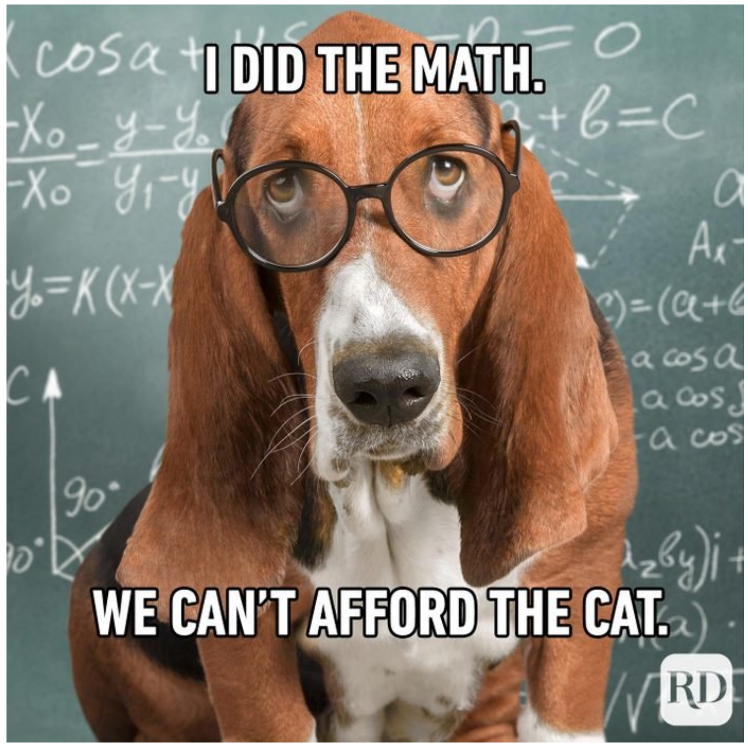
No Handout today

Exam 1 Review



EECS 203: Discrete Mathematics

Lecture 12

Lec 11 Outline: Exam 1 Review

- Check-in: *How do we feel about this first exam?*
- Session Goals
- Key reminders & logistics
- Top tips & Things to know
- Opportunities to clarify conceptual questions & practice on the topics of the exam

Check-in:
How do we feel about this first exam?

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How *prepared* do you feel for the upcoming exam?

A. I absolutely got this!
B. I believe I will be okay with a bit more study
C. I have a lot to do to be prepared!
D. Yikes! I am worried!

As of today, how much *time and effort* have you invested in preparing for the exam?

A. I have been studying consistently and attended (or watched) the weekend review sessions
B. I have been looking at old exam problems and doing a few when I have time
C. I have been focusing on completing homeworks, not much time for additional study
D. I haven't even started to think about the exam yet!

Session goals: *How this session will support your exam preparation and confidence*

This session has been designed as complement to the weekend review sessions and test strategies session. Today we will review key information and strategies that will support a successful exam and practice with the topics and problems you feel most in need of review. By the end of the session you should feel ready to:

- Describe the general structure of the exam
- Describe different types of problems you might encounter in the exam
- Plan a set of steps to tackle the upcoming exam, based on a clear understanding of the concepts and procedures assessed in the exam

Reminders & Logistics

What should I expect on the exam?

- 13 Multiple-Choice Questions
  - 4 points each
  - Single-answer multiple choice
  - Multiple-answer multiple choice
    - 0, 1, 2, 3, 4, or 5 correct answers
    - Partial credit awarded
  - On **Canvas**
- 6 Free Response Questions
  - 8 points each
  - Partial credit awarded
  - Show your work!
  - Question prompts on Canvas
  - Upload answers to **Gradescope**
- Covers Lectures 1-9
  - Through Natural Deduction
  - No induction

Reminders & Logistics

When/where is the exam?

- Tomorrow: Weds, Feb. 16
- Exam Timing: Starts 7pm, 2 hours
  - +15m for breaks/interruptions
  - Your Canvas quiz will auto-submit at 9:15pm
  - +15 minutes to upload to Gradescope
  - Hard deadline for gradescope submission: 9:30. YOU WILL LOSE POINTS AT 9:31
- You can take the test from anywhere
  - Optional: take exam from a North Campus classroom, if you filled out the request form

Exam Logistics

- Zoom Helpline
  - https://umich.zoom.us/j/93850208854
  - Meeting ID: **938 5020 8854**
  - Joining #: +1 312 626 6799. Write this down!
  - Not in AA? Look up your joining #: https://umich.zoom.us/u/ads3iwNVIv
- Use Helpline for:
  - Clarifying questions about the test
  - Logistical issues
  - IF YOU HAVE GRADESCOPE UPLOAD ISSUES, GET ON THE HELPLINE TO LET US KNOW **BEFORE** 9:30
- Note Sheet:
  - A little extra credit for uploading an 8.5” x 11” **note sheet** to Gradescope by tonight.

