 6.5. Although equation (3) is less accurate than equation (2), yet it serves as a reasonably good approximation for obtaining the starting values at the mesh points.

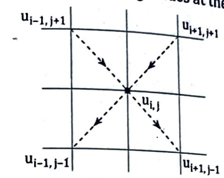


Figure 6.5

Now, to find the initial values of u at the interior mesh points, we first use the diagonal five-point formula (3) and compute $u_{3,3}$, $u_{2,4}$, $u_{4,4}$, $u_{4,2}$ and $u_{2,2}$ in this order. Thus, we get,

$$u_{3,3} = \frac{1}{4}(b_{1,5} + b_{5,1} + b_{5,5} + b_{1,1})$$

$$u_{2,4} = \frac{1}{4}(b_{1,5} + u_{3,3} + b_{3,5} + b_{1,3})$$

$$u_{4,4} = \frac{1}{4}(b_{3,5} + b_{5,3} + b_{3,5} + u_{3,3})$$

$$u_{4,2} = \frac{1}{4}(u_{3,3} + b_{5,1} + b_{3,1} + b_{5,3})$$

$$u_{2,2} = \frac{1}{4}(b_{1,3} + b_{3,1} + u_{3,3} + b_{1,1})$$

The values at the remaining interior points *i.e.*, $u_{2,3}$, $u_{3,4}$, $u_{4,3}$ and $u_{3,2}$ are computed by the standard five-point formula (2). Thus, we get,

$$u_{2,3} = \frac{1}{4}(b_{1,3} + u_{3,3} + u_{2,4} + u_{2,2})$$

$$u_{3,4} = \frac{1}{4}(u_{2,4} + u_{4,4} + b_{3,5} + u_{3,3})$$

$$u_{4,3} = \frac{1}{4}(u_{3,3} + b_{5,3} + u_{4,4} + u_{4,2})$$

$$u_{3,2} = \frac{1}{4}(u_{2,2} + u_{4,2} + u_{3,3} + u_{3,1})$$

Having found all the nine values of $u_{i,j}$ once, their accuracy is improved by either of the following iterative methods. In each case, the method is repeated until the difference between two consecutive iterates becomes negligible.

i) Jacobi's Method

Denoting th n^{th} iterative value of $u_{i,j}$ by $u^{n}_{i,j}$, the iterative formula to solve (2) is, ... (4)

$$u^{(n+1)}_{i,j} = \frac{1}{4}[u^{(n)}_{i-1,j} + u^{(n)}_{i+1} + u^{(n)}_{i,j+1} + u^{(n)}_{i,j-1}]$$

It gives improved values of $u_{i,j}$ at the interior mesh points and is called the point of Jacobi's formula.

ii) Gauss-Seidel Method

In this method, the iteration formula is,

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n+1)} + u_{i,j-1}^{(n)}\right] \quad \text{.... (5)}$$

It utilizes the latest derivative value available and scans the mesh points symmetrically from left to right along successive rows.

The accuracy of calculations depends on the mesh seize *i.e.*, smaller the h, the better the accuracy. But if h is too small, it may increase rounding off errors and also increases the labor of computation.

D. Solution of Poisson' Equation

Here, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y)$ ..... (1)

Its method of solution is similar to that of the Laplace equation. Here the standard five-point formula for (1) takes the form,

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh) \quad \text{..... (2)}$$

By applying (2) at each interior mesh points, we arrive at linear equations in the nodal values $u_{i,j}$. These equations can be solved by the Gauss-seidal method.

E. Parabolic Equations

The one-dimensional heat conduction equation $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ is a well known example of parabolic partial differential equations. The solution of this equation is a temperature function u(x, t) which is defined for values of x from 0 to 1 and for values of time t from 0 to ∞. The solution is not defined in a closed domain but advances in an open-ended region from initial values, satisfying the prescribed boundary conditions.

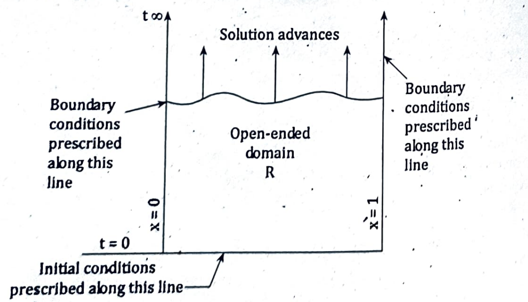


Figure 6.6

In general, the study of pressure waves in a fluid, propagation of heat and unsteady state problems lead to parabolic type of equations.

Solution of One Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{..... (1)}$$

where, $c^2 = \dfrac{k}{s\rho}$ is the diffusivity of the substance (cm²/sec)

Schmidt Method

Consider a rectangular mesh in the x-t plane with spacing h along x-direction and k along time t direction. Denoting a mesh point (x, t) = (ih, jk) as simply i, j

We have,

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k}$$

and, $$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

Replacing these in equation (1), we get,

$$u_{i,j+1} - u_{i,j} = \frac{kc^2}{h^2}\left[u_{i,j} - 2u_{i,j} + u_{i+1,j}\right]$$

or, $$u_{i,j+1} = \alpha u_{i-1,j} + (1 - 2\alpha) u_{i,j} + \alpha u_{i+1,j} \quad \text{.... (2)}$$

where, $\alpha = \dfrac{kc^2}{h^2}$ is the mesh ratio parameter

This formula enables us to determine the value of u at the $(i, j+1)^{th}$ mesh point in terms of the known function values at the points x_{i-1}, x_i and x_{i+1} at the instant t_j. It is a relation between the function values at the two time levels j + 1 and j and is called a two level formula. In schematic form equation (2) is shown in figure 6.7.

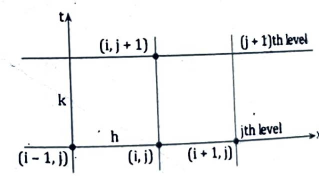


Figure 6.7

Hence, equation (2) is called the Schmidt explicit formula which is valid only for $0 < \alpha \leq 12$.

Solution of Two Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{.... (1)}$$

The methods employed for the solution of one dimensional heat equation can be readily extended to the solution of equation (1).

Consider the square region $0 \le x \le y \le a$ and assume that u is known at all points within and on the boundary of this square.

If h is the step-size, then a mesh point $(x, y, t) = (ih, jh, nl)$ may be denoted as simply (i, j, n). Replacing the derivatives in (1) by their finite difference approximations, we get,

$$\frac{u_{i,j,n+1} - u_{i,j,n}}{l} = \frac{c^2}{h^2}[(u_{i-1,j,n} - 2u_{i,j,n} + u_{i+1,j,n} + (u_{i,j-1,n} - 2u_{i,j,n} + u_{i,j+1,n})]$$

i.e., $$u_{i,j,n+1} = u_{i,j,n} + \alpha(u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j-1,n} - 4u_{i,j,n}) \quad \text{..... (2)}$$

where, $\alpha = \frac{lc^2}{h^2}$

This equation needs the five points available on the n^{th} plane.

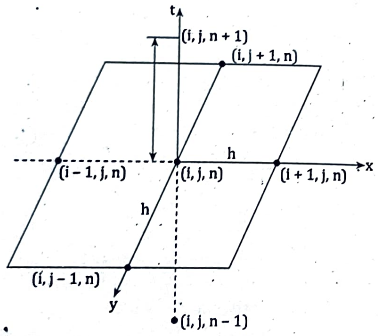


Figure 6.8

The computation process consists of point-by-point evaluation in the $(n + 1)^{th}$ plane using the points on the n^{th} plane. It is followed by plane by plane evaluation. This method is known as alternating direction explicit method.

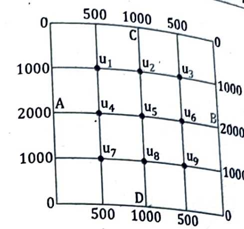
F. Hyperbolic Equations

The wave equation $\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is the simplest example of hyperbolic partial differential equations. Its solution is the displacement function $u(x, t)$ defined for values of x from 0 to 1 and for t from 0 to ∞, satisfying the initial and boundary conditions. In the case of hyperbolic equations, however, we have two initial conditions and two boundary conditions.

Such equations arise from connective type of problems in vibrations, wave mechanics and gas dynamics.

Example 6.1

Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in figure.



Solution:

Let, u_1, u_2, u_3 up to u_9 be the values of u at the interior mesh points. Since the boundary values of u are symmetrical about AB,

∴ $u_7 = u_1, u_8 = u_2, u_9 = u_3$

Also the values of u being symmetrical about CD

$u_3 = u_1, u_6 = u_4, u_9 = u_7$

Thus it is sufficient to find the values of u_1, u_2, u_4 and u_5

Now, we find their initial values in the following order

$$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) \text{[using standard 5 point formula]}$$
$$= 1500$$
$$u_1 = \frac{1}{4}(0 + 1500 + 1000 + 2000) \quad \text{[using diagonal 5 point formula]}$$
$$= 1125$$
$$u_2 = \frac{1}{4}(1125 + 1125 + 1000 + 1500) \quad \text{[using standard 5 point formula]}$$
$$\approx 1188$$
$$u_4 = \frac{1}{4}(2000 + 1125 + 1500 + 1125) \text{[using standard 5 point formula]}$$
$$\approx 1438$$

Now, we carryout the iteration process using the standard formulae:

$$u_1^{n+1} = \frac{1}{4}[1000 + u_2^n + 500 + u_4^n]$$
$$u_2^{n+1} = \frac{1}{4}[u_1^{n+1} + u_1^n + 1000 + u_5^n]$$
$$u_4^{n+1} = \frac{1}{4}[u_1^{n+1} + u_5^n + 2000 + u_1^n]$$
$$u_5^{n+1} = \frac{1}{4}[u_4^{n+1} + u_2^{n+1} + u_4^n + u_2^n]$$

First iteration, put n = 0

$$u_1^1 = \frac{1}{4}(1000 + 1188 + 500 + 1438) \approx 1032$$
$$u_2^1 = \frac{1}{4}(1032 + 1125 + 1000 + 1500) = 1164$$
$$u_4^1 = \frac{1}{4}(2000 + 1500 + 1032 + 1125) = 1414$$
$$u_5^1 = \frac{1}{4}(1414 + 1438 + 1164 + 1188) = 1301$$

Second iteration, put n = 1

$$u_1^2 = \frac{1}{4}(1000 + 1164 + 500 + 1414) = 1020$$

$$u_2^2 = \frac{1}{4}(1020 + 1032 + 1000 + 1301) = 1088$$

$$u_4^2 = \frac{1}{4}(2000 + 1301 + 1020 + 1032) = 1338$$

$$u_5^2 = \frac{1}{4}(1338 + 1414 + 1088 + 1164) = 1251$$

Third iteration, put n = 2

$$u_1^3 = \frac{1}{4}(1000 + 1088 + 500 + 1338) = 982$$

$$u_2^3 = \frac{1}{4}(982 + 1020 + 1000 + 1251) = 1063$$

$$u_4^3 = \frac{1}{4}(2000 + 1251 + 982 + 1020) = 1313$$

$$u_5^3 = \frac{1}{4}(1313 + 1338 + 1063 + 1088) = 1201$$

Fourth iteration, put n = 3

$$u_1^4 = \frac{1}{4}(1000 + 1063 + 500 + 1313) = 969$$

$$u_2^4 = \frac{1}{4}(969 + 982 + 1000 + 1201) = 1038$$

$$u_4^4 = \frac{1}{4}(2000 + 1201 + 969 + 982) = 1288$$

$$u_5^4 = \frac{1}{4}(1288 + 1313 + 1038 + 1063) = 1176$$

Similarly,

$u_1^5 = 957$, $u_2^5 = 1026$, $u_4^5 = 1276$, $u_5^5 = 1157$
$u_1^6 = 951$, $u_2^6 = 1016$, $u_4^6 = 1266$, $u_5^6 = 1146$
$u_1^7 = 946$, $u_2^7 = 1011$, $u_4^7 = 1260$, $u_5^7 = 1138$
$u_1^8 = 943$, $u_2^8 = 1007$, $u_4^8 = 1257$, $u_5^8 = 1134$
$u_1^9 = 941$, $u_2^9 = 1005$, $u_4^9 = 1255$, $u_5^9 = 1131$
$u_1^{10} = 940$, $u_2^{10} = 1003$, $u_4^{10} = 1253$, $u_5^{10} = 1129$
$u_1^{11} = 939$, $u_2^{11} = 1002$, $u_4^{11} = 1252$, $u_5^{11} = 1128$

There is a negligible difference between the values obtained in the tenth and eleventh iterations.

Hence,

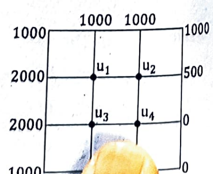
$u_1 = 939$, $u_2 = 1002$,
$u_4 = 1252$, $u_5 = 1128$

Example 6.2

Given the values of u(x, y) on the boundary of the square in the figure, Evaluate the function u(x, y) satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal points of this figure by

a) Jacobi's method

b) Gauss-Seidel method



Solution

To get the initial values of u_1, u_2, u_3, u_4, we assume that $u_4 = 0$. Then,

$u_1 = \frac{1}{4}(1000 + 0 + 1000 + 2000) = 1000$ [Diag. formula]

$u_2 = \frac{1}{4}(1000 + 500 + 1000 + 0) = 625$ [Standard formula]

$u_3 = \frac{1}{4}(2000 + 0 + 1000 + 500) = 875$ [Standard formula]

$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375$ [Standard formula]

a) We carry out the successive iterations, using Jacobi's formulae;

Itn.	$u_1 = \frac{1}{4}(3000 + u_2 + u_3)$	$u_2 = \frac{1}{4}(u_1 + 1500 + u_4)$	$u_3 = \frac{1}{4}(2500 + u_1 + u_4)$	$u_4 = \frac{1}{4}(u_2 + u_3)$
1	$\frac{1}{4}(3000+625+875) = 1125$	$\frac{1}{4}(1000+1500+375) = 719$	$\frac{1}{4}(2500+1000+375) = 969$	375
2	1172	750	1000	422
3	1188	774	1024	438
4	1200	782	1032	450
5	1204	788	1038	454
6	1206.5	790	1040	456.5
7	1208	791	1041	458
8	1208	791.5	1041.5	458

There is no significant difference between 7th and 8th iteration values.

Hence, $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$, $u_4 = 458$

b) We carry out the successive iterations, using Gauss-Seidel formulae

Itn.	$u_1 = \frac{1}{4}(3000 + u_2 + u_3)$	$u_2 = \frac{1}{4}(u_1 + 1500 + u_4)$	$u_3 = \frac{1}{4}(2500 + u_1 + u_4)$	$u_4 = \frac{1}{4}(u_2 + u_3)$
1	$\frac{1}{4}(3000 + 625 + 875) = 1125$	$\frac{1}{4}(1125 + 1500 + 375) = 750$	$\frac{1}{4}(2500 + 375 + 1125) = 1000$	$\frac{1}{4}(1000 + 750) = 438$
2	1188	782	1032	454
3	1204	789	1040	458
4	1207	791	1041	458
5	1208	791.5	1041.5	458.25

There is no significant difference between last two iterations

Hence, $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$ and $u_4 = 458$

Example 6.3

Solve the Poisson equation $u_{xx} + u_{yy} = -81xy$, $0 < x < 1$, $0 < y < 1$ given that ... **y) = 0** ... **, u(1, y) = 100, u(x, 1) = 100 and $h = \frac{1}{3}$.**

Second iteration, put n = 1

$$u_1^2 = \frac{1}{4}(1000 + 1164 + 500 + 1414) = 1020$$

$$u_2^2 = \frac{1}{4}(1020 + 1032 + 1000 + 1301) = 1088$$

$$u_4^2 = \frac{1}{4}(2000 + 1301 + 1020 + 1032) = 1338$$

$$u_5^2 = \frac{1}{4}(1338 + 1414 + 1088 + 1164) = 1251$$

Third iteration, put n = 2

$$u_1^3 = \frac{1}{4}(1000 + 1088 + 500 + 1338) = 982$$

$$u_2^3 = \frac{1}{4}(982 + 1020 + 1000 + 1251) = 1063$$

$$u_4^3 = \frac{1}{4}(2000 + 1251 + 982 + 1020) = 1313$$

$$u_5^3 = \frac{1}{4}(1313 + 1338 + 1063 + 1088) = 1201$$

Fourth iteration, put n = 3

$$u_1^4 = \frac{1}{4}(1000 + 1063 + 500 + 1313) = 969$$

$$u_2^4 = \frac{1}{4}(969 + 982 + 1000 + 1201) = 1038$$

$$u_4^4 = \frac{1}{4}(2000 + 1201 + 969 + 982) = 1288$$

$$u_5^4 = \frac{1}{4}(1288 + 1313 + 1038 + 1063) = 1176$$

Similarly,

$u_1^5 = 957$, $u_2^5 = 1026$, $u_4^5 = 1276$, $u_5^5 = 1157$

$u_1^6 = 951$, $u_2^6 = 1016$, $u_4^6 = 1266$, $u_5^6 = 1146$

$u_1^7 = 946$, $u_2^7 = 1011$, $u_4^7 = 1260$, $u_5^7 = 1138$

$u_1^8 = 943$, $u_2^8 = 1007$, $u_4^8 = 1257$, $u_5^8 = 1134$

$u_1^9 = 941$, $u_2^9 = 1005$, $u_4^9 = 1255$, $u_5^9 = 1131$

$u_1^{10} = 940$, $u_2^{10} = 1003$, $u_4^{10} = 1253$, $u_5^{10} = 1129$

$u_1^{11} = 939$, $u_2^{11} = 1002$, $u_4^{11} = 1252$, $u_5^{11} = 1128$

There is a negligible difference between the values obtained in the tenth and eleventh iterations.

Hence,

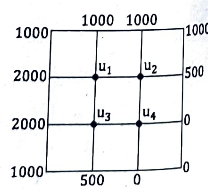
$u_1 = 939$, $u_2 = 1002$,
$u_4 = 1252$, $u_5 = 1128$

Example 6.2

Given the values of u(x, y) on the boundary of the square in the figure, Evaluate the function u(x, y) satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal points of this figure by

a) Jacobi's method

b) Gauss-Seidel method



Solution

To get the initial values of u_1, u_2, u_3, u_4, we assume that $u_4 = 0$. Then,

$$u_1 = \frac{1}{4}(1000 + 0 + 1000 + 2000) = 1000 \quad \text{[Diag. formula]}$$

$$u_2 = \frac{1}{4}(1000 + 500 + 1000 + 0) = 625 \quad \text{[Standard formula]}$$

$$u_3 = \frac{1}{4}(2000 + 0 + 1000 + 500) = 875 \quad \text{[Standard formula]}$$

$$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375 \quad \text{[Standard formula]}$$

a) We carry out the successive iterations, using Jacobi's formulae;

Itn.	$u_1 = \frac{1}{4}(3000 + u_2 + u_3)$	$u_2 = \frac{1}{4}(u_1 + 1500 + u_4)$	$u_3 = \frac{1}{4}(2500 + u_1 + u_4)$	$u_4 = \frac{1}{4}(u_2 + u_3)$
1	$\frac{1}{4}(3000+625+875) = 1125$	$\frac{1}{4}(1000+1500+375) = 719$	$\frac{1}{4}(2500+1000+375) = 969$	375
2	1172	750	1000	422
3	1188	774	1024	438
4	1200	782	1032	450
5	1204	788	1038	454
6	1206.5	790	1040	456.5
7	1208	791	1041	458
8	1208	791.5	1041.5	458

There is no significant difference between 7th and 8th iteration values.

Hence, $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$, $u_4 = 458$

b) We carry out the successive iterations, using Gauss-Seidel formulae

Itn.	$u_1 = \frac{1}{4}(3000 + u_2 + u_3)$	$u_2 = \frac{1}{4}(u_1 + 1500 + u_4)$	$u_3 = \frac{1}{4}(2500 + u_1 + u_4)$	$u_4 = \frac{1}{4}(u_2 + u_3)$
1	$\frac{1}{4}(3000 + 625 + 875) = 1125$	$\frac{1}{4}(1125 + 1500 + 375) = 750$	$\frac{1}{4}(2500 + 375 + 1125) = 1000$	$\frac{1}{4}(1000 + 750) = 438$
2	1188	782	1032	454
3	1204	789	1040	458
4	1207	791	1041	458
5	1208	791.5	1041.5	458.25

There is no significant difference between last two iterations

Hence, $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$ and $u_4 = 458$

Example 6.3

Solve the Poisson equation $u_{xx} + u_{yy} = -81xy$, $0 < x < 1$, $0 < y < 1$ given that $u(0, y) = 0$ [illegible], $u(1, y) = 100$, $u(x, 1) = 100$ and $h = \frac{1}{3}$.

Second iteration, put n = 1

$$u_1^2 = \frac{1}{4}(1000 + 1164 + 500 + 1414) = 1020$$

$$u_2^2 = \frac{1}{4}(1020 + 1032 + 1000 + 1301) = 1088$$

$$u_4^2 = \frac{1}{4}(2000 + 1301 + 1020 + 1032) = 1338$$

$$u_5^2 = \frac{1}{4}(1338 + 1414 + 1088 + 1164) = 1251$$

Third iteration, put n = 2

$$u_1^3 = \frac{1}{4}(1000 + 1088 + 500 + 1338) = 982$$

$$u_2^3 = \frac{1}{4}(982 + 1020 + 1000 + 1251) = 1063$$

$$u_4^3 = \frac{1}{4}(2000 + 1251 + 982 + 1020) = 1313$$

$$u_5^3 = \frac{1}{4}(1313 + 1338 + 1063 + 1088) = 1201$$

Fourth iteration, put n = 3

$$u_1^4 = \frac{1}{4}(1000 + 1063 + 500 + 1313) = 969$$

$$u_2^4 = \frac{1}{4}(969 + 982 + 1000 + 1201) = 1038$$

$$u_4^4 = \frac{1}{4}(2000 + 1201 + 969 + 982) = 1288$$

$$u_5^4 = \frac{1}{4}(1288 + 1313 + 1038 + 1063) = 1176$$

Similarly,

$u_1^5 = 957$, $u_2^5 = 1026$, $u_4^5 = 1276$, $u_5^5 = 1157$
$u_1^6 = 951$, $u_2^6 = 1016$, $u_4^6 = 1266$, $u_5^6 = 1146$
$u_1^7 = 946$, $u_2^7 = 1011$, $u_4^7 = 1260$, $u_5^7 = 1138$
$u_1^8 = 943$, $u_2^8 = 1007$, $u_4^8 = 1257$, $u_5^8 = 1134$
$u_1^9 = 941$, $u_2^9 = 1005$, $u_4^9 = 1255$, $u_5^9 = 1131$
$u_1^{10} = 940$, $u_2^{10} = 1003$, $u_4^{10} = 1253$, $u_5^{10} = 1129$
$u_1^{11} = 939$, $u_2^{11} = 1002$, $u_4^{11} = 1252$, $u_5^{11} = 1128$

There is a negligible difference between the values obtained in the tenth and eleventh iterations.

