Chapter

5

SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

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5.1 INTRODUCTION OF INITIAL AND BOUNDARY VALUE PROBLEMS

The solution of an ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary functions of x. Such a solution of a differential equation is known as the closed or finite form of solution. In the absence of such a solution, we have recourse to numerical methods of solution.

let us consider the first order differential equation,

$$\frac{dy}{dx} = f(x, y) \text{ given } y(x_0) = y_0 \quad \text{..... (1)}$$

to study the various numerical methods of solving such equations. In most of these methods, we replace the differential equations by a difference equation and then solve it. These methods yields solutions either as a power series in x from which the values of y can be found by direct substitution or a set of values of x and y. The methods of Picard and Taylor series belong to the former class of solution. In these methods, y in equation (1) is approximated by a truncated series, each term, of which is a function of x. The information about the curve at one point is utilized and the solution is not iterated. As such, these are referred to as single-step methods.

The methods of Euler, Range-Kutta, Milne, etc belong to the latter class of solutions. In these methods, the next point on the curve is evaluated in short steps ahead, by performing iterations until sufficient accuracy is achieved. As such, these methods are called step-by-step methods.

Euler and Runge-Kutta methods are used for computing y over a limited range of x-values whereas Milne and Adams methods may be applied for finding y over a wider range of x-values which are found by Picard's Taylor series or Runge-Kutta methods.

Initial and Boundary Conditions

An ordinary differential equation of the n^{th} order is of the form,

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{d^2x}, \ldots\ldots, \frac{d^ny}{d^nx}\right) = 0 \quad \text{---- (2)}$$

Its general solution contains n arbitrary constants and is of the form,

$$\phi\,(x, y, c_1, c_2, \ldots\ldots, c_n) = 0 \quad \text{---- (3)}$$

To obtain its particular solution, n conditions must be given so that the constants $c_1, c_2, \ldots\ldots, c_n$ can be determined.

If these conditions are prescribed at one point only (say: x_0), then the differential equation together with the conditions constitute an initial value problem of the n^{th} order.

If the conditions are prescribed at two or more points, then the problem is termed as boundary value problem.

5.2 PICARD'S METHOD

Consider the first order equation,

$$\frac{dy}{dx} = f(x, y) \quad \ldots\ldots (1)$$

It is required to find that particular solution of (1) which assumes the value y_0 when $x = x_0$.

On integrating (1) between limits, we get,

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx$$

or, $$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx \quad \ldots\ldots (2)$$

This is an integral equation equivalent to (1), for it contains the unknown y under the integral sign.

As a first approximation y_1 to the solution, we put,

$y = y_0$ in $f(x, y)$ and integrate (2), giving,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

For a second approximation y_2, we put $y = y_1$ in $f(x, y)$ and integrate (2) giving,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y)\, dx$$

Similarly, a third approximation is,

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

Continuing this process, we get, $y_4, y_5, y_6, \ldots\ldots y_n$, where,

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

Hence this method gives a sequence of approximations $y_1, y_2, y_3, \ldots\ldots\ldots$, each giving a better result than the preceding one.

Example 5.1

Find the value of y for x = 0.1 by Picard's method, given that,

$$\frac{dy}{dx} = \frac{y - x}{y + x}, \; y(0) = 1$$

Solution:

We have,

$$y = 1 + \int_0^x \frac{y - x}{y + x}\, dx$$

1^{st} approximation:

Put $y = 1$ in the integrand giving

$$y_1 = 1 + \int_0^x \frac{y - x}{y + x}\, dx = 1 + \int_0^x \left(-1 + \frac{2}{1 + x}\right) dx$$

$$= 1 + \left[-x + 2 \log (1 + x)\right]_0^x$$

$$= 1 - x + 2 \log (1 + x)$$

2^{nd} approximation:

Put $y = 1 - x + 2 \log (1 + x)$ in the integrand giving,

$$y_2 = 1 + \int_0^x \frac{1 - x + 2 \log (1 + x) - x}{1 - x + 2 \log (1 + x) - x}\, dx$$

$$= 1 + \int_0^x \left[1 - \frac{2x}{1 + 2\log (1 + x)}\right] dx$$

which is very difficult to integrate.

Hence we use the first approximation and taking $x = 0.1$, we get,

$$y(0.1) = 1 - 0.1 + 2 \log (1.1) = 0.9828$$

5.3 TAYLOR'S SERIES METHOD

Consider the first order equation,

$$\frac{dy}{dx} = f(x, y) \qquad \ldots\ldots (1)$$

Differentiating (1) with respect to x, we get,

$$\frac{d_y^2}{d_x^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \qquad \text{..... (2)}$$

i.e., $y'' = f_x + f_y f'$

Differentiating this successively, we can get y''', y^{iv} etc

Putting $x = x_0$ and $y = 0$, the values of $(y')_0$, $(y'')_0$ and $= y_0$ can be obtained.

Hence the Taylor series

$$y = y_0 + (x - x_0)(y') + (x - x_0)^2 (y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \text{.........} \qquad \text{..... (3)}$$

Gives the values of y for every value of x for which (3) converges.

On finding the value y_1 for $x = x_i$ from (3), y', y'' etc can be evaluated at $x = x_1$ by means of (1), (2) etc. Then y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (3).

> **NOTE:**
> This is a single step method and works well so long as the successive derivatives can be calculated easily. If (x, y) is somewhat complicated and the calculation of higher order derivatives becomes tedious, the Taylor's method cannot be used significantly. This is the main drawback of this method. However, it is useful for finding starting values for the application of powerful methods like Runge-Kutta, Milne method.

Example 5.2

Solve $y' = x + y$, $y(0) = 1$ by Taylor's series method. Hence find the values of y at $x = 0.1$ and $x = 0.2$.

Solution:

Differentiating successively, we get, $\qquad [\because y(0) = 1]$

$y' = x + y$	$y'(0) = 1$
$y'' = 1 + y'$	$y''(0) = 2$
$y''' = y''$	$y'''(0) = 2$, etc

Now, Taylor's series is,

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x + x_0)^2}{2!}(y'')_0 + \frac{(x + x_0)^3}{3!}(y''')_0 + \text{........}$$

Here, $x_0 = 0$, $y_0 = 1$

$$\therefore \quad y = 1 + x(1) + \frac{x^2}{2} \times 2 + \frac{x^3}{3!} \times 2 + \frac{x^4}{4!} \times 4 \text{ ....}$$

$$\text{Hence, } y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{3!} + \text{.........}$$

$$= 1.1103$$

$$\text{and, } \quad y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{6} + \text{.........}$$

$$= 1.2427$$

Example 5.3

Find by Taylor's method, the values of the y at x = 0.1 and x = 0.2 to five places of decimals from $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$

Solution:

$$\frac{dy}{dx} = x^2 y - 1$$

Differentiating successively, we get,

$$y' = x^2 y - 1, \qquad (y')_0 = -1 \qquad [\because y(0) = 1]$$
$$y'' = 2xy + x^2y', \qquad (y'')_0 = 0$$
$$y''' = 2y + 4xy' + x\,2y'', \qquad (y''')_0 = 2$$
$$y^{iv} = 6y' + 6xy'' + x\,2y''', \qquad (y^{iv})_0 = -6 \text{ etc}$$

Replacing these values in the Taylor series, we get,

$$y = 1 + x(-1) + \frac{x^2}{2}(0) + \frac{(x^3)}{3!} \times 2 + \frac{x^4}{4!}(-6) + \dots\dots\dots$$
$$= 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots\dots$$

$$y(0.1) = 1 - 0.1 + \frac{0.1^3}{3} - \frac{0.1^4}{4} = 0.90033$$

Hence, $y(0.1) = 0.90033$ and $y(0.2) = 0.80227$

Example 5.4

Solve by Taylor series method of third order equation $\frac{dy}{dx} = \frac{x^3 + xy^2}{e^x}$, $y(0) = 1$ for y at x = 0.1 x = 0.2 and x = 0.3.

Solution:

We have,

$$y' = (x^3 + xy^2)\,e^{-x}, \quad y'(0) = 0$$

Differentiating successively and replacing x = 0 and y = 1

$$y'' = (x^3 + xy^2)(-e^{-x}) + (3x^2 + y^2 + x2y \cdot y')\,e^{-x}$$
$$= (-x^3 - xy^2 + 3x^2 + y^2 + 2xyy')\,e^{-x}; \quad y''(0) = 1$$
$$y''' = (-x^3 - xy^2 + 3x^2 + y^2 + 2xyy')(-e^{-x}) + \{-3x^2 - (y^2 + x\,2y\,y')$$
$$+ 6x + 2yy + 2[yy' + x(y^{12} + yy'')]\}\,e^{-x}, \quad y'''(0) = -2$$

Replacing these values in the Taylor's series, we have,

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \dots\dots\dots$$
$$= 1 + x(0) + \frac{x^2}{2}(1) + \frac{x^3}{6}(-2) + \dots\dots\dots = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \dots\dots\dots$$

Hence, $y(0.1) = 1 + \frac{1}{2}(0.1)^2 - \frac{1}{3}(0.1)^3 = 1.005$

$$y(0.2) = 1 + \frac{1}{2}(0.2)^2 - \frac{1}{3}(0.2)^3 = 1.017$$

$$y(0.3) = 1 + \frac{1}{2}(0.3)^2 - \frac{1}{3}(0.3)^3 = 1.036$$

5.4 THE EULER METHOD

Consider the equation, ... (1)

$$\frac{dy}{dx} = f(x, y)$$

given that $y(x_0) = y_0$. Its curve of solution through $P(x_0, y_0)$ is shown dotted in figure 5.1. Now, we have to find the ordinate of any other point Q on this curve.

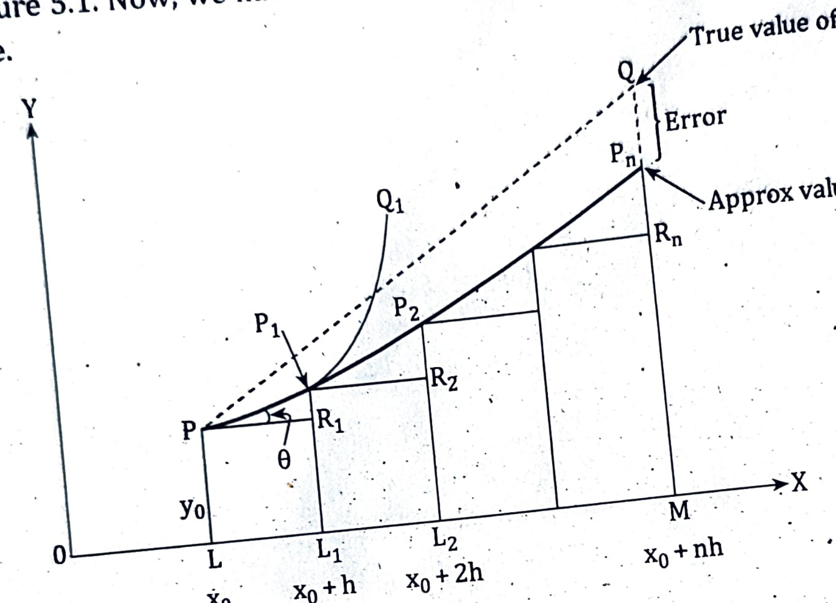


Figure 5.1

Let us divide LM into n-sub-intervals each of width h at L_1, L_2, ..., so that h is quite small. In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$ then,

$$y_1 = L_1P_1 = LP + R_1P_1 = y_0 + PR_1 \tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_p = y_0 + hf(x_0, y_0)$$

Let P_1Q_1 be the curve of solution (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$. Then, ... (2)

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

Repeating this process n times, we finally reach on an approximation MP_n of MQ given by,

$$y_n = y_{n-1} + hf(x_0 + \overline{n-1}h, y_{n-1})$$

This is Fuller's method of finding an approximate solution of (1).

> **NOTE:**
> In Euller's method, we approximate the curve of solution by the tangent in each interval *i.e.*, by a sequence of short lines. Unless h is small, the error is bound to be quite significant. This sequence of lines may also deviate considerably from the curve of solution. As such, the method is very slow and hence there is a modification of this method which is given in next section.

Example 5.5

Using Euler's method, find an approximate value of y corresponding to $x = 1$ given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$.

Solution:

Given that;

$$\frac{dy}{dx} = x + y$$

We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows.

x	y	$x + y = \frac{dy}{dx}$	Old $y + 0.1\left(\frac{dy}{dx}\right)$	New y
0.1	1.00	1.00	1.00 + 0.1 (1.00)	1.10
0.1	1.10	1.20	1.10 + 0.1 (1.20)	1.22
0.2	1.22	1.42	1.22 + 0.1 (1.42)	1.36
0.3	1.36	1.66	1.36 + 0.1 (1.66)	1.53
0.4	1.53	1.93	1.53 + 0.1 (1.93)	1.72
0.5	1.72	2.22	1.72 + 0.1 (2.22)	1.94
0.6	1.94	2.54	1.94 + 0.1 (2.54)	2.19
0.7	2.19	2.89	2.19 + 0.1 (2.89)	2.48
0.8	2.48	3.29	2.48 + 0.1 (3.29)	2.81
0.9	2.81	3.71	2.81 + 0.1 (3.71)	3.18
1.0	3.18			

Thus the required approximation value of $y = 3.18$

5.5 MODIFIED EULER'S METHOD OR HUEN'S METHOD

In Euler's method, the curve of solution in the interval LL_1 is approximated by the tangent at P (Figure 5.1) such that at P_1, we have,

$$y_1 = y_0 + h\, f(x_0, y_0) \qquad \text{..... (1)}$$

Then the slope of the curve of solution through P_1,

$$\left[\text{i.e.,} \left(\frac{dy}{dx}\right) P_1 = f(x_0 + h, y_1)\right] \qquad \text{..... (5)}$$

is computed and the tangent at P_1 to P_1Q_1 is drawn meeting the ordinate through L_2 in,

$$P_2(x_0 + 2h, y_2)$$

Now, we find a better approximation y_1^{1} of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and P_1, *i.e.,* $y_1^1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1)]$

As the slope of the tangent at P_1 is not known, we take y_1 as found in (1) by Euler's method and insert it on RHS of (2) to obtain the first modified value y_1. Again (2) is applied and we find a still better value $y_{1(2)}$ corresponding to L_1 as,

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})]$$

We repeat this step, until two consecutive values of y agree. This is then taken as the starting point for the next interval L_1L_2. Once y_1 is obtained to a desired degree of accuracy, y corresponding to L_2 is found from (1).

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

and a better approximation $y_1^{(1)}$ is obtained from (2)

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]$$

We repeat this step until y_2 becomes stationary. Then we proceed to calculate $y_3^{(2)}$, as above and so on. This is the modified Euler's method which gives great improvement in accuracy over the original method.

Example 5.6

Using modified Euler's method, find an approximate value of y when x = 0.3 given that $\frac{dy}{dx} = x + y$ and y = 1 when x = 0.

Solution:

The various calculations are arranged as followings taking h = 0.1

x	x + y = y'	Mean slope	Old y + 0.1 (mean slope) = New y
0.0	0 + 1	–	1.00 + 0.1 × 1.00 = 1.10
0.1	0.1 + 1.1	$\frac{1}{2}(1 + 1.2)$	1.00 + 0.1 (1.1) = 1.11
0.1	0.1 + 1.11	$\frac{1}{2}(1 + 1.21)$	1.00 + 0.1 (1.105) = 1.1105
0.1	0.1 + 1.1105	$\frac{1}{2}(1 + 1.2105)$	1.00 + 0.1 (1.1052) = 1.1105

Since the last two values are equal, we take y(0.1) = 1.1105.

x	x + y = y'	Mean slope	Old y + 0.1 (mean slope) = New y
0.1	1.2105	–	1.1105 + 0.1 (1.2105) = 1.2316
0.2	0.2 + 1.2316	$\frac{1}{2}(1.12105+1.4316)$	1.1105 + 0.1 (1.3211) = 1.2426
0.2	0.2 + 1.2426	$\frac{1}{2}(1.2105+1.4426)$	1.1105 + 0.1 (1.3266) = 1.2432
0.2	0.2 + 1.2432	$\frac{1}{2}(1.2105+1.4432)$	1.1105 + 0.1 (1.3268) = 1.2432

Since the last two values are equal, we take y(0.2) = 1.2432

x	x + y = y'	Mean slope	Old y + 0.1 (mean slope) = New y
0.2	0.3 + 1.3875	–	1.2432 + 0.1 (1.4432) = 1.3875
0.3	0.3 + 1.3875	$\frac{1}{2}(1.4432 + 1.6875)$	1.2432 + 0.1 (1.5654) = 1.3997
0.3	0.3 + 1.3997	$\frac{1}{2}(1.4432 + 1.6997)$	1.2432 + 0.1 (1.5715) = 1.4003
0.3	0.3 + 1.4003	$\frac{1}{2}(1.4432 + 1.7003)$	1.2432 + 0.1 (1.5718) = 1.4004
0.3	0.3 + 1.4004	$\frac{1}{2}(1.4432 + 1.7004)$	1.2432 + 0.1(1.5718) = 1.4004

Since the last two values are equal, we take, y(0.3) = 1.4004

Hence, y(0.3) = 1.4004 approximately.

