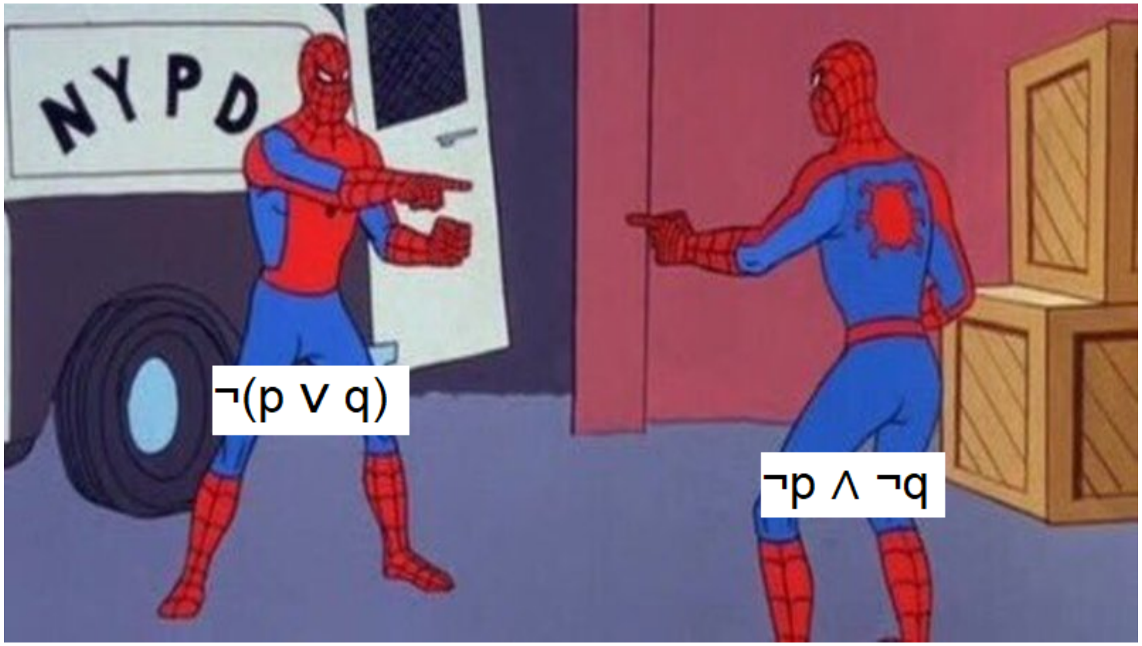
Logical Equivalences
EECS 203
Lecture 6



Grading Groupwork

- We mistakenly had old (less clear) guidance on how to grade your groupwork. A new version has been uploaded.
- If you have already submitted, no need to resubmit
- If not, please fill out a table similar to the one below, showing what rubric items you received and what you didn't
- The updated homework document has a premade table for the assignment

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	Total:
Exponents	+0.5	+0	+0.5	+0	+0.5	+0.5	+0.5		2.5/3.5
Logarithms	+0.5	+0.5	+0	+0	+0.5	+0.5	+0.5	+0.5	3/4
Equations 1	+1	+0							1/2
Equations 2	+1	+1							2/2
Total:									8.5/11.5

- An example grading of (a slightly old version of) homework 0 has been uploaded to Canvas
- Reminder: we will give you a little extra credit if you include positive messages

"Unless"…

- Our usage of "unless" is inconsistent/ambiguous
- In logic: $\boxed{\boldsymbol{a} \textbf{ unless } \boldsymbol{b} \equiv \boldsymbol{a} \vee \boldsymbol{b}}$
- "$\underbrace{\text{You can't board a plane from Detroit to Delhi}}_{a}$ unless $\underbrace{\text{you have a passport}}_{b}$."
  - if you don't have a passport, you can't board the plane
  - if you do have a passport, you may or may not be able to board the plane. Because having a passport doesn't guarantee that you can board the plane. For example, you also need a plane ticket.

Outline

- **Translation**
  - **Domain Restrictions**
  - Quantifier Scoping
- Logic Puzzles
- Intro to Logical Equivalence
- Logical Equivalence Rules
  - DeMorgan’s Laws
  - Distributive Laws
  - Table of Other Equivalences
  - Implication Breakout
- Tautologies and Contradictions

Domain Restrictions

S(x) = "x is a student in this class"

C(x) = "x has studied calculus"

"Every student in this class has studied calculus."

Is this translation correct? $\forall x\ [\ S(x) \land C(x)\ ]$ ***(domain = all people)***

Domain Restrictions

S(x) = “x is a student in this class”

C(x) = “x has studied calculus”

“Every student in this class has studied calculus.”

Is this translation correct? $\forall x\ [\ S(x) \land C(x)\ ]$ ***(domain = all people)***

NO! This says: “Everyone is a student in this class who has studied Calculus.”

Use **if/then** with **universal quantifiers** to make a proposition about only **some** of the domain.

$$\forall x\ [\ S(x) \rightarrow C(x)\ ]$$

Domain Restrictions

S(x) = “x is a student in this class”

C(x) = “x has studied calculus”

“**Some** student in this class has studied calculus.”

