Compound Propositions and Predicates

EECS203 Lecture 4

Outline

These slides on Canvas:

Files > Lecture Slides & Handouts > Annotated Slides (Diaz)

- **Interesting/fun Proofs (from Lecture 3)**
- Growth Mindset
- Compound Propositions
  - Negation/And/Or
  - Truth Tables
  - If-Then/If and only if
- Predicates
  - Instantiation, For all, There exists
  - Nested Quantifiers

Proof Styles

L3 Handout

The 4 proof styles discussed today:

1. **Direct Proofs**: Proceed without any of the following styles

2. **Proof by _________________** : To prove “if p, then q”, instead prove the logically equivalent statement “if _________ , then _________”

3. **Proof by ______________** : Assume the **negation**, and try to prove something false.

4. **Proof by ______________ :** Break the situation down into cases, and show that each case leads to the desired conclusion.

Which Style Should I Use?

Handout

1. **Proof by Contrapositive**: Flagged by an "if p , then q" piece of the proposition that you find difficult to prove directly.

2. **Proof by Contradiction**: Often (but not always) flagged by "there do not exist…" at beginning of proposition

3. **Proof by Cases:** Often (but not always) flagged by a "p or q" piece of the proposition

Irrationality

Definition:

- A number **x** is **"rational"** if there exist two integers $\boldsymbol{a}, \boldsymbol{b}$ with $\boldsymbol{x} = \frac{\boldsymbol{a}}{\boldsymbol{b}}$.
- Otherwise, it is **"irrational."**

Proposition:
$\sqrt{2}$ is irrational.

Expanded Proposition:
There do not exist two integers a, b with $\frac{a}{b} = \sqrt{2}$.

"There do not exist" proposition – think proof by contradiction!

This proof will be harder in two ways than what we're used to:

- More steps in the logic
- We will use a **lemma:** reuse a statement proved previously (without redoing the proof)

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- Otherwise, it is **"irrational."**

Proposition: $\sqrt{2}$ is irrational.

- **We will use a proof by contradiction. Seeking contradiction,**

- **Assume that $\sqrt{2}$ is rational**. So there exist integers a, b with $\frac{\mathrm{a}}{\mathrm{b}} = \sqrt{\mathbf{2}}$, where a,b have **no common factors**

Irrationality

Proposition: $\sqrt{2}$ is irrational.

Definition:

- A number $\mathbf{x}$ is **"rational"** if there exist two integers $\boldsymbol{a}, \boldsymbol{b}$ with $\boldsymbol{x} = \frac{\boldsymbol{a}}{\boldsymbol{b}}$.
- Otherwise, it is **"irrational."**

- **We will use a proof by contradiction. Seeking contradiction,**

- **Assume that $\sqrt{\mathbf{2}}$ is rational**. So there exist integers a, b with **no common factors** and $\frac{\mathrm{a}}{\mathrm{b}} = \sqrt{\mathbf{2}}$.
- $\frac{a}{b} = \sqrt{2}$, so $a^2 = 2b^2$, so a^2 is even

Irrationality

Proposition: $\sqrt{2}$ is irrational.

Definition:

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- **Assume that $\sqrt{2}$ is rational**. So there exist integers a, b with **no common factors** and $\frac{\mathrm{a}}{\mathrm{b}} = \sqrt{\mathbf{2}}$.

- $\frac{a}{b} = \sqrt{2}$, so $a^2 = 2b^2$, so a^2 is even
- By Lemma, ***a* is even**

Lemma:
For all integers **x**, if $\mathbf{x}^2$ is even, then **x** is even.

Irrationality

Proposition: $\sqrt{2}$ is irrational.

