Lec 8: Natural Deduction

Quick Feedback Survey

- tinyurl.com/203W22Diaz

Lessons from Homework 1

- Make sure to match pages to questions
  - For Homework 1, you can request a regrade for this, but not for future homeworks.

- Homework 1 Question 4, we didn't remind you that all questions need work shown for credit
  - We decided the most fair way to resolve this is a one-time grant of full credit with no work
  - We have already graded everyone's submission again. If we missed yours, submit a regrade
  - We may not always state explicitly to show your work, but moving forward, know that the policy is that supporting work is required (stated on page 1 of each homework assignment)

- To ease grading, please make sure pages in your pdf are upright, not rotated

- Clearly label each problem (e.g. larger numbers, numbers to the left of the margin, etc.)

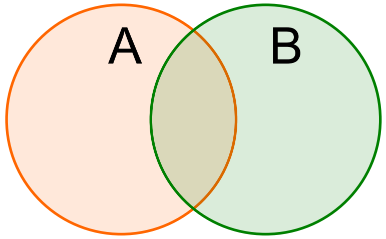
Exam 1 Planning

	Sunday	Monday	Tuesday	Wednes.	Thursday	Friday	Saturday
This week			Today	2/2: Practice Exams released	HW 3 due		
2/6-2/12				2/9: Practice Exam Solutions released	HW 4 due		Review session #1: 1-4 pm
2/13-2/19	Review session #2: 1-4 pm		*Lecture 12: Exam Review*	**2/16: Exam 1: 7-9pm**			
2/20-2/26					HW 5 due		

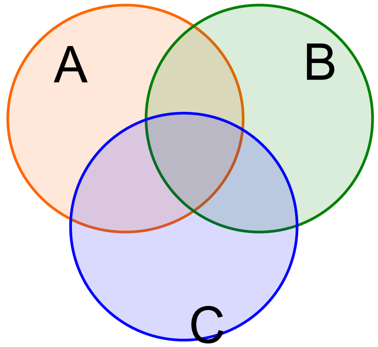
- Exam 1 covers: Modular arithmetic, Proofs, Logic, Sets, Natural Deduction (HW 1-4)
  - Does NOT cover mathematical induction
- See "Exam 1 Information" doc – coming soon to Canvas - for more information

Inclusion-Exclusion Principle

$$|A \cup B| = |A| + |B| - |A \cap B|$$



$$|A \cup B \cup C| = |A|+|B|+|C|$$
$$-|A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$



Original Question: How many multiples of 2, 3, or 5 from 1 to 300?

Now $S_m = \{x \in \mathbb{Z} \mid 1 \leq x \leq \mathbf{300} \text{ and } x \equiv 0 \ (mod\ m)\}$

What is $|S_2 \cup S_3 \cup S_5|$?

Lemma: If a, b are relatively prime, then $S_a \cap S_b = S_{a \cdot b}$

First Inclusion:

$$|S_2| + |S_3| + |S_5| = \frac{300}{2} + \frac{300}{3} + \frac{300}{5} = 150 + 100 + 60 = 310$$

Second Exclusion:

$$-|S_{2\cdot3}| - |S_{2\cdot5}| - |S_{3\cdot5}| = -\frac{300}{6} - \frac{300}{10} - \frac{300}{15} = -50 - 30 - 20 = -100$$

Third Inclusion:

$$|S_{2\cdot3\cdot5}| = \frac{300}{30} = 10$$

$$|S_2 \cup S_3 \cup S_5| = 310 - 100 + 10 = \mathbf{220}$$

Learning Objectives: Lec 8

After today's lecture (and the associated readings, discussion, & homework), you should be able to:

- Solve natural deduction problems using the 12 rules covered here
- Understand why the 12 natural deduction rules covered in this lecture are valid
  - You do **not** need to memorize the list of rules
- Understand the meaning of assumption boxes and how to use them
  - Including scoping rules
- Understand the meaning of deriving F in a natural deduction problem

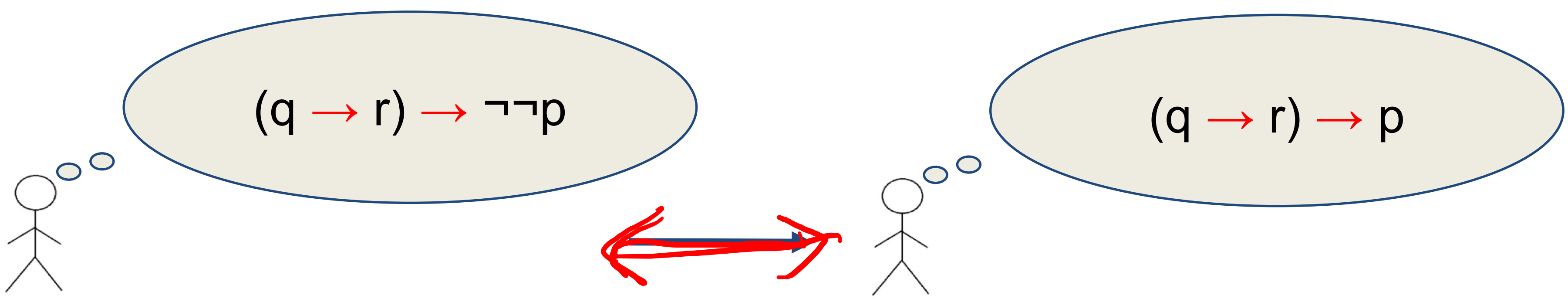
Outline

- **Intro to Natural Deduction**
- The Straightforward Rules
- Assumption Boxes
- Rules Using False

