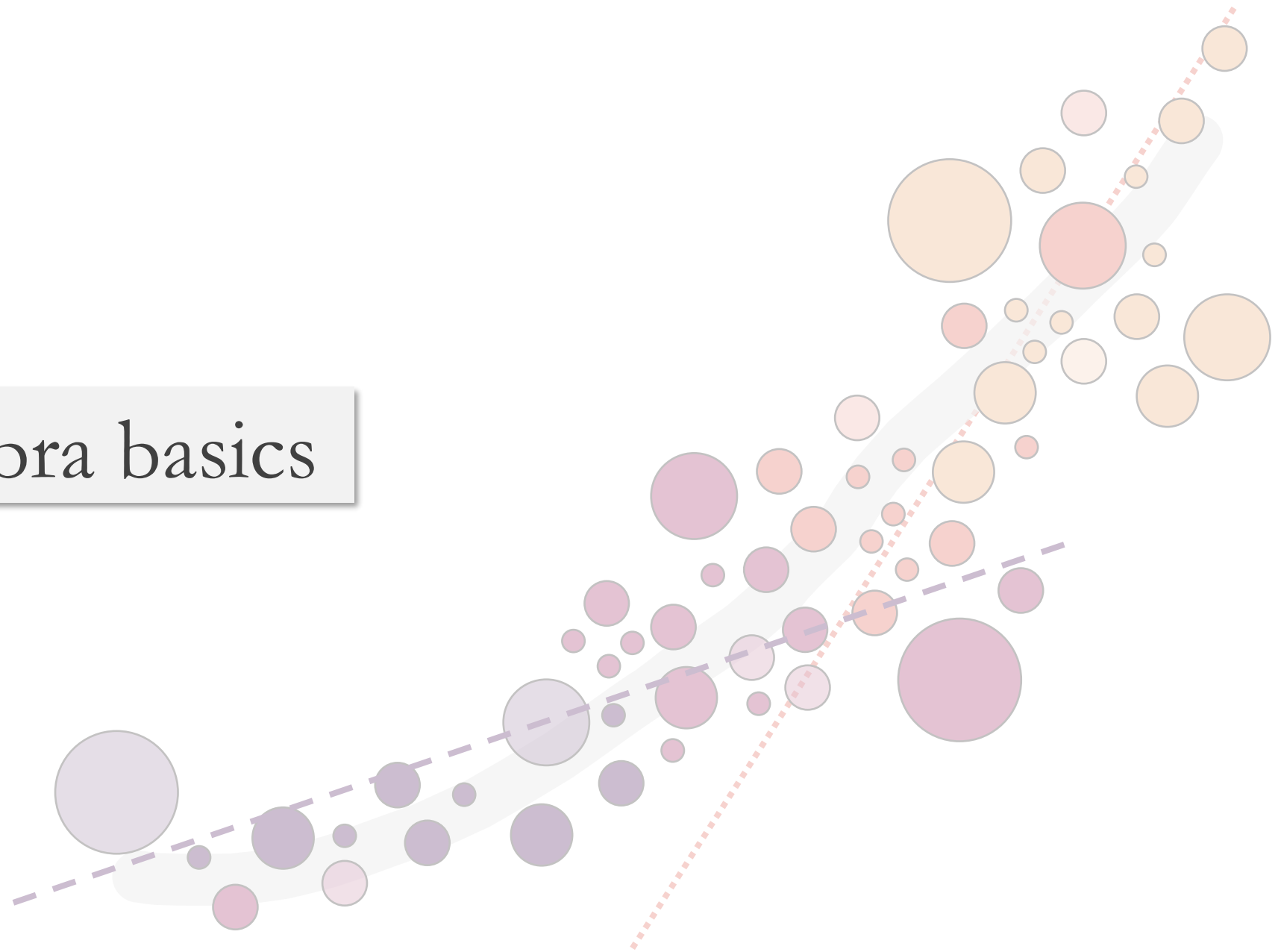


Linear algebra basics



Working with vectors

A 3D column vector

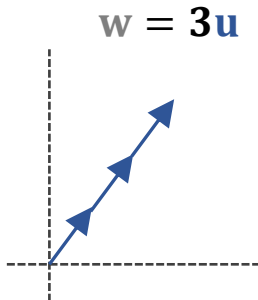
$$\mathbf{x} = x = \vec{x} = \begin{bmatrix} 2.0 \\ 5.0 \\ 7.0 \end{bmatrix}$$

