L11: More Induction & Strong Induction

EECS 203: Discrete Mathematics
Lecture 11

Learning Objectives: Induction 2

After today's lecture (and this week's readings, discussion & homework), you should be able to:

- **Know technical vocab:** weak induction, strong induction
- **Write and understand proofs by mathematical induction and by strong induction**
  - Explain the role and necessity of the base case and the inductive step
  - Determine the necessary base case and correctly prove it
    - For strong induction: determine the number of bases cases and prove them
  - Determine the inductive hypothesis
  - Apply the inductive hypothesis as part of the inductive step
  - Prove the inductive step

- **Determine whether a given induction proof is valid, and if not, determine where it fails**

- **Use induction to prove claims related to:** equalities, inequalities, divisibility, and other statements (sets, geometry, etc.)

Lecture 11 Outline

- **(More) Mathematical Induction**
  - **Recap** & finish previous lectures' examples, as needed
  - Example: A Divisibility proof
  - Questionable Inductive Proof: Horses
  - Well-Ordering
  - Example: Towers of Hanoi (time permitting)

- Strong Induction
  - Example: Stamps
  - Guide for Strong Induction Proofs
  - Example: Piles of Stones

Mathematical Induction ≈ Dominos

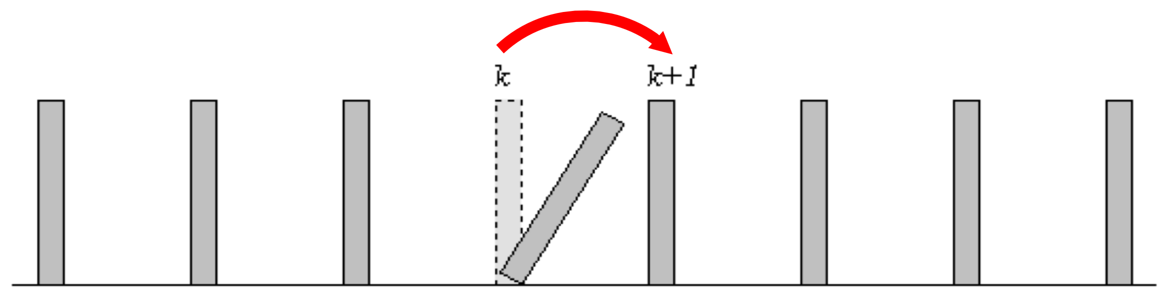
Let $P(n)$ be a predicate.

Goal: Prove that $\boldsymbol{P(n)}$ is true for all $\boldsymbol{n} \in \mathbf{N}$

Step 1: “Inductive Step”

If you can knock down one domino, then you can knock down the next one.

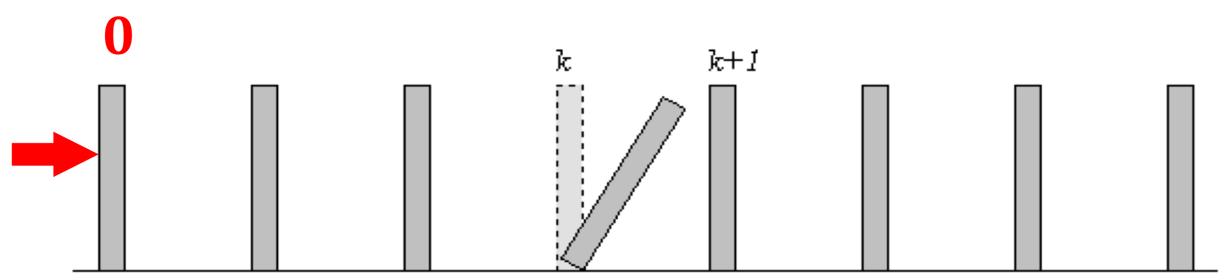
For any $\boldsymbol{k} \in \mathbf{N}$,
$\boldsymbol{P(k) \to P(k+1)}$



Step 2: “Base Case”

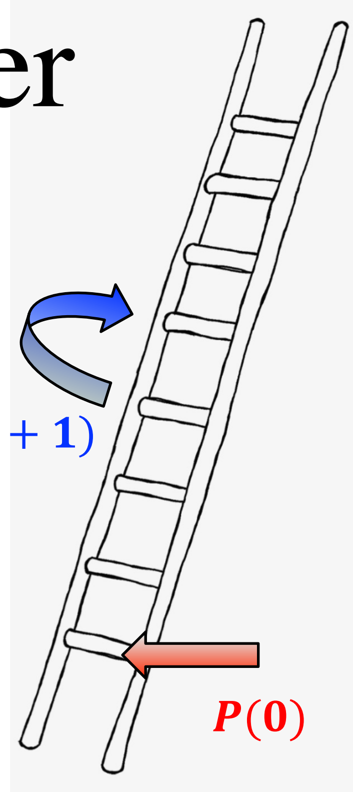
You can knock down the first domino

$\boldsymbol{P(0)}$



Therefore, you can knock down all dominos.

Induction ≈ Climbing the Ladder



We want to show that $\forall n \geq 0$ $P(n)$ is true.

- Think of the non-negative integers as a ladder.
- 0, 1, 2, 3, 4, 5, 6, . . .

- From *each* ladder step, you can reach the *next*.
  "Inductive step"

  $P(0) \rightarrow P(1)$, $P(1) \rightarrow P(2)$, $P(2) \rightarrow P(3)$, . . .

  $$\forall k \geq 0 \quad P(k) \rightarrow P(k+1)$$

- You can *get on* the ladder (at the bottom): **"Base case"**

  $P(0)$

- Then, by mathematical induction, you can climb up the whole ladder:

  $$\forall n \geq 0 \quad P(n)$$

Guide for [weak] Induction Proofs

- Restate the claim you are trying to prove

- **Base case**: Prove the claim holds for the "first" value of n
  - Prove $P(n_0)$ is true

- **Inductive Step**: Prove that $P(k) \rightarrow P(k+1)$ for an arbitrary integer k in the desired range.
  - Let k be an arbitrary integer with $k \geq n_0$
  - Assume $P(k)$
  - Show that $P(k+1)$ holds

$$\boxed{\text{Equivalently: Show } P(k-1) \rightarrow P(k)}$$

- Conclusion: explain that you've proven the desired claim.

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Divisibility Example

- For integers a, b **$a|b$ = "*a* divides *b*"**
  - $a \mid b$ iff b is a multiple of a.
  - $a \mid b$ iff b is divisible by a.
  - $a \mid b$ iff $am = b$ for some $m \in \mathbb{Z}$
- Example:
  - $3 \mid (5^3\text{-}5)$ because $(5^3\text{-}5) = 120 = 3(40)$.

- Prove $\forall n \in \mathbb{N} \;\; 3|n^3 - n$.

Divisibility Example

Prove $\forall \boldsymbol{n} \in \mathbb{N} \;\; 3 \mid n^3 - n$

Claim: $\forall n \in \mathbb{N} \;\; 3 | n^3 - n$

Base Case: $n = 0$.

- *Thoughts*:
  - Prove P(0).
  - Does $3 \mid 0^3 - 0$?
  - $3 \mid 0^3 - 0 \leftrightarrow 3 \mid 0 \leftrightarrow$ "3 divides 0"

    $\leftrightarrow$ 0 is a multiple of 3 $\leftrightarrow$ $\exists m \in \mathbb{Z}$ such that $3m = 0$
  - $m = 0$ solves this.
- $3 \mid 0^3 - 0$ because $0^3 - 0 = 3m$ for $m = 0$.

Divisibility Example

Prove $\forall \boldsymbol{n} \in \mathbb{N}\ \ 3 \mid n^3 - n$

Claim: $\forall n \in \mathbb{N}\ \ 3 | n^3 - n$

Base Case: $n = 0$: $3 \mid 0^3 - 0$ because $0^3 - 0 = 3m$ for $m = 0$, so $P(0)$ holds.

Inductive Step:

Divisibility Example

Prove $\forall \boldsymbol{n} \in \mathbb{N}\ \ 3 \mid n^3 - n$

Claim: $\forall n \in \mathbb{N}\ \ 3|n^3 - n$

Base Case: $n = 0$: $3 \mid 0^3 - 0$ because $0^3 - 0 = 3m$ for $m = 0$, so $P(0)$ holds.

