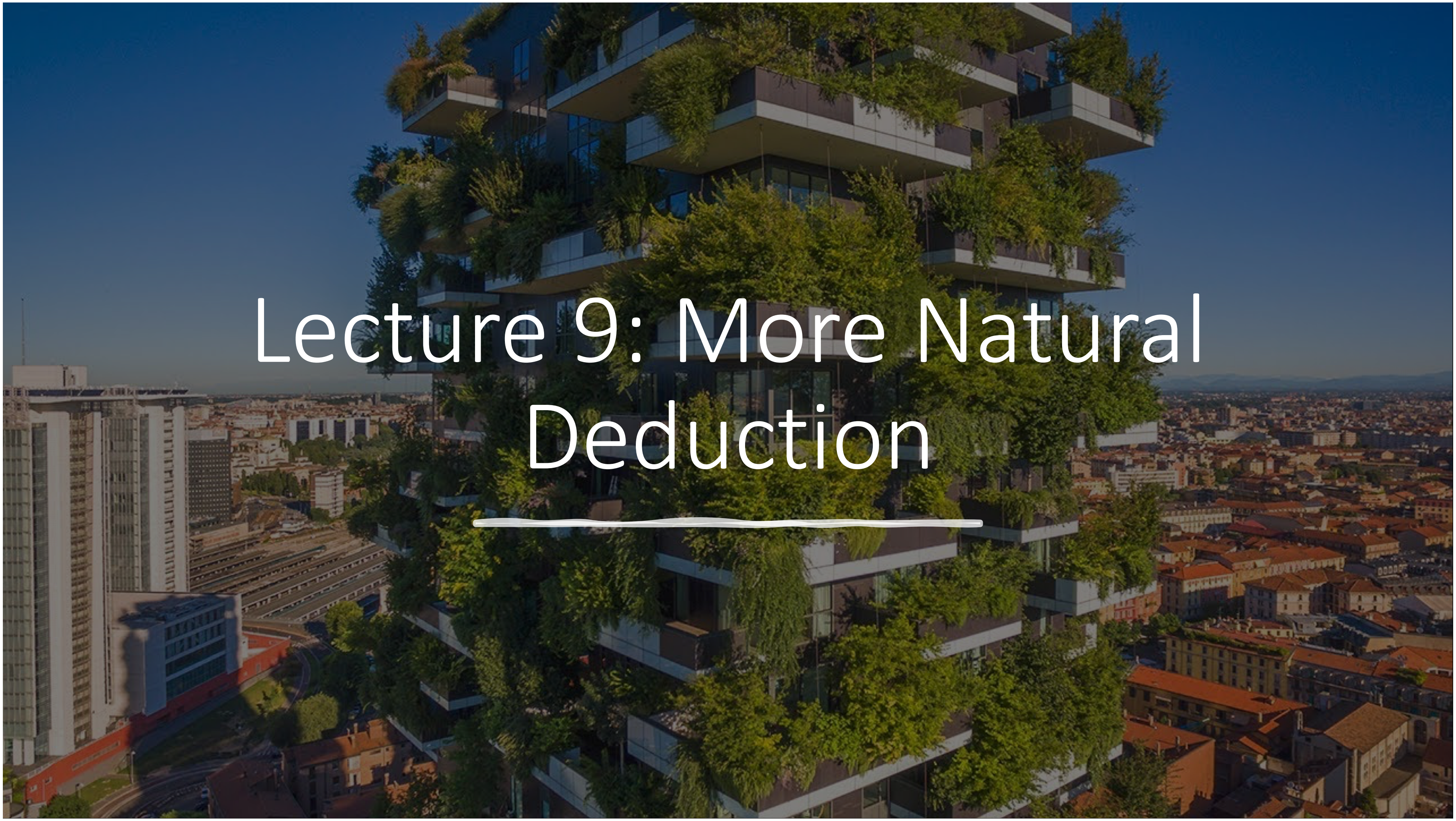
Lecture 9: More Natural Deduction



Learning Objectives: Lec 9

After today's lecture (and the associated readings, discussion, & homework), you should be able to:

- Solve natural deduction problems using all 16 rules
  - Including the 4 new quantifier rules introduced in this lecture
- Understand why the 16 natural deduction rules covered in this lecture are valid
  - You do **not** need to memorize the list of rules
- Understand how to introduce/use variables in a natural deduction proof, for use with the 4 new quantifier rules

Recap

- **Natural Deduction:** a list of rules used to *grow a knowledge pool* worth of statements, phrased using Boolean logic, until a desired statement is reached.