Hence,

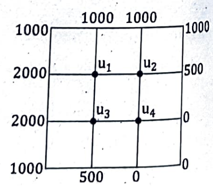
$u_1 = 939$, $u_2 = 1002$,
$u_4 = 1252$, $u_5 = 1128$

Example 6.2

Given the values of u(x, y) on the boundary of the square in the figure, Evaluate the function u(x, y) satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal points of this figure by

a) Jacobi's method

b) Gauss-Seidel method



Solution

To get the initial values of u_1, u_2, u_3, u_4, we assume that $u_4 = 0$. Then,

$u_1 = \frac{1}{4}(1000 + 0 + 1000 + 2000) = 1000$ [Diag. formula]

$u_2 = \frac{1}{4}(1000 + 500 + 1000 + 0) = 625$ [Standard formula]

$u_3 = \frac{1}{4}(2000 + 0 + 1000 + 500) = 875$ [Standard formula]

$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375$ [Standard formula]

a) We carry out the successive iterations, using Jacobi's formulae;

Itn.	$u_1 = \frac{1}{4}(3000 + u_2 + u_3)$	$u_2 = \frac{1}{4}(u_1 + 1500 + u_4)$	$u_3 = \frac{1}{4}(2500 + u_1 + u_4)$	$u_4 = \frac{1}{4}(u_2 + u_3)$
1	$\frac{1}{4}(3000+625+875) = 1125$	$\frac{1}{4}(1000+1500+375) = 719$	$\frac{1}{4}(2500+1000+375) = 969$	375
2	1172	750	1000	422
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8	1208	791.5	1041.5	458

There is no significant difference between 7th and 8th iteration values.

Hence, $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$, $u_4 = 458$

b) We carry out the successive iterations, using Gauss-Seidel formulae

Itn.	$u_1 = \frac{1}{4}(3000 + u_2 + u_3)$	$u_2 = \frac{1}{4}(u_1 + 1500 + u_4)$	$u_3 = \frac{1}{4}(2500 + u_1 + u_4)$	$u_4 = \frac{1}{4}(u_2 + u_3)$
1	$\frac{1}{4}(3000 + 625 + 875) = 1125$	$\frac{1}{4}(1125 + 1500 + 375) = 750$	$\frac{1}{4}(2500 + 375 + 1125) = 1000$	$\frac{1}{4}(1000 + 750) = 438$
2	1188	782	1032	454
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There is no significant difference between last two iterations

Hence, $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$ and $u_4 = 458$

Example 6.3

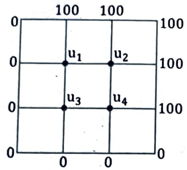
Solve the Poisson equation $u_{xx} + u_{yy} = -81xy$, $0 < x < 1$, $0 < y < 1$ given that $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = 100$, $u(x, 1) = 100$ and $h = \frac{1}{3}$.

Solution:

Given that;

$$u_{xx} + u_{yy} = -81\,xy$$

From the given boundary, the figure can be illustrated as,



Here $h = \frac{1}{3}$

The standard five point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$
$$= h^2\,[-81\,(ih \cdot jh)]$$
$$= h^4\,(-81)ij$$
$$= -ij \qquad \text{..... (1)}$$

For u_1 $(i = 1, j = 2)$

$$0 + u_2 + u_3 + 100 - 4u_1 = -2$$

i.e., $\quad -4u_1 + u_2 + u_3 = -102$ ..... (2)

For u_2 $(i = 2, j = 2)$

$$u_1 + 100 + u_4 + 100 - 4u_2 = -4$$

i.e., $\quad u_1 - 4u_2 + u_4 = -204$ ..... (3)

For u_3 $(i = 1, j = 1)$

$$0 + u_4 + 0 + u_1 - 4u_3 = -1$$

i.e., $\quad u_1 - 4u_3 + u_4 = -1$ ..... (4)

For u_4 $(i = 2, j = 1)$

$$u_3 + 100 + u_2 - 4u_4 = -2$$

i.e., $\quad u_2 + u_3 - 4u_4 = -102$ ..... (5)

Subtracting (5) from (2),

$$-4u_1 + 4u_4 = 0$$

i.e., $\quad u_1 = u_4$

Then (3) becomes

$$2u_1 - 4u_2 = -240 \qquad \text{..... (6)}$$

and, (4) becomes

$$2u_1 - 4u_3 = -1 \qquad \text{..... (7)}$$

Now, 4 × equation (2) + equation (6) gives,

$$-14u_1 + 4u_3 = -612 \qquad \text{..... (8)}$$

and, (7) + (8) gives

$$-12\,u_1 = -613$$

Thus,

$$u_1 = \frac{613}{12} = 51.0833 = u_4$$

From (6), $u_2 = \frac{1}{2}(u_1 + 102) = 76.5477$

From (7), $u_3 = \frac{1}{2}\left(u_1 + \frac{1}{2}\right) = 25.7916$

Example 6.4

Solve the boundary value problem $u_t = u_{xx}$ under the conditions $u(0, t) = u(1, t) = 0$ and $u(x, 0) = \sin px$, $0 \le x \le$ using the Schmidt method (take $h = 0.2$ and $\alpha = \frac{1}{2}$).

Solution:

Since $h = 0.2$ and $\alpha = \frac{1}{2}$

$\therefore \quad \alpha = \frac{k}{h^2}$ gives $k = 0.02$

Since $\alpha = \frac{1}{2}$, we use the Bendre-Schmidt relation

$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j}) \qquad \text{— (1)}$$

we have, $u(0, 0) = 0$, $u(0.2, 0)$ $\sin\frac{\pi}{5} = 0.5875$

$$u(0.4, 0) = \sin\frac{2\pi}{5} = 0.9511,\ u(0.6, 0) = \sin\frac{3\pi}{5} = 0.951$$

$$u(0.8, 0) = 0.5875,\ u(1, 0) = \sin\pi = 0$$

The values of u at the mesh points can be obtained by using the recurrence relation (1) as shown in the table below.

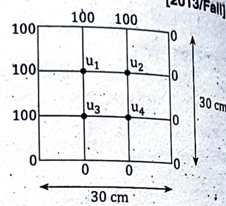
x →		0	0.2	0.4	0.6	0.8	1.0
t ↓ 0	i / j	0	1	2	3	4	5
	0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	1	0	0.4756	0.7695	0.7695	0.4756	0
0.04	2	0	0.3848	0.6225	0.6225	0.3848	0
0.06	3	0	0.3113	0.5036	0.5036	0.3113	0
0.08	4	0	0.2518	0.4074	0.4074	0.2518	0
0.1	5	0	0.2037	0.3296	0.3296	0.2037	0

BOARD EXAMINATION SOLVED QUESTIONS

1. **The steady state two dimensional heat flow in a metal plate of size 30 × 30 cm is defined by $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$. Two adjacent sides are placed at 100° C and other side at 0° C. Find the temperature at inner points, assuming the grid size of 10 × 10 cm.** **[2013/Fall]**

Solution:

The metal plate can be drawn as,



Given that;

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Let the inner points be defined as u_1, u_2, u_3 and u_4. Now using standard five point formula. We have,

$$u_1 = \frac{1}{4}(100 + 100 + u_2 + u_3)$$

$$= \frac{1}{4}(200 + u_2 + u_3)$$

$$u_2 = \frac{1}{4}(0 + 100 + u_1 + u_4) = \frac{1}{4}(100 + u_1 + u_4)$$

$$u_3 = \frac{1}{4}(0 + 100 + u_1 + u_4) = \frac{1}{4}(100 + u_1 + u_4)$$

$$u_4 = \frac{1}{4}(0 + 0 + u_2 + u_3) = \frac{1}{4}(u_2 + u_3)$$

To obtain the values let initial values of

$u_1 = 0$, $u_2 = 0$, $u_3 = 0$ and $u_4 = 0$ then

Using gauss Siedal method of iteration in tabular form

Itn.	$u_1 = \frac{1}{4}(200 + u_2 + u_3)$	$u_2 = \frac{1}{4}(100 + u_1 + u_4)$	$u_3 = \frac{1}{4}(100 + u_1 + u_4)$	$u_4 = \frac{1}{4}(u_2 + u_3)$
1	$\frac{1}{4}(200 + 0 + 0)$ = 50	$\frac{1}{4}(100 + 50 + 0)$ = 37.5	$\frac{1}{4}(100 + 50 + 0)$ = 37.5	$\frac{1}{4}(37.5 + 37.5)$ = 18.75
2	68.75	46.875	46.875	23.437
3	73.437	49.218	49.218	24.409
4	74.609	49.804	49.804	24.902
5	74.902	49.951	49.951	24.975
6	74.975	49.987	49.987	24.993
7	74.993	49.996	49.996	24.998
8	74.998	49.999	49.999	24.9995
9	74.9995	49.9997	49.9997	24.9998

Here the values of u_1, u_2, u_3 and u_4 are correct up to 3 decimal places

Hence, the required temperature at inner points are;

$u_1 = 74.9995 \approx 75°C$

$u_2 = 49.9997 \approx 50°C$

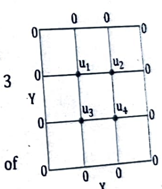
$u_3 = 49.9997 \approx 50°C$

and, $u_4 = 24.9998 \approx 25°C$

> ***NOTE:***
> *Procedure to iterate in programmable calculator:*
> *Let,* $A = u_1$, $B = u_2$, $C = u_3$, $D = u_4$
> **Step 1:** Set the following in calculator;
> $A = \frac{200 + B + C}{4}$: $B = \frac{100 + A + D}{4}$: $C = \frac{100 + A + D}{4}$: $D = \frac{B + C}{4}$
> **Step 2:** Press CALC then,
> enter the value of B? then press =
> enter the value of C? then press =
> enter the value of D? then press =
> **Step 3:** Now press = only, again and again to get the values for the respective row for each column.
> **Step 4:** The values are updated automatically so continue pressing = till the required number of iterations.

2. **Solve the Poisson equation $\nabla^2 f = 2x^2 y^2$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with f = 0 on the boundary and h = 1.** **[2013/Spring, 2014/Spring, 2018/Spring]**

Solution:



Given that;

$$\nabla^2 f = 2x^2 y^2 \quad \text{..... (1)}$$

Also the square domain of $0 \le x \le 3$ and $0 \le y \le 3$ with f = 0 on the boundary.

It is illustrated in figure as,

Here, let u_1, u_2, u_3 and u_4 be the initial nodes of Poisson equation and replacing $\nabla^2 f$ by difference equation with $x = ih$, $y = jk$ where, $(h = k = 1)$

Then, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = (2i^2 j^2)(1)^2$

For node u_1, put $i = 1$, $j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 2(1)^2(2)^2$$

or, $0 + u_2 + u_3 + 0 - 4u_1 = 8$

or, $u_2 + u_3 - 4u_1 = 8$

or, $u_1 = \frac{1}{4}(u_2 + u_3 - 8)$

Likewise,

For node u_2, put i = 2, j = 2

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 2\,(2)^2(2)^2$$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = 32$

$$u_2 = \frac{1}{4}(u_1 + u_4 - 32) \quad \text{.....(2)}$$

For node, u_4 put i = 2, j = 1

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 2\,(2)^2(1)^2$$

or, $u_3 + 0 + 0 + u_2 - 4u_4 = 8$

or, $u_3 + u_2 - 4u_4 = 8$

or, $u_4 = \frac{1}{4}(u_3 + u_2 - 8)$

or, $u_4 = u_1$

Equation (2) becomes

$$u_2 = \frac{1}{4}(2u_1 - 32) = \frac{1}{2}(u_1 - 16)$$

For node u_3, put i = 1, j = 1

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 2(1)^2(1)^2$$

or, $0 + u_4 + 0 + u_1 - 4u_3 = 2$

or, $u_3 = \frac{1}{4}(u_1 + u_4 - 2)$

or, $u_3 = \frac{1}{4}(2u_1 - 2)$

or, $u_3 = \frac{1}{2}(u_1 - 1)$

Now, let initial guess for u_1, u_2, u_3 and u_4 be 0. Then using Gauss Seidel method of iteration in tabular form.

Iteration	$u_1 = \frac{1}{4}(u_2 + u_3 - 8) = u_4$	$u_2 = \frac{1}{2}(u_1 - 16)$	$u_3 = \frac{1}{2}(u_1 - 1)$
1	$\frac{1}{4}(0 + 0 - 8) = -2$	$\frac{1}{2}(-2 - 16) = -9$	$\frac{1}{2}(-2 - 1) = -1.5$
2	-4.625	-10.312	-2.812
3	-5.281	-10.640	-3.140
4	-5.445	-10.722	-3.222
5	-5.486	-10.743	-3.243
6	-5.496	-10.748	-3.248
7	-5.499	-10.749	-3.249
8	-5.499	-10.749	-3.249

Hence the required values of nodes are

$$u_1 = u_4 = -5.499 \approx -5.5$$
$$u_2 = -10.749 \approx -10.75$$
$$u_3 = -3.249 \approx -3.25$$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$

Set the following in calculator;

$$A = \frac{B + C - 8}{4} : B = \frac{A - 16}{2} : C = \frac{A - 1}{2}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

3. Torsion on a square bar of size 9 cm × 9 cm subject to twisting is governed by $\nabla^2 u = -4$ with Dirichlet boundary condition of u(x, y) = 0 and h = 1. Calculate the steady state temperatures at interior points. Assume a grid size of 3 cm × 3 cm. Iterate until the minimum difference at any point is correct to two decimal places by applying gauss Gauss point is correct to two decimal places by applying gauss Seidel method. [2014/Fall]

Solution:

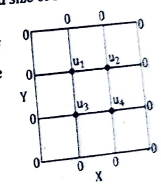
Given that;

$$\nabla^2 u = -4 \quad \text{.....(1)}$$

with u(x, y) = 0 and h = 1

Torsion on a square bar of size 9 cm × 9 cm with grid size of 3 cm × 3 cm

It is illustrated in figure as:



Let u_1, u_2, u_3 and u_4 be the internal points of Poisson equation and replacing $\nabla^2 u$ by difference equation with x = ih, y = jk where (h = k = 1)

Then, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -4\,(1)^2$

For node u_1, put i = 1, j = 2

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4\,u_{1,2} = -4$$

or, $0 + u_2 + u_3 + 0 - 4u_1 = -4$

or, $u_2 + u_3 - 4u_1 = -4$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 4)$

For node u_4, put i = 2, j = 1

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4\,u_{2,1} = -4$$

or, $u_3 + 0 + 0 + u_2 - 4\,u_4 = -4$

or, $u_3 + u_2 - 4u_4 = -4$

or, $u_4 = \frac{1}{4}(u_3 + u_2 + 4)$

or, $u_4 = u_1$

For node u_2, put i = 2, j = 2

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = -4$$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = -4$

or, $u_2 = \frac{1}{4}(u_1 + u_4 + 4)$

or, $u_2 = \frac{1}{2}(u_1 + 2)$ [$\because u_1 = u_4$]

For node u_3, put $i = 1$ $j = 1$

$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = -4$

or, $0 + u_4 + 0 + u_1 - 4u_3 = -4$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 4)$

or, $u_3 = \frac{1}{2}(u_1 + 2)$

or, $u_3 = u_2$

Now, let the initial guess for u_1, u_2, u_3 and u_4 be 0.

Then using Gauss Seidel method of iteration in tabular form,

Iteration	$u_1 = u_4 = \frac{1}{2}(u_2 + 2)$	$u_2 = u_3 = \frac{1}{2}(u_1 + 2)$
1	$\frac{1}{2}(0 + 2) = 1$	1.5
2	1.75	1.875
3	1.9375	1.9688
4	1.9844	1.9922
5	1.9961	1.9980

Here, the obtained values are correct up to two decimal places.

Hence the required steady state temperatures at interior points are

$u_1 = u_2 = u_3 = u_4 = 1.99 \approx 2$.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = u_1 = u_4$, $B = u_2 = u_3$
Set the following in calculator;

$A = \frac{B+2}{2} : B = \frac{A+2}{2}$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

4. Solve the Poisson equation $\nabla^2 f = (2 + x^2 y)$, over the square domain of $0 \le x \le 3$ and $0 \le y \le 3$ with f = 0 on the boundary and h = 1.

[2015/Fall]

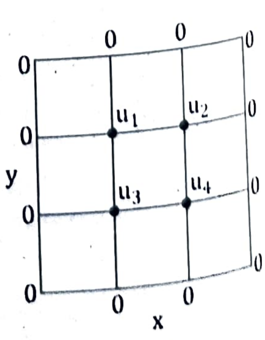
Solution:

Given that;

$\nabla^2 f = 2 + x^2 y$

Over the square domain of $0 \le x \le 3$ and $0 \le y \le 3$ with f = 0 on the boundary.

It is illustrated on the figure as:



Let u_1, u_2, u_3 and u_4 be the interior points and using Poisson formula with $x = ih$, $y = jk$ where ($h = k = 1$)

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} - 4u_{i,j} = (2 + i^2 j)(1)^2$$

Now, for interior point u_1, put $i = 1, j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 2 + (1)^2 2$$

or, $0 + u_2 + u_3 + 0 - 4u_1 = 4$

or, $u_1 = \frac{1}{4}(u_2 + u_3 - 4)$

For interior point u_4, put $i = 2, j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 2 + (2)^2 \cdot 1$$

or, $u_3 + 0 + 0 + u_2 - 4u_4 = 6$

or, $u_4 = \frac{1}{4}(u_2 + u_3 - 6)$

For interior point u_2, put $i = 2, j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 2 + (2)^2 \cdot 2$$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = 10$

or, $u_2 = \frac{1}{4}(u_1 + u_4 - 10)$

For interior point u_3, put $i = 1, j = 1$

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 2 + (1)^2 \cdot 1$$

or, $0 + u_4 + 0 + u_1 - 4u_3 = 3$

or, $u_3 = \frac{1}{4}(u_1 + u_4 - 3)$

Now, let the initial guess for u_1, u_2, u_3 and u_4 be 0.

Then using Gauss Seidel method of iteration in tabular form,

Iteration	$u_1 = \frac{1}{4}(u_2 + u_3 - 4)$	$u_2 = \frac{1}{4}(u_1 + u_4 - 10)$	$u_3 = \frac{1}{4}(u_1 + u_4 - 3)$	$u_4 = \frac{1}{4}(u_2 + u_3 - 6)$
1	$\frac{1}{4}(0 - 4) = -1$	$\frac{1}{4}(-1 + 0 - 10)$ $= -2.75$	$\frac{1}{4}(-1 + 0 - 3)$ $= -1$	$\frac{1}{4}(-2.75 - 1 - 6)$ $= -2.437$
2	-1.9375	-3.5936	-1.8436	-2.8593
3	-2.3593	-3.8047	-2.0547	-2.9648
4	-2.4649	-3.8574	-2.1074	-2.9912
5	-2.4912	-3.8706	-2.1206	-2.9978
6	-2.4978	-3.8739	-2.1239	-2.9995
7	-2.4995	-3.8747	-2.1248	-2.9999
8	-2.4999	-3.8749	-2.1250	-3.0000
9	-2.5000	-3.8750	-2.1250	-3.0000
10	-2.5000	-3.8750	-2.1250	-3.0000

Here, the obtained values are values are correct up to 4 decimal places.

Hence the required interior points are,

$u_1 = -2.5$

$u_2 = -3.875$

$u_3 = -2.125$

and, $u_4 = -3$

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = u_1, B = u_2, C = u_3, D = u_4$
> Set the following in calculator;
> $$A = \frac{B + C - 4}{4} : B = \frac{A + D - 10}{4} : C = \frac{A + D - 3}{4} : D = \frac{B + C - 6}{4}$$
> Now press CALC and enter the initial value of B, C and D and continue pressing = only for the required number of iterations.

5. Solve the Poisson equation $\nabla^2 f = 2x^2 + y$, over the square domain $1 \le x \le 3$, $1 \le, \le 3$ with $f = 1$ on the boundary. Take $h = k = 1$. [2015/Spring]

Solution:

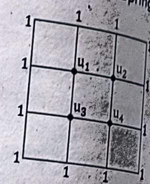
Given that;

$$\nabla^2 f = 2x^2 + y$$

Over the square domain $1 \le x \le 3, 1 \le y \le 3$

With $f = 1$ on the boundary.

It is illustrated in figure as:



Let u_1, u_2, u_3 and u_4 be the interior points and using Poisson formula with $x = ih, y = jk$ where, $(h = k = 1)$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4\,u_{i,j} = (2\,i^2 + j)\,(1)^2$$

Now for interior point u_1, put $i = 1, j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 2\,(1)^2 + 2$$

or, $1 + u_2 + u_3 + 1 - 4u_1 = 4$

or, $u_2 + u_3 - 4u_1 = 2$

or, $u_1 = \frac{1}{2}(u_2 + u_3 - 2)$

For interior point u_2, put $i = 2, j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 2(2)^2 + 2$$

or, $u_1 + 1 + u_4 + 1 - 4u_2 = 10$

or, $u_2 = \frac{1}{4}(u_1 + u_4 - 8)$

For interior point u_3, put $i = 1, j = 1$

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 2(1)^2 + 1$$

or, $1 + u_4 + 1 + u_1 - 4u_3 = 3$

or, $u_3 = \frac{1}{4}(u_1 + u_4 - 1)$

For interior point u_4, put $i = 2, j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 2\,(2)^2 + 1$$

or, $u_3 + 1 + 1 + u_2 - 4u_4 = 9$

or, $u_4 = \frac{1}{4}(u_2 + u_3 - 7)$

Now let the initial guess for u_1, u_2, u_3 and u_4 be 0.