$F \rightarrow \boxed{?} = T$

Is this translation correct? $\exists x\ [\ S(x) \rightarrow C(x)\ ]$ ***(domain = all people)***

Domain Restrictions

S(x) = "x is a student in this class"

C(x) = "x has studied calculus"

"**Some** student in this class has studied calculus."

Is this translation correct? $\exists x\ [\ S(x) \rightarrow C(x)\ ]$ ***(domain = all people)***

NO! Remember: if-then with existential quantifiers is unintuitive

(This is satisfied by any x with $\neg S(x)$, because $F \rightarrow T$ and $F \rightarrow F$)

Use **and** with **existential quantifiers** to make a proposition about only **some** of the domain.

$$\exists x\ [\ S(x) \land C(x)\ ]$$

Outline

- Translation
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  - **Quantifier Scoping**
- Logic Puzzles

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Handout

Scoping

Translate the following logical statements into English.

Do the two statements have the same meaning? ___________

B(x, y) = "x buys a y"

Logic	**English**
$\forall x\, [\, B(x, \text{umbrella}) \lor B(x, \text{raincoat})\,]$	
$[\forall x\, B(x, \text{umbrella})] \lor [\forall x\, B(x, \text{raincoat})]$	

Scoping

$B(x, y)$ = "x buys a y"

Translate: ***(domain = all people)***

$\forall x\, [B(x,\text{umbrella}) \vee B(x,\text{raincoat})]$

$[\forall x\, B(x,\text{umbrella})] \vee [\forall x\, B(x,\text{raincoat})]$

Notice: the **scope** of the variable **x** ends at the parentheses, so we can reuse the name!

Scoping

$B(x, y)$ = "x buys a y"

Translate: ***(domain = all people)***

$\forall x\,[B(x,\text{umbrella}) \lor B(x,\text{raincoat})]$

$[\forall x\, B(x,\text{umbrella})] \lor [\forall y\, B(y,\text{raincoat})]$

More common: use different variable names, just to avoid confusion

Scoping

B(x, y) = "x buys a y"

Translate: ***(domain = all people)***

$\forall x\ [B(x,\text{umbrella}) \vee B(x,\text{raincoat})]$

"Everyone buys an umbrella or a raincoat."

$[\forall x\ B(x,\text{umbrella})] \vee [\forall y\ B(y,\text{raincoat})]$

"Everyone buys an umbrella,
or everyone buys a raincoat."

"Variable scope:" the part of the expression in which a variable created by a for-all/there-exists exists is still active.

This matters! It causes these two propositions to have different meanings.

Scoping

L(x) = "x laughs at bad jokes"
P(x, y) = "x is a parent of y"
"Everyone who is the parent of someone laughs at bad jokes"

(A) $\forall x[\, (\exists y\, \boxed{P(x,y)}) \rightarrow L(x)\,]$

(B) $\forall x \exists y\, \boxed{[\, P(x,y) \rightarrow L(x)\,]}$ ← does not require all parents to laugh at bad jokes

$\underbrace{F \rightarrow F}_{T}$

Scoping

L(x) = "x laughs at bad jokes"
P(x, y) = "x is a parent of y"
"Everyone who is the parent of someone laughs at bad jokes"

For all people, if they are the parent of someone, then they laugh at bad jokes.

(A) $\forall x [(\exists y \, P(x,y)) \rightarrow L(x)]$

(B) $\forall x \exists y \, [P(x,y) \rightarrow L(x)]$

If/Then directly inside an existential quantifier is a red flag! Probably non-meaningful behavior

Outline

- Translation
  - Domain Restrictions
  - Quantifier Scoping
- **Logic Puzzles**

- Intro to Logical Equivalence
- Logical Equivalence Rules
  - DeMorgan's Laws
  - Distributive Laws
  - Table of Other Equivalences
  - Implication Breakout
- Tautologies and Contradictions

A Puzzle

You are on an island where there are only two kinds of people:

- **Liars**, who always lie,
- **Truth-tellers**, who always tell the truth.

You meet an native, and ask: “are you a truth-teller?” A clap of thunder masks their response, so you say: “Excuse me, I couldn't hear you: did you say you were a truth-teller?” They answer “No, I said I was a liar.”

Is the native a liar or a truth-teller?

Logic Puzzle

You: Are you a truth-teller?

Answer 1: ???

You: Did you say you were a truth-teller?

Answer 2: No, I said I was a liar

In the beginning, we aren't sure whether the native is a truth teller or a liar

N	A1	A2
TT		
L		

Logic Puzzle

You: Are you a truth-teller?

Answer 1: ???

You: Did you say you were a truth-teller?

Answer 2: No, I said I was a liar

In either case, their first answer is yes

N	A1	A2
TT	*Yes*	
L	*Yes*	

Logic Puzzle

You: Are you a truth-teller?

Answer 1: ???