Inductive Step:

- Consider an arbitrary integer k, with $k \geq 0$.
- Assume $3|k^3 - k$
- Want to show $3|(k+1)^3 - (k+1)$

Divisibility Example

Prove $\forall \boldsymbol{n} \in \mathbb{N} \;\; 3 \mid n^3 - n$

Claim: $\forall n \in \mathbb{N} \;\; 3 | n^3 - n$

Base Case: $n = 0$: $3 \mid 0^3 - 0$ because $0^3 - 0 = 3m$ for $m = 0$, so $P(0)$ holds.

Inductive Step:

- Consider an arbitrary integer k, with $k \geq 0$.
- Assume $3 | k^3 - k$
- Want to show $3 | (k+1)^3 - (k+1)$

From our inductive hypothesis, $k^3 - k = 3a$ for some integer a.

We want to show that $(k+1)^3 - (k+1) = 3b$ for some integer b.

$(k+1)^3 - (k+1) =$

Divisibility Example

Prove $\forall \boldsymbol{n} \in \mathbb{N} \;\; 3 \mid n^3 - n$

Claim: $\forall n \in \mathbb{N} \;\; 3|n^3 - n$

Base Case: $n = 0$: $3 \mid 0^3 - 0$ because $0^3 - 0 = 3m$ for $m = 0$, so $P(0)$ holds.

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From our inductive hypothesis, $k^3 - k = 3a$ for some integer a.

We want to show that $(k+1)^3 - (k+1) = 3b$ for some integer b.

$$\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\
&= k^3 - k + 3k^2 + 3k \\
&= 3a + 3k^2 + 3k && \text{by IH} \\
&= 3(a + k^2 + k) \\
&= 3b && (\text{let } b = a + k^2 + k)
\end{aligned}$$

So $(k+1)^3 - (k+1)$ is 3 times an integer, and thus $3|(k+1)^3 - (k+1)$

Therefore, by mathematical induction, $3|n^3 - n$ for all $n \in \mathbb{N}$.

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A Questionable "Proof"

- **Claim:** All horses are the same color.
- **Proof.** By **induction**.

Let **P(n)** = "In any set of $\boldsymbol{n}$ horses, all horses in the set have the same color."

- We want to prove $\forall n \geq 1 \;\; P(n)$.

- We'll do the Base Case first, then handle the Inductive Step

A Questionable “Proof”

Claim: $P(n)$ for all $n \geq 1$, where $P(n)$ is the predicate
“In any set of n horses, all horses in the set have the same color.”

Base case: Prove the claim holds for the “first” value of n

Inductive Step: Prove that $P(k) \rightarrow P(k+1)$ for an arbitrary integer k in the desired range.

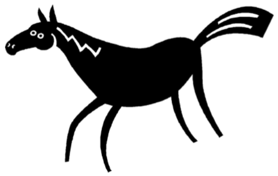
Conclusion: explain that you’ve proven the desired claim.

A Questionable "Proof"

Claim: $P(n)$ for all $n \geq 1$, where $P(n)$ is the predicate
"In any set of n horses, all horses in the set have the same color."

Base case: Prove **P(1):** "In any set of 1 horse, all horses have the same color."

- **P(1)** is true because there is only 1 horse, and that horse is the same color as itself.



Inductive Step: Prove that $P(k) \rightarrow P(k+1)$ for an arbitrary integer k in the desired range.

Conclusion: explain that you've proven the desired claim.

A Questionable "Proof"

Claim: $P(n)$ for all $n \geq 1$, where $P(n)$ is the predicate "In any set of n horses, all horses in the set have the same color."

Inductive Step: Prove that $P(k) \rightarrow P(k+1)$ for an arbitrary integer k in the desired range.

- Let k be arbitrary positive integer.
- Assume $P(k)$

Assume: In any set of k horses, all horses in the set have the same color"

- Show $P(k+1)$ holds

Want to show: In any set of $k+1$ horses, all horses in the set have the same color"

A Questionable "Proof"

Claim: $P(n)$ for all $n \geq 1$, where $P(n)$ is the predicate
"In any set of n horses, all horses in the set have the same color."

Inductive Step: Let k be an arbitrary positive integer.

Assume **P(k)**: "In any set of $\boldsymbol{k}$ horses, all horses in the set have the same color"

Want to show **P(k+1)**: "In any set of $\boldsymbol{k+1}$ horses, all horses …"

- Now consider an arbitrary set of k+1 horses
  - **{Butterscotch, Bojack, Hollyhock, … , Racer, Spirit}**

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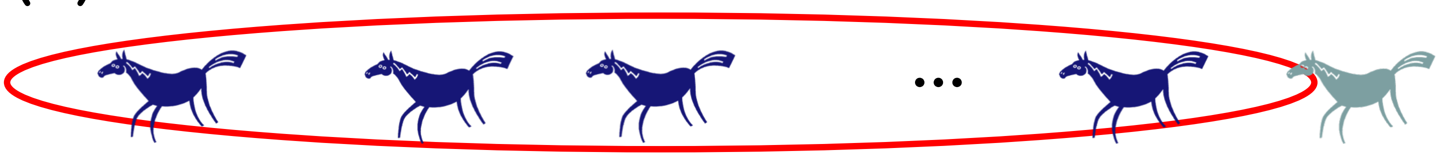
- Now consider an arbitrary set of k+1 horses
  - **{Butterscotch, Ginger, Hollyhock, … , Racer, Spirit}**

- By **`P(k):`** The **first k** horses in the list all have the same color:



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Want to show **P(k+1)**: "In any set of ***k*** **+ 1** horses, all horses …"

- Now consider an arbitrary set of k+1 horses
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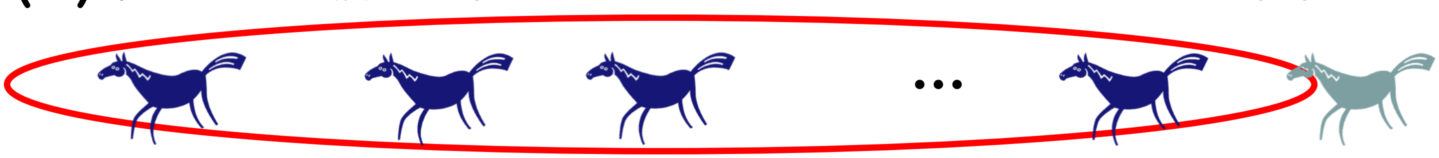
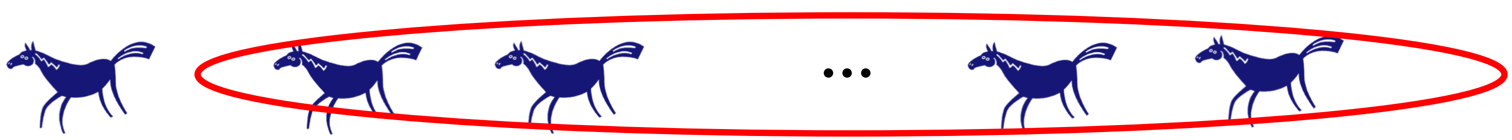


- By **`P(k):`** The **last k** horses in the list all have the same color:



- So all **k+1** horses have the same color as **Ginger**, and therefore the same color as each other.

Question

- What was wrong with that proof?

A. Mathematical induction only applies to numbers, not horses.

B. The Base Case reasoning was incorrect.

C. The Inductive Step reasoning was incorrect.

D. The Base Case and Inductive Step are correct, but you've put them together wrong.

E. Nothing was wrong.

The Problem with the Horses

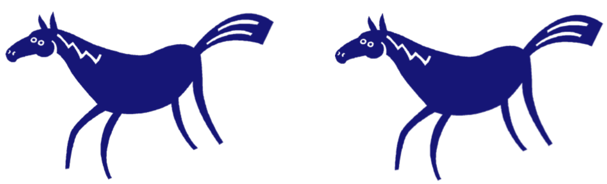
Claim: P(n) = “In any set of **n** horses, all horses in the set have the same color.” for all $\boldsymbol{n} \geq \mathbf{1}$

– **{Butterscotch, Bojack, Hollyhock, … , Racer, Spirit}**

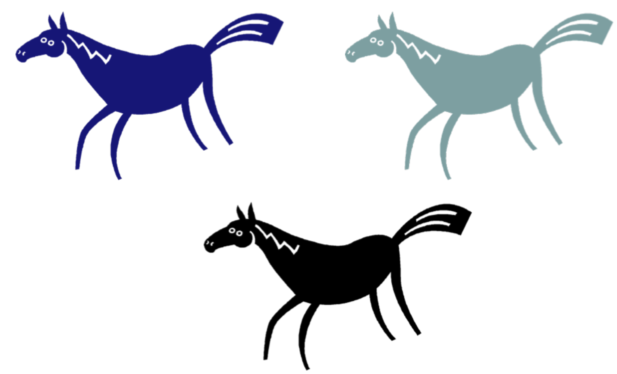
- By **`P(k)`** : The **first k** horses in the list all have the same color.
- By **`P(k)`** : The **last k** horses in the list all have the same color.