Reminders & Logistics

Submitting:

- You can submit to Gradescope as many times as you want. Only the last one will count. Don't submit after 9:30!
- Your Canvas quiz will auto-submit itself at 9:15 (if you haven't submitted it previously).
- If your have trouble with your Canvas quiz, disable your adblocker.

What Resources Can I Use?

- Note Sheet
  - But you must make it yourself
  - Can include screenshots, pictures, can be typed or handwritten
- Any class materials
  - Textbook
  - Lecture slides
  - Notes you took

What Resources **Can't** I Use?

- No other humans
  - Either in-person or on the internet
- No googling the question prompt
- No posting answers/hints anywhere yourself

Top Tips

- Do the practice exams
  - And read the Common Mistakes listed in the solutions
- Manage your time during the exam
  - Do the questions you know first
  - Come back to questions you’re unsure of later
- Show all your work for Free Response questions
  - Partial credit is available
- Ask if you want clarification during the exam
  - Use the Zoom Helpline: **https://umich.zoom.us/j/93850208854**

Things to know

- Review the posted learning objectives
- You will **not** usually be given definitions or rules on the test itself: good to have on a reference sheet
  - Definitions
  - Logical equivalence rules
  - Natural deduction rules
- Word proof strategies
  - How to prove something step-by-step, using the definitions
  - Know your proof styles (contrapositive, contradiction, cases)
  - Mix and match approaches (e.g., cases can be used within a proof by contradiction)

You've got this!

- Good luck!! We have confidence in you
- Take care of yourself
  - Sleep, eat, take a walk, recharge…

- Reminder: this is just one test
  - … in one course
  - … in your whole life

- You are a multidimensional person with your own set of interests, circumstances, and challenges
  - you $\neq$ your exam score

Opportunities to Practice

Opportunities to Practice: You choose the topic(s)!

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1. Working with multiple moduli in the same problem (e.g., $x \equiv a \pmod{m}$ and $y \equiv b \pmod{p}$)
2. Translation (with quantifiers): English to Logic
3. Translation (with quantifiers): Logic to English
4. Is the multi-quantified predicate true or false? (e.g., $\forall x \exists y\ [x = y^2]$)
5. Proving logical equivalence
6. Natural Deduction without quantifiers
7. Natural Deduction with quantifiers
8. Word Proofs

Working with multiple moduli in the same problem

Which of the answer options must be true, given:

- $x \equiv 3 \pmod{12}$ → $x = 3 + 12k$
- $y \equiv 11 \pmod{21}$ → $y = 11 + 21m$
- $z \equiv 3 \pmod{4}$ → $z = 3 + 4n$

Multiple answers are possible

A. $x+y \equiv 2 \pmod{3}$

B. $x+z \equiv 3 \pmod{4}$

$3+4j$

$$x+z = 3+12k + 3+4n = 6 + 12k + 4n \equiv 2 \pmod{4}$$

C. $x-y \equiv -8 \pmod{12}$

$$x-y = 3+12k-(11+21m) \equiv -8 + 12k - 21m \equiv -8 - 9m \pmod{12}$$

D. $xy \equiv 12 \pmod{21}$

$$xy = (3+12k)(11+21m) = 33 + 63m + 12\cdot 11k + 12\cdot 21km \equiv 12 + 131k \pmod{21}$$

E. $xz \equiv 1 \pmod{4}$

Working with multiple moduli in the same problem

Which of the answer options must be true, given:

- $x \equiv 3 \pmod{12}$
- $y \equiv 11 \pmod{21}$
- $z \equiv 3 \pmod{4}$

Multiple answers are possible

A. $x+y \equiv 2 \pmod{3}$

B. $x+z \equiv 3 \pmod{4}$

C. $x-y \equiv -8 \pmod{12}$

D. $xy \equiv 12 \pmod{21}$

E. $xz \equiv 1 \pmod{4}$

Quick Intuition:

- Since 12, 21 are multiples of 3, we know x, y (mod 3)
- Since 12, 4 are multiples of 4, we know x, z (mod 4)

Working with multiple moduli in the same problem

Which of the answer options must be true, given:

- $x \equiv 3 \pmod{12}$
- $y \equiv 11 \pmod{21}$
- $z \equiv 3 \pmod{4}$

Multiple answers are possible

A. $x+y \equiv 2 \pmod{3}$

$x = 3 + 12k_1$ for some integer k_1
$y = 11 + 21k_2$ for some integer k_2

So $x + y = 14 + 3(4k_1 + 7k_2) \equiv 2\ (mod\ 3)$

Working with multiple moduli in the same problem

Which of the answer options must be true, given:

- x ≡ 3 (mod 12)
- y ≡ 11 (mod 21)
- z ≡ 3 (mod 4)

Multiple answers are possible

E. xz ≡ 1 (mod 4)

$x = 3 + 12k_1$ for some integer k_1
$z = 3 + 4k_3$ for some integer k_3

So $xz = 9 + 4(9k_1 + 3k_3 + 36k_1k_2) \equiv 1\ (mod\ 4)$

Working with multiple moduli in the same problem

Which of the answer options must be true, given:

- $x \equiv 3 \pmod{12}$
- $y \equiv 11 \pmod{21}$
- $z \equiv 3 \pmod{4}$

Multiple answers are possible

B. [wrong] $x+z \equiv 3 \pmod{4}$

We have enough information to compute $x + z\ (mod\ 4)$

…but it's 2, not 3.