You: Did you say you were a truth-teller?

Answer 2: No, I said I was a liar

But they didn't say Yes in A2

N	A1	A2
~~TT~~	~~*Yes*~~	~~*Yes*~~
L	*Yes*	*No*

Native must be a liar

Logic Puzzle

You meet natives Anand, Blanca and Carol.

- You: Anand, are you a liar?
- Anand: [drowned out by a clap of thunder]
- You: Blanca, what did Anand say?
- Blanca: Anand said he’s a liar.
- Carol: Don’t believe Blanca, she’s lying!
- Carol: Also, Anand is a liar.
- **Which one(s) are liars?**

Handout

Logic Puzzle

Q1: “Anand, are you a liar?”
Anand: “????”

Q2: “Blanca, what did Anand say?”
Blanca: “He said yes.”

Q3: “Is Blanca a liar?”
Carol: “Yes.”

Q4: “Is Anand a liar?”
Carol: “Yes.”

Anand	Blanca	Carol	Q1 (Anand)	Q2 (Blanca)	Q3 (Carol 1)	Q4 (Carol 2)
TT	TT	TT				
TT	TT	L				
TT	L	TT				
TT	L	L				
L	TT	TT				
L	TT	L				
L	L	TT				
L	L	L				

Logic Puzzle

Q1: "Anand, are you a liar?"
Anand: "No." *(Infer answer)*

Q2: "Blanca, what did Anand say?"
Blanca: "He said yes." ***(Blanca is a liar.)***

Q3: "Is Blanca a liar?"
Carol: "Yes." ***(Carol is a truth teller.)***

Q4: "Is Anand a liar?"
Carol: "Yes." ***(Anand is a liar.)***

Anand	Blanca	Carol	Q1 (Anand)	Q2 (Blanca)	Q3 (Carol 1)	Q4 (Carol 2)
~~TT~~	~~TT~~	~~TT~~	~~**No**~~	~~*No*~~		
~~TT~~	~~TT~~	~~L~~	~~**No**~~	~~*No*~~		
~~TT~~	~~L~~	~~TT~~	~~**No**~~	~~**Yes**~~	~~**Yes**~~	~~*No*~~
~~TT~~	~~L~~	~~L~~	~~**No**~~	~~**Yes**~~	~~*No*~~	
~~L~~	~~TT~~	~~TT~~	~~**No**~~	~~*No*~~		
~~L~~	~~TT~~	~~L~~	~~**No**~~	~~*No*~~		
L	L	TT	**No**	**Yes**	**Yes**	**Yes**
~~L~~	~~L~~	~~L~~	~~**No**~~	~~**Yes**~~	~~*No*~~	

Wrapup

- Can turn English/word problems into “logic equations”

- Next time: manipulating these equations **algebraically**, instead of having to write out truth tables.

Logical Equivalences
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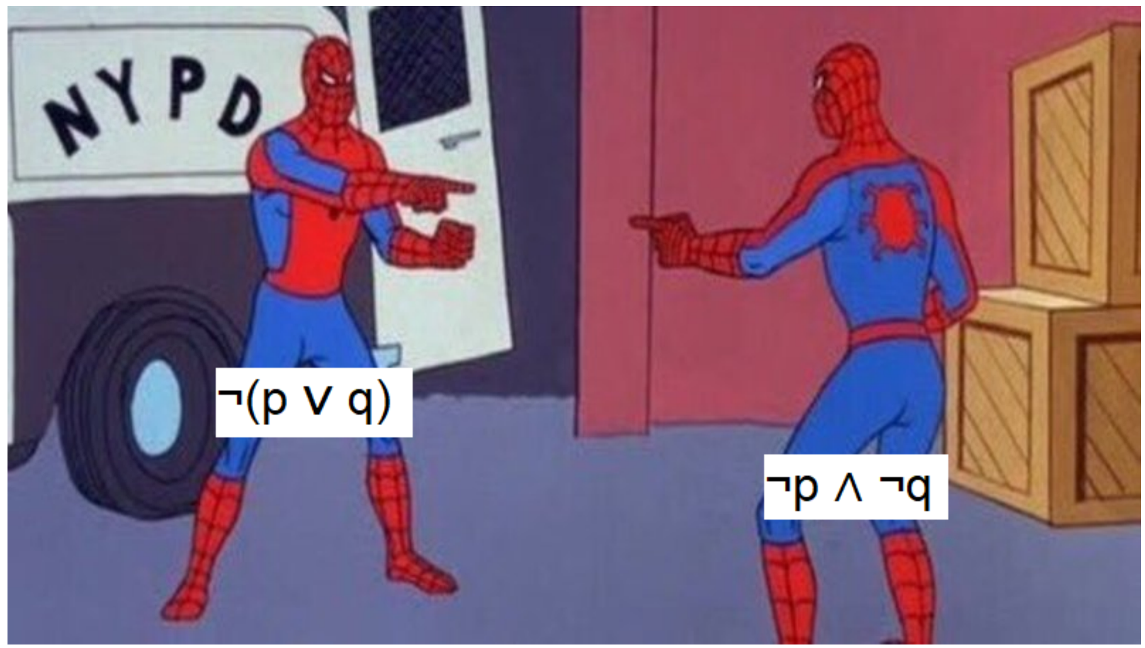