Definition:

- A number **x** is **"rational"** if there exist two integers $\boldsymbol{a}, \boldsymbol{b}$ with $\boldsymbol{x} = \frac{\boldsymbol{a}}{\boldsymbol{b}}$.
- Otherwise, it is **"irrational."**

- **We will use a proof by contradiction. Seeking contradiction,**

- **Assume that $\sqrt{2}$ is rational**. So there exist integers a, b with **no common factors** and $\frac{a}{b} = \sqrt{\mathbf{2}}$.
- $\frac{a}{b} = \sqrt{2}$, so $a^2 = 2b^2$, so a^2 is even
- By Lemma, ***a* is even**
- There exists an integer k for which $a = 2k$, so $a^2 = 4k^2$

Irrationality

Proposition: $\sqrt{2}$ is irrational.

Definition:

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- Otherwise, it is **"irrational."**

- **We will use a proof by contradiction. Seeking contradiction,**

- **Assume that $\sqrt{2}$ is rational**. So there exist integers a, b with **no common factors** and $\frac{a}{b} = \sqrt{\mathbf{2}}$.
- $\frac{a}{b} = \sqrt{2}$, so $a^2 = 2b^2$, so a^2 is even
- By Lemma, ***a* is even**
- There exists an integer k for which $a = 2k$, so $a^2 = 4k^2$
- We have $2b^2 = a^2 = 4k^2$

Irrationality

Proposition: $\sqrt{2}$ is irrational.

Definition:

- A number **x** is **"rational"** if there exist two integers $\boldsymbol{a}, \boldsymbol{b}$ with $\boldsymbol{x} = \frac{\boldsymbol{a}}{\boldsymbol{b}}$.
- Otherwise, it is **"irrational."**

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- **Assume that $\sqrt{2}$ is rational**. So there exist integers a, b with **no common factors** and $\frac{\mathrm{a}}{\mathrm{b}} = \sqrt{\mathbf{2}}$.
- $\frac{a}{b} = \sqrt{2}$, so $a^2 = 2b^2$, so a^2 is even
- By Lemma, ***a* is even**
- There exists an integer k for which $a = 2k$, so $a^2 = 4k^2$
- We have $2b^2 = a^2 = 4k^2$
- So $2k^2 = b^2$, so by Lemma, ***b* is even**

Lemma:
For all integers **x**, if $\mathbf{x^2}$ is even, then **x** is even.

Irrationality

Proposition: $\sqrt{2}$ is irrational.

Definition:

- A number **x** is **"rational"** if there exist two integers $\boldsymbol{a}, \boldsymbol{b}$ with $\boldsymbol{x} = \frac{\boldsymbol{a}}{\boldsymbol{b}}$.
- Otherwise, it is **"irrational."**

- **We will use a proof by contradiction. Seeking contradiction,**

- **Assume that $\sqrt{\mathbf{2}}$ is rational**. So there exist integers a, b with **no common factors** and $\frac{a}{b} = \sqrt{\mathbf{2}}$.
- $\frac{a}{b} = \sqrt{2}$, so $a^2 = 2b^2$, so a^2 is even
- By Lemma, ***a* is even**
- There exists an integer k for which $a = 2k$, so $a^2 = 4k^2$
- We have $2b^2 = a^2 = 4k^2$
- So $2k^2 = b^2$, so by Lemma, ***b* is even**

If the contradiction is at all non-obvious, or the pieces are scattered around the proof, explain it a little.

- **This completes the contradiction**, since we have proved that **a** is even, **b** is even, and **a, b** have no common factors.

The Reveal

Definition: A number $\boldsymbol{x}$ is **"rational"** if there exist two integers $\boldsymbol{a}, \boldsymbol{b}$ with $\boldsymbol{x} = \frac{\boldsymbol{a}}{\boldsymbol{b}}$. Otherwise, x is **"irrational."**

Proposition:
There exist irrational numbers $\boldsymbol{x}, \boldsymbol{y}$ such that $\boldsymbol{x}^{\boldsymbol{y}}$ is rational.