Recap

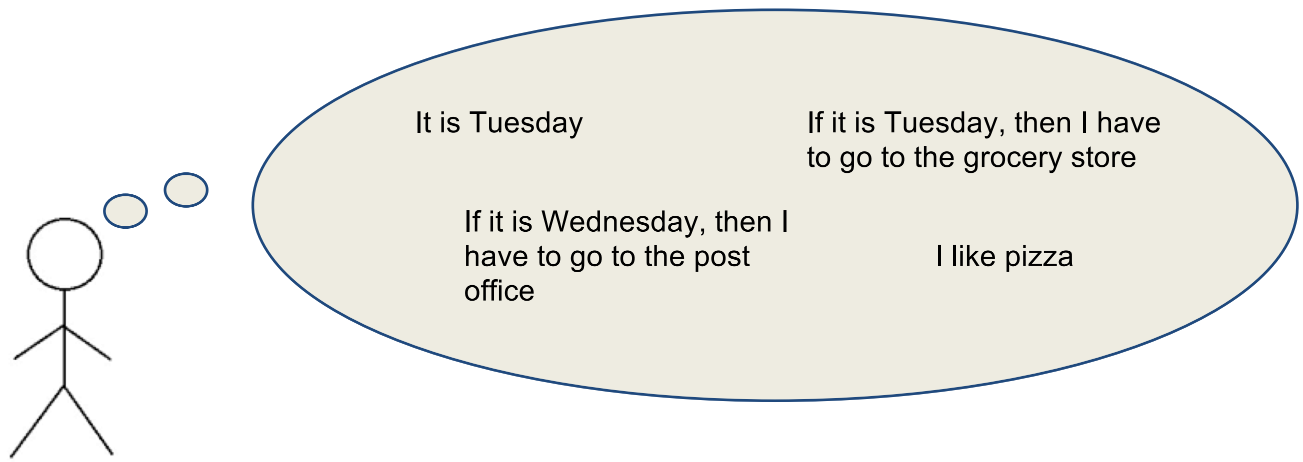
Last Week: Simplifying Knowledge with **Boolean Algebra**

- You “know” **one expression** at a time
- Use rules to replace parts of the expression with other **logically equivalent (↔)** statements with the same truth value



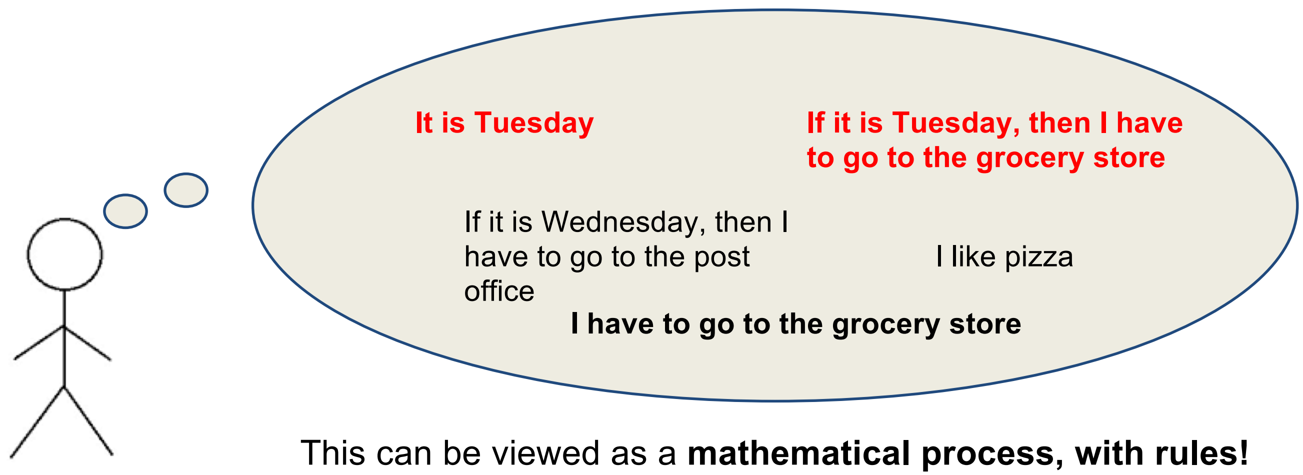
Recap

- Another thing we can do with logic is **add to a knowledge pool**



Recap

- Another thing we can do with logic is **add to a knowledge pool**



This can be viewed as a **mathematical process, with rules!**
More on this today…