Does $\boldsymbol{P(2)} \rightarrow \boldsymbol{P(3)}$?

Assume true for $k = 2$



Then for $k = 3$



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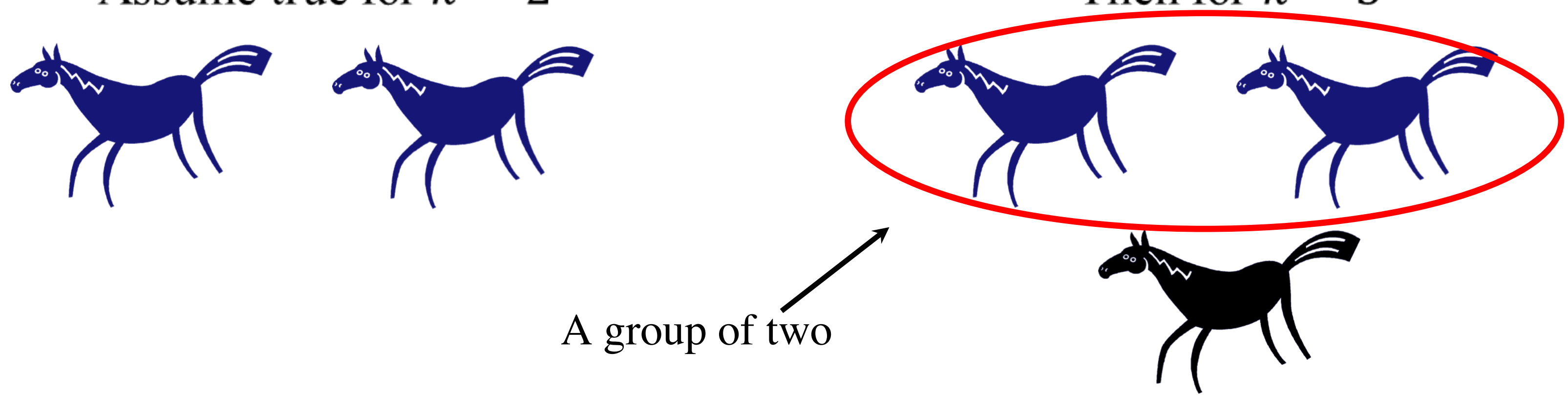

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Does $P(2) \rightarrow P(3)$? **YES**

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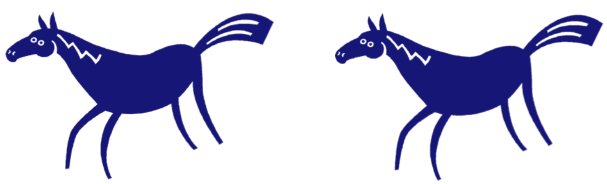

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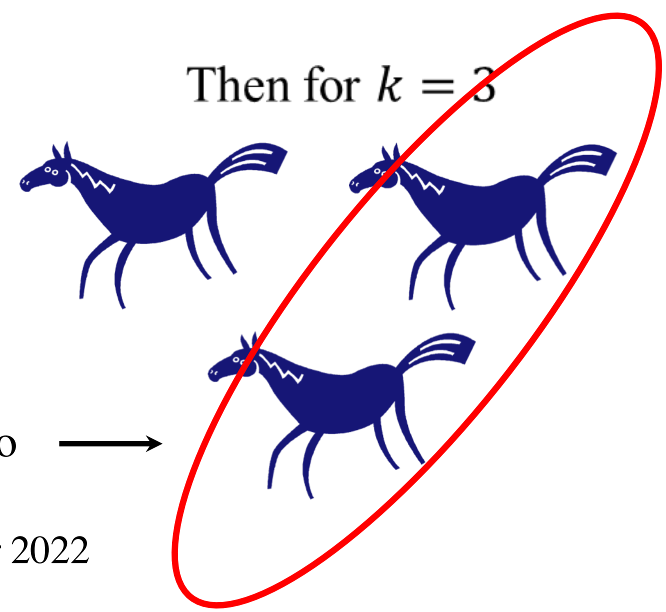

Another group of two →

The Problem with the Horses

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  - By **`P(k)`** : The **first k** horses in the list all have the same color.
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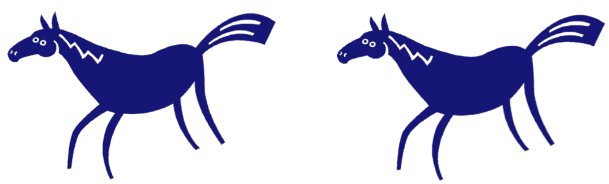
Does $P(2) \rightarrow P(3)$? **YES**

By similar reasoning:
$P(3) \rightarrow P(4)$, $P(4) \rightarrow P(5)$, …

Assume true for $k = 2$



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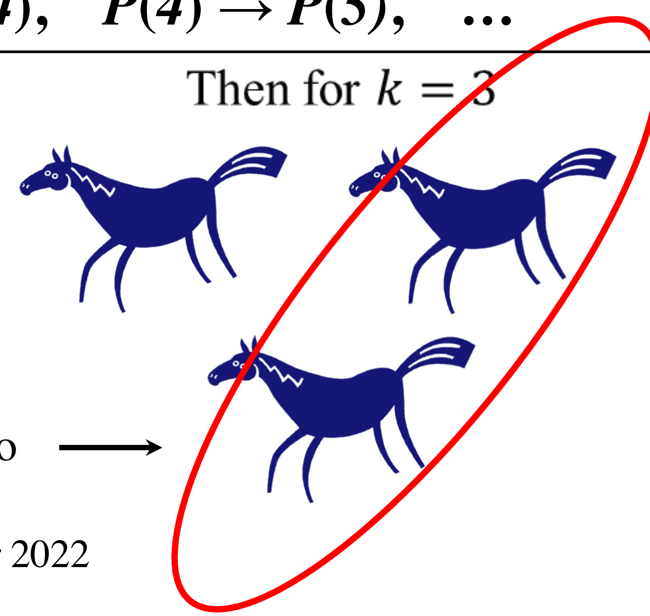

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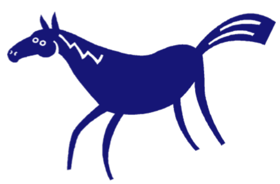
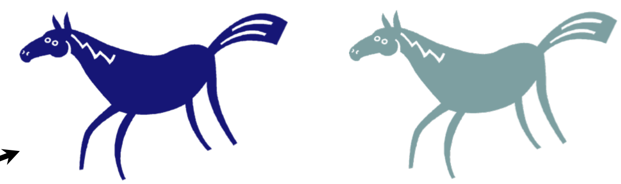
- By **`P(k)`** : The **first k** horses in the list all have the same color.
- By **`P(k)`** : The **last k** horses in the list all have the same color.

Does $P(1) \rightarrow P(2)$? **NO!**

Start with $k = 1$



Then for $k = 2$



A set of one horse with the same color

Another set of one horse with the same color

The Problem with the Horses

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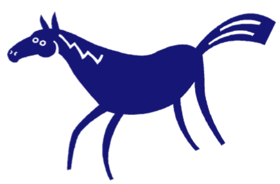

A set of one horse with the same c

In our "proof" we proved:

- $P(1)$
- $P(2) \rightarrow P(3)$, $P(3) \rightarrow P(4)$, $P(4) \rightarrow P(5)$, …
- But $P(1) \rightarrow P(2)$ is completely false!
  - We can't knock over the second domino
  - We can't get to the second rung of the ladder

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Intro to Well-Ordering

- **So Far:** Induction over **N** or $\mathbf{Z}^+$ or similar

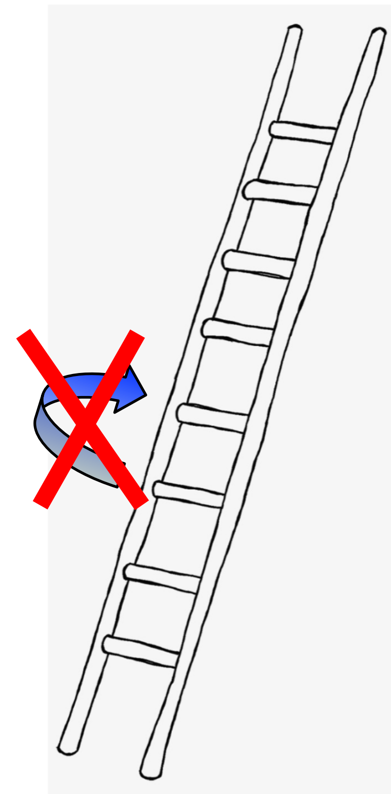
  Claim: **P(n)** for all integers $\boldsymbol{n} \geq \mathbf{0}$

- Can we do induction over **nonnegative reals**?