Then using Gauss Seidel method of iteration in tabular form

Itn.	$u_1 = \frac{1}{4}(u_2+u_3-2)$	$u_2 = \frac{1}{4}(u_1+u_4-8)$	$u_3 = \frac{1}{4}(u_1+u_4-1)$	$u_4 = \frac{1}{4}(u_2+u_3-7)$
1	-0.5	-2.1250	-0.3750	-2.3750
2	-1.1250	-2.8750	-1.1250	-2.7500
3	-1.5000	-3.0625	-1.3125	-2.8438
4	-1.5938	-3.1094	-1.3594	-2.8672
5	-1.6172	-3.1211	-1.3711	-2.8731
6	-1.6231	-3.1240	-1.3741	-2.8745
7	-1.6245	-3.1248	-1.3748	-2.849
8	-1.6249	-3.1250	-1.3750	-2.8750
9	-1.6250	-3.1250	-1.3750	-2.8750

Here, the obtained values are correct up to 4 decimal places

Hence the required interior points are

$u_1 = -1.6250$

$u_2 = -3.1250$

$u_3 = -1.3750$

and, $u_4 = -2.8750$

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = u_1, B = u_2, C = u_3, D = u_4$
> Set the following in calculator;
> $$A = \frac{B + C - 2}{4} : B = \frac{A + D - 8}{4} : C = \frac{A + D - 1}{4} : D = \frac{B + C - 7}{4}$$
> Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

6. Given the Poisson's equation. $\Delta^2 f = -10\,(x^2 + y^2 + 10)$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with Dirichlet boundary condition of $f(x, y) = 0$ and $h = 1$. Calculate the steady state temperatures at the interior nodes by using Gauss Seidel method. [2016/Fall, 2018/Fall]

Solution:

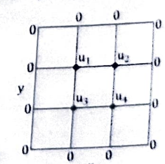
Given that;

$$\Delta^2 f = -10\,(x^2 + y^2 + 10)$$

Over the square domain $0 \le x \le 3$ and $0 \le y \le 3$

With Dirichlet boundary condition of $f(x, y) = 0$

It is illustrated in figure as:



Let, u_1, u_2, u_3, u_4 be the interior nodes and using Poisson formula with $x = ih$, $y = jk$ where $(h = k = 1)$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10\,(i^2 + j^2 + 10)\cdot(1)^2$$

Now, for interior node u_1, put i = 1, j = 2

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = -10\,((1)^2 + (2)^2 + 10)$$

or, $0 + u_2 + u_3 + 0 - 4u_1 = -150$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 150)$

For interior node u_2, put i = 2, j = 2

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = -10\,((2)^2 + (2)^2 + 10)$$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = -180$

or, $u_1 = \frac{1}{4}(u_1 + u_4 + 180)$

For interior node u_3, put i = 1, j = 1

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = -10[(1)^2 + (1)^2 + 10]$$

or, $0 + u_4 + 0 + u_1 - 4u_3 = -120$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 120)$

For interior node u_4, put i = 2, j = 1

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = -10\,[(2)^2 + (1)^2 + 10]$$

or, $u_3 + 0 + 0 + u_2 - 4u_4 = -150$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 150)$

Here, $u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 150)$

so, $u_2 = \frac{1}{2}(u_1 + 90)$ and $u_3 = \frac{1}{2}(u_1 + 60)$

Now, let initial Guess for u_1, u_2, u_3 and u_4 be 0.

Now, solving the equations by the Gauss Seidel method,

Iteration	$u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 150)$	$u_2 = \frac{1}{2}(u_1 + 90)$	$u_3 = \frac{1}{2}(u_1 + 60)$
1	37.5	63.75	48.75
2	65.625	77.8125	62.8125
3	72.6563	81.3281	66.3282
4	74.4141	82.2070	67.2071
5	74.8535	82.4268	67.4268
6	74.9634	82.4817	67.4817
7	74.9909	82.4954	67.4955
8	74.9977	82.4989	67.4989
9	74.9994	82.4997	67.4997
10	74.9999	82.4999	67.4999

Hence the required steady state temperatures at the interior nodes are

$u_1 = u_4 = 75$

$u_2 = 82.5$

and, $u_3 = 67.5$.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$
Set the following in calculator;

$$A = \frac{B + C + 150}{4} : B = \frac{A + 90}{2} : C = \frac{A + 60}{2}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

7. **Solve the parabolic equation $2f_{xx}(x, t) = f_t(x, t)$, $0 \le t \le 1.5$ and given initial condition $f(x, 0) = 50(4 - x)$, $0 \le x \le 4$ with boundary condition $f(0, t) = 0 = f(4, t)$ $0 \le t \le 1.5$.** **[2017/Fall]**

Solution:

Given that;

$f_t(x, t) = 2f_{xx}(x, t)$

We have the parabolic equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where, c^2 is the diffusivity of the substance

$c^2 = 2$

Let, $h = 1 \rightarrow$ Spacing along x-direction, $0 \le x \le 4$

Let, $k = 0.5 \rightarrow$ Spacing along time, t-direction, $0 \le t \le 1.5$

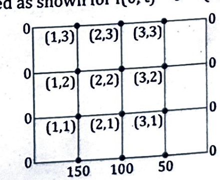
Now, solving the parabolic equation using Schmidt method.

We have,

$$\alpha = \frac{kc^2}{h^2} = \frac{0.5 \times 2}{1^2} = 1$$

Here, α lies between $0 < \alpha \le 12$ which satisfies the condition

The figure is illustrated as shown for $f(0, t) = 0 = f(4, t)$



Here, boundary values for

$u_{1,0} = 50(4 - x) = 50(4 - 1) = 150$

$u_{2,0} = 50(4 - 2) = 100$

$u_{3,0} = 50(4 - 3) = 50$

From Schmidt's formula, we have,

$$u_{i,j+1} = \alpha\, u_{i-1,j} + (1 - 2\alpha)\, u_{i,j} + \alpha\, u_{i+1,j}$$

Substituting the value of $\alpha = 1$

$$u_{i,j+1} = u_{i-1,j} - u_{i,j} + u_{i+1,j}$$

Now, for $i = 1, 2, 3$ and $j = 0$

$$u_{1,1} = [u_{0,0} - u_{1,0} + u_{2,0}] = 0 - 150 + 100 = -50$$
$$u_{2,1} = [u_{1,0} - u_{2,0} + u_{3,0}] = 150 - 100 + 50 = 100$$
$$u_{3,1} = [u_{2,0} - u_{3,0} + u_{4,0}] = 100 - 50 + 0 = 50$$

For $i = 1, 2, 3$ and $j = 1$

$$u_{1,2} = [u_{0,1} - u_{1,1} + u_{2,1}] = 0 + 50 + 100 = 150$$
$$u_{2,2} = [u_{1,1} - u_{2,1} + u_{3,1}] = -50 - 100 + 50 = -100$$
$$u_{3,2} = [u_{2,1} - u_{3,1} + u_{4,1}] = 100 - 50 + 0 = 50$$

For $i = 1, 2, 3$ and $j = 2$

$$u_{1,3} = [u_{0,2} - u_{1,2} + u_{2,2}] = 0 - 150 + (-100) = -250$$
$$u_{2,3} = [u_{1,2} - u_{2,2} + u_{3,2}] = 150 + 100 + 50 = 300$$
$$u_{3,3} = [u_{2,2} - u_{3,2} + u_{4,2}] = -100 - 50 + 0 = -150$$

8. **Given the Poisson's equation $\nabla^2 u = -10\,(x^2 + y^2 + 10)$ over the square domain such that $0 \le x \le 3$ and $0 \le y \le 3$ with Dirichlet boundary condition of $u(x, y) = 0$. Calculate the steady state temperature at interior points by suing successive over relaxation method up to 5th iteration. Assume $h = k = 1$.**

Solution: **[2017/Spring]**

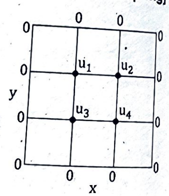
Given that;

$$\nabla^2 u = -10\,(x^2 + y^2 + 10)$$

Over the square domain; $0 \le x \le 3$ and $0 \le y \le 3$

With Dirichlet boundary condition of $u(x, y) = 0$

It is illustrated in figure as:



Let u_1, u_2, u_3 and u_4 be the interior points and using Poisson formula with $x = ih$, $y = jk$ where $(h = k = 1)$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10\,(i^2 + j^2 + 10) \cdot (1)^2$$

Now, for interior point u_1, put $i = 1, j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = -10\,[(1)^2 + (2)^2 + 10]$$

or, $0 + u_2 + u_3 + 0 - 4u_1 = -150$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 150)$

For interior node u_2, put $i = 2, j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = -10\,[(2)^2 + (2)^2 + 10]$$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = -180$

or, $u_2 = \frac{1}{4}(u_1 + u_4 + 180)$

For interior node u_3, put $i = 1, j = 1$

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4\,u_{1,1} = -10\,[(1)^2 + (1)^2 + 10]$$

or, $0 + u_4 + 0 + u_1 - 4u_3 = -120$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 120)$

For interior node u_4, put $i = 2, j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = -10\,[(2)^2 + (1)^2 + 10]$$

or, $u_3 + 0 + 0 + u_2 - 4u_4 = -150$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 150)$

Here, $u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 150)$

so, $u_2 = \frac{1}{2}(u_1 + 90)$ and $u_3 = \frac{1}{2}(u_1 + 60)$

Now, using successive over relaxation method

We have,

$$x_i^{n+1} = (1 - w)\,x_i^n + w\,[\text{Gauss Seidel iteration}]$$

Here, w is relaxation parameter which value lies from $0 < w < 2$ for convergence reason.

Lets choose $w = 1.25$

$$x_i^{n+1} = -0.25x_i^n + 1.25\,[\text{Gauss Seidel iteration}]$$

Now, the equations are formed as

$$u_1^{n+1} = -0.25\,u_1^n + \frac{1.25}{4}(u_2^n + u_3^n + 150)$$

$$u_2^{n+1} = -0.25\,u_2^n + \frac{1.25}{4}(u_1^{n+1} + u_4^n + 180)$$

$$u_3^{n+1} = -0.25\,u_3^n + \frac{1.25}{4}(u_1^{n+1} + u_4^n + 120)$$

$$u_4^{n+1} = -0.25\,u_4^n + \frac{1.25}{4}(u_2^{n+1} + u_3^{n+1} + 150)$$

Here, $u_1^{n+1} = u_4^{n+1}$

Then, $u_1^{n+1} = -0.25\,u_1^n + 0.3125\,(u_2^n + u_3^n + 150) = u_4^{n+1}$

$u_2^{n+1} = -0.25\,u_2^n + 0.3125\,(u_1^{n+1} + u_4^n + 180)$

$u_3^{n+1} = -0.25\,u_3^n + 0.3125\,(u_1^{n+1} + u_4^n + 120)$

Let the initial guess for u_1, u_2, u_3 and u_4 be 0.

Now, 1st iteration,

For $n = 0$

$$u_4^1 = u_1^{(1)} = -0.25\,u_1^0 + 0.3125\,(u_2^0 + u_3^0 + 150)$$
$$= -0.25 \times 0 + 0.3125\,(0 + 0 + 150)$$
$$= 46.875$$
$$u_2^1 = -0.25\,u_2^0 + 0.3125\,(u_1^1 + u_4^0 + 180)$$
$$= 0 + 0.3125\,(46.875 + 0 + 180)$$
$$= 70.8984$$

Substituting the value of $\alpha = 1$

$$u_{i,j+1} = u_{i-1,j} - u_{i,j} + u_{i+1,j}$$

Now, for $i = 1, 2, 3$ and $j = 0$

$$u_{1,1} = [u_{0,0} - u_{1,0} + u_{2,0}] = 0 - 150 + 100 = -50$$
$$u_{2,1} = [u_{1,0} - u_{2,0} + u_{3,0}] = 150 - 100 + 50 = 100$$
$$u_{3,1} = [u_{2,0} - u_{3,0} + u_{4,0}] = 100 - 50 + 0 = 50$$

For $i = 1, 2, 3$ and $j = 1$

$$u_{1,2} = [u_{0,1} - u_{1,1} + u_{2,1}] = 0 + 50 + 100 = 150$$
$$u_{2,2} = [u_{1,1} - u_{2,1} + u_{3,1}] = -50 - 100 + 50 = -100$$
$$u_{3,2} = [u_{2,1} - u_{3,1} + u_{4,1}] = 100 - 50 + 0 = 50$$

For $i = 1, 2, 3$ and $j = 2$

$$u_{1,3} = [u_{0,2} - u_{1,2} + u_{2,2}] = 0 - 150 + (-100) = -250$$
$$u_{2,3} = [u_{1,2} - u_{2,2} + u_{3,2}] = 150 + 100 + 50 = 300$$
$$u_{3,3} = [u_{2,2} - u_{3,2} + u_{4,2}] = -100 - 50 + 0 = -150$$

8. **Given the Poisson's equation $\nabla^2 u = -10\,(x^2 + y^2 + 10)$ over the square domain such that $0 \le x \le 3$ and $0 \le y \le 3$ with Dirichlet boundary condition of $u(x, y) = 0$. Calculate the steady state temperature at interior points by suing successive over relaxation method up to 5th iteration. Assume $h = k = 1$.** **[2017/Spring]**

Solution:

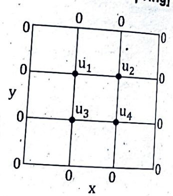
Given that;

$$\nabla^2 u = -10\,(x^2 + y^2 + 10)$$

Over the square domain; $0 \le x \le 3$ and $0 \le y \le 3$

With Dirichlet boundary condition of $u(x, y) = 0$

It is illustrated in figure as:



Let u_1, u_2, u_3 and u_4 be the interior points and using Poisson formula with $x = ih$, $y = jk$ where $(h = k = 1)$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10\,(i^2 + j^2 + 10) \cdot (1)^2$$

Now, for interior point u_1, put $i = 1, j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = -10\,[(1)^2 + (2)^2 + 10]$$

or, $0 + u_2 + u_3 + 0 - 4u_1 = -150$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 150)$

For interior node u_2, put $i = 2; j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = -10\,[(2)^2 + (2)^2 + 10]$$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = -180$

or, $u_2 = \frac{1}{4}(u_1 + u_4 + 180)$

For interior node u_3, put $i = 1, j = 1$

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4\,u_{1,1} = -10\,[(1)^2 + (1)^2 + 10]$$

$0 + u_4 + 0 + u_1 - 4u_3 = -120$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 120)$

or, For interior node u_4, put $i = 2, j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = -10\,[(2)^2 + (1)^2 + 10]$$

$u_3 + 0 + 0 + u_2 - 4u_4 = -150$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 150)$

or, Here, $u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 150)$

so, $u_2 = \frac{1}{2}(u_1 + 90)$ and $u_3 = \frac{1}{2}(u_1 + 60)$

Now, using successive over relaxation method

We have,

$$x_i^{n+1} = (1 - w)\,x_i^n + w\,[\text{Gauss Seidel iteration}]$$

Here, w is relaxation parameter which value lies from $0 < w < 2$ for convergence reason.

Lets choose $w = 1.25$

$$x_i^{n+1} = -0.25x_i^n + 1.25\,[\text{Gauss Seidel iteration}]$$

Now, the equations are formed as

$$u_1^{n+1} = -0.25\,u_1^n + \frac{1.25}{4}(u_2^n + u_3^n + 150)$$
$$u_2^{n+1} = -0.25\,u_2^n + \frac{1.25}{4}(u_1^{n+1} + u_4^n + 180)$$
$$u_3^{n+1} = -0.25\,u_3^n + \frac{1.25}{4}(u_1^{n+1} + u_4^n + 120)$$
$$u_4^{n+1} = -0.25\,u_4^n + \frac{1.25}{4}(u_2^{n+1} + u_3^{n+1} + 150)$$

Here, $u_1^{n+1} = u_4^{n+1}$

Then, $u_1^{n+1} = -0.25\,u_1^n + 0.3125\,(u_2^n + u_3^n + 150) = u_4^{n+1}$

$u_2^{n+1} = -0.25\,u_2^n + 0.3125\,(u_1^{n+1} + u_4^n + 180)$

$u_3^{n+1} = -0.25\,u_3^n + 0.3125\,(u_1^{n+1} + u_4^n + 120)$

Let the initial guess for u_1, u_2, u_3 and u_4 be 0.

Now, 1st iteration,

For $n = 0$

$u_4^1 = u_1^{(1)} = -0.25\,u_1^0 + 0.3125\,(u_2^0 + u_3^0 + 150)$

$= -0.25 \times 0 + 0.3125\,(0 + 0 + 150)$

$= 46.875$

$u_2^1 = -0.25\,u_2^0 + 0.3125\,(u_1^1 + u_4^0 + 180)$

$= 0 + 0.3125\,(46.875 + 0 + 180)$

$= 70.8984$

$u_3^1 = -0.25\, u_3^0 + 0.3125\, (u_1^1 + u_4^0 + 120)$
$= 0 + 0.3125\, (46.875 + 0 + 120)$
$= 52.1484$

Likewise,

2nd iteration, n = 1

$u_1^2 = -0.25\, u_1^1 + 0.3125\, (u_1^2 + u_3^1 + 150)$
$= 73.6084$

$u_2^2 = -0.25\, u_2^1 + 0.3125\, (u_1^2 + u_4^1 + 180)$
$= 76.1765$

$u_3^2 = -0.25\, u_3^1 + 0.3125\, (u_1^2 + u_4^1 + 120)$
$= 62.1140$

$u_4^2 = u_1^2 = 73.6084$

3rd iteration, n = 2

$u_1^3 = -0.25\, u_1^2 + 0.3125\, (u_2^2 + u_3^2 + 150)$
$= 71.6887$

$u_2^3 = -0.25\, u_2^2 + 0.3125\, (u_1^3 + u_4^2 ++ 180)$
$= 82.6112$

$u_3^3 = -0.25\, u_3^2 + 0.3125\, (u_1^3 + u_4^2 + 120)$
$= 67.3768$

$u_4^3 = u_1^3 = 71.6887$

4th iteration

n = 3 then

$u_1^4 = -0.25\, u_1^3 + 0.3125\, (u_2^3 + u_3^3 + 150)$
$= 75.8241$

$u_2^4 = -0.25\, u_2^3 + 0.3125\, (u_1^4 + u_4^3 + 180)$
$= 81.6950$

$u_3^4 = -0.25\, u_3^3 + 0.3125\, (u_1^4 + u_4^3 + 120)$
$= 66.7536$

$u_4^4 = u_1^4 = 75.8241$

5th iteration, n = 4

$u_1^5 = -0.25\, u_1^4 + 0.3125\, (u_2^4 + u_3^4 + 150)$
$= 74.3092$

$u_2^5 = -0.25\, u_2^4 + 0.3125\, (u_1^5 + u_4^4 + 180)$
$= 82.7429$

$u_3^5 = -0.25\, u_3^4 + 0.3125\, (u_1^5 + u_4^4 + 120)$
$= 67.7283$

$u_4^5 = u_1^5 = 74.3092$

Hence the required steady state temperature at interior points are

$u_1 = u_4 = 74.3092$
$u_2 = 82.7429$
$u_3 = 67.7283$

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$
Set the following in calculator;
X = −0.25A + 0.3125(150 + B + C) : Y = −0.25B + 0.3125(180 + X + A) :
M = −0.25C + 0.3125(120 + X + A)
Press CALC and enter the initial value of A, B and C and continue pressing = only for the required row for each column.
Update the values of A?, B? and C? when asked again.

9. **Given the Poisson's equation $\Delta^2 f = 4x^2 y^2$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with Dirichlet boundary condition of f(x, y) = 100 and h = k = 1. Calculate the steady state temperatures at the interior nodes by using Gauss Seidel method. Iterate until the successive values at any point is correct to two decimal places.** [2019/Fall]

Solution:

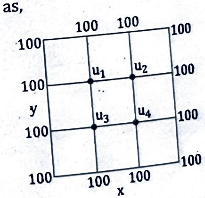
Given that;

$\Delta^2 f = 4x^2 y^2$

Over the square domain $0 \le x \le 3$ and $0 \le y \le 3$

With Dirichlet boundary condition of f(x, y) = 100

It is illustrated in figure as,



Let u_1, u_2, u_3 and u_4 be the interior nodes of Poisson's equation with x = ih, y = jk where h = k = 1

Then, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = (4\, i^2 j^2)\, (1)^2$

For node μ_1, put i = 1, j = 2

$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 4(1)^2\, (2)^2$

or, $100 + u_2 + u_3 + 100 - 4u_1 = 16$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 184)$

For node u_2, put i = 2, j = 2

$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 4\, (2)^2\, (2)^2$

or, $u_1 + 100 + u_4 + 100 - 4u_2 = 64$

or, $u_2 = \frac{1}{4}(u_1 + u_4 + 136)$

For node u_3, put $i = 1, j = 1$

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 4(1)^2 (1)^2$$

or, $100 + u_4 + 100 + u_1 - 4u_3 = 4$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 196)$

For node u_4, put $i = 2, j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 4(2)^2 (1)^2$$

or, $u_3 + 100 + 100 + u_2 - 4u_4 = 16$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 184)$

Here, $u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 184)$ then,

$$u_2 = \frac{1}{2}(u_1 + 68)$$

$$u_3 = \frac{1}{2}(u_1 + 98)$$

Let the initial guess for u_1, u_2, u_3, u_4 be zero.

Now, using Gauss Seidel method in tabular form,

Iteration	$u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 184)$	$u_2 = \frac{1}{2}(u_1 + 68)$	$u_3 = \frac{1}{2}(u_1 + 98)$
1	46	57	72
2	78.25	73.125	88.125
3	86.3125	77.1563	92.1563
4	88.3281	78.1641	93.1641
5	88.8320	78.4160	934160
6	88.9580	78.4790	93.4790
7	88.9895	78.4948	93.4948
8	88.9974	78.4987	93.4987

Hence the required values of temperatures at interior nodes are

$u_1 = u_4 = 88.9974$

$u_2 = 78.4987$

and, $u_3 = 93.4987$

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$
Set the following in calculator;

$$A = \frac{B + C + 184}{4} : B = \frac{A + 68}{2} : C = \frac{A + 98}{2}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

10. **Solve the Poisson's equation $u_{xx} + u_{yy} = 243(x^2 + y^2)$ over a square domain $0 \le x \le 1$, $0 \le y \le 1$ with step size $h = \frac{1}{3}$ with $u = 100$ on the boundary.** [2019/Spring]

Solution:

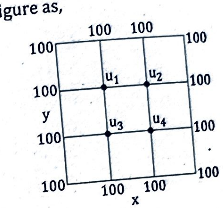
Given that;

$$u_{xx} + u_{yy} = 243(x^2 + y^2)$$

Over a square domain $0 \le x \le 1, 0 \le y \le 1$

With $u = 100$ on the boundary.

It is illustrated in the figure as,



Let u_1, u_2, u_3 and u_4 be the interior nodes of Poisson's equation and replacing $u_{xx} + u_{yy}$ by difference equation with $x = ih$, $y = jk$ where $h = k = \frac{1}{3}$.