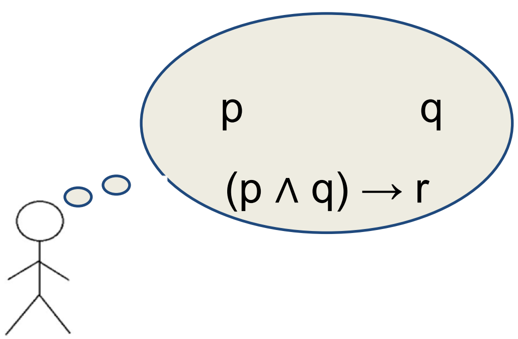
This Time

This Week: Gaining Knowledge with **Natural Deduction**

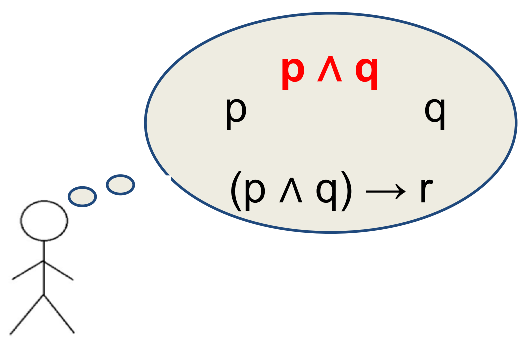
- A metaphor for proofs
- “Knowledge pool” represented as **list of propositions in Boolean logic**
- Use rules to **add** new statements to your knowledge pool, if they are implied ($\rightarrow$) by some of the statements you already have

Note: Natural Deduction is not in Rosen. Read the Natural Deduction Handouts we published on Canvas instead.

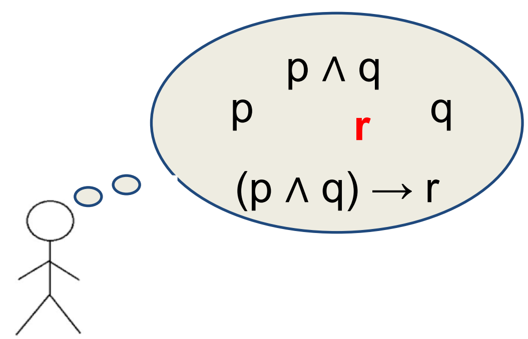
(3 statements known)



(4 statements known)



(5 statements known)



Natural Deduction Overview

A sample natural deduction problem looks like this:

PROBLEM:

$$\frac{t \quad \neg r \quad (t \wedge \neg r) \rightarrow w}{\therefore w}$$

Statements above the line = **"premises"**

You **begin** with these statements in your list of known statements **for free.**

In this example, there are three premises.

Statement below the line = **"(desired) conclusion"**

Your **goal** is to eventually add the conclusion to your list of known statements.

There is always only one conclusion.

Natural Deduction Overview

A sample natural deduction problem looks like this:

PROBLEM:

$t \quad \neg r \quad (t \land \neg r) \rightarrow w$

$\therefore w$

t = it is Tuesday

r = it is raining

w = I will take a walk

You usually won't get flavor like this

Your solution looks like this:

1.	t	premise
2.	$\neg r$	premise
3.	$(t \land \neg r) \rightarrow w$	premise
	Intermediate steps go here	***name of "rule" used to get step***
n.	w	***rule name***

Natural Deduction Overview

You get a list of 16 “rules” to form intermediate steps

rules mimic common reasoning steps used by humans

For each of the basic operators & quantifiers: $\land$, $\lor$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$, there is

- one rule to *introduce* the operator, and
- one rule to *eliminate* the operator

If your current list of statements has two statements of the forms $p \rightarrow q$ and p (where p, q can stand for any expressions)

example rule:

$\rightarrow$-elimination

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Then you are allowed to write q on the next line

Natural Deduction Overview

You get a list of 16 “rules” to form intermediate steps

rules mimic common reasoning steps used by humans

Note that while logical equivalences are valid in any proof at all, we will not allow them in natural deduction proofs. This is because we want to teach and test you on all the natural deduction rules, but logical equivalences can be used to circumvent that. This means either:

- We give you nasty complex problems and expect you to use equivalences fully
- We give you simpler problems, but say you cannot use equivalences

statements has two statements of the forms $p \rightarrow q$ and p (where p, q can stand for any expressions)

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Then you are allowed to write q on the next line

Natural Deduction

Every rule can be justified by a tautology:

$\rightarrow$-elimination is valid because

$$((p \rightarrow q) \wedge p) \rightarrow q$$

is a tautology.

"Regardless of p,q, if (p $\rightarrow$ q) and p are both true, then q must be true as well."

example rule:

$\rightarrow$-elimination

$$\begin{array}{l} \quad p \rightarrow q \\ \quad p \\ \hline \therefore q \end{array}$$

Plan for today

- Cover *most* of the natural deduction rules (12/16 of them)
- Understand why these rules are valid
- Do some practice natural deduction problems

Why study this?