What do you think? (A) Proposition is true (B) Proposition is false

The Reveal

Definition: A number $\boldsymbol{x}$ is **"rational"** if there exist two integers $\boldsymbol{a}, \boldsymbol{b}$ with $\boldsymbol{x} = \frac{\boldsymbol{a}}{\boldsymbol{b}}$. Otherwise, x is **"irrational."**

Proposition:
There exist irrational numbers $\boldsymbol{x}, \boldsymbol{y}$ such that $\boldsymbol{x}^{\boldsymbol{y}}$ is rational.

What do you think? **(A) Proposition is true** (B) Proposition is false

- Strategy: **proof by cases**
- The magic of the proof is picking a clever case breakdown - after that, it's easy!

The Reveal

Theorem: There exist irrational numbers $\boldsymbol{x}, \boldsymbol{y}$ such that $\boldsymbol{x}^{\boldsymbol{y}}$ is rational.

Lemma: $\sqrt{2}$ is irrational.

The Reveal

Theorem: There exist irrational numbers $\boldsymbol{x}, \boldsymbol{y}$ such that $\boldsymbol{x}^{\boldsymbol{y}}$ is rational.

- Consider the number $\sqrt{2}^{\sqrt{2}}$. This number is either rational or irrational.
- **We split into two cases accordingly.**

- **Case 1: Assume that $\sqrt{\mathbf{2}}^{\sqrt{\mathbf{2}}}$ is rational.**
  - In this case, consider $x = y = \sqrt{2}$.
  - By **Lemma**, both x, y are irrational

Lemma: $\sqrt{2}$ is irrational.

The Reveal

Theorem: There exist irrational numbers $\boldsymbol{x}, \boldsymbol{y}$ such that $\boldsymbol{x^y}$ is rational.

- Consider the number ${\sqrt{2}}^{\sqrt{2}}$.  This number is either rational or irrational.
- **We split into two cases accordingly.**

- **Case 1: Assume that ${\sqrt{2}}^{\sqrt{2}}$ is rational.**
  - In this case, consider $x = y = \sqrt{2}$.
  - By **Lemma**, both x, y are irrational
  - By **assumption,** $\boldsymbol{x^y}$ is rational.
  - So the theorem holds.

Lemma: $\sqrt{2}$ is irrational.

The Reveal

Theorem: There exist irrational numbers $\boldsymbol{x}, \boldsymbol{y}$ such that $\boldsymbol{x}^{\boldsymbol{y}}$ is rational.

- Consider the number $\sqrt{2}^{\sqrt{2}}$. This number is either rational or irrational.
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- **Case 1: Assume that $\sqrt{\mathbf{2}}^{\sqrt{\mathbf{2}}}$ is rational.**
  - In this case, consider $x = y = \sqrt{2}$.
  - By **Lemma**, both x, y are irrational
  - By **assumption,** $\boldsymbol{x}^{\boldsymbol{y}}$ is rational.
  - So the theorem holds.

- **Case 2: Assume that $\sqrt{\mathbf{2}}^{\sqrt{\mathbf{2}}}$ is irrational.**
  - In this case, consider $x = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2}$.

The Reveal

Theorem: There exist irrational numbers $\boldsymbol{x}, \boldsymbol{y}$ such that $\boldsymbol{x^y}$ is rational.

- Consider the number $\sqrt{2}^{\sqrt{2}}$. This number is either rational or irrational.
- **We split into two cases accordingly.**

- **Case 1: Assume that $\sqrt{2}^{\sqrt{2}}$ is rational.**
  - In this case, consider $x = y = \sqrt{2}$.
  - By **Lemma**, both x, y are irrational
  - By **assumption,** $\boldsymbol{x^y}$ is rational.
  - So the theorem holds.

- **Case 2: Assume that $\sqrt{2}^{\sqrt{2}}$ is irrational.**
  - In this case, consider $x = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2}$.
  - By **assumption, x** is irrational.
  - By **lemma**, y is irrational

Lemma: $\sqrt{2}$ is irrational.