Learning Objectives: Lec 6

After today's lecture (and the associated readings, discussion, & homework), you should be able to:

- **Know Technical Vocab:** logical equivalence, tautology, contradiction
- Know rules: DeMorgan's, distributive, implication breakout
- Recognize other basic logical equivalence rules (see table)
- Show a logical equivalence using a truth table
- Show a logical equivalence by applying a chain of logical equivalence rules
- Show that a compound proposition is a tautology, a contradiction, or neither
- Understand the relationship between tautologies and proof styles

Outline

- **Intro to Logical Equivalence**
- Logical Equivalence Rules
  - DeMorgan's Laws
  - Distributive Laws
  - Table of Other Equivalences
  - Implication Breakout
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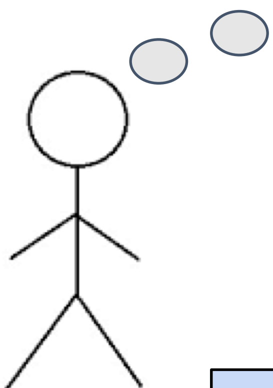
Goals

Humans use logic to understand the world. What does that process actually look like?



I need to go to the grocery store and the post office today, unless I decide to order a pizza, but even then I still need to go to the post office.

means the same thing as

I need to go to the post office, and I need to go to the grocery store or order a pizza.

Answer #1: Simplifying Knowledge – this is our topic today

Translation Recap

Last Time: translations of English into logic

$c =$ "my nose is cold" $\qquad$ $h=$"I am happy"

1. *If my nose is cold then I am unhappy.*

2. *If I'm happy then my nose isn't cold.*

3. *My nose is not cold or I'm unhappy.*

Consider the Truth Tables

Handout

1. *If my nose is cold then I am unhappy.* $c \to \neg h$
2. *If I'm happy then my nose isn't cold.* $h \to \neg c$
3. *My nose is not cold or I'm unhappy.* $\neg c \vee \neg h$

$c =$ "my nose is cold" $h=$"I am happy"

c	h	$\neg c$	$\neg h$	$c \to \neg h$ (1)	$h \to \neg c$ (2)	$\neg c \vee \neg h$ (3)
T	T	F	F	F	F	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

same truth-table means logically equiv.

Consider the Truth Tables

1. *If my nose is cold then I am unhappy.*
2. *If I'm happy then my nose isn't cold.*
3. *My nose is not cold or I'm unhappy.*

$c = \text{"my nose is cold"}$ $h = \text{"I am happy"}$

These three statements have the same truth table!
(That makes sense – don't they **mean the same thing**?)

We say that they are **logically equivalent**.

Some notations for logical equivalence:

- "p, q are logically equivalent"
- "p if and only if q"
- "p is necessary and sufficient for q"
- "p iff q"
- $p \equiv q$
- $p \leftrightarrow q$

Logical Equivalence

Two compound propositions A, B are **logically equivalent** if they have the same truth value for any instantiation of their input truth values

One way to show logical equivalence: write out the truth tables, and check that they're the same.

p	q	$p \wedge q$	$p \wedge \neg q$	$(p \wedge q) \vee (p \wedge \neg q)$
T	T			
T	F			
F	T			
F	F			

Logical Equivalence

Two compound propositions A, B are **logically equivalent** if they have the same truth value for any instantiation of their input truth values

One way to show logical equivalence: write out the truth tables, and check that they're the same.

p	q	$p \land q$	$p \land \neg q$	$(p \land q) \lor (p \land \neg q)$
T	T	T		
T	F	F		
F	T	F		
F	F	F		

Logical Equivalence

Two compound propositions A, B are **logically equivalent** if they have the same truth value for any instantiation of their input truth values

One way to show logical equivalence: write out the truth tables, and check that they're the same.

p	q	$p \land q$	$p \land \neg q$	$(p \land q) \lor (p \land \neg q)$
T	T	T	F	
T	F	F	T	
F	T	F	F	
F	F	F	F	

Logical Equivalence

Two compound propositions A, B are **logically equivalent** if they have the same truth value for any instantiation of their input truth values

One way to show logical equivalence: write out the truth tables, and check that they're the same.

p	q	$p \wedge q$	$p \wedge \neg q$	$(p \wedge q) \vee (p \wedge \neg q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	F	F
F	F	F	F	F

Logical Equivalence

Two compound propositions A, B are **logically equivalent** if they have the same truth value for any instantiation of their input truth values

One way to show logical equivalence: write out the truth tables, and check that they're the same.

p	q	$p \land q$	$p \land \neg q$	$(p \land q) \lor (p \land \neg q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	F	F
F	F	F	F	F

p is logically equivalent to $(p \land q) \lor (p \land \neg q)$

Logical Equivalence

Two compound propositions A, B are **logically equivalent** if they have the same truth value for any instantiation of their input truth values