Example 5.7

Solve the following by Euler's modified method

$$\frac{dy}{dx} = \log(x + y),\ y(0) = 2$$

at x = 1.2 and 1.4 with h = 0.2

Solution:

The various calculations are arranged as follows

x	log(x + y) = y'	Mean slope	Old y + 0.2 (mean slope) = New y
0.0	log(0+2)	–	2+0.2(0.301)=2.0602
0.2	log(0.2+2.0602)	$\frac{1}{2}(0.310+0.3541)$	2+0.2(0.3276)=2.0655
0.2	log(0.2+2.0655)	$\frac{1}{2}(0.301+0.3552)$	2+0.2(0.3281)=2.0656

x	log(x + y) = y'	Mean slope	Old y + 0.2 (mean slope) = New y
0.2	0.3552	–	2.0656+0.2(0.3552)=2.1366
0.4	log(0.4+2.1366)	$\frac{1}{2}(0.3552+0.4042)$	2.056+0.2(0.3797)=2.1415
0.4	log(0.4+2.1415)	$\frac{1}{2}(0.3552+0.4051)$	2.0656+0.2(0.3801)=2.1416

x	log(x + y) = y'	Mean slope	Old y + 0.2 (mean slope) = New y
0.4	0.4051	–	2.1416+0.2(0.4051)=2.2226
0.6	log(0.6+2.2226)	$\frac{1}{2}(0.4051+0.4506)$	2.1416+0.2(0.4279)=2.2272
0.6	log(0.6+2.2272)	$\frac{1}{2}(0.4051+0.4514)$	2.1416+0.2(0.4282)=2.2272
0.6	0.4514	–	2.2272+0.2(0.4514)=2.3175
0.8	log(0.8+2.3175)	$\frac{1}{2}(0.4514+0.4938)$	2.2272+0.2(0.4726)=2.3217

x	log(x + y) = y'	Mean slope	Old y + 0.2 (mean slope) = New y
0.8	log(0.8+2.3217)	$\frac{1}{2}$(0.4514+0.4913)	2.2272+0.2(0.4727)=2.3217
0.8	0.4943	-	2.3217+0.2(0.4943)=2.4206
1.0	log(1+2.4206)	$\frac{1}{2}$(0.4943+0.5341)	2.3217+0.2(0.5142)=2.4245

x	log(x + y) = y'	Mean slope	Old y + 0.2 (mean slope) = New y
1.0	log(1+2.4245)	$\frac{1}{2}$(0.4943+0.5346)	2.3217+0.2(0.5144)=2.4245
1.0	0.5346	-	2.425+0.2(0.5346)=2.5314
1.2	log(1.2+2.5314)	$\frac{1}{2}$(0.5346+0.5719)	2.4245+0.2(0.5532)=2.5351

x	log(x + y) = y'	Mean slope	Old y + 0.2 (mean slope) = New y
1.2	log(1.2+2.5351)	$\frac{1}{2}$(0.5346+0.5123)	2.4245+0.2(0.5534)=2.5351
1.2	0.5723	-	2.5351+0.2(0.5723)=2.6496

x	log(x + y) = y'	Mean slope	Old y + 0.2 (mean slope) = New y
1.4	log(1.4+2.6496)	$\frac{1}{2}$(0.5723+0.6074)	2.5351+0.2(0.5898)=2.6531
1.4	log(1.4+2.6531)	$\frac{1}{2}$(0.5723+0.6078)	2.5351+0.2(0.5900)=2.6531

Hence, y (1.2) = 2.5351 and dy (1.4) = 2.6531 approximately.

5.6 RUNGE'S METHOD

Consider the differential equation,

$$\frac{dy}{dx} = f(x, y), \; y(x_0) = y_0 \qquad \text{.....(1)}$$

Clearly the slope of the curve through $P(x_0, y_0)$ is $f(x_0, y_0)$

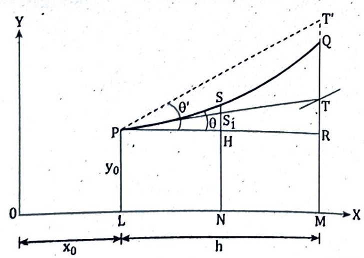


Figure 5.2

Integrating both sides of equation (1) from (x_0, y_0) to $(x_0 + h, y_0 + k)$, we have,

$$\int_0^{y_0+k} dy = \int_{x_0}^{x_0+h} f(x, y)\, dx \qquad \text{.....(2)}$$

To evaluate the integral on the right, we take N as the midpoint of LM and find the values of f(x, y) $\left(i.e., \frac{dy}{dx}\right)$ at the point x_0, $x_0 + \frac{h}{2}$, $x_0 + h$. For this purpose, we first determine the values of y at these points.

Let the ordinate through N cut the curve PQ in S and the tangent PT in S_1.

The value of y_s is given by the point S_1,

$$\therefore \quad y_s = NS_1 = LP + HS_1 = y_0 + PH \tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_P$$

$$= y_0 + \frac{h}{2} f(x_0, y_0) \qquad \text{.....(3)}$$

Also, $y_T = MT = LP + RT = y_0 + PR \tan\theta = y_0 + hf(x_0 + y_0)$

Now the value of y_Q at $x_0 + h$ is given by the point T" where the line through P drawn with slope at $T(x_0 + h_1, y_T)$ meets MQ.

$$\therefore \quad \text{Slope at } T = \tan\theta' = f(x_0 + h_1, y_T)$$

$$= f[x_0 + h, y_0 + hf(x_0, y_0)]$$

$$\therefore \quad y_Q = R + RT = y_0 + PR \tan\theta' \qquad \text{.....(4)}$$

$$= y_0 + hf[x_0 + h, y_0 + hf(x_0, y_0)]$$

Thus, the value of f(x, y) at $P = f(x_0, y_0)$

the value of f(x, y) at $S = f\left(x_0 + \frac{h}{2}, y_s\right)$

the value of (x, y) at $Q = (x_0 + h_1, y_0)$

where, y_s and y_Q are given by (3) and (4)

Hence from (2), we get, [By Simpson's rule]

$$k = \int_{x_0}^{x_0+h} f(x, y)\, dx = \frac{h}{6}[f_P + 4f_S + f_Q] \qquad \text{.....(5)}$$

$$= \frac{h}{6}\left[f(x_0 + y_0) + f\left(x_0 + \frac{h}{2}, y_s\right) + (x_0 + h, y_Q)\right]$$

which gives a sufficiently accurately value of k and also $y = y_0 + k$.

The repeated application of (5) gives the values of y for equispaced points.

Working Rule to Solve by Runge's Method

Calculate successively;

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k' = hf(x_0 + h, y_0 + k_1)$$

and, $k_3 = hf(x_0 + h, y_0 + k')$

Finally compute,

$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

which gives the required approximate value as $y_1 = y_0 + k$.

Note that k is the weighted mean of k_1, k_2 and k_3.

Example 5.8

Apply Runge's method to find an approximate value of y when x = 0.2, given that $\frac{dy}{dx} = x + 1$ and y = 1 when x = 0.

Solution:

Given that;

$$\frac{dy}{dx} = x + 1$$

$$y = 1 \text{ when } x = 0$$

We have,

$$x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$$

$\therefore \quad k_1 = hf(x_0, y_0) = 0.2\ (1) = 0.20$

$\therefore \quad k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2\ f(0.1, 1.1) = 0.240$

$\therefore \quad k' = hf(x_0 + h, y_0 + k_1) = 0.2f\ (0.2, 1.2) = 0.280$

$\therefore \quad k_3 = hf(x_0 + h, y_0 + k') = 0.2f\ (0.1, 1.28) = 0.296$

$\therefore \quad k = \frac{1}{6}(k_1 + 4k_2 + k_3) = \frac{1}{6}(0.20 + 0.960 + 0.296) = 0.2426$

Hence the required approximate value is 1.2426

5.7 RUNGE-KUTTA METHOD

Runge-Kutta method do not require the calculations of higher order derivatives and give greater accuracy. The Runge-Kutta formulae possess the advantage of requiring only the function values at some selected points. These methods agree with Taylor's series solution up to the term in h^r where r differs from method to method and is called the order of that method.

A. First order R-K Method

From Euler's method,

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0' \qquad [\because y' = f(x,y)]$$

Expanding by Taylor's series,

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \ldots\ldots\ldots$$

It follows that the Euler's method agrees with the Taylor's series solution up to the term in h.

Hence Euler's method is the Runge-Kutta method of the first order.

B. Second order R-K Method

The modified Euler's method gives,

$$y_1 = y + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1)] \qquad \ldots\ldots (1)$$

Replacing $y_1 = y_0 + hf(x_0, y_0)$ on the right hand side of (1), we get,

$$y_1 = y_0 + \frac{h}{2}[f_0 + f(x_0 + h); y_0 + hf_0] \qquad \ldots\ldots (2)$$

where, $f_0 = (x_0, y_0)$

Expanding L.H.S. by Taylor's series, we get,

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots\ldots\ldots \qquad \ldots\ldots (3)$$

Expanding $f(x_0 + h + y_0, hf_0)$ by Taylor's series for a function of two various, (2) gives,

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + \left\{f_0 = (x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + hf_0\left(\frac{\partial f}{\partial y}\right)_0 + O(h^2)\right\}\right]$$

$$= y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left\{\left(\frac{\partial f}{\partial x}\right)_0 + \left(\frac{\partial f}{\partial y}\right)_0\right\} + O(h^3)\right]$$

$$\left[\because \frac{df(x,y)}{dx} = \frac{\partial f}{\partial x} + \frac{f\partial f}{\partial y}\right]$$

$$= y_0 + hf_0 + \frac{h^2}{2}f_0' + O(h^3) \qquad \ldots\ldots (4)$$

$$= y_0 + hy_0' + \frac{h^2}{2!}y_0'' + O(h^3)$$

where, $O(h^2)$ means terms contianing second and higher powers of h" and is read as order of h^2.

Comparing (3) and (4), it follows that the modified Euler's method agrees with the Taylor's series solution up to the term in h^2.

Hence the modified Euler's method is the Runge-Kutta method of the second order.

The second order Rung-Kutta formula is,

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where, $k_1 = hf(x_0, y_0)$ and $k_2 = hf(x_0 + h, y_0 + k)$

C. Third order R-K Method

Runge's method is the Runge-Kutta method of the third order.

$\therefore$ The third order Runge-kutta formula is,

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where, $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{1}{2}h_0,\ y_0 + \frac{1}{2}k_1\right)$$

and, $k_3 = hf(x_0 + h, y_0 + k')$

where, $k' = k_3 = hf(x_0 + h, y_0 + k_1)$

D. Fourth order R-K Method

This method is most commonly used and is often referred to as the Runge-Kutta method only.

Working rule for finding the increment of k of y corresponding to an increment h of x. By Runge-Kutta method from,

$\frac{dy}{dx} = f(x, y), y(x_0)$ is as follows;

Calculate successively, $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right)$$

and, $k_4 = hf(x_0 + h, y_0 + k_3)$

Finally,

Compute $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

which gives the required approximate value as $y_1 = y_0 + k$

> **NOTE:** One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

Example 5.9

Apply the Runge-Kutta fourth order method to find an approximate value of y when x = 0.2 given that $\frac{dy}{dx} = x + y$ and y = 1 when x = 0.

Solution:

$$\frac{dy}{dx} = x + y$$

Here;

$x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

$\therefore$ $k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.20$

$$k_2 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.240$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right) = 0.2 \times f(0.1, 1.12) = 0.2440$$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) = 0.2888$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore$ $= \frac{1}{6}(0.20 + 0.480 + 0.4880 + 0.2888)$

$= \frac{1}{6} \times 1.4568$

$= 0.2428$

Hence the required approximate value of y = 1.2428

Example 5.10

Apply the Runge-Kutta method to find the approximate value of y for x = 0.2, in steps 0.1, if $\frac{dy}{dx} = x + y^2$, y = 1 where, x = 0.

Solution:

Given that;

$f(x, y) = x + y^2$

Here, we take h = 0.1 and carry out the calculations in two steps

Step I:

$x_0 = 0, y_0 = 0, h = 0.1$

$\therefore$ $k_1 = hf(x_0, y_0) = 0.1\ f(0, 1) = 0.10$

$$k_2 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right) = 0.1\ f(0.05, 1.1) = 0.1152$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right) = 0.1\ f(0.05, 1.1152) = 0.1168$$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1\ f(0.1, 1.1168) = 0.1347$

$\therefore$ $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.10 + 0.2304 + 0.2336 + 0.1347)$

$= 0.1165$

giving $(0.1) = y_0 + k = 1.1165$

Step II:

$x_1 = x_0 + h = 0.1, y_1 = 1.1165, h = 0.1$

$\therefore$ $k_1 = hf(x_1, y_2) = 0.1\ f(0.1, 1.1165) = 0.1347$

$$k_2 = hf\left(x_0 + \frac{1}{2}h,\ y_1 + \frac{1}{2}k_1\right) = 0.1\ f(0.15, 1.1838) = 0.1551$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_1 + \frac{1}{2}k_2\right) = 0.1\ f(0.15, 1.194) = 0.1576$$

$k_4 = hf(x_1 + h, y_2 + k_3) = 0.1\ f(0.2, 1.1576) = 0.1823$

$\therefore$ $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1571$

Hence, $y(0.2) = y_1 + k = 1.2736$

5.8 SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

The simultaneous differential equations of the type,

$$\frac{dy}{dx} = f(x, y, z) \qquad .....(1)$$

$$\frac{dz}{dx} = \phi(x, y, z) \qquad .....(2)$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by the methods discussed in the preceding sections, especially Picard's or Runge-Kutta methods.

i) Picard's method gives

$$y_1 = y_0 + \int f(x, y_0, z_0)\, dx, \quad z_1 = z_0 + \int \phi(x, y_0, z_0)\, dx$$

$$y_2 = y_0 + \int f(x, y_1, z_1)\, dx, \quad z_2 = z_0 + \int \phi(x, y_1, z_1)\, dx$$

$$y_3 = y_0 + \int f(x, y_2, z_2)\, dx, \quad z_3 = z_0 + \int \phi(x, y_2, z_2)\, dx$$

and so on.

ii) Taylor's method is used as follows

If h be the step-size, $y_1 = y(x_0 + h)$ and $z_1 = z(x_0 + h)$. Then Taylor's algorithm for (1) and (2) gives,

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \qquad .....(3)$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \qquad .....(4)$$

Differentiating (1) and (2) successively, we get y", z". So the values y_0', y_0'', y_0''' and z_0', z_0'', z_0''' .... are known.

Replacing these values in (3) and (4), we obtain y_1, z_1 for the next step.

Similarly, we have the algorithms,

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \qquad .....(5)$$

$$z_2 = z_0 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \qquad .....(6)$$

Since y_1 and z_1 are known, we can calculate y_1', y_1'', ... and z_1', z_1'', ...

Replacing these in (5) and (6), we get y_2 and z_2.

Proceeding further, we can calculate the other values of y and z step by step.

iii) Runge-Kutta method is applied as follows

Starting at (x_0, y_0, z_0) and taking the step-sizes for x, y, z to be h, k, l respectively, the Runge-kutta method gives,

$$k_1 = hf(x_0, y_0, z_0)$$

$$l_1 = h\,\phi(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$k_4 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_3, z_0 + \frac{1}{2}l_3\right)$$

$$l_4 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_3, z_0 + \frac{1}{2}l_3\right)$$

Hence, $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

To compute y_2 and z_2, we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in above formula.

Example 5.11

Solve the differential equations

$$\frac{dy}{dx} = 1 + xz, \quad \frac{dz}{dx} = -xy \text{ for } x = 0.3$$

Using the fourth order Runge Kutta method. Initial values are x = 0, y = 0 and z = 1.