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jk)$$

Then, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = \frac{1}{9} \times 243\left(\frac{i^2}{9} + \frac{j^2}{9}\right)$

or, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 3(i^2 + j^2)$

Now, for node u_1, put $i = 1, j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 3(1^2 + 2^2)$$

or, $100 + u_2 + u_3 + 100 - 4u_1 = 15$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 185)$

For node u_2, put $i = 2, j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 3[(2)^2 + (2)^2]$$

or, $u_1 + 100 + u_4 + 100 - 4u_2 = 24$

or, $u_2 = \frac{1}{4}(u_1 + u_4 + 176)$

For node u_3, put $i = 1, j = 1$

$$u_{0,1}\ u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 3[(1)^2 + (1)^2]$$

or, $100 + u_4 + 100 + u_1 - 4u_3 = 6$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 194)$

For node u_4, put $i = 2, j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 3[(2)^2 + (1)^2]$$

or, $u_1 + 100 + u_4 + 100 - 4u_2 = 64$

or, $u_2 = \frac{1}{4}(u_1 + u_4 + 136)$

For node u_3, put i = 1, j = 1

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 4(1)^2(1)^2$$

or, $100 + u_4 + 100 + u_1 - 4u_3 = 4$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 196)$

For node u_4, put i = 2, j = 1

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 4(2)^2(1)^2$$

or, $u_3 + 100 + 100 + u_2 - 4u_4 = 16$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 184)$

Here, $u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 184)$ then,

$u_2 = \frac{1}{2}(u_1 + 68)$

$u_3 = \frac{1}{2}(u_1 + 98)$

Let the initial guess for u_1, u_2, u_3, u_4 be zero.

Now, using Gauss Seidel method in tabular form,

Iteration	$u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 184)$	$u_2 = \frac{1}{2}(u_1 + 68)$	$u_3 = \frac{1}{2}(u_1 + 98)$
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Hence the required values of temperatures at interior nodes are

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and, $u_3 = 93.4987$

NOTE:

Procedure to iterate in programmable calculator:

Let, A = u_1 = u_4, B = u_2, C = u_3

Set the following in calculator;

$$A = \frac{B + C + 184}{4} : B = \frac{A + 68}{2} : C = \frac{A + 98}{2}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

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Solution:

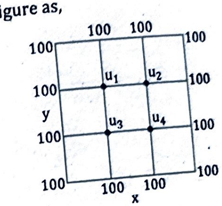
Given that;

$u_{xx} + u_{yy} = 243(x^2 + y^2)$

Over a square domain $0 \le x \le 1, 0 \le y \le 1$

With u = 100 on the boundary.

It is illustrated in the figure as,



Let u_1, u_2, u_3 and u_4 be the interior nodes of Poisson's equation and replacing $u_{xx} + u_{yy}$ by difference equation with x = ih, y = jk where $h = k = \frac{1}{3}$.

Then, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jk)$

or, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = \frac{1}{9} \times 243\left(\frac{i^2}{9} + \frac{j^2}{9}\right)$

or, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 3(i^2 + j^2)$

Now, for node u_1, put i = 1, j = 2

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 3(1^2 + 2^2)$$

or, $100 + u_2 + u_3 + 100 - 4u_1 = 15$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 185)$

For node u_2, put i = 2, j = 2

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 3[(2)^2 + (2)^2]$$

or, $u_1 + 100 + u_4 + 100 - 4u_2 = 24$

or, $u_2 = \frac{1}{4}(u_1 + u_4 + 176)$

For node u_3, put i = 1, j = 1

$$u_{0,1} \; u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 3[(1)^2 + (1)^2]$$

or, $100 + u_4 + 100 + u_1 - 4u_3 = 6$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 194)$

For node u_4, put i = 2, j = 1

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 3[(2)^2 + (1)^2]$$

or, $u_3 + 100 + 100 + u_2 - 4u_4 = 15$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 185)$

Here, $u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 185)$ and

$u_2 = \frac{1}{2}(u_1 + 88)$, $u_3 = \frac{1}{2}(u_1 + 97)$

Let the initial guess for u_1, u_2, u_3 and u_4 be zero.

Now, using Gauss Seidel method in tabular form,

Iteration	$u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 185)$	$u_2 = \frac{1}{2}(u_1 + 88)$	$u_3 = \frac{1}{2}(u_1 + 97)$
1	46.250	67.125	71.625
2	80.938	84.469	88.969
3	89.610	88.805	93.305
4	91.778	89.889	94.389
5	92.320	90.160	94.660
6	92.455	90.228	94.728
7	92.489	90.244	94.745
8	92.497	90.249	94.749
9	92.499	90.250	94.750
10	92.500	90.250	94.750

Hence the required values of interior points are,

$u_1 = u_4 = 92.5$, $u_2 = 90.25$ and $u_3 = 94.75$

NOTE:

Procedure to iterate in programmable calculator:

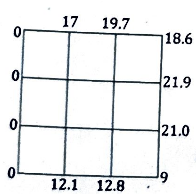
Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$

Set the following in calculator;

$$A = \frac{B + C + 185}{4} : B = \frac{A + 88}{2} : C = \frac{A + 97}{2}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

11. **Solve the Poisson equation $\nabla^2 f = 4x^2 y + 3xy^2$, over the square domain $x \le 3$, $1 \le y \le 3$, with f on the boundary is given in figure below. Take $h = k = 1$.** [2020/Fall]

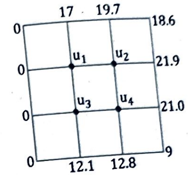


Solution:

Given that;

$\nabla^2 f = 4x^2 y + 3x y^2$

Over the square domain $x \le 3$, $1 \le y \le 3$ with f on the boundary



Let u_1, u_2, u_3 and u_4 be the interior nodes of Poisson's equation and replacing $\nabla^2 f$ by difference equation with $x = ih$, $y = jk$ where, $(h = k = 1)$

Then,

$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = (1)^2 (4i^2 j + 3ij^2)$

or, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 4i^2 j + 3ij^2$

Now, for node, u_1, put $i = 1, j = 2$

$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 4(1)^2(2) + 3(1)(2)^2$

or, $0 + u_2 + u_3 + 17 - 4u_1 = 20$

or, $u_1 = \frac{1}{4}(u_2 + u_3 - 3)$

Now for node u_2, put $i = 2, j = 2$

$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 4(2)^2(2) + 3(2)(2)^2$

or, $u_1 + 21.9 + u_4 + 19.7 - 4u_2 = 56$

or, $u_2 = \frac{1}{4}(u_1 + u_4 - 14.4)$

For node u_3, put $i = 1, j = 1$

$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 4(1)^2(1) + 3(1)(1)^2$

or, $0 + u_4 + 12.1 + u_1 - 4u_3 = 7$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 5.7)$

For node u_4, put $i = 2, j = 1$

$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 4(2)^2(1) + 3(2)(1)^2$

or, $u_3 + 21 + 12.8 + u_2 - 4u_4 = 22$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 11.8)$

Let the initial guess for u_1, u_2, u_3 and u_4 be zero.

Now, using Gauss Seidel method in tabular form,

Itn.	$u_1 = \frac{1}{4}(u_2 + u_3 - 3)$	$u_2 = \frac{1}{4}(u_1 + u_4 - 14.4)$	$u_3 = \frac{1}{4}(u_1 + u_4 + 5.7)$	$u_4 = \frac{1}{4}(u_2 + u_3 + 11.8)$
1	-0.75	-3.788	1.238	2.312
2	-1.388	-3.369	1.656	2.522
3	-1.178	-3.264	1.761	2.574
4	-1.126	-3.238	1.787	2.587
5	-1.113	-3.231	1.794	2.591
6	-1.109	-3.230	1.796	2.592
7	-1.109	-3.229	1.796	2.592
8	-1.108	-3.229	1.796	2.592

Hence the required values of interior points are

$u_1 = -1.108$

$u_2 = -3.229$

$u_3 = 1.796$

and, $u_4 = 2.592$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1$, $B = u_2$, $C = u_3$, $D = u_4$

Set the following in calculator;

$$A = \frac{B + C - 3}{4} : B = \frac{A + D - 14.4}{4} : C = \frac{A + D + 5.7}{4} : D = \frac{B + C + 11.8}{4}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

12. Write short notes on: Laplacian equation. **[2013/Fall, 2013/Spring, 2016/Fall, 2016/Spring]**

Solution: See the topic 6.2 'C'.

13. Write short notes on; Hyperbolic equations. **[2015/Spring]**

Solution: See the topic 6.2. 'F'.

14. Write short notes on: Laplace method for partial differential. **[2017/Fall, 2018/Fall]**

Solution: See the topic 6.2 'C'.

15. Write short notes on: Parabolic equation. **[2017/Spring]**

Solution: See the topic 6.2.'E'.

16. Write short notes on Elliptical equations. **[2017/Spring]**

Solution: See the topic 6.2. 'B'.

ADDITIONAL QUESTION SOLUTION

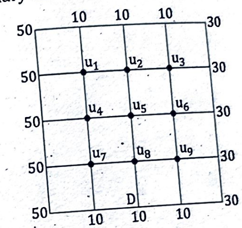
1. **Solve the elliptic equation $\nabla^2 u = 0$ in the square plate of size 8 cm × 8 cm if the boundary values are given 50 on one side of the plate and 30 on its opposite side. On the other sides the values are given 10. Assume the square grids of size 2 cm × 2 cm.**

Solution:

Given that;

Elliptic equation $\nabla^2 u = 0$.

From the given boundary values, the figure can be illustrated as,



Let the inner points be defined as $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8$ and u_9 as shown

Then we find the first initial values as

$u_5 = \frac{1}{4}[50 + 10 + 30 + 10]$ (Using standard 5-point formula)

$= 25$

$u_1 = \frac{1}{4}[10 + 50 + 50 + 25]$ (Using diagonal 5-point formula)

$= 33.75$

Likewise,

$u_3 = \frac{1}{4}[10 + 30 + 30 + 25]$ (Using diagonal 5-point formula)

$= 23.75$

$u_2 = \frac{1}{4}[10 + u_1 + u_3 + u_5]$ (Using standard 5-point formula)

$= \frac{1}{4}[10 + 33.75 + 23.75 + 25]$

$= 23.125$

$u_7 = \frac{1}{4}[10 + 50 + 50 + 25]$ (Using diagonal 5-point formula)

$= 33.75$

$u_9 = \frac{1}{4}[10 + 30 + 30 + 25]$ (Using diagonal 5-point formula)

$= 23.75$

$u_4 = \frac{1}{4}[50 + u_1 + u_5 + u_7]$ (Using standard 5-point formula)

$= \frac{1}{4}[50 + 33.75 + 25 + 33.75]$

$= 35.625$

$u_6 = \frac{1}{4}[30 + u_3 + u_5 + u_9]$ (Using standard 5-point formula)

$= \frac{1}{4}[30 + 23.75 + 25 + 23.75] = 25.625$

$u_8 = \frac{1}{4}[u_5 + u_7 + u_9 + 10]$ (Using standard 5-point formula)

$= \frac{1}{4}[25 + 33.75 + 23.75 + 10]$

$= 23.125$

Now, we can carry out Gauss Seidel iteration using standard 5-point formula.

Iteration 1, put n = 0 at

$u_1^{n+1} = \frac{1}{4}[10 + 50 + u_2^n + u_4^n] = \frac{1}{4}[60 + u_2^0 + u_4^0]$

$\therefore \quad u_1^1 = \frac{1}{4}[60 + 23.125 + 35.625] = 29.6875$

$u_2^{n+1} = \frac{1}{4}[10 + u_1^{n+1} + u_3^n + u_5^n] = \frac{1}{4}[10 + u_1^1 + u_3^0 + u_5^0]$

$\therefore \quad u_2^1 = \frac{1}{4}[10 + 29.6875 + 23.75 + 25] = 22.1094$

$u_3^{n+1} = \frac{1}{4}[10 + 30 + u_2^{n+1} + u_6^n] = \frac{1}{4}[40 + u_2^1 + u_6^0]$

$\therefore \quad u_3^1 = \frac{1}{4}[40 + 22.1094 + 25.625] = 21.9336$

$u_4^{n+1} = \frac{1}{4}[u_1^{n+1} + u_5^n + u_7^n + 50] = \frac{1}{4}[u_1^1 + u_5^0 + u_7^0 + 50]$

$\therefore \quad u_4^1 = \frac{1}{4}[29.6875 + 25 + 33.75 + 50] = 34.6094$

$u_5^{n+1} = \frac{1}{4}[u_2^{n+1} + u_4^{n+1} + u_6^n + u_8^n] = \frac{1}{4}[u_2^1 + u_4^1 + u_6^0 + u_8^0]$

$\therefore \quad u_5^1 = \frac{1}{4}[29.6875 + 34.6094 + 25.625 + 23.125] = 28.2617$

$u_6^{n+1} = \frac{1}{4}[u_3^{n+1} + u_5^{n+1} + u_9^n + 30] = \frac{1}{4}[u_3^1 + u_5^1 + u_9^0 + 30]$

$\therefore \quad u_6^1 = \frac{1}{4}[21.9336 + 28.2617 + 23.75 + 30] = 25.9863$

$u_7^{n+1} = \frac{1}{4}[u_4^{n+1} + 50 + 10 + u_8^n] = \frac{1}{4}[u_4^1 + 60 + u_8^0]$

$\therefore \quad u_7^1 = \frac{1}{4}[34.6094 + 60 + 23.125] = 29.4336$

$u_8^{n+1} = \frac{1}{4}[u_5^{n+1} + u_7^{n+1} + 10 + u_9^n] = \frac{1}{4}[u_5^1 + u_7^1 + 10 + u_9^0]$

$\therefore \quad u_8^1 = \frac{1}{4}[28.2617 + 29.4336 + 10 + 23.75] = 22.8613$

$u_9^{n+1} = \frac{1}{4}[u_6^{n+1} + u_8^{n+1} + 10 + 30] = \frac{1}{4}[u_6^1 + u_8^1 + 40]$

$\therefore \quad u_9^1 = \frac{1}{4}[25.9863 + 22.8613 + 40] = 22.2119$

Likewise, put n = 1 and carry out the iterations

$u_1^2 = 29.1797$ $\quad u_2^2 = 22.3438$

$u_3^2 = 22.0825$ $\quad u_4^2 = 34.2188$

$u_5^2 = 26.3526$ $\quad u_6^2 = 25.1618$

$u_7^2 = 29.2700$ $\quad u_8^2 = 21.9586$

$u_9^2 = 21.7801$

Put n = 2 and carry out the iterations

$u_1^3 = 29.1407$ $\quad u_2^3 = 21.8940$

$u_3^3 = 21.7640$ $\quad u_4^3 = 33.6908$

$u_5^3 = 25.6763$ $\quad u_6^3 = 24.8051$

$u_7^3 = 28.9124$ $\quad u_8^3 = 21.5922$

$u_9^3 = 21.5993$

Similarly, the iterations are carried out upto required significant differences for the inner points.

2. **Solve the elliptic equation $u_{xx} + u_{yy} = 0$ on the square mesh bounded by $0 \le x \le 3$, $0 \le y \le 3$. The boundary values are $u(x, 0) = 10$, $u(x, 3) = 90$, $0 \le x \le 3$ and $u(0, y) = 70$, $u(3, y) = 0$, $0 < y < 3$.**

Solution:

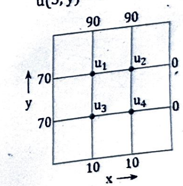
Given the elliptic equation

$u_{xx} + u_{yy} = 0$

Now, using the boundary values provided, the figure can be illustrated as

$u(x, 0) = 10$, $\quad u(0, 4) = 70$

$u(x, 3) = 90$, $\quad u(3, y) = 0$



Let the inner points be defined as u_1, u_2, u_3 and u_4.

Now, using standard five point formula

We have,

$u_1 = \frac{1}{4}(70 + 90 + u_2 + u_3) = \frac{1}{4}(160 + u_2 + u_3)$

$$u_2 = \frac{1}{4}(0 + 90 + u_1 + u_4) = \frac{1}{4}(90 + u_1 + u_4)$$

$$u_3 = \frac{1}{4}(u_1 + 70 + 10 + u_4) = \frac{1}{4}(80 + u_1 + u_4)$$

$$u_4 = \frac{1}{4}(0 + 10 + u_2 + u_3) = \frac{1}{4}(10 + u_2 + u_3)$$

To obtain the values, let initial guess be,

$u_1 = 0$, $u_2 = 0$, $u_3 = 0$ and $u_4 = 0$ then,

Using Gauss Seidel method of iteration in tabular form,

Itn.	$u_1 = \frac{1}{4}(160 + u_2 + u_3)$	$u_2 = \frac{1}{4}(90 + u_1 + u_4)$	$u_3 = \frac{1}{4}(80 + u_1 + u_4)$	$u_4 = \frac{1}{4}(10 + u_2 + u_3)$
1	$\frac{1}{4}(160 + 0 + 0) = 40$	$\frac{1}{4}(90 + 40 + 0) = 32.5$	$\frac{1}{4}(80 + 40 + 0) = 30$	$\frac{1}{4}(10 + 32.5 + 30) = 18.125$
2	55.6250	40.9375	38.4375	22.3438
3	59.8438	43.0469	40.5469	23.3984
4	60.8984	43.5742	41.0742	23.6621
5	61.1621	43.7061	41.2061	23.7280
6	61.2280	43.7390	41.2390	23.7445
7	61.2445	43.7473	41.2473	23.7486
8	61.2486	43.7493	41.2493	23.7497
9	61.2497	43.7498	41.2498	23.7499
10	61.2499	43.7500	41.2500	23.7500
11	61.2500	43.7500	41.2500	23.7500

Hence the required values of interior points are

$u_1 = 61.25$

$u_2 = 43.75$

$u_3 = 41.25$

and, $u_4 = 23.75$

POKHARA UNIVERSITY

Level: Bachelor
Semester: Fall
Year: 2021

Programme: BE
Full Marks: 100

Course Numerical Methods
Pass Marks: 45

Time: 3 hrs.

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

Attempt all the questions.

1. a) Solve $x^3 + x^2 - 3x - 3 = 0$ by secant method up to 8^{th} iteration. Assume that the error should be less than 10^{-4}.

Solution:

$x^3 + x^2 - 3x - 3 = 0$ (Correct mp 10^{-4}, 8 iteration)

$F(x) = x^3 + x^2 - 3x - 3$

Let $a = -1.5$

$b = -2.5$

Note: It has 3 different roots so you can choose other initial guess as well.

N	A	b	F(a)	F(b)	x	F(x)
1	−1.5	−2.5	0.375	−4.875	−1.571428	0.303206
2	−2.5	−1.571428	−4.875	0.3032077	−1.625800	0.223269
3	−1.571428	−1.625800	0.3032077	0.223269	1.77762	−0.124490
4.	−1.625800	1.777662	0.223269	−0.124489	−1.723298	0.021873
5.	−1.777662	−1.723298	− 0.124489	0.021875	−1.731423	0.001590
6.	−1.723298	−1.731423	0.021875	0.001590	− 1.732060	−0.000023
7.	−1.731423	−1.732060	0.001590	− 0.000023	−1.732050	0.0000002424
8.	−1.732060	−1.732050	−0.000023	0.00000020	−1.732050	-13.14×10^{-12}

∴ The required root is −1.732050,

b) Find the root of the equation $\log x - \cos x = 0$ correct to three decimal placed by using N–R method.

Solution:

$F(x) = \log x - \cos x = 0$

$F(x) = \log x - \cos x$

$F'(x) = \frac{1}{x} + \sin x$ [take base as e]

Let $x_0 = 1$

N	x	F(x)	F'(x)	$x_{i+1} = x_i - \frac{F(x_i)}{F'(x_i)}$
1	1	−0.540302	1.841470	1.293407
2	1.293407	−0.016565	1.734925	1.302955
3	1.302955	−0.0000155	1.731830	1.302964
4	1.302964	−0.0000000212	1.731827	1.302964

∴ Required root is 1.302964

POKHARA UNIVERSITY

Level: Bachelor
Programme: BE
Course Numerical Methods

Semester: Fall

Year: 2021
Full Marks: 100
Pass Marks: 45
Time: 3 hrs.

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

Attempt all the questions.

1. a) Solve $x^3 + x^2 - 3x - 3 = 0$ by secant method up to 8^{th} iteration. Assume that the error should be less than 10^{-4}.

Solution:

$x^3 + x^2 - 3x - 3 = 0$ (Correct mp 10^{-4}, 8 iteration)

$F(x) = x^3 + x^2 - 3x - 3$

Let $a = -1.5$

$b = -2.5$

Note: It has 3 different roots so you can choose other initial guess as well.

N	A	b	F(a)	F(b)	x	F(x)
1	-1.5	-2.5	0.375	-4.875	-1.571428	0.303206
2	-2.5	-1.571428	-4.875	0.3032077	-1.625800	0.223269
3	-1.571428	-1.625800	0.3032077	0.223269	1.77762	-0.124490
4.	-1.625800	1.777662	0.223269	-0.124489	-1.723298	0.021873
5.	-1.777662	-1.723298	-0.124489	0.021875	-1.731423	0.001590
6.	-1.723298	-1.731423	0.021875	0.001590	-1.732060	-0.000023
7.	-1.731423	-1.732060	0.001590	-0.000023	-1.732050	0.00000002424
8.	-1.732060	-1.732050	-0.000023	0.00000020	-1.732050	-13.14×10^{-12}

∴ The required root is −1.732050,

b) Find the root of the equation logx − cosx = 0 correct to three decimal placed by using N–R method.