Working with multiple moduli in the same problem

Which of the answer options must be true, given:

- $x \equiv 3 \pmod{12}$
- $y \equiv 11 \pmod{21}$
- $z \equiv 3 \pmod{4}$

Multiple answers are possible

C. [wrong] $x-y \equiv -8 \pmod{12}$

D. [wrong] $xy \equiv 12 \pmod{21}$

Not enough information to compute $x - y\ (mod\ 12)$ or $xy\ (mod\ 21)$

To be formal, we would need to give an example of x, y satisfying the given mods, but where C,D don't hold.

Translation: English to Logic

Given $L(x)$ is "x likes EECS 203' and $K(x, y)$ is "x knows y," translate the following statement from English to logic: "There is someone who knows everyone who likes EECS 203"

(a) $\forall x \exists y (L(x) \rightarrow K(y, x))$

(b) $\forall x \exists y (L(x) \leftrightarrow K(y, x))$

(c) $\forall x \exists y (L(x) \wedge K(y, x))$

(d) $\exists x \forall y (L(y) \rightarrow K(x, y))$

(e) $\exists x \forall y (L(y) \wedge K(x, y))$

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(c) $\forall x \exists y (L(x) \wedge K(y, x))$

(d) $\exists x \forall y (L(y) \to K(x, y))$

(e) $\exists x \forall y (L(y) \wedge K(x, y))$

Quick Intuition:

There are two quantifiers going on:

- "There is someone" → there exists
- "Everyone who likes EECS 203" → for all **with restricted domain**

Translation: English to Logic

Given $L(x)$ is "x likes EECS 203' and $K(x, y)$ is "x knows y," translate the following statement from English to logic: "There is someone who knows everyone who likes EECS 203"

(d) $\exists x \forall y (L(y) \rightarrow K(x, y))$

There is someone such that

for all people

if that person likes EECS 203,

(domain restriction)

then the original person knows them

Translation: English to Logic

Given $L(x)$ is "x likes EECS 203' and $K(x, y)$ is "x knows y," translate the following statement from English to logic: "There is someone who knows everyone who likes EECS 203"

$$\text{(a)}\ \forall x \exists y (L(x) \rightarrow K(y, x))$$

For all people,

There is someone such that

then the original person knows that person

if the original person likes EECS 203,
(domain restriction)

"Everyone who likes EECS 203 knows someone." [wrong]

Translation: English to Logic

Given $L(x)$ is "x likes EECS 203' and $K(x,y)$ is "x knows y," translate the following statement from English to logic: "There is someone who knows everyone who likes EECS 203"

$$\text{(c)}\ \forall x \exists y (L(x) \wedge K(y,x))$$

For all people,

There is someone such that

That person likes 203
(domain restriction)

And they know the original person

"Everyone knows someone who likes EECS 203." [wrong]

Translation: English to Logic

Given $L(x)$ is "x likes EECS 203' and $K(x, y)$ is "x knows y," translate the following statement from English to logic: "There is someone who knows everyone who likes EECS 203"

(b) $\forall x \exists y (L(x) \leftrightarrow K(y, x))$

Implication directly inside "there exists," using the existentially-quantified variable, is a huge red flag!

Probably any direct translation will be junk

Translation: English to Logic

Given $L(x)$ is "x likes EECS 203' and $K(x, y)$ is "x knows y," translate the following statement from English to logic: "There is someone who knows everyone who likes EECS 203"

(e) $\exists x \forall y (L(y) \wedge K(x, y))$

"There exists someone who knows everyone. Also, everyone likes EECS 203."

"And" directly inside "for all" **could** be right, but it's not the right way to restrict a domain.

Translation: Logic to English

Let $T(x,y)$ mean that student x likes cuisine y, where the domain for x consists of all students at your school and the domain for y consists of all cuisines. Express the following statement as an English sentence.

$$\forall x \forall z \exists y \left[\, x \neq z \rightarrow \big(\neg T(x,y) \vee \neg T(z,y)\big) \,\right]$$

Translation: Logic to English

Let *T(x,y)* mean that student *x* likes cuisine *y*, where the domain for *x* consists of all students at your school and the domain for *y* consists of all cuisines. Express the following statement as an English sentence.

$$\forall x \forall z \exists y \, [\, x \neq z \rightarrow \left(\neg T(x,y) \vee \neg T(z,y)\right) \,]$$

For any two distinct students,
(domain restriction)

Translation: Logic to English

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$$\forall x \forall z \exists y \, [\, x \neq z \rightarrow \big(\neg T(x,y) \vee \neg T(z,y)\big) \,]$$

There is a cuisine

For any two distinct students,
(domain restriction)

Translation: Logic to English

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$$\forall x \forall z \exists y\, [\, x \neq z \rightarrow (\neg T(x,y) \vee \neg T(z,y))\,]$$

There is a cuisine

For any two distinct students, (domain restriction)

That one of them dislikes

Translation: Logic to English

Let $T(x,y)$ mean that student x likes cuisine y, where the domain for x consists of all students at your school and the domain for y consists of all cuisines. Express the following statement as an English sentence.

$$\forall x \forall z \exists y \, [\, x \neq z \rightarrow \left(\neg T(x,y) \vee \neg T(z,y)\right) \,]$$

There is a cuisine

For any two distinct students,
(domain restriction)

That one of them dislikes

"There do not exist two students who both like every cuisine."
(Optional - equivalent by DeMorgan)

Is the quantified predicate T or F?