Solution:

Here;

$$f(x, y, z) = 1 + xz, \ \phi(x, y, z) = -xy$$

$$x_0 = 0, y_0 = 0, z_0 = 1$$

Let us take h = 0.3,

$\therefore$

$$k_1 = hf(x_0, y_0, z_0) = 0.3\,f(0, 0, 1) = 0.3\,(1 + 0) = 0.3$$

$$l_1 = h\phi(x_0, y_0, z_0) = 0.3\,(-0 \times 0) = 0$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= 0.3f(0.15, 0.15, 1)$$

$$= 0.3\,(1 + 0.15) = 0.345$$

$$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= 0.3\,[-(0.15(0.15)]$$

$$= -0.00675$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{k_2}{2}k_1, z_0 + \frac{1}{2}\right)$$
$$= 0.3\,f(0.15, 0.1725, 0.996625)$$
$$= 0.3\,[1 + 0.996625 \times 0.15]$$
$$= 0.34485$$

$$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{k}{2}, z_0 + \frac{1}{2}\right)$$
$$= 0.3\,[-(0.15)(0.1725)]$$
$$= -0.007762$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.3f(0.3, 0.34485, 0.99224)$$
$$= 0.3893$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.3\,[-(0.3)(0.34485)]$$
$$= -0.03104$$

Hence, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

i.e., $y(0.3) = 0 + \frac{1}{6}[0.3 + 2 \times (0.345) + 2 \times (0.34485) + 0.3893]$

$= 0.34483$

and, $z(x + h) = y_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$z(0.3) = 1 + \frac{1}{6}[0 + 2(-0.00675) + (0.0077625) + (-0.03104)]$

$= 0.98999$

5.9 SECOND ORDER DIFFERENTIAL EQUATIONS

Consider the second order differential equation,

$$\frac{d^2y}{dx^2} = \left(x, y, \frac{dy}{dx}\right)$$

By writing $\frac{dy}{dx} = z$, it can be reduced to two first order simultaneous differential equations.

$$\frac{dy}{dx} = z, \frac{dz}{dx} f(x, y, z)$$

These equations can be solved as explained above.

Example 5.12

Using the Runge-Kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$ correct to 4 decimal places. Initial conditions are $x = 0$, $y = 1$, $y' = 0$.

Solution:

Let, $\frac{dy}{dx} = f(x, y, z) = z$

Then, $\frac{dy}{dx} = xz^2 - y^2 = \phi(x, y, z)$

We have,

$x_0 = 0, y_0 = 1, z_0 = 0, h = 0.2$

$\therefore$ Runge-Kutta formulae becomes,

$$k_1 = hf(x_0, y_0, z_0) = 0.2 \times 0 = 0$$
$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$
$$= 0.2(-0.1) = -0.02$$
$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$
$$= 0.2(-0.0999) = -0.02$$
$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.2(-0.1958) = -0.0392$$

$\therefore$ $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.0199$

Now,

$$l_1 = hf(x_0, y_0, z_0) = 0.2(-1) = -0.2$$
$$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$
$$= -0.2(-0.999) = -0.1998$$
$$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$
$$= 0.2(-0.9791) = -0.1958$$
$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.2(0.9527)$$
$$= -0.1905$$
$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$
$$= \frac{1}{6}(-0.2 - 0.1998 \times 2 - 2 \times 0.1958 - 0.1905)$$
$$= -0.1970$$

Hence at $x = 0.2$,

$y = y_0 + k = 1 - 0.0199 = 0.9801$

$y' = z = z_0 + l = 0 - 0.1970 = -0.1970$

5.10 BOUNDARY VALUE PROBLEMS

Such a problem requires the solution of a differential equation in a region R subject to the various conditions on the boundary of R. Practical applications give rise to many such problems.

A. Shooting Method or Marching Method

In this method, the given boundary value problem is first transformed to an initial value problem. Then this initial value problem is solved by Taylor's series method or Runge-Kutta method etc. Finally, the given boundary value problem is solved. The approach in this method is quite simple.

Consider the boundary value problem,

$$y''(x) = y(x); \; y(a) = A, y(b) = B \quad \text{..... (1)}$$

One condition is $y(a) = A$ and let us assume that $y'(a) = m$ which represents the slope. We start with two initial guesses for m, then find the corresponding value of $y(b)$ using any initial value method.

Let the two guesses be m_0, m_1 so that the corresponding values of $y(b)$ are $y(m_0, b)$ and $y(m_1, b)$. Assuming that the values of m and y (b) are linearly related, we obtain the better approximation m_2 for m from the relation.

$$\frac{m_2 - m_1}{y(b) - y(m_1, b)} = \frac{m_1 - m_0}{y(m_1, b) - y(m_0, b)}$$

This gives,

$$m_2 = m_1 - (m_1 - m_0)\frac{y(m, b) - y(b)}{y(m_1, b) - y(m_0, b)} \quad \text{..... (2)}$$

Now, we solve the initial value problem,

$$y''(x) = y(x), y(a) = A, y'(a) = m_2$$

and obtain the solution $y(m_2, b)$

To obtain a better approximation m_3 for m, we again use the linear relation (2) with $[m_1, y(m_1, b)]$ and $[m_2, y(m_2, b)]$. This process is repeated until the value of $y(m_i, b)$ agrees with $y(b)$ to desired accuracy.

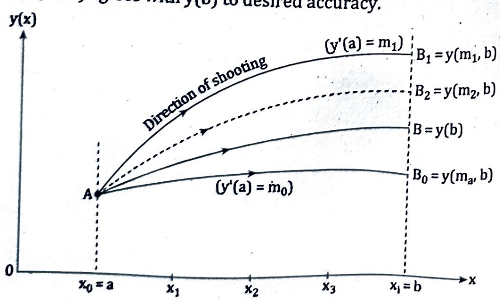


Figure 5.3

> **NOTE:**
> Shooting method is quite slow in practice. Also, this method is quite tedious to apply to higher order boundary value problems.

Example 5.13

Using the shooting method, solve the boundary value problem.

$$y''(x) = y(x) \; ; \; y(0) = 0 \text{ and } y(1) = 1.17$$

Solution:

Let the initial guesses for $y'(0) = m$ be $m_0 = 0.8$

and, $m_1 = 0.9$. Then $y''(x) = y(x); y(0) = 0$ gives,

$$y'(0) = m$$
$$y''(0) = y(0) = 0$$
$$y'''(0) = y'(0) = m$$
$$y^{iv}(0) = y''(0) = 0$$
$$y^{v}(0) = y'''(0) = m$$
$$y^{vi}(0) = y^{iv}(0) = 0$$

and so on.

Replacing these values in the Taylor's series, we have,

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \text{.........}$$

$$= m\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \text{.........}\right)$$

$$\therefore \quad y(1) = m\,(1 + 0.1667 + 0.0083 + 0.0002 + \text{.........})$$

$$= m(1.175)$$

For $m_0 = 0.8$, $y(m_0, 1) = 0.8 \times 1.175 = 0.94$

For $m_1 = 0.9$, $y(m_1, 1) = 0.9 \times 1.175 = 1.057$

Hence a better approximation for m, *i.e.*, m_2 is given by,

$$m_2 = m_1 - (m_1 - m_0)\frac{y(m_1, 1) - y(1)}{y(m_1, 1) - y(m_0, 1)}$$

$$= 0.9 - (0.1)\left(\frac{1.057 - 1.175}{1.057 - 0.94}\right)$$

$$= 0.9 + 0.10085 = 1.00085$$

Which is closer to the exact value of $y'(0) = 0.996$

Solving the initial value problem

$$y''(x) = y(x), y(0) = 0, y'(0) = m_2$$

Taylor's series solution is given by,

$$y(m_2, 1) = m_2\,(1.175) = 1.1759$$

Hence the solution at $x = 1$ is $y = 1.176$ which is close to the exact value of $y(1) = 1.17$.

B. Finite Difference Method

In this method, the derivatives appearing in the differential equation and the boundary conditions are replaced by their finite-difference approximation and the resulting linear system of equations are solved by any standard procedure. These roots are the values of the required solution at the pivotal points.

The finite difference approximations to the various derivatives are derived as under.

If y(x) and its derivatives are single-valued continuous functions of x then by Taylor's expansion.

We have,

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \text{.........} \quad \text{..... (1)}$$

and, $$y(x-h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \text{.........} \quad \text{..... (2)}$$

Equation (1) gives,

$$y'(x) = \frac{1}{h}[y(x+h) - y(x)] - \frac{h}{2}y''(x) - \text{.........}$$

i.e., $$y'(x) = \frac{1}{h}[y(x+h) - y(x)] + O(h)$$

which is the forward difference approximation of y'(x) with an error of the order h.

Similarly, (2) gives,

$$y'(x) = \frac{1}{h}[y(x) - y(x-h)] + O(h)$$

Which is the backward difference approximation of y'(x) with an error of the order h.

Subtracting (2) from (1), we get,

$$y'(x) = \frac{1}{2h}[y(x+h) - y(x-h)] + O(h^2)$$

which is the central-difference approximation of y'(x) with an error of the order h^2. Clearly, this central difference approximations and hence should be preferred.

Adding (1) and (2), we get,

$$y''(x) = \frac{1}{h^2}[y(x+h) - 2y(x) + y(x-h)] + O(h^2)$$

which is the central difference approximation to higher derivatives.

Hence the working expressions for the central difference approximations to the first four derivatives of y_i are as under;

$$y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1}) \quad \text{..... (3)}$$

$$y_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \quad \text{..... (4)}$$

$$y_i''' = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}) \quad \text{..... (5)}$$

$$y_i^{iv} = \frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) \quad \text{..... (6)}$$

NOTE: The accuracy of this method depends on the size of the sub-interval h and also on the order of approximation. As we reduce h, the accuracy improves but the number of equations to be solved also increases.

Example 5.14

Solve the equation y" = x + y with the boundary conditions y(0) = y (1) = 0.

Solution:

Divide the interval (0, 1) into four sub-intervals so that $h = \frac{1}{4}$ and the pivot points are at $x_0 = 0$,

$$x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4} \text{ and } x_4 = 1$$

Then the differential equation is approximated as,

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] = x_i + y_i$$

or, $$16\,y_{i+1} - 33y_i + 16\,y_{i-1} = x_i \;;\; i = 1, 2, 3 \text{.........}$$

Using $y_0 = y_4 = 0$, we get the system of equations

or, $$16\,y_2 - 33\,y_1 = \frac{1}{4}$$

or, $$16\,y_3 - 33\,y_2 + 16\,y_1 = \frac{1}{2}$$

or, $$-33\,y_3 + 16\,y_2 = \frac{3}{4}$$

Their solution gives,

$y_1 = -0.03488$

$y_2 = -0.05632$

$y_3 = -0.05003$

BOARD EXAMINATION SOLVED QUESTIONS

1. Solve $\frac{dy}{dx} = y - \frac{2x}{y}$, y(0) = 1 in the range $0 \le x \le 0.2$ by using (i) Euler's method and (ii) Huen's method. Comment on the results. Take h = 0.2. [2013/Fall]

Solution:

$$\frac{dy}{dx} = y - \frac{2x}{y} \text{ and } y(0) = 1$$

$$\Rightarrow x_0 = 0 \text{ and } y_0 = 1$$

Also, h = 0.2, $0 \le x \le 0.2$

i) From Euler's method,

$$f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1 - \frac{2(0)}{1} = 1$$

Now,

$$y_1 = y_{new} = y_0 + hf(x_0, y_0)$$

or, $y_{new} = y_{old} + h\frac{dy}{dx} = 1 + 0.2(1)$

$\therefore$ $y_1 = 1.2$

iii) From Huen's method or modified Euler's method

h = 0.2

Solving in tabular form

S.N.	x	$\frac{dy}{dx} = y - \frac{2x}{y}$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0	$1 - \frac{2(0)}{1} = 1$	-	1 + 0.2 × 1 = 1.2
2	0.2	$1.2 - \frac{2(0.2)}{1.2}$ = 0.8667	$\frac{1}{2}(1 + 0.8667)$ = 0.9333	1 + 0.2 × 0.9333 = 1.1866
3	0.2	$1.1866 - \frac{2(0.2)}{1.1866}$ = 0.8495	$\frac{1}{2}(1 + 0.8495)$ = 0.9247	1 + 0.2 × 0.9247 = 1.1849

Here the last two values are equal at $y_1 = 1.1849$.

The result from Euler's method is 1.2 and from Huen's method is 1.1849 which shows better result and we prefer Huen's method or modified Euler's method.

2. Using Runge Kutta method of order 4, solve the equation $\frac{d^2y}{dx^2} = 6xy^2 + y$, y(0) = 1 and y'(0) = 0 to find y(0.2) and y'(0.2). Take h = 0.2. [2013/Fall]

Solution:

$$\frac{d^2y}{dx^2} - 6xy^2 - y = 0$$

or, $y'' - 6xy^2 - y = 0$ ..... (1)

Also, $y(0) = 1$ and $y'(0) = 0$ and h = 0.2

Let, $y' = z = f_1(x, y, z)$ ..... (A)

so, $y'' = z'$, then equation (1) becomes

$z' = 6xy^2 + y = f_2(x, y, z)$ ..... (B)

Given that;

$y(0) = 1$ $\Rightarrow x_0 = 0,\ y_0 = 1$

and, $y'(0) = 0 = z_0$

Now, using RK method to find increment value of k and l

$k_1 = hf_1(x_0, y_0, z_0)$ at equation (A)

$= hf_1(0, 1, 0)$

$= 0$

$l_1 = hf_2(x_0, y_0, z_0)$ at equation (B)

$= hf_2(0, 1, 0)$

$= (6(0)(1)^2 + 1)\,0.2$

$= 0.2$

Likewise,

$$k_2 = hf_1\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2},\ z_0 + \frac{l_1}{2}\right)$$

$$= 0.2f_1\left(0 + \frac{0.2}{2},\ 1 + \frac{0}{2},\ 0 + \frac{0.2}{2}\right)$$

$= 0.2 \times 0.1$

$= 0.02$

$l_2 = hf_2(0.1, 1, 0.1)$

$= 0.2\,[6(0.1)(1)^2 + 1]$

$= 0.32$

$$k_3 = hf_1\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$

$= 0.2f_1(0.1, 1.01, 0.16)$

$= 0.2 \times 0.16$

$= 0.032$

$l_3 = hf_2(0.1, 1.01, 0.16)$

$= 0.2\,[6(0.1)(1.01)^2 + 1.01] = 0.324$

$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= 0.2f_1(0.2, 1.032, 0.324)$

$= 0.2 \times 0.324 = 0.064$

$l_4 = hf_2(0.2, 1.032, 0.324) = 0.462$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0 + 0.064 + 2(0.02 + 0.032)]$$

$$= 0.028$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$

$$= \frac{1}{6}[0.2 + 0.462 + 2(0.32 + 0.324)]$$

$$= 0.325$$

Now,

$$y_1 = y_0 + k = 1 + 0.028 = 1.028$$

and, $z_1 = z_0 + l = 0 + 0.325 = 0.325$

are the required answer for $y'(0.2)$ and $y(0.2)$.

3. **Use the Runge-Kutta 4th order method to estimate $y(0.2)$ of the following equation with $h = 0.1$**

$$y'(x) = 3x + \frac{1}{2}y,\ y(0) = 1$$ **[2013/Spring]**

Solution:

Given that,

$$y'(x) = 3x + 0.5y$$

and, $y(0) = 1$

$\rightarrow$ $x_0 = 0,\ y_0 = 1,\ h = 0.1$

Now, using RK method to find increment on k

$$k_1 = hf(x_0, y_0)$$

$$= 0.1f(0, 1)$$

$$= 0.1[3(0) + 0.5(1)]$$

$$= 0.05$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

$$= 0.1f\left(0 + \frac{0.1}{2},\ 1 + \frac{0.05}{2}\right)$$

$$= 0.1f(0.05, 1.025)$$

$$= 0.0662$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$

$$= 0.1f(0.05, 1.033)$$

$$= 0.0666$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1f(0.1, 1.0666)$$

$$= 0.0833$$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0.05 + 0.0833 + 2(0.0662 + 0.0666)]$$

$$= 0.0664$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$\therefore$ $y_1 = y_0 + k = 1 + 0.0664 = 1.0664$

Again, $x_1 = 0.1,\ y_1 = 1.0664,\ h = 0.1$

Then,

$$k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.0664) = 0.0833$$

$$k_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right)$$

$$= 0.1f(0.15, 1.1080)$$

$$= 0.1004$$

$$k_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right)$$

$$= 0.1f(0.15, 1.1166)$$

$$= 0.1008$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= 0.1f(0.2, 1.1672)$$

$$= 0.1183$$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0.0833 + 0.1183 + 2(0.1004 + 0.1008)]$$

$$= 0.1006$$

So, $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$$y_2 = y_1 + k = 1.0664 + 0.1006 = 1.167$$

is the required estimated value of $y(0.2)$.

