Chapter

3

NUMERICAL DIFFERENTIATION AND INTEGRATION

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3.1 NUMERICAL DIFFERENTIATION

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of values (x_i, y_i). To compute, $\frac{dy}{dx}$ we first replace the exact relation $y = f(x)$ by the best interpolating polynomial $y = \phi(x)$ and then differentiate the latter as many times as we desire. The choice of the interpolation formula to be used depend, will depend on the assigned value of x at which $\frac{dy}{dx}$ is desired.

If the values of x are equispaced and $\frac{dy}{dx}$ is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula. For values near the middle of the table, $\frac{dy}{dx}$ is calculated by means of Stirling's or Bessel's formula. If the values of x are not equispaced, we use Lagrange's formula or Newton's divided difference formula to represent the function.

Hence corresponding to each the interpolation formula, we can derive a formula for finding the derivative.

3.2 FORMULA FOR DERIVATIVES

Consider the function $y = f(x)$ which is tabulated for the values $x_i (= x_0 + ih)$, $i = 0, 1, 2, \ldots\ldots, n$.

A. Derivatives using Newton's Forward Difference Formula

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

Differentiating both sides with respect to p, we have,

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \ldots\ldots$$

Since, $p = \dfrac{(x-x_0)}{h}$

Hence, $\dfrac{dp}{dx} = \dfrac{1}{h}$

Now,

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \ldots\ldots\right] \quad \ldots\ldots (1)$$

At $x = x_0$, $p = 0$. Hence Putting $p = 0$

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\ldots\right] \quad \ldots\ldots (2)$$

Again differentiating (1) with respect to x, we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dp}\right)\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p^2-36p+22}{4!}\Delta^4 y_0 + \ldots\ldots\right]\frac{1}{h}$$

Putting $p = 0$, we obtain,

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \ldots\ldots\right] \quad \ldots\ldots (3)$$

Similarly,

$$\left(\frac{d^3y}{dx^3}\right) = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\ldots\right]$$

Otherwise;

We know that, $1 + \Delta = E = e^{hD}$

$\therefore \quad hD = \log(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\ldots$

or, $$D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \dots\dots\dots\right]$$

and, $$D^2 = \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \dots\dots\dots\right]^2$$

$$= \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 + \dots\dots\dots\right]$$

and, $$D^2 = \frac{1}{h^2}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \dots\dots\dots\right]$$

Now, applying the above identities to y_0, we get,

Dy_0 *i.e.*,

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\Delta y_0 - \frac{1}{2}\left[\Delta^2 y_0 \frac{1}{3}\Delta^2 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \dots\dots\dots\right] \quad \dots\dots (4)$$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \dots\dots\dots\right]$$

and, $$\left(\frac{d^3y}{dx^3}\right) = \frac{1}{h^3}\left[\Delta^2 y_0 - \frac{3}{2}\Delta^4 y_0 + \dots\dots\dots\right]$$

which are same as (2), (3) and (4) respectively.

B. Derivatives using Newton's Backward Difference Formula

Newton's backward interpolation is,

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots\dots\dots$$

Differentiating both sides with respect to p, we get,

$$\frac{dy}{dp} = \nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \dots\dots\dots$$

Since, $p = \dfrac{x - x_n}{h}$,

Hence, $\dfrac{dp}{dx} = \dfrac{1}{h}$

Now,

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_0 + \dots\dots\dots\right] \quad \dots\dots (5)$$

At $x = x_n$, $p = 0$

Hence, putting $p = 0$, we get,

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \dots\dots\dots\right] \dots (6)$$

Differentiating equation (5), with respect to x, we have,

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx^3}$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \ldots\ldots\right]$$

Putting p = 0, we get,

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\Delta^4 y_n + \frac{5}{6}\Delta^5 y_n + \frac{137}{180}\Delta^6 y_n + \ldots\ldots\right] \quad \text{---- (7)}$$

Similarly,

$$\left(\frac{d^3y}{dx^3}\right) = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\Delta^4 y_n + \ldots\ldots\right] \quad \text{---- (8)}$$

Otherwise:

We know,

$$1 - \nabla = E^{-1} = e^{-hD}$$

$$\therefore \quad hD = \log(1-\nabla) = -\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{3}\nabla^4 + \ldots\ldots\right]$$

or, $$D = \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \ldots\ldots\right]$$

$$\therefore \quad D^2 = \frac{1}{h^2}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{2}\nabla^3 + \ldots\ldots\right]^2$$

$$= \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \ldots\ldots\right]$$

Similarly,

$$D^2 = \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + \ldots\ldots\right]$$

Applying these identities to y_n, we get,

Dy_n *i.e.*,

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{2}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \ldots\ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\ldots\right]$$

and, $$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^2 y_n + \frac{3}{2}\Delta^4 y_n + \ldots\ldots\right]$$

which are same as (6), (7) and (8).

C. Derivatives using Stirling's Central Difference formula

Stirling's formula is,

$$y_p = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + \ldots\ldots$$

Differentiating both sides with respect to p, we get,

$$\frac{dy}{dp} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{2p}{2!}\Delta^2 y_{-1} + \frac{3p^2 - 1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{4p^3 - 2p}{4!}\Delta^3 y_{-2} + \dots\dots\dots$$

Since, $p = \dfrac{x - x_0}{h}$

$$\therefore \quad \frac{dp}{dx} = \frac{1}{h}$$

Now,

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$$

$$= \frac{1}{h}\left[\left(\frac{y_0 + \Delta y_{-1}}{26}\right) + p\Delta^2{}_{y-1} + \frac{3p^2 - 1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)\right.$$

$$\left. + \frac{2p^3 - p}{12}\Delta^4 y_{-2} + \dots\dots\dots\right]$$

At x = 0, p = 0. Hence putting p = 0, we get,

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \dots\dots\dots\right] \quad \dots\dots (9)$$

Similarly,

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + \dots\dots\dots\right] \quad \dots\dots (10)$$

Example 3.1

The following data gives the velocity of a particle for twenty seconds at an interval of five seconds. Find the initial acceleration using the entire data.

Time, t(sec)	0	5	10	15	20
Velocity, v(m/s)	3	14	69	228	?

Solution:

The difference table is,

n = 5

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
0	0				
3					
5	3		8		
11		36		36	
10	14		44		24
55		60			
15	69		104		
159					
20	228				

An initial acceleration *i.e.,* $\left(\frac{dv}{dt}\right)$ at t = 0 is required, we use Newton's forward formula.

$$\left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 v_0 + \dots\dots\dots\right]$$

$$\therefore \quad \left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{5}\left[3 - \frac{1}{2}(8) + \frac{1}{3} \times 36 - \frac{1}{4} \times 24\right]$$

$$= \frac{1}{5}(3 - 4 + 12 - 6)$$

$$= 1$$

Hence the initial acceleration is 1 m/sec².

3.3 NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated values of the integrand f(x) is called numerical integration. This process when applied to a function of a single variable, is known as quadrature.

The problem of numerical integration, like that of numerical differentiation, is solved by representing f(x) by an interpolation formulation and then integrating it between the given limits. In this way, we can derive quadrature formula for approximate integration of a function defined by a set of numerical values

3.4 NEWTON-COTES QUADRATURE FORMULA

Let, $I = \int_a^b f(x)\,dx$

where, f(x) takes the values $y_0, y_1, y_2, \dots\dots\dots, y_n$ for $x = x_0, x_1, x_2, \dots\dots\dots, x_n$.

Let us divide the interval (a, b) into n sub-intervals of width h so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \dots\dots\dots, x_n = x_0 + nh = b$. Then,

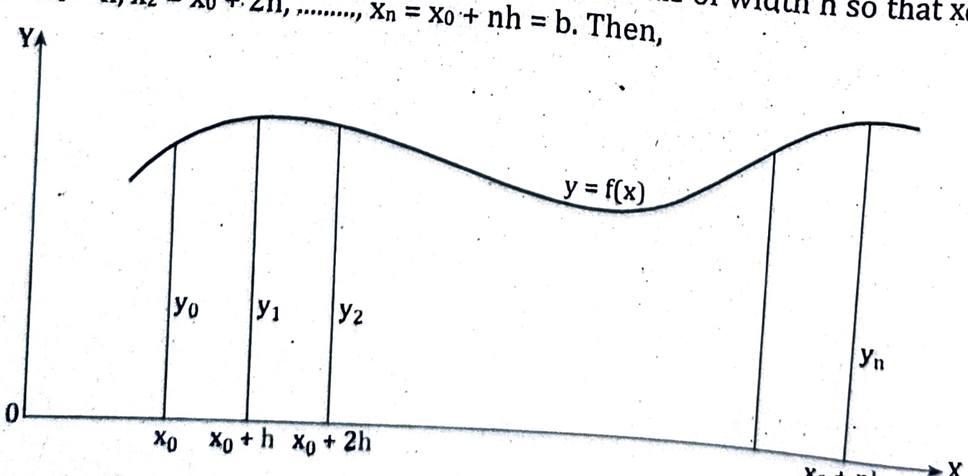

Figure 3.1

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= h\int_0^n f(x_0 + rh)\, dr,$$

Putting $x = x_0 + rh$, $dx = hdr$

$$= h\int_0^n \Big[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0$$

$$+ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 y_0\Big]$$

$$+ \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 y_0 + \dots\dots\Big] dr$$

[By Newton's interpolation formula]

Integrating term by term, we get,

$$\int_{x_0}^{x_0+nh} f(x)\, dx = nh\Big[y_0 + \frac{n}{2} y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0$$

$$+ \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!}$$

$$+ \left(\frac{n^4}{6} - 2n^4 + \frac{34n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{5!}$$

$$+ \left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{225n^3}{4} + \frac{274n^2}{3} - 60n\right)\frac{\Delta^6 y_0}{6!} + \dots\dots\Big] \quad \dots\dots (1)$$

This is known as Newton's cotes quadrature formula. From this general formula, we deduce the following important quadrature rules by taking $n = 1, 2, 3, \dots\dots$

I. Trapezoidalo Rule

Putting $n = 1$ in equation (1) and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line *i.e.*, a polynomial of first order so that differences of order higher than first becomes zero, we get,

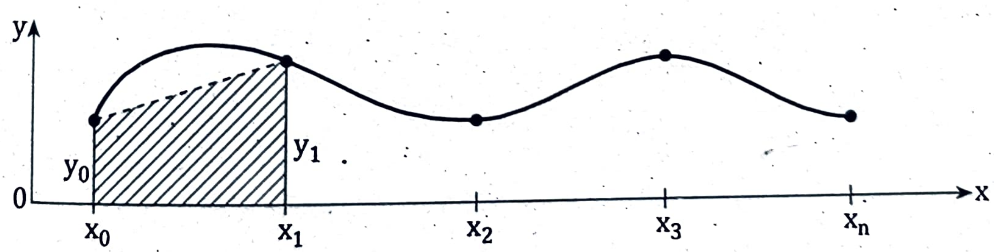

Figure 3.2

Here; $$\int_{x_0}^{x_0+h} f(x)\, dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}(y_0 + y_1)$$

Similarly,

$$\int_{x_0+h}^{x_0+h} f(x)\, dx = h\left(y_1 + \frac{1}{2}\Delta y_1\right) = \frac{h}{2}(y_1 + y_2)$$

....

$$\int_{x_0+(n-1)}^{x_0+nh} f(x)\, dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding these n integrals, we get,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \dots\dots + y_{n-1})] \quad \dots (2)$$

This is known as the trapezoidal rule.

> **NOTE:**
> The area of each strips (trapezium) is found separately. Then the area under the curve and the ordinates at x_0 and x_n is approximately equal to the sum of the areas of the n trapeziums.