Solution:

$F(x) = \log x - \cos x = 0$

$F(x) = \log x - \cos x$

$F'(x) = \frac{1}{x} + \sin x$ [take base as e]

Let $x_0 = 1$

N	x	F(x)	F'(x)	$x_i + 1 = x_i - \frac{F(x_i)}{F'(x_i)}$
1	1	-0.540302	1.841470	1.293407
2	1.293407	-0.016565	1.734925	1.302955
3	1.302955	-0.0000155	1.731830	1.302964
4	1.302964	-0.0000000212	1.731827	1.302964

∴ Required root is 1.302964

2. a) Define interpolation. From the following table, estimate the number of students who passed marks between 40 and 45:

Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of students	30	40	50	38	31

Solution:

We have to find cumulative frequency

Below x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	30				
		40			
Below 50	70		10		
		50		-22	
Below 60	120		-12		27
		38		5	
Below 70	158		-7		
		31			
Below 80	189				

Let $x_p = 45$, $x_0 = 40$, $h = 10$

We know,

$$p = \frac{x_p - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

Also, using Newton forward interpolate

$$y_p = y_0 + p \cdot \Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$= 30 + 0.5 \times 40 + \frac{0.5(0.5-1)}{2!} \times 10 + \frac{0.5(0.5-1)(0.5-2)}{3!} \times (-22) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!} \times 27$$

On solving

$= 46.3203125 \approx 47$

So No. of students with marks less than 45 = 47

∴ No. of students with marks less than 40 = 30

∴ Students with marks between 40 – 45 = 47 – 30 = 17

b) Fit cubic polynomial equations to the given data set and find the value of (3.7) and f (7.5).

x	2	4	7	9
f(x)	1	2	1	2

Solution:

Since it is cubic polynomial i.e.,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Taking Σ on b.s.

$$\Sigma y = n a_0 + a_1 \Sigma x + a_2 \Sigma x^2 + a_3 \Sigma x^3 \quad \text{..... (i)}$$

Taking x on b.s.

$$\Sigma xy = a_0 \Sigma x + a_1 \Sigma x^2 + a_2 \Sigma x^3 + a_3 \Sigma x^4 \quad \text{..... (ii)}$$

$$\Sigma x^2 y = a_0 \Sigma x^2 + a_1 \Sigma x^3 + a_2 \Sigma x^4 + a_3 \Sigma x^5 \quad \text{..... (iii)}$$

$$\Sigma x^3 y = a_0 \Sigma x^3 + a_1 \Sigma x^4 + a_2 \Sigma x^5 + a_3 \Sigma x^6 \quad \text{..... (iv)}$$

x	Y	xy	x^2	x^3	x^4	x^5	x^6	x^2y	x^3y
2	1	2	4	8	16	32	64	4	8
4	2	8	16	64	256	1024	4096	32	128
7	1	7	49	343	2401	16807	117649	49	343
9	2	18	81	729	6561	59049	531441	162	1458

∴ $\Sigma x = 22$ $\Sigma x^2 = 150$ $\Sigma x^2 y = 247$ $\Sigma x^6 = 653250$

$\Sigma y = 6$ $\Sigma x^3 = 114$ $\Sigma x^3 y = 1937$

$\Sigma xy = 35$ $\Sigma x^4 = 9234$ $\Sigma x^5 = 76912$

Now equation (i), (ii) and (iv) becomes

$$6 = 4 a_0 + 22 a_1 + 150 a_2 + 1144 a_3$$

$$35 = 22 a_0 + 150 a_2 + 1144 a_2 + 9234 a_3$$

$$247 = 150 a_0 + 1144 a_1 + 9234 a_2 + 76912 a_3$$

$$1937 = 1144 a_0 + 9234 a_1 + 76912 a_2 + 653250 a_3$$

On solving

$$a_0 = -4 \; a_1 = \frac{163}{42} \; a_2 = \frac{-11}{14} \; a_3 = \frac{1}{21}$$

∴ Required cubic polynomial is

$$y = -4 + \frac{163}{42}x \frac{-11}{14}x^2 + \frac{1}{21}x^3$$

Now y (3.7)

$$y = -4 + \frac{163}{42}(3.7) - \frac{1.1}{14}(3.7)^2 + \frac{1}{21} \times (3.7)^3 = \frac{7053}{3500} = 2.01514$$

Similarly,

$$y' = \frac{163}{42} - \frac{2 \times 11}{14}x + \frac{3}{21}x^2 = \frac{163}{42} - \frac{11}{7}x + \frac{1}{7}x^2$$

$$y'(7.5) = \frac{163}{42} - \frac{11}{7} \times 7.5 + \frac{1}{7}(7.5)^2 = \frac{935}{84} = 11.130952$$

3. a) Integrate the following function by using Trapezoidal rule, Simpson's $\frac{1}{3}$ rule and Simpson $\frac{3}{8}$ rule. Take n = 6 $\int_0^{\frac{\pi}{2}} \sin x \, dx$.

Solution:

$$\int_0^{\frac{\pi}{2}} \sin x dx$$

$$I = \int_0^{\frac{\pi}{2}} \sin x dx$$

$a = 0$

$b = \frac{\pi}{2}$

$$h = \frac{b - a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

F(x) = sin x in radian

X	0	$\frac{\pi}{12}$	$\frac{2\pi}{12} - \frac{\pi}{6}$	$\frac{3\pi}{12} = \frac{\pi}{4}$	$\frac{3\pi}{12} = \frac{\pi}{3}$	$\frac{5\pi}{17}$	$\frac{6\pi}{12} = \frac{\pi}{2}$
$\frac{y}{f(x)}$	0	0.258	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	0.965	1

i) **Trapezoidal rule**

$$\int_{x_0}^{x_0+nh} f(x)\, dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1})]$$

$$\int_0^{\frac{\pi}{2}} \sin xdx = \frac{h}{2}[y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{12 \times 2}\left[(0 + 1) + 2\left(\frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} + 0.965\right)\right]$$

$= 0.9938$

Using Simpson $\frac{1}{3}$

$$\int_0^{\frac{\pi}{2}} \sin xdx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{12 \times 3}\left[(0 + 1) + 4\left(0.258 + \frac{1}{\sqrt{2}} + 0.965\right) + 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)\right]$$

$= 0.9993$

Step Simpson $\frac{3}{8}$

$$\int_0^{\frac{\pi}{2}} \sin xdx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_5)\ 2(y_3)]$$

$$= \frac{3 \times \pi}{12 \times 8}\left[(0 + 1) + 3\left(0.258 + \frac{1}{2} + \frac{\sqrt{3}}{2} + 0.965\right) + 2\left(\frac{1}{\sqrt{2}}\right)\right]$$

$= 0.9995$

b) Integrate the given integral $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin x}} dx$

Using gauss quadrature formula for n = 2 and n = 3

Solution:

Using gauss quadrates 1st of all we need to change the limit from $\left(0, \frac{\pi}{2}\right)$ to (-1, 1)

We know,

$a = 0, b = \frac{\pi}{2}$

$\therefore \quad x = \left(\frac{b - a}{2}\right) z + \left(\frac{b + a}{2}\right)$

or, $\quad x = \frac{\left(\frac{\pi}{2} - 0\right)}{2} z + \frac{\frac{\pi}{2} + 0}{2}$

or $\quad x = \frac{\pi}{4} z + \frac{\pi}{4} = \frac{\pi}{4}(z + 1)$

or $\quad dx = \frac{\pi}{4} dz$

So,

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin x}} \int_{-1}^{1} \frac{\cos\left(\frac{\pi}{4}(z + 1)\right)}{\sqrt{1 + \sin\left(\frac{\pi}{4}(z + 1)\right)}} \times \frac{\pi}{4} dz$$

$$\therefore \quad F(z) = \frac{\pi}{4} \frac{\cos\left(\frac{\pi}{4}(z + 1)\right)}{\sqrt{1 + \sin\left(\frac{\pi}{4}(z + 1)\right)}}$$

For n = 2

$$\int_{-1}^{1} F(z) \cdot dz = \omega_1 F(z_1) + \omega_2 F(z_2)$$

$\omega_1 = \omega_2 = 1$

$z_1 = \frac{-1}{\sqrt{3}}\ z_2 = \frac{1}{\sqrt{3}}$

$$\therefore \quad \int_{-1}^{1} F(z) \cdot dz = F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{4}\left(\frac{\cos\left(\frac{\pi}{4}\left(1 - \frac{1}{\sqrt{3}}\right)\right)}{\sqrt{1 + \sin\left(\frac{\pi}{4}\left(1 - \frac{1}{\sqrt{3}}\right)\right)}} + \frac{\cos\left(\frac{\pi}{4}\left(1 + \frac{1}{\sqrt{3}}\right)\right)}{\sqrt{1 + \sin\left(\frac{\pi}{4}\left(1 + \frac{1}{\sqrt{3}}\right)\right)}}\right)$$

$= \frac{\pi}{4}((0.82104469572) + (0.23364672459))$

$= 0.82835270446$

For n = 3

$$\int_{-1}^{1} F(z)\cdot dz = \omega_1 F(z_1) + \omega_2 \text{ of } (z_2) + \omega_3 F(z_3)$$

$$z_1 = -\sqrt{\frac{3}{5}} \qquad \omega_1 = \omega_3 = \frac{5}{9}$$

$$z_2 = 0 \qquad \omega_2 = \frac{8}{9}$$

$$z_3 = +\sqrt{\frac{3}{5}}$$

$$\int_{-1}^{1} F(z)\cdot dz = \frac{8}{9}F(0) + \frac{5}{9}\left(F\left(-\sqrt{\frac{3}{5}}\right) + F\left(\sqrt{\frac{3}{5}}\right)\right)$$

$$= \frac{8}{9}(0.54119610015) + \frac{5}{9}(0.9076\,8490509 + 0.1250166757)$$

$$= 1.05478630058 \times \frac{\pi}{4}$$

$$= 0.82842722325$$

4. a) Find the inverse of the matrix, using Gauss Jordan method.

$$A\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & -5 \\ -2 & -4 & -4 \end{bmatrix}$$

Solution:

Using gauss jordun method

$[A : I] \rightarrow [I : A^{-1}]$

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & -5 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1 \; R_3 \leftarrow R_3 + 2R_1$

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -9 & -1 & 1 & 0 \\ 0 & 0 & 4 & 2 & 0 & 1 \end{array}\right]$$

$R_3 \leftarrow \frac{R_3}{4}$

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -9 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 1/4 \end{array}\right]$$

$R_2 \leftarrow R_2 + 9R_3 \; R_1 \leftarrow R_1 - 4R_3$

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 7/2 & 1 & 9/4 \\ 0 & 0 & 1 & 1/2 & 0 & 1/4 \end{array}\right]$$

$R_1 \leftarrow R_1 - 2R_2$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -8 & -2 & -11/2 \\ 0 & 1 & 0 & 7/2 & 1 & 9/4 \\ 0 & 0 & 1 & 1/2 & 0 & 1/4 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -8 & -2 & -11/2 \\ 7/2 & 1 & 9/4 \\ 1/2 & 0 & 1/4 \end{bmatrix}$$

b) Find the largest Eigen-value and the corresponding Eigen-vector of the following square matrix using power method.

$$\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

Solution:

Let Eigen value = λ°

$$\text{Eigen vector } X^\circ = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

1st iteration

$$AX^\circ = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix}$$

$$= 25\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix}$$

$$= \lambda^1 X^1$$

2nd iteration

$$AX^1 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 2.32 \end{bmatrix}$$

$$= 25.2\begin{bmatrix} 1 \\ 0.0444 \\ 0.09206 \end{bmatrix}$$

$$= \lambda^2 X^2$$

3rd iteration

$$AX^2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0444 \\ 0.09206 \end{bmatrix} = \begin{bmatrix} 25.228571 \\ 1.133333 \\ 2.368253 \end{bmatrix}$$

$$= 25.228571\begin{bmatrix} 1 \\ 0.044922 \\ 0.093871 \end{bmatrix}$$

$$= \lambda^3 X^3$$

4^{th} iteration

$$AX^3 = \begin{bmatrix}25 & 1 & 2\\ 1 & 3 & 0\\ 2 & 0 & 4\end{bmatrix}\begin{bmatrix}1\\ 0.044922\\ 0.093871\end{bmatrix} = \begin{bmatrix}25.23266\\ 1.1347678\\ 2.375484\end{bmatrix}$$

$$= 25.23266\begin{bmatrix}1\\ 0.0449721\\ 0.094143\end{bmatrix}$$

$$= \lambda^4 X^4$$

5^{th} iteration

$$AX^4 = \begin{bmatrix}25 & 1 & 2\\ 1 & 3 & 0\\ 2 & 0 & 4\end{bmatrix}\begin{bmatrix}1\\ 0.0449721\\ 0.094143\end{bmatrix} = \begin{bmatrix}25.2332581\\ 1.1349163\\ 2.376572\end{bmatrix}$$

$$= 25.2332581\begin{bmatrix}1\\ 0.0449770\\ 0.0941841\end{bmatrix}$$

$$= \lambda^5 X^5$$

6^{th} iteration

$$AX^5 = \begin{bmatrix}25 & 1 & 2\\ 1 & 3 & 0\\ 2 & 0 & 4\end{bmatrix}\begin{bmatrix}1\\ 0.0449770\\ 0.0941841\end{bmatrix} = \begin{bmatrix}25.233345\\ 1.134931\\ 2.376736\end{bmatrix}$$

$$= 25.233345\begin{bmatrix}1\\ 0.044977\\ 0.09419028\end{bmatrix}$$

∴ Largest Eigent values 25.233345

$$\text{Corresponding Eigen vector} = \begin{bmatrix}1\\ 0.044977\\ 0.09419028\end{bmatrix}$$

5. a) Solve the following set of equations by using LU Crout method.

$$3x + 2y + z = 10$$
$$2x + 3y + 2z = 14$$
$$x + 2y + 3z = 14$$

Solution:

$$3x + 2y + z = 10$$
$$2x + 3y + 2z = 14$$
$$x + 2y + 3z = 14$$

We know,

$$AX = B$$

$$A = \begin{bmatrix}3 & 2 & 1\\ 2 & 3 & 2\\ 1 & 2 & 3\end{bmatrix} X = \begin{bmatrix}x\\ y\\ z\end{bmatrix} B = \begin{bmatrix}10\\ 14\\ 14\end{bmatrix}$$

Let $A = LU$ using crout

$$A = \begin{bmatrix}a & 0 & 0\\ b & c & 0\\ d & e & f\end{bmatrix}\begin{bmatrix}1 & g & h\\ 0 & 1 & i\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}3 & 2 & 1\\ 2 & 3 & 2\\ 1 & 2 & 3\end{bmatrix} = \begin{bmatrix}a & ag & ah\\ b & bg + c & bh + ci\\ d & dg + e & dh + ei + f\end{bmatrix}$$

$a = 3, b = 2$ $\quad d = 1$ $\quad dg + e = 2$ $\quad bg + c = 3$

∴ $Ag = 2$ $\quad ah = 1$ $\quad e = \frac{4}{3}$ $\quad c = \frac{5}{3}$

∴ $g = \frac{2}{3}$ $\quad h = \frac{1}{3}$

$bh + ci = 2$ $\quad dh + ei + f = 3$

∴ $i = \frac{4}{5}$ $\quad F = \frac{8}{5}$

So,

$$L = \begin{bmatrix}3 & 0 & 0\\ 2 & 5/3 & 0\\ 1 & 4/3 & 8/5\end{bmatrix}$$

$$U = \begin{bmatrix}1 & 2/3 & 1/3\\ 0 & 1 & 4/5\\ 0 & 0 & 1\end{bmatrix}$$

$$AX = B$$
$$LUX = B$$

∴ $LZ = B$ where $Z = \begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}$

$$\begin{bmatrix}3 & 0 & 0\\ 2 & 5/3 & 0\\ 1 & 4/3 & 8/5\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} = \begin{bmatrix}10\\ 14\\ 14\end{bmatrix}$$

Using forward substitution

$3z_1 = 10$ $\quad 2z_1 + \frac{5}{3}z_2 = 14$

∴ $z_1 = \frac{10}{3}$ $\quad z_2 = \frac{22}{5}$

$$z_1 + \frac{4}{3}z_2 + \frac{8}{5}z_3 = 14$$

∴ $z_3 = 3$

Now, $UX = Z$ using backward substitution

$$\begin{bmatrix}1 & 2/3 & 1/3\\ 0 & 1 & 4/5\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}10/3\\ 22/5\\ 3\end{bmatrix}$$

$\therefore \quad z = 3 \qquad\qquad x + \frac{2}{3}y + \frac{z}{3} = \frac{10}{3}$

$y + \frac{4}{5}z = \frac{22}{5} \qquad\qquad x = 1$

$\therefore \quad y = 2$

$\therefore \quad x = 1, y = 2, z = 3$

b) Apply R-K-4 method to solve y(0.2) for the given equation $\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y$ given that y = 1 and $\frac{dy}{dx} = 0$ when x = 0.

Solution:

$$\frac{d^2y}{dx^2} + \frac{xdy}{dx} - y = 0$$

$y = 1$ at $x = 0$

$y' = 0$

$y(0.2) = ?$

Let, $\frac{dy}{dx} = z = F_1(x, y, z)$

Then,

$z' + xz - y = 0$

$z' = y - xz = F_2(x, y, z)$

Using initial condition

$y(0) = 1 \qquad y'(0) = 0$

$\therefore \quad x_0 = 0, y_0 = 1 \qquad z_0 = 0$

Now,

$k_1 = F_1(z_0, y_0, z_0) = F_1(0, 1, 0) = 0 \qquad l_1 = F_2(0, 1, 0) = 1 - 0 = 1$

$k_2 = F_1\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \times k_1, z_0 + \frac{h}{2} \times l_1\right) = F_1(0.1, 1, 0.1) = 0.1 \qquad l_2 = F_2(0.1, 1, 0.1) = 1 - 0.1 \times 0.1 = 0.99$

$k_3 = F_1\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \times k_2, z_0 + \frac{h}{2} \times l_2\right) = F_1(0.1, 1.01, 0.099) = 0.099 \qquad l_3 = F_2(0.1, 1.01, 0.099) = 1.01 - 0.1 \times 0.099 = 1.0001$

$k_4 = F_1(x_0 + h, y_0 + h \times k_3, z_0 + h \times l_3) = F_1(0.2, 1.0198, 0.20002) = 0.20002 \qquad l_4 = F_2(0.2, 01.0198, 0.2002) = 1.0198 - 0.2 \times 0.20002 = 0.979796$

$$\therefore \quad k = \frac{k_1 + 2(k_2 + k_3) + k_4}{6} = 0.09967$$

$$l = \frac{l_1 + 2(l_2 + l_3) + l_4}{6} = 0.99333$$

$y(0.2) = y_0 + h \times k = 1 + 0.2 \times 0.09967 = 1.019934$

$y'(0.2) = z(0.2) = z_0 + h \times l = 0 + 0.2 \times 0.99333 = 0.198666$

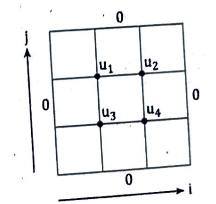
6. a) In a square bar with dimension of 3 inch × 3 inch, torsion function, ϕ, can be obtained from the following P.D.E: $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2$ where $\phi = 0$ on the outer boundary of the bar's cross-section. Subdivided the region into nine equal squares to form a mesh and find the values of ϕ in the interior nodes.

Solution:

Given equation P.D.E. is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2 \qquad \phi = 0$$

Let U_1, U_2, U_3 and U_4 be the values of 4 at the 4 mesh points as shown in figure.



We know,

$U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{i,j} = -2$

For U_1 $(i = 1, j = 2)$

$\therefore \quad U_1 = \frac{U_2 + U_3 + 2}{4}$

For U_2 $(i = 2, j = 2)$

$\therefore \quad U_2 = \frac{U_1 + U_4 + 2}{4}$

For U_3 $(i = 1, j = 1)$

$U_3 = \frac{U_1 + U_4 + 2}{4}$

For U_4 $(i = 2, j = 1)$

$U_4 = \frac{U_2 + U_3 + 2}{y}$

Form here we can see that $U_1 = U_4$ and $U_2 = U_3$, replacing U_4 by U_1 and U_3 by U_2

Also,

$$\therefore \quad U_1 = \frac{2U_2 + 2}{4} = \frac{U_2 + 1}{2} \qquad U_2 = \frac{2U_1 + 2}{4}$$

$\therefore \quad \dfrac{U_1 + 1}{2}$

Using gauss seidal method

N	U_1	U_2
0	0	0
1	0.5	0.75
2	0.875	0.9375
3	0.96875	0.984375
4	0.9921875	0.9990234375
5	0.998046875	0.9990234375
6	0.99951171875	0.99975585938
7	0.99987792969	0.9993896484
8	0.99996948242	0.99998474121
9	0.99999237061	0.99999611853

$\therefore \quad U_1 \approx 1 \quad U_2 \approx 1$

So, $\quad U_1 = U_2 = U_3 = U_4 = 1$

b) Consider second order initial value problem $y'' - 4y' + 2y = e^t \sin(t)$ with $y(0) = 0.4$ and $y'(0) = -0.6$, using Heun's find value of $y(0.2)$ and $y'(0.2)$.

Solution:

$y'' - 4y' + 2y = e^t \sin(f)$

$y(0) = 0.4\ y'(0) = -0.6$ 250 At start in $y(0.2) = ?$

Let $\dfrac{dy}{dt} = z = F1(t, y, z)\ y'$ $\qquad y_{(0-2)} = ?$

$z' - 4z + 2y = (e^t \sin(t))$ $\qquad y'_{(0-2)} = ?$

$\therefore \quad z' = e^t \sin(t) + 4z - 2y = F_2(t, y, z)$

Using initial condition

$y(0) = 0.4 \qquad y'(0) = 0.6$

$\therefore \quad t_0 = 0, = y_0 = 0.4 \qquad z_0 = -0.6$

Using Hens formula

$M_1(1) = F_1(t_0, y_0, z_0) \qquad M_1(2) = F_2(0, 0.4, 0.6)$

$= F_1(0, 0.4, -0.6) \qquad = e^{\circ}\sin(0) + 4 \times (-0.6) - 2 \times 0.4$

$= -0.6 \qquad = -3.2$

$M_2(1) = F_1(t_0 + h, y_0 + h \times M_1(1), z_0 + h \times M_1(2))$

$= F_1(0.2, 0.28, -1.24)$

$= -1.24$

$M_2(2) = F_2(0.2, 0.28, -1.24)$

$= e^{0.2} \sin(0.2) + 4 \times (-1.24) - 2 \times 0.28$

$= -5.277344$

$\therefore \quad M(1) = \dfrac{M_1(1) + (M_2(1))}{2} = -0.92$

$M(2) = \dfrac{M_1(2) + M_2(2)}{2} = -4.238672$

$y(0.2) = y_0 + h \times M(1)$

$= 0.4 + 0.2 \times (-0.92)$

$= 0.216$

$y'(0.2) = z_0 + h \times M(2)$

$= -0.6 + 0.2 \times (-4.238672)$

$= -1.4477344$

7. **Write short notes on: (Any two)** [2×5]

a) **Taylor's series for solving ODE**

b) **Ill-conditioned system**

c) **Classify the partial differential equation $U_{xx} + 2U_{xy} + U_{yy} = 0$**

Answer: See the theory part

POKHARA UNIVERSITY

Level: Bachelor Semester: Spring Year: 2021
Programme: BE Full Marks: 100
Course Numerical Methods Pass Marks: 45
Time: 3 hrs.

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

Attempt all the questions.

1. a) What is the difference between the bisection method and false position even though both are bracketing methods? Find the real root of the given non-linear equation correct up to three decimal place using Newton Raphson method.