PROBLEM:
$a \rightarrow b \qquad b \rightarrow c$
$\therefore a \rightarrow c$

1.	$a \rightarrow b$	premise
2.	$b \rightarrow c$	premise
3.	a	assumption
4.	b	$\rightarrow$-elim(1, 3)
5.	c	$\rightarrow$-elim(2, 4)
6.	$a \rightarrow c$	$\rightarrow$-intro(3-5)

Rules for Operators

Introduction Rules

$\wedge$-introduction

$$\begin{array}{l} p \\ q \\ \hline \therefore\ p \wedge q \end{array}$$

$\neg$-introduction

$$\begin{array}{l} \boxed{\begin{array}{c} p \\ \vdots \\ F \end{array}} \\ \hline \therefore\ \neg p \end{array}$$

$\vee$-introduction

$$\begin{array}{l} p \\ \hline \therefore\ p \vee q \end{array}$$

$\rightarrow$-introduction

$$\begin{array}{l} \boxed{\begin{array}{c} p \\ \vdots \\ q \end{array}} \\ \hline \therefore\ p \rightarrow q \end{array}$$

$\leftrightarrow$-introduction

$$\begin{array}{l} \boxed{\begin{array}{c} p \\ \vdots \\ q \end{array}} \ \boxed{\begin{array}{c} q \\ \vdots \\ p \end{array}} \\ \hline \therefore\ p \leftrightarrow q \end{array}$$

Elimination Rules

$\wedge$-elimination

$$\begin{array}{l} p \wedge q \\ \hline \therefore\ p \end{array}$$

$\neg$-elimination

$$\begin{array}{l} p \\ \neg p \\ \hline \therefore\ F \end{array}$$

$\vee$-elimination

$$\begin{array}{l} p \vee q \\ \boxed{p} \ \boxed{q} \\ \hline \therefore\ r \end{array}$$

F-elimination

$$\begin{array}{l} F \\ \hline \therefore\ p \end{array}$$

$\neg\neg$-elimination

$$\begin{array}{l} \neg\neg p \\ \hline \therefore\ p \end{array}$$

$\rightarrow$-elimination

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

$\leftrightarrow$-elimination

$$\begin{array}{l} p \leftrightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

These are 12/16 of the rules (covered last time)

Missing: how do you introduce/eliminate **quantifiers?**

Outline

- **“There Exists” rules**
  - **∃-introduction, ∃-elimination**
  - Sample problem 1
- “For All” rules
  - ∀-elimination
  - Sample problem 2
  - ∀-introduction
  - Sample problem 3
- A Corner Case
- Natural Deduction solving advice

2/16: "There Exists" rules

∃-elimination
$\exists x\ P(x)$
$\therefore\ P(x_0)$

This rule **creates a new variable**, which stands for some **particular** element of the domain

Here we name the variable x_0, but you can name it anything you like – must be different than any other variables in scope!

P(x) can be any predicate.

Note: if you took EECS 203 in a previous term, we used a more complicated version of this rule

∃-introduction
$P(c)$
$\therefore\ \exists x\ P(x)$

This rule **uses an existing variable**:
c can be any previously-created member of the domain.

Lecture 9 Handout: Natural Deduction, part 2

Handout

2/16: "There Exists" rules

∃-elimination

$$\frac{\exists x\ P(x)}{\therefore\ P(x_0)}$$

∃-introduction

$$\frac{P(c)}{\therefore\ \exists x\ P(x)}$$

Does this rule:
1. create a new variable, or
2. use an existing domain element?