II. Simpson's One-third Rule

Putting n = 2 in equation (1) above and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola *i.e.*, a polynomial of the second order so that difference of order higher than the second vanish, we get,

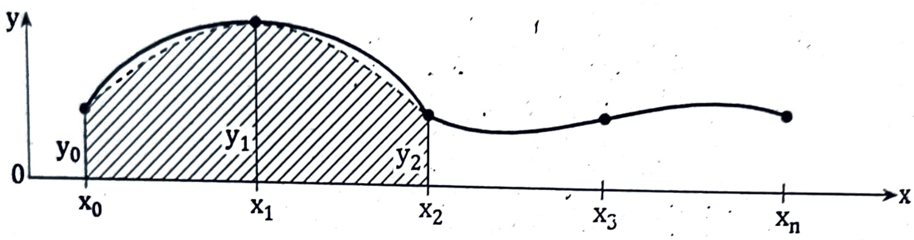

Figure 3.3

Here,

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

....

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n), \text{ n being even.}$$

Adding all these integrals, we have when n is even,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \dots\dots + y_{n-1})$$
$$+ 2(y_2 + y_4 + \dots\dots + y_{n-2})] \quad \dots (3)$$

This is known as the Simpson's one-third rule or simply Simpson's rule and is most commonly used.

> **NOTE:**
> While applying (3), the given interval must be divided into an even number of equal subintervals, since we find the area of two strips at a time.

III. Simpson's Three-eight Rule

Putting n = 3 in (1) above and taking the curve through (x_i, y_i); i = 0, 1, 2, 3 as a polynomial of the third order so that differences above the third order vanish, we get,

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

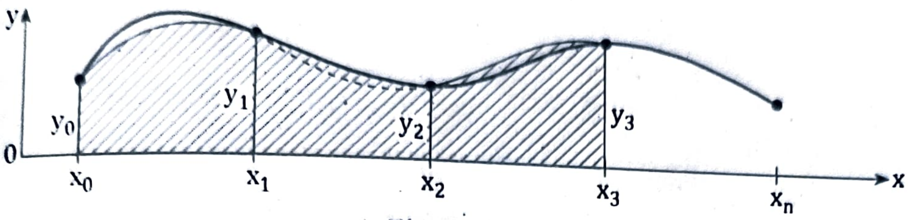

Figure 3.4

Similarly,

$$\int_{x_0+3h}^{x_0+5h} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all such expressions from x_0 to $x_0 + nh$, where n is a multiple of 3, we get,

$$\int_{x_0}^{x_0+h} f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots\dots\dots + y_{n-1})$$

$$+ 2(y_3 + y_0 + \dots\dots\dots + y_{n-3})] \qquad \dots\dots (4)$$

> **NOTE:**
> While applying equation (4), the number of sub-intervals should be taken as a multiple of 3.

Example 3.2

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

i) Trapezoidal rule

ii) Simpson's $\frac{1}{3}$ rule

iii) Simpson's $\frac{3}{8}$ rule

Solution:

Divide the interval (0, 6) into six parts, each of width h = 1. The values of $f(x) = \frac{1}{1+x^2}$ are given below;

x	0	1	2	3	4	5	6
f(x)	1	0.5	0.2	0.1	0.0588	0.0385	0.027
= y	y_0	y_1	y_2	y_3	y_4	y_5	y_6

i) By trapezoidal

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}[(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)]$$

$$= 1.4108$$

ii) By Simpson's $\frac{1}{3}$ rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(1 + 0.27) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)]$$

$$= 1.3662$$

iii) By Simpson's $\frac{3}{8}$ rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}[(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2 \times 0.1]$$

$$= 1.3571$$

Example 3.3

Evaluate the integral $\int_0^1 \frac{x^2}{1+x^3}dx$ by using Simpson's $\frac{1}{3}$ rule. Compare the error with the exact value.

Solution:

Let us divide the internal (0, 1) into 4 equal parts so that $h = \frac{1-0}{4} = 0.25$.

Taking $y = \frac{x^2}{(1+x^3)}$, we have,

x	0	0.25	0.50	0.75	1.00
y	0	0.06153	0.22222	0.39560	0.5
	y_0	y_1	y_2	y_3	y_4

By Simpson's $\frac{1}{3}$ rule, we have,

$$\int_0^1 \frac{x^2}{1+x^3}dx = \frac{h}{3}[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)]$$

$$= \frac{0.25}{3}[(0 + 0.5) + 2 \times 0.22222) + 4(0.06153 + 0.3956)]$$

$$= \frac{0.25}{3}[0.5 + 0.44444 + 1.82852]$$

$$= 0.23108$$

Also,

$$\int_0^1 \frac{x^2}{1+x^3}dx = \frac{1}{3}|\log(1+x^3)|_0^1$$

$$= \frac{1}{3}\log_e(2) = 0.23108$$

Hence the error = 0.23108 – 0.23105 = –0.00003

Example 3.4

Use trapezoidal rule to evaluate $\int_0^1 x^3\,dx$ considering five sub-intervals.

Solution:

Given that,

$$I = \int_0^1 x^3\,dx$$

Also, a = 0, b = 1, sub-intervals = 5, intervals (n) = 5 − 1 = 4

Then,

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

Now table is created at the interval of 0.25 from 0 to 1.

x	0	0.25	0.5	0.75	1
y	0	0.0156	0.125	0.4219	1
	y_0	y_1	y_2	y_3	y_4

Now, using trapezoidal rule,

$$I = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.25}{2}[0 + 1 + 2\,(0.0156 + 0.125 + 0.4219)]$$

$$= 0.256$$

Also, $I_{abs} = \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1^4}{4} - 0 = 0.25$

Example 3.5

Evaluate $\int_0^1 \frac{dx}{1+x}$ applying

i) **Trapezoidal rule**

ii) **Simpson's $\frac{1}{3}$ rule**

iii) **Simpson's $\frac{3}{8}$ rule**

Solution:

Given that;

$$I = \int_0^1 \frac{dx}{1+x}$$

Also, a = 0, b = 1, Taking n = 5

$$h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$$

Now, table is created at the interval of 0.2 from 0 to 1.

x	0	0.2	0.4	0.6	0.8	1
y	1	0.8333	0.7143	0.625	0.5556	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

i) By trapezoidal rule,

$$I = \frac{h}{2}[y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4)]$$

$$= \frac{0.2}{2}[1 + 0.5 + 2(0.8333 + 0.7143 + 0.625 + 0.5556)]$$

$$= 0.6956$$

ii) By Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + y_5 + 4(y_1 + y_3) + 2(y_2 + y_4)]$$

$$= \frac{0.2}{3}[1 + 0.5 + 4(0.8333 + 0.625) + 2(0.7143 + 0.5556)]$$

$$= 0.6582$$

iii) By Simpson's $\frac{3}{8}$ rule,

$$I = \frac{h}{3}[y_0 + y_5 + 3(y_1 + y_2 + y_4) + 2y_3]$$

$$= \frac{3 \times 0.2}{8}[1 + 0.5 + 3(0.8333 + 0.7143 + 0.5556) + 2(0.625)]$$

$$= 0.6795$$

Also, $I_{abs} = \int_0^1 \frac{dx}{1+x} = 0.6931$

Example 3.6

Given that;

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
log x	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Evaluate $\int_4^{5.2} \log x \, dx$ by,

a) Trapezoidal rule

b) Simpson's $\frac{1}{3}$ rule

c) Simpson's $\frac{3}{8}$ rule

Solution:

Given that;

$$I = \int_4^{5.2} \log x \cdot dx$$

From the given table, n = 6

so, $h = \frac{b-a}{n} = \frac{5.2-4}{6} = 0.2$

Simply we can find the h from table as 4.2 − 4 = 0.2

Now, from the table we have,

$y_0 = 1.3863 \qquad y_3 = 1.5261$

$y_1 = 1.4351 \qquad y_4 = 1.5686$

$y_2 = 1.4816 \qquad y_5 = 1.6094$

$y_6 = 1.6487$

Now, by Trapezoidal rule,

$$I = \frac{h}{2}[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)$$

$$= \frac{0.2}{2}[1.3863 + 1.6487 + 2(1.4351 + 1.4816 + 1.5261 + 1.5686 + 1.6094)]$$

$$= 1.8277$$

By Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{0.2}{3}[1.3863 + 1.6487 + 4(1.4351 + 1.5261 + 1.6094) + 2(1.4816 + 1.5686)]$$

$$= 1.8279$$

By Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3(0.2)}{8}[1.3863 + 1.6487 + 3(1.4351 + 1.4816 + 1.5686 + 1.6094) + 2(1.5261)]$$

$$= 1.8278$$

Also, $I_{abs} = \int_4^{5.2} \log x \, dx = 1.8278$

3.5 ERRORS IN QUADRATURE FORMULA

The error in the quadrature formula is given by,

$$E = \int_a^b y \, dx - \int_a^b p(x) \, dx$$

where, p(x) is the polynomial representing the function y = f(x), in the interval [a, b].

I. Error in Trapezoidal Rule

Expanding y = f(x) around $x = x_0$ by Taylor's series, we get,

$$y = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \ldots\ldots\ldots \qquad \ldots\ldots (1)$$

$$\therefore \quad \int_{x_0}^{x_0+h} y\,dx = \int_{x_0}^{x_0+h} \left[y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \ldots\ldots\ldots \right] dx \qquad \ldots(2)$$

$$= y_0 h + \frac{h^2}{2!} y_0' + \frac{h^3}{3!} y_0'' + \ldots\ldots\ldots$$

Also, A = area of the first trapezium in the interval

$$[x_0, x_1] = \frac{1}{2} h(y_0 + y_1) \qquad \ldots(3)$$

Putting $x = x_0 + h$ and $y = y_1$ in equation (1), we get,

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \ldots\ldots\ldots$$

Replacing this value of y_1 in (3), we get,

$$A_1 = \frac{1}{2} h \left[y_0 + y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \ldots\ldots\ldots \right] \qquad \ldots(4)$$

$$= hy_0 + \frac{h^2}{2!} y_0' + \frac{h^3}{2 \times 2!} y_0'' + \ldots\ldots\ldots$$

$\therefore$ Error in the interval $[x_0, x_1] = \int_{x_1}^{x_2} y\,dx - A_1$

$$= \frac{1}{3!} - \frac{1}{2.2!} h^3 y_0'' + \ldots\ldots\ldots$$

$$= -\frac{h^3}{12} y_0'' + \ldots\ldots\ldots$$

i.e., principal part of the error in $[x_0, x_1] = -\frac{h^3}{12} y_0''$

Hence the total error, $E = -\frac{h^3}{12} [y_0'' + y_1'' + \ldots\ldots\ldots + y_{n-1}'']$

Assuming that $y''(X)$ is the largest of n quantities, $y_0'', y_1'', \ldots\ldots\ldots, y_{n-1}''$, we get,

$$E < -\frac{nh^3}{12} y''(X) = -\frac{(b-a)h^2}{12} y''(X) \qquad [\because nh = b - a] \ldots\ldots (5)$$

Hence the error in the trapezoidal rule is of the order h^2.

II. Error in Simpson's $\frac{1}{3}$ Rule $= -\left(\frac{b-a}{180}\right) h^4 y^{iv}(X)$

Assuming the $y^{iv}(X)$ is the largest of

$y_0^{iv}, y_2^{iv}, \ldots\ldots\ldots, y_{2n-2}^{iv}$

i.e., the error in Simpson's $\frac{1}{3}$ rule is of the order h^4.

III. Error in Simpson's $\frac{3}{8}$ Rule $= -\frac{3h^5}{80} y^{iv}$

3.6 ROMBERG'S INTEGRATION

Romberg integration method is named after Werner Romberg. This method is an extrapolation formula of the trapezoidal rule for integration. It provides a better approximation of the integral by reducing the true error. We compute the value of the integral with a number of step lengths using the same method. Usually, we start with a coarse step length, then reduce the step lengths are recomputed the value of the integral. The sequence of these values converges to the exact value of the integral. Romberg method uses these values of the integral obtained with various step lengths, to refine the solution such that the new values are of higher order. That is, as if the results are obtained using a higher order method than the order of the method used. The extrapolation method is derived by studying the error of the method that is being used.

Romberg's method provides a simple modification to the quadrature formulae for finding their better approximations. As an illustrations, let us improve upon the value of the integral,

$$I = \int_a^b f(x)\,dx$$

by the trapezoidal rule.