  Claim: **P(n)** for all real numbers $\boldsymbol{n} \geq \mathbf{0}$

Inductive Step:

- Assume I'm on some step of the ladder
- Want to show I can get to the *next* step



Say I'm on the step for **n** = 1.7. What is *the next* real number?

- There is none

There is no "next step" on the ladder.

So, induction over nonnegative reals doesn't work .

Well-Ordering = When **does** Induction work?

- A set A with ordering "$\prec$" is **well-ordered** iff every non-empty subset $S \subseteq A$ has a *least element* (according to $\prec$)
  - For any $S \subseteq A$ and $a, b \in S$: $\boldsymbol{a}$ **is "less than"** $\boldsymbol{b}$ **iff** $\boldsymbol{a \prec b}$**.**
  - $\mathbf{Z}^+$ is well-ordered under the usual ordering $<$
    - Every non-empty subset of $\mathbf{Z}^+$ has a least element
  - $\boldsymbol{R}^+ \cup \{\mathbf{0}\}$, the nonnegative reals, are **not** well-ordered under the usual ordering $<$
    - There are subsets of $\boldsymbol{R} \cup \{\mathbf{0}\}$ that do not have a least element

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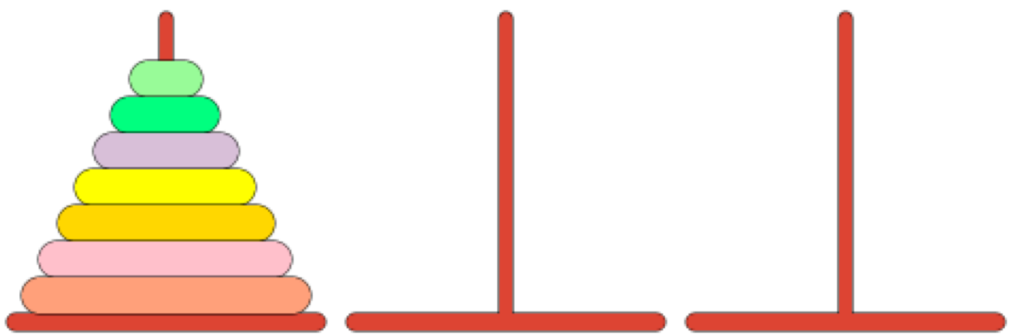
An Inductive Logic Puzzle

- Tower of Hanoi rules of the game.
  - *n* discs initially on peg 1.
  - Can only move one disc at a time.
  - A disc can only be put on a larger disc or an empty peg.
  - GOAL: move all discs from peg 1 to peg 3.



Start

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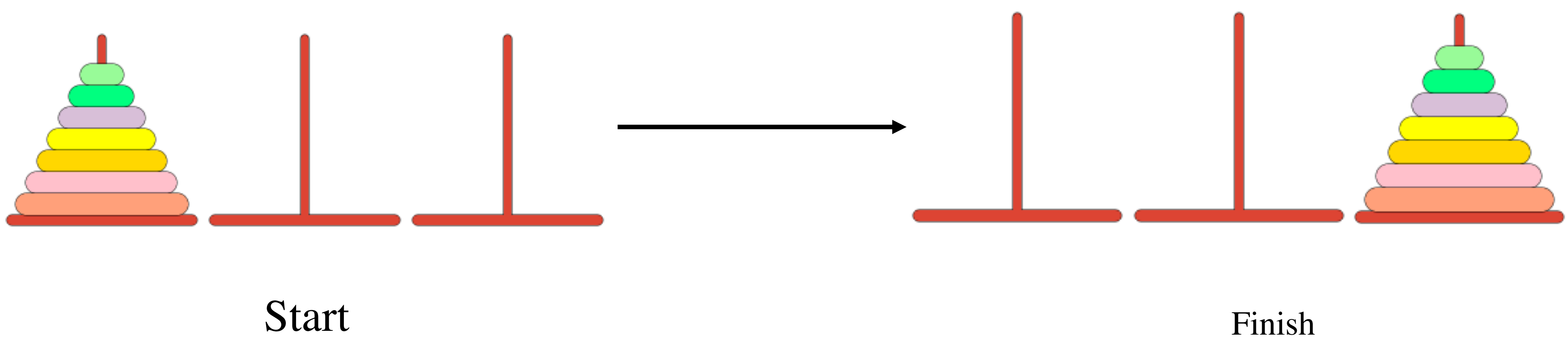

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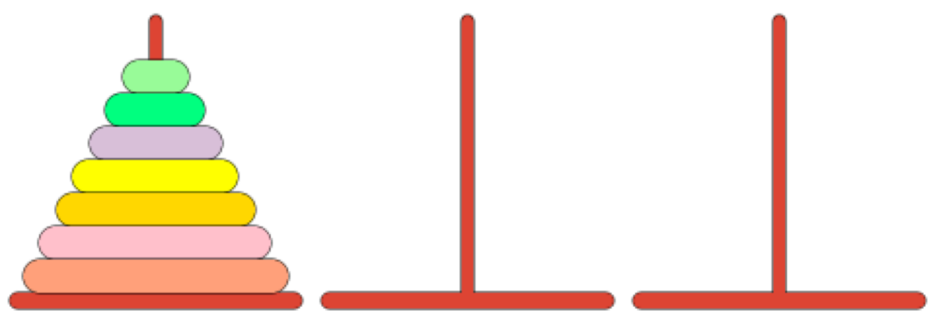

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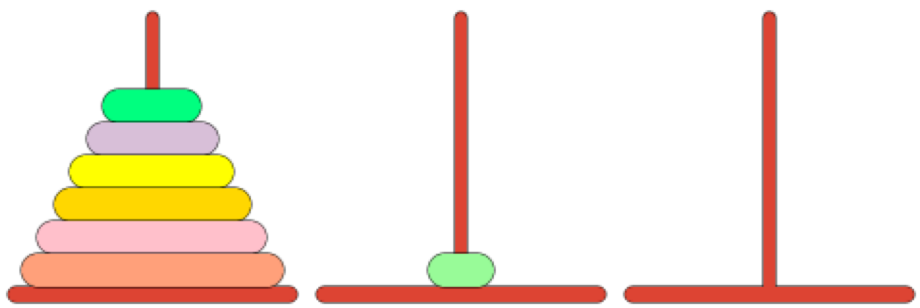

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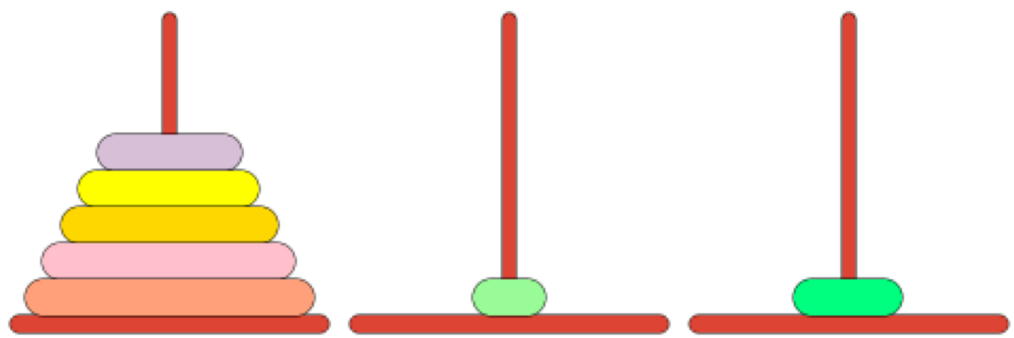
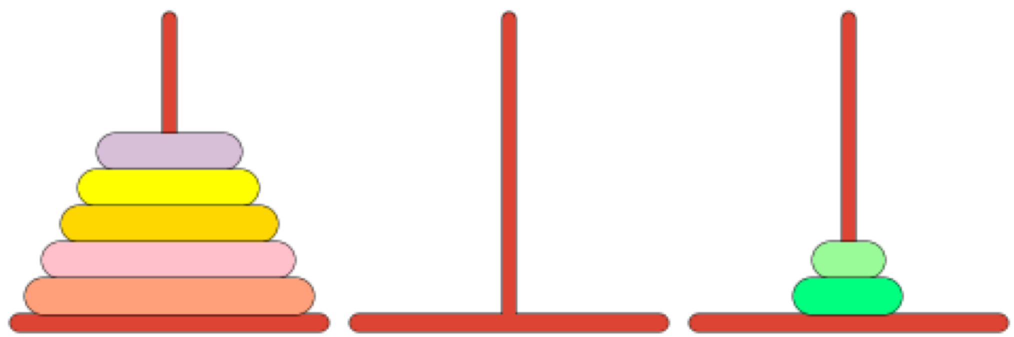


An Inductive Logic Puzzle

- Tower of Hanoi rules of the game.
  - ***n*** discs initially on peg 1.
  - Can only move one disc at a time.
  - A disc can only be put on a larger disc or an empty peg.
  - GOAL: move all discs from peg 1 to peg 3.