The Reveal

Theorem: There exist irrational numbers $\boldsymbol{x}, \boldsymbol{y}$ such that $\boldsymbol{x^y}$ is rational.

- Consider the number $\sqrt{2}^{\sqrt{2}}$.  This number is either rational or irrational.
- **We split into two cases accordingly.**

- **Case 1: Assume that $\sqrt{2}^{\sqrt{2}}$ is rational.**
  - In this case, consider $x = y = \sqrt{2}$.
  - By **Lemma**, both x, y are irrational
  - By **assumption, $\boldsymbol{x^y}$** is rational.
  - So the theorem holds.

- **Case 2: Assume that $\sqrt{2}^{\sqrt{2}}$ is irrational.**
  - In this case, consider $x = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2}$.
  - By **assumption, x** is irrational.
  - By **lemma**, y is irrational
  - By algebra: $x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^2 = 2.$
  - 2 is rational, so the theorem holds.

Since the theorem holds in either case, it is proved.

Growth vs Fixed Mindset

https://www.youtube.com/watch?v=NWv1VdDeoRY

- A video summarizing the science and sociology behind all this
- **Tl;dw**: Focus on effort rather than "innate ability" leads to better performance + more fun

Distinguishing Between a Fixed and Growth Mindset

Fixed: abilities and intelligence are innate and practice does not change these traits…"I can't draw"

VS.

images from www.drawright.com

Growth: abilities and intelligence can develop through hard work and practice…"I've never really learned to draw."

Distinguishing Between a Fixed and Growth Mindset

Fixed: abilities and intelligence are innate and practice does not change these traits…"I can't draw"

vs.

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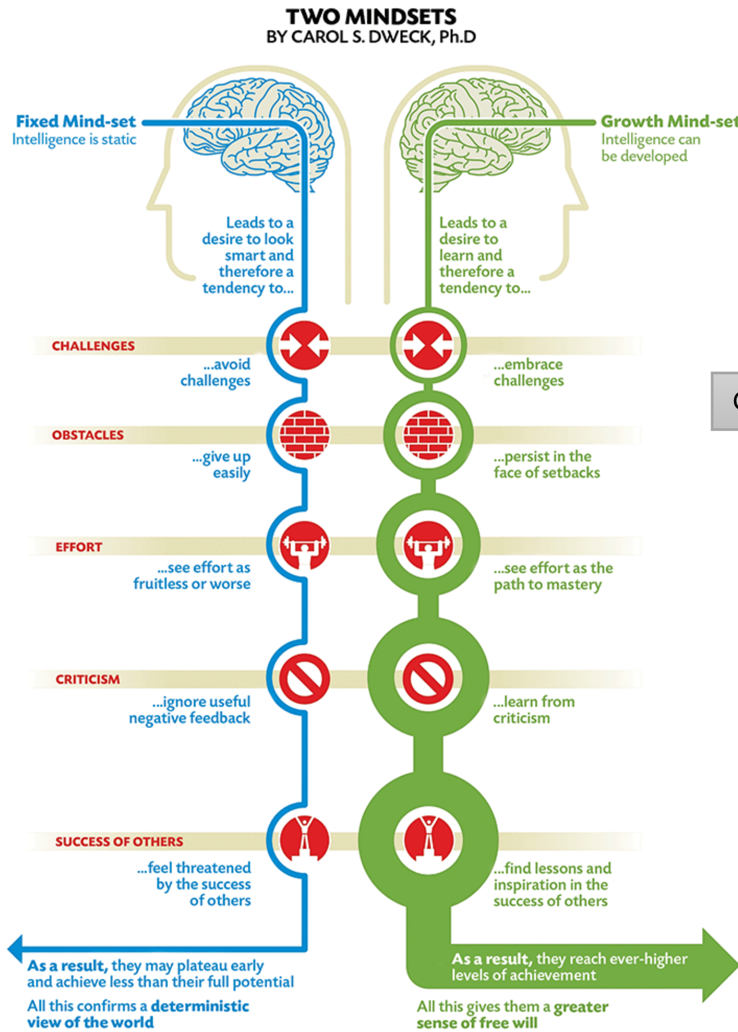
Growth: abilities and intelligence can develop through hard work and practice…"I've never really learned to draw."