If I_1, I_2 are the values of I with sub-intervals of width h_1, h_2 and E_1, E_2 their corresponding errors, respectively, then,

$$E_1 = -\frac{(b-a)\,h^2}{12}\,y''(X)$$

$$E_2 = -\frac{(b-a)^2\,h^2}{12}\,y''(\bar{X})$$

Since, $y''(\bar{X})$ is also the largest value of y'', we can reasonably assume that $y''(X)$ and $y''(\bar{X})$ are very nearly equal.

$$\therefore \quad \frac{E_1}{E_2} = \frac{h_1^2}{h_2^2} \text{ or } \frac{E_1}{E_2 - E_1} = \frac{h_1^2}{h_2^2 - h_1^2} \qquad \text{..... (1)}$$

Now,

$$\text{Since } I = I_1 + E_1 = I_2 + E_2 \qquad \text{..... (2)}$$

$$\therefore \quad E_2 - E_1 = I_1 - I_2$$

From (1) and (2), we get,

$$E_1 = \frac{h_1^2}{h_2^2 - h_1^2}(I_1 - I_2)$$

$$\text{Hence, } I = I_1 + E_1 = I_1 + \frac{h_1^2}{h_2^2 - h_1^2}(I_1 - I_2)$$

$$\text{i.e., } \quad I = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2} \qquad \text{..... (3)}$$

which is a better approximation of I.

To evaluate I systematically, we take $h_1 = h$ and $h_2 = \frac{1}{2}h$.

So that (3) gives,

$$I = \frac{I_1\left(\frac{h}{2}\right)^2 - I_2 h_2^2}{\left(\frac{h}{2}\right)^2 - h^2} = \frac{4I_2 - I_1}{3}$$

i.e., $$I\left(h, \frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right] \qquad \text{.... (4)}$$

Now, we use trapezoidal rule several times successively halving h and apply (4) to each per pair of values as per the following scheme.

$I(h)$			
	$I\left(h, \frac{h}{2}\right)$		
$I\left(\frac{h}{2}\right)$		$I\left(h, \frac{h}{2}, \frac{h}{4}\right)$	
	$I\left(\frac{h}{2}, \frac{h}{4}\right)$		$I\left(h, \frac{h}{2}, \frac{h}{4}, \frac{h}{8}\right)$
$I\left(\frac{h}{4}\right)$		$I\left(\frac{h}{2}, \frac{h}{4}, \frac{h}{8}\right)$	
	$I\left(\frac{h}{4}, \frac{h}{8}\right)$		
$I\left(\frac{h}{8}\right)$			

The computation is continued until successive values are close to each other. This method is called Richardson's deferred approach to the limit and its systematic refinement is called Romberg's method.

Example 3.7

Evaluate $\int_0^{0.5} \left(\frac{x}{\sin x}\right) dx$ correct to three decimals places using Romberg's method.

Solution:

Taking h = 0.25, 0.125, 0.0625 respectively, let us evaluate the given integral by using Simpson's $\frac{1}{3}$ rule.

i) When h = 0.25, the values of $y = \frac{x}{\sin x}$ are,

x	0	0.25	0.5
y	1	1.0105	1.0429
	y_0	y_1	y_2

$\therefore$ By Simpson's rule,

$$I = \frac{h}{3}[(y_0 + y_2) + 4y_1]$$

$$= \frac{0.25}{3}[(1 + 1.0429) + 1.0105]$$

$$= 0.5071$$

ii) When h = 0.125, the values of y are,

x	0	0.125	0.25	0.375	0.5
y	1	1.0026	1.0105	1.1003	1.0429
	y_0	y_1	y_2	y_3	y_4

$\therefore$ By Simpson's rule,

$$I = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{0.125}{3}[(1 + 1.0429) + 4(1.0026 + 1.1003) + 2(1.0105)]$$

$$= 0.5198$$

iii) When h = 0.0625, the values of y are,

x	0	0.0625	0.125	0.1875	0.25	0.3125	0.1875	0.4375	0.5
y	1	0.0006	1.0026	1.0059	1.0157	1.0165	1.1003	1.0326	1.0429
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$\therefore$ By Simpson's rule,

$$I = \frac{h}{3}[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{0.0625}{3}[(1 + 1.0429) + 4(1.0006 + 1.0059 + 1.0165 + 1.0326)$$

$$+ 2(1.0026 + 1.0105 + 1.1003)]$$

$$= 0.510253$$

Using Romberg's formulae, we get

$$I = \left(h, \frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right] = 0.5241$$

$$I = \left(\frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right] = 0.5070$$

$$I\left(h, \frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}, \frac{h}{4}\right) - I\left(h, \frac{h}{2}\right)\right] = 0.5013$$

Hence,

$$\int_0^{0.5} \left(\frac{x}{\sin x}\right) dx = 0.501$$

Example 3.8

Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using the Romberg's method.

Solution:

Given that;

$$I = \int_0^2 \frac{dx}{x^2+4}$$

Here, a = 0, b = 2

i) Taking h = 1 and creating interval of 1 from 0 to 2

x	0	1	2
y	0.25	0.2	0.125
	y_0	y_1	y_2

Now, using Trapezoidal rule,

$$I(1) = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{1}{2}[0.25 + 0.125 + 2(0.2)] = 0.3875$$

ii) Taking h = 0.5 and creating interval of 0.5 from 0 to 2

x	0	0.5	1	1.5	2
y	0.25	0.2353	0.2	0.16	1.125
	y_0	y_1	y_2	y_3	y_4

Now, using trapezoidal rule,

$$I(0.5) = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3]$$

$$= \frac{0.5}{2}[0.25 + 0.125 + 2(0.2353 + 0.2 + 0.16]$$

$$= 0.3914$$

iii) Taking h = 0.25 and creating interval of 0.25 from 0 to 2

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
y	0.25	0.2462	0.2353	0.2192	0.2	0.1798	0.16	0.1416	0.125

Now, using trapezoidal rule,

$$I(0.25) = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.25}{2}[0.25 + 0.125 + 2(0.2462 + 0.2353 + 0.2192 + 0.2 + 0.1798 + 0.16 + 0.1416)]$$

$$= 0.3924$$

Now, optimizing values by Romberg integration,

$$I(1, 0.5) = \frac{1}{3}[4I(0.5) - I(1)]$$

$$= \frac{1}{3}[4(0.3914) - 0.3875]$$

$$= 0.3927$$

$$I(0.5, 0.25) = \frac{1}{3}[4I(0.25) - I(0.5)]$$

$$= \frac{1}{3}[4(0.3924) - 0.3914]$$

$$= 0.3927$$

$$I(1, 0.5, 0.25) = \frac{1}{3}[4I(0.5, 0.25) - I(1,0.5)]$$

$$= \frac{1}{3}[4(0.3927) - 0.3927]$$

$$= 0.3927$$

Hence the value of $\int_0^2 \frac{dx}{x^2 + 4} = 0.3927$

3.7 GAUSSIAN INTEGRATION

Gauss derived a formula which uses the same number of functional values but with different Gauss formula is expressed as,

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) + \ldots\ldots\ldots + w_n f(x_n)$$

$$= \sum_{i=1}^{n} w_i f(x_i) \qquad \ldots\ldots (1)$$

where, w_i and x_i are called the weights and abscissae, respectively. The abscissae and weights are symmetrical with respect to the middle point of the interval. There being 2n unknowns in (1), 2n relations between them are necessary so that the formula is exact for all polynomials of degree not exceeding 2n – 1. Thus, we consider,

$$f(x) = c_0 + c_1x + c_2x + \ldots\ldots\ldots + c_{2n-1}\,x^{2n-1} \qquad \ldots\ldots (2)$$

Then, (1) gives,

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (c_0 + c_1x + c_2x + \ldots\ldots\ldots + c_{2n-1}\,x^{2n-1})\,dx \qquad \ldots\ldots (3)$$

$$= 2c_0 + \frac{2}{3}c_2 + \frac{2}{5}c_4 + \ldots\ldots\ldots$$

Putting $x = x_i$ in (2), we get,

$$f(x_i) = c_0 + c_1x_i + c_2x_i^2 + c_2x_i^3 + \ldots\ldots\ldots + c_{2n-1}\,x_i^{2n-1}$$

Substituting these values on the right hand side of (1), we get,

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx = {} & w_1 (c_0 + c_1x_1 + c_2x_1^2 + c_3x_1^3 + \ldots\ldots\ldots + c_{2n}\,x_1^{2n-1}) \\ & + w_2 (c_0 + c_1x_2 + c_2x_2^2 + c_3x_2^3 + \ldots\ldots\ldots + c_{2n-1}\,x_2^{2n-1}) \\ & + w_3 (c_0 + c_1x_3 + c_2x_3^2 + c_3x_3^2 + \ldots\ldots\ldots + c_{2n-1}\,x_3^{2n-1}) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + w_n (c_0 + c_1x_n + c_2x_n^2 + c_3x_n^3 + \ldots\ldots\ldots + c_{2n-1}\,x_n^{2n-1})\end{aligned}$$

$$= c_0 (w_1 + w_2 + w_3 + \ldots\ldots + w_n) \ldots\ldots + c_1(w_1x_1 + w_2x_2 + w_3x_3 + \ldots\ldots + w_nx_n) + c_2 (w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + \ldots\ldots + w_nx_n^2) + \ldots\ldots + c_{2n-1} (w_1x_1^{2n-1} + w_2x_2^{2n-1} + w_3x_3^{2n-1} + \ldots\ldots + w_nx_n^{2n-1}) \quad \text{---(4)}$$

But the equation (3) and (4) are identical for all values of c_i hence comparing coefficients of c_i, we get 2n equations in 2n unknowns in 2n unknowns w_i and x_i ($i = 1, 2, 3, \ldots\ldots, n$).

$$\left.\begin{aligned} &w_1 + w_2 + w_3 + \ldots\ldots + w_n = 2 \\ &w_1x_1 + w_2x_2 + w_3x_3 + \ldots\ldots + w_nx_n = 0 \\ &w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + \ldots\ldots + w_nx_n^2 = \frac{2}{3} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &w_1x_1^{2n-1} + w_2x_2^{2n-1} + w_3x_n^{2n-1} + \ldots\ldots + w_nx_n^{2n-1} = 0 \end{aligned}\right\} \quad \text{---(5)}$$

The solution of above equations is extremely complicated. It can however be shown that x_i are the zeros of the $(n + 1)^{th}$ Legendre polynomial.

Gauss formula for n = 2

$$\int_{-1}^{1} f(x)\, dx = w_1 f(x_1) + w_2 f(x_2)$$

Then the equation (5) becomes,

$$w_1 + w_2 = 2$$

$$w_1x_1 + w_2x_2 = 0$$

$$w_1x_1^2 + w_2x_2^2 = \frac{2}{3}$$

$$w_1x_1^3 + w_2x_2^3 = 0$$

On solving, we get,

$$w_1 = w_2 = 1,\ x_1 = \frac{-1}{\sqrt{3}} \text{ and } x_2 = \frac{1}{\sqrt{3}}$$

Thus, gauss formula for n = 2 is,

$$\int_{-1}^{1} f(x)\, dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \quad \ldots\ldots (6)$$

which gives the correct values of the integral of f(x) in the range (−1, 1) for any function upto third order. Equation (6) is also called as Gauss-Legendre formula.

Gauss formula for n = 3 is,

$$\int_{-1}^{1} f(x)\, dx = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\frac{\sqrt{3}}{5}\right) + f\left(\frac{\sqrt{3}}{5}\right)\right]$$

which is exact for polynomials upto degree 5.

$$= c_0 (w_1 + w_2 + w_3 + \ldots\ldots + w_n) \ldots\ldots + c_1(w_1x_1 + w_2x_2 + w_3x_3 + \ldots\ldots + w_nx_n) + c_2 (w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + \ldots\ldots + w_nx_n^2)$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots +$$
$$+ c_{2n-1} (w_1x_1^{2n-1} + w_2x_2^{2n-1} + w_3x_3^{2n-1} + \ldots\ldots + w_nx_n^{2n-1}) \quad \ldots (4)$$

But the equation (3) and (4) are identical for all values of c_i, hence comparing coefficients of c_i, we get 2n equations in 2n unknowns in 2n unknowns w_i and $x_i (i = 1, 2, 3, \ldots\ldots, n)$.

$$\left.\begin{aligned} &w_1 + w_2 + w_3 + \ldots\ldots + w_n = 2 \\ &w_1x_1 + w_2x_2 + w_3x_3 + \ldots\ldots + w_nx_n = 0 \\ &w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + \ldots\ldots + w_nx_n^2 = \frac{2}{3} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &w_1x_1^{2n-1} + w_2x_2^{2n-1} + w_3x_n^{2n-1} + \ldots\ldots + w_nx_n^{2n-1} = 0 \end{aligned}\right\} \quad \ldots (5)$$

The solution of above equations is extremely complicated. It can however, be shown that x_i are the zeros of the $(n + 1)^{th}$ Legendre polynomial.