An Inductive Logic Puzzle

- **Claim**: The Tower of Hanoi Puzzle has a solution.
- **For all $\boldsymbol{n} \in \mathbb{N}, \boldsymbol{P}(\boldsymbol{n})$:** The Tower of Hanoi puzzle has a solution with $\boldsymbol{n}$ discs initially on peg 1.

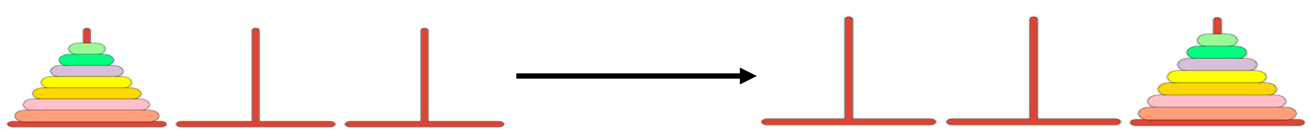
Which of these is the best base case for the induction?

(A) When the puzzle has 0 discs, the start and end configurations are the same.

(B) When the puzzle has 1 disc, you can just move it directly from peg 1 to peg 3.

(C) These both work.

(D) I have no idea.



An Inductive Logic Puzzle

- **<u>Claim</u>**: The Tower of Hanoi Puzzle has a solution.
- **For all $\boldsymbol{n} \in \mathbb{N}, \boldsymbol{P}(\boldsymbol{n})$:** The Tower of Hanoi puzzle has a solution with $\boldsymbol{n}$ discs initially on peg 1.

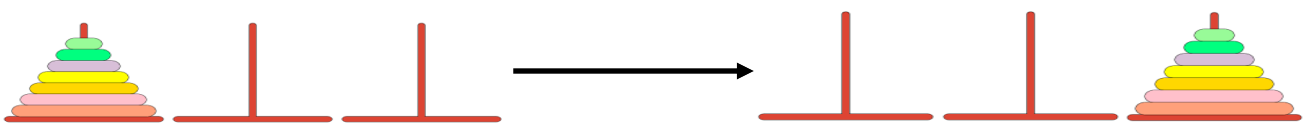
Which of these is the best base case for the induction?

(A) When the puzzle has 0 discs, the start and end configurations are the same. (Easiest.)

(B) When the puzzle has 1 disc, you can just move it directly from peg 1 to peg 3.

(C) These both work.

(D) I have no idea.



An Inductive Logic Puzzle

- **<u>Claim</u>**: The Tower of Hanoi Puzzle has a solution.
- **For all $\boldsymbol{n} \in \mathbb{N}, \boldsymbol{P(n)}$:** The Tower of Hanoi puzzle has a solution with $\boldsymbol{n}$ discs initially on peg 1.

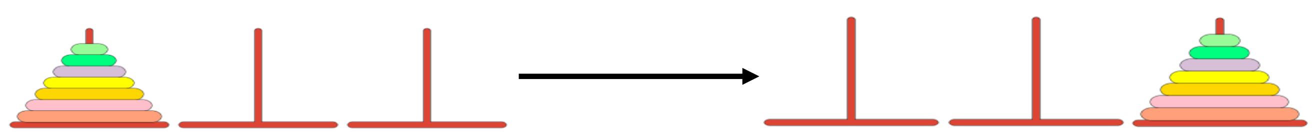
Which of these is the best base case for the induction?

(A) When the puzzle has 0 discs, the start and end configurations are the same. (Easiest.)

(B) When the puzzle has 1 disc, you can just move it directly from peg 1 to peg 3. (Would be fine if we start from n=1.)

(C) These both work.

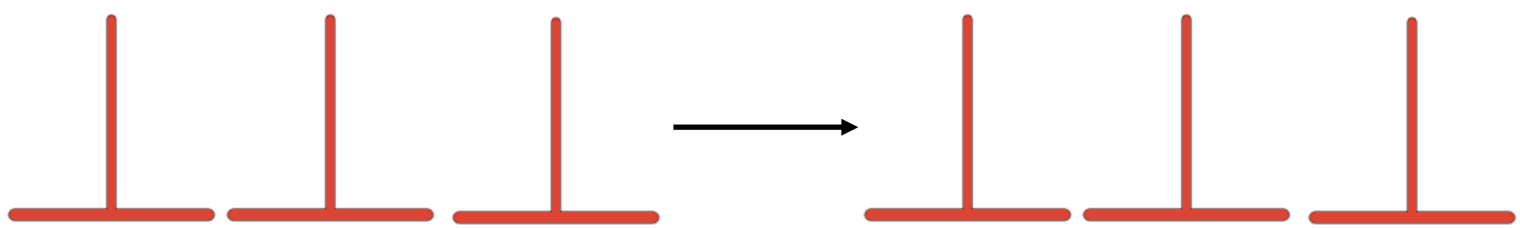
(D) I have no idea.



An Inductive Logic Puzzle

- **Claim**: The Tower of Hanoi Puzzle has a solution.
- **For all $n \in \mathbb{N}, P(n)$:** The Tower of Hanoi puzzle has a solution with n discs initially on peg 1.

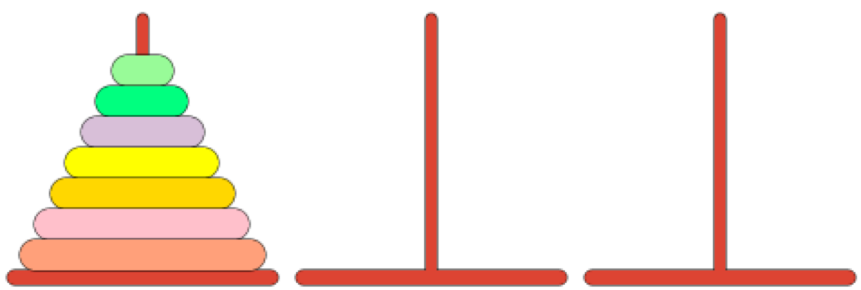
Base Case: When **n = 0**, there are no discs, and the start configuration = the end configuration. We solved the puzzle instantly!



An Inductive Logic Puzzle

- **For all $\boldsymbol{n} \in \mathbb{N}, \boldsymbol{P(n)}$:** The Tower of Hanoi puzzle has a solution with $\boldsymbol{n}$ discs initially on peg 1.
- **Base Case** ✓

Inductive Step:
Assume P(k): the puzzle has a solution for k discs.
Want to show P(k+1): the puzzle has a solution for $k + 1$ discs.



An Inductive Logic Puzzle

Handout

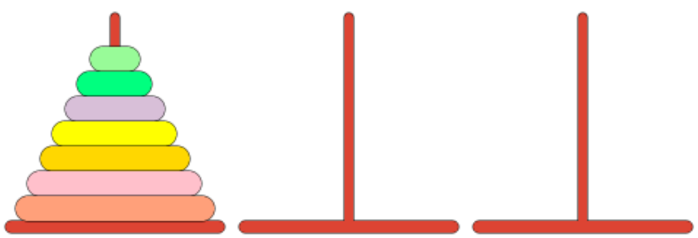
- **For all $n \in \mathbb{N}, P(n)$:** The Tower of Hanoi puzzle has a solution with n discs initially on peg 1.
- **Proof: Base Case : P(_____)**

When n = ______, start configuration ________ end configuration

Inductive Step:

Assume P(k): the puzzle has a solution for k discs.