Which of the following statements are true? The domain of x, y, z is $\mathbb{R}$.

(a) $\exists x \forall y (x^y = -1)$ False

(b) $\forall x \exists y (x \cdot y = 1)$ $\Rightarrow$ doesn't work for $x = 0$

(c) $\forall x ((x \neq 0) \rightarrow \exists y (x^y = 1))$ True

(d) $\exists x \exists y ((x^2 \neq y^2) \wedge \forall z (xz - yz = 0))$ False

(e) $\forall x \forall y ((x - y = y - x) \rightarrow (x = y))$ True

d)

a) $\exists x \forall y \quad x^y = -1$

Is the quantified predicate T or F?

$$\text{(a)}\ \exists x \forall y (x^y = -1)$$

False: no matter which integer you pick for x, there are some values of y where $x^y \neq -1$.

Is the quantified predicate T or F?

(b) $\forall x \exists y (x \cdot y = 1)$

False: Counterexample: consider $x = 0$…

Is the quantified predicate T or F?

$$\text{(c) } \forall x((x \neq 0) \rightarrow \exists y(x^y = 1))$$

True.

For all nonzero real numbers…

Is the quantified predicate T or F?

$$\text{(c)}\ \forall x((x \neq 0) \rightarrow \exists y(x^y = 1))$$

True.

For all nonzero real numbers…

There is a value y such that $x^y = 1$

Example: $y = 0$

Is the quantified predicate T or F?

(d) $\exists x \exists y((x^2 \neq y^2) \wedge \forall z(xz - yz = 0))$

False.

There exist two numbers that do not have the same square

(implies $x \neq y$ and $x \neq -y$)

Is the quantified predicate T or F?

(d) $\exists x \exists y((x^2 \neq y^2) \wedge \forall z(xz - yz = 0))$

False.

There exist two numbers that do not have the same square

(implies $x \neq y$ and $x \neq -y$)

And such that, no matter which number z you choose, $xz - yz = 0$

(rewrite as $(x-y)z = 0$ → only true if $x = y$)

Is the quantified predicate T or F?

(e) $\forall x \forall y((x - y = y - x) \to (x = y))$

True.

For any two numbers with $x - y = y - x$,

Is the quantified predicate T or F?

(e) $\forall x \forall y((x - y = y - x) \rightarrow (x = y))$

True.

For any two numbers with $x - y = y - x$,

$$x = y.$$

true by algebra:

$$x - y = y - x$$
$$2x = 2y$$
$$x = y$$

Is the quantified predicate T or F?

Which of the following statements are true? The domain of x, y, z is $\mathbb{R}$.

(a) $\exists x \forall y (x^y = -1)$

(b) $\forall x \exists y (x \cdot y = 1)$

(c) $\forall x ((x \neq 0) \rightarrow \exists y (x^y = 1))$

(d) $\exists x \exists y ((x^2 \neq y^2) \wedge \forall z (xz - yz = 0))$

(e) $\forall x \forall y ((x - y = y - x) \rightarrow (x = y))$

Logical Equivalence Proof

Use logical equivalences to show that $\neg\big(p \vee (\neg p \wedge q)\big) \equiv \neg p \wedge \neg q$.

Logical Equivalence Proof

Use logical equivalences to show that $\neg\big(p \vee (\neg p \wedge q)\big) \equiv \neg p \wedge \neg q$.

$\neg\big(p \vee (\neg p \wedge q)\big) \equiv \neg p \wedge \neg(\neg p \wedge q)$	DeMorgan's
$\equiv \neg p \wedge (p \vee \neg q)$	DeMorgan's
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	Distributive
$\equiv F \vee (\neg p \wedge \neg q)$	Negation law
$\equiv \neg p \wedge \neg q$	Identity law

Natural Deduction without Quantifiers

Using only the 12 rules of Natural Deduction, prove:

$$\frac{(p \wedge \neg q)}{\neg(p \rightarrow q)}$$

Natural Deduction without Quantifiers

Using only the 12 rules of Natural Deduction, prove:

$$\frac{(p \wedge \neg q)}{\neg(p \rightarrow q)}$$

1.	$p \wedge \neg q$	Premise
?.	$\neg(p \rightarrow q)$	$\neg$-intro

$\rightarrow$-introduction

$\boxed{p \; \vdots \; q}$

$\therefore \; p \rightarrow q$

Natural Deduction without Quantifiers

Using only the 12 rules of Natural Deduction, prove:

$$\frac{(p \wedge \neg q)}{\neg(p \to q)}$$

1.	$p \wedge \neg q$	Premise
2.	$p \to q$	assume
?.	F	
?.	$\neg(p \to q)$	$\neg$-intro