Transpose

$$\mathbf{x}^T = [2.0 \quad 5.0 \quad 7.0]$$

Scalar multiplication

$$\lambda = 2.0$$
$$\lambda \mathbf{x} = \begin{bmatrix} 4.0 \\ 10.0 \\ 14.0 \end{bmatrix}$$



A 4D row vector

$$\mathbf{y} = [3.0 \quad -5.0 \quad 11.0 \quad 2.0]$$

Vector addition

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 9 \end{bmatrix}$$

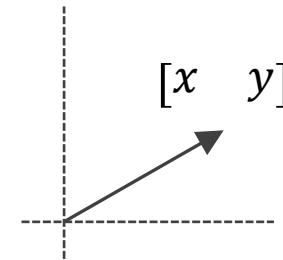
Two vectors can be added together only if they have the same dimensionality and the same orientation.

Vector magnitude (norm)

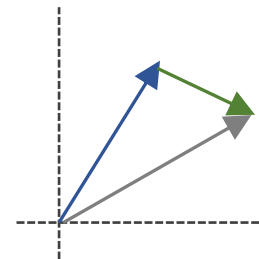
$$\|\mathbf{v}\| = \sqrt{\sum_{i=1}^n v_i^2}$$

Geometric interpretation

$\mathbf{x} \in \mathbb{R}^3$ ← dimension



$$\mathbf{w} = \mathbf{u} + \mathbf{v}$$

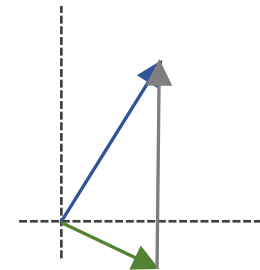


Algebraic interpretation

$$[x_1 \quad x_2 \quad x_3]$$

An ordered list of number

$$\mathbf{w} = \mathbf{u} - \mathbf{v}$$



Unit vector

$$\|\mathbf{v}\| = 1$$

$$\hat{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|} \mathbf{v}$$

Any non unit vector has an associated unit vector.

Euclidean distance

$$d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_3 - v_3)^2}$$

Vector algebra

Commutative Property

$$\mathbf{u} + \mathbf{v} = (\mathbf{v} + \mathbf{u})$$

Additive Associative Property

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

Zero Property

$$\mathbf{u} + \mathbf{0} = \mathbf{u} \Leftrightarrow \mathbf{u} - \mathbf{u} = \mathbf{0}$$

Unit Rule

$$\mathbf{1u} = \mathbf{u}$$

Vector Distributive Property

$$s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}$$

Scalar Distributive Property

$$(s + t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}$$

Zero Multiplicative Property

$$\mathbf{0u} = \mathbf{0}$$



Let's practice all this in the first module of the following [Grasple course](#) (lesson 1 & 2).

Vector Dot Products

Vector *Dot* (also inner or *scalar*) Product

Commutative Property

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

Associative Property

$$s(\mathbf{u} \cdot \mathbf{v}) = (s\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (s\mathbf{v})$$

Distributive Property

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$$

Unit Rule Zero Property

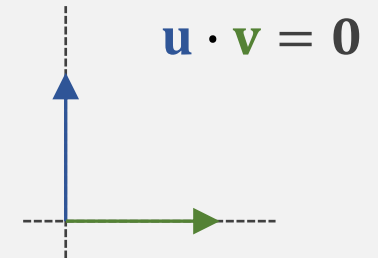
$$1\mathbf{u} = \mathbf{u}$$

$$\mathbf{u} \cdot \mathbf{0} = 0$$

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle = [u_1 v_1 + u_2 v_2 + \cdots + u_k v_k] = \sum_{i=1}^k u_i v_i$$

The dot product can be interpreted as a measure of similarity or mapping between two vectors.

Orthogonal Vectors Have a Zero Dot Product



Other kinds of products

$$\mathbf{u} \odot \mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} \odot \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 20 \\ 14 \end{bmatrix}$$

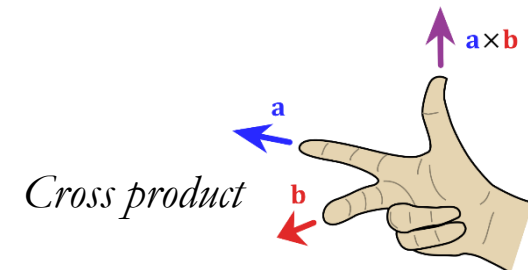
Hadamard multiplication

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} au & av \\ bu & bv \\ cu & cv \end{bmatrix}$$

Outer product



Let's practice all this in the first module of the following [Grasple course](#) (lesson 3).