Want to show P(k+1): the puzzle has a solution for $k+1$ discs.

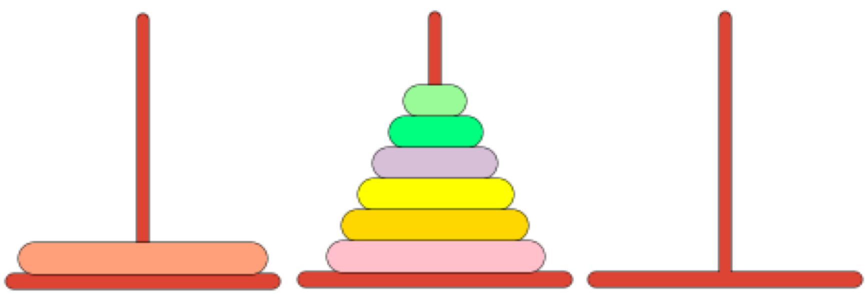


1. Use **inductive hypothesis** to move ______________________________ over one.
2. Move ______________________ to peg 3.
3. Use **inductive hypothesis** to move ______________________________ to peg 3.

An Inductive Logic Puzzle

- **For all $\boldsymbol{n} \in \mathbb{N}, \boldsymbol{P}(\boldsymbol{n})$:** The Tower of Hanoi puzzle has a solution with $\boldsymbol{n}$ discs initially on peg 1.
- **Base Case** ✓

Inductive Step:
Assume P(k): the puzzle has a solution for k discs.
Want to show P(k+1): the puzzle has a solution for $k + 1$ discs.

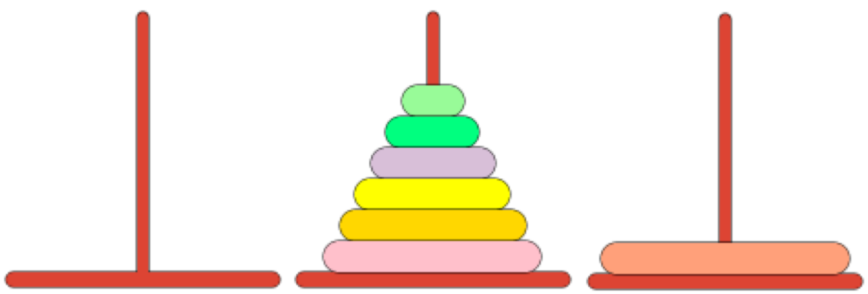


1. Use **inductive hypothesis** to move top **k-1** discs over one.

An Inductive Logic Puzzle

- **For all $\boldsymbol{n} \in \mathbb{N}, \boldsymbol{P}(\boldsymbol{n})$:** The Tower of Hanoi puzzle has a solution with $\boldsymbol{n}$ discs initially on peg 1.
- **Base Case** ✓

Inductive Step:
Assume P(k): the puzzle has a solution for k discs.
Want to show P(k+1): the puzzle has a solution for $k+1$ discs.



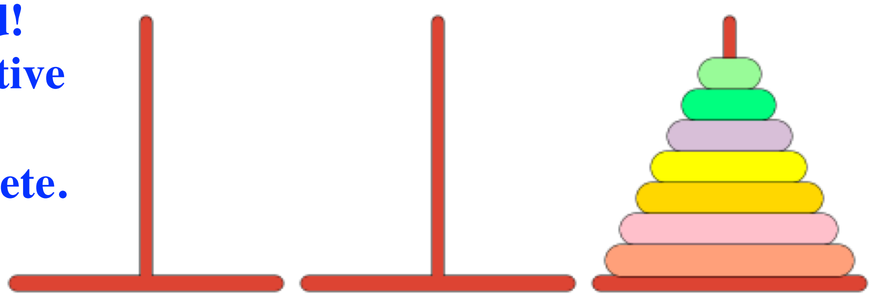
1. Use **inductive hypothesis** to move top **k-1** discs over one.
2. Move biggest disc to peg 3.

An Inductive Logic Puzzle

- **For all $\boldsymbol{n} \in \mathbb{N}, \boldsymbol{P}(\boldsymbol{n})$:** The Tower of Hanoi puzzle has a solution with $\boldsymbol{n}$ discs initially on peg 1.
- **Base Case** ✓

Inductive Step:
Assume P(k): the puzzle has a solution for k discs.
Want to show P(k+1): the puzzle has a solution for $k+1$ discs.

Solved! Inductive Step complete.



1. Use **inductive hypothesis** to move top **k-1** discs over one.
2. Move biggest disc to peg 3.
3. Use **inductive hypothesis** to move top **k-1** discs to peg 3.

Unwinding Induction

We Just Proved:

- You can (easily) solve **n=0** discs.
- If you can solve **n=0** discs, then you can solve **n=1** discs.
- If you can solve **n=1** discs, then you can solve **n=2** discs.
- ...

- By the time you're thinking about **k+1** discs, you have already solved **0, 1, 2, … , k** discs.
  - Induction: uses the **k** disc solution.
  - Next: a stronger version of induction, that uses **all** previous solutions!

Lecture 11 Outline

- (More) Mathematical Induction
  - Recap & finish previous lectures' examples, as needed
  - Example: A Divisibility proof
  - Questionable Inductive Proof: Horses
  - Well-Ordering
  - Example: Towers of Hanoi (time permitting)

- **Strong Induction**
  - **Example: Stamps**
  - Guide for Strong Induction Proofs
  - Example: Piles of Stones

Stamps

- What is the largest cent-value that **cannot** be formed using only 3-cent and 5-cent stamps?
  - (A) 2
  - (B) 4
  - (C) 7
  - (D) 8
  - (E) 11

Stamps

- What is the largest cent-value that **cannot** be formed using only 3-cent and 5-cent stamps?
  - (A) 2
  - (B) 4
  - **(C) 7** <= Correct answer
  - (D) 8
  - (E) 11

Stamps

- Let $P(n) =$ "n cents can be formed using 3-cent and 5-cent stamps."

- **Claim**: $\forall n \geq 8\ P(n)$.
- **Proof by strong induction:**

Strong Induction as Dominos

Handout

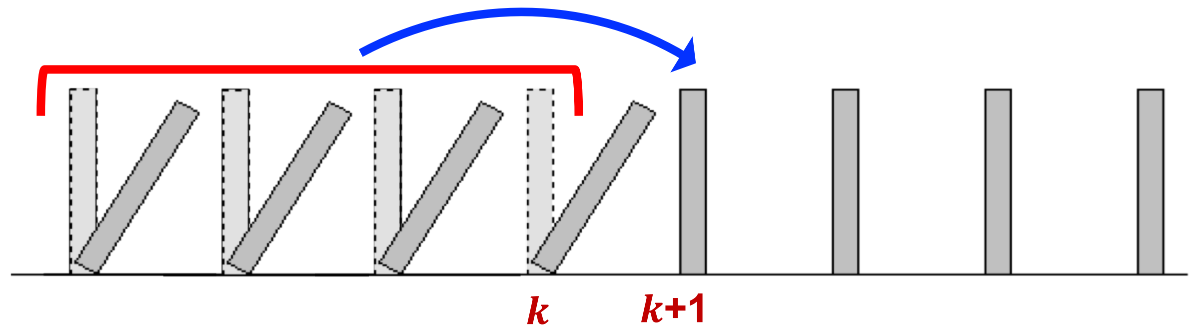
Let $P(n)$ be a predicate.

Goal: Prove that $\boldsymbol{P(n)}$ is true for all $\boldsymbol{n \in Z^+}$

Step 1: [Strong] Inductive Step

If you can knock down *all the previous* dominos, then you can knock down the **k+1**st one.