$\to$-introduction

$$\frac{\boxed{\begin{array}{c} p \\ \vdots \\ q \end{array}}}{\therefore\ p \to q}$$

Natural Deduction without Quantifiers

Using only the 12 rules of Natural Deduction, prove:

$$\frac{(p \land \neg q)}{\neg(p \to q)}$$

1.	$p \land \neg q$	Premise
2.	$p \to q$	assume
?.	F	¬-elim
?.	$\neg(p \to q)$	¬-intro

¬-elimination
p ¬p
∴ F

Natural Deduction without Quantifiers

Using only the 12 rules of Natural Deduction, prove:

$$\frac{(p \wedge \neg q)}{\neg(p \to q)}$$

1.	$p \wedge \neg q$	Premise
2.	$p \to q$	assume
?.	q	
?.	$\neg q$	
?.	F	¬-elim
?.	$\neg(p \to q)$	¬-intro

¬-elimination

p
¬p
∴ F

Natural Deduction without Quantifiers

Using only the 12 rules of Natural Deduction, prove:

$$\frac{(p \wedge \neg q)}{\neg(p \to q)}$$

1.	$p \wedge \neg q$	Premise
2.	$p \to q$	assume
?.	q	
?.	$\neg q$	∧-elim(1)
?.	F	¬-elim
?.	$\neg(p \to q)$	¬-intro

∧-elim

$$\frac{p \wedge q}{\therefore q}$$

Natural Deduction without Quantifiers

Using only the 12 rules of Natural Deduction, prove:

$$\frac{(p \wedge \neg q)}{\neg(p \to q)}$$

1.	$p \wedge \neg q$	Premise
2.	$p \to q$	assume
3.	$\boldsymbol{p}$	**∧-elim(1)**
4.	$\boldsymbol{q}$	**→-elim(2,3)**
5.	$\neg q$	∧-elim(1)
6.	F	¬-elim (4,5)
7.	$\neg(p \to q)$	¬-intro (2-6)

→-elimination

$$\frac{\begin{array}{l} p \to q \\ p \end{array}}{\therefore q}$$

Natural Deduction with Quantifiers

$$\begin{array}{rl} & \exists x(P(x) \to Q(x)) \\ & \underline{\forall x(P(x) \vee \neg P(x))} \\ \therefore & \exists x[\neg P(x)] \vee \exists x[Q(x)] \end{array}$$

Natural Deduction with Quantifiers

$$\exists x(P(x) \to Q(x))$$
$$\forall x(P(x) \vee \neg P(x))$$
$$\therefore \exists x \neg P(x) \vee \exists x Q(x)$$

∃ elim → elim
∀ elim ∨ elim ¬ elim

∃ intro ¬ intro
∃ intro ∨ intro

1. $\exists x\ [P(x) \to Q(x)]$ Premise
2. $\forall x\ [P(x) \vee \neg P(x)]$ Premise

?. $\exists x\ \neg P(x) \vee \exists x\ Q(x)$

Natural Deduction with Quantifiers

$$\begin{array}{rl} & \exists x(P(x) \rightarrow Q(x)) \\ & \forall x(P(x) \vee \neg P(x)) \\ \hline \therefore & \exists x \neg P(x) \vee \exists x Q(x) \end{array}$$

1.	$\exists x\ [P(x) \rightarrow Q(x)]$	Premise
2.	$\forall x\ [P(x) \vee \neg \mathrm{P}(\mathrm{x})]$	Premise
3.	$P(x_0) \rightarrow Q(x_0)$	$\exists$−elim(1)
?.	$\exists x\ \neg \mathrm{P}(\mathrm{x}) \vee \exists x\ Q(x)$	

∃-elimination

$$\begin{array}{rl} & \exists x\ P(x) \\ \hline \therefore & P(x_0) \end{array}$$

Natural Deduction with Quantifiers

$$\begin{array}{l} \exists x(P(x) \rightarrow Q(x)) \\ \underline{\forall x(P(x) \vee \neg P(x))} \\ \therefore\ \exists x \neg P(x) \vee \exists x Q(x) \end{array}$$

1.	$\exists x\ [P(x) \rightarrow Q(x)]$	Premise
2.	$\forall x\ [P(x) \vee \neg P(x)]$	Premise
3.	$P(x_0) \rightarrow Q(x_0)$	∃-elim(1)
4.	$P(x_0) \vee \neg P(x_0)$	∀-elim(2)

- Eliminating quantifiers often a good start
- ∃-elim before ∀-elim to create variables