Most proofs have two components:

Find a **way to look at the problem** that helps you understand why the proposition is true

Write a **step-by-step logical receipt** of your understanding

This will not be as important in ND as in word proofs

Punchline of Natural Deduction:

- The receipt itself can be viewed as a rigorous, mathematical object obeying fixed rules

Outline

- Intro to Natural Deduction
- **The Straightforward Rules**
- Assumption Boxes
- Rules Using False

The Straightforward Rules

→-elimination

$$\begin{array}{l} p \rightarrow q \\ \underline{p \qquad\quad} \\ \therefore q \end{array}$$

∧-introduction

$$\begin{array}{l} p \\ \underline{q \qquad\quad} \\ \therefore p \land q \end{array}$$

∧-elimination

$$\begin{array}{l} \underline{p \land q} \\ \therefore q \end{array} \quad \text{or} \quad \begin{array}{l} \underline{p \land q} \\ \therefore p \end{array}$$

∨-introduction

$$\begin{array}{l} \underline{p \qquad} \\ \therefore p \lor q \end{array} \quad \text{or} \quad \begin{array}{l} \underline{q \qquad} \\ \therefore p \lor q \end{array}$$

order of **p, q** matters here

The Straightforward Rules

→-elimination

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

∧-introduction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

↔-elimination

$$\begin{array}{l} p \leftrightarrow q \\ \boxed{p} \\ \hline \therefore q \end{array} \quad \text{or} \quad \begin{array}{l} p \leftrightarrow q \\ q \\ \hline \therefore p \end{array}$$

∧-elimination

$$\begin{array}{l} p \wedge q \\ \hline \therefore q \end{array}$$

∨-introduction

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

¬¬-elimination

$$\begin{array}{l} \neg\neg p \\ \hline \therefore p \end{array}$$

okay to use these four with **p, q** in either order

Sample Natural Deduction Problem

PROBLEM:

$t \quad \neg r \quad (t \wedge \neg r) \rightarrow w$

$\therefore w$

Statements above the line = **premises**

Statement below the line = **desired conclusion**

1. t — premise
2. $\neg r$ — premise
3. $(\underbrace{t \wedge \neg r}_{p}) \rightarrow \underbrace{w}_{q}$ — premise
4. $\underbrace{t \wedge \neg r}_{p}$ — $\wedge$-intro (1, 2)
5. w — $\rightarrow$ elim (3, 4)

Sample Natural Deduction Problem

PROBLEM:

$t \quad \neg r \quad (t \wedge \neg r) \rightarrow w$

$\therefore w$

reminder:

$\wedge$-introduction

p

q

$\therefore p \wedge q$

1. t premise
2. $\neg r$ premise
3. $(t \wedge \neg r) \rightarrow w$ premise
4. $t \wedge \neg r$ $\wedge$-intro(1, 2)

?. w ???

Name of rule + previous lines that it uses

Sample Natural Deduction Problem

PROBLEM:

$t \quad \neg r \quad (t \wedge \neg r) \rightarrow w$

$\therefore w$

reminder:

$\rightarrow$-elimination

$p \rightarrow q$

p

$\therefore q$

1.	t	premise
2.	$\neg r$	premise
3.	$(t \wedge \neg r) \rightarrow w$	premise
4.	$t \wedge \neg r$	$\wedge$-intro(1, 2)
5.	w	$\rightarrow$-elim(3, 4)

Got to desired conclusion - stop!

Sample Natural Deduction Problem

PROBLEM:

$t \quad \neg r \quad (t \land \neg r) \rightarrow w$

$\therefore w$

reminder:

$\rightarrow$-elimination

$p \rightarrow q$

p

$\therefore q$

1.	t	premise
2.	$\neg r$	premise
3.	$(t \land \neg r) \rightarrow w$	premise
4.	$t \land \neg r$	$\land$-intro(1, 2)
5.	w	$\rightarrow$-elim(3, 4)

This is your whole answer: no need for a concluding sentence or anything

Outline

- Intro to Natural Deduction
- The Straightforward Rules
- **Assumption Boxes**
- Rules Using False

Assumption Boxes

In **word proofs**, it was sometimes helpful to **temporarily assume something** and see what happened under this assumption.

Proposition
For all integers **x**, if **x** is even, then **x+2** is even.

Proof:

- Let x be an arbitrary integer.
- Assume x is even; there is an integer k with $x = 2k$.
- So $x + 2 = 2k + 2 = 2(k + 1)$
- $k + 1$ is an integer, so $x + 2$ is even

Prove "if … then …" statement by assuming the **if** part, and proving the **then** part.

We put a box around the scope where we were making an assumption about x.

Assumption Boxes

In **word proofs**, it was sometimes helpful to **temporarily assume something** and see what happened under this assumption.

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Proof:

- Let x be an arbitrary integer.
- Assume x is even; there is an integer k with $x = 2k$.
- So $x + 2 = 2k + 2 = 2(k + 1)$
- $k + 1$ is an integer, so $x + 2$ is even
- **If x is even, then x+2 is even**

p
q
$p \to q$

Then explicitly "convert" the assumption box to an if-then statement

Assumption Boxes

$\to$-introduction

$$\boxed{\begin{array}{c} p \\ \vdots \\ q \end{array}}$$

$$\therefore p \to q$$

If you can **assume** p and reach the statement q (under this assumption),

Then it is safe to add $p \to q$ to your knowledge base

Some rules include **assumption boxes**.

- You can start an assumption box any time you want.
- First line is any proposition you want, and the justification is **"assumption"**
- To finish the problem, you need your conclusion **outside all assumption boxes**.
- This means you should only start an assumption box when **you have a plan to get out**

Natural Deduction with Assumption Boxes

PROBLEM:

$a \rightarrow b \qquad b \rightarrow c$

$\therefore\ a \rightarrow c$

maybe use:

$\rightarrow$elim x2

$\rightarrow$intro

1.	$a \rightarrow b$	premise
2.	$b \rightarrow c$	premise
?.	$a \rightarrow c$	???