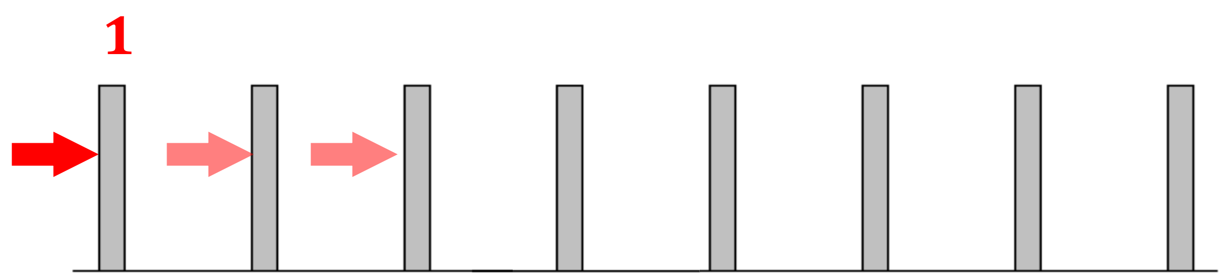
$[\boldsymbol{P(1) \land P(2) \land \cdots \land P(k)}] \rightarrow \boldsymbol{P(k+1)}$



Step 2: Base Case(s)

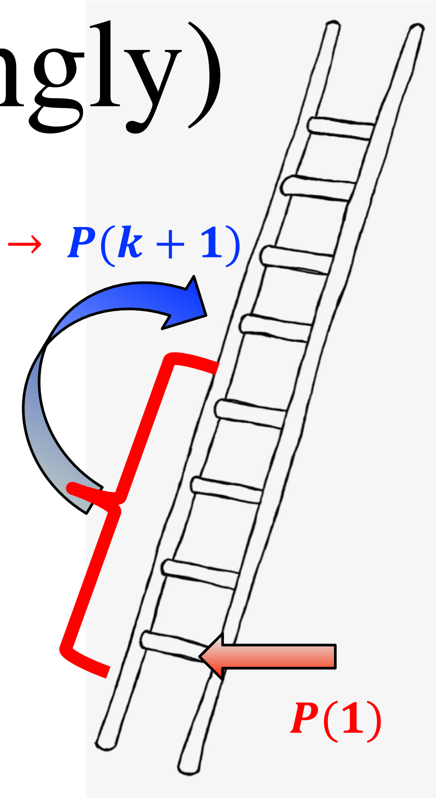
You can knock down the first domino(s) $\boldsymbol{P(1)}$

Possibly also:
$\boldsymbol{P(2)}$ **and** $\boldsymbol{P(3)}$ **and more**



Therefore, you can knock down all dominos.

Climbing the Ladder (Strongly)



- We want to show that $\forall n{\geq}1\ P(n)$ is true.
  - Think of the positive integers as a ladder.
  - 1, 2, 3, 4, 5, 6, . . .
- You can reach the *bottom* of the ladder:
  - $P(1)$
- Given *all lower* steps, you can reach the *next*.
  - $P(1) \rightarrow P(2),\quad P(1) \wedge P(2) \rightarrow P(3),\quad \ldots$
  - $\forall k{\geq}1\ [P(1) \wedge \ldots \wedge P(k)] \rightarrow P(k{+}1)$
- Then, by strong induction:
  - $\forall n{\geq}1\ P(n)$

Weak vs. Strong Induction

- To prove that $P(n)$ is true for all positive n.
- Induction:

$P(1)$	Basis step
$\forall k \in \mathbf{Z}^+ \; [P(k) \to P(k+1)]$	Inductive step
$\forall n \in \mathbf{Z}^+ \;\; P(n)$	Conclusion

- **Strong** induction:

$P(1)$	Basis step
$\forall k \in \mathbf{Z}^+ \; [P(1) \wedge \cdots \wedge P(k) \to P(k+1)]$	Inductive step
$\forall n \in \mathbf{Z}^+ \;\; P(n)$	Conclusion

Stamps Proof

P(n): "n cents can be made using 3- and 5-cent stamps"

Claim: P(n) for all $n \geq 8$

Base Case(s)

- P(8) and maybe more…
- We'll come back to this after the inductive step

(Strong) Inductive step

Let $k \geq n_1$ be an arbitrary integer

Assume P(j) for all $8 \leq j \leq k-1$

- "all cent values from 8 to k-1 can be made"

Want to show: P(k)

- "k cents can be made"

In logic: $[P(8) \wedge P(9) \wedge \cdots \wedge P(k-1)] \rightarrow P(k)$

Stamps Proof

P(n): "n cents can be made using 3- and 5-cent stamps"

Claim: P(n) for all $n \geq 8$

(Strong) Inductive step

- Let k be an integer with $k \geq$ ____

Stamps Proof

P(n): "n cents can be made using 3- and 5-cent stamps"

Claim: P(n) for all $n \geq 8$

(Strong) Inductive step

- Let k be an integer with $k \geq$ ____
- Assume $P(j)$ is true for all $8 \leq j \leq k-1$.
  - $P(j)$: "j cents can be made using …"
- Want to show: $P(k)$
  - $P(k)$: "k cents can be made using …"

We need to relate $P(k)$ back to some value covered by the inductive hypothesis.

Stamps Proof

P(n): "n cents can be made using 3- and 5-cent stamps"

Claim: P(n) for all $n \geq 8$

(Strong) Inductive step

- Let k be an integer with $k \geq$ ____
- Assume $P(j)$ is true for all $8 \leq j \leq k-1$.
  - $P(j)$: "j cents can be made using …"
- Want to show: $P(k)$
  - $P(k)$: "k cents can be made using …"

This works as long as $k-3$ is in the "magic range" covered by the inductive hypothesis.

- $P(k-3)$ is true, by IH (because $8 \leq k-3 \leq k-1$)
  - i.e., we can make $k-3$ cents
- To make k cents: first make $k-3$ cents, then add an additional 3-cent stamp.
- Therefore, $P(k)$ is true.

Stamps Proof

P(n): "n cents can be made using 3- and 5-cent stamps"

Claim: P(n) for all $n \geq 8$

(Strong) Inductive step.

- Let k be an integer with $k \geq \mathbf{11}$
- Assume $P(j)$ is true for all $8 \leq j \leq k-1$.
  - $P(j)$: "j cents can be made using …"
- Want to show: $P(k)$
  - $P(k)$: "k cents can be made using …"
- $P(k-3)$ is true, by IH (because $8 \leq k-3 \leq k-1$)
  - i.e., we can make $k-3$ cents
- To make k cents: first make $k-3$ cents, then add an additional 3-cent stamp.
- Therefore, $P(k)$ is true.

Stamps Proof

P(n): "n cents can be made using 3- and 5-cent stamps"

Claim: P(n) for all $n \geq 8$

(Strong) Inductive step. ✓

Base cases:

- P(8) is true because 8 = 5 + 3
- P(9) is true because 9 = 3 + 3 + 3
- P(10) is true because 10 = 5 + 5

Therefore, by strong induction, $P(n)$ holds for all $n \geq 8$.

- **Number of base cases** depends on the problem *and* your implementation of the inductive step.
- Ex: we could had made **k-5** cents + a **5**-cent stamp in the inductive step, but then we would need **5** base cases.

Lecture 11 Outline

- (More) Mathematical Induction
  - Recap & finish previous lectures' examples, as needed
  - Example: A Divisibility proof
  - Questionable Inductive Proof: Horses
  - Well-Ordering
  - Example: Towers of Hanoi (time permitting)

- Strong Induction
  - Example: Stamps
  - **Guide for Strong Induction Proofs**
  - Example: Piles of Stones

Guide for **Strong** Induction Proofs

- Restate the claim you are trying to prove

- **Inductive Step**:

- **Base case(s)**:

- Conclusion: explain that you've proven the desired claim.

Guide for **Strong** Induction Proofs

- Restate the claim you are trying to prove

Equivalently: Show
$[\,P(n_0) \wedge P(n_0+1) \wedge \cdots \wedge P(\boldsymbol{k})\,] \rightarrow P(\boldsymbol{k}+\mathbf{1})$

- **Inductive Step**: Prove that for an arbitrary integer k in the desired range,

$$[\,P(\boldsymbol{n_0}) \wedge P(n_0+1) \wedge \cdots \wedge P(\boldsymbol{k}-\mathbf{1})\,] \;\rightarrow\; P(\boldsymbol{k})$$

  – Let k be an arbitrary integer with $k \geq$ ____
  value depends on the proof
  – Assume $P(j)$ is true for all $\boldsymbol{n_o} \leq j \leq \boldsymbol{k}-\mathbf{1}$
  – Show that $P(\boldsymbol{k})$ holds

- **Base case(s)**: Prove the claim holds for the "first" **value(s)** of n
  – Prove $P(n_0)$ is true
  – May also need to prove $P(n_0+1)$ and more, depending on the inductive step
- Conclusion: explain that you've proven the desired claim.