∀-elimination

$$\frac{\forall x\ P(x)}{\therefore\ P(c)}$$

?. $\exists x\ \neg P(x) \vee \exists x\ Q(x)$

Natural Deduction with Quantifiers

$$\begin{array}{rl} & \exists x(P(x) \rightarrow Q(x)) \\ & \underline{\forall x(P(x) \vee \neg P(x))} \\ \therefore & \exists x \neg P(x) \vee \exists x Q(x) \end{array}$$

1.	$\exists x\ [P(x) \rightarrow Q(x)]$	Premise
2.	$\forall x\ [P(x) \vee \neg \mathrm{P}(\mathrm{x})]$	Premise
3.	$P(x_0) \rightarrow Q(x_0)$	$\exists$-elim(1)
4.	$P(x_0) \vee \neg P(x_0)$	$\forall$-elim(2)

If this one holds, then $\exists x\ Q(x)$

If this one holds, then $\exists x\ \neg \mathrm{P}(\mathrm{x})$

?. $\exists x\ \neg \mathrm{P}(\mathrm{x}) \vee \exists x\ Q(x)$

Natural Deduction with Quantifiers

$$\begin{array}{rl} & \exists x(P(x) \to Q(x)) \\ & \underline{\forall x(P(x) \vee \neg P(x))} \\ \therefore & \exists x \neg P(x) \vee \exists x Q(x) \end{array}$$

1.	$\exists x \, [P(x) \to Q(x)]$	Premise
2.	$\forall x \, [P(x) \vee \neg \mathrm{P(x)}]$	Premise
3.	$P(x_0) \to Q(x_0)$	∃-elim(1)
4.	$P(x_0) \vee \neg P(x_0)$	∀-elim(2)
?.	$\exists x \, \neg \mathrm{P(x)} \vee \exists x \, Q(x)$	∨-elim

∨-elimination

$$\begin{array}{l} p \vee q \\ \boxed{\begin{array}{l} p \\ \vdots \\ r \end{array}} \quad \boxed{\begin{array}{l} q \\ \vdots \\ r \end{array}} \\ \hline \therefore \; r \end{array}$$

Natural Deduction with Quantifiers

$$\begin{array}{l} \exists x(P(x) \to Q(x)) \\ \underline{\forall x(P(x) \lor \neg P(x))} \\ \therefore\ \exists x \neg P(x) \lor \exists x Q(x) \end{array}$$

1.	$\exists x\, [P(x) \to Q(x)]$	Premise
2.	$\forall x\, [P(x) \lor \neg P(x)]$	Premise
3.	$P(x_0) \to Q(x_0)$	∃-elim(1)
4.	$P(x_0) \lor \neg P(x_0)$	∀-elim(2)
5.	$P(x_0)$	Assumption
6.	$Q(x_0)$	→elim (3,5)
7.	$\exists x\, Q(x)$	∃ intro (6)
?.8	$\exists x \neg P(x) \lor \exists x\, Q(x)$	∨ intro (7)
?.9.	$\neg P(x_0)$	Assumption
10.	$\exists x \neg P(x)$	∃ intro (9)
?.11.	$\exists x \neg P(x) \lor \exists x\, Q(x)$	∨ intro (10)
?.12	$\exists x \neg P(x) \lor \exists x\, Q(x)$	∨-elim (4, 5-8, 9-11)

∨-elimination

$$\begin{array}{c} p \lor q \\ \boxed{\begin{array}{c} p \\ \vdots \\ r \end{array}} \quad \boxed{\begin{array}{c} q \\ \vdots \\ r \end{array}} \\ \hline \therefore\ r \end{array}$$

Natural Deduction with Quantifiers

$$\begin{array}{l} \exists x(P(x) \to Q(x)) \\ \forall x(P(x) \lor \neg P(x)) \\ \hline \therefore\ \exists x \neg P(x) \lor \exists x Q(x) \end{array}$$

1.	$\exists x\ [P(x) \to Q(x)]$	Premise
2.	$\forall x\ [P(x) \lor \neg P(x)]$	Premise
3.	$P(x_0) \to Q(x_0)$	$\exists$-elim(1)
4.	$P(x_0) \lor \neg P(x_0)$	$\forall$-elim(2)
5.	$P(x_0)$	Assumption
6.	$Q(x_0)$	$\to$-elim(4,5)
?.	$\exists x\ \neg P(x) \lor \exists x\ Q(x)$	
?.	$\neg P(x_0)$	Assumption
?.	$\exists x\ \neg P(x) \lor \exists x\ Q(x)$	
?.	$\exists x\ \neg P(x) \lor \exists x\ Q(x)$	$\lor$-elim

$\to$-elimination

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

Natural Deduction with Quantifiers

$$\begin{array}{l} \exists x(P(x) \to Q(x)) \\ \underline{\forall x(P(x) \vee \neg P(x))} \\ \therefore\ \exists x \neg P(x) \vee \exists x Q(x) \end{array}$$