Cross product

Working with matrices

dimension \swarrow

$$\mathbf{X}_{3 \times 2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

\searrow row \times column

$$\mathbf{X}_{i,j}^T = \mathbf{X}_{j,i}$$

$$x_{12} = 2$$

Square matrix

$$\begin{bmatrix} 1 & 3 & 3 & 4 \\ 5 & 6 & 7 & 9 \\ 9 & 2 & 3 & 0 \\ 4 & 3 & 7 & 8 \end{bmatrix}$$

Upper triangle

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 8 & 5 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Symetric matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 8 & 5 & 6 \\ 3 & 5 & 1 & 7 \\ 4 & 6 & 7 & 8 \end{bmatrix}$$

Diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Identity matrix

$$\mathbf{I} = \mathbf{I}_{4 \times 4} = \mathbf{I}(4) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix vector multiplication

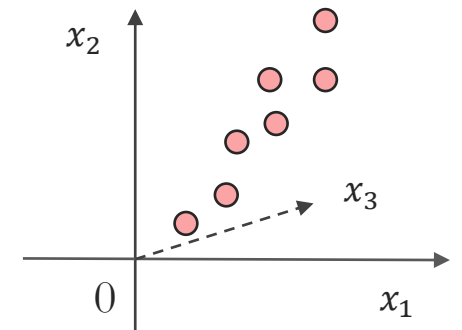
A matrix can be right-multiplied by a column vector but not a row vector, and it can be left-multiplied by a row vector but not a column vector.

$$\mathbf{X}\boldsymbol{\beta} = \mathbf{Y}$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 5 & 6 & 7 \\ 9 & 2 & 3 \\ 4 & 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} = \begin{bmatrix} X_{1,*} \cdot B \\ X_{2,*} \cdot B \\ X_{3,*} \cdot B \\ X_{4,*} \cdot B \end{bmatrix}$$

$$\mathbf{X}\boldsymbol{\beta} = \mathbf{Y}$$

$$\begin{bmatrix} 1 & 5 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 & 9 & 4 \\ 3 & 6 & 2 & 3 \\ 3 & 7 & 3 & 7 \end{bmatrix} = [X_{*,1} \cdot B \quad X_{*,2} \cdot B \quad X_{*,3} \cdot B]$$



Matrix multiplication

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} \begin{matrix} b_{12} \\ b_{22} \\ b_{23} \end{matrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

\mathbf{A} $(n \times k)$ \mathbf{B} $(k \times m)$ $=$ \mathbf{C} $(n \times m)$

conformability

size of the resulting matrix

Two matrices are **conformable** for multiplication if the number of columns in the first matrix matches the number of rows in the second matrix.

$$\sum_{i=1}^k a_{1i}b_{i1} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$



Let's practice all this in the first module of the following [Grasple course](#) (lesson 1, 2 & 3).

Matrix inverse

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \Leftrightarrow \det(\mathbf{A}) \neq 0$$

The inverse of a matrix \mathbf{A} is another matrix that multiplies \mathbf{A} to produce the identity matrix.

Matrix determinant

$$\det(X) = |X| = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11}x_{22} - x_{12}x_{21}$$

for 2×2 matrices only

Find the inverse of a 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbf{A}^{-1} = \det(\mathbf{X})^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We need to “cancel” a matrix in order to solve problems that can be expressed in the form $\mathbf{Ax} = \mathbf{b}$, where \mathbf{A} and \mathbf{b} are known quantities, and we want to solve for \mathbf{x} .

$$\begin{aligned}\mathbf{Ax} &= \mathbf{b} \\ \mathbf{A}^{-1}\mathbf{Ax} &= \mathbf{A}^{-1}\mathbf{b} \\ \mathbf{Ix} &= \mathbf{A}^{-1}\mathbf{b} \\ \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b}\end{aligned}$$



Let's practice all this in the first module of the following [Grasple course](#) (lesson 4, 5 & 6).