4. **Solve the following equation by Picard's method.**

$$y'(x) = x^2 + y^2,\ y(0) = 0 \text{ and estimate } y(0.1), y(0.2) \text{ and } y(1).$$ **[2013/Spring]**

Solution:

Given that;

$y'(x) = x^2 + y^2$, $y(0) = 0$

$\rightarrow$ $x_0 = 0,\ y_0 = 0$

Now, using Picard's method

$$y = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx = 0 + \int_0^x (x^2 + y^2)\, dx$$

First approximation, put y = 0 in the integrand

$$y_1 = 0 + \int_0^x (x^2 + 0^2)\, dx$$

$$= \int_0^x (x^2)\, dx = \left[\frac{x^3}{3}\right]_0^x = \frac{x^3}{3}$$

Second approximation, put $y = \frac{x^3}{3}$ in the integrand

$$y_2 = 0 + \int_0^x \left[x^2 + \left(\frac{x^3}{3}\right)^2\right] dx$$

$$= \int_0^x \left(x^2 + \frac{x^6}{9}\right) dx$$

$$= \left[\frac{x^3}{3} + \frac{x^7}{63}\right]_0^x$$

$$= \frac{x^3}{3} + \frac{x^7}{63}$$

Further processing of this task is difficult from here so we stop at

$$y_2 = \frac{x^3}{3} + \frac{x^7}{63}$$

Now, using the second approximation and taking

x = 0.1, 0.2 and 1

We have,

$$y(0.1) = \frac{(0.1)^3}{3} + \frac{(0.1)^7}{63} = 0.000033$$

$$y(0.2) = \frac{(0.2)^3}{3} + \frac{(0.2)^7}{63} = 0.0026$$

$$y(1) = \frac{(1)^3}{3} + \frac{(1)^7}{63} = 0.3492$$

5. Given: $\frac{dy}{dx} = \frac{2x + e^x}{x^2 + xe^x}$; y(1) = 0. Solve for y at x = 1.04, by using Euler's method (take h = 0.01) [2014/Fall]

Solution:

Given that;

$$\frac{dy}{dx} = \frac{2x + e^x}{x^2 + xe^x}$$

$y(1) = 0$, $h = 0.01$

$x_0 = 1$, $y_0 = 0$

Using Euler's method, in tabular form

S.N.	x	y	$\frac{dy}{dx} = \frac{2x + e^x}{x^2 + xe^x}$	$y_{new} = y_{old} + h\frac{dy}{dx}$
1	1	0	1.268	0 + 0.01 (1.268) = 0.0126
2	1.01	0.0126	1.256	0.0126 + 0.01 (1.256) = 0.0251
3	1.02	0.0251	1.244	0.0375
4	1.03	0.0375	1.231	0.0498
5	1.04	0.0498		

Hence the required solution at x = 1.04 for y is 0.0498.

6. Solve $\frac{dy}{dx} = 1 + xz$, $\frac{dz}{dx} = -xy$ for y(0.6) and z(0.6), given that y = 0, z = 1 at x = 0 by using Heun's method. Assume, h = 0.3. [2014/Fall]

Solution:

$\frac{dy}{dx} = 1 + xz$, $x_0 = 0$, $y_0 = 0$, $h = 0.3$

and, $\frac{dz}{dx} = -xy$, $x_0 = 0$, $y_0 = 0$, $h = 0.3$

Using Heun's method, solving in tabular form

S.N.	x	$\frac{dy}{dx} + 1 + xz$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0	1 + (0) (1)	–	0 + 0.3 × 1 = 0.3
2	0.3	1 + (0.3) (1) = 1.3	$\frac{1 + 1.3}{2} = 1.15$	0 + 0.3 × 1.15 = 0.345
3	0.3	1 + (0.3) (0.9865) = 1.295	$\frac{1 + 1.295}{2} = 1.147$	0 + 0.3 × 1.147 = 0.344
4	0.3	1 + (0.3) (0.9847) =1.295	$\frac{1 + 1.295}{2} = 1.147$	0 + 0.3 × 1.147 = 0.344

Here, the last two values are equal at $y_1 = 0.344$

S.N.	x	$\frac{dy}{dx} = -xy$	Mean slope	$z_{new} = z_{old} + h$ (mean slope)
1	0	(–0) (0)	–	1 + 0.3 × 0 = 1
2	0.3	–(0.3) (0.3) = –0.09	$\frac{0 - 0.09}{2} = -0.045$	1 + 0.3 × –0.045 = 0.9865
3	0.3	–(0.3) (0.344) = –0.103	$\frac{0 - 0.103}{2} = -0.051$	1 + 0.3 × –0.051 = 0.9847
4	0.3	–(0.3) (0.344) = –0.103	$\frac{0 - 0.103}{2} = -0.051$	1 + 0.3 × –0.051 = 0.9847

Here, the last two values are equal at $z_1 = 0.9847$.

NOTE:

Use both tables to use the value of y_{new} and z_{new} to calculate $\frac{dy}{dx}$ and $\frac{dz}{dx}$.

Again,

S.N.	x	$\frac{dy}{dx} + 1 + xz$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0.3	$1 + (0.3)(0.9847)$ $= 1.295$	-	$0.344 + 0.3(1.295)$ $= 0.732$
2	0.6	$1 + (0.6)(0.9538)$ $= 1.572$	$\frac{1.295 + 1.572}{2}$ $= 1.433$	$0.344 + 0.3(1.433)$ $= 0.773$
3	0.6	$1 + (0.6)(0.899)$ $= 1.539$	$\frac{1.295 + 1.539}{2}$ $= 1.417$	$0.344 + 0.3(1.417)$ $= 0.769$
4	0.6	$1 + (0.6)(0.9)$ $= 1.54$	$\frac{1.295 + 1.54}{2}$ $= 1.417$	$0.344 + 0.3(1.417)$ $= 0.769$

Here, the last two values are equal at $y_2 = 0.769$.

S.N.	x	$\frac{dy}{dx} = -xy$	Mean slope	$z_{new} = z_{old} + h$ (mean slope)
1	0.3	$-(0.3)(0.344)$ $= -0.103$	-	$0.9847 + 0.3(-0.103)$ $= 0.9538$
2	0.6	$-(0.6)(0.773)$ $= -0.463$	$\frac{-0.103 - 0.463}{2}$ $= -0.283$	$0.9847 + 0.3(-0.283)$ $= 0.899$
3	0.6	$-(0.6)(0.769)$ $= -0.461$	$\frac{-0.103 - 0.461}{2}$ $= -0.282$	$0.9847 + 0.3(-0.282)$ $= 0.900$
4	0.6	$-(0.6)(0.769)$ $= -0.461$	$\frac{-0.103 - 0.461}{2}$ $= -0.282$	$0.9847 + 0.3(-0.282)$ $= 0.9$

Here, the last two values are equal at $z_2 = 0.9$.

Hence, the required values of $y(0.6) = 0.769$ and $z(0.6) = 0.9$.

7. Using R-K fourth order method, solve the given differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6$, $y(0) = 0$, $y'(0) = 1$ with $h = 0.2$ for $y(0.4)$? **[2014/Spring]**

Solution:

Given that;

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6$$

or, $y'' + 2y' - 3y = 6$ ..... (1)

Also, $y(0) = 0$, $y'(0) = 1$, $h = 0.2$

→ $x_0 = 0$, $y_0 = 0$

Let, $y' = z = f_1(x, y, z)$

so, $y'' = z'$ then (1) becomes

$z' = 6 + 3y - 2z = f_2(x, y, z)$

Subject to

$y'(0) = 1$

→ $z_0 = 1$

Now, using RK method to find increments,

$$k_1 = hf_1(x_0, y_0, z_0)$$
$$= 0.2f_1(0, 0, 1) = 0.2$$
$$l_1 = hf_2(x_0, y_0, z_0)$$
$$= 0.2[6 + 3(0) - 2(1)] = 0.8$$
$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= 0.2f_1(0.1, 0.1, 1.4) = 0.28$$
$$l_2 = hf_2(0.1, 0.1, 1.4)$$
$$= 0.2[6 + 3(0.1) - 2(1.4)] = 0.7$$
$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= 0.2f_1(0.1, 0.14, 1.35) = 0.27$$
$$l_3 = hf_2(0.1, 0.14, 1.35) = 0.744$$
$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.2f_1(0.2, 0.27, 1.744) = 0.348$$
$$l_4 = hf_2(0.2, 0.27, 1.744) = 0.664$$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0.2 + 0.348 + 2(0.28 + 0.27)] = 0.274$$
$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$
$$= \frac{1}{6}[0.8 + 0.664 + 2(0.7 + 0.744)] = 0.725$$

Then, $y_1 = y_0 + k = 0 + 0.274 = 0.274$

$z_1 = z_0 + l = 1 + 0.725 = 1.725$

Again, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$y_1 = 0.274$

$z_1 = 1.725$

Using RK method to find increment on k and l.

$$k_1 = hf_1(x_1, y_1, z_1)$$
$$= 0.2f_1(0.2, 0.274, 1.725)$$
$$= 0.345$$
$$l_1 = hf_2(x_1, y_1, z_1)$$
$$= 0.2 (0.2, 0.274, 1.725)$$
$$= 0.674$$
$$k_2 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$$
$$= 0.2f_1(0.3, 0.4465, 2.062)$$
$$= 0.412$$
$$l_2 = hf_2(0.3, 0.4465, 2.062)$$
$$= 0.643$$
$$k_3 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$$
$$= 0.2f_1(0.3, 0.48, 2.046)$$
$$= 0.409$$
$$l_3 = hf_2(0.3, 0.48, 2.046)$$
$$= 0.669$$
$$k_4 = hf_1(x_1 + h, y_1 + k_3, z_1 + l_3)$$
$$= 0.2f_1(0.4, 0.683, 2.394)$$
$$= 0.478$$
$$l_4 = hf_2(0.4, 0.683, 2.394)$$
$$= 0.652$$

Then, $$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0.345 + 0.478 + 2(0.412 + 0.409)$$
$$= 0.410$$
$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$
$$= \frac{1}{6}[0.674 + 0.652 + 2(0.643 + 0.669)]$$
$$= 0.658$$

Now, $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$\therefore$ $y_2 = y(0.4) = y_1 + k = 0.274 + 0.410 = 0.684$

8. **Given the boundary value problem: $y'' = 6x$ with $y(1) = 2$ and $y(2) = 9$. Solve it in the interval (1, 2) by using RK method of second order (take, $h = 0.5$ and guess value = 3.25)** [2014/Spring]

Solution:

Given that;

$$y'' = 6x \quad \text{.... (1)}$$
$$y(1) = 2$$
$$y(2) = 9$$
$$h = 0.5$$

Let, $y' = z = f_1(x, y, z)$

$y'' = z'$

So equation (1) becomes,

$$z' = 6x = f_2(x, y, z)$$

Subjected to

$$y(1) = 2 = z(1)$$

Initial guess value = 3.25

Now, from RK method of second order

Iteration 1:

$x_0 = 1, \quad y_0 = 2, \quad z_0 = 2$

$$k_1 = hf_1(x_0, y_0, z_0)$$
$$= 0.5f_1(1, 2, 2)$$
$$= 0.5 \times 2$$
$$= 1$$
$$l_1 = hf_2(x_0, y_0, z_0)$$
$$= 0.5 \times 6 \times 1$$
$$= 3$$
$$k_2 = hf_1(x_0 + h, y_0 + k_1, z_0 + l_1)$$
$$= 0.5f_1(1.5, 3, 5)$$
$$= 0.5 \times 5$$
$$= 2.5$$
$$l_2 = hf_2(1.5, 3, 5)$$
$$= 0.5 \times 6 \times 1.5$$
$$= 4.5$$

Then, $$k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(1 + 2.5) = 1.75$$
$$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(4.5 + 3) = 3.75$$

So, $y_1 = y_0 + k = 2 + 1.75 = 3.75$

$z_1 = z_0 + l = 2 + 3.75 = 5.75$

$x_1 = x_0 + h = 1 + 0.5 = 1.5$

Again, iteration 2:

$$k_1 = hf_1(x_1, y_1, z_1)$$
$$= 0.5f_1(1.5, 3.75, 5.75)$$
$$= 0.5 \times 5.75$$
$$= 2.875$$
$$l_1 = hf_2(1.5, 3.75, 5.75)$$
$$= 0.5 \times 6 \times 1.5$$
$$= 4.5$$
$$k_2 = hf_1(1.5 + 0.5, 3.75 + 2.875, 5.75 + 4.5)$$
$$= 0.5f_1(2, 6.62, 10.25)$$
$$= 0.5 \times 10.25$$
$$= 5.125$$
$$l_2 = hf_2(2, 6.62, 10.25)$$
$$= 0.5 \times 6 \times 2$$
$$= 6$$

so, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(2.875 + 5.125) = 4$

$$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(4.5 + 6) = 5.25$$

Then, $x_2 = x_1 + h = 1.5 + 0.5 = 2$

$$y_2 = y_1 + k = 3.75 + 4 = 7.75$$
$$z_2 = z_1 + l = 5.75 + 5.25 = 11$$

Thus, we obtain, $y(2) = 7.75 < y(2) = 9$ and can be further denoted as $y_B = B = 9$ giving $B_1 = 7.75$.

9. Using Euler's method solve the given differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6$, $y(0) = 0$, $y'(0) = 1$ with $h = 0.2$ for $y(0.4) = ?$ [2015/Fall]

Solution:

Given that;

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6 \quad \text{..... (1)}$$

or, $y'' + 2y' - 3y = 6$ ..... (A)

Let, $y' = \frac{dy}{dx} = z$

Then, $y'' = z'$

So, equation (1) becomes

$$z' + 2z - 3y = 6 \quad \text{..... (B)}$$

or, $z' = 6 + 3y - 2z$

Subject to

$y(0) = 0 \rightarrow x_0 = 0, \; y_0 = 0$

$y'(0) = 1 \rightarrow z_0 = 1$ at $h = 0.2$

Now, using Euler's method

$$y_1 = y(0.2) = y_0 + h\frac{dy_0}{dx_0} = 0 + 0.2(z_0) = 0.2 \times 1 = 0.2$$
$$z_1 = z(0.2) = z_0 + hz'(x_0) \text{ from equation (B)}$$
$$= 1 + 0.2(6 + 3y_0 - 2z_0)$$
$$= 1 + 0.2[6 + 3(0) - 2(1)] = 1.8$$

Again, $y(0.4) = y_1 + hy'(x_1) = y_1 + h\frac{dy_1}{dx_1} = y_1 + h(z_1)$

$$= 0.2 + 0.2(1.8) = 0.56$$

$\therefore$ $y(0.4) = 0.56$ is the required solution.