1.	$a \rightarrow b$	premise
2.	$b \rightarrow c$	premise
3.	a	assumption
4.	b	$\rightarrow$elim (1,3)
5.	c	$\rightarrow$elim (2,4)
6.	$a \rightarrow c$	$\rightarrow$intro (3-5)

Natural Deduction with Assumption Boxes

PROBLEM:

$a \rightarrow b$ $\quad$ $b \rightarrow c$

$\therefore a \rightarrow c$

$\rightarrow$-introduction

reminder

$$\begin{array}{|c|}\hline p \\ \vdots \\ q \\ \hline \end{array}$$

$\therefore p \rightarrow q$

1.	$a \rightarrow b$	premise
2.	$b \rightarrow c$	premise
3.	a	assumption
?.	c	???
?.	$a \rightarrow c$	$\rightarrow$-intro

Natural Deduction with Assumption Boxes

PROBLEM:

$a \rightarrow b \qquad b \rightarrow c$

$\therefore a \rightarrow c$

reminder

$\rightarrow$-elimination

$p \rightarrow q$

p

$\therefore q$

1.	$a \rightarrow b$	premise
2.	$b \rightarrow c$	premise
3.	a	assumption
4.	b	$\rightarrow$-elim(1, 3)
?.	c	???
?.	$a \rightarrow c$	$\rightarrow$-intro

Natural Deduction with Assumption Boxes

PROBLEM:

$a \rightarrow b \qquad b \rightarrow c$

$\therefore a \rightarrow c$

$\rightarrow$-introduction

reminder

$$\boxed{\begin{array}{c} p \\ \vdots \\ q \end{array}}$$

$\therefore p \rightarrow q$

1.	$a \rightarrow b$	premise
2.	$b \rightarrow c$	premise
3.	a	assumption
4.	b	$\rightarrow$-elim(1, 3)
5.	c	$\rightarrow$-elim(2, 4)
6.	$a \rightarrow c$	$\rightarrow$-intro(3-5)

write the **range** of a box to cite it

Assumption Boxes

$\rightarrow$-introduction

$$\boxed{\begin{array}{c} p \\ \vdots \\ q \end{array}}$$

$$\therefore p \rightarrow q$$

$\leftrightarrow$ -introduction

$$\boxed{\begin{array}{c} p \\ \vdots \\ q \end{array}} \quad \boxed{\begin{array}{c} q \\ \vdots \\ p \end{array}}$$

$$\therefore \; p \leftrightarrow q$$

(+ two more rules that use assumption boxes, to follow shortly)

Assumption Boxes

Proposition:
For all integers x, we have $x^2 \equiv 0$ or $x^2 \equiv 1 \pmod 3$

- Let **x** be an arbitrary integer.
- **We will use a proof by cases**. By mod arithmetic, we have $x \equiv 0$, $x \equiv 1$, or $x \equiv 2 \pmod 3$

$p \vee q \vee s$

Assumption Boxes

Proposition:
For all integers x, we have $x^2 \equiv 0$ or $x^2 \equiv 1 \pmod 3$

- Let **x** be an arbitrary integer.
- **We will use a proof by cases**. By mod arithmetic, we have $x \equiv 0$, $x \equiv 1$, or $x \equiv 2 \pmod 3$

- **Case 1:** Assume $x \equiv 0$.
- Then $x^2 \equiv 0$, and the proposition is true.

Assumption boxes show up in proof by cases

Assumption Boxes

Proposition:
For all integers x, we have $x^2 \equiv 0$ or $x^2 \equiv 1 \pmod 3$

- Let **x** be an arbitrary integer.
- **We will use a proof by cases**. By mod arithmetic, we have $x \equiv 0$, $x \equiv 1$, or $x \equiv 2 \pmod 3$

- **Case 1:** Assume $x \equiv 0$.
- Then $x^2 \equiv 0$, and the proposition is true.

- **Case 2:** Assume $x \equiv 1$.
- Then $x^2 \equiv 1$, and the proposition is true.

Assumption boxes show up in proof by cases

Assumption Boxes

Proposition:
For all integers x, we have $x^2 \equiv 0$ or $x^2 \equiv 1 \pmod 3$

- Let **x** be an arbitrary integer.
- **We will use a proof by cases**. By mod arithmetic, we have $x \equiv 0$, $x \equiv 1$, or $x \equiv 2 \pmod 3$

- **Case 1:** Assume $x \equiv 0$.
- Then $x^2 \equiv 0$, and the proposition is true.

- **Case 2:** Assume $x \equiv 1$.
- Then $x^2 \equiv 1$, and the proposition is true.

- **Case 3:** Assume $x \equiv 2$.
- Then $x^2 \equiv 4 \equiv 1$, and the proposition is true.

Assumption boxes show up in proof by cases

Assumption Boxes

Proposition:

For all integers x, we have $x^2 \equiv 0$ or $x^2 \equiv 1 \pmod 3$

- Let **x** be an arbitrary integer.
- **We will use a proof by cases**. By mod arithmetic, we have $x \equiv 0$, $x \equiv 1$, or $x \equiv 2 \pmod 3$

$p \vee q \vee s$

- **Case 1:** Assume $x \equiv 0$.
- Then $x^2 \equiv 0$, and the proposition is true.