Gauss formula for n = 2

$$\int_{-1}^{1} f(x)\, dx = w_1 f(x_1) + w_2 f(x_2)$$

Then the equation (5) becomes,

$$w_1 + w_2 = 2$$
$$w_1x_1 + w_2x_2 = 0$$
$$w_1x_1^2 + w_2x_2^2 = \frac{2}{3}$$
$$w_1x_1^3 + w_2x_2^3 = 0$$

On solving, we get,

$$w_1 = w_2 = 1,\ x_1 = \frac{-1}{\sqrt{3}} \text{ and } x_2 = \frac{1}{\sqrt{3}}$$

Thus, gauss formula for n = 2 is,

$$\int_{-1}^{1} f(x)\, dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \quad \ldots (6)$$

which gives the correct values of the integral of f(x) in the range (−1, 1) for any function upto third order. Equation (6) is also called as Gauss-Legendre formula.

Gauss formula for n = 3 is,

$$\int_{-1}^{1} f(x)\, dx = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\frac{\sqrt{3}}{5}\right) + f\left(\frac{\sqrt{3}}{5}\right)\right]$$

which is exact for polynomials upto degree 5.

Gauss formula imposes a restriction on the limits of integration to be form -1 to 1.

In general, the limits of the integral $\int_a^b f(x)\,dx$ are changed to -1 to 1 by means of the transformation,

$$x = \frac{1}{2}(b-a)\,u + \frac{1}{2}(b+a)$$

Example 3.9

Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ using Gauss formula for n = 2 and n = 3.

Solution:

i) Gauss formula for n = 2 is,

$$I = \int_{-1}^{1} \frac{dx}{1+x^2} = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

where, $f(x) = \frac{dx}{1+x^2}$

$$\therefore \quad I = \frac{1}{1+\left(\frac{-1}{\sqrt{3}}\right)^2} + \frac{1}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{3}{4} + \frac{3}{4} = 1.5$$

ii) Gauss formula for n = 3 is,

$$I = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(\frac{-\sqrt{3}}{5}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

where, $f(x) = \frac{1}{1+x^2}$

$$\text{Hence, } I = \frac{8}{9} + \frac{5}{9}\left(\frac{5}{8} + \frac{5}{8}\right) = \frac{8}{9} + \frac{50}{72} = 1.5833$$

BOARD EXAMINATION SOLVED QUESTIONS

1. **Evaluate the integral $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx$. Compare the result in both conditions for Simpson $\frac{1}{3}$ and $\frac{3}{8}$ rule.** **[2013/Fall]**

Solution:

Given that;

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx$$

$$a = 0, \quad b = \frac{\pi}{2}$$

Taking n = 6,

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

Now, table is created at the interval of $\frac{\pi}{12}$ from 0 to $\frac{\pi}{2}$.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	0	0.508	0.707	0.840	0.930	0.982	1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now, by Simpson's $\frac{1}{3}$ rule

$$I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{3 \times 12}[0 + 1 + 4(0.508 + 0.840 + 0.982) + 2(0.707 + 0.930)]$$

$$= 1.186$$

Again, by Simpson's $\frac{3}{8}$ rule

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\pi}{8 \times 12}[0 + 1 + 3(0.508 + 0.707 + 0.930 + 0.982) + 2(0.840)]$$

$$= 1.184$$

and, Absolute value of I

$$I_{abs} = \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx = 1.198$$

NOTE: Use calculator to directly obtain the absolute value in radian mode.

Now,

Error by Simpson $\frac{1}{3}$ rule = $|1.186 - 1.198| = 0.012$

Error by Simpson $\frac{3}{8}$ rule = $|1.184 - 1.198| = 0.014$

Here, the error by Simpson $\frac{1}{3}$ rule is less than Simpson $\frac{3}{8}$ rule.

2. Evaluate the integral $I = \int_0^6 \frac{1}{1+x^2}\,dx$. Compare the absolute error in both conditions for Simpson $\frac{1}{3}$ rule and Simpson $\frac{3}{8}$ rule. [2013/Spring]

Solution:

Given that;

$$I = \int_0^6 \frac{1}{1+x^2}\,dx$$

$a = 0, b = 6$

Let, $n = 6$ then

$$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$$

Now, Table is created at the interval of 1 from 0 to 6

Formulating the table,

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	0.058	0.038	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[1 + 0.027 + 4(0.5 + 0.1 + 0.038) + 2(0.2 + 0.058)]$$

$$= 1.365$$

By Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$= \frac{3}{8}[1 + 0.027 + 3(0.5 + 0.2 + 0.058 + 0.038) + 2(0.1)]$$

$$= 1.355$$

Now, Absolute value of I,

$$I = \int_0^6 \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_0^6 = 1.405$$

Now,

$$\text{Error by Simpson } \frac{1}{3} \text{ rule} = |1.405 - 1.365| = 0.04$$

$$\text{Error by Simpson } \frac{3}{8} \text{ rule} = |1.405 - 1.355| = 0.05$$

Here, the error by Simpson $\frac{1}{3}$ rule is less than Simpson $\frac{3}{8}$ rule.

3. Find the integral value $I = \int_0^1 \frac{dx}{1+x^2}$ correct to three decimal place by using Romberg integration. **[2013/Spring, 2018/Spring]**

Solution:

Given that;

$$I = \int_0^1 \frac{dx}{1+x^2}$$

Here, $a = 0, b = 1$

i) Taking $h = 0.5$ and creating interval of 0.5 from 0 to 1.

x	0	0.5	1
y = f(x)	1	0.8	0.5
	y_0	y_1	y_2

Now, using trapezoidal rule,

$$I(0.5) = \frac{h}{2}[y_0 + y_2 + 2y_1]$$

$$= \frac{0.5}{2}[1 + 0.5 + 2(0.8)]$$

$$= 0.775$$

ii) Taking $h = 0.25$ and creating interval of 0.25 from 0 to 1.

x	0	0.25	0.5	0.75	1
y	1	0.9411	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

Now, using trapezoidal rule,

$$I(0.25) = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.25}{2}[1 + 0.5 + 2(0.9411 + 0.8 + 0.64)]$$

$$= 0.7827$$

iii) Taking $h = 0.125$

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.9411	0.8767	0.8	0.7191	0.64	0.5663	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Now, using Trapezoidal rule,

$$I(0.125) = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.125}{2}[1 + 0.5 + 2(0.9846 + 0.9411 + 0.8767 + 0.8 + 0.7191 + 0.64 + 0.5663)]$$

$$= 0.7847$$

Now, optimizing values by Romberg integration,

$$I(0.5, 0.25) = \frac{1}{3}[4I(0.25) - I(0.5)]$$

$$= \frac{1}{3}[4 \times 0.7827 - 0.775]$$

$$= 0.7852$$

$$I(0.25, 0.125) = \frac{1}{3}[4I(0.125) - I(0.25)]$$

$$= \frac{1}{3}[4 \times 0.7847 - 0.7827]$$

$$= 0.7853$$

$$I(0.5, 0.25, 0.125) = \frac{1}{3}[4I(0.25, 0.125) - I(0.5, 0.25)]$$

$$= 0.7853$$

Hence the value of integral $\int_0^1 \frac{dx}{1+x^2} = 0.7853$

Also, $\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(x)\Big|_0^1 = 0.7853$

Table of obtained values;

$$\left.\begin{matrix} I(0.5) \\ I(0.25) \end{matrix}\right\} I(0.5, 0.25)$$

$$\left.\begin{matrix} I(0.25) \\ I(0.125) \end{matrix}\right\} I(0.25, 0.125)$$

$$\left.\begin{matrix} I(0.5, 0.25) \\ I(0.25, 0.125) \end{matrix}\right\} I(0.5, 0.25, 0.125)$$

4. The following table gives the displacement, x (cms) of an object at various of time, t(seconds). Find the velocity and acceleration of the object at t = 1.6 sec. Using suitable interpolation method. **[2014/Fall]**

T	1.0	1.2	1.4	1.6	1.8
X	9.0	9.5	10.2	11.0	13.2

Solution:

Creating the difference table from given data

x = T	y = x	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1.0	9.0				
		0.5			
1.2	9.5		0.2		
		0.7		−0.1	
1.4	10.2		0.1		1.4
		0.8		1.3	
1.6	11.0		1.4		
		2.2			
1.8	13.2				

Here the data of T is equispaced and t = 1.6 sec is near the end of the table, so using Newton's backward formula for numerical differentiation.

$h = 1.8 - 1.6 = 0.2$

Now, at t = 1.6 sec

From numerical differentiation, using Newton's backward formula,

$$\left(\frac{dy}{dx}\right)_{1.6} = y' = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3}\right]$$

$$= \frac{1}{0.2}\left[0.8 + \frac{0.1}{2} + \frac{-0.1}{3}\right]$$

= 4.083 cm/s is the required velocity of an object

Now, for acceleration

$$y'' = \frac{1}{h^2}[\nabla^2 y_n + \nabla^3 y_n] = \frac{1}{0.2^2}[0.1 + -0.1]$$

$\therefore$ $y'' = 0$ cm/s² is the required acceleration of an object.

5. Evaluate the integral $\int_0^{\pi} (1 + 3\cos^2 x)\, dx$ by,

i) Trapezoidal rule

ii) Simpson's $\frac{3}{8}$ rule, taking number of intervals (n) = 6

[2014/Spring]

Solution:

Given that;

$$I = \int_0^{\pi} (1 + 3\cos^2 x)\, dx$$

$n = 6$

Also,

$a = 0, b = \pi$

Then,

$$h = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

Now, table is created at the interval of $\frac{\pi}{6}$ from 0 to π

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
y	4	3.25	1.75	1	1.75	3.25	4
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

i) By trapezoidal rule,

$$I = \frac{h}{2}[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{2 \times 6}[4 + 4 + 2(3.25 + 1.75 + 1 + 1.75 + 3.25)]$$

$\therefore$ $I = 7.8539$

ii) By Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\pi}{8 \times 6}[4 + 4 + 3(3.25 + 1.75 + 1.75 + 3.25) + 2(1)]$$

$= 7.8539$

Also,

$$I_{abs} = \int_0^{\pi} (1 + 3\cos^2 x)\, dx = \int_0^{\pi} 1 + \frac{3}{2}(\cos 2x + 1) = 7.8539$$

6. Evaluate the integral $I = \int_0^{\frac{\pi}{2}} \sin x\, dx$ for n = 6 and compare the result in both conditions for Simpson $\frac{1}{3}$ and $\frac{3}{8}$ rule. **[2015/Fall]**

Solution:

Given that;

$$I = \int_0^{\frac{\pi}{2}} \sin x\, dx$$

$a = 0,\ b = \frac{\pi}{2},\ n = 6$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

Now, creating table at the interval of $\frac{\pi}{12}$ from 0 to $\frac{\pi}{2}$

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	0	0.258	0.5	0.707	0.866	0.965	1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now, By Simpson's $\frac{1}{3}$ rule

$$I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{3 \times 12}[0 + 1 + 4(0.258 + 0.707 + 0.965) + 2(0.5 + 0.866)]$$

$$= 0.9993$$

Again, by Simpson's $\frac{3}{8}$ rule

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\pi}{8 \times 12}[0 + 1 + 3(0.258 + 0.5 + 0.866 + 0.965) + 2(0.707)]$$

$$= 0.9995$$

and, $I_{abs} = \int_0^{\frac{\pi}{2}} \sin x \cdot dx = [-\cos x]_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = 1$

Now, Error by Simpson $\frac{1}{3}$ rule $= |1 - 0.9993| = 0.0007$

Error by Simpson $\frac{3}{8}$ rule $= |1 - 0.9995| = 0.0005$

Here, the error by Simpson $\frac{1}{3}$ rule is more than Simpson $\frac{3}{8}$ rule, so Simpson $\frac{3}{8}$ rule is more accurate.