Does this rule:
1. create a new variable, or
2. use an existing domain element?

Outline

- **“There Exists” rules**
  - ∃-introduction, ∃-elimination
  - **Sample problem 1**
- “For All” rules
  - ∀-elimination
  - Sample problem 2
  - ∀-introduction
  - Sample problem 3
- A Corner Case
- Natural Deduction solving advice

Handout

ND with "There Exists" rules

optional

Problem
$\exists x\, [A(x) \wedge B(x)]$
$\therefore \exists x\, A(x)$

∃ elim ✓

∧ elim ✓

∃ intro ✓

1. $\exists x\, [A(x) \wedge B(x)]$ premise
2. $A(x_0) \wedge B(x_0)$ ∃ elim (1, x_0)
3. $A(x_0)$ ∧ elim (2)
4. $\exists x\, A(x)$??? ∃ intro (3)

ND with “There Exists” rules

Problem

$\exists x\, A(x) \wedge B(x)$

$\therefore\ \exists x\, A(x)$

1. $\exists x\, A(x) \wedge B(x)$ premise
2. $A(x_0) \wedge B(x_0)$ $\exists$-elim(1)

?. $\exists x\, A(x)$???

Creating new variable x_0 which represents some element of the domain

Using $P(x)=A(x) \wedge B(x)$

$\exists$-elimination

$\exists x\ P(x)$

$\therefore\ P(x_0)$

ND with "There Exists" rules

Problem

$\exists x\, A(x) \wedge B(x)$

$\therefore \exists x\, A(x)$

1. $\exists x\, A(x) \wedge B(x)$ premise
2. $A(x_0) \wedge B(x_0)$ $\exists$-elim(1)
3. $A(x_0)$ $\wedge$-elim(2)

?. $\exists x\, A(x)$???

$\wedge$-elimination

$p \wedge q$

$\therefore q$

ND with "There Exists" rules

Problem

$\exists x\, A(x) \wedge B(x)$

$\therefore \exists x\, A(x)$

1. $\exists x\, A(x) \wedge B(x)$ premise
2. $A(x_0) \wedge B(x_0)$ $\exists$-elim(1)
3. $A(x_0)$ $\wedge$-elim(2)
4. $\exists x\, A(x)$ $\exists$-intro(3)

$\exists$-introduction

$P(c)$

$\therefore \exists x\, P(x)$

Outline

- "There Exists" rules
  - ∃-introduction, ∃-elimination
  - Sample problem 1
- **"For All" rules**
  - **∀-elimination**
  - Sample problem 2
  - ∀-introduction
  - Sample problem 3
- A Corner Case
- Natural Deduction solving advice

1/16: ∀-elimination

∀-elimination

$$\frac{\forall x\ P(x)}{\therefore\ P(c)}$$

Here c stands for a variable **already known to be in the domain**.

- Maybe c is a variable created by a previous rule, such as ∃-elimination
- Maybe we were told the domain, and we use a known domain element
  - **Example:** If domain = $\mathbb{Z}$, then we could write P(5)

Rules Involving Quantifiers

∃-elimination ★

$$\frac{\exists x\, P(x)}{\therefore\ P(x_0)}$$

∃-introduction

$$\frac{P(c)}{\therefore\ \exists x\, P(x)}$$

★ Creates a **new** variable

Uses an **existing** domain element

∀-elimination

$$\frac{\forall x\, P(x)}{\therefore\ P(c)}$$

(one more still to come)

Outline

- "There Exists" rules
  - ∃-introduction, ∃-elimination
  - Sample problem 1
- **"For All" rules**
  - ∀-elimination
  - **Sample problem 2**
  - ∀-introduction
  - Sample problem 3
- A Corner Case
- Natural Deduction solving advice

Quantifier ND

Handout

- Premises:
  i. "Everyone who has a bacterial infection takes antibiotics."
  ii. "Someone has a bacterial infection."
- Conclusion:
  - Someone takes antibiotics.