Solution: See the theory part

b) Define error and write its different types in numerical methods with examples. If x = 1.350253 is rounded off to four significant digits.

Solution:

Using symmetric round off

$x = 1.350253$

0.1350253×10^1

4 significant digit s

$(0.1350 + 0.0000253) \times 10^1$

$(0.1350 + 0/253\times 10^{-4}) \times 10^1$

$Fx = 0.1350$

$9x = 0.253$

Since, $9x = 0.253 < 0.5$ so no change or no round off

So, Approx of $x = Fx \times 10^1$

$= 0.1350 \times 10^1$

$= 1.350$

Given true value = 1.350253

$\therefore$ Absolute error = 1.350253 – 1.350 = 0.000253

Relative error = 1.8737×10^{-4}

2. a) By using least square method find the straight line that best fit the following data:

x	1	2	3	4	5
y	14	27	40	55	68

Solution:

x	1	2	3	4	5
y	14	27	40	55	68

We know the equation of st Line $y = a + bx$

$\Sigma y = na + b\Sigma x$ ..... (i)

$\therefore$ $\Sigma xy = a\Sigma x + b\Sigma x^2$ ..... (ii)

X	y	xy	x²
1	14	14	1
2	27	54	4
3	40	120	16
4	55	220	9
5	68	340	25
Σx = 15	Σy = 204	Σxy = 748	Σx² = 55

Substituting the value n (i) and (ii)

$204 = 5a + 15b$

$748 = 15a + 55b$

On solving we get

$a = 0$

$b = \frac{68}{5}$

$\therefore$ $\boxed{y = 0 + \frac{68}{5}x = \frac{68}{5}x}$

b) Find the cubic spline interpolation formula for the following data:

x	1	2	3	4	5
f(x)	1	0	1	0	1

Solution:

X	1	2	3	4	5
y	1	0	1	0	1

Cubic spline formula

$$F(x) = \frac{(x_i - x)^3}{6h} M_{i-1} + \frac{(x - x_{i-1})^3}{6h} M_i + \frac{(x_i - x)}{h}\left(y_1 - \frac{h^2}{6} M_1\right) \quad \text{..... (i)}$$

We have, ... (ii)

$$M_{i-1} + 4M_i + M_{i-1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$$

Here,

$h = 1, n = 4$

$M_0 = 0, M_4 = 0$

Substitute i = 1 in equation (ii)

$$M_0 + 4M_1 + M_2 = \frac{6}{h^2}(y_{i-1} - 2y_{i-1})$$

or, $0 - 4M_1 + M_2 = \frac{6}{1} \cdot (1 - 2.0 + 1)$

or, $4M_1 + M_2 = 12$

Substitute i = 2 in equation (ii)

$$M_1 + 4M_2 + M_3 = \frac{6}{h^2}(y_1 - 2y_2 + y_3)$$

or, $M_1 + 4M_2 + M_3 = \frac{6}{1} \cdot (0 - 2.1 + 0)$

or, $M_1 + 4M_2 + M_3 = -12$

Substitute i = 3 in equation (ii)

$$M_2 + 4M_3 + M_4 = \frac{6}{h^2}(y_2 - 2y_3 + y_4)$$

or, $M_2 + 4M_3 + 0 = \frac{6}{1} \cdot (1 - 2.0 + 1)$

or, $M_2 + 4M_3 = 12$

Solving these 3 equations using elimination method

Total equations are 3

$4M_1 + M_2 + 0M_3 = 12$ ..... (i)

$M_1 + 4M_2 + M_3 = -12$ ..... (ii)

$0M_1 + M_2 + 4M_3 = 12$ ..... (iii)

Select the equations (i) and (ii), and eliminate the variable M_1

$4M_1 + M_2 = 12 \times 1 \rightarrow 4M_1 + M_2 = 12$

$M_1 + 4M_2 + M_3 = -12 \times 4 \rightarrow 4M_1 + 16M_2 + 4M_3 = -48$

$-15M_2 - 4M_3 = 60$ ..... (iv)

Select the equations (iii) and (iv) and eliminate the variable M_2.

$M_2 + 4M_3 = 12 \times 15 \rightarrow 15M_2 + 60M_3 = 180$

$-15M_2 - 4M_3 = 60 \rightarrow -15M_2 - 4M_3 = 60$

$56M_3 = 240$ ..... (v)

Now use back substitution method from (v)

$56M_3 = 240$

or, $M_3 = \frac{240}{56} = 4.2857$

From equation (iii)

$M_2 + 4M_3 = 12$

or, $M_2 + 4(4.2857) = 12$

or, $M_2 + 17.1429 = 12$

or, $M_2 = 12 - 17.1429 = -5.1429$

From equation (ii)

$M_1 + 4M_2 + M_3 = -12$

or, $M_1 + 4(-5.1429) + (4.2857) = -12$

or, $M_1 - 16.2857 = -12$

or, $M_1 = -12 + 16.2857 = 4.2857$

Solution using elimination method

$M_1 = 4.2857$

$M_2 = -5.1429$

$M_3 = 4.28\,57$

Substitute i = 1 in equation (i). We get cubic spline in 1st interval $[x_0, x_1] = [1, 2]$

$$f_1(x) = \frac{(x_1 - x)^3}{6h} M_0 + \frac{(x - x_0)^3}{6h} M_1 + \frac{(x_1 - x)}{h}\left(y_0 - \frac{h^2}{6} M_0\right) + \frac{(x - x_0)}{h}\left(y_1 - \frac{h^2}{6} M_1\right)$$

$$f_2(x) = \frac{(2 - x)^3}{6} 0 + \frac{(x - 1)^3}{6} 4.2857 + \frac{(2 - x)}{1}\left(1 - \frac{1}{6} \cdot 0\right) + \frac{(x - 1)}{1}\left(0 - \frac{1}{6} \cdot 4.2857\right)$$

$f_1(x) = 0.7143x^3 - 2.1428x^2 + 0.4286x + 2$, for $1 \cdot \le x \le 2$

Substitute i = 2 in equation (1). We get cubic spline in 2nd interval $[x_1, x_2] = [2, 3]$

$$f_2(x) = \frac{(x_2 - x)^3}{6h} M_1 + \frac{(x - x_1)^3}{6h} M_2 + \frac{(x_2 - x)}{h}\left(y_1 - \frac{h^2}{6} M_1\right) + \frac{(x - x_1)}{h}\left(y_2 - \frac{h^2}{6} M_2\right)$$

$$f_2(x) = \frac{(3 - x)^3}{6} \cdot 4.2857 + \frac{(x - 2)^3}{6} \cdot -5.1429 + \frac{(3 - x)}{1}\left(0 - \frac{1}{6} \cdot 4.2857\right) + \frac{(x - 2)}{1}\left(1 - \frac{1}{6} \cdot -5.1429\right)$$

$f_2(x) = -1.574x^3 + 11.5714x^2 - 27x + 20.2857$, for $2 \le x \le 3$

Substitute i = 3 in equation (1), we get cubic spline in 3rd interval $[x_2 - x_3] = [3, 4]$.

$$f_3(x) = \frac{(x_3 - x)^3}{6h} M_2 + \frac{(x - x_2)^3}{6h} M_3 + \frac{(x_3 - x)}{h}\left(y_2 - \frac{h^2}{6} M_2\right) + \frac{(x - x_2)}{h}\left(y_3 - \frac{h^2}{6} M_3\right)$$

$$f_3(x) = \frac{(4 - x)^3}{6} \cdot -5.1429 + \frac{(x - 3)^3}{6} \cdot 4.2857 + \frac{(4 - x)}{1}\left(1 - \frac{1}{6} \cdot -5.1429\right) + \frac{(x - 3)}{1}\left(0 - \frac{1}{6} \cdot 4.2857\right)$$

$f_3(x) = 1.5714x^3 - 16.7144x^2 + 57.8574x - 64.5718$ for $3 \le x \le 4$

Substitute i ≐ 4 in equation (1), we get cubic spline in 4th interval $[x_3, x_4] = [4, 5]$

$$f_4(x) = \frac{(x_4 - x)^3}{6h} M_3 + \frac{(x - x_3)^3}{6h} M_4 + \frac{(x_4 - x)}{h}\left(y_3 - \frac{h^2}{6} M_3\right) + \frac{(x - x_3)}{h}\left(y_4 - \frac{h^2}{6} M_4\right)$$

$$f_4(x) = \frac{(5-x)^3}{6}\cdot 4.2857 + \frac{(x-4)^3}{6}\cdot 0 + \frac{(5-x)}{1}\left(0 - \frac{1}{6}\cdot 4.2857\right) + \frac{(x-4)}{1}\left(1 - \frac{1}{6}\cdot 0\right)$$

$$f_4(x) = -0.7143x^3 + 10.7142x^2 + 10.71142x^2 - 51.857x + 81.714$$

for $4 \le x \le 5$

3. a) Evaluate $\int_4^{5.2}$ logx dx from the following data

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
y	1.3863	1.4351	1.48 16	1.5261	1.5686	1.6094	1.6487

By using

i) Trapezoidal method

ii) Simpson $\frac{1}{3}$ method

iii) Simpson $\frac{3}{8}$ method

Solution:

$$\int_4^{5.2} \log x \cdot dx$$

X	4.5	4.2	4.4	4.6	4.8	5.0	5.2
y	1.3863	1.4351	1.4816	1.5261	1.56 86	1.6094	1.6487
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Using trapezoidal method

$$\int_4^{5.2} .\log x \cdot dx = \frac{h}{2}[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4)]$$

$$= \frac{0.2}{2}[1.3863 + 1.6487 + 2(1.4351 + 1.4816 + 1.5261 + 1.5686 + 1.6094)]$$

$$= \frac{0.2}{2}[18.2766]$$

$$= 1.82766$$

Using Simpson $\frac{1}{3}$ rule

$$\int_4^{5.2} \log x \cdot dx = \frac{h}{3}[y_0 + y_6 + 2\,9y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{0.2}{3}[1.3863 + 1.6487 + 2(1.4816 + 1.5686) + 4(1.4531 + 1.5261 + 1.698]$$

$$= \frac{0.2}{3}[27.4]$$

$$= 1.8278533$$

Using Simpson $\frac{3}{8}$ rule

$$\int_4^{5.2} \log(x)\,dx = \frac{3h}{8}[y_0 + y_6 + 2y_3 + 3(y_1 + y_2 + y_4 + y_5]$$

$$= \frac{x \times 0.2}{8}[1.3863 + 1.6487 + 2 \times 1.5261 + 3(1.4351 + 1.4816 + 1.5686 + 1.6094)]$$

$$= \frac{3 \times 0.2}{8}[24.3713]$$

$$= 1.8278475$$

b) Evaluate $\int_0^2 \frac{x^2 + 2x + 1}{1 + (x+1)^4}$ dx by using Gaussian integration formula for n = 2, n = 3 and compare their values with exact solution.

Solution:

$$\int_0^2 \frac{x^2 + 2x + 1}{1 + (x+1)^4} Dx$$

$\eta = 2, \eta = 3$ Gaussian integration

We have to change the limit from

(0, 2) to (–1, 1)

Here a = 0, b =?

So,

$$x = \left(\frac{b-a}{2}\right)z + \left(\frac{b+a}{2}\right)$$

$$x^2 + 2x + 1 = (x+1)^2$$

$$x = \left(\frac{2-0}{2}\right)z + \left(\frac{2+0}{2}\right)$$

$$\therefore \quad x = z + 1$$

$$dx = dz$$

$$\int_0^2 \frac{x^2 + 2x + 1}{1 + (x+1)^4}\ \int_{-1}^1 \frac{(z+1+1)^2}{1 + (z+1+1)^4} Dz$$

$$= \int_{-1}^1 \frac{(z+2)^2}{1 + (z+2)^4} Dz$$

For n = 2

$$\int_{-1}^1 F(z)\cdot dz = \omega_1 F(z_1) + \omega_2 F(z_2)$$

$$\omega_1 = \omega_2 = 1$$

$$z_1 = \frac{-1}{\sqrt{3}}\ z_2 = \frac{1}{\sqrt{3}}$$

$$F(z) = \frac{(z+2)^2}{1 + (z+2)^4}$$

$$\int_{-1}^{1} F(z)\cdot dz = F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\left(2-\frac{1}{\sqrt{3}}\right)^2}{1+\left(2\frac{-1}{\sqrt{3}}\right)^4} + \frac{\left(2+\frac{1}{\sqrt{3}}\right)^2}{1+\left(\frac{2+1}{\sqrt{3}}\right)^4}$$

$$= 0.54434185098$$

For $\eta = 3$

$$\int_{-1}^{1} F(z)\cdot dz = \omega_1 F(z_1) + \omega_2 F(z_2) + \omega_3 F(z_3)$$

$$\omega_1 = \omega_3 = \frac{5}{9} \qquad z_1 = -\sqrt{\frac{3}{5}}$$

$$\omega_2 = \frac{8}{9} \qquad z_3 = \sqrt{\frac{3}{5}}\ z_2 = 0$$

$$\int_{-1}^{1} F(z)\cdot dz = \frac{8}{9}F(0) + \frac{5}{9}\left(F\left(-\sqrt{\frac{3}{5}}\right) + F\left(\sqrt{\frac{3}{5}}\right)\right)$$

$$= \frac{8}{9}\left(\frac{22}{1+24}\right) + \frac{5}{9}\left(\frac{\left(2-\sqrt{\frac{3}{5}}\right)^2}{1+\left(2-\sqrt{\frac{3}{5}}\right)^4} + \frac{\left(2+\sqrt{\frac{3}{5}}\right)^2}{1+\left(2+\sqrt{\frac{3}{5}}\right)^4}\right)$$

$$= 0.53642219$$

4. a) Solve the following system of equations by using relaxation method correct to two decimal places.

$9x - y + 2z = 9$

$x + 10y - 2z = 15$

$2x - 2y - 13z = -17$

Solution:

$9x - y + 2z = 9$

$x + 10y - 2z = 15$

$2x - 2y - 13z = -17$

The residuals are given by

$R_x = 9 - 9x + y - 2z$

$R_y = 15 - x - 10y + 2z$

$R_z = -17 - 2x + 2y + 13z$

Operation table

Residuals	δR_x	δR_y	δR_z
Increments			
δx	-9	-1	-2
δy	1	-10	2
δz	-2	2	13

Relaxation table

N	Residuals Increments	Rx	Ry	Rz	x	y	z
1.	$x = y = z = 0$	9	15	-17	0	0	0
2.	$\delta z = -\frac{17}{13} = 1.30769$	6.38462	17.61538	-0.00003	0	0	1.30760
3.	$\delta y = -\frac{17.61538}{-10} = 1761538$	8.146158	0	3.5230460	0	1.761538	1.30769
4.	$\delta x = -\frac{8.146158}{-9} = 0.9051286$	0.0000006	-0.90512886	1.712788	0.901286	1.761538	1.30769
5.	$\delta z = -\frac{1.712788}{13} = -0.1317529$	0.2635064	-1.16863	0.000011	0.901286	1.761538	1.1759371
6.	$\delta y = -\frac{1.16863}{-10} = -0.116863$	0.1466434	-0.0000044	-0.2337249	0.901286	1.644675	1.1759371
7.	$\delta z = -\frac{0.2337249}{13} = 0.0179788$	0.1106858	0.03595	-0.0000005	0.901286	1.644675	1.1939159
8.	$\delta x = \frac{0.1106858}{-9} = 0.012298$	-0.0008062	0.0235652	-0.0247765	0.9175166	1.644675	1.1939159
9.	$\delta z = \frac{-0.0247765}{13} = 0.00190588$	-0.0046164	0.0273754	-0.0000102	0.9775166	1.644675	1.195821

$x = 0.9175160$

$y = 1.644675$

$z = 1.195821$

b) Using Dollittle LU decomposition method, solve the following system equations.

$3x + 2y + z = 10$

$2x + 3y + 2z = 14$

$x + 2y + 3z = 14$

Solution:

Coloe the following problem using LU decomposition method (either do little or rout)

$3x + 2y + z = 10,\ 2x + 3y + 2z = 14,\ x + 2y + 3z = 14$

The given system of linear equation be uniform in matrix formal

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 14 \end{pmatrix}$$

$AX = B$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Let, $LU = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

or, $$\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{11}L_{21} & U_{12}L_{21} + U_{22} & U_{13}L_{21} + U_{23} \\ U_{11}L_{31} & U_{12}L_{21} + U_{22}L_{32} & U_{13}L_{21} + U_{33} \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

Equating corresponding element B

$u_{11} = 73$ $\quad U_{12} = 2$ $\quad U_{13} = 1$ $\quad U_{22} = \frac{5}{3}$

$L_{21} = \frac{2}{3}$ $\quad L_{31} = \frac{1}{3}$ $\quad U_{23} = \frac{4}{3}$ $\quad L_{32} = \frac{4}{5}$

Now,

$L_2 = B$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 14 \end{pmatrix}$$

$z_1 = 10$

$\frac{2}{3}z_1 + z_2 = 14$

$\frac{1}{3}z_1 + \frac{4}{5}z_2 + z_3 = 14$

$\therefore$ $z_1 = 10 \; z_2 = \frac{22}{3} \; z_3 = \frac{24}{5}$

Again,

$UX = z$

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 22/3 \\ 24/5 \end{pmatrix}$$

$\therefore$ $3x + 2y + 2 = 10$

$\frac{5}{3}y + \frac{4}{3}z = \frac{22}{3}$

$\frac{8}{5}z = \frac{2y}{5}$

$\therefore$ $z = 3$

$y = 2$

$\therefore$ $x = 1$

5. a) Use runge-kutta of order four to find the solution of the given differential equation at x = 1.5 taking a step size of h = 0.25.

$\frac{dy}{ax} + 2y = x^2, y(1) = 5$

Solution:

$\frac{dy}{dx} + 2y = x^2$ $\quad y(1) = 5$

$F(x, y) = x^2 - 2y$ $\quad h = 0.25$

$\therefore$ $y(1.5) = ?$

Using initial condition

$y(1) = 5$

$y(x_0) = y_0$

$\therefore$ $x_0 = 1, y_0 = 5$

1st iteration

$M_1 = F(x_0, y_0)$

$= F(1, 5)$

$= 1 - 10,$

$= -9$

$M_2 = \left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \times M_1\right)$

$= F\left(1 + \frac{0.25}{2}, 5 + \frac{0.25}{2} \times (-9)\right)$

$= F(1.125, 3.875)$

$= 1.125^2 - 2 \times 3.875$

$= -6.484375$

$M_3 = F\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \times m_2\right)$

$= F\left(1.125, \frac{5 + 0.25}{2} \times (-6.484375)\right)$

$= F(1.125, 4.189453)$

$= -7.113281$

$M_4 = F(x_0 + h, y_0 + h \times m_3)$

$= F(1 + 0.25 \times (-7.113281)$

$= F(1.25, 3.22167975)$

$= 1.25^2 - 3.22167975 \times 2$

$= -4.8808595$

$\therefore$ $M = \frac{M_1 + 2(M_2 + M_3) + M_4}{6} = -6.846028$

$y(1.25) = y_0 + h \times m = 5 + 0.25 \times (-6.846028) = 3.288493$

$y(1.25) = 3.288493$

$x_1 = 1.25$

$y_1 = 3.288493$

2nd Iteration

$M_1 = F(x_1, y_1)$

$= F(1.25, 3.288493)$

$= 1.25^2 - 2 \times 3.288493$

$= -5.014486$

$M_2 = F\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} \times m_1\right)$

$= F\left(1.25 + \frac{0.25}{2}, 3.288493 + \frac{0.25}{2} \times (-5.14486)\right)$

$= F(1.375, 2.60168225)$

$= 1.375^2 - 2 \times 2.66168225$

$= -3.43273925$

$M_3 = F\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} \times M_2\right)$

$= F\left(1.375, 3.288493 + \frac{0.25}{2} \times (-3.43273925)\right)$

$= F(1.375, 2.85940059375)$

$= (1.375^2 - 2 \times 2.85940059375) - 3.82817586$

$M_4 = F(x_1 + h, y_1 + h \times M_3)$

$= F(1.25 + 0.25, 3.288493 + 0.25 \times -3.82817586)$

$= F(1.5, 2.331449035) = -2.4128977$

$\therefore \quad M = \frac{M_1 + 2(M_2 + M_3) + M_4}{6} = -3.65820232$

$\therefore \quad y(1.5) = y_1 + h \times m$

$= 3.288493 + 0.25 \times (-3.65820232)$

$= 2.37394242$

Summary

N	m_1	m_2	m_3	m_4	m	y
1.	-9	-6.484375	-7.113281	-4.8808595	-6.846028	3.288493
2.	-5.014486	-3.43273925	-3.82817586	-2.4128977	-3.65820232	-2.37394242

b) Find the solution of the given ordinary differential equation at x = 0.5 using the step size of h = 0.25 using Heun's method.

$$\frac{dy}{dx} + 0.4y = 3e^{-x}, \; y(1) = 5$$

Solution:

$\frac{dy}{dx} + 0.4y = 3e^{-x}$

$y(0) = 5$

$y(0.5) = 8$

$h = 0.25$

Using initial condition

$y(0) = 5$

$x_0 = 5$

N	x_i	y_i	m_1	m_2	$y_{i+1} = y_i + h \times m$
1	0.	5	1	0.2364023	5.154550
2	0.25	5.154550	0.2745823	-0.26968	5.1551620
3	0.5	5.1551620	-0.2424728	-0.6207178	5.04726316

6. a) Determine the steady-state heart distribution in a thin square metal plate with dimensions 0.5 by 0.5 m using n = m = 4. Two adjacent boundaries are held at 0°C, and the heat on the other boundaries increases linearly from °C at one corner to 100°C where the sides meet.