7. Use following table of data to estimate velocity at t = 7 sec

Time, t(s)	5	6	7	8	9
Distance Travelled, s(t) (km)	10.0	14.5	19.5	25.5	32.0

Hint: velocity is first derivative of s(t). **[2015/Spring]**

Solution:

Creating difference table

t = x	y = s(t)	1st diff	2nd diff	3rd diff	4th diff
5	10.0				
		4.5			
6	14.5		0.5		
		5		0.5	
7	19.5		1		−1
		6		−0.5	
8	25.5		0.5		
		6.5			
9	32.0				

Now, to estimate velocity at t = 7 sec which lies at the mid of table.

Using Stirling's central difference formula,

We have,

$$y_p = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2 - 1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \dots\dots$$

$$= x_0 + ph$$

Differentiating with respect to p, we get,

$$\frac{dy_p}{dx} = \frac{\Delta y_0 + \Delta y_{-1}}{2} + p\Delta^2 y_{-1} + \frac{3p^2 - 1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \dots\dots$$

and, $\frac{dx}{dp} = h$

Then,

$$\frac{dy_p}{dx} = \frac{dy_p}{dp} \times \frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} + p\Delta^2 y_{-1} + \frac{3p^2 - 1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \dots\dots\right]$$

At $x = x_0$, $p = 0$,

so, $\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \dots\dots\right]$

Now,

$$s'(t) = \frac{d(s(t))}{dt} = \left(\frac{dy}{dx}\right)_7 = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right]$$

> **NOTE:**
> Formula is placed according to the data available in difference table *i.e.*, Δy_0 and Δy_{-1} are present but not for other $\Delta^3 y_{-1}$, $\Delta^3 y_{-2}$ etc for t = 7.

or, $s'(t) = \frac{1}{1}\left[\frac{6 + 5}{2}\right]$

∴ $s'(t) = 5.5$ km/s is the required velocity

8. Evaluate the integral $I = \int_0^{10} \exp\left(\frac{-1}{1 + x^2}\right) dx$, using gauss quadrature formula with n = 2 and n = 3. **[2016/Fall]**

Solution:

Given that;

$$I = \int_0^{10} f(x)\, dx$$

where, $f(x) = \exp\left(\frac{-1}{1 + x^2}\right)$

Using gauss quadrature formula with n = 2 and n = 3 since limit a = 0 and b = 10 is not from −1 to 1, so using,

$$x = \frac{1}{2}(b - a)u + \frac{1}{2}(b + a)$$

or, $x = \frac{1}{2}(10 - 0)\,u + \frac{1}{2}(10 + 0)$

$\therefore$ $x = 5u + 5$ ....(1)

Differentiating on both sides

$dx = 5\,du$ ....(2)

Then, substituting the values form (1) and (2) to I,

$$I = \int_{-1}^{1} \exp\left(\frac{-1}{1 + (5u+5)^2}\right) 5du$$

Now,

i) Gauss formula for n = 2 is

$$I = \int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= 5\exp\left[\frac{-1}{1 + \left(5\left(\frac{-1}{\sqrt{3}}\right) + 5\right)^2}\right] + 5\exp\left[\frac{-1}{1 + \left(\frac{5}{\sqrt{3}} + 5\right)^2}\right]$$

$$= 4.164 + 4.921 = 9.085$$

Then,

ii) Gauss formula for n = 3 is,

$$I = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9}\left(5\exp\left(\frac{-1}{1 + (0+5)^2}\right)\right)$$

$$+ \frac{5}{9}\left[5\exp\left(\frac{-1}{1 + \left(5\left(-\sqrt{\frac{3}{5}}\right) + 5\right)^2}\right)\right]$$

$$+ 5\exp\left[5\exp\left(\frac{-1}{1 + \left(5\left(\sqrt{\frac{3}{5}}\right) + 5\right)^2}\right)\right]$$

$$= 4.276 + 4.531 = 8.807$$

9. **Evaluate the integral $\int_0^{0.6} e^{x^2}\,dx$, using Simpson $\frac{1}{3}$ rule and Simpson $\frac{1}{8}$ rule, dividing the interval into six parts.** **[2016/Spring]**

Solution:

Given that;

$$I = \int_0^{0.6} e^{x^2}\,dx,$$

$a = 0$, $b = 0.6$ and $n = 6$

Then,

$$h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1$$

Now, table is created at the interval of 0.1 from 0 to 0.6.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
y	1	1.010	1.040	1.094	1.173	1.284	1.433
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now, by Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{0.1}{3}[1 + 1.433 + 4(1.010 + 1.094 + 1.284) + 2(1.04 + 1.173)]$$

$$= 0.68036$$

Again, by Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3 \times 0.1}{8}[1 + 1.433 + 3(1.010 + 1.040 + 1.173 + 1.284) + 2(1.094)]$$

$$= 0.68032$$

Also, $I_{abs} = \int_0^{0.6} e^{x^2}\,dx = 0.68049$

10. **Estimate the following integrals by,**

i) **Simpson's $\frac{3}{8}$ method**

ii) **Simpson's $\frac{1}{3}$ method and compare the result** **[2017/Fall]**

$$\int_2^1 \frac{e^x\,dx}{x} \text{ (Assume } n = 4)$$

Solution:

Given that;

$$I = \int_2^1 \frac{e^x}{x}\,dx$$

$a = 2$, $b = 1$, $n = 4$

Then,

$$h = \frac{b-a}{n} = \frac{1-2}{4} = -0.25$$

Now, creating table at the interval of (−0.25) from 2 to 1.

x	2	1.75	1.5	1.25	1
y	3.694	3.288	2.987	2.792	2.718
	y_0	y_1	y_2	y_3	y_4

Now, by Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2(y_2)]$$

$$= \frac{-0.25}{3}[3.694 + 2.718 + 4(3.288 + 2.792) + 2(2.987)]$$

$$= -3.0588$$

And, by Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_4 + 3(y_1 + y_2) + 2y_3]$$

$$= \frac{3 - 0.25}{8}[3.694 + 2.718 + 3(3.288 + 2.987) + 2(2.792)]$$

$$= -2.8894$$

Then, $I_{abs} = \int_2^1 \frac{e^x}{x}dx = -3.0591$

Now, Error by Simpson $\frac{1}{3}$ rule $= |-3.0591 + 3.0588| = 0.0003$

Error by Simpson $\frac{3}{8}$ rule $= |-3.0591 + 2.8894| = 0.1697$

So, Simpson's $\frac{1}{3}$ rule is more accurate.

11. Apply Romberg's method to evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin x}}dx$$

[2017/Fall]

Solution:

Given that;

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin x}}dx$$

$a = 0, b = \frac{\pi}{2}$

i) Taking $h = \frac{\pi}{4}$ and creating interval of $\frac{\pi}{4}$ from 0 to $\frac{\pi}{2}$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
y	1	0.541	0
	y_0	y_1	y_2

Now, using trapezoidal rule

$$I\left(\frac{\pi}{4}\right) = \frac{h}{2}[y_0 + y_2 + 2y_1]$$

$$= \frac{\pi}{2 \times 4}[1 + 0 + 2(0.541)] = 0.8175$$

ii) Taking $h = \frac{\pi}{8}$

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y	1	0.785	0.541	0.275	0
	y_0	y_1	y_2	y_3	y_4

$$I\left(\frac{\pi}{8}\right) = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{\pi}{2 \times 16}[1 + 0 + 2(0.785 + 0.541 + 0.275)]$$

$$= 0.8250$$

iii) Taking $h = \frac{\pi}{16}$

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$	$\frac{5\pi}{16}$	$\frac{3\pi}{8}$	$\frac{7\pi}{16}$	$\frac{\pi}{2}$
y	1	0.897	0.785	0.667	0.541	0.410	0.275	0.138	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$$I = \left(\frac{\pi}{16}\right) = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{\pi}{2 \times 16}[1 + 0 + 2(0.897 + 0.785 + 0.667 + 0.541 + 0.410 + 0.275 + 0.138)]$$

$$= 0.8272$$

Now, optimizing values by Romberg Integration

$$I\left(\frac{\pi}{4}, \frac{\pi}{8}\right) = \frac{1}{3}\left[4I\left(\frac{\pi}{8}\right) - I\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{3}[4 \times 0.8250 - 0.8175] = 0.8275$$

$$I\left(\frac{\pi}{8}, \frac{\pi}{16}\right) = \frac{1}{3}\left[4I\left(\frac{\pi}{16}\right) - I\left(\frac{\pi}{8}\right)\right]$$

$$= \frac{1}{3}[4(0.8279) - (0.8250)] = 0.8279$$

$$I\left(\frac{\pi}{4}, \frac{\pi}{8}, \frac{\pi}{16}\right) = \frac{1}{3}\left[4I\left(\frac{\pi}{8}, \frac{\pi}{16}\right) - I\left(\frac{\pi}{4}, \frac{\pi}{8}\right)\right]$$

$$= \frac{1}{3}[4 \times 0.8272 - 0.8275] = 0.8280$$

Hence the value of integral $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin x}}dx = 0.8280$

Also, $I_{abs} = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin x}}dx = 0.8284$

12. **A slider in a machine moves along a fixed straight rod 9 + s distance x (cm) along the rod is given below for various values of time t seconds. Find the velocity and the acceleration of the slider when t = 0.2.** **[2017/Spring]**

t	0	0.1	0.2	0.3
x	30.13	31.62	32.87	33.95

Solution:

Creating difference table from given data

x = t	y = x	1st diff	2nd diff	3rd diff
0	30.13			
		1.49		
0.1	31.62		−0.24	
		1.25		0.07
0.2	32.87		−0.17	
		1.08		
0.3	33.95			

Here, the data of t is equispaced and t = 0.2 lies near the end of the table so using Newton's backward formula for numerical differentiation.

$$h = 0.3 - 0.2 = 0.1$$

Now, at t = 0.2

From, numerical differentiation using Newton's backward formula

$$y' = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2}\right] = \frac{1}{0.1}\left[1.25 + \frac{-0.24}{2}\right]$$

∴ $y' = 11.3$ cm/s is the required velocity of an object.