Domain: people
Predicates:
- A(x) = "x takes antibiotics"
- B(x) = "x has a bacterial infection"

premises:

i. $\forall x\,[B(x) \to A(x)]$

ii. $\exists x\; B(x)$

conclusion:

$\exists x\; A(x)$

Quantifier ND

- Premises:
  i. "Everyone who has a bacterial infection takes antibiotics."
  ii. "Someone has a bacterial infection."
- Conclusion:
  - Someone takes antibiotics.

Domain: people
Predicates:
- $A(x)$ = "x takes antibiotics"
- $B(x)$ = "x has a bacterial infection"

premises:

$\forall x\ [B(x) \rightarrow A(x)]$

$\exists x\ B(x)$

conclusion:

$\exists x\ A(x)$

Handout

This is the version of the proof that **DOES NOT** work

Quantifier ND

PROBLEM:	
$\forall x\,(B(x) \rightarrow A(x))$	$\exists x\, B(x)$
$\therefore \exists x\, A(x)$	

1. $\forall x\,[B(x) \rightarrow A(x)]$ premise
2. $\exists x\, B(x)$ premise
3. $B(x_0) \rightarrow A(x_0)$ $\forall$ elim(1)
4. $B(\cancel{x_0}\, x_1)$ $\exists$ elim (2)
5. $A(x_0)$ $\rightarrow$ elim(3,4)

$\rightarrow$ elim doesn't work.

6. $\exists x\, A(x)$ $\exists$intro (5) ???

introduced a new variable NOT ALLOWED

Quantifier ND

PROBLEM:

$\forall x\, (B(x) \rightarrow A(x))$ $\exists x\, B(x)$

$\therefore \exists x\, A(x)$

1. $\forall x\, [B(x) \rightarrow A(x)]$ — premise
2. $\exists x\, B(x)$ — premise
3. $B(x_0) \rightarrow A(x_0)$ — $\forall$-elim(1)
4. $B(x_0)$ — $\exists$-elim(2)
5. ~~$A(x_0)$ — $\rightarrow$-elim(3, 4)~~

?. $\exists x\, A(x)$ — ???

$\exists$-elim always creates a new variable, so this is a new variable that we also called x_0. That's a confusing choice of name

Quantifier ND

PROBLEM:

$\forall x\,(B(x) \rightarrow A(x))$ $\quad$ $\exists x\, B(x)$

$\therefore \exists x\, A(x)$

1.	$\forall x\,[B(x) \rightarrow A(x)]$	premise
2.	$\exists x\, B(x)$	premise
3.	$B(x_0) \rightarrow A(x_0)$	$\forall$-elim(1)
4.	$B(x_1)$	$\exists$-elim(2)
~~5.~~	$\cancel{A(x_0)}$	~~$\rightarrow$-elim(3, 4)~~
?.	$\exists x\, A(x)$	???

Now it's a lot more obvious why $\rightarrow$-elim is not valid

$\rightarrow$-elimination

$p \rightarrow q$

p

$\therefore q$

Quantifier ND

PROBLEM:

$\forall x\,(B(x) \rightarrow A(x))$ $\quad$ $\exists x\, B(x)$

$\therefore \exists x\, A(x)$

1.	$\forall x\,[B(x) \rightarrow A(x)]$	premise
2.	$\exists x\, B(x)$	premise
3.	$B(x_0)$	$\exists$ elim (2)
4.	$B(x_0) \rightarrow A(x_0)$	$\forall$ elim (1)
5.	$A(x_0)$	$\rightarrow$ elim (3,4)
~~?~~ 6	$\exists x\, A(x)$	~~???~~ $\exists$ intro (5)

$\exists$-elimination

$\exists x\, P(x)$

$\therefore P(x_0)$

Quantifier ND

PROBLEM:

$\forall x\, (B(x) \rightarrow A(x))$ $\quad$ $\exists x\, B(x)$

$\therefore\ \exists x\, A(x)$

1.	$\forall x\, [B(x) \rightarrow A(x)]$	premise
2.	$\exists x\, B(x)$	premise
3.	$B(x_0)$	$\exists$-elim(2)
4.	$B(x_0) \rightarrow A(x_0)$	$\forall$-elim(1)

$\forall$-elim uses an existing domain element
(c, created on previous line)

?.	$\exists x\, A(x)$	???