Another way to show logical equivalence: apply **rules of Boolean algebra**

Double Negation Law:
p is logically equivalent to $\neg\neg p$

$(q \vee r) \rightarrow \neg\neg p$

is logically equivalent to

$(q \vee r) \rightarrow p$

Numeric Algebra vs. Boolean Algebra

Numeric Algebra:

You can replace part of an expression with something new on the next line, if you're sure that the old and new expressions **represent equal numbers under any possible variable settings**.

$$5 + x(y^2 + 3)$$
$$5 + xy^2 + 3x$$

"distributive law:"
these are equal

Boolean Algebra:

You can replace part of an expression with something new on the next line, if you're sure that the old and new expressions **represent equivalent truth values under any possible variable settings.**

$$(q \vee r) \rightarrow \neg\neg p$$
$$(q \vee r) \rightarrow p$$

"double negation:"
these are equivalent

Outline

- Intro to Logical Equivalence
- **Logical Equivalence Rules**
  - **DeMorgan's Laws**
  - Distributive Laws
  - Table of Other Equivalences
  - Implication Breakout
- Tautologies and Contradictions

DeMorgan's Laws

s = "I will go to the store."
p = "I will go to the park."

Proposition 1:
"It's not true that (I will go to the store or I will go to the park)."
$\neg (s \vee p)$

Proposition 2:
"I will not go to the store and I will not go to the park."
$(\neg s) \wedge (\neg p)$

DeMorgan's Laws

Prop 1:
"It's not true that (I will go to the store or I will go to the park)."

Prop 2:
"I will not go to the store and I will not go to the park."

s	p	**¬ (s ∨ p)**
T	T	**F**
T	F	**F**
F	T	**F**
F	F	**T**

s	p	**(¬ s) ∧ (¬ p)**
T	T	**F**
T	F	**F**
F	T	**F**
F	F	**T**

Same truth table!
These are **logically equivalent**.

DeMorgan’s Laws

In Regular Algebra:

- You can distribute a negative sign over a sum.
- **$-(x+y) = (-x) + (-y)$**

Analog in Boolean Algebra:

- **DeMorgan’s Law #1:** You can distribute a **not** over an **or**
- $\neg(x \lor y)$ and $(\neg x) \land (\neg y)$ are logically equivalent

Negate both terms, also switch the **or** to an **and**

DeMorgan’s Laws

DeMorgan’s Law #1: You can distribute a **not** over an **or**

$$\neg(x \lor y) \equiv (\neg x) \land (\neg y)$$

DeMorgan’s Law #2: You can distribute a **not** over an **and**

$$\neg(x \land y) \equiv (\neg x) \lor (\neg y)$$

Negate both terms, also switch the **and** to an **or**

“It’s not true that (I will go to the store and I will go to the park).”
“I will not go to the store or I will not go to the park.”

DeMorgan In Action (Words)

"It's not true that (I will go to the store or I will go to the park)."
"I will not go to the store and I will not go to the park."

"It's not true that ($2+7=10$ or $3^2=8$)."
"$2+7\neq10$ and $3^2\neq8$."

"It's not true that ($2+7=9$ and $3^2=8$)."
"$2+7\neq9$ or $3^2\neq8$."

DeMorgan promises that each pair of statements is **logically equivalent.**

DeMorgan over Quantifiers

- Recall: existential quantifiers can be interpreted as a **chain of “or”s**
- Recall: universal quantifiers can be interpreted as a **chain of “and”s**

“There exists a city that is the capital of Michigan”

$\exists$**c Capital(c)**

“Capital(Acme) $\vee$ Capital(Ada) $\vee$ … $\vee$ Capital(Zilwaukee)”

What’s the **negation**?

DeMorgan over Quantifiers

- Recall: existential quantifiers can be interpreted as a **chain of "or"s**
- Recall: universal quantifiers can be interpreted as a **chain of "and"s**

"There **does not exist** a city that is the capital of Michigan"

$\neg\exists c\ \text{Capital}(c)$

"$\neg$[Capital(Acme) $\vee$ Capital(Ada) $\vee$ … $\vee$ Capital(Zilwaukee)]"

(apply DeMorgan's Law)

" $\neg$Capital(Acme) $\wedge$ $\neg$Capital(Ada) $\wedge$ … $\wedge$ $\neg$Capital(Zilwaukee)"

$\forall c\ \neg\text{Capital}(c)$

"For all cities, that city is **not** the capital of Michigan"

DeMorgan over Quantifiers

- Recall: existential quantifiers can be interpreted as a **chain of “or”s**
- Recall: universal quantifiers can be interpreted as a **chain of “and”s**

“I got every problem on the homework right.”

R(p) = I got problem p right

$\forall p\ R(p)$

“R(problem 1) $\wedge$ R(problem 2) $\wedge$ … $\wedge$ R(problem n)”

What’s the **negation**?