10. Solve the following differential equation within $0 \le x \le 0.5$ using RK 4th order method. $20\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4y = 5$, $y(0) = 0$, $y'(0) = 0$. Take $h = 0.25$. [2015/Fall]

Solution:

Given that;

$$20\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4y = 5$$

Let, $\frac{dy}{dx} = y' = z = f_1(x, y, z)$

Then, $\frac{d^2y}{dx^2} = y'' = z' = f_2(x, y, z)$

or, $20z' + 2z - 4y = 5$

or, $z' = \frac{5 - 2z + 4y}{20} = f_2(x, y, z)$

Subject to

$y(0) = 0 \rightarrow x_0 = 0, y_0 = 0$

$y'(0) = 0 \rightarrow z_0 = 0$

and, $h = 0.25$

Now, by RK 4th order method

$$k_1 = hf_1(x_0, y_0, z_0)$$
$$= 0.25f_1(0, 0, 0)$$
$$= 0$$
$$l_1 = hf_2(x_0, y_0, z_0)$$
$$= 0.25\left(\frac{5 - 0 + 0}{20}\right)$$
$$= 0.0625$$
$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= hf_1(0.125, 0, 0.03125)$$
$$= 0.25 \times 0.03125$$
$$= 0.00781$$

$$l_2 = hf_2\,(0.125, 0, 0.03125)$$
$$= 0.25 \times \left(\frac{5 - 2(0.03125) + 4(0)}{20}\right)$$
$$= 0.06171$$
$$k_3 = hf_1\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$
$$= 0.25\,(0.125, 0.0039, 0.0308)$$
$$= 0.0077$$
$$l_3 = hf_2\,(0.125, 0.0039, 0.0308)$$
$$= 0.0619$$
$$k_4 = hf_1\,(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.25f_1\,(0.25, 0.0077, 0.0619)$$
$$= 0.0154$$
$$l_4 = hf_2\,(0.25, 0.0077, 0.0619)$$
$$= 0.0613$$

Then, $$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0 + 0.0154 + 2(0.00781 + 0.0077)]$$
$$= 0.0077$$
$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$
$$= \frac{1}{6}[0.0625 + 0.0613 + 2(0.06171 + 0.0619)]$$
$$= 0.0618$$

so, $$x_1 = x_0 + h = 0 + 0.25 = 0.25$$
$$y_1 = y_0 + k = 0 + 0.0077 = 0.0077$$
$$z_1 = z_0 + l = 0 + 0.0618 = 0.0618$$

Again, $$k_1 = hf_1\,(x_1, y_1, z_1)$$
$$= 0.25f_1\,(0.25, 0.0077, 0.0618)$$
$$= 0.25 \times 0.0618$$
$$= 0.0154$$
$$l_1 = hf_2\,(x_1, y_1, z_1)$$
$$= 0.613$$
$$k_2 = hf_1\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2},\ z_1 + \frac{l_1}{2}\right)$$
$$= 0.25f_1\,(0.375, 0.0154, 0.0924)$$
$$= 0.0231$$
$$l_2 = hf_2\,(0.375, 0.0154, 0.0924)$$
$$= 0.609$$

$$k_3 = hf_1\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2},\ z_1 + \frac{l_2}{2}\right)$$
$$= 0.25f_1\,(0.375, 0.0192, 0.0922)$$
$$= 0.0230$$
$$l_3 = hf_2\,(0.375, 0.0192, 0.0922)$$
$$= 0.0611$$
$$k_4 = hf_1\,(x_1 + h, y_1 + k_3, z_1 + l_3)$$
$$= 0.25f_1\,(0.5, 0.0307, 0.1229)$$
$$= 0.0307$$
$$l_4 = hf_2\,(0.5, 0.0307, 0.1229)$$
$$= 0.0609$$

Now,
$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0.0154 + 0.0307 + 2(0.0231 + 0.0230)]$$
$$= 0.0229$$
$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$
$$= \frac{1}{6}[0.0613 + 0.0609 + 2(0.0609 + 0.0611)]$$
$$= 0.0610$$

Then, $$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$$
$$y_2 = y_1 + k = 0.0077 + 0.0229 = 0.0306$$
$$z_2 = z_1 + l = 0.0618 + 0.0610 = 0.1228$$

11. Solve the following differential equation within $0 \le x \le 0.5$ using RK 4th order method. $10\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 5$, $y(0) = 0$, $y'(0) = 0$. **[2015/Spring]**

Take h = 0.25.

Solution:

Given that;

$$10\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4y = 5$$

Let, $$\frac{dy}{dx} = y' = z = f_1\,(x, y, z)$$

Then, $$\frac{d^2y}{dx^2} = y'' = z' = f_2\,(x, y, z)$$

or, $$z' = \frac{5 - 2z + 4y}{10} = f_2\,(x, y, z)$$

Subject to

$$y(0) = 0 \rightarrow x_0 = 0, y_0 = 0$$
$$y'(0) = 0 \rightarrow z_0 = 0$$

and, $h = 0.25$

Now, by RK 4th order method,

$$k_1 = hf_1(x_0, y_0, z_0)$$
$$= 0.25f_1(0, 0, 0)$$
$$= 0$$
$$l_1 = hf_2(x_0, y_0, z_0)$$
$$= 0.25\left(\frac{5 - 2(0)0 + 4(0)}{10}\right)$$
$$= 0.125$$
$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= 0.25f_1(0.125, 0, 0.0625)$$
$$= 0.25 \times 0.0625$$
$$= 0.0156$$
$$l_2 = hf_2(0.125, 0, 0.0625)$$
$$= 0.25 \times \left(\frac{5 - 2(0.0625) + 4(0)}{10}\right)$$
$$= 0.1218$$
$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= 0.25f_1(0.125, 0.0078, 0.0609)$$
$$= 0.0152$$
$$l_3 = hf_2(0.125, 0.0078, 0.0609)$$
$$= 0.1227$$
$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.25f_1(0.25, 0.0152, 0.1227)$$
$$= 0.0306$$
$$l_4 = hf_2(0.25, 0.0152, 0.1227)$$
$$= 0.1203$$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0 + 0.0306 + 2(0.0156 + 0.0152)]$$
$$= 0.0153$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$
$$= \frac{1}{6}[0.125 + 0.1203 + 2(0.1218 + 0.1227)]$$
$$= 0.1223$$

So,
$$x_1 = x_0 + h = 0 + 0.25 = 0.25$$
$$y_1 = y_0 + k = 0 + 0.0153 = 0.0153$$
$$z_1 = z_0 + l = 0 + 0.01223 = 0.1223$$

Again,
$$k_1 = hf_1(x_1, y_1, z_1)$$
$$= 0.25f_1(0.25, 0.0153, 0.1223)$$
$$= 0.0305$$
$$l_1 = hf_2(0.25, 0.0153, 0.1223)$$
$$= 0.1204$$
$$k_2 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$$
$$= 0.25f_1(0.375, 0.0305, 0.1825)$$
$$= 0.0456$$
$$l_2 = hf_2(0.375, 0.0305, 0.1825)$$
$$= 0.1189$$
$$k_3 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$$
$$= 0.25f_1(0.375, 0.0381, 0.1817)$$
$$= 0.0454$$
$$l_3 = hf_2(0.375, 0.0381, 0.1817)$$
$$= 0.1197$$
$$k_4 = hf_1(x_1 + h, y_1 + k_3, z_1 + l_3)$$
$$= 0.25f_1(0.5, 0.0607, 0.242)$$
$$= 0.0605$$
$$l_4 = hf_2(0.5, 0.0607, 0.242)$$
$$= 0.1189$$

Then,
$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0.0305 + 0.0605 + 2(0.0456 + 0.0454)]$$
$$= 0.0455$$
$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$
$$= \frac{1}{6}[0.1204 + 0.1189 + 2(0.1189 + 0.1197)]$$
$$= 0.1194$$

Now,

$$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$$
$$y_2 = y_1 + k = 0.0153 + 0.0455 = 0.0608$$
$$z_2 = z_1 + l = 0.1223 + 0.1194 = 0.2417$$

12. Solve the given differential equation by RK 4th order method y" – xy' + y = 0 with initial condition y(0) = 3, y'(0) = 0 for y(0.2) taking h = 0.2. [2016/Fall]

Solution:

Given that;

$$y'' - xy' + y = 0 \quad \text{.....(1)}$$

Let $y'' = z'$ and $y' = z$

So equation (1) becomes

$$z' - xz + y = 0$$

or, $z' = xz - y$

We have,

$$y' = z = f_1(x, y, z)$$

and, $z' = xz - y = f_2(x, y, z)$

Subject to

$$y(0) = 3 \rightarrow x_0 = 0, y_0 = 3$$
$$y'(0) = 0 \rightarrow z_0 = 0$$

Taking h = 0.2.

Now, using RK 4th order method,

$$k_1 = hf_1(x_0, y_0, z_0)$$
$$= 0.2f_1(0, 3, 0)$$
$$= 0.2 \times 0$$
$$= 0$$

$$l_1 = hf_2(x_0, y_0, z_0)$$
$$= 0.2f_2(0, 3, 0)$$
$$= 0.2(0 \times 0 - 3)$$
$$= -0.6$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= 0.2f_1(0.1, 3, -0.3)$$
$$= -0.06$$

$$l_2 = hf_2(0.1, 3, -0.3)$$
$$= -0.606$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= 0.2f_1(0.1, 2.97, -0.303)$$
$$= -0.0606$$

$$l_3 = hf_2(0.1, 2.97, -0.303)$$
$$= -0.6$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.2f_1(0.2, 2.9394, -0.6)$$
$$= -0.12$$

$$l_4 = hf_2(0.2, 2.9394, -0.6)$$
$$= -0.6118$$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0 + (-0.12) + 2(-0.06 - 0.0606)]$$
$$= -0.0602$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$
$$= \frac{1}{6}[-0.6 - 0.6118 + 2(-0.606 - 0.6)]$$
$$= -0.6039$$

Then, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$\therefore$ $y_1 = y(0.2) = y_0 + k = 3 - 0.0602 = 2.9398$

13. Solve the differential equation y' = x + y using approximate method within 0 ≤ x ≤ 0.2 with initial condition y(0) = 1 and stepsize h = 0.1. [2016/Fall]

Solution:

Given that;

$$y' = x + y, \quad 0 \le x \le 0.2$$

Subject to

$$y(0) = 1 \text{ at } h = 0.1$$

$\rightarrow$ $x_0 = 0, y_0 = 1$

Now, using modified Euler's method

Solving in tabular form

S.N.	x	$\frac{dy}{dx} = x + y$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0	0 + 1	–	$1 + 0.1 \times 1 = 1.1$
2	0.1	0.1 + 1.1 = 1.2	$\frac{1 + 1.2}{2} = 1.1$	$1 + 0.1 \times 1.1 = 1.11$
3	0.1	0.1 + 1.11 = 1.21	$\frac{1 + 1.21}{2} = 1.105$	$1 + 0.1 \times 1.105 = 1.1105$
4	0.1	0.1 + 1.1105 = 1.2105	$\frac{1 + 1.2105}{2} = 1.1052$	$1 + 0.1 \times 1.1052 = 1.1105$

Here, last two values are equal at $y_1 = 1.1105$.

S.N.	x	$\frac{dy}{dx} = x + y$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
5	0.1	$0.1 + 1.1105 = 1.2105$	–	$1.1105 + 0.1 \times 1.2105 = 1.2315$
6	0.2	$0.2 + 1.2315 = 1.4315$	$\frac{1.2105 + 1.4315}{2} = 1.3210$	1.2426
7	0.2	1.4426	1.3265	1.2431
8	0.2	1.4431	1.3268	1.2431

Here, last two values are equal at $y_2 = 1.2431$.

Hence the required solution within $0 \le x \le 0.2$ are,

$$x_0 = 0, \quad y_0 = 1$$
$$x_1 = 0.1, \quad y_1 = 1.1105$$
and, $$x_2 = 0.2, \quad y_2 = 1.2431$$

14. Employ Taylor's method to obtain approximate value of y at x = 0.2 for the differential equation. **[2016/Spring]**

$$\mathbf{y' = 2y + e^x, \; y(0) = 0}$$

Solution:

We have,

$$y' = 2y + e^x \quad \text{and} \quad y(0) = 0$$

Then, $y'(0) = 2y(0) + e^0 = 2(0) + 1 = 1$

Now, differentiating successively and substituting

= 0 and $y_0 = 0$ we get,

$$y'' = 2y' + e^x, \quad y''(0) = 2y'(0) + e^0 = 2(1) + 1 = 3$$
$$y''' = 2y'' + e^x, \quad y'''(0) = 2y''(0) + 1 = 2(3) + 1 = 7$$
$$y^{iv} = 2y''' + e^x, \quad y^{iv}(0) = 2y'''(0) + 1 = 2(7) + 1 = 15$$

and so on.

Now, putting these values in the Taylor's series. We have,

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots\ldots$$
$$= 0 + x(1) + \frac{x^2}{2}(3) + \frac{x^3}{6}(7) + \frac{x^4}{24}(15) + \ldots\ldots$$
$$= x + \frac{3x^2}{2} + \frac{7x^2}{6} + \frac{5}{8}x^4 + \ldots\ldots$$

Hence, $y(0.2) = 0.2 + \frac{3(0.2)^2}{2} + \frac{7(0.2)^3}{6} + \frac{5(0.2)^4}{8} + \ldots\ldots$

$\therefore \quad y(0.2) = 0.2703$

15. Using Runge-Kutta second order method, solve the differential equation $y'' = xy' - y$; $y(0) = 3$, $y'(0) = 0$ for $x = 0, 0.2, 0.4$. **[2016/Spring]**

Solution:

Given that;

$$y'' = xy' - y \quad \ldots\ldots (1)$$

Let $y' = z$

Then, $y'' = z'$

So equation (1) becomes

$$z' = xz - y = f_2(x, y, z)$$

and, $y' = z = f_1(x, y, z)$

Subject to

$$y(0) = 3 \rightarrow x_0 = 0, \quad y_0 = 3$$
$$y'(0) = 0 \rightarrow z_0 = 0$$

Taking $h = 0.2$

Now, using Runge-Kutta second order method

$$k_1 = hf_1(x_0, y_0, z_0) = 0.2f_1(0, 3, 0) = 0.2(0) = 0$$
$$l_1 = hf_2(x_0, y_0, z_0) = 0.2f_2(0, 3, 0) = 0.2[0(0) - 3] = -0.6$$
$$k_2 = hf_1(x_0 + h, y_0 + k_1, z_0 + l_1) = 0.2f_1(0.2, 3, -0.6) = -0.12$$
$$l_2 = 0.2f_2(0.2, 3, -0.6) = -0.624$$

Then, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}[0 + (-0.12)] = -0.06$

$$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(-0.6 - 0.624) = -0.612$$

and, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$$y_1 = y_0 + k = 3 + (-0.06) = 2.94$$
$$z_1 = z_0 + l = 0 - 0.612 = -0.612$$

Again, $k_1 = hf_1(x_1, y_1, z_1) = 0.2f_1(0.2, 2.94, -0.612) = -0.1224$

$$l_1 = hf_2(0.2, 2.94, -0.612) = -0.6124$$

$$k_2 = hf_1(x_1 + h, y_1 + k_1, z_1 + l_1)$$
$$= 0.2f_1(0.4, 2.8176, -1.2244)$$
$$= -0.2448$$
$$l_2 = hf_2(0.4, 2.8176, -1.2244)$$
$$= -0.6614$$

Then, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(-0.1224 - 0.2448) = -0.1836$

$$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(-0.6124 - 0.6614) = -0.6369$$

and, $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$$y_2 = y_1 + k = 2.94 - 0.1836 = 2.7564$$
$$z_2 = z_1 + l = -0.612 - 0.6369 = -1.2489$$

16. Solve the differential equation $y' = y + \sin x$ using appropriate method within $0 \le x \le 0.2$ with initial condition $y(0) = 2$ and step size = 0.1.

[2017/Fall]

Solution:

Given that;

$$y' = y + \sin x \quad , \quad 0 \le x \le 0.2$$

and, $y(0) = 2$

$\rightarrow$ $x_0 = 0, \quad y_0 = 2$

Taking step size h = 0.1

Now, using Euler's method for solving the differential equation. We have,

$$y_{new} = y_{old} + h\frac{dy}{dx} = y_{old} + hf(x, y)$$

en, $y_1 = y_0 + hf(x_0, y_0)$

$$= 2 + 0.1[2 + \sin(0)]$$
$$y_1 = 2.2$$
$$y_2 = y_1 + hf(x_1, y_1)$$
$$= 2.2 + 0.1[2.2 + \sin(0.1)]$$

$\therefore$ $y_2 = 2.429$

and, $y_3 = y_2 + hf(x_2, y_2)$

$$= 2.429 + 0.1[2.429 + \sin(0.2)]$$

$\therefore$ $y_3 = 2.691$

17. Apply RK-4 method to solve y(0.2) for the equation $\frac{d^2y}{dx^2} = x\frac{dy}{dx} - y$ given that y = 1 and $\frac{dy}{dx} = 0$ when x = 0. (Assume h = 0.2)

[2017/Fall, 2017/Spring]

Solution:

Given that;

$$\frac{d^2y}{dx^2} = x\frac{dy}{dx} - y$$

or, $y'' = xy' - y = 0$ ..... (1)

Let, $y' = z$

Then, $y'' = z'$

So, equation (1) becomes

$$z' = xz - y = f_2(x, y, z)$$

and, $y' = z = f_1(x, y, z)$

Also,

$$x_0 = 0, \quad y_0 = 1, \quad z_0 = 0$$

At $h = 0.2$

Now, using RK-4 method

$$k_1 = hf_1(x_0, y_0, z_0)$$
$$= 0.2f_1(0, 1, 0)$$
$$= 0.2 \times 0$$
$$= 0$$
$$l_1 = hf_2(0, 1, 0)$$
$$= 0.2[0(0) - 1]$$
$$= -0.2$$
$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= 0.2f_1(0.1, 1, -0.1)$$
$$= -0.02$$
$$l_2 = hf_2(0.1, 1, -0.1)$$
$$= 0.2[0.1(-0.1) - 1]$$
$$= -0.202$$
$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= 0.2f_1(0.1, 0.99, -0.101)$$
$$= -0.0202$$
$$l_3 = hf_2(0.1, 0.99, -0.101)$$
$$= -0.2$$
$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.2f_1(0.2, 0.979, -0.2)$$
$$= -0.04$$
$$l_4 = hf_2(0.2, 0.979, -0.2)$$
$$= -0.203$$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0 - 0.04 + 2(-0.02 - 0.0202)]$$
$$= -0.02006$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$

$$= \frac{1}{6}[-0.2 - 0.203 + 2(-0.202 - 0.2)]$$

$$= -0.2011$$

Then, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$y_1 = y_0 + k = 1 - 0.02006 = 0.9799$

$z_1 = z_0 + l = 0 - 0.2011 = -0.2011$

Hence, $y(0.2) = 0.9799$ is the required solution.