Image from
http://www.nigelholmes.com/gallery

Fixed Mindset

Growth Mindset



Compound Propositions and Predicates

EECS203 Lecture 4

Learning Objectives: Lec 4

After today's lecture (and the associated readings, discussion, & homework), you should be able to:

- **Know Technical Vocab:** Compound proposition, truth table, predicate, instantiation, universal quantifier, existential quantifier, nested quantifiers
- Know truth tables for not, and, or, if-then, if-and-only-if
- Figure out truth tables for compound propositions
- Interpret and prove basic propositions using nested quantifiers
- Understand when you can and can't switch the order of nested quantifiers while preserving meaning

Outline

- **Compound Propositions**
  - **Negation/And/Or**
  - Truth Tables
  - If-Then/If and only if
- Predicates
  - Instantiation, For all, There exists
  - Nested Quantifiers

Where we're going next

- We **aren't** heading towards new proofs/styles for awhile.
- We **are** going to look back at the proofs in the last three lectures in a new way
- Headline: You might have noticed that **many propositions are made out of smaller propositions**.

The very first thing we proved:

For all integers **x**, if **x** is even, then **x+2** is even.

Entire "if … then …" part is **also** a proposition (once x is specified)

Proposition (once x is specified)

Truth Tables

Boolean Algebra: we work with **truth values {T, F}**, and we have:

- Negation (**not x** to get the "opposite" of **x**)
- Basic **operators** that combine two truth values into one **(and, or,** *(more later)***)**

Since there are only 2 truth values, we can **completely list** the input/outputs of all operators

This side lists **all possibilities** for the truth value of **p**

p	not p
T	F
F	T

This side lists the corresponding truth value for the negation in each case.

Negations

“not p” sometimes written with a symbol: “¬p”

Every proposition has a negation

If we know the truth value of p, we also know the truth value of ***not*** $\boldsymbol{p}$.

This side lists **all possibilities** for the truth value of **p**

p	not p
T	F
F	T

This side lists the corresponding truth value for the negation in each case.

Lec 4 Handout: Compound Propositions & Predicates

p	not p
T	
F	

p	q	p and q
T	T	
T	F	
F	T	
F	F	

p	q	p or q
T	T	
T	F	
F	T	
F	F	

The ONLY time that
“if p, then q” is **false** is:

____

____

p	q	if p then q
T	T	
T	F	
F	T	
F	F	

p	q	p if and only if q
T	T	
T	F	
F	T	
F	F	

AND as a joiner

Given two propositions, you can join them together with “AND” and get a third proposition.

The forecast says it will rain today.

I will bring my umbrella.

The forecast says it will rain today **and** I will bring my umbrella.

“p and q” sometimes written with a symbol: “p∧q”

AND as a joiner

Given two propositions, you can join them together with “AND” and get a third proposition.

p q

p **and** q

Important feature: the truth value of “p and q” is predictable from the truth values of p, q individually.

List of all possibilities for the **truth values** of p, q

p	q	p and q
T	T	
T	F	
F	T	
F	F	

"p and q" sometimes written with a symbol: "p∧q"

AND as a joiner

Given two propositions, you can join them together with "AND" and get a third proposition.

p → p **and** q ← q

Important feature: the truth value of "p and q" is predictable from the truth values of p, q individually.

List of all possibilities for the **truth values** of p, q

p	q	p and q
T	T	T
T	F	F
F	T	F
F	F	F

"2+2=4 and 4 is even." (→ row T, T, T)

"3+3=4 and 4 is even." (→ row F, T, F)

OR as a joiner

Given two propositions, you can join them together with “OR” and get a third proposition.