Solution:

$$\frac{\partial^2 U}{\partial x^2}(x, y) + \frac{\partial^2 U}{\partial y^2}(x, y) = 0$$

We know,

$$U_{i,j} = \frac{1}{4}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1})$$

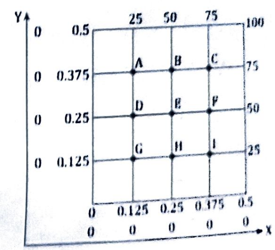
From question we can concludes that

$U(0, y) = 0$

$U(x, 0) = 0$

$U(x, 0.5) = 200x$

$U(0.5, y) = 2\,wy$



$A = \frac{1}{4}(24 + B + D)$

$B = \frac{1}{4}(A + 50 + C + F)$

$C = \frac{1}{4}(B + 75 + 75 + F)$

$D = \frac{1}{4}(A + E + G)$

$E = \frac{1}{4}(D + B + F + H)$

$F = \frac{1}{4}(E + C + 50 + I)$

$G = \frac{1}{4}(D + H)$

$H = \frac{1}{4}(G + E + I)$

$I = \frac{1}{4}(H + F + 25)$

Solve using gauss

$A = 18.75$

$B = 37.5$

$E = 25$

$F = 37.5$

C = 56.25 G = 6.25
D = 12.5 H = 12.5
I = 18.75

b) **The following table gives the corresponding values of pressure and specific value of superheated steam.**

v	2	4	6	8
p	105	42.07	25.3	16.7

i) **Find the rate of change of pressure with respect to volume when v = 2.**

ii) **Find the rate of change of volume with respect to pressure when p = 105.**

Solution:

v	2	4	6	8
p	105	42.07	25.3	16.7

$\frac{dp}{dv} = ?\ \frac{dv}{dp} = ?$

Difference table

v	p	Δp	$\Delta^2 p$	$\Delta^3 p$
2	105			
4	42.07	− 62.93		
6	25.3	− 16.77	46.16	
8	16.7	-8.6	8. 17	-37 .99

Here,

$h = 2\ (v_1 - v_0)$

$$p = \frac{v - v_0}{h} = \frac{2-2}{2} = 0$$

Using Newton forward interpolation formula

At p = 105

$$\frac{dp}{dv} = \frac{1}{h}\left[\Delta p_0 + \frac{1}{2}(2p-1)\Delta^2 p_0 + \frac{1}{6}(3p^2 - 6p + 2)\Delta^3 p_0\right]$$

$$= \frac{1}{2}\left[\Delta p_0 - \frac{1}{2}\Delta^2 p_0 + \frac{1}{3}\Delta^3 p_0\right]$$

$$= \frac{1}{2}\left[-62.93 - \frac{1}{2}(64.16) + \frac{1}{3}(-37.99)\right]$$

$= -49.3367$ MPa kg/m³

Also,

$$\left(\frac{dv}{dp}\right)_{p=1.5} = \frac{1}{\frac{dp}{dv}} = \frac{1}{-49.3367} = -0.020268 \text{ m}^3/\text{MPa kg}$$

7. **Write short notes on: (Any two)**
 a) **Ill-conditioned systems**
 b) **Laplacian equation**
 c) **Classification of second order partial differential equation**

Answer: See the theory part

POKHARA UNIVERSITY

Semester: Fall

Level: Bachelor Year: 2022
Programme: BE Full Marks: 100
Course Numerical Methods Pass Marks: 45
Time: 3 hrs.

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

Attempt all the questions.

1\. a) **Find the root of the equation $f(x) = x^2 - 4x - 10$ correct to three decimal places by using false position method.**

Solution:

$F(x) = x^2 - 4x - 10$

Using false position method

Let $a = -1$
$b = -2$
$F(a) = -5(-ve)$
$F(b) = 2\ (+ve)$

n	a	b	F(a)	F(b)	x	$F(\lambda_n)$
1	−1	−2	−5	2	−1.714285	-0.204086
2	−1.714285	−2	−0.204086	2	−1.74074071	-0.00685
3	−1.74074071	−2	−0.006858	2	−1.7416267	-0.000228
4	−1.7416267	−2	−0.0002289	2	−1.741656	-0.00000766
5	−1.741656	−2	−0.00000766	2	−1.7416573	-0.000000346331
					1.7416573	

which is correct upto 3 decimal places

b) **Estimate the root of the equation $f(x) = xe^x - \cos x$ using Newton Raphson method correct to three decimal places.** [8]

Solution:

Here,

$F(x) = xe^x - \cos x$

$$F(x) = x \cdot \frac{de^x}{dx} + e^x \cdot \frac{dx}{dx} - \frac{d\cos}{dx}$$

$= x \cdot e^x + e^x + \sin x$

Let $x_0 = 0.1$

N	x_i	$F(\lambda_i)$	$F'(x_i)$	$X_{i1} = x_i - \frac{F(x_i)}{F'(x_i)}$
1.	0.1	− 0.884487	1.315521	0.772347
2.	0.772347	0.955734	4.534668	0.561585
3.	0.561585	0.1382988	3.270688	0.519300

4.	0.519300	0.004698	3.049988	0.517759
5.	0.517759	0.0000049	3.042132	0.517757
6.	0.517757	−0.0000011	3.042121	0.517757

∴ 0.517757

2. a) From the following table estimate the number of student who obtained marks between 40 and 45. **[7]**

Marks	30 – 40	40 – 50	50 – 60	60 – 70
No. of students	31	42	51	35

Solution:

Finding cumulative frequency for table as below.

Marks less then x	40	50	60	70
Cumulative frequency	31	73	124	159

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_1 = 40$	31			
		42		
50	73		9	
		51		−25
60	124		−16	
		35		
70	159			

For x = 45 y =?

Using Newton's forward interpolation method

$$h = 10 \quad p = \frac{45-40}{10} = 0.5$$

$$y_{45} = y_0 + p\,\Delta y_0 + \frac{P(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$= 31 + 0.5 \times 42 + \frac{0.5\,(0.5-1)}{2!}\,9 + \frac{0.5\,(0.5-1)\,(0.5-2)}{3!} \times (-25)$$

$$= 49.3125 \approx 49$$

$$y_{45} = 49$$

So, No. of students with marks less than 45 = 49

And no. of student less than 40 = 31

∴ Students with marks between

40 – 45 – 49 – 31 = 18

b) Form the following data given in the table below evaluate f(2.5) by using lagrange method. **[8]**

x	1	2	4	5	7
f(x)	1	1.414	1.732	2.00	2.6

Solution:

Use Lagrange method to find f(2.5) from following data.

x	1	2	4	5	7
f(x)	1	1.414	1.732	2.00	26

Here,

$x_0 = 1$	$y_0 = 1 = f(4)$
$x_1 = 2$	$y_1 = 1.414$
$x_2 = 4$	$y_2 = 1.732$
$x_3 = 5$	$y_3 = 2.00$
$x_4 = 7$	$y_2 = 2.6$

By Lagrange's formula: we have:

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \times y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_2)(x_3-x_4)} \times y_3 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_1-x_2)(x_4-x_3)} \times y_4$$

$$y = F(x) = \frac{(x-2)(x-4)(x-5)(x-7)}{(1-2)(1-4)(1-5)(1-7)} \times 1 + \frac{(x-1)(x-4)(x-5)(x-7)}{(2-1)(2-4)(2-5)(2-7)} \times 1.474 + \frac{(x-1)(x-2)(x-5)(x-7)}{(4-1)(4-2)(4-5)(4-7)} \times 1.732 + \frac{(x-1)(x-2)(x-4)(x-7)}{(5-1)(5-2)(5-4)(5-7)} \times 2 + \frac{(x-1)(x-2)(x-4)(x-5)}{(7-1)(7-2)(7-4)(7-5)} \times 26$$

$$y = f(2.5) = \frac{(2.5-2)(2.5-5)(2.5-7)}{(1-2)(1-4)(1-5)(1-7)} \times 1 + \frac{(2.5-1)(2.5-4)(2.5-5)(2.5-7)}{(2-1)(2-4)(2-5)(2-7)} \times 1.414 + \frac{(2.5-1)(2.5-2)(2.5-5)(2.5-7)}{(4-1)(4-2)(4-5)(4-7)} \times 2 + \frac{(2.5-1)(2.5-4)(2.5-5)}{(7-1)(7-2)(7-4)(7-5)} \times 2.6$$

$$y = f(2.5) = -0.117188 + 1193063 + 0.811875 - 0.421875 + 0.040625$$

$$y = f(2.5) = 1.5065$$

3. a) Evaluate $\int_0^6 \frac{1}{x}\,dx$ by using Gaussian integration formula for n = 3 and compare the value with exact solution.

$\int_1^5 \frac{1}{x} dx$, **using Gaussion Integration**

Solution:

1st of all we need to change the limit from (1, 5) to (−1, 1)

Here,

$$a = 1, b = 5$$

$$x = \left(\frac{b-a}{2}\right) z + \left(\frac{b+a}{2}\right)$$

$$x = \left(\frac{5-1}{2}\right) z + \left(\frac{5+1}{2}\right)$$

$$x = 2z + 3$$

$$\therefore \quad dx = 2dz$$

$$\therefore \quad \int_1^5 \frac{1}{x} dx = \int_1^1 \frac{1}{2z+3} x.dz$$

$$F(z) = \frac{2}{2z+3}$$

For n = 3

$$\int_{-1}^{1} F(z) \cdot dz = \omega_1 F(z_1) + \omega_2 F(z_1) + \omega_3 F(z_3)$$

$$z_1 = -\sqrt{\frac{3}{5}} \qquad \omega_1 = \omega_3 = \frac{5}{9}$$

$$z_2 = 0 \qquad \omega_2 = \frac{8}{9}$$

$$z_3 = \sqrt{\frac{3}{5}}$$

$$\int_{-1}^{1} F(z) \cdot dz = \frac{8}{9} F(0) + \frac{5}{9}\left(F\left(-\sqrt{\frac{3}{5}}\right) + F\left(\sqrt{\frac{3}{5}}\right)\right)$$

$$= \frac{8}{9} \times \frac{2}{3} + \frac{5}{9}\left(\frac{2}{2 \times \left(-\sqrt{\frac{5}{5}^3}\right)} + \frac{2}{2\left(\sqrt{\frac{3}{5}}\right) + 3}\right)$$

$$= 1.6026936$$

Exact and from calculation = 1.60943791

Approx = 1.6026936

∴ erm = 0.0060744 }1

b) Use the Romberg integration to find the solution correct up to three decimal places.

$$I = \int_0^1 \frac{1}{1+x^2} Dx$$

Use Romberg's method to compute $\int_0^1 \frac{dx}{1+x^2}$

Correct to 4 decimal places

Solution:

We take h = 0.5, 0.25 and 0.125 successively and evaluate the given integral *using* Trapezoidal rule.

When h = 0.5, the values of $y = (1+x^2)^{-1}$ are

x	0	0.5	1.0
y	1	0.8	0.5

$$I = \frac{0.5}{2}[1 + 2 \times 0.8 + 0.5] = 0.775$$

When h = 0.25 the value of $y = (1+x^2)^{-1}$ are

x	0	0.25	0.5	0.75	1.0
y	1	0.9412	0.8	0.64	0.5

$$I = \frac{0.25}{2}[(1 + 0.5) + 2(0.9412 + 0.8 + 0.64)] = 0.7828$$

When h = 0.125, we find that I = 0.7848

Thus,

$$I(h) = 0.7750$$

$$I\left(\frac{h}{2}\right) = 0.7828$$

$$I\left(\frac{h}{4}\right) = 0.7848$$

Now,

$$I\left(h\frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right]$$

$$= \frac{1}{3}[3.1312 - 0.775]$$

$$= 0.7854$$

$$I\left(\frac{h}{2}\frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right]$$

$$= \frac{1}{3}[3.1312 - 0.775]$$

$$= 0.7854$$

$$I\left(\frac{h}{2}\frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right]$$

$$= \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I\frac{h}{2}\right]$$

$$= \frac{1}{3}[3.1392 - 0.7828]$$

$$= 0.7855$$

$$I\left(h\frac{h}{2}\frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\frac{h}{4}\right)\right]$$

$$= \frac{1}{3}[3.142 - 0.9.7854]$$

$$= 0.7855$$

4. a) Find the solution of the given simultaneous linear equation using Gauss seidel method.

$$6x_1 - 2x_2 + x_3 = 11$$
$$-2x_1 + 7x_2 + 2x_3 = 5$$
$$x_1 + 2x_2 - 5x_3 = -1$$

Solution:

$$6x_1 - 2x_2 + x_3 = 11 \quad \text{.....(i)}$$
$$-2x_1 + 7x_2 + 2x_3 = 3 \quad \text{.....(ii)}$$
$$x_1 + 2x_2 - 5x_3 = 1 \quad \text{.....(iii)}$$

Given equation are already diagonally dominals

From equation (i)

$$x_1 = \frac{11 + 2x_2 - x_3}{6}$$

From equation (ii)

$$x_2 = \frac{5 + 2x_1 - 2x_3}{7}$$

From equation (iii)

$$-x_3 = \frac{-1 - x_1 - 2x_2}{5}$$

$$\therefore \frac{1 + x_1 + 2xz}{5}$$

Now using gauss seid-1

N	x_1	x_2	x_3
0	0	0	0
1	1.833333	1.238095	1.061904
2	2.069047	1.002040	1.014625
3	1.998242	0.995319	0.997776
4	1.998810	1.000295	0.999880
5	2.000118	1.000068	1.000050
6	2.000014	0.999989	0.999998
7	1.99996	0.999999	0.999999
8	1.999999	1.000000	1.000000
	≈ 2	≈ 1	≈ 1
	$= x_1 = 2$	$x_2 = x_3 = 1$	

b) Solve the following system of equations using Crout method.
$x + y + z = 4$, $x + 4y + 3z = 8$, $x + 6y + 2z = 6$

Solution:

$$x + y + z = 4$$
$$x + 4y + 3z = 8$$
$$x + 6y + 2z = 6$$

We known,

$$AX = B$$

$$\therefore A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 3 \\ 1 & 6 & 2 \end{bmatrix} x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix}$$

Let $A - LU$

Where,

$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} U = \begin{bmatrix} g & h & i \\ 0 & J & K \\ 0 & 0 & L \end{bmatrix}$$

Using rout

$$U = \begin{bmatrix} 1 & h & i \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

Now,

$$A = LU$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 3 \\ 1 & 6 & 2 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & h & i \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & ah & ai \\ b & bh + c & bi + ck \\ d & dh + e & di + ek + f \end{bmatrix}$$

Now

$a = 1, b = 1,$
$ah = 1$
$\therefore h = 1$
$bh + c = 4$
$1 + c = 4$
$= c = 3$

$dh + e = 6$

$1 + e = 6$

$d = 1$
$a_I = 1$
$i = 1$
$b_I + ck = 3$
$1 + 3k = 3$
$3k = 2$
$k = \frac{2}{3}$
$di + ek + f = 2$
$1 + 5 \times \frac{2}{3} + f = 2$
$1 + \frac{10}{3} + f = 2$
$f = \frac{-7}{3}$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 5 & -7/3 \end{bmatrix}$$

$$0 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Ax = B$$
$$LUX = B$$

Let $UX = z$

$\therefore$ $LZ = B$

Where, $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 5 & -7/3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix}$$

Using formed substitution:

$z_1 = 4$

$z_1 + 3z_2 = 8$

$4 + 3z_2 = 8$

$z_2 = \frac{4}{3}$

$z_1 + 5z_2 \frac{-7}{3} z_3 = 8$

$4 + 5 \times \frac{4}{3} - \frac{7}{3} z_3 = 8$

$\frac{-7}{3} z_3 = -\frac{14}{3} = z_3 = 2$

Now,

$UX = Z$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 4/3 \\ 2 \end{bmatrix}$$

Using backward substitution

$z = 2$

$y + \frac{2}{3} z = \frac{4}{3}$

$y + \frac{2}{3} \times 2 = \frac{4}{3}$

$\therefore$ $y = 0$

$x + y + z = 4$

$x + 0 + 2 = 4$

$= x = 2$

$\therefore$ $x = 2$

$y = 0$

$z = 2$

5. a) Using the Euler's (R–K 1st order method) find an approximate value of y corresponding to x = 1, given that $\frac{dy}{dx} = x + y$ and y = 1. When x = 0, h = 0.1.

Solution:

$\frac{dy}{dx} = x + y$

$y = 1$ when $x = 0$

i.e., $y(0) = 1$

$h = 0.1$

$F(x, y) = x + y$ $\quad y(1) = ?$

Using initial condition $y(0) = ?$ $X_0 = 0, y_0 = 1$

N	x_i	y_i	$y_{i+1} = y_i + h \times F(x_i, y_i)$
1	0	1	1.1
2	0.1	1.1	1.22
3	0.2	1.22	1.362
4	0.3	1.362	1.5282
5	0.4	1.5282	1.72102
6	0.5	1.72102	1943122
7	0.6	1.943122	2.1974342
8	0.7	2.1974342	2.48717762
9	0.8	2.48717762	2.81 5895382
10	0.9	2.815895382	3.1874849202
11	1	3.1874 849202	3.6062334122

$\therefore$ $y(1) = 3.606233412222$

b) Apply Euler's method to approximate value of y(0.3) for the differential equation.

$$\frac{dy}{dx} = y + x, \; y(0) = 1$$

Solution:

$\frac{dy}{dx} = y + x$

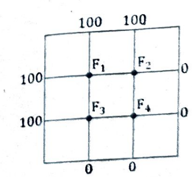
$y(0.3) = ?$

Similar to 5(a) do up to x = 0.3

$y(0.3) = 1.5282$

6. a) Torsion on a square bar of size 15 cm × 15 cm. If two of the sides are held at 100° C and the other two sides are held at 0°C. Calculate the steady state temperature at interior points. Assume a grid size of 5 cm × 5 cm.

Solution:



Let F_1, F_2, F_3 and F_4 are the 4 interior points whose study state temp is to determine.

We know 5 point formal

$$F_{i-1,j} + F_{i+1,j} + F_{i,j-1} + F_{i,j+1} - 4F_{i,j} = F(x, y)$$

$$\therefore \quad F_1 = \frac{1}{4}(100 + 100 + F_2 + F_3) = \frac{1}{4}(200 + F_2 + F_3)$$

$$F_2 = \frac{1}{4}(F_1 + 100 + F_4)$$

$$F_3 = \frac{1}{4}(1\omega + F_1 + F_4)$$

$$F_4 = \frac{1}{4}(F_2 + F_3)$$

Here,

$F_2 = F_3$

Using Gauss seidal method

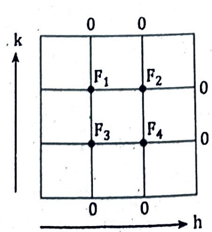
N	F_1	F_2	F_3
0	0	0	0
1	50	37.5	18.75
2	68.75	46.875	23.4375
3	73.4375	49.21875	24,609375
4	74.609375	49.8046875	24.9023437
5	74.90234375	49.95 1171875	24.975585 93
6	74.9755859375	49 .98779296875	24.9938964843
7	74.99389648438	49.99694824219	24.99847412
8	74.998 47412109	49.99923706055	24.99961853
9	74.9996 1853027	49.99980926514	24.9999046325
10	74.99990463257	49.99995231628	24.999976158

$\therefore \quad F_1 \approx 75$

$F_2 = F_3 \approx 50$

$F_4 \approx 25$

b) **Solve the Poisson equation $\nabla^2 f = 2x^2 + y$, over the square domain $1 \le x \le 4$, $1 \le y \le 4$, with f = 0 on the boundary. Take step size in x and y, h = k = 1.**



Solution:

Given,

$$\Delta^2 F = 2x^2 + y$$

or $\quad F_{xx} + F_{yy} = 2x^2 + 1$

At point F_1 (x = 1, y = 2)

$$\therefore \quad F_1 = \frac{1}{4}(F_2 + F_3 - 4)$$

At point F_2 (x = 2, y = 2)

$$\therefore \quad F_2 = \frac{1}{4}(F_1 + F_4 - 10)$$

At point F_3 (x = 1, y = 1)

$$F_3 = \frac{1}{4}(F_1 + F_4 - 3)$$

At point F_4 (x = 2, y = 1)

$$F_4 = \frac{1}{4}(F_2 + F_3 - 9)$$

Now, solving these equations by gauss siedac method

Number of incretion	F_1	F_2	F_3	F_4
1	0	0	0	0
2	-1	-2.75	-1	-3.187
3	-1.937	-3.781	-2.031	-3.703
4	-2.453	-4.039	-2.289	-3.832
5.	-2.582	-4.103	-2.353	-3.864
6	-2.614	-4.119	-2.369	-3.872
7	-2.622	-4.123	-2.373	-3.874
8	-2.624	-4.124	-2.374	-3.874
8	-2.624	-4.124	-2.374	-3.874

Hence,

$U_1 = -2.624$

$U_2 = -4.124$

$U_3 = -2.374$

$U_4 = -3.874$

7. **Write short notes on: (Any two)** **[2x5]**

a) **Ill-conditioned and well-conditioned systems**

b) **Error in numerical method**

c) **Cubic Spline**

Answer: See the theory part

POKHARA UNIVERSITY

Level: Bachelor | Semester: Spring | Year: 2023
Programme: BE | Full Marks: 100
Course Numerical Methods | Pass Marks: 45
Time: 3 hrs.

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

Attempt all the questions.

1. a) Explain in brief the errors in numerical calculations.
Solution: See the theory part

b) Find a root of $3x + \sin x - e^x = 0$ using
i) One of the bracketing methods and
ii) One of the non-bracketing methods
$\log(x) = \cos x = 0$

Solution:
Here,

$$3x + \sin x - e^x = 0$$

i) Using bracketing methods
Bisection method
Let $a = 0$
$b = 1$
$f(a) = -1$ (−ve)
$f(b) = 1.1231$ (+ve) [Root line between 0 and 1]

N	a	b	x_n	$f(x_n)$
1	0	1	0.5	0.330704
2	0	0.5	0.25	−0.286621
3	0.25	0.5	0.375	0.036281
4	0.25	0.25	0.3125	−0.12189
5	0.3125	0.375	0.34375	−0.04195
6	0.34375	0.375	0.359375	−0.00261
7	0.359375	0.375	0.3671875	0.01688
8	0.359375	0.3671875	0.36328125	0.007146
9	0.359375	0.36328125	0.361328125	0.002266
10	0.359375	0.361328125	0.360351	−0.00017
11	0.360351	0.361328	0.360839	0.001045
12	0.360351	0.360839	0.360595	0.000433
13	0.360351	0.360839	0.360595	0.00433

∴ 0.360595

ii) Using non-bracketing methods

Newton raphson

$F(x) = 3x + \sin x - e^x$

$F'(x) = 3 + 0 \cos x - e^x$

N.	x_i	F(x)	F'(x)	$x_{i+1} = x_i - \frac{F(x_i)}{F'(x_i)}$
1	0	−1	3	0.333333
2	0.333333	−0.0684177	2.5493445	0.3601701
3	0.3601701	−0.0006279	2.5022625	0.360421168
4	0.36042168	−0.000000056625	2.50181.424	0.36042170

∴ 0.360 42170

2. a) From the data given below, find the number of students whose weight is between 60 to 70.

Weight in lbs	0 – 40	40 – 60	60 – 80	80 – 100	100 – 120
No. of students	250	120	100	70	50

Solution:

Weight in lbs	0 – 40	40 – 60	60 – 80	80 – 100	100 – 120
No. of std	250	120	100	70	50

(60 – 70) = ?