Now, for acceleration

$$y'' = \frac{1}{h^2}[\nabla^2 y_n] = \frac{1}{0.1^2} \times -0.24$$

∴ $y'' = -24$ cm/s²

is the required acceleration of an object

13. **The velocity 'v' of a particle at a distance 's' from a point on its path is given is given by the following table.**

s(m)	0	10	20	30	40	50	60
v(m/s)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 metres by using Simpson's $\frac{1}{3}$ rule and Simpson's $\frac{3}{8}$ rule. **[2017/ Spring]**

Solution:

We have,

$$v = \frac{ds}{dt}$$

$$dt = \frac{1}{v}ds = y \cdot ds \;\boxed{\Rightarrow y = \frac{1}{v}}$$

On integration,

$$t = \int_0^{60} y \cdot ds$$

Here; $a = 0, b = 60, n = 6$

so, $h = \frac{60 - 0}{6} = 10$

Creating table at the interval of 10 from 0 to 60.

x = s	0	10	20	30	40	50	60
$y = \frac{1}{v}$	$\frac{1}{47}$	$\frac{1}{58}$	$\frac{1}{64}$	$\frac{1}{65}$	$\frac{1}{61}$	$\frac{1}{52}$	$\frac{1}{38}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now, by Simpson's $\frac{1}{3}$ rule,

$$\therefore \quad I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{10}{3}\left[\frac{1}{47} + \frac{1}{38} + 4\left(\frac{1}{58} + \frac{1}{65} + \frac{1}{52}\right) + 2\left(\frac{1}{64} + \frac{1}{61}\right)\right] = 1.063$$

Again, by Simpson's $\frac{3}{8}$ rule

$$\therefore \quad I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3 \times 10}{8}\left[\frac{1}{47} + \frac{1}{38} + 3\left(\frac{1}{58} + \frac{1}{64} + \frac{1}{61} + \frac{1}{52}\right) + 2\left(\frac{1}{65}\right)\right] = 1.064 \text{ s}$$

14. **Evaluate the integral $I = \int_0^{\frac{\pi}{2}} (1 + 3\cos 2x)\,dx$. Compare the result in both conditions for Simpson $\frac{1}{3}$ and $\frac{3}{8}$ rule.** **[2018/Fall]**

Solution:

Given that:

$$I = \int_0^{\frac{\pi}{2}} (1 + 3\cos 2x)\,dx$$

$$a = 0, b = \frac{\pi}{2}, n = 6$$

$$h = \frac{b - a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

Now, table is created at the interval of $\frac{\pi}{12}$

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	4	3.598	2.5	1	−0.5	−1.598	−2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now, by Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{3 \times 12}[4 + (-2) + 4(3.598 + 1 - 1.598) + 2(2.5 - 0.5)]$$

$$= 1.57079$$

Again, by Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3)]$$

$$= \frac{3\pi}{8 \times 12}[4 + (-2) + 3(3.598 + 2.5 - 0.5 - 1.598) + 2(1)]$$

$$= 1.57079$$

Also, $I_{abs} = \int_0^{\frac{\pi}{2}} (1 + 3\cos 2x)\, dx = 1.57079$

Now, Error by Simpson $\frac{1}{3}$ rule $= |1.57079 - 1.57079| = 0$

Error by Simpson $\frac{3}{8}$ rule $= |1.57079 - 1.57079| = 0$

Hence, the Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule is accurate with zero error.

15. From the following table of values of x and y, obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for x = 1.2.

x	1.0	1.2	1.4	1.6	1.8
y	2.7183	3.3201	4.0552	4.9530	6.0496

[2018/Spring]

Solution:

Creating difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	2.7183				
		0.6018			
1.2	**3.3201**		0.1333		
		0.7351		0.0294	
1.4	4.0552		**0.1627**		0.0067
		0.8978		**0.0361**	
1.6	4.9530		0.1988		
		1.0966			
1.8	6.0496				

Here, the data of x is equispaced and x = 1.2 lies near the starting of table so using Newton's forward formula for numerical differentiation.

$$h = 1.2 - 1.0 = 0.2$$

Now, at x = 1.2.

From numerical differentiation, using Newton's forward formula

$$\frac{dy}{dx} = y' = \frac{1}{h}\left[\Delta y_n - \frac{\Delta^2 y_n}{2} + \frac{\Delta^3 y_n}{3}\right]$$

$$= \frac{1}{0.2}\left[0.7351 - \frac{0.1627}{2} + \frac{0.0361}{3}\right]$$

$$\therefore \quad y' = 3.328$$

Again, for $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = y'' = \frac{1}{h^2}[\Delta^2 y_n - \Delta^3 y_n]$$

$$= \frac{1}{0.2^2}[0.1627 - 0.0361]$$

$$\therefore \quad y'' = 3.165$$

16. The following data gives corresponding values of pressure 'p' and specific volume 'v' of steam.

p	105	42.7	25.3	16.7	13
v	2	4	6	8	10

Find the rate of change of volume when pressure is 105 and 13.

[2018/Fall]

Solution:

As the values of p are not equispaced, we use Newton's divided difference formula.

The divided difference table is

	x = p	y = v	1st diff	2nd diff	3rd diff	4th diff
x_0	105	2				
			−0.0321			
				0.0010		
x_1	42.7	4			**-3.96×10^{-5}**	
			−0.1149			**7.06×10^{-6}**
				0.0045		
x_2	25.3	6			-6.90×10^{-4}	
			−0.2325			
				0.0250		
x_3	16.7	8				
			−0.5405			
x_4	13	10				

Now, Newton's divided formula for the 1st derivative.

We get,

$$f'(x) = \frac{dv}{dp} = [x_0, x_1] + (2x - x_0 - x_1)[x_0, x_1, x_2]$$
$$+ [3x^2 - 2x(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_2x_0][x_0, x_1, x_2, x_3]$$
$$+ [4x^3 - 3x^2(x_0 + x_1 + x_2 + x_3)$$
$$+ 2x(x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 + x_1x_3 + x_0x_2)$$
$$- (x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_0x_1x_3)][x_0, x_1, x_2, x_3, x_4]$$

Now, when pressure is 105

$$\frac{dv}{dp}_{at\ p=105} = -0.0321 + (2(105) - 105 - 42.7)(0.0010)$$
$$+ [3(105)^2 - 2(105)(105 + 42.7 + 25.3) + (105 \times 42.7)$$
$$+ (42.7 \times 25.3) + (25.3 \times 105)](-3.96 \times 10^{-5})$$
$$+ [4(105)^3 - 3(105)^2(105 + 42.7 + 25.3 + 16.7)$$
$$+ 2(105)(105 \times 42.7 + 42.7 \times 25.3 + 25.3 \times 16.7$$
$$+ 16.7 \times 105 + 42.7 \times 16.7 + 105 \times 25.3)$$
$$- (105 \times 42.7 \times 25.3 + 42.7 \times 25.3 \times 16.7$$
$$+ 25.3 \times 16.7 \times 105 + 105 \times 42.7 \times 16.7)](7.06 \times 10^{-6})$$
$$= 2.9289$$

Similarly when pressure is 13, using x = 13 in the formula, we get,

$$\frac{dV}{dp}_{at\ p=13} = -0.6689$$

17. Evaluate $\int_{-2}^{2} \frac{x}{x + 2e^x} dx$ by using trapezoidal, Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule with n = 6. **[2019/Fall]**

Solution:

Given that;

$$I = \int_{-2}^{2} \frac{x\,dx}{x + 2e^x}$$

$$a = -2, b = 2, n = 6$$

Then,

$$h = \frac{b-a}{n} = \frac{2+2}{6} = \frac{2}{3}$$

Now, table is created at the interval of $\frac{2}{3}$ from –2 to 2.

x	–2	$\frac{-4}{3}$	$\frac{-2}{3}$	0	$\frac{2}{3}$	$\frac{4}{3}$	2
y	1.156	1.653	–1.850	0	0.146	0.149	0.119
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now, by trapezoidal rule,

$$I = \frac{h}{2}[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$
$$= \frac{2}{2 \times 3}[1.156 + 0.119 + 2(1.653 - 1.850 + 0 + 0.146 + 0.149)] = 0.490$$

Again, by Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$
$$= \frac{2}{3 \times 3}[1.56 + 0.119 + 4(1.653 + 0 + 0.149) + 2(-1.850 + 0.146)]$$
$$= 1.1277$$

And, by Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$
$$= \frac{3 \times 2}{8 \times 3}[1.156 + 0.119 + 3(1.653 - 1.850 + 0.146 + 0.149) + 2 \times 0]$$
$$= 0.3922$$

18. Using three-point Gaussian Quadrature formula, evaluate, **[2019/Fall]**

$$\int_0^1 \frac{dx}{1+x}$$

Solution:

Given that;

$$I = \int_0^1 \frac{dx}{1+x}$$

Using gauss quadrature formula with n = 3.

Since limit a = 0 and b = 1 is not from –1 to 1 so using,

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$$

or, $$x = \frac{1}{2}(1-0)u + \frac{1}{2}(1+0) \quad ..... (1)$$

∴ $$x = \frac{u}{2} + \frac{1}{2} \quad ..... (2)$$

Differentiating on both sides

$$dx = \frac{du}{2}$$

Now, substituting the values from (1) and (2) to I,

$$I = \int_{-1}^{1} \frac{\frac{du}{2}}{1 + \left(\frac{u}{2} + \frac{1}{2}\right)} = \int_{-1}^{1} \frac{du}{3+u}$$

Now, Gauss formula for n = 3 is

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{5}{3}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9} \times \frac{1}{3} + \frac{5}{9}\left[\frac{1}{3 - \sqrt{\frac{3}{5}}} + \frac{1}{3 + \sqrt{\frac{3}{5}}}\right]$$

$\therefore$ $I = 0.69312$

19. The following table gives the velocity of a velocity at various points of time.

Time, t(seconds)	1	2	4	5
Velocity, v(m/s)	0.25	1	2.2	4

Find the acceleration of the vehicle at t = 1.1 second and t = 2.5 second using any suitable differential formula. **[2019/Spring]**

Solution:

As the values of time are not equispaced, we use Newton's divided difference formula.

The divided difference table is

	x = t	y = v	1st diff	2nd diff	3rd diff
x_0	1	0.25			
			0.75		
x_1	2	1		−0.05	
			0.6		0.1125
x_2	4	2.2		0.4	
			1.8		
x_3	5	4			

From Newton's divided formula for the 1st derivative, we get,

$$f'(x) = [x_0, x_1] + (2x - x_0 - x_1)[x_0, x_1, x_2] + [3x^2 - 2x(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_2x_0][x_0, x_1, x_2, x_3]$$

Now, when t = 1.1

$$f'(x)_{1.1} = 0.75 + [2(1.1) - 1 - 2](-0.05) + [3(1.1)^2 - 2(1.1)(1 + 2 + 4) + (1)(2) + (2)(4) + (1)(4)](0.1125)$$

$$= 0.75 + 0.04 + 0.2508$$

$\therefore$ $f'(x)_{1.1} = 1.0408$ is the required acceleration in m/s^2

Again, when t = 2.5

$$f'(x)_{2.5} = 0.75 + 2(2.5) - 1 - 2)(-0.05) + [3(2.5)^2 - 2(2.5)(1 + 2 + 4) + (1)(2) + (2)(4) + (1)(4)](0.1125)$$

$$= 0.75 - 0.1 - 0.2531$$

$= 0.3969\ m/s^2$ is the required acceleration.

20. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin u}{u} du$ by using trapezoidal, Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule with n = 6. **[2019/Spring]**

Solution:

Given that;

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin u}{u} du$$

$$a = 0, b = \frac{\pi}{2}, n = 6$$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

Now, table is created at the interval of $\frac{\pi}{12}$ from 0 to $\frac{\pi}{2}$

x = u	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	1	0.988	0.954	0.9	0.826	0.737	0.636
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

NOTE:

At $x = u = 0$, $\frac{\sin u}{u} = \frac{0}{0}$, so we use L-Hopital's rule for 0.

Rest of the values are normally calculated.