1.	$\exists x\ [P(x) \to Q(x)]$	Premise
2.	$\forall x\ [P(x) \vee \neg P(x)]$	Premise
3.	$P(x_0) \to Q(x_0)$	∃-elim(1)
4.	$P(x_0) \vee \neg P(x_0)$	∀-elim(2)
5.	$P(x_0)$	Assumption
6.	$Q(x_0)$	→-elim(4,5)
7.	$\exists x\ Q(x)$	∃-intro(6)
?.	$\exists x\ \neg P(x) \vee \exists x\ Q(x)$	
?.	$\neg P(x_0)$	Assumption
?.	$\exists x\ \neg P(x) \vee \exists x\ Q(x)$	
?.	$\exists x\ \neg P(x) \vee \exists x\ Q(x)$	∨-elim

∃-introduction

$$\begin{array}{c} P(c) \\ \hline \therefore\ \exists x\ P(x) \end{array}$$

Natural Deduction with Quantifiers

$$\begin{array}{l} \exists x(P(x) \to Q(x)) \\ \underline{\forall x(P(x) \lor \neg P(x))} \\ \therefore\ \exists x \neg P(x) \lor \exists x Q(x) \end{array}$$

1.	$\exists x\,[P(x) \to Q(x)]$	Premise
2.	$\forall x\,[P(x) \lor \neg P(x)]$	Premise
3.	$P(x_0) \to Q(x_0)$	∃-elim(1)
4.	$P(x_0) \lor \neg P(x_0)$	∀-elim(2)
5.	$P(x_0)$	Assumption
6.	$Q(x_0)$	→-elim(4,5)
7.	$\exists x\, Q(x)$	∃-intro(6)
8.	$\exists x\, \neg P(x) \lor \exists x\, Q(x)$	∨-intro(7)
9.	$\neg P(x_0)$	Assumption
?.	$\exists x\, \neg P(x) \lor \exists x\, Q(x)$	
?.	$\exists x\, \neg P(x) \lor \exists x\, Q(x)$	∨-elim

∨-introduction

$$\begin{array}{l} p \\ \hline \therefore p \lor q \end{array}$$

Natural Deduction with Quantifiers

$$\begin{array}{l} \exists x(P(x) \rightarrow Q(x)) \\ \underline{\forall x(P(x) \vee \neg P(x))} \\ \therefore\ \exists x \neg P(x) \vee \exists x Q(x) \end{array}$$

1.	$\exists x\ [P(x) \rightarrow Q(x)]$	Premise
2.	$\forall x\ [P(x) \vee \neg P(x)]$	Premise
3.	$P(x_0) \rightarrow Q(x_0)$	$\exists$-elim(1)
4.	$P(x_0) \vee \neg P(x_0)$	$\forall$-elim(2)
5.	$P(x_0)$	Assumption
6.	$Q(x_0)$	$\rightarrow$-elim(4,5)
7.	$\exists x\ Q(x)$	$\exists$-intro(6)
8.	$\exists x\ \neg P(x) \vee \exists x\ Q(x)$	$\vee$-intro(7)
9.	$\neg P(x_0)$	Assumption
10.	$\exists x\ \neg P(x)$	$\exists$-intro(9)
?.	$\exists x\ \neg P(x) \vee \exists x\ Q(x)$	$\vee$-elim

$\exists$-introduction

$$\frac{P(c)}{\therefore\ \exists x\ P(x)}$$

Natural Deduction with Quantifiers

$$\begin{array}{l} \exists x(P(x) \to Q(x)) \\ \underline{\forall x(P(x) \lor \neg P(x))} \\ \therefore\ \exists x \neg P(x) \lor \exists x Q(x) \end{array}$$

1.	$\exists x\ [P(x) \to Q(x)]$	Premise
2.	$\forall x\ [P(x) \lor \neg P(x)]$	Premise
3.	$P(x_0) \to Q(x_0)$	$\exists$-elim(1)
4.	$P(x_0) \lor \neg P(x_0)$	$\forall$-elim(2)
5.	$P(x_0)$	Assumption
6.	$Q(x_0)$	$\to$-elim(4,5)
7.	$\exists x\ Q(x)$	$\exists$-intro(6)
8.	$\exists x\ \neg P(x) \lor \exists x\ Q(x)$	$\lor$-intro(7)
9.	$\neg P(x_0)$	Assumption
10.	$\exists x\ \neg P(x)$	$\exists$-intro(9)
11.	$\exists x\ \neg P(x) \lor \exists x\ Q(x)$	$\lor$-intro(10)
12.	$\exists x\ \neg P(x) \lor \exists x\ Q(x)$	$\lor$-elim

$\lor$-introduction

$$\begin{array}{l} \underline{\ \ p\ \ } \\ \therefore p \lor q \end{array}$$

Natural Deduction with Quantifiers

$$\begin{array}{l} \exists x(P(x) \rightarrow Q(x)) \\ \underline{\forall x(P(x) \vee \neg P(x))} \\ \therefore\ \exists x \neg P(x) \vee \exists x Q(x) \end{array}$$