We have to find cumulative free

Below x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
Below 60	370		−20		
		100		−10	
Below 80	470		−30		20
		70		10	
Below 100	540		−20		
		50			
Below 120	590				

Let $x_p = 70$ $\quad h = 20$

$x_0 = 60$ $\quad p = \frac{x_p - x_0}{h} = \frac{70 - 60}{20} = \frac{10}{20} = 0.5$

Using Newton forward interpolation.

$$y_p = y_0 + p\Delta y_0 + p(p-1)\,\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

$$= 370 + 0.5 \times 100 + \frac{0.5\,(0.5-1)}{2!}(-30) + \frac{0.5\,(0.5-1)\,(0.5-2)}{6} \times \ldots\ldots$$

$424.375 \approx 424$

No. of std. with weight less than 70 = 424

No. of std. with weight less than 60

∴ No. of std. between 60 – 70 = 424 – 370 = 54

b) **Using the method of least squares, fit the curves $ax^2 + \frac{b}{x}$ to the following data.**

x	1	2	3	4
y	–1.52	0.96	8.88	7.66

x	2	4	7	9
F(x)	1	2	1	2

Solution:

Let the n points are given by $(x_1 - y_1) \cdot (x_2, y_2), (x_3, y_3)........ (x_n, y_n)$. The error of estimate for the i^{th} point (x_i, y_i) is $e_i = \left[y_i - ax_i^2 - \left(\frac{b}{x_i}\right)\right]$.

By the principle of least square the values of a and b are such so that the sum of the square of error S. viz.

$$S = \sum_{i=1}^{n} e_i^2 = \sum\left(y_i - ax_i^2 - \frac{b}{x}\right)^2 \text{ is minimum}$$

Therefore the normal equations are given by

$$\frac{\partial s}{\partial a} = 0 \quad \frac{\partial s}{\partial b} = 0$$

or $$\sum_{i=1}^{n} y_i x_i^2 = a\sum_{i=1}^{n} x_i^4 + b\sum_{i=1}^{n} x_i \text{ and } \sum_{i=1}^{n} \frac{y_i}{x_i} = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} \frac{1}{x_i^2}$$

These are the required least square equations

X	y	x^2	x^4	$\frac{1}{x}$	$\frac{1}{x^2}$	yx^2	$\frac{y}{x}$
1	–1.51	1	1	1	1	– 1.51	– 1.51
2	0.99	4	16	0.5	0.25	3.96	0.495
3	3.88	9	81	0.3333	0.1111	34.92	1.2933
4	7.66	16	256	0.25	0.0625	122.56	1. 0943
10			354		1.4236	159.93	1.1933

Putting the values in the above least square equations we get

$$159.93 = 354a + 10b \text{ and } 1.1933 = 10a + 1.4236b$$

Solving these we get a = 0.509 and b = –2.04

Therefore, the equation of the curve fitted to the above data is $y = 0.509x^2 - \frac{2.04}{x}$

3. a) **Use Romberg's method, to compute $\int_0^2 \frac{e^x + \sin x}{1 + x^2} dx$ correct up to two decimal places. Integrate using**

I. Trapezoidal II. Simpson's $\frac{1}{3}$ rule III. Simpson $\frac{3}{8}$ rule for n = 6

Solution:

Here,

$$h = \frac{b-a}{n}$$

Taking n = 2

$$\therefore \quad h = \frac{b-a}{n} = \frac{2-0}{2} = 1$$

Taking h = 1, 0.5, 0.25

Here,

$$f(x) = y = \frac{e^x + \sin x}{1 + x^2}$$

i) When h = 1

x	0	1	2
y	1	1. 777988	1.659671

$$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{1}{2}[1 + 1.659671 + 2 \times 1.77988] = 3.10972$$

ii) When h = 0.5

x_i	0	0.5	1	1.5	2
y_i	1	1.70252	1.77988	1.68590	1.659671

$$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.5}{2}[1 + 1.659671 + 2(1.70252 + 1.77988 + 1.68590)]$$

$$\therefore \quad I_2 = 3.24906$$

iii) When h = 0.25

x_i	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
y_i	1	1.44134	1.70252	1.79112	1.77988	1.7342	1.68590	1.6587	1.659671

$$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.25}{2}[1 + 1.659671 + 2(1.44134 + 1.70252 + 1.79112$$

$$+ 1.77\,988 + 1.73242 + 1.68590 + 1.6587)]$$

$$\therefore \quad I_3 = 3.28043$$

Now,

$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 3.24906 + \frac{3.24906 - 3.10972}{3} = 3.2955$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 3.28043 + \frac{3.28043 - 3.28043 - 3.24906}{3} = 3.29088$$

$$I = I_5 + \frac{1}{3}(I_5 - I_4) = 3.29088 + \frac{3.29088 - 3.2955}{3} = 3.2893$$

$$\therefore \int_0 \frac{e^x + \sin x}{1 + x^2}\,dx = 3.28934$$

b) Estimate approximate derivative of $f(x) = x^2$ at $x = 1$ for $h = 0, 0.2, 0.05, 0.01$. Use first order difference method and find the respective error.

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin x}}$$

Solution:

A forward difference formula is,

$$F'(x) = \frac{f(x + h) - f(x)}{h}$$

$$F'(x) = \frac{f(1 + 0.2) - f(1)}{0.2} = \frac{1.2^2 - 1^2}{0.2} = 2.2$$

$$F'(x) = x', f'(x) = 2x, f''(x) = 2$$

h	F'(x)	Error $= \frac{h}{2} F''(x) = \frac{h}{2} x^2$
0.2	2.2	0.2
0.1	2.1	0.1
0.05	2.05	0.05
0.01	2.01	0.01

4. a) Apply the factorization method to solve the equation $3x + 2y + 7z = 4$; $2x + 3y + z = 5$; $3x + 4y + z = 7$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & -5 \\ -2 & -4 & -4 \end{bmatrix}$$

Solution:

$$3x + 2y + 7z = 4$$
$$2x + 3y + z = 5$$
$$3x + 4y + z = 7$$

We know,

$$AX = B$$

$$A = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

Let A = LU (using Do – little)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \quad 0 = \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} d & e & f \\ ad & ae + g & aF + h \\ bd & be + cg & bF + ch + i \end{bmatrix}$$

$\therefore$ $d = 3, e = 2, F = 7$ $\qquad ae + g = 3$

$\therefore \quad ad = 2$

$\therefore \quad a = \frac{2}{3}$

$af + h = 1$

$\frac{2}{3} \times 7 + h = 1$

$\therefore \quad h = -\frac{11}{3}$

$be + cg = 4$

$1.2 + c \cdot \frac{5}{3} = 4$

$2 + \frac{5c}{3} = 4$

$\frac{5c}{3} = 2$

$\therefore \quad c = \frac{6}{5}$

$\frac{2}{3} \times 2 + g = 3.$

$g = \frac{5}{3}$

$bd = 3$

$b = 1$

$bF + ch + i = 1$

$1 \times 7 + \frac{6}{5} \times \left(\frac{-11}{3}\right) + i = 1$

$i = 1 - 7 + \frac{22}{5}$

$i = -\frac{8}{5}$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} U = \begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}$$

$AX = B$

$LUX = B$

Let $\quad UX = Z$

$\therefore \quad LZ = B$

Where, $Z = \begin{bmatrix} x_1 \\ y_2 \\ z_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

$\therefore \quad z_1 = 4$

$\frac{2}{3} z_1 + z_2 = 5$

$\therefore \quad z_2 = 5 - \frac{8}{3} = \frac{7}{3}$

$z_1 + \frac{6}{5} z_2 + z_3 = 7$

$4 + \frac{6}{5} \times \frac{7}{3} + z_3 = 7$

$z_3 = 7 - \frac{34}{5} = \frac{1}{5}$

Now,

$UX = Z$

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

$$\frac{-8}{5}z = \frac{1}{5}$$

$$= z = -\frac{1}{8}$$

$$= -0.125$$

$$\frac{5}{3}y - \frac{11}{3}z = \frac{7}{3}$$

$$\frac{5}{3}y - \frac{11}{3}\left(\frac{-1}{8}\right) = \frac{7}{3}$$

$$\frac{5}{3}y = \frac{7}{3} - \frac{11}{24}$$

$$\frac{5y}{3} = \frac{15}{8}$$

$$\therefore \quad y = \frac{9}{8} = 1.125$$

$$3x + 2y + 7z = 4$$

$$3x + 2 \times \frac{9}{8} + 7 \times \left(\frac{-1}{8}\right) = 4$$

$$3x + \frac{9}{4} - \frac{7}{8} = 4$$

$$\therefore \quad x = \frac{7}{8}$$

$$\therefore \quad x = \frac{7}{8} = 0.875$$

$$y = \frac{9}{8}$$

$$z = \frac{-1}{8}$$

b) Using SOR method, solve the following system of $4x + y + 2z = 4$: $3x + 5y + z = 7$; $x + y + 3z = 3$

$$\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

Solution:

$$4x + y + 2z = 4$$
$$3x + 5y + z = 7$$
$$x + y + 3z = 3$$

The residuals are given by

$$R_x = 4 - 4x - y - 2z$$
$$R_y = 7 - 3x - 5y - z$$
$$R_z = 3 - x - y - 3z$$

Operation table

Residuals increment	δR_x	δR_y	δR_z
δ_x	-4	-3	-1
δ_y	-1	-5	-1
δ_z	-2	-1	-3

Relaxation table:

N	Residuals increments
	$\delta x = \delta y = \delta z = 0$
1	$\delta y = -\frac{7}{-5} = 1.4$
2	$\delta x = \frac{2.6}{-4} = 0.65$
3	$\delta y = \frac{1.95}{-5} = -0.39$
4	$\delta z = -\frac{1.31}{-3} = 0.4467$
5	$\delta x = \frac{-05.34}{-4} = -0.1258$
6	$\delta z = \frac{-0.1257}{-3} = 0.0419$
7	$\delta y = -\frac{-0.1112}{-5} = -0.2224$
8	$z = \frac{-0.05176}{4} = -0.01294$
9	$\delta y = \frac{-0.08882}{-5} = 0.017764$
10	$\delta z = \frac{-0.027416}{-3} = 0.009138$
11	

$\therefore$ $x = 0.51126$

$y = 0.99332$

$z = 0.4977$

5. a) Find the lar the matrix.

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix}$$

Relaxation table:

N	Residuals Increments	R_x	R_y	R_z	x	y	z
1	$\delta x = \delta y = \delta z = 0$	4	7	3	0	0	0
2	$\delta y = -\frac{7}{-5} = 1.4$	2.6	0	1.6	0	1.4	0
3	$\delta x = \frac{2.6}{-4} = 0.65$	0	−1.95	0.95	0.65	1.4	0
4	$\delta y = \frac{1.95}{-5} = -0.39$	0.39	0	1.34	0.65	1.01	0
5	$\delta z = -\frac{1.31}{-3} = 0.4467$	− 0.5034	− 04467	− 0.0001	0.65	1.01	0.4467
6	$\delta x = \frac{-05.34}{-4} = -0.1258$	− 0.0002	− 0.0693	0.1257	0.5242	1.01	0.446
7	$\delta z = \frac{-0.1257}{-3} = 0.0419$	− 0.084	− 0.112	0	0.3242	1.01	0.4886
8	$\delta y = -\frac{-0.1112}{-5} = -0.2224$	− 0.05176	0.05	0.03224	0.5242	0.97776	0.4886
9	$z = \frac{-0.05176}{4} = -0.01294$	0	0.08882	0.04518	0.51126	0.97776	0.4886
10	$\delta y = \frac{-0.08882}{-5} = 0.017764$	−0.017746	0	0.027416	0.51926	0.995524	0.4886
11	$\delta z = \frac{-0.027416}{-3} = 0.009138$	− 0.033964	− 0.0099	0.000116	0.51126	0.995524	0.4977

$\therefore$
$$x = 0.51126 \approx 0.5$$
$$y = 0.993324 \approx 1$$
$$z = 0.4977 \approx 0.5$$

5. a) Find the largest Eigen value and the corresponding Eigen vector of the matrix. **[8]**

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$ **using power method**

Solution:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Relaxation table:

N	Residuals Increments	R_x	R_y	R_z	x	y	z
1	$\delta x = \delta y = \delta z = 0$	4	7	3	0	0	0
2	$\delta y = -\frac{7}{-5} = 1.4$	2.6	0	1.6	0	1.4	0
3	$\delta x = \frac{2.6}{-4} = 0.65$	0	−1.95	0.95	0.65	1.4	0
4	$\delta y = \frac{1.95}{-5} = -0.39$	0.39	0	1.34	0.65	1.01	0
5	$\delta z = -\frac{1.31}{-3} = 0.4467$	− 0.5034	− 04467	− 0.0001	0.65	1.01	0.4467
6	$\delta x = \frac{-05.34}{-4} = -0.1258$	− 0.0002	− 0.0693	0.1257	0.5242	1.01	0.446
7	$\delta z = \frac{-0.1257}{-3} = 0.0419$	− 0.084	− 0.112	0	0.3242	1.01	0.4886
8	$\delta y = -\frac{-0.1112}{-5} = -0.2224$	− 0.05176	0.05	0.03224	0.5242	0.97776	0.4886
9	$z = \frac{-0.05176}{4} = -0.01294$	0	0.08882	0.04518	0.51126	0.97776	0.4886
10	$\delta y = \frac{-0.08882}{-5} = 0.017764$	−0.017746	0	0.027416	0.51926	0.995524	0.4886
11	$\delta z = \frac{-0.027416}{-3} = 0.009138$	− 0.033964	− 0.0099	0.000116	0.51126	0.995524	0.4977

$\therefore$ $x = 0.51126 \approx 0.5$

$y = 0.993324 \approx 1$

$z = 0.4977 \approx 0.5$

5. a) Find the largest Eigen value and the corresponding Eigen vector of the matrix. **[8]**

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$ **using power method**

Solution:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Let the largest Eigen value be λ and corresponding Eigen vector $x^\circ = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

1st iteration

$$AX^\circ = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} -0.5 \\ 1 \\ -0.5 \end{bmatrix}$$

$$= \lambda^1 X^1$$

2nd iteration

$$AX^1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -0.5 \\ 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$$

$$= 3 \begin{bmatrix} -0.6667 \\ 1 \\ -0.6667 \end{bmatrix}$$

$$= \lambda^2 X^2$$

3rd iteration

$$AX^2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -0.6667 \\ 1 \\ 0.6667 \end{bmatrix} = \begin{bmatrix} -2.3334 \\ 3.3334 \\ -2.3334 \end{bmatrix}$$

$$= 3.3334 \begin{bmatrix} -0.700 \\ 1 \\ \ldots\ldots \end{bmatrix}$$

$$= \lambda^2 X^3$$

4th iteration

$$AX^3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -0.7 \\ 1 \\ -0.7 \end{bmatrix}$$

$$= 3.4 \begin{bmatrix} -0.7058 \\ 1 \\ -0.7058 \end{bmatrix}$$

$$= \lambda^4 X^4$$

5th iteration

$$AX^4 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -0.7058 \\ 1 \\ 0.7058 \end{bmatrix}$$

$$= 3.4117 \begin{bmatrix} -0.7069 \\ 1 \\ -0.7069 \end{bmatrix}$$

$$= \lambda^5 X^5$$

6^{th} iteration

$$AX^5 = 3.4138\begin{bmatrix}-0.7070\\1\\-0.7070\end{bmatrix} = \lambda^6 X^6$$

7^{th} iteration

$$AX^6 = 3.4142\begin{bmatrix}-0.7071\\1\\-0.7071\end{bmatrix}$$

Answer 3.4142

$$\begin{bmatrix}-0.7071\\1\\-0.7071\end{bmatrix}$$

b) **Using the R–K 1st order method, find an approximate value of y corresponding to x = 1, given that $\frac{dy}{dx} = \frac{y-x}{y+x}$ and y = 1. When x = 0, and h = 0.02.**

Solution:

$$\frac{dy}{dx} = \frac{y-x}{y+x} y = 1 \text{ where } x = 0$$

$h = 0.02$

$y = ?$ When $x = 1$

Let use $h = 0.2$ $y(1) = ?$

Using initial condition

$y(0) = 1$

$\therefore$ $x_0 = 0, y_0 = 1$

We know RK 1^{st} order method,

$y_{i+1} = y_i + hf(x_i, y_i)$

n	x_i	y_i	$y_{i+1} = y_i + h \times (x_i, y_i)$
1	0	1	1.2
2	0.2	1.2	1.342857
3	0.4	1.342857	1.451053
4	0.6	1.451053	1.534039
5	0.8	1.534039	1.596937
6	1	1.596937	1.642909

$\therefore$ $y(1) = 1 - 642902$

6. a) **Using the R–K method of fourth order, solve for y at x = 1.2, 1.4, from $\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}$ given $x_0 = 1$, $y_0 = 0$.**

Solution:

$$\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x} \quad x_0 = 1, y_0 = 0$$

$y = 3$ at $x = 1.2$ and 1.4

Using initial condition

$x_0 = 1, y_0 = 0$

Let $h = 0.2$

1st iteration

$$M_1 = F(x_0, y_0) = F(1, 0) = 0.731058$$

$$M_2 = F\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} M_1\right)$$

$$= F\left(1 + \frac{0.2}{2}, 0 + \frac{0.2}{2} \times M_1\right)$$

$$= F(1.1, 0.0731058)$$

$$= 0.701061$$

$$M_3 = F\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \times M\right)$$

$$= F\left(1.1, 0 + \frac{0.2}{2} \times 0.701061\right)$$

$$= F(1.1, 0.0701061)$$

$$= 0.699599$$

$$M_4 = F(x_0 + h, y_0 + h \times M_3)$$

$$= (1 + 0.2, 0 + 0.2 \times 0.699599)$$

$$= F(1.2, 0.1399198)$$

$$= 0.67400993257$$

$$\therefore \quad M = \frac{m_1 + 2(m_2 + m_3) + m_4}{6} = 0.70106453$$

$$\therefore \quad y_1 = y_0 + h \times m$$

$$= 0 + 0.2 \times 0.70106465$$

$$= 0.1402129$$

$$\therefore \quad y_1 = 0.1402129$$

$$x_1 = 1.2$$

2nd iteration

$$m_1 = F(x_1, y_1)\ 0.1402129$$

$$= F(1.2, 0.674139)$$

$$m_2 = F\left(1.2 + \frac{0.2}{2} 0.70106465 + \frac{0.2}{2} \times 0.650730\right)$$

$$= 0.6515589$$

$$m_3 = F\left(x_1 + \frac{h}{2} y_1 + \frac{h}{2} \times m_2\right)$$

$$= 0.650650$$

$$m_4 = F(x_1 + h, y_1 + h \times M_3)$$

$$= F(1.2 + 0.2, 0.70106465 + 0.2 \times 0.650650)$$

$$= 0.630088$$

$\therefore \quad m = \dfrac{m_1 + z(m_2 + m_3) + m_4}{6} = 0.65144\ 16$

$\therefore \quad y_2 = y_1 + h \times m = 0.270$

$\therefore \quad y(1.2) = 0.0873058$

$y(1.4) = 0.270511$

Summer

n	m_1	m_2	m_3	m_4	x	y
1	0.731058	0.701061	0.699599	0.6740099	0.701064	0.1402299
2	0.674139	0.65155 89	0.65050	0.630088	0.6514410	0.270501195

b) **Solve the elliptic equation $U_{xx} + U_{yy} = 0$ over a square mesh of side four units satisfying the following boundary conditions; $u(0, y) = 0$ for $0 \le y \le 4$, $u(4, y) = 12 + y$ for $0 \le y \le 4$; $u(x, 0) = 3x$ for $0 \le x \le 4$, $u(x, 4) = x^2$ for $0 \le 4$**

Solution:

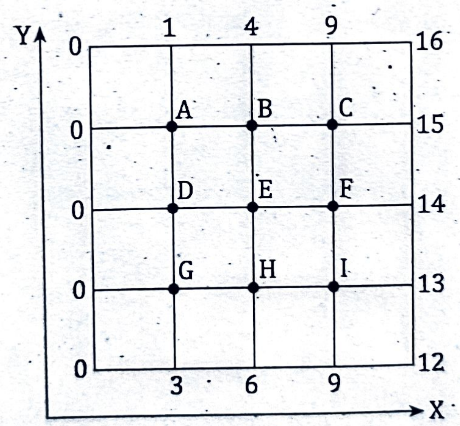
$U_{xx} + U_{yy} = 0$

$4(0, y) = 0$

$4(4, y) = 12 + y$

$4(x, 0) = 3x$

$4(x, y) = x^2$



With the help of boundary condition fig can be drawn

$A = \frac{1}{4}(1 + B + D)$ $\qquad B = \frac{1}{4}(A + 4 + c + E)$

$C = \frac{1}{4}(B + 9 + 15 + F)$ $\qquad D = \frac{1}{4}(A + E + h)$

$E = \frac{1}{4}(D + B + F + H)$ $\qquad F = \frac{1}{4}(E + C + 14 + I)$

$G = \frac{1}{4}(D + H + 3)$ $\qquad H = \frac{1}{4}(G + E + I + 6)$

$I = \frac{1}{4}(H + F + 13 + 9)$

N	A	B	C	D	E	F	G	H	I
0	0	0	0	0	0	0	0	0	0
1	0.25	1.0625	6.265625	0.0625	0.28125	5.13671875	0.765625	1.76171875	7.22469375
2	0.53125	2.76953125	7.9765625	0.39453125	2.515625	7.929199	1.2890625	4.257312	8.54663
3	1.041015	3.883300	8.953125	1.21425	4.3203125	8.955017	2.1171875	5.2460327	9.05026
4	1.52368	4.69927	9.41357	1.990295	5.22265	9.42162	2.55908	5.70800	9.2824
5	1.92239	5.13965	9.64031	2.426033	5.67328	9.64913	2.78350	5.93493	9.3960
6	2.14142	5.36389	9.75325	2.64968	5.89941	9.76217	2.899615	6.04789	9.4525
7	2.25339	5.47651	9.80967	2.76224	6.01220	9.81859	2.95243	6.104310	9.4807
8	2.30968	5.53289	9.83787	2.81860	6.06860	9.84680	2.98073	6.1325	9.4945
9	2.3378	5.5610	9.8519	2.8468	6.0968	9.8609	2.9948	6.1466	9.5018
10	2.3519	5.5751	9.8590	2.8609	6.1109	9.8679	3.0018	6.1536	9.5054

7. Write short notes on: (Any two)

a) Shooting method

b) Algorithm of gauss Jordan method

c) Algorithm of fixed point iteration method

Answer: See the theory part