18. Solve the given differential equation by RK 4th order method $y'' - x^2y' - 2xy = 0$ with initial condition $y(0) = 1$ $y'(0) = 0$, for $y(0.1)$ taking $h = 0.1$. **[2018/Fall]**

Solution:

Given that;

$$y'' - x^2y' - 2xy = 0 \quad \text{....(1)}$$

Let, $y' = z$

Then, $y'' = z'$

So, equation (1) becomes

$$z' = x^2z + 2xy = f_2(x, y, z)$$

and, $y' = z = f_1(x, y, z)$

Subject to

$y(0) = 1 \rightarrow x_0 = 0, \quad y_0 = 1$

$y'(0) = 0 \rightarrow z_0 = 0$

Taking $h = 0.1$

Now, using RK-4th method

$$k_1 = hf_1(x_0, y_0, z_0) = 0.1f_1(0, 1, 0) = 0.1 \times 0 = 0$$

$$l_1 = hf_2(0, 1, 0) = 0.1[0^2(0) + 2(0)(1)] = 0$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = 0.1f_1(0.05, 1, 0) = 0$$

$$l_2 = hf_2(0.05, 1, 0) = 0.01$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = 0.1f_1(0.05, 1, 0.005) = 0.0005$$

$$l_3 = hf_2(0.05, 1, 0.005) = 0.010$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.1f_1(0.1, 1.0005, 0.010) = 0.001$$

$$l_4 = hf_2(0.1, 1.0005, 0.010) = 0.020$$

Then,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)] = \frac{1}{6}[0 + 0.001 + 2(0 + 0.0005)] = 0.00033$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)] = \frac{1}{6}[0 + 0.02 + 2(0.01 + 0.01)] = 0.01$$

Now,

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$\therefore$ $y_1 = y(0.1) = y_0 + k = 1 + 0.00033 = 1.00033$

$\therefore$ $z_1 = z_0 + l = 0 + 0.01 = 0.01$

19. Solve the differential equation $y' = y - \frac{2x}{y}$ using appropriate method within $0 \leq x \leq 0.2$ with initial conditions $y(0) = 1$ and step size $h = 0.1$. **[2018/Fall]**

Solution:

Given that;

$$y' = y - \frac{2x}{y}, \quad 0 \leq x \leq 0.2$$

and, $y(0) = 1$

$\rightarrow$ $x_0 = 0$; $y_0 = 1$

Step size $= h = 0.1$

Now, using Euler's method

$$f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1 - \frac{2(0)}{1} = 1$$

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1 \times (1) = 1.1$$

Again, $f(x_1, y_1) = y_1 - \frac{2x_1}{y_1} = 1.1 - \frac{2(0.1)}{1.1} = 0.918$

$y_2 = y_1 + hf(x_1, y_1) = 1.1 + 0.1 \times 0.918 = 1.1918$

Hence, the required solutions are

$\therefore \quad x_0 = 0 \quad , \quad y_0 = 1$

$\therefore \quad x_1 = 0.1 \quad , \quad y_1 = y(0.1) = 1.1$

$\therefore \quad x_2 = 0.2 \quad , \quad y_2 = y(0.2) = 1.1918$

20. Use the Runge-Kutta 4^th^ order to solve $10\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$ for the interval $0 \le x \le 0.4$ with $h = 0.1$. **[2018/Spring]**

Solution:

Given that;

$$10\frac{dy}{dx} = x^2 + y^2 \quad , \quad 0 \le x \le 0.4$$

or, $$y' = \frac{x^2 + y^2}{10}$$

Subjected to

$y(0) = 1 \rightarrow x_0 = 0, \quad y_0 = 1$

Taking $h = 0.1$

Now, using Runge-Kutta 4th order method

$$k_1 = hf(x_0, y_0) = 0.1f(0, 1) = 0.1\left(\frac{0^2 + 1^2}{10}\right) = 0.01$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f(0.05, 1.005) = 0.0101$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f(0.05, 1.00505) = 0.0101$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.0101) = 0.0103$$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$

$= \frac{1}{6}[0.01 + 0.0103 + 2(0.0101 + 0.0101)]$

$= 0.01011$

so, $x_1 = x_0 + h = 0 + 0.1 = 0.1$

$\therefore \quad y_1 = y_0 + k = 1 + 0.01011 = 1.01011$

Again, $k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.01011) = 0.0103$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1f(0.15, 1.0152) = 0.0105$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1f(0.15, 1.0153) = 0.01053$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1f(0.2, 1.0206) = 0.0108$$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)] = 0.0105$

so, $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$\therefore \quad y_2 = y_1 + k = 1.01011 + 0.0105 = 1.0206$

Again,

$$k_1 = hf(x_2, y_2) = 0.1f(0.2, 1.0206) = 0.0108$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.1f(0.25, 1.026) = 0.0111$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.1f(0.25, 1.0261) = 0.0111$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = 0.1f(0.3, 1.0317) = 0.0115$$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)] = 0.0111$

so, $x_3 = x_2 + h = 0.3$

$\therefore \quad y_3 = y_2 + k = 1.0206 + 0.0111 = 1.0317$

Again,

$$k_1 = hf(x_3, y_3) = 0.1f(0.3, 1.0317) = 0.0115$$

$$k_2 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right) = 0.1f(0.35, 1.0374) = 0.0119$$

$$k_3 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right) = 0.1f(0.35, 1.0376) = 0.0119$$

$$k_4 = hf(x_3 + h, y_3 + k_3) = 0.1f(0.4, 1.0436) = 0.0124$$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)] = 0.0119$

so, $x_4 = x_3 + h = 0.4$

$\therefore \quad y_4 = y_3 + k = 1.0317 + 0.0119 = 1.0436.$

21. Solve the boundary value problem

$y''(x) = y(x)$,

$y(0) = 0$ and $y(1) = 1.1752$ by shooting method, **[2018/Spring]**

taking $m_0 = 0.8$ and $m_1 = 0.9$

Solution:

Given that;

$m_0 = 0.8$ and $m_1 = 0.9$ be initial guess for $y'(0) = m$

Then, using shooting method,

$y'' = y(x) \quad , \quad y(0) = 0$ gives

$y'(0) = m \quad , \quad y''(0) = y(0) = 0$

$y'''(0) = y'(0) = m \quad , \quad y^{iv}(0) = y''(0) = 0$

$y^{v}(0) = y'''(0) = m \quad , \quad y^{vi}(0) = y^{iv}(0) = 0$

and so on.

Putting these values in the Taylor's series. We have,

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots\ldots$$

$$= m\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \ldots\ldots\right)$$

$\therefore$ $y(1) = m(1 + 0.1667 + 0.0083 + 0.0002 + \ldots\ldots$

$= m\,(1.175)$

For $m_0 = 0.8$, $y(m_0, 1) = 0.85 \times 1.175 = 0.94$

For $m_1 = 0.9$, $y(m_1, 1) = 0.9 \times 1.175 = 1.057$

So, for better approximation of m,

$$m_2 = m_1 - (m_1 - m_0)\frac{y(m_1, 1) - y(1)}{y(m_1, 1) - y(m_0, 1)}$$

$$= 0.9 - (0.1)\frac{1.057 - 1.175}{1.057 - 0.94}$$

$$= 0.9 + 0.10085$$

$$= 1.00085$$

Here, $m_2 = 1.00085$ is closer to the exact value of $y'(0) = 0.996$.

We know solve the initial value problem

$y''(x) = y(x), y(0) = 0, y'(0) = m_2$

Taylor's series solution is given by

$y(m_2, 1) = m_2\,(1.175) = 1.00085 \times 1.175 = 1.17599$

Hence, the solution at $x = 1$ is $y = 1.176$ which is close to the exact value of $y(1) = 1.1752$.

22. Use Picard's method to approximate the value of y when x = 0.1, x = 0.2 and x = 0.4, given that y = 1 at x = 0 and $\frac{dy}{dx} = 1 + xy$ correct to three decimal places. (Use upto second approximation) [2019/Fall]

Solution:

Given that;

$$\frac{dy}{dx} = 1 + xy = f(x, y)$$

and, $x_0 = 0$, $y_0 = 1$

Using Picard's method, we have,

$$y = y_0 + \int_{x_0}^{x} f(x, y)\,dx$$

First approximation, put $y = 1$ in the integrand

$$y_1 = 1 + \int_0^x [1 + x(1)]\,dx = 1 + \left[x + \frac{x^2}{2}\right]_0^x = 1 + x + \frac{x^2}{2}$$

Second approximation, put $y = 1 + x + \frac{x^2}{2}$ in the integrand

$$y_2 = 1 + \int_0^x \left[1 + x\left(1 + x + \frac{x^2}{2}\right)\right] dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right) dx$$

$$= 1 + \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right]_0^x$$

$\therefore$ $y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$

Now, using the first approximation and taking

$x = 0.1, 0.2, 0.4$

We have,

$y_1(0.1) = 1 + x + \frac{x^2}{2} = 1 + 0.1 + \frac{(0.1)^2}{2} = 1.105$

$y_1(0.2) = 1.06$

$y_1(0.4) = 1.24$

Now, using the second approximation and taking

$x = 0.1, 0.2, 0.4$

We have,

$y_2(0.1) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} = 1.1053$

$y_2(0.2) = 1.2228$

$y_2(0.4) = 1.5045$

Also, the exact solution of $y' = 1 + xy$ is e^x

$y(0) = e^0 = 1$

$y(0.1) = e^{0.1} = 1.1051$

$y(0.2) = e^{0.2} = 1.221$

$y(0.4) = e^{0.4} = 1.492$

Here, $y(0.1) = 1.105$ is correct upto 3 decimal places.

For $y(0.2)$ using $y(0.1) = 1.105$ as initial value.

First approximation, put $y = 1.105$ in the integrand

$$y_1 = 1.105 + \int_{0.1}^{x} [1 + x(1.105)]dx$$

$$= 1.105 + \left[x + \frac{x^2}{2}(1.105)\right]_{0.1}^{x}$$

$$= 1.105 + x + 0.5525x^2 - 0.1 - 0.0055$$

$$= 0.999 + x + 0.5525x^2$$

Second approximation, put $y = 0.999 + x + 0.5525x^2$ in the integrand

$$y_2 = 1.105 + \int_{0.1}^{x} [1 + x(0.999 + x + 0.5525x^2)]dx$$

$$= 1.105 + \int_{0.1}^{x} [1 + 0.999x + x^2 + 0.5525x^3]dx$$

$$= 1.105 + \left[x + \frac{0.999x^2}{2} + \frac{x^3}{3} + \frac{0.5525x^4}{4}\right]_{0.1}^{x}$$

$$= 1.105 + x + 0.499x^2 + 0.333x^3 + 0.1381x^4 - 0.1 - \frac{0.999(0.1)^2}{2} - \frac{(0.1)^3}{3} - \frac{0.5525(0.1)^4}{4}$$

$$= 0.999 + x + 0.499x^2 + 0.333x^3 + 0.1381x^4$$

Now, using the second approximation and taking $x = 0.2, 0.4$

We have,

$$\therefore \quad y(0.2) = 0.999 + 0.2 + 0.499(0.2)^2 + 0.333(0.2)^3 + 0.1381(0.2)^4$$

$$= 1.2218$$

$$\therefore \quad y(0.4) = 1.5036$$

Here, $y(0.2) = 1.2218$ is correct upto three decimal places compared to exact solution.

For, $y(0.4)$, using $y(0.2) = 1.2218$ as initial value.

First approximation, put $y = 1.2218$ in the integrand.

$$y_1 = 1.2218 + \int_{0.2}^{x} [1 + x(1.2218)]dx$$

$$= 1.2218 + \left[x + \frac{1.2218x^2}{2}\right]_{0.2}^{x}$$

$$= 1.2218 + x + 0.6109x^2 - 0.2 - 0.0244$$

$$= 0.9974 + x + 0.6109x^2$$

Second approximation, put $y = 0.9974 + x + 0.6109x^2$ in the integrand.

$$y_2 = 1.2218 + \int_{0.2}^{x} [1 + x(0.9974 + x + 0.6109x^2)]\,dx$$

$$= 1.2218 + \left[x + \frac{0.9974x^2}{2} + \frac{x^3}{3} + \frac{0.6109x^4}{4}\right]_{0.2}^{x}$$

$$= 0.9989 + x + \frac{0.9974x^2}{2} + \frac{x^3}{3} + \frac{0.6109x^4}{4}$$

Now, using the second approximation and taking

$x = 0.4$

We have,

$$y(0.4) = 0.9989 + 0.4 + \frac{0.9974}{2}(0.4)^2 + \frac{(0.4)^3}{3} + \frac{0.6109}{4}(0.4)^4$$

$$\therefore \quad y(0.4) = 1.5039$$

Here, $y(0.4) = 1.5039$ is correct upto 3 decimal places.

Thus, $y(0.1) = 1.105$

$y(0.2) = 1.221$

$y(0.4) = 1.503$

23. Using Runge-Kutta method of second order (RK-2), obtain a solution of the equation $y'' = y + xy'$ with initial condition $y(0) = 1$, $y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$, taking $h = 0.1$. **[2019/Fall]**

Solution:

Given that;

$$y'' = xy' + y \quad \text{.....(1)}$$

Let, $y' = z$

Then, $y'' = z'$

So, equation (1) becomes

$$z' = xz + y = f_2(x, y, z)$$

and, $y' = z + f_1(x, y, z)$

Subject to

$y(0) = 1 \rightarrow x_0 = 0, \quad y_0 = 1$

$y'(0) = 0 \rightarrow z_0 = 0$

Taking $h = 0.1$

Now, using Runge-Kutta method of second order,

$$k_1 = hf_1(x_0, y_0, z_0) = 0.1f_1(0, 1, 0) = 0.1 \times 0 = 0$$

$$l_1 = hf_2(0, 1, 0) = 0.1(0(0) + 1) = 0.1$$

$$k_2 = hf_1(x_0 + h, y_0 + k_1, z_0 + l_1) = 0.1f_1(0.1, 1, 0.1) = 0.01$$

$$l_2 = hf_2(0.1, 1, 0.1) = 0.101$$

Then,

$$k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0 + 0.01) = 0.005$$

$$l = \frac{1}{2}(l_1 + l_4) = \frac{1}{2}(0.1 + 0.101) = 0.1005$$

So,

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$\therefore \quad y_1 = y_0 + k = 1 + 0.005 = 1.005$$

$$\therefore \quad z_1 = z_0 + l = 0 + 0.1005 = 0.1005$$

Again, $k_1 = hf_1(x_1, y_1, z_1)$

$= 0.1f_1(0.1, 1.005, 0.1005)$

$= 0.01$

$l_1 = hf_2(0.1, 1.005, 0.1005)$
$= 0.1015$
$k_2 = hf_1(x_1 + h, y_1 + k_1, z_1 + l_1)$
$= 0.1f_1(0.2, 1.015, 0.202)$
$= 0.020$
$l_2 = hf_2(0.2, 1.015, 0.202)$
$= 0.1055$

Then, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.01 + 0.02) = 0.015$

$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(0.1015 + 0.1055) = 0.1035$

Hence,

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$\therefore \quad y_2 = y_1 + k = 1.005 + 0.015 = 1.02$

$\therefore \quad z_2 = z_1 + l = 0.1005 + 0.1035 = 0.204$

24. Solve the given differential equation by Heun's method $y'' - y' - 2y = 3e^{2x}$ with initial condition $y(0) = 0$, $y'(0) = -2$ for $y(0.2)$ taking $h = 0.1$ [2019/Spring]

Solution:

Given that;

$$y'' - y' - 2y = 3e^{2x} \quad \text{..... (1)}$$

Let, $y' = z$

Then, $y'' = z'$

So, equation (1) becomes

$z' - z - 2y = 3e^{2x}$

and, $z' = z + 2y + 3e^{2x}$

Subject to

$y(0) = 1 \quad \rightarrow x_0 = 0, \quad y_0 = 0$

$y'(0) = -2 \quad \rightarrow z_0 = -2$

Taking $h = 0.1$

Now, using Heun's method or modified Euler's method solving in tabular form.