DeMorgan over Quantifiers

- Recall: existential quantifiers can be interpreted as a **chain of "or"s**
- Recall: universal quantifiers can be interpreted as a **chain of "and"s**

"I didn't get every problem on the homework right."

R(p) = I got problem p right

$\neg\forall p\, R(p)$

"$\neg[R(\text{problem } 1) \wedge R(\text{problem } 2) \wedge \ldots \wedge R(\text{problem } n)]$"

(apply DeMorgan's)

"$\neg R(\text{problem } 1) \vee \neg R(\text{problem } 2) \vee \ldots \vee \neg R(\text{problem } n)$"

$\exists p\, \neg R(p)$

"There exists a problem on the homework that I didn't get right."

Lecture ~~5~~ 6 Handout: Logical Equivalences

Handout

Distributive Laws:

$p \lor (q \land r) \equiv$ ____________

$p \land (q \lor r) \equiv$ ____________

DeMorgan's Laws:

$\neg(p \lor q) \equiv \neg p \land \neg q$

$\neg(p \land q) \equiv \neg p \lor \neg q$

DeMorgan's Laws with Quantifiers:

$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$

$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

"Implication breakout" rule:

$p \to q \equiv$ ____________

Which of the following ALWAYS has the same truth value as $p \to q$?

A) Converse: $q \to p$

B) Inverse: $\neg p \to \neg q$

C) Contrapositive: $\neg q \to \neg p$

DeMorgan’s Laws (recap)

Negation on the outside	Negation on the inside
$\neg(x \lor y)$	$(\neg x) \land (\neg y)$
$\neg(x \land y)$	$(\neg x) \lor (\neg y)$
$\neg\exists x\ P(x)$	$\forall x\ \neg P(x)$
$\neg\forall x\ P(x)$	$\exists x\ \neg P(x)$

In each row in this table, the left side is logically equivalent to the right side.

Handout

An Exercise

P(x, y, z), Q(x, y, z) are unknown predicates

Simplify:

$$\neg \exists x \, \forall y \, \exists z \, [\neg P(x, y, z) \vee \neg Q(x, y, z)]$$

P(x, y, z), Q(x, y, z) are unknown predicates

An Exercise

Simplify:

$\neg \exists x\ \forall y\ \exists z\ [\neg P(x, y, z) \lor \neg Q(x, y, z)]$

$\equiv \forall x \neg \forall y\ \exists z\ [\neg P(x, y, z) \lor \neg Q(x, y, z)]$

justification

DeMorgan (quantifier)

P(x, y, z), Q(x, y, z) are unknown predicates

An Exercise

Simplify:

$\neg \exists x \forall y \exists z [\neg P(x, y, z) \lor \neg Q(x, y, z)]$

$\equiv \forall x \neg \forall y \exists z [\neg P(x, y, z) \lor \neg Q(x, y, z)]$ DeMorgan (quantifier)

$\equiv \forall x \exists y \neg \exists z [\neg P(x, y, z) \lor \neg Q(x, y, z)]$ DeMorgan (quantifier)

P(x, y, z), Q(x, y, z) are unknown predicates

An Exercise

Simplify:

$\neg \exists x \forall y \exists z [\neg P(x, y, z) \lor \neg Q(x, y, z)]$

$\equiv \forall x \neg \forall y \exists z [\neg P(x, y, z) \lor \neg Q(x, y, z)]$ DeMorgan (quantifier)

$\equiv \forall x \exists y \neg \exists z [\neg P(x, y, z) \lor \neg Q(x, y, z)]$ DeMorgan (quantifier)

$\equiv \forall x \exists y \forall z \neg [\neg P(x, y, z) \lor \neg Q(x, y, z)]$ DeMorgan (quantifier)

P(x, y, z), Q(x, y, z) are unknown predicates

An Exercise

Simplify:

$\neg \exists x \forall y \exists z [\neg P(x, y, z) \vee \neg Q(x, y, z)]$

$\equiv \forall x \neg \forall y \exists z [\neg P(x, y, z) \vee \neg Q(x, y, z)]$ DeMorgan (quantifier)

$\equiv \forall x \exists y \neg \exists z [\neg P(x, y, z) \vee \neg Q(x, y, z)]$ DeMorgan (quantifier)

$\equiv \forall x \exists y \forall z \neg[\neg P(x, y, z) \vee \neg Q(x, y, z)]$ DeMorgan (quantifier)

$\equiv \forall x \exists y \forall z [P(x, y, z) \wedge Q(x, y, z)]$ DeMorgan (normal)

Outline

- Intro to Logical Equivalence
- **Logical Equivalence Rules**
    - DeMorgan's Laws
    - **Distributive Laws**
    - Table of Other Equivalences
    - Implication Breakout
- Tautologies and Contradictions

Distributive Laws

s = "I will go to the store."
p = "I will go to the park."
c = "I will go to class."