Lecture 11 Outline

- (More) Mathematical Induction
  - Recap & finish previous lectures' examples, as needed
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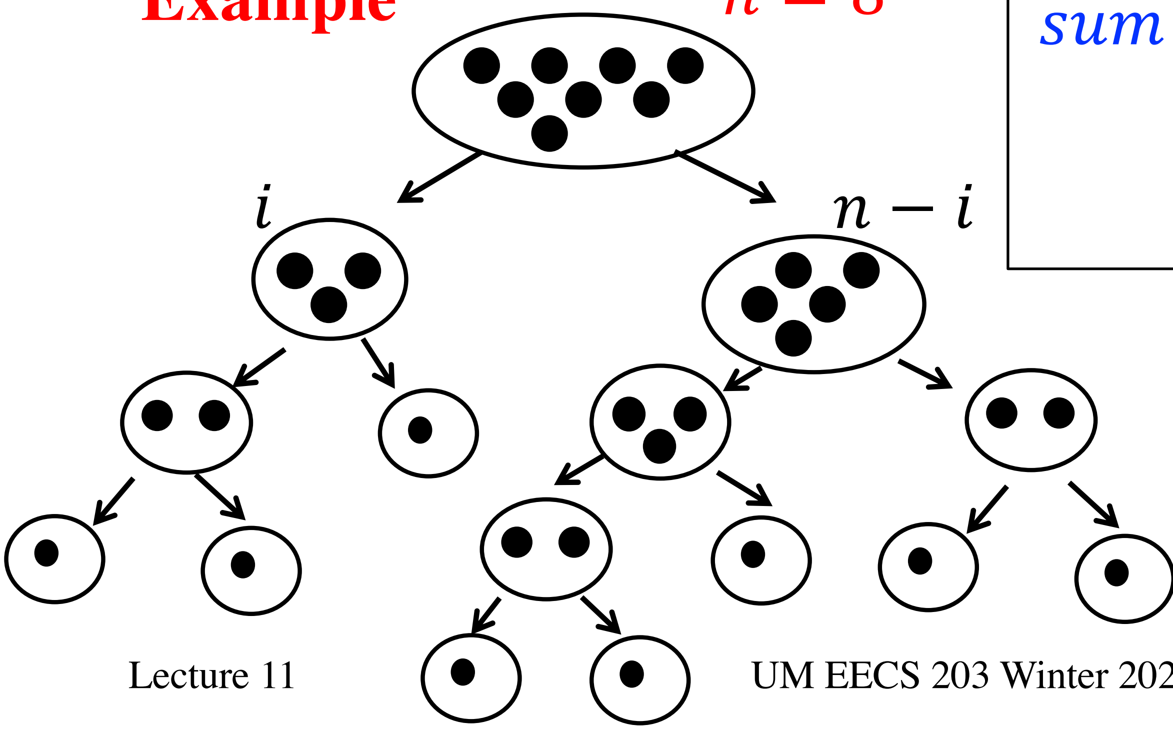
- Strong Induction
  - Example: Stamps
  - Guide for Strong Induction Proofs
  - **Example: Piles of Stones**

Piles of stones

Claim: $\forall n \geq 1\ P(n)$.
$P(n)$: with n stones,
$$sum(n) = \frac{n(n-1)}{2}$$

- Suppose you begin with a pile of n stones.
  - Split the pile into two smaller piles of size i and n-i.
  - Multiply together the number of stones in each of the two smaller piles and write down the number $i(n$-$i)$.
  - Repeat until you get n piles of one stone each.
- $\boldsymbol{P(n)}$**:** sum of the numbers your wrote down is $\frac{n(n-1)}{2}$

Example $n = 8$



$$\begin{aligned} sum &= 15 + 2 + 6 + 1 + 2 + 1 + 1 \\ &= 28 \\ &= \frac{8(8-1)}{2} \end{aligned}$$

$(3 \cdot 5)$

$+(2 \cdot 1) + (3 \cdot 2)$

$+(1 \cdot 1) + (2 \cdot 1) + (1 \cdot 1)$

$+(1 \cdot 1)$

Proof.. Piles of stones

Claim: $\forall n \geq 1\ P(n)$.

$P(n)$: with n stones,

$$sum(n) = \frac{n(n-1)}{2}$$

Inductive step: Assume $P(j)\ \ \forall j\ \ 1 \leq j \leq k-1$

- Want to show $P(k)$, i.e., $sum(k) = \frac{k(k-1)}{2}$

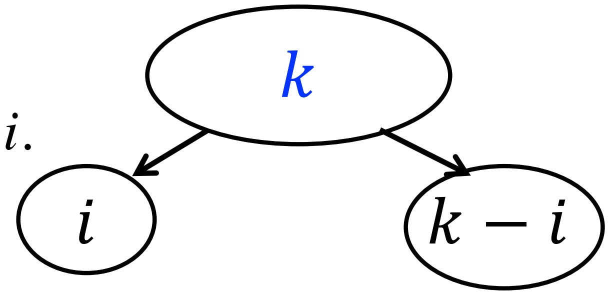
k

Proof.. Piles of stones

Claim: $\forall n \geq 1\ P(n)$.
$P(n)$: with n stones,
$$sum(n) = \frac{n(n-1)}{2}$$

Inductive step: Assume $P(j)\ \forall j\ 1 \leq j \leq k-1$

- Want to show $P(k)$, i.e., $sum(k) = \frac{k(k-1)}{2}$
  - Take a pile of size k.
  - Divide the pile into two smaller piles of size i and k-i.



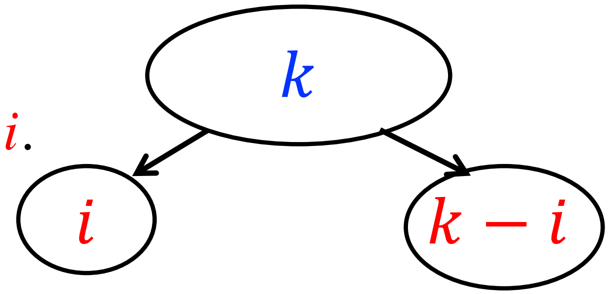
$$sum(k) = i(k-i) + sum(i) + sum(k-i)$$

Proof.. Piles of stones

Claim: $\forall n \geq 1\ P(n)$.
$P(n)$: with n stones,
$$sum(n) = \frac{n(n-1)}{2}$$

Inductive step: Assume $P(j)\ \ \forall j\ \ 1 \leq j \leq k-1$

- Want to show $P(k)$, i.e., $sum(k) = \frac{k(k-1)}{2}$
  - Take a pile of size k.
  - Divide the pile into two smaller piles of size i and k-i.



$$sum(k) = i(k-i) + sum(i) + sum(k-i)$$
$$= i(k-i) + \frac{i(i-1)}{2} + \frac{(k-i)(k-i-1)}{2}$$

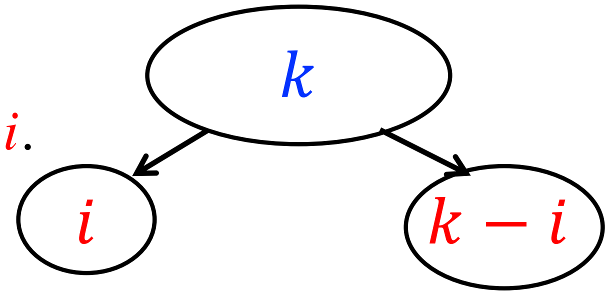
(Inductive Hypothesis)

Proof.. Piles of stones

Claim: $\forall n \geq 1\ P(n)$.
$P(n)$: with n stones,
$sum(n) = \frac{n(n-1)}{2}$

Inductive step: Assume $P(j)\ \ \forall j\ \ 1 \leq j \leq k-1$

- Want to show $P(k)$, i.e., $sum(k) = \frac{k(k-1)}{2}$
  - Take a pile of size k.
  - Divide the pile into two smaller piles of size i and k-i.



$$sum(k) = i(k-i) + sum(i) + sum(k-i)$$

$$= i(k-i) + \frac{i(i-1)}{2} + \frac{(k-i)(k-i-1)}{2}$$ (Inductive Hypothesis)

$$= \frac{1}{2}[2i(k-i) + i(i-1) + (k-i)(k-i-1)]$$

$$= \frac{1}{2}[\ i(k-i) + i(i-1) + i(k-i) + (k-i)(k-i-1)]$$

$$= \frac{1}{2}[\ i(k-1) + (k-i)(k-1)]$$

$$= \frac{1}{2}[(k-1)(k-i+i)]$$

$$= \frac{k(k-1)}{2}$$

Base case: $n = 1$

$sum(1) = 0 = \frac{1(1-1)}{2}$

By strong induction, $P(n)$ holds $\forall n \geq 1$.

Wrap up

- You’ve learned a powerful tool: proof by induction and by strong induction

- We’ll see more “counting-style” proofs in the next few weeks