$\forall$-elimination

$\forall x\, P(x)$

$\therefore\ P(c)$

Quantifier ND

PROBLEM:

$\forall x\, (B(x) \rightarrow A(x))$ $\quad\quad$ $\exists x\, B(x)$

$\therefore \exists x\, A(x)$

1. $\forall x\, [B(x) \rightarrow A(x)]$ premise
2. $\exists x\, B(x)$ premise
3. $B(x_0)$ $\exists$-elim(2)
4. $B(x_0) \rightarrow A(x_0)$ $\forall$-elim(1)
5. $A(x_0)$ $\rightarrow$-elim(3, 4)

?. $\exists x\, A(x)$???

$\rightarrow$-elimination

$p \rightarrow q$

p

$\therefore q$

Quantifier ND

PROBLEM:

$\forall x\,(B(x) \rightarrow A(x))$ $\quad$ $\exists x\, B(x)$

$\therefore \exists x\, A(x)$

1. $\forall x\,[B(x) \rightarrow A(x)]$ — premise
2. $\exists x\, B(x)$ — premise
3. $B(x_0)$ — $\exists$-elim(2)
4. $B(x_0) \rightarrow A(x_0)$ — $\forall$-elim(1)
5. $A(x_0)$ — $\rightarrow$-elim(3, 4)
6. $\exists x\, A(x)$ — $\exists$-intro(5)

$\exists$-intro uses an existing domain element (x_0 again)

$\exists$-introduction

$$\frac{P(c)}{\therefore \exists x\, P(x)}$$

Outline

- “There Exists” rules
  - ∃-introduction, ∃-elimination
  - Sample problem 1
- **“For All” rules**
  - ∀-elimination
  - Sample problem 2
  - **∀-introduction**
  - Sample problem 3
- A Corner Case
- Natural Deduction solving advice

1/16 (last one!): ∀-introduction

∀-introduction

$$\boxed{\begin{array}{l} x_0 \quad \text{arbitrary} \\ \quad\vdots \\ \quad P(x_0) \end{array}}$$

$$\therefore \forall x\, P(x) \longrightarrow \text{Forall } x,\ P(x)$$

Let x be an arbitrary elemt. of the Domain

⋮

$P(x_0)$

New use for boxes:

- Create a new **variable** x_0 at the top.
- Like saying: "Let x_0 be **any arbitrary** element of the domain."
- The box marks the range where x_0 is in scope.

Handout

Rules Involving Quantifiers

∃-elimination ★

$\exists x\, P(x)$

$\therefore\ P(x_0)$

∃-introduction ⬣

$P(c)$

$\therefore\ \exists x\, P(x)$

★ Creates a **new** variable

⬣ Uses an **existing** domain element

∀-elimination ⬣

$\forall x\, P(x)$

$\therefore\ P(c)$

∀-introduction ★

$$\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ P(x_0) \end{array}}$$

$\therefore\ \forall x\, P(x)$

Outline

- “There Exists” rules
  - ∃-introduction, ∃-elimination
  - Sample problem 1
- **“For All” rules**
  - ∀-elimination
  - Sample problem 2
  - ∀-introduction
  - **Sample problem 3**
- A Corner Case
- Natural Deduction solving advice

1/16 (last one!): ∀-introduction

Handout

∀-introduction

x_0

⋮

$P(x_0)$

∴ ∀x P(x)

Does this rule:

1. create a new variable, or
2. use an existing domain element?