$\boxed{p \; \vdots \; r}$

- **Case 2:** Assume $x \equiv 1$.
- Then $x^2 \equiv 1$, and the proposition is true.

$\boxed{q \; \vdots \; r}$

- **Case 3:** Assume $x \equiv 2$.
- Then $x^2 \equiv 4 \equiv 1$, and the proposition is true.

$\boxed{s \; \vdots \; r}$

- **Since the proposition is true in any case, it is true.**

$\therefore r$

Last step: explicitly decode boxes into a conclusion

Assumption Boxes

"Proof by (two) cases"

∨-elimination

$$\dfrac{p \lor q \qquad \boxed{\begin{matrix} p \\ \vdots \\ r \end{matrix}} \qquad \boxed{\begin{matrix} q \\ \vdots \\ r \end{matrix}}}{\therefore\ r}$$

??

Either p or q is true (these are the two cases)

Case 1: Assume p is true. [Then we can prove r]

Case 2: Assume q is true. [Then we can prove r]

Then r must be true

Assumption Boxes

→-introduction

$$\begin{array}{l} \boxed{\begin{array}{c} p \\ \vdots \\ q \end{array}} \\ \hline \therefore p \rightarrow q \end{array}$$

↔ -introduction

$$\begin{array}{l} \boxed{\begin{array}{c} p \\ \vdots \\ q \end{array}} \quad \boxed{\begin{array}{c} q \\ \vdots \\ p \end{array}} \\ \hline \therefore \quad p \leftrightarrow q \end{array}$$

∨-elimination

$$\begin{array}{c} p \vee q \\ \boxed{\begin{array}{c} p \\ \vdots \\ r \end{array}} \quad \boxed{\begin{array}{c} q \\ \vdots \\ r \end{array}} \\ \hline \therefore \quad r \end{array}$$

(+ one more assumption box rule coming, but it uses additional concepts we need to discuss first)

Wrong Natural Deduction with Assumption Boxes

PROBLEM:

$a \rightarrow b \qquad b \rightarrow c$

$\therefore c$

1.	$a \rightarrow b$	premise
2.	$b \rightarrow c$	premise
3.	a	assumption
4.	b	$\rightarrow$-elim(1, 3)

This is not actually possible (the premises do not imply the conclusion). So what's wrong with the following argument?

(same start as last time)

Wrong Natural Deduction with Assumption Boxes

PROBLEM:

$$\frac{a \rightarrow b \qquad b \rightarrow c}{\therefore c}$$

reminder

$\rightarrow$-elimination

$$\frac{\begin{array}{l} p \rightarrow q \\ p \end{array}}{\therefore q}$$

1.	$a \rightarrow b$	premise
2.	$b \rightarrow c$	premise
3.	a	assumption
4.	b	$\rightarrow$-elim(1, 3)
5.	c	$\rightarrow$-elim(2, 4)

Scoping:

- Statements in box derived **under some assumption**.
- So you can't refer back to **individual** statements in the box, once you're outside the box and no longer making these assumptions!
- You can only refer to **the box itself**

Outline

- Intro to Natural Deduction
- The Straightforward Rules
- Assumption Boxes
- **Rules Using False**

Rules Using False

Some rules involve F *(literally stands for the truth value False)*

¬-elimination

$$\begin{array}{l} p \\ \neg p \\ \hline \therefore F \end{array}$$

If you have two contradictory statements in your list, then you can add "False" to your list of *true* statements!

- What?

This can show up when you **have made contradictory assumptions**.

- (Rare) You were given contradictory premises
- (More common) **You're inside an assumption box** where you assumed something that contradicts the premises

Rules Using False

Some rules involve F *(literally stands for the truth value False)*

¬-elimination

$$\begin{array}{l} p \\ \neg p \\ \hline \therefore F \end{array}$$

F

$\underbrace{p \land \neg p}_{F}$

Some ways you might interpret adding F to your knowledge pool:

- A marker of: "in the current scope, the assumptions I have made (premises + assumption boxes) are contradictory"
- Every statement you write down is ***implied*** by the assumptions you have made (in scope). Recall how $\boxed{\mathbf{F} \rightarrow \mathbf{F}}$ in logic. If your assumptions are equivalent to **F**, it's true that they imply **F**.

T

Proofs by Contradiction

Proposition:

There do not exist integers **a, b** such that $\mathbf{18a + 6b = 1}$.

- **We will use a proof by contradiction. Seeking contradiction,**
- Assume there exist integers a, b such that $18a + 6b = 1$.

Assumption boxes **and** deriving F shows up in proof by contradiction

Proofs by Contradiction

Proposition:

There do not exist integers **a, b** such that $\mathbf{18a + 6b = 1}$.

- **We will use a proof by contradiction. Seeking contradiction,**
- Assume there exist integers a, b such that $18a + 6b = 1$.
- So $3a + b = \frac{1}{6}$
- Since a, b are integers, $3a + b$ is an integer
- So $\frac{1}{6}$ is an integer. *(Our assumption led to something false!)*

Assumption boxes **and** deriving F shows up in proof by contradiction

Proofs by Contradiction

Proposition:

There do not exist integers **a, b** such that $\mathbf{18a + 6b = 1}$.