Proposition 1:
"I will go to class, and also I will go to the store or I will go to the park."
$c \wedge (s \vee p)$

Proposition 2:
"Either I will go to class and I will go to the store, or I will go to class and I will go to the park."
$(c \wedge s) \vee (c \wedge p)$

Distributive Laws

In Regular Algebra:

- You can distribute a multiplicative factor over a sum.
- **$c(x+y) = (cx) + (cy)$**

Analog In Boolean Algebra:

- **Distributive Law #1:** You can distribute an **or** over an **and**
- $z \lor (x \land y)$ and $(z \lor x) \land (z \lor y)$ are logically equivalent

Apply "**or z**" to both terms in the **and**

Distributive Laws

Distributive Law #1: How to distribute an **or** over an **and**.

$$z \lor (x \land y) \equiv (z \lor x) \land (z \lor y)$$

Distributive Law #2: How to distribute an **and** over an **or**.

$$z \land (x \lor y) \equiv (z \land x) \lor (z \land y)$$

Apply "**and z**" to both terms in the **or**

Outline

- Intro to Logical Equivalence
- **Logical Equivalence Rules**
  - DeMorgan’s Laws
  - Distributive Laws
  - **Table of Other Equivalences**
  - Implication Breakout
- Tautologies and Contradictions

Table of Logical Equivalences

See Rosen 1.3, Tables 6, 7, 8

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Handout

An Exercise

Simplify: $(\neg(\underbrace{\neg p \wedge q}_{s}) \vee r) \wedge (\underbrace{\neg p \wedge q}_{s})$. ***Include justification for each step***

$\equiv (\neg s \vee r) \wedge s$	Define $s \equiv \neg p \wedge q$
$\equiv \underbrace{(\neg s \wedge s)} \vee (r \wedge s)$	Distrib. Law
$\equiv F \vee (r \wedge s)$	Negation Law
$\equiv r \wedge s$	Identity Law
$\equiv r \wedge \neg p \wedge q$	Plug in $s \equiv \neg p \wedge q$

An Exercise

Simplify: $(\neg(\neg p \wedge q) \vee r) \wedge (\neg p \wedge q)$

$\equiv (\neg s \vee r) \wedge s$

Define s as $(\neg p \wedge q)$

An Exercise

Simplify: $(\neg(\neg p \land q) \lor r) \land (\neg p \land q)$

$\equiv (\neg s \lor r) \land s$

$\equiv (\neg s \land s) \lor (r \land s)$

Define s as $(\neg p \land q)$

Distributive Law

An Exercise

Simplify: $(\neg(\neg p \wedge q) \vee r) \wedge (\neg p \wedge q)$

$\equiv (\neg s \vee r) \wedge s$

$\equiv (\neg s \wedge s) \vee (r \wedge s)$

$\equiv F \vee (r \wedge s)$

Define s as $(\neg p \wedge q)$

Distributive Law

"Negation Law"

An Exercise

Simplify: $(\neg(\neg p \wedge q) \vee r) \wedge (\neg p \wedge q)$

$\equiv (\neg s \vee r) \wedge s$

$\equiv (\neg s \wedge s) \vee (r \wedge s)$

$\equiv F \vee (r \wedge s)$

$\equiv r \wedge s$

Define s as $(\neg p \wedge q)$

Distributive Law

"Negation Law"

"Identity Law"

An Exercise

Simplify: $(\neg(\neg p \land q) \lor r) \land (\neg p \land q)$

$\equiv (\neg s \lor r) \land s$

$\equiv (\neg s \land s) \lor (r \land s)$

$\equiv F \lor (r \land s)$

$\equiv r \land s$

$\equiv r \land \neg p \land q$

Define s as $(\neg p \land q)$

Distributive Law

"Negation Law"

"Identity Law"

Substitute back in for s

Outline

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Implication Breakout

$$\boxed{p \to q \equiv \neg p \vee q}$$

r = "It's raining."
g = "The grass is wet."

Proposition 1:
"If it's raining, then the grass is wet."

$r \quad r \to g$

Proposition 2:
"Either it's not raining, or the grass is wet."

$\neg\ r \quad \neg r \vee g$

Implication Breakout

Prop 1:
"If it's raining, then the grass is wet."

Prop 2:
"Either it's not raining, or the grass is wet."

r	g	$r \rightarrow g$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Same truth table!
These are **logically equivalent**.

r	g	$\neg r \vee g$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Lecture 5 Handout: Logical Equivalences

Handout

Distributive Laws:

$p \vee (q \wedge r) \equiv$ ____________

$p \wedge (q \vee r) \equiv$ ____________

DeMorgan's Laws:

$\neg(p \vee q) \equiv$ ____________

$\neg(p \wedge q) \equiv$ ____________

DeMorgan's Laws with Quantifiers:

$\neg\exists x\, P(x) \equiv$ ______________

$\neg\forall x\, P(x) \equiv$ ______________

"Implication breakout" rule:

$p \rightarrow q \equiv$ $\neg p \vee q$

Which of the following ALWAYS has the same truth value as $p \rightarrow q$?