Problem

∀x∀y(P(x)→Q(y))   ∃xP(x)

∴ ∀yQ(y)

1.	∀x∀y(P(x)→Q(y))	premise
2.	∃xP(x)	premise
3.	y_0	arbitrary
4.	$P(x_0)$	∃elim(2)
5.	$\forall y[P(x_0) \to Q(y)]$	∀elim(1, x_0)
6.	$P(x_0) \to Q(y_0)$	∀elim(5, y_0)
7.	$Q(y_0)$	→elim(6,4)
~~?.~~ 8.	∀yQ(y)	~~?????~~ ∀intro(3–7)

ND with ∀-intro – Alternate Solution

Problem

$\forall x \forall y(P(x) \rightarrow Q(y))$ $\quad\quad$ $\exists x P(x)$

$\therefore \forall y Q(y)$

1.	$\forall x \forall y(P(x) \rightarrow Q(y))$	premise
2.	$\exists x P(x)$	premise
3.	$P(x_0)$	∃-elim(2)
4.	$\forall y(P(x_0) \rightarrow Q(y))$	∀-elim(1)
5.	y_0	arbitrary
6.	$P(x_0) \rightarrow Q(y_0)$	∀-elim(5)
?.	$Q(y_0)$	
?.	$\forall y Q(y)$	∀-intro

ND with ∀-intro – Alternate Solution

Problem

$\forall x \forall y(P(x) \rightarrow Q(y))$ $\quad$ $\exists x P(x)$

$\therefore \forall y Q(y)$

1.	$\forall x \forall y(P(x) \rightarrow Q(y))$	premise
2.	$\exists x P(x)$	premise
3.	$P(x_0)$	∃-elim(2)
4.	$\forall y(P(x_0) \rightarrow Q(y))$	∀-elim(1)
5.	y_0	arbitrary
6.	$\quad P(x_0) \rightarrow Q(y_0)$	∀-elim(5)
7.	$\quad Q(y_0)$	→-elim(4, 6)
8.	$\forall y Q(y)$	∀-intro(5-7)

Outline

- "There Exists" rules
  - ∃-introduction, ∃-elimination
  - Sample problem 1
- "For All" rules
  - ∀-elimination
  - Sample problem 2
  - ∀-introduction
  - Sample problem 3
- **A Corner Case**
- Natural Deduction solving advice

A Corner Case in Quantifier Rules

Problem?

$$\forall x\, P(x)$$

$$\therefore\ \exists x\, P(x)$$

Can Natural Deduction rules solve this?

A Corner Case in Quantifier Rules

Problem?
$\forall x\ P(x)$
$\therefore\ \exists x\ P(x)$

1. $\forall x\ P(x)$ Premise

?. $\exists x\ P(x)$?????

A Corner Case in Quantifier Rules

Problem?
$\forall x\ P(x)$
$\therefore\ \exists x\ P(x)$

1. $\forall x\ P(x)$ Premise
2. ?

?. $\exists x\ P(x)$?????

What could even go here?
$\forall$-elim can only be used **with an existing domain element**.

A Corner Case in Quantifier Rules

Problem?
$\forall x\ P(x)$
$\therefore\ \exists x\ P(x)$

This is unsolvable …

Because the premises don't necessarily imply the conclusion!

Why not?

Empty Domains

- The **domain** for “for all” or “there exists” can be any set.
- That includes the **empty set {}**.
- What is the truth value of “For all x in {}, P(x)?” True
- What is the truth value of “There exists x in {} such that P(x)?” False

A Corner Case in Quantifier Rules

Problem?

$$\forall x\ P(x)$$

$$\therefore\ \exists x\ P(x)$$

Over **empty domain**, this is true

Over **empty domain**, this is false

$T \not\to F$

That's why **you need to know a domain element** to start this ND problem:
It's important that the domain isn't empty

Outline

- "There Exists" rules
  - ∃-introduction, ∃-elimination
  - Sample problem 1
- "For All" rules
  - ∀-elimination
  - Sample problem 2
  - ∀-introduction
  - Sample problem 3
- A Corner Case
- **Natural Deduction solving advice**

Solving Strategies

lead to

- Always try to understand **why** the premises ~~follow from~~ the conclusion before expressing this with ND rules.
  - Deriving forward from the premises without a plan usually doesn't work

- Previous problem included a **solving backwards** strategy
  - This is sometimes really useful if you're stuck!