1.	$\exists x\, [P(x) \rightarrow Q(x)]$	Premise
2.	$\forall x\, [P(x) \vee \neg \mathrm{P}(\mathrm{x})]$	Premise
3.	$P(x_0) \rightarrow Q(x_0)$	∃-elim(1)
4.	$P(x_0) \vee \neg P(x_0)$	∀-elim(2)
5.	$P(x_0)$	Assumption
6.	$Q(x_0)$	→-elim(4,5)
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9.	$\neg P(x_0)$	Assumption
10.	$\exists x\, \neg P(\mathrm{x})$	∃-intro(9)
11.	$\exists x\, \neg \mathrm{P}(\mathrm{x}) \vee \exists x\, Q(x)$	∨-intro(10)
12.	$\exists x\, \neg \mathrm{P}(\mathrm{x}) \vee \exists x\, Q(x)$	∨-elim(4,5-8,9-11)

Cleanup:
Fill in missing line #s

Word Proofs

Let x and y be real numbers. Prove the following statement, and state which proof method you used.

"If $\underbrace{x - 2y \text{ is rational and } xy \text{ is irrational}}_{p}$, then $\underbrace{x \text{ and } y \text{ are both irrational}}_{q}$."

- *You may assume that the sum and product of rationals is rational.*
- *You may also assume that the sum and product a rational and an irrational is irrational.*

Could do: proof by contradiction: $\neg(p \to q) \equiv p \wedge \neg q$

Assume not, i.e. $x-2y$ is rational and xy is irrational, and x and y are not both irration.

then cases

- or -

Proof by contrapositive: $\neg q \to \neg p$

"If x and y are not both irrational, then $x-2y$ is irrational OR xy is rational."

equiv: If at least one of x and y are rational,

- OR - Direct proof w/ 4 Cases: x, y both rat. x, y both irr. x rat, y irr. x irr, y rat.

Word Proofs

Let x and y be real numbers. Prove the following statement, and state which proof method you used.

"If $x - 2y$ is rational and xy is irrational, then x and y are both irrational."

- *You may assume that the sum and product of rationals is rational.*
- *You may also assume that the sum and product a rational and an irrational is irrational.*

Proof by Contrapositive. We will prove:
"If x or y is rational, then $x - 2y$ is irrational or xy is rational."

Word Proofs

Let x and y be real numbers. Prove the following statement, and state which proof method you used.

"If $x - 2y$ is rational and xy is irrational, then x and y are both irrational."

- *You may assume that the sum and product of rationals is rational.*
- *You may also assume that the sum and product a rational and an irrational is irrational.*

Proof by Contrapositive. We will prove:
"If x or y is rational, then $x - 2y$ is irrational or xy is rational."

Assume x or y is rational. There are three cases:

- x, y both rational $\longrightarrow$ xy rational
- x rational and y irrational $\longrightarrow$ $x-2y$ is irr.
- x irrational and y rational $\longrightarrow$ $x-2y$ is irr.

Word Proofs

Let x and y be real numbers. Prove the following statement, and state which proof method you used.

"If $x - 2y$ is rational and xy is irrational, then x and y are both irrational."

- *You may assume that the sum and product of rationals is rational.*
- *You may also assume that the sum and product a rational and an irrational is irrational.*

Proof by Contrapositive. We will prove:
"If x or y is rational, then $x - 2y$ is irrational or xy is rational."

Case 1: Assume x, y both rational

Then xy is a product of rational numbers, so xy is rational.

Word Proofs

Let x and y be real numbers. Prove the following statement, and state which proof method you used.

"If $x - 2y$ is rational and xy is irrational, then x and y are both irrational."

- *You may assume that the sum and product of rationals is rational.*
- *You may also assume that the sum and product a rational and an irrational is irrational.*

Proof by Contrapositive. We will prove:
"If x or y is rational, then $x - 2y$ is irrational or xy is rational."

Case 2: Assume x rational, y irrational

Then $x - 2y$ is a sum/product of a rational and an irrational, so it is irrational.

Word Proofs

Let x and y be real numbers. Prove the following statement, and state which proof method you used.

"If $x - 2y$ is rational and xy is irrational, then x and y are both irrational."

- *You may assume that the sum and product of rationals is rational.*
- *You may also assume that the sum and product a rational and an irrational is irrational.*

Proof by Contrapositive. We will prove:
"If x or y is rational, then $x - 2y$ is irrational or xy is rational."

Case 3: Assume x irrational, y rational

Then $x - 2y$ is a sum/product of a rational and an irrational, so it is irrational.

Word Proofs

Let x and y be real numbers. Prove the following statement, and state which proof method you used.

"If $x - 2y$ is rational and xy is irrational, then x and y are both irrational."

- *You may assume that the sum and product of rationals is rational.*
- *You may also assume that the sum and product a rational and an irrational is irrational.*

Proof by Contrapositive. We will prove:
"If x or y is rational, then $x - 2y$ is irrational or xy is rational."

Full Proof: Assume x or y is rational. There are three cases:

- x, y both rational
- x rational and y irrational
- x irrational and y rational

[case analysis on previous slides]

In any case $x - 2y$ is rational or xy is irrational.