- **We will use a proof by contradiction. Seeking contradiction,**
- Assume there exist integers a, b such that $18a + 6b = 1$.
- So $3a + b = \frac{1}{6}$
- Since a, b are integers, $3a + b$ is an integer
- So $\frac{1}{6}$ is an integer. *(Our assumption led to something false!)*
- **This completes the contradiction.**

P

F

$\therefore \neg P$

Explicitly decode assumption box into conclusion

3/16: The Rules Using False

¬-elimination

$$\begin{array}{l} p \\ \neg p \\ \hline \therefore F \end{array}$$

¬-introduction

$$\begin{array}{l} \boxed{\begin{array}{l} p \\ \vdots \\ F \end{array}} \\ \hline \therefore \neg p \end{array}$$

If you can assume **p** and reach **False** ...

... Then this implies **¬p.** This is exactly proof by contradiction!

Exercise using False

"Modus Tollens"

$a \rightarrow b$

$\neg b$

$\therefore \neg a$

(Sort of like contrapositive)

maybe use:

- $\rightarrow$ elim ✓
- $\neg$ elim ✓
- $\neg$ intro ✓

1.	$a \rightarrow b$	premise
2.	$\neg b$	premise
3.	a	assume
4.	b	$\rightarrow$ elim (1, 3)
5.	F	$\neg$ elim (2, 4)
6.	$\neg a$	$\neg$ intro (3–5)

Exercise using False

"Modus Tollens"

$a \rightarrow b$

$\neg b$

$\therefore \neg a$

$\neg$-introduction

p
$\vdots$
F

$\therefore \neg p$

1.	$a \rightarrow b$	premise
2.	$\neg b$	premise
3.	a	assumption
?.	F	???
?.	$\neg a$	$\neg$-intro

Exercise using False

“Modus Tollens”

$a \rightarrow b$

$\neg b$

$\therefore \neg a$

$\rightarrow$-elimination

$p \rightarrow q$

p

$\therefore q$

1.	$a \rightarrow b$	premise
2.	$\neg b$	premise
3.	a	assumption
4.	b	$\rightarrow$-elim (1, 3)
?.	F	???
?.	$\neg a$	$\neg$-intro

Exercise using False

"Modus Tollens"

$a \rightarrow b$

$\neg b$

$\therefore \neg a$

1.	$a \rightarrow b$	premise
2.	$\neg b$	premise
3.	a	assumption
4.	b	$\rightarrow$-elim (1, 3)
5.	F	$\neg$-elim (4, 2)
6.	$\neg a$	$\neg$-intro

$\neg$-elimination

p

$\neg p$

$\therefore F$

Exercise using False

"Modus Tollens"

$a \rightarrow b$

$\neg b$

$\therefore \neg a$

$\neg$-introduction

p

$\vdots$

F

$\therefore \neg p$

1.	$a \rightarrow b$	premise
2.	$\neg b$	premise
3.	a	assumption
4.	b	$\rightarrow$-elim (1, 3)
5.	F	$\neg$-elim (4, 2)
6.	$\neg a$	$\neg$-intro (3-5)

3/16: The Rules Using False

¬-elimination
p $\neg p$ ——— $\therefore F$

¬-introduction
$\boxed{\begin{array}{l} p \\ \vdots \\ F \end{array}}$ ——— $\therefore \neg p$

F-Elimination
F ——— $\therefore p$

$F \to F \equiv T$

$F \to T \equiv T$

If you have a contradiction, you can conclude **anything you want**.

- In current scope, your assumptions are already contradictory
- **F** → **p** in logic, regardless of the truth value of **p**

Natural Deduction with "False" Derivations

PROBLEM:

$$\frac{a \vee b \qquad \neg a}{\therefore\ b}$$

$\vee$ elim

$\neg$ elim

1.	$a \vee b$	premise
2.	$\neg a$	premise
3.	a	assume
4.	F	$\neg$ elim (2, 3)
5.	b	F elim (4)
6.	b	assume
	~~$\therefore$ b~~	~~???~~
7.	b	$\vee$ elim (1, 3-5, 6)

Natural Deduction with "False" Derivations

PROBLEM:

a ∨ b   ¬a

∴ b

That looks weird……

There's a few ways we could get this box

1.	a ∨ b	premise
2.	¬a	premise
3.	a	assumption
?.	b	???
?.	b	assumption
?.	b	???
?.	b	∨-elim

Natural Deduction with “False” Derivations

PROBLEM:

$a \vee b \quad \neg a$

$\therefore b$

Use our ND rules to make something just to break it back

1.	$a \vee b$	premise
2.	$\neg a$	premise
3.	a	assumption
?.	b	???
?.	b	assumption
?.	$b \wedge b$	$\wedge$-intro
?.	b	$\wedge$-elim
?.	b	$\vee$-elim

Natural Deduction with “False” Derivations

PROBLEM:

$a \vee b$ $\neg a$

$\therefore b$

Just cite the previous line. We already know it's true (in this assumption box)