Now, by trapezoidal rule,

$$I = \frac{h}{2}[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{24}[1 + 0.636 + 2(0.988 + 0.954 + 0.9 + 0.826 + 0.737)]$$

$$= 1.367$$

Again, by Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{36}[1 + 0.636 + 4(0.988 + 0.9 + 0.737) + 2(0.954 + 0.826)]$$

$$= 1.369$$

And, by Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\pi}{96}[1 + 0.636 + 3(0.988 + 0.954 + 0.826 + 0.737) + 2(0.9)]$$

$$= 1.369$$

21. Use Gauss-Legendre 2-point and 3 point formula to evaluate;

$$\int_{0.5}^{1.5} e^{x^2}\,dx$$

[2019/Spring]

Solution:

Given that;

$$I = \int_{0.5}^{1.5} e^{x^2}\,dx$$

Since limit a = 0.5 and b = 1.5 is not from −1 to 1

so, $x = \frac{1}{2}(b-a)\,u + \frac{1}{2}(b+a)$

or, $x = \frac{1}{2}(1.5-0.5)\,u + \frac{1}{2}(1.5+0.5)$

or, $x = \frac{u}{2} + 1$ ....(1)

Differentiating on both sides

$$dx = \frac{du}{2} \quad \text{....(2)}$$

Then, substituting the values from (1) and (2) to I,

$$I = \int_{-1}^{1} \frac{e^{(\frac{u}{2}+1)^2}}{2}\,du$$

Now,

i) Gauss formula for n = 2 is

$$I = \int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{e^{(-\frac{1}{2\sqrt{3}}+1)^2}}{2} + \frac{e^{(\frac{1}{2\sqrt{3}}+1)^2}}{2}$$

$$= 0.829 + 2.631$$

$$= 3.46$$

ii) Gauss formula for n = 3 is

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9}\left(\frac{e^{(0+1)^2}}{2}\right) + \frac{5}{9}\left[\frac{e^{(-\frac{1}{2}\sqrt{3/5}+1)^2}}{2} + \frac{e^{(\frac{1}{2}\sqrt{3/5}+1)^2}}{2}\right]$$

$$= 1.208 + 2.307$$

$\therefore$ I = 3.515

22. Obtain divided difference table for the given data set **[2019/Fall]**

e	−1	2	5	7
y	−8	3	1	12

Solution:

Creating the divided difference table

x	y	1st diff	2nd diff	3rd diff
−1	−8			
		$\frac{3+8}{2+1} = 3.667$		
2	3		$\frac{-0.667-3.667}{5+1} = -0.722$	
		$\frac{1-3}{5-2} = -0.667$		$\frac{1.233+0.722}{7+1} = 0.244$
5	1		$\frac{5.5+0.667}{7-2} = 1.233$	
		$\frac{12-1}{7-5} = 5.5$		
7	12			

23. Integrate the given integral using Romberg integration, **[2020/Fall]**

$$\int_{1}^{2} \frac{1}{1+x^3}\,dx$$

Solution:

Given that;

$$I = \int_{1}^{2} \frac{1}{1+x^3}\,dx$$

Here, a = 1, b = 2

i) Taking h = 0.5

x	1	1.5	2
y	0.5	0.228	0.111
	y_0	y_1	y_2

Now using Trapezoidal rule

$$I(0.5) = \frac{h}{2}[y_0 + y_2 + 2y_1]$$

$$= \frac{0.5}{2}[0.5 + 0.111 + 2(0.228)] = 0.266$$

ii) Taking h = 0.25

x	1	1.25	1.5	1.75	2
y	0.5	0.338	0.228	0.157	0.111
	y_0	y_1	y_2	y_3	y_4

Now, using Trapezoidal rule

$$I(0.25) = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.25}{2}[0.5 + 0.111 + 2(0.338 + 0.228 + 0.157)] = 0.257$$

iii) Taking h = 0.125

x	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
y	0.5	0.412	0.338	0.277	0.228	0.188	0.157	0.131	0.111
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Now, using Trapezoidal rule

$$I(0.125) = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.125}{2}[0.5 + 0.111 + 2(0.412 + 0.338 + 0.277 + 0.228 + 0.118 + 0.157 + 0.131)]$$

$$= 0.254$$

Now, optimizing values by Romberg Integration

$$I(0.5, 0.25) = \frac{1}{3}[4\,I(0.25) - I(0.5)]$$

$$= \frac{1}{3}[4(0.257) - 0.266]$$

$$= 0.254$$

$$I(0.25, 0.125) = \frac{1}{3}[4I(0.125) - I(0.25)]$$

$$= \frac{1}{3}[4(0.254) - 0.257]$$

$$= 0.253$$

$$I(0.5, 0.25, 0.125) = \frac{1}{3}[4I(0.25, 0.125) - I(0.5, 0.25)]$$

$$= \frac{1}{3}[4(0.253) - 0.254]$$

$$= 0.252$$

Hence the value of integral $\int_1^2 \frac{1}{1+x^3}dx = 0.252$

Also, $I_{abs} = \int_1^2 \frac{1}{1+x^3}dx = 0.2543$

24. Compute the integral using Gaussian 3-point formula.

$$\int_2^5 \frac{e^x + \sin x}{1+x^2}dx$$

Solution: **[2020/Fall]**

Given that;

$$I = \int_2^5 \frac{e^x + \sin x}{1+x^2}dx$$

Since limit a = 2 and b = 5 is not from –1 to 1,

so, $x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$

or, $x = \frac{1}{2}(5-2)u + \frac{1}{2}(5+2)$

or, $x = \frac{3}{2}u + \frac{7}{2}$ ..... (1)

Differentiating on both sides, we get,

$dx = \frac{3}{2}du$ ..... (2)

Then, substituting the values from (1) and (2) to I,

$$I = \int_{-1}^{1} \frac{e^{\frac{3u+7}{2}} + \sin\left(\frac{3u+7}{2}\right)}{1+\left(\frac{3u+7}{2}\right)^2}\cdot\frac{3}{2}du$$

Now, using Gaussian 3-point formula,

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9}\left[\frac{e^{(7/2)} + \sin^{(7/2)}}{1+\left(\frac{7}{2}\right)^2}\cdot\frac{3}{2}\right] + \frac{5}{9}\left[\left(\frac{3}{2}\cdot\frac{e^{\frac{-3\sqrt{3/5}+7}{2}} + \sin\left(\frac{-3\sqrt{3/5}+7}{2}\right)}{1+\left(\frac{-3\sqrt{3/5}+7}{2}\right)^2}\right)\right]$$

$$+\left[\left(\frac{3}{2}\cdot\frac{e^{\frac{3\sqrt{3/5}+7}{2}} + \sin\left(\frac{3\sqrt{3/5}+7}{2}\right)}{1+\left(\frac{3\sqrt{3/5}+7}{2}\right)^2}\right)\right]$$

$$= 3.297 + 5.271$$

$\therefore$ $I = 8.568$

25. Write short notes on Romberg integration.

[2013/Fall, 2015/Fall, 2015/Spring]

Solution: See the topic 3.6.

iii) Taking h = 0.125

x	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
y	0.5	0.412	0.338	0.277	0.228	0.188	0.157	0.131	0.111
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Now, using Trapezoidal rule

$$I(0.125) = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.125}{2}[0.5 + 0.111 + 2(0.412 + 0.338 + 0.277 + 0.228 + 0.118 + 0.157 + 0.131)]$$

$$= 0.254$$

Now, optimizing values by Romberg Integration

$$I(0.5, 0.25) = \frac{1}{3}[4\,I(0.25) - I(0.5)]$$

$$= \frac{1}{3}[4(0.257) - 0.266]$$

$$= 0.254$$

$$I(0.25, 0.125) = \frac{1}{3}[4I\,(0.125) - I(0.25)]$$

$$= \frac{1}{3}[4(0.254) - 0.257]$$

$$= 0.253$$

$$I\,(0.5, 0.25, 0.125) = \frac{1}{3}[4I(0.25, 0.125) - I(0.5, 0.25)]$$

$$= \frac{1}{3}[4(0.253) - 0.254]$$

$$= 0.252$$

Hence the value of integral $\int_1^2 \frac{1}{1+x^3}dx = 0.252$

Also, $I_{abs} = \int_1^2 \frac{1}{1+x^3}dx = 0.2543$

24. Compute the integral using Gaussian 3-point formula.

$$\int_2^5 \frac{e^x + \sin x}{1+x^2}dx$$

[2020/Fall]

Solution:

Given that;

$$I = \int_2^5 \frac{e^x + \sin x}{1+x^2}dx$$

Since limit a = 2 and b = 5 is not from −1 to 1,

so, $x = \frac{1}{2}(b-a)\,u + \frac{1}{2}(b+a)$

or, $x = \frac{1}{2}(5-2)\,u + \frac{1}{2}(5+2)$

or, $x = \frac{3}{2}u + \frac{7}{2}$ ..... (1)

Differentiating on both sides, we get,

$dx = \frac{3}{2}du$ ..... (2)

Then, substituting the values from (1) and (2) to I,

$$I = \int_{-1}^{1} \frac{e^{\frac{3u+7}{2}} + \sin\left(\frac{3u+7}{2}\right)}{1+\left(\frac{3u+7}{2}\right)^2} \cdot \frac{3}{2}du$$

Now, using Gaussian 3-point formula,

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9}\left[\frac{e^{(7/2)} + \sin^{(7/2)}}{1+\left(\frac{7}{2}\right)^2} \cdot \frac{3}{2}\right] + \frac{5}{9}\left[\left(\frac{3}{2} \cdot \frac{e^{\frac{-3\sqrt{3/5}+7}{2}} + \sin\left(\frac{-3\sqrt{3/5}+7}{2}\right)}{1+\left(\frac{-3\sqrt{3/5}+7}{2}\right)^2}\right)\right.$$

$$\left. + \left(\frac{3}{2} \cdot \frac{e^{\frac{3\sqrt{3/5}+7}{2}} + \sin\left(\frac{3\sqrt{3/5}+7}{2}\right)}{1+\left(\frac{3\sqrt{3/5}+7}{2}\right)^2}\right)\right]$$

$$= 3.297 + 5.271$$

∴ $I = 8.568$

25. Write short notes on Romberg integration. **[2013/Fall, 2015/Fall, 2015/Spring]**

Solution: See the topic 3.6.

ADDITIONAL QUESTION SOLUTION

1. Evaluate $\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx$ using Gaussian 3-point formula.

Solution:

Given that;

$$I = \int_0^{\frac{\pi}{2}} e^{\sin x}\,dx$$

Since limit a = 0 and b = $\frac{\pi}{2}$ is not from −1 to 1.

so, $x = \frac{1}{2}(b-a)\,u + \frac{1}{2}(b+a)$

or, $x = \frac{1}{2}\left(\frac{\pi}{2} - 0\right)u + \frac{1}{2}\left(\frac{\pi}{2} + 0\right)$

or, $x = \frac{\pi}{4}u + \frac{\pi}{4}$ ..... (1)

Differentiating on both sides, we get,

$$dx = \frac{\pi}{4}\,du \quad \text{..... (2)}$$

Then, substituting the values from (1) and (2) to I

$$I = \int_{-1}^{1} e^{\sin\frac{\pi}{4}(u+1)} \cdot \frac{\pi}{4}\,du$$

Now, using Gaussian 3-point formula

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9}\left[\frac{\pi}{4} \cdot e^{\sin\frac{\pi}{4}}\right] + \frac{5}{9}\left[\left(\frac{\pi}{4} \cdot e^{\sin\frac{\pi}{4}(-\sqrt{3/5}+1)}\right) + \left(\frac{\pi}{4} \cdot e^{\sin\frac{\pi}{4}(\sqrt{3/5}+1)}\right)\right]$$

$$= 1.4159 + 1.6880$$

$\therefore$ $I = 3.1039$

2. Estimate the value of cos (1.74) from the following data:

x	1.7	1.74	1.78	1.82	1.86
sin x	0.9916	0.9857	0.9781	0.9691	0.9584

Solution:

Here the data of x are equispaced and x = 1.74 lies near the starting of table so using Newton's forward formula for numerical differentiation.

Creating the difference table,

x	y = sin x	Δy	Δ²y	Δ³y	Δ⁴y
1.7	0.9916				
		−0.0059			
1.74	0.9857		−0.0017		
		−0.0076		0.0003	
1.78	0.9781		−0.0014		−0.0006
		−0.0090		−0.0003	
1.82	0.9691		−0.0017		
		−0.0107			
1.86	0.9584				

h = 1.74 − 1.7 = 0.04

Now, at x = 1.74,

From Newton's forward formula for numerical differentiation

$$\frac{dy}{dx} = y' = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3}\right]$$

$$= \frac{1}{0.04}\left[-0.0076 + \frac{0.0014}{2} - \frac{0.0003}{3}\right]$$

$$= -0.1750$$

Hence, cos (1.74) = −0.1750

3. Find f'(3) from the following table:

x	2	4	8	12	16
f(x)	20	23	30	35	40

Solution:

Here, the data of x are not equispaced, we shall use Newton's divided difference formula.