S.N.	x	$y' = z$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0	-2	–	$0 + 0.1 \times (-2) = -0.2$
2	0.1	-1.9	$\frac{-2 - 1.9}{2} = -1.95$	$0 + 0.1 \times (-1.95) = -0.195$
3	0.1	-1.882	$\frac{-2 - 1.882}{2} = -1.94$	$0 + 0.1 \times (-1.94) = -0.194$
4	0.1	-1.881	$\frac{-2 - 1.882}{2} = -1.94$	$0 + 0.1 \times (-1.94) = -0.194$

Here, the last two values are equal at $y_1 = -0.194$.

S.N.	x	$z' = z + 2y + 3e^{2x}$	Mean slope	$z_{new} = z_{old} + h$ (mean slope)
1	0	$-2 + 2(0) + 3e^{2(0)} = 1$	-	$-2 + 0.1 \times 1 = -1.9$
2	0.1	$-1.9 + 2(-0.2) + 3e^{2(0.1)} = 1.364$	$\frac{1 + 1.364}{2} = 1.18$	$-2 + 0.1 \times 1.18 = -1.882$
3	0.1	$-1.88 + 2(-0.195) + 3e^{2(0.1)} = 1.394$	$\frac{1 + 1.394}{2} = 1.19$	$-2 + 0.1 \times 1.19 = -1.881$
4	0.1	$-1.88 + 2(-0.194) + 3e^{2(0.1)} = 1.396$	$\frac{1 + 1.396}{2} = 1.19$	$-2 + 0.1 \times 1.19 = -1.881$

Here, the last two values are equal at $z_1 = -1.881$.

Again,

S.N.	x	$y' = z$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0.1	-1.881	–	$-0.194 + 0.1 \times (-1.881) = -0.382$
2	0.2	-1.741	$\frac{-1.881 - 1.741}{2} = -1.811$	$-0.194 + 0.1 \times (-1.811) = -0.375$
3	0.2	-1.712	$\frac{-1.881 - 1.712}{2} = -1.796$	$-0.194 + 0.1 \times (-1.796) = -0.373$
4	0.2	-1.710	$\frac{-1.881 - 1.710}{2} = -1.795$	$-0.194 + 0.1 \times (-1.795) = -0.373$

Here, the last two values are equal at $y_2 = -0.373$

S.N.	x	$z' = z + 2y + 3e^{2x}$	Mean slope	$z_{new} = z_{old} + h$ (mean slope)
1	0	$-1.881 + 2(-0.194) + 3e^{2(0.1)} = 1.395$	-	$-1.881 + 0.1(1.395) = -1.741$
2	0.1	$-1.741 + 2(-0.382) + 3e^{2(0.2)} = 1.970$	$\frac{1.395 + 1.970}{2} = 1.682$	$-1.881 + 0.1(1.682) = -1.712$
3	0.1	$-1.712 + 2(-0.375) + 3e^{2(0.2)} = 2.013$	$\frac{1.395 + 2.013}{2} = 1.704$	$-1.881 + 0.1(1.704) = -1.710$
4	0.1	$-1.71 + 2(-0.373) + 3e^{2(0.2)} = 2.019$	$\frac{1.395 + 2.019}{2} = 1.707$	$-1.881 + 0.1(1.707) = -1.710$

Here, the last two values are equal at $z_2 = -1.710$.

Hence, the required solution of $y(0.2) = -0.373$.

25. Solve $y' = y + e^x$, $y(0) = 0$ for $y(0.2)$ and $y(0.4)$ by RK-4th order method. [2019/Spring]

Solution:

Given that;

$y' = y + e^x$

$y(0) = 0 \quad \rightarrow x_0 = 0, \quad y_0 = 0$

Taking h = 0.2

Now, using RK-4th order method

$k_1 = hf(x_0, y_0) = 0.2f(0, 0) = 0.2(0 + e^0) = 0.2$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f(0.1, 0.1) = 0.241$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2f(0.1, 0.120) = 0.245$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 0.245) = 0.293$

Then,

$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$

$= \frac{1}{6}[0.2 + 0.293 + 2(0.241 + 0.245)]$

$= 0.244$

so, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$\therefore \quad y_1 = y_0 + k = 0 + 0.244 = 0.244$

Again,

$k_1 = hf(x_1, y_1) = 0.2f(0.2, 0.244) = 0.293$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2f(0.3, 0.39) = 0.347$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2f(0.3, 0.417) = 0.353$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 0.597) = 0.417$

Then,

$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$

$= \frac{1}{6}[0.293 + 0.417 + 2(0.347 + 0.353)]$

$= 0.351$

so, $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$\therefore \quad y_2 = y_1 + k = 0.244 + 0.351 = 0.595$

Hence, y(0.2) = 0.244 and y(0.4) = 0.595 are the required solutions.

26. Applying Runge-Kutta fourth order method to find an approximate value of y when x = 0.3 given that: $y' = 2.5y + e^{0.3x}$ with an initial y(0) = 1, taking h = 0.3 **[2020/Fall]**

Solution:

Given that;

$y' = 2.5y + e^{0.3x}$

$y(0) = 1 \quad \rightarrow x_0 = 0, \quad y_0 = 1$

$h = 0.3$

Now, using Runge-Kutta fourth other method

$k_1 = hf(x_0, y_0)$

$= 0.3f(0, 1)$

$= 0.3[2.5(1) + e^{0.3\times0}]$

$= 1.05$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.3f(0.15, 1.525)$

$= 0.3[2.5(1.525) + e^{0.3(0.15)}]$

$= 1.457$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.3f(0.15, 1.728)$

$= 1.609$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.3f(0.3, 2.609)$

$= 2.285$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$

$= \frac{1}{6}[1.05 + 2.285 + 2(1.457 + 1.609)]$

$= 1.577$

Now,

$x_1 = x_0 + h = 0 + 0.3 + 0.3$

$\therefore \quad y_1 = y(0.3) = y_0 + k = 1 + 1.577 = 2.577$

27. Solve the Boundary value problem (BVP) using shooting method by dividing into four sub-interval employing Euler's method.

$$y'' + 2y' - y = x$$

Subjective to boundary condition y(1) = 2 and y(2) = 4. **[2020/Fall]**

Solution:

Given that;

$y' + 2y' - y = x \quad \text{.....(1)}$

Let $y' = z$

Then, $y'' = z'$

So equation (1) becomes,

$x' + 2z - y = x$

or, $z' = x + y - 2z = f_2(x, y, z)$

and, $y' = z = f_1(x, y, z)$

Subject to

$y(1) = 2 \quad \rightarrow x_0 = 1, \quad y_0 = 2$

Assuming

$y'(1) = 4 \quad \rightarrow z_0 = 4$

And having four subintervals, $h = 0.25$

Now, using shooting method by employing Euler's method

At, $i = 0$, $x_0 = 1$, $y_0 = 2$, $z_0 = 4$, $h = 0.25$

$$\begin{aligned} y_1 &= y_0 + hf_1(x_0, y_0, z_0) \\ &= 2 + 0.25f_1(1, 2, 4) \\ &= 2 + 0.25 \times 4 = 3 \end{aligned}$$

$$\begin{aligned} z_1 &= z_0 + hf_2(x_0, y_0, z_0) \\ &= 4 + 0.25f_2(1, 2, 4) \\ &= 4 + 0.25(1 + 2 - 2 \times 4) \\ &= 2.75 \end{aligned}$$

At, $i = 1$, $x_1 = x_0 + h = 1.25$, $y_1 = 3$, $z_1 = 1.25$, $h = 0.25$

$$\begin{aligned} y_2 &= y_1 + hf_1(x_1, y_1, z_1) \\ &= 3 + 0.25f_1(1.25, 3, 2.75) \\ &= 3 + 0.25(2.75) \\ &= 3.687 \end{aligned}$$

$$\begin{aligned} z_2 &= z_1 + hf_2(x_1, y_1, z_1) \\ &= 2.75 + 0.25f_2(1.25, 3, 2.75) \\ &= 2.75 + 0.25(1.25 + 3 - 2(2.75) \\ &= 2.437 \end{aligned}$$

At, $i = 2$, $x_2 = 1.5$, $y_2 = 3.687$, $z_2 = 2.43$, $h = 0.25$

$$\begin{aligned} y_3 &= y_2 + hf_1(x_2, y_2, z_2) \\ &= 3.687 + 0.25f_1(1.5, 3.687, 2.437) \\ &= 4.296 \end{aligned}$$

$$\begin{aligned} z_3 &= z_2 + hf_2(1.5, 3.687, 2.437) \\ &= 2.515 \end{aligned}$$

At, $i = 3$, $x_3 = 1.75$, $y_3 = 4.296$, $z_3 = 2.515$, $h = 0.25$

$$\begin{aligned} y_4 &= y_3 + hf_1(x_3, y_3, z_3) \\ &= 4.296 + 0.25f_1(1.75, 4.296, 2.515) \\ &= 4.924 \end{aligned}$$

$$\begin{aligned} z_4 &= z_3 + hf_2(x_3, y_3, z_3) \\ &= 2.769 \end{aligned}$$

Here, given $y(2) = 4$

and we obtain $y(2) = y_4 = 4.924$ which is greater than 4.

So, we choose $y'(0) = 1 = z_0$ and carry out the process

At, $i = 0$, $x_0 = 1$, $y_0 = 2$, $z_0 = 1$, $h = 0.25$

$$\begin{aligned} y_1 &= y_0 + hf_1(x_0, y_0, z_0) \\ &= 2 + 0.25f_1(1, 2, 1) \\ &= 2.25 \end{aligned}$$

$$\begin{aligned} z_1 &= z_0 + hf_2(1, 2, 1) \\ &= 1.25 \end{aligned}$$

At, $i = 1$, $x_1 = 1.25$, $y_1 = 2.25$, $z_1 = 1.25$, $h = 0.25$

$$\begin{aligned} y_2 &= y_1 + hf_1(x_1, y_1, z_1) \\ &= 2.562 \end{aligned}$$

$$\begin{aligned} z_2 &= z_1 + hf_2(x_1, y_1, z_1) \\ &= 1.5 \end{aligned}$$

At, $i = 2$, $x_2 = 1.5$, $y_2 = 2.562$, $z_2 = 1.5$, $h = 0.25$

$$\begin{aligned} y_3 &= y_2 + hf_1(x_2, y_2, z_2) \\ &= 2.937 \end{aligned}$$

$$\begin{aligned} z_3 &= z_2 + hf_2(x_2, y_2, z_2) \\ &= 1.765 \end{aligned}$$

At, $i = 3$, $x_3 = 1.75$, $y_3 = 2.937$, $z_3 = 1.765$, $h = 0.25$

$$\begin{aligned} y_4 &= y_3 + hf_1(x_3, y_3, z_3) \\ &= 3.378 \end{aligned}$$

$$\begin{aligned} z_4 &= z_3 + hf_2(x_3, y_3, z_3) \\ &= 2.054 \end{aligned}$$

Here, we obtain,

$y_4 = y(2) = 3.378$ at $y'(0) = 1$

Also, we have,

$y_4 = y(2) = 4.924$ at $y'(0) = 4$

So for better approximation

$P_1 = y'(0) = 4$, $\quad Q_1 = y(2) = 4.924$

$P_2 = y'(0) = 1$, $\quad Q_2 = y(2) = 3.378$

Then to obtain $y(2) = 4 = Q$

$$\begin{aligned} P &= P_1 + \frac{P_2 - P_1}{Q_2 - Q_1}(Q - Q_1) \\ &= 4 + \frac{1 - 4}{3.378 - 4.924}(4 - 4.924) \\ &= 2.206 \end{aligned}$$

So, now using $y'(0) = 2.206 = z_0$ and continuing the process.

At, $i = 0$, $x_0 = 1$, $y_0 = 2$, $z_0 = 2.206$, $h = 0.25$

$$\begin{aligned} y_1 &= y_0 + hf_1(x_0, y_0, z_0) \\ &= 2.551 \end{aligned}$$

$$\begin{aligned} z_1 &= z_0 + hf_2(x_0, y_0, z_0) \\ &= 1.853 \end{aligned}$$

At, i = 1, $x_1 = 1.25, y_1 = 2.551, z_1 = 1.853, h = 0.25$

$$y_2 = y_1 + hf_1(x_1, y_1, z_1) = 3.014$$

$$z_2 = z_1 + hf_2(x_1, y_1, z_1) = 1.876$$

At, i = 2, $x_2 = 1.5, y_2 = 3.014, z_2 = 1.876, h = 0.25$

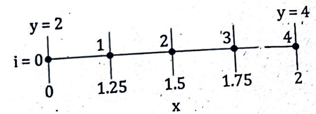
$$y_3 = y_2 + hf_1(x_2, y_2, z_2) = 3.483$$

$$z_3 = z_2 + hf_2(x_2, y_2, z_2) = 2.066$$

At, i = 3, $x_3 = 1.75, y_3 = 3.483, z_3 = 2.066, h = 0.25$

$$y_4 = y_3 + hf_1(x_3, y_3, z_3) = 3.995$$

$$z_4 = z_3 + hf_2(x_3, y_3, z_3) = 2.341$$



Here, we obtain $y_4 = y(2) = 3.995$ which is close to the exact value of $y(2) = 4$.
Hence, the solution at $x = 2$ is $y = 3.995$.

28. Write short notes on: Finite differences. **[2020/Fall]**

Solution: See the topic 5.10 'B'.

29. Write short notes on: Picard's iterative formula. **[2020/Fall]**

Solution: See the topic 5.2.

30. Write short notes on: Solution of 2nd order differential equation. **[2016/Fall]**

Solution: See the topic 5.9.

31. Write short notes on: Boundary value problem. **[2017/Spring]**

Solution: See the topic 5.10.

A boundary value problem is a system of ordinary differential equations with solution and derivative values specified at more than one point. Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two point boundary value problem. In the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions. Boundary value problem arise in several branches of physics as any physical differential equation will have them.

Boundary value problems are similar to initial value problems. A boundary value problem has conditions specified at the extremes ("boundaries") of the independent variable in the equation whereas on initial value problem has all of the conditions specified at the same value of the independent variable (and that value is at the lower boundary of the domain, thus the term "initial value). A boundary value is a data value that corresponds to a minimum or maximum input, internal or output value specified for a system or component.

32. Write short notes on: algorithm for second order Runge-Kutta (RK-2) method. **[2020/Fall]**

Solution:

1. Define function $f(x, y)$
2. Get values of x_0, y_0, h, x_n
   where, x_0 is starting value of x *i.e.*, x_0, x_n is the value of x for which y is to be determined.
3. If $x = x_n$ then go to step 7
   else
   $k_1 = h \times f(x, y)$
   $k_2 = h \times f(x + h, y + k_1)$
4. Compute $k = \left(\frac{k_1 + k_2}{2}\right)$ and,
   $x = x + h$
   $y = y + k$
5. Display x and y
6. Go to step 3
7. Stop.

33. Write short notes on: Taylor series for solving ordinary differential equations. **[2015/Spring]**

Solution: See the topic 5.3.

34. Write short notes on: Algorithm for Euler methods. **[2018/Spring]**

1. Define function $df(x, y)$ *i.e.*, dy/dx
2. Get values of x_0, y_0, h, x
   where, x_0 is x_{n+0}
   x_1 is x_{n+1}
3. Assign $x_1 = x_0$ and $y_1 = y_0$
4. If $x_1 > x$, then go to step 7
   else
   Compute $y_1 += h \times df(x_1, y_1)$
   and, $x_1 += h$ *i.e.*, $x_1 = x_1 + h$
5. Display x_1 and y_1
6. Go to step 4.
7. Stop.