1.	$a \vee b$	premise
2.	$\neg a$	premise
3.	a	assumption
?.	b	???
X.	b	assumption
?.	b	Line X
?.	b	$\vee$-elim

Natural Deduction with “False” Derivations

PROBLEM:

$a \lor b \qquad \neg a$

$\therefore b$

End the box right away
- Starts with b
- Ends with b

1.	$a \lor b$	premise
2.	$\neg a$	premise
3.	a	assumption
?.	b	???
?.	b	assumption
?.	b	$\lor$-elim

Natural Deduction with “False” Derivations

PROBLEM:

a ∨ b ¬a

∴ b

reminder:

¬-elimination

p
¬p

F

1.	a ∨ b	premise
2.	¬a	premise
3.	a	assumption
4.	F	¬-elim(2, 3)
?.	b	???
?.	b	assumption
?.	b	∨-elim

Natural Deduction with “False” Derivations

PROBLEM:

$$\frac{a \vee b \qquad \neg a}{\therefore\ b}$$

reminder:

F-Elimination

$$\frac{F}{\therefore\ p}$$

1.	$a \vee b$	premise
2.	$\neg a$	premise
3.	a	assumption
4.	F	$\neg$-elim(2, 3)
5.	b	F-elim(4)
6.	b	assumption
7.	b	$\vee$-elim

Natural Deduction with “False” Derivations

PROBLEM:

a ∨ b   ¬a

∴ b

∨-elimination

p ∨ q

p … r   q … r

∴ r

1.	a ∨ b	premise
2.	¬a	premise
3.	a	assumption
4.	F	¬-elim(2, 3)
5.	b	F-elim(4)
6.	b	assumption
7.	b	∨-elim (1,3-5,6-6)

Reached desired conclusion (outside all assumption boxes)

6/16: The Straightforward Rules

→-elimination

$p \to q$
p
$\therefore q$

∧-introduction ★

p
q
$\therefore p \land q$

↔-elimination ★

$p \leftrightarrow q$
p
$\therefore q$

∧-elimination ★

$p \land q$
$\therefore q$

∨-introduction ★

p
$\therefore p \lor q$

¬-elimination

$\neg\neg p$
$\therefore p$

★ okay to use these four rules with **p, q** in either order

3/16: The Assumption Box Rules

$\rightarrow$-introduction

$$\boxed{\begin{array}{c} p \\ \vdots \\ q \end{array}}$$

$\therefore\ p \rightarrow q$

$\leftrightarrow$ -introduction

$$\boxed{\begin{array}{c} p \\ \vdots \\ q \end{array}} \quad \boxed{\begin{array}{c} q \\ \vdots \\ p \end{array}}$$

$\therefore\ p \leftrightarrow q$

$\vee$-elimination

$$p \vee q$$

$$\boxed{\begin{array}{c} p \\ \vdots \\ r \end{array}} \quad \boxed{\begin{array}{c} q \\ \vdots \\ r \end{array}}$$

$\therefore\ r$

3/16: The Rules Using False

¬-elimination

$$\frac{\begin{array}{l} p \\ \neg p \end{array}}{\therefore F}$$

¬-introduction

$$\frac{\boxed{\begin{array}{l} p \\ \vdots \\ F \end{array}}}{\therefore \neg p}$$

F-Elimination

$$\frac{F}{\therefore p}$$

4/16: Quantifier Rules

Introduction Rules

$\wedge$-introduction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

$\vee$-introduction

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

$\rightarrow$-introduction

$$\begin{array}{c} \boxed{\begin{array}{c} p \\ \vdots \\ q \end{array}} \\ \hline \therefore p \rightarrow q \end{array}$$

$\leftrightarrow$-introduction

$$\begin{array}{c} \boxed{\begin{array}{c} p \\ \vdots \\ q \end{array}} \boxed{\begin{array}{c} q \\ \vdots \\ p \end{array}} \\ \hline \therefore p \leftrightarrow q \end{array}$$

$\neg$-introduction

$$\begin{array}{c} \boxed{\begin{array}{c} p \\ \vdots \\ F \end{array}} \\ \hline \therefore \neg p \end{array}$$

Elimination Rules

$\wedge$-elimination

$$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$

$\neg\neg$-elimination

$$\begin{array}{l} \neg\neg p \\ \hline \therefore p \end{array}$$

$\rightarrow$-elimination

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

$\vee$-elimination

$$\begin{array}{c} p \vee q \\ \boxed{\begin{array}{c} p \\ \vdots \\ r \end{array}} \boxed{\begin{array}{c} q \\ \vdots \\ r \end{array}} \\ \hline \therefore r \end{array}$$

$\leftrightarrow$-elimination

$$\begin{array}{l} p \leftrightarrow q \\ p \\ \hline \therefore q \end{array}$$

$\neg$-elimination

$$\begin{array}{l} p \\ \neg p \\ \hline \therefore F \end{array}$$

F-elimination

$$\begin{array}{l} F \\ \hline \therefore p \end{array}$$

Wrapup

- Solving Natural Deduction problems is a skill! It takes practice to find combinations of rules that go where you want.

- Finding a solution yourself is harder than following a solution someone else came up with.

- **Next Time:** Natural deduction with quantifiers involved, and some strategies for solving problems on your own.