Then, creating difference table

x	y = f(x)	1st diff	2nd diff	3rd diff	4th diff
2	20				
		1.5			
4	23		0.0417		
		1.75		−0.0104	
8	30		−0.0625		0.0011
		1.25		0.0052	
12	35		0		
		1.25			
16	40				

Now, using Newton's divided difference formula,

$$f'(x) = [x_0, x_1] + (2x - x_0 - x_1)\,[x_0, x_1, x_2]$$
$$+ [3x^2 - 2x\,(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_2x_0][x_0, x_1, x_2, x_3]$$

$$+ [4x^3 - 3x^2(x_0 + x_1 + x_2 + x_3) + 2x(x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 + x_1x_3 + x_0x_2) - (x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_0x_1x_3)][x_0, x_1, x_2, x_3, x_4]$$

At $x = 3$,

$$= 1.5 + (6 - 2 - 4)(0.0417) + [27 - 6(2 + 4 + 8) + 8 + 32 + 16](-0.0104) + [108 - 27(2 + 4 + 8 + 12) + 6(8 + 32 + 96 + 24 + 48 + 16) - (64 + 384 + 192 + 96)](0.0011)$$

$$= 1.5 + 0 + 0.0104 + 0.0154$$

$\therefore \quad f'(3) = 1.5258$

4. Evaluate $\int_2^4 e^{-x^2} dx$ using 2-point Gauss Legendre method.

Solution:

Given that;

$$I = \int_2^4 e^{-x^2} dx$$

Since limit a = 2 and b = 4 is not from −1 to 1, so using,

$$x = \frac{1}{2}(b - a)u + \frac{1}{2}(b + a)$$

or, $x = \frac{1}{2}(4 - 2)u + \frac{1}{2}(4 + 2)$

or, $x = u + 3$ ..... (1)

Differentiating on both sides, we get,

$dx = du$ ..... (2)

Substituting the values from (1) and (2) to I,

$$I = \int_{-1}^{1} e^{-(u+3)^2} du$$

Now, using 2-point Gauss Legendre method

$$I = \int_{-1}^{1} e^{-(u+3)^2} du$$

$$= f\left(-\sqrt{\frac{1}{3}}\right) + f\left(\sqrt{\frac{1}{3}}\right)$$

$$= e^{-\left(\frac{-1}{\sqrt{3}}+3\right)^2} + e^{-\left(\frac{1}{\sqrt{3}}+3\right)^2}$$

$$= 0.0028$$

5. Evaluate the following using Simpson's $\frac{1}{3}$ rule. (take h = 0.2)

$$\int_0^2 \frac{4e^x}{1 + x^3} dx$$

Solution:

Given that;

$$I = \int_0^2 \frac{4e^x}{1 + x^3} dx$$

$h = 0.2$

Table is created at the interval of 0.2 from 0 to 2.

x	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
y	4	4.8468	5.6084	5.9938	5.8877	5.4366	4.8682	4.3325	3.8878	3.5419	3.2840
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

Now, using Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + y_{10} + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$

$$= \frac{0.2}{3}[(4 + 3.2840 + 4(4.8468 + 5.9938 + 5.4366 + 4.3325 + 3.5419) + 2(5.6084 + 5.8877 + 4.8682 + 3.8878)]$$

$\therefore \quad I = 9.6263$

Also,

$$I_{abs} = \int_0^2 \frac{4e^x}{1 + x^3} dx = 9.62615$$

6. Evaluate $\int_0^2 f(x)\, dx$, for the function $f(x) = e^x + \sin 2x$ using composite Simpson's $\frac{3}{8}$ formula taking step h = 0.4.

Solution:

Given that;

$$I = \int_0^2 f(x)\, dx = \int_0^2 e^x + \sin 2x\, dx$$

$h = 0.4$

Table is created at the interval of 0.4 from o to 2,

x	0	0.4	0.8	1.2	1.6	2
y	1	2.2092	3.2251	3.9956	4.8747	6.6323
	y_0	y_1	y_2	y_3	y_4	y_5

Now, using Simpson's $\frac{3}{8}$ formula,

$$I = \frac{3h}{8}[y_0 + y_5 + 3(y_1 + y_2 + y_4) + 2y_3]$$

$$= \frac{3(0.4)}{8}[1 + 6.6323 + 3(2.2092 + 3.2251 + 4.8947) + 2(3.9956)]$$

$\therefore \quad I = 6.9916$

Also, $I_{abs} = \int_0^2 e^x + \sin 2x\, dx = \left[e^x - \frac{\cos 2x}{2}\right]_0^2 = 7.2159$

7. Evaluate the following integral using Romberg method corrected to two decimal places.

$$\int_0^2 \frac{e^x + \sin x}{1 + x^2} dx$$

Solution:

Given that;

$$I = \int_0^2 \frac{e^x + \sin x}{1 + x^2}\,dx$$

Here, a = 0 and b = 2

i) Taking h = 1 and creating interval of 1 from 0 to 2.

x	0	1	2
y	1	1.7799	1.6597
	y_0	y_1	y_2

Now, using Trapezoidal rule,

$$I(1) = \frac{h}{2}[y_0 + y_2 + 2y_1]$$

$$= \frac{1}{2}[1 + 1.6597 + 2(1.7799)]$$

$$= 3.1098$$

ii) Taking h = 0.5 and creating interval of 0.5 from 0 to 2

x	0	0.5	1	1.5	2
y	1	1.7025	1.7799	1.6859	1.6597
	y_0	y_1	y_2	y_3	y_4

Now, using Trapezoidal rule,

$$I(0.5) = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.5}{2}[1 + 1.6597 + 2(1.7025 + 1.7799 + 1.6859)]$$

$$= 3.2491$$

iii) Taking h = 0.25 and creating interval of 0.25 from 0 to 2

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
y	1	1.4413	1.7025	1.7911	1.7799	1.7324	1.6859	1.6587	1.6597
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Now, using Trapezoidal rule,

$$I(0.25) = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.25}{2}[1 + 1.6597 + 2(1.4413 + 1.7025 + 1.7911 + 1.7799 + 1.7324 + 1.6859 + 1.6587)]$$

$$= 3.2804$$

Now, optimizing values by Romberg Integration,

$$I(1, 0.5) = \frac{1}{3}[4I(0.5) - I(1)]$$

$$= \frac{1}{3}[4(3.2491) - 3.1098]$$

$$= 3.2955$$

$$I(0.5, 0.25) = \frac{1}{3}[4I(0.25) - I(0.5)]$$

$$= \frac{1}{3}[4(3.2804) - 3.2491]$$

$$= 3.2908$$

$$I(1, 0.5, 0.25) = \frac{1}{3}[4I(0.5, 0.25) - I(1, 0.5)]$$

$$= \frac{1}{3}[4(3.2908) - 3.2955]$$

$$= 3.2892 \approx 3.290$$

Hence, the value of integral is 3.290.

8. **The distance travelled by a vehicle at intervals of 2 minutes are given as follows:**

Time (min)	2	4	6	8	10	12
Distance (km)	0.25	1	2.2	4	6.5	8.5

Evaluate the velocity and acceleration of the vehicle at t = 3 minutes.

Solution:

Here, the data of time is equispaced and t = 3 min lies near the starting of table. So, we use Newton's forward formula for numerical differentiation.

Creating difference table

x = time	y = distance	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2	0.25					
		0.75				
			0.45			
4	1			0.15		
		1.2			-0.05	
			0.6			-1.25
6	2.2			0.1		
		1.8			-1.3	
			0.7			
8	4			-1.2		
		2.5				
			-0.5			
10	6.5					
		2				
12	8.5					

NOTE:
We cannot use the Newton's forward differentiation formula directly because t = 3 is not available in the table.

We have,

$$x = x_0 + ph \text{ at } x = 3, x_0 = 2, h = 4 - 2 = 2$$

or, $3 = 2 + 2p$

$\therefore$ $p = 0.5$

Let, $x = x_0 + ph$ --- (1)

And, using Newton's forward interpolation formula,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y_0 \quad \text{--- (2)}$$

Now, differentiating (1) and (2) with respect to p, we get,

$$\frac{dx}{dp} = h$$

$$\frac{dy_p}{dp} = 0 + \Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6}\Delta^3 y_0 + \frac{(4p^3-18p^2-22p-6)}{24}\Delta^4 y_0 + \frac{(5p^4-40p^3+105p^2-100p+24)}{120}\Delta^5 y_0$$

Then,

$$y_p' = \frac{dy_p}{dp} \times \frac{dp}{dx} = \frac{dy_p}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6}\Delta^3 y_0 + \frac{(4p^3-18p^2-22p-6)}{24}\Delta^4 y_0 + \frac{(5p^4-40p^3+105p^2-100p+24)}{120}\Delta^5 y_0\right]$$

Substituting the values, we obtain,

$$y_p' = \frac{1}{2}[0.75 + 0 - 0.0063 - 0.0021 + 0.0462]$$

$\therefore$ $y_p' = 0.3939$ is the required velocity at $t = 3$ minutes.

Now, for acceleration differentiating y_p' with respect to p, we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\cdot\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2}\Delta^2 y_0 + \frac{(6p-6)}{6}\Delta^3 y_0 + \frac{(12p^2-36p+22)}{24}\Delta^4 y_0 + \frac{(20p^3-120p^2+210p-100)}{120}\Delta^5 y_0\right]$$

$$= \frac{1}{2^2}[0.45 - 0.075 - 0.0146 + 0.2344]$$

$\therefore$ $y_p'' = 0.1487$ is the required acceleration at $t = 3$ minutes

9. A rod is rotating in a plain. The following table gives the angle in radians (q) through which the rod has turned for various values of time in seconds (t). Find the angular velocity and angular acceleration t = 0.2.

t	0	0.2	0.4	0.6	0.8
θ	0	0.122	0.493	0.123	2.022

Solution:

Here, the data of time is equispaced and $t = 0.2$ lies near the starting of table so we use Newton's forward differentiation formula.

Creating difference table:

$x = t$	$y = \theta$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0				
		0.122			
0.2	0.122		0.249		
		0.3710		−0.99	
0.4	0.493		−0.741		4
		−0.37		3.01	
0.6	0.123		2.269		
		1.899			
0.8	2.022				

$h = 0.2 - 0 = 0.2$

At $t = 0.2$,

From numerical differentiation, using Newton's forward formula.

$$\frac{dy}{dx} = y' = \frac{1}{h}\left[\Delta y_n - \frac{\Delta^2 y_n}{2} + \frac{\Delta^3 y_n}{3}\right]$$

$$= \frac{1}{0.2}\left[0.3710 - \frac{0.741}{2} + \frac{3.01}{3}\right]$$

$\therefore$ $y' = 8.7242$ is the required angular velocity.

Again, for $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = y'' = \frac{1}{h^2}[\Delta^2 y_n - \Delta^3 y_n]$$

$$= \frac{1}{0.2^2}[-0.741 - 3.01]$$

$\therefore$ $y'' = -98.775$ is the required angular acceleration.

10. Evaluate $\int_0^{1.4} (\sin x^3 + \cos x^2)\,dx$ using Gaussian 3-point formula.

Solution:

Given that;

$$I = \int_0^{1.4} (\sin x^3 + \cos x^2)\,dx$$

Since limit a = 0 and b = 1.4 is not from −1 to 1,

so, $x = \frac{1}{2}(b - a)\,u + \frac{1}{2}(b + a)$

or, $x = \frac{1}{2}(1.4 - 0)\,u + \frac{1}{2}(1.4 + 0)$

or, $x = 0.7u + 0.7$ ..... (1)

Differentiating on both sides, we get,

$$dx = 0.7\,du \quad \text{..... (2)}$$

Substituting the values from (1) and (2) to I,

$$I = \int_{-1}^{1} (\sin (0.7u + 0.7)^3 + \cos (0.7u + 0.7)^2 \,(0.7)\, du$$

Now, using Gaussian 3-point formula

$$I = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{5}{3}}\right)\right]$$

$$= \frac{8}{9}[(\sin 0.7^3 + \cos 0.7^2)(0.7)] + \frac{5}{9}\left[\left(0.7\left(\sin\left(0.7\left(-\sqrt{\frac{3}{5}}\right) + 0.7\right)^3 + \cos\left(0.7\left(-\sqrt{\frac{3}{5}}\right) + 0.7\right)^2\right) + \left(0.7\left(\sin\left(0.7\left(\sqrt{\frac{3}{5}}\right) + 0.7\right)^3 + \cos\left(0.7\left(\sqrt{\frac{3}{5}}\right) + 0.7\right)^2\right)\right]$$

$$= 0.5303 + \frac{5}{9}(0.6854 + 0.6665)$$

$\therefore$ $I = 1.2813$