ADDITIONAL QUESTION SOLUTION

1. Solve $y' = \frac{y}{x^2 + y^2}$, $y(0) = 1$ using RK-2 method in the range of 0, 0.5, 1.

Solution:

Given that;

$$y' = \frac{y}{x^2 + y^2} = f(x, y)$$

Subject to

$y(0) = 1 \quad \rightarrow x_0 = 0$ and $y_0 = 1$

in the range of 0, 0.5, 1, so taking $h = 0.5$

Now, using RK-2 method

$$k_1 = hf(x_0, y_0)$$
$$= 0.5 \times f(0, 1)$$
$$= 0.5 \times \left(\frac{1}{0^2 + 1^2}\right)$$
$$= 0.5$$
$$k_2 = hf(x_0 + h, y_0 + k_1)$$
$$= 0.5 \times f(0.5, 1.5)$$
$$= 0.5 \times \left(\frac{1}{0.5^2 + 1.5^2}\right)$$
$$= 0.3$$

Then,

$$k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.5 + 0.3) = 0.4$$

so, $x_1 = x_0 + h = 0 + 0.5 = 0.5$

$y_1 = y_0 + k = 1 + 0.3 = 1.3$

Again,

$$k_1 = hf(x_1, y_1)$$
$$= 0.5 \times f(0.5, 1.3)$$
$$= 0.5 \times \left(\frac{1.3}{0.5^2 + 1.3^2}\right)$$
$$= 0.3351$$
$$k_2 = hf(x_1 + h, y_1 + k_1)$$
$$= 0.5 \times f(1, 1.6351)$$
$$= 0.5 \times \left(\frac{1.6351}{1^2 + 1.6351^2}\right)$$
$$= 0.2226$$

Then,

$$k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.3351 + 0.2226) = 0.2788$$

so, $x_2 = x_1 + h = 0.5 + 0.5 = 1$

$y_2 = y_1 + k = 1.3 + 0.2788 = 1.5788$

2. Solve the BVP: $y'' + 3y' = y + x^2$, $y(0) = 2$, $y(2) = 5$ at $x = 0.5, 1, 1.5$ using finite difference method.

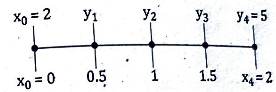
Solution:

Given that;

$$y'' + 3y' = y + x^2 \qquad \text{..... (1)}$$

$y(0) = 2$ and $y(2) = 5$

and, $h = 0.5$



Now, from finite difference approximation, we have,

$$\frac{dy}{dx} = y' = \frac{1}{2h}[y_{i+1} - y_{i-1}]$$

$$\frac{d^2y}{dx^2} = y'' = \frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}]$$

Now using the approximated value in equation (1),

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] + \frac{3}{2h}[y_{i+1} - y_{i-1}] = y_i + x_i^2$$

Put $i = 1$, at $h = 0.5$

$$\frac{1}{0.5^2}(y_2 - 2y_1 + y_0) + \frac{3}{2 \times 0.5}(y_2 - y_0) = y_1 + x_1^2$$

or, $4y_2 - 8y_1 + 4y_0 + 3y_2 - 3y_0 = y_1 + x_1^2$

Substituting the values of y_0 and x_1

$$4y_2 - 8y_1 + 4(2) + 3y_2 - 3(2) = y_1 + (0.5)^2$$

or, $7y_2 - 9y_1 = -1.75 \qquad \text{..... (A)}$

Again,

Put $i = 2$,

$$4(y_3 - 2y_2 + y_1) + 3(y_3 - y_1) = y_2 + x_2^2$$

or, $7y_3 - 9y_2 + 4y_1 - 3y_1 = x_2^2$

Substituting the values

$$y_1 + 7y_3 - 9y_2 = 1^2$$

or, $y_1 - 9y_2 + 7y_3 = 1 \qquad \text{..... (B)}$

Again,

Put i = 3,

$$4(y_4 - 2y_3 + y_2) + 3(y_4 - y_2) = y_3 + x_3^2$$

or, $4y_4 - 8y_3 + 4y_2 - 3y_4 - 3y_2 = y_3 + x_3^2$

Substituting the values

or, $4(5) - 8y_3 - y_3 + 4y_2 - 3y_2 + 3(5) = (1.5)^2$

or, $y_2 - 9y_3 = -32.75$ ..... (C)

Now solving the equations (A), (B) and (C), we get,

$$y_1 = 2.7716$$
$$y_2 = 3.3134$$
$$y_3 = 4.0070$$

Hence, the required solutions are;

$x_1 = 0.5$, $y_1 = 2.7716$

$x_2 = 1$, $y_2 = 3.3134$

$x_3 = 1.5$, $y_3 = 4.0070$

3. Solve the following boundary value problem using the finite difference method by dividing the interval into four sub-intervals.

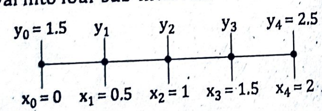
$$y'' = e^x + 2y' - y;\ y(0) = 1.5,\ y(2) = 2.5$$

Solution:

Given that;

$$y'' = e^x + 2y' - y \quad \text{..... (1)}$$
$$y(0) = 1.5,\ y(2) = 2.5$$

Dividing the interval into four sub-intervals



Here, $h = 0.5$

Now, for finite difference approximation, we have

$$\frac{dy}{dx} = y' = \frac{1}{2h}[y_{i+1} - y_{i-1}]$$

$$\frac{d^2y}{dx^2} = y'' = \frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}]$$

Now using the approximated value in equation (1),

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] = e^{x_i} + \frac{2}{2h}[y_{i+1} - y_{i-1}] - y_i$$

Put i = 1, at h = 0.5

$$\frac{1}{0.5^2}(y_2 - 2y_1 + y_0) = e^{x_1} + \frac{2}{2 \times 0.5}(y_2 - y_0) - y_1$$

or, $4y_2 - 8y_1 + 4y_0 = e^{x_1} + 2y_2 - 2y_0 - y_1$

Substituting the values

$$2y_2 - 7y_1 = e^{0.5} - 2(1.5) - 4(1.5)$$

or, $2y_2 - 7y_1 = -7.3512$ ..... (A)

Put i = 2,

$$4(y_3 - 2y_2 + y_1) = e^{x_2} + 2(y_3 - y_1) - y_2$$

or, $4y_3 - 8y_2 + 4y_1 = e^{x_2} + 2y_3 - 2y_1 - y_2$

Substituting the values

$$2y_3 - 7y_2 + 6y_1 = e^1$$

or, $6y_1 - 7y_2 + 2y_3 = 2.7183$ ..... (B)

Put i = 3,

$$4(y_4 - 2y_3 + y_2) = e^{x_3} + 2(y_4 - y_2) - y_3$$

or, $4y_4 - 8y_3 + 4y_2 = e^{x_3} + 2y_4 - 2y_2 - y_3$

or, $2y_4 - 7y_3 + 6y_2 = e^{x_3}$

Substituting the values

or, $6y_2 - 7y_3 = e^{1.5} - 2(2.5)$

or, $6y_2 - 7y_3 = -0.5183$ ..... (C)

Now solving the equations (A), (B) and (C), we get,

$$y_1 = 1.3487$$
$$y_2 = 1.0447$$
$$y_3 = 0.9695$$

4. Solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ using RK-4 method, for y(0.4)

Given, y(0) = 1, h = 0.2

Solution:

We have,

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2} = f(x, y)$$

Subject to

$y(0) = 1 \quad \rightarrow x_0 = 0, \quad y_0 = 1$

At h = 0.2

Now, using RK-4 method

$$k_1 = hf(x_0, y_0)$$
$$= 0.2f(0, 1)$$
$$= 0.2 \times \left(\frac{1^2 - 0^2}{1^2 + 0^2}\right)$$
$$= 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$
$$= 0.2f(0.1, 1.1)$$

$$= 0.2 \times \left(\frac{1.1^2 - 0.1^2}{1.1^2 + 0.1^2}\right)$$
$$= 0.1967$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.2f(0.1, 1.0983)$$
$$= 0.1967$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= 0.2f(0.2, 1.1967)$$
$$= 0.1891$$

Then,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0.2 + 0.1891 + 2(0.1967 + 0.1967)]$$
$$= 0.1959$$

so, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$\therefore$ $y_1 = y_0 + k = 1 + 0.1959 = 1.196$

Again,

$$k_1 = hf(x_1, y_1)$$
$$= 0.2f(0.2, 1.196)$$
$$= 0.1891$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.2f(0.3, 1.2906)$$
$$= 0.1795$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.2f(0.3, 1.2858)$$
$$= 0.1793$$
$$k_4 = hf(x_1 + h, y_1 + k_3)$$
$$= 0.2f(0.4, 1.3753)$$
$$= 0.1688$$

Then,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0.1891 + 0.1688 + 2(0.1795 + 0.1793)]$$
$$= 0.1792$$

so, $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$\therefore$ $y_2 = y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752$

5. Solve the following simultaneous differential equations using RK second order method at x = 0.1 and 0.2

$\frac{dy}{dx} = xz + 1$; $\frac{dz}{dx} = -xy$ with initial conditions $y(0) = 0$, $z(0) = 1$

Solution:

Given that;

$$\frac{dy}{dx} = y' = 1 + xz = f_1(x, y, z)$$

and, $$\frac{dz}{dx} = z' = -xy = f_2(x, y, z)$$

Subject to

$y(0) = 0 \quad \rightarrow x_0 = 0, \quad y_0 = 0$

$z(0) = 1 \quad \rightarrow z_0 = 1$

At $h = 0.1$

Now, using Runge-Kutta method of second order

$$k_1 = hf_1(x_0, y_0, z_0)$$
$$= 0.1f_1(0, 0, 1)$$
$$= 0.1 \times [1 + 0 \times (1)]$$
$$= 0.1$$
$$l_1 = 0.1f_2(x_0, y_0, z_0)$$
$$= 0.1f_2(0, 0, 1)$$
$$= 0.1 \times (-0 \times 0)$$
$$= 0$$
$$k_2 = hf_1(x_0 + h, y_0 + k_1, z_0 + l_1)$$
$$= 0.1f_1(0.1, 0.1, 1)$$
$$= 0.1 \times (1 + 0.1 \times 1)$$
$$= 0.11$$
$$l_2 = hf_2(0.1, 0.1, 1)$$
$$= 0.1 \times (-0.1 \times 0.1)$$
$$= -0.001$$

Then, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.1 + 0.11) = 0.105$

$$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}[0 + (-0.001)] = -0.0005$$

so, $x_1 = x_0 + h = 0 + 0.1 = 0.1$

$y_1 = y_0 + k = 0 + 0.105 = 0.105$

$z_1 = z_0 + l = 1 - 0.0005 = 0.9995$

Again,

$$k_1 = hf_1(x_1, y_1, z_1)$$
$$= 0.1f_1(0.1, 0.105, 0.9995)$$
$$= 0.1 \times (1 + 0.1 \times 0.9995)$$
$$= 0.11$$

$$l_1 = hf_2\,(0.1, 0.105, 0.9995)$$
$$= -0.0011$$
$$k_2 = hf_1\,(x_1 + h, y_1 + k_1, z_1 + l_1)$$
$$= 0.1f_1\,(0.2, 0.215, 0.9984)$$
$$= 0.12$$
$$l_2 = hf_2\,(0.2, 0.215, 0.9984)$$
$$= -0.0043$$

Then,

$$k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.11 + 0.12) = 0.115$$

$$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(-0.0011 - 0.0043) = -0.0027$$

Hence,

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$
$$\therefore \quad y_2 = y_1 + k = 0.105 + 0.115 = 0.22$$
$$\therefore \quad z_2 = z_1 + l = 0.9995 - 0.0027 = 0.9968$$

6. Solve $\frac{dy}{dx} = \log(x + y)$, $y(0) = 2$ for $x = 0.8$ taking $h = 0.1$ using Euler's method.

Solution:

We have,

$$\frac{dy}{dx} = \log(x + y)$$

Subject to

$$y(0) = 2 \quad \rightarrow x_0 = 0, \quad y_0 = 2$$

Taking $h = 0.1$

Now, using Euler's method in tabular form

S.N.	x	y	$\frac{dy}{dx} = \log(x+y)$	$y_{new} = y_{old} + h\frac{dy}{dx}$
1	0	2	log(0 + 2) = 0.30102	2 + 0.1(0.30102) = 2.03010
2	0.1	2.03010	0.32840	2.06294
3	0.2	2.06294	0.35467	2.09840
4	0.3	2.09840	0.37992	2.13639
5	0.4	2.13639	0.40421	2.17681
6	0.5	2.17681	0.42761	2.21957
7	0.6	2.21957	0.45018	2.26458
8	0.7	2.26458	0.47196	2.31177
9	0.8	2.31177		

Hence the required approximate value is 2.31177 for $x = 0.8$.

7. **Solve the following by Euler's modified method:**

$$\frac{dy}{dx} = \log(x + y),\ y(0) = 2 \text{ at } x = 1.2 \text{ and } 1.4 \text{ with } h = 0.2$$

Solution:

Given that;

$$\frac{dy}{dx} = \log(x + y)$$

Subject to

$$y(0) = 2 \quad \rightarrow x_0 = 0, \quad y_0 = 2$$

At $h = 0.2$

Now, solving in tabular form

S.N.	x	$\frac{dy}{dx} = \log(x+y)$	Mean slope	$y_{new} = y_{old} + h(\text{mean slope})$
1	0	log(0+2)=0.301	–	2+0.2(0.301)=2.0602
2	0.2	log(0.2+2.0602)	$\frac{1}{2}$(0.301+0.3541)	2+0.2(0.3276)=2.0655
3	0.2	log(0.2+2.0655)	$\frac{1}{2}$(0.301+0.3552)	2+0.2(0.3281)=2.0656

Here, last two values are equal at $y_1 = 2.0656$.

S.N.	x	$\frac{dy}{dx} = \log(x+y)$	Mean slope	$y_{new} = y_{old} + h(\text{mean slope})$
4	0.2	0.3552	–	2.0656+0.2(0.3552)=2.1366
5	0.4	log(0.4+2.1366)	$\frac{1}{2}$(0.3552+0.4042)	2.0656+0.2(0.3797)=2.1415
6	0.4	log(0.4+2.1415)	$\frac{1}{2}$(0.3552+0.4051)	2.0656+0.2(0.3801)=2.1416

Here, last two values are equal at $y_2 = 2.1416$.

S.N.	x	$\frac{dy}{dx} = \log(x+y)$	Mean slope	$y_{new} = y_{old} + h(\text{mean slope})$
7	0.4	0.4051	–	2.1416+0.2(0.4051)=2.2226
8	0.6	log(0.6+2.2226)	$\frac{1}{2}$(0.4051+0.4506)	2.1416+0.2(0.4279)=2.2272
9	0.6	log(0.6+2.2272)	$\frac{1}{2}$(0.4051+0.4514)	2.1416+0.2(0.4282)=2.2272

Here, last two values are equal at $y_3 = 2.2272$.

S.N.	x	$\frac{dy}{dx} = \log(x+y)$	Mean slope	$y_{new} = y_{old} + h(\text{mean slope})$
10	0.6	0.4514	–	2.2272+0.2(0.4514)=2.3175
11	0.8	log(0.8+2.3175)	$\frac{1}{2}$(0.4514+0.4938)	2.2272+0.2(0.4726)=2.3217
[illegible]	0.8	log(0.8+2.3217)	$\frac{1}{2}$(0.4514+0.4943)	2.2272+0.2(0.4727)=2.3217

Here, last two values are equal at $y_4 = 2.3217$.

S.N.	x	$\frac{dy}{dx} = \log(x+y)$	Mean slope	$y_{new} = y_{old} + h(\text{mean slope})$
13	0.8	0.4943	–	2.3217+0.2(0.4943)=2.4206
14	1	log(1+2.4206)	$\frac{1}{2}(0.4943+0.5341)$	2.3217+0.2(0.5142)=2.4245
15	1	log(1+2.4245)	$\frac{1}{2}(0.4943+0.5346)$	2.3217+0.2(0.5144)=2.4245

Here, last two values are equal at $y_5 = 2.4245$.

S.N.	x	$\frac{dy}{dx} = \log(x+y)$	Mean slope	$y_{new} = y_{old} + h(\text{mean slope})$
16	1	0.5346	–	2.4245+0.2(0.5346)=2.5314
17	1.2	log(1.2+2.5314)	$\frac{1}{2}(0.5346+0.5719)$	2.4245+0.2(0.5532)=2.5351
18	1.2	log(1.2+2.5351)	$\frac{1}{2}(0.5346+0.5723)$	2.4245+0.2(0.5534)=2.5351

Here, last two values are equal at $y_6 = 2.5351$.

S.N.	x	$\frac{dy}{dx} = \log(x+y)$	Mean slope	$y_{new} = y_{old} + h(\text{mean slope})$
19	1.2	0.5723	–	2.5351+0.2(0.5723)=2.6496
20	1.4	log(1.4+2.6496)	$\frac{1}{2}(0.5723+0.6074)$	2.5351+0.2(0.5898)=2.6531
21	1.4	log(1.4+2.6531)	$\frac{1}{2}(0.5723+0.6078)$	2.5351+0.2(0.5900)=2.6531

Here, last two values are equal at $y_7 = 2.6531$.

Hence, $y(1.2) = 2.5351$ and $y(1.4) = 2.6531$ are the required approximated values.