A) Converse: $q \rightarrow p$

B) Inverse: $\neg p \rightarrow \neg q$

C) Contrapositive: $\neg q \rightarrow \neg p$

Implication Breakout

- **Solving Strategy:** replace implications $p \rightarrow q$ with $\neg p \vee q$
  - leaving only $\neg$, $\vee$, $\wedge$
  - then apply other laws to simplify further
- This trick doesn't have a standard name. *("Implication Breakout" is not in Rosen)*

"If it's Saturday, then I can sleep in."
"Either it's not Saturday, or I can sleep in."

Handout

An Exercise

Simplify: $\neg\, \exists x\, [P(x) \longrightarrow \neg Q(x)]$

$p \rightarrow q \equiv \neg p \vee q$

$\equiv \neg \exists x\, [\neg P(x) \vee \neg Q(x)]$ Implication Breakout

$\equiv \neg \exists x\, \neg (P(x) \wedge Q(x))$ DeMorgan's

$\equiv \forall x\, \neg \neg (P(x) \wedge Q(x))$ DMs for Quants.

$\equiv \forall x\, [P(x) \wedge Q(x)]$ $\neg\neg$ law

An Exercise

Simplify: $\neg \exists x\, [P(x) \longrightarrow \neg Q(x)]$

$\equiv \forall x \neg [P(x) \longrightarrow \neg Q(x)]$

DeMorgan (quantifier)

An Exercise

Simplify: $\neg \exists x \, [P(x) \longrightarrow \neg Q(x)]$

$\equiv \forall x \neg [P(x) \longrightarrow \neg Q(x)]$

$\equiv \forall x \neg [\neg P(x) \vee \neg Q(x)]$

DeMorgan (quantifier)

Implication Breakout

An Exercise

Simplify: $\neg \exists x \, [P(x) \longrightarrow \neg Q(x)]$

$\equiv \forall x \neg [P(x) \longrightarrow \neg Q(x)]$

$\equiv \forall x \neg [\neg P(x) \lor \neg Q(x)]$

$\equiv \forall x \, [P(x) \land Q(x)]$

DeMorgan (quantifier)

Implication Breakout

DeMorgan (normal)

Outline

- Intro to Logical Equivalence
- Logical Equivalence Rules
  - DeMorgan's Laws
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  - Implication Breakout
- **Tautologies and Contradictions**

Tautologies and Contradictions

Some compound propositions are **always true** or **always false** regardless of their inputs.

Tautology: A compound proposition that is **always true**, regardless of its input values

- A tautology is logically equivalent to “True”

Example: $p \vee q \vee \neg p$ is a tautology

$$p \vee q \vee \neg p \equiv T$$

Contradiction: A compound proposition that is **always false**, regardless of its input values

- A contradiction is logically equivalent to “False”

Example: $p \wedge q \wedge \neg p$ is a contradiction

$$p \wedge q \wedge \neg p \equiv F$$

Logical Equivalence and Tautologies

A compound proposition is called a "**tautology**" if it is logically equivalent to "True." *Equivalently: it is true for any truth values of its inputs*

Example:

A(p, q) = (p ∧ q) ∨ (p ∧ ¬q)
B(p, q) = p

p	q	A(p, q)	B(p, q)	A↔B
T	T	T	T	T
T	F	T	T	T
F	T	F	F	T
F	F	F	F	T

A↔B is a tautology

Handout

Tautologies and Contradictions

A compound proposition is a **tautology** if

___________________________________.

Give an example of a tautology.

A compound proposition is a **contradiction** if

___________________________________.

Give an example of a contradiction.

Exercises

Handout

These tautologies have special significance! What do they **mean**?

1. $\underbrace{(\neg q \rightarrow \neg p)} \rightarrow \underbrace{(p \rightarrow q)}$

2. $(\neg p \rightarrow F) \rightarrow p$ *Capital F stands for the truth value "false," not a different variable*

3. $[(a \vee b) \wedge (a \rightarrow p) \wedge (b \rightarrow p)] \rightarrow p$

Exercises

These tautologies have special significance! What do they **mean**?

1. $(\neg q \to \neg p) \to (p \to q)$

"Proof by contrapositive works"
Regardless of p, q, if we prove $(\neg q \to \neg p)$, that implies $(p \to q)$

2. $(\neg p \to F) \to p$ *Capital F stands for the truth value "false," not a different variable*

"Proof by contradiction works"
Regardless of p, if we can assume $\neg p$ and prove that something false is true, that implies p

3. $[(a \vee b) \wedge (a \to p) \wedge (b \to p)] \to p$

"Proof by cases works"
For any a, b, p, if at least one of a, b is true, and either case implies p, that implies p unconditionally.

Wrapup

- Logical equivalence = mathematically proving that two differently-phrased statements really mean the same thing
- Tautologies = show that proof styles work! Any tautology would imply a valid proof style, even though contrapositive, contradiction, cases tend to be the most useful ones
- **Next Time:** More practice with this, in the context of a new discrete structure: **sets**