Natural Deduction Solving

Best solving strategy: **work both forwards and backwards**.

always start with the premises

PROBLEM

¬b   a→b

∴ ¬a

1. ¬b   premise
2. a→b   premise

what can we conclude based on the premises?
Nothing yet, in this case

?.   ¬a   ???

always end with the conclusion

what will the justification for the conclusion probably be?

Which natural deduction rule can produce ¬a?

Natural Deduction Solving

Best solving strategy: **work both forwards and backwards**.

reasonable guess:

¬-introduction

p
⋮
F

∴ ¬ p

1. ¬b premise
2. a→b premise

?. ¬a ¬-intro(??)

The **missing ingredient** is an assumption box of this type.
So the entire rest of the proof is a box.

Natural Deduction Solving

Best solving strategy: **work both forwards and backwards**.

reasonable guess:

¬-introduction

$\boxed{\begin{array}{l} p \\ \vdots \\ F \end{array}}$

$\therefore \neg p$

1.	¬b	premise
2.	a→b	premise
3.	a	assumption
?.	F	
?.	¬a	¬-intro(3-?)

Natural Deduction Solving

Best solving strategy: **work both forwards and backwards**.

only guess:

¬-elimination

$$\frac{p \quad \neg p}{F}$$

1.	¬b	premise
2.	a→b	premise
3.	a	assumption
?.	F	?????
?.	¬a	¬-intro(3-?)

Keep working backwards: which natural deduction rule can produce F?

Natural Deduction Solving

Best solving strategy: **work both forwards and backwards**.

only guess:

¬-elimination

$$\frac{p \quad \neg p}{F}$$

1.	¬b	premise
2.	a→b	premise
3.	a	assumption
?.	F	¬-elim(?, ?)
?.	¬a	¬-intro(3-?)

Decision: what will be our p?

We'll use **p=b**

Missing ingredient: b *(we already have* ***¬b****)*

Natural Deduction Solving

Best solving strategy: **work both forwards and backwards**.

1.	¬b	premise
2.	a→b	premise
3.	a	assumption
?.	b	?????
?.	F	¬-elim(?, 1)
?.	¬a	¬-intro(3-?)

Now how can we justify b?

Natural Deduction Solving

Best solving strategy: **work both forwards and backwards**.

→-elimination

$p \rightarrow q$

p

$\therefore q$

1.	¬b	premise
2.	a→b	premise
3.	a	assumption
?.	b	?????
?.	F	¬-elim(?, 1)
?.	¬a	¬-intro(3-?)

Now how can we justify b?

Natural Deduction Solving

Best solving strategy: **work both forwards and backwards**.

→-elimination

$p \rightarrow q$

p

$\therefore q$

1.	¬b	premise
2.	a→b	premise
3.	a	assumption
?.	b	→-elim(2, 3)
?.	F	¬-elim(?, 1)
?.	¬a	¬-intro(3-?)

Done! Cleanup step: fill in missing ? line numbers

Natural Deduction Solving

Best solving strategy: **work both forwards and backwards**.

$\rightarrow$-elimination
$p \rightarrow q$
p
$\therefore q$

1.	$\neg b$	premise
2.	$a \rightarrow b$	premise
3.	a	assumption
4.	b	$\rightarrow$-elim(2, 3)
5.	F	$\neg$-elim(4, 1)
6.	$\neg a$	$\neg$-intro(3-5)

Wrapup

- **This is the end of the content eligible for Exam 1!**
  - Natural deduction is fair game for exam 1
  - Next lecture is not tested until exam 2

- **This is the end of the "logic" unit of the course**
  - We'll still think about proofs, and why things are true
  - But we'll focus more on new *reasons (proof styles)* why things are true, rather than their logical foundations