The number 5 is even.

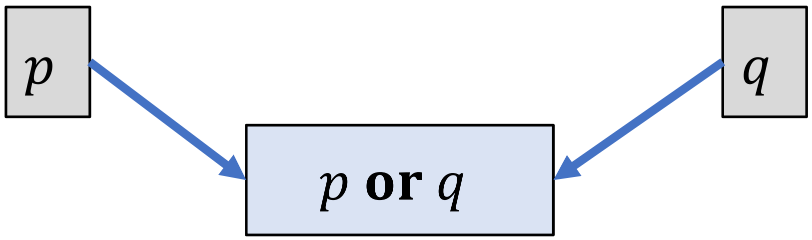
The number 5 is positive.

The number 5 is even **or** the number 5 is positive.

“p or q” sometimes written with a symbol: “p∨q”

OR as a joiner

Given two propositions, you can join them together with “OR” and get a third proposition.



List of all possibilities for the **truth values** of p, q

p	q	p or q
T	T	
T	F	
F	T	
F	F	

“p or q” sometimes written with a symbol: “p∨q”

OR as a joiner

Given two propositions, you can join them together with “OR” and get a third proposition.

p → p **or** q ← q

p	q	p or q
T	T	???
T	F	T
F	T	T
F	F	F

List of all possibilities for the **truth values** of p, q

English is ambiguous!
“I will go to the store or I will go to the park” - does that mean I might do both?

“2+2=4 or 4 is odd.”

“3+3=4 or 4 is odd.”

"p or q" sometimes written with a symbol: "p∨q"

OR as a joiner

Given two propositions, you can join them together with "OR" and get a third proposition.

p $\qquad$ q

p **or** q

p	q	p or q
T	T	**T**
T	F	T
F	T	T
F	F	F

List of all possibilities for the **truth values** of p, q

Convention in logic: "or" always includes the possibility that **both** are true.

"2+2=4 or 4 is odd."

"3+3=4 or 4 is odd."

Outline

- Compound Propositions
  - Negation/And/Or
  - **Truth Tables**
  - If-Then/If and only if

- Predicates
  - Instantiation, For all, There exists
  - Nested Quantifiers

Truth Tables

- You can combine {not, and, or} (and others) **repeatedly** to create arbitrarily complex compound propositions out of input propositions

- **Truth Table:** Table that lists all possible truth values of the input variables on the left, and the corresponding truth value of a compound proposition on the right.

Exercises

Complete the truth tables.

p	q	not (p and q)
T	T	
T	F	
F	T	
F	F	

p	q	q and not (p or q)
T	T	
T	F	
F	T	
F	F	

Exercises

Complete the truth tables.

Handout

not (p and q)

p	q	
T	T	
T	F	
F	T	
F	F	

q and not (p or q)

p	q	
T	T	
T	F	
F	T	
F	F	

Outline

- Compound Propositions
  - Negation/And/Or
  - Truth Tables
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- Predicates
  - Instantiation, For all, There exists
  - Nested Quantifiers

If-Then as a joiner

We’ve seen another way to combine propositions:

I bring an umbrella today.

I will stay dry.

If bring an umbrella today, **then** I will stay dry.

“if p, then q” sometimes written with a symbol: “p→q”

If-Then as a joiner

Given two propositions, you can join them together with “if … then …” .

p q

If p**, then** q

List of all possibilities for the **truth values** of p, q

p	q	If p then q
T	T	??
T	F	??
F	T	??
F	F	??

What is the truth value of:

“If I am a walrus, then $0 = 1$”?

“if p, then q” sometimes written with a symbol: “p→q”

If-Then as a joiner

Given two propositions, you can join them together with “if … then …” .

p

q

If p, **then** q

If bring an umbrella today, **then** I will stay dry.

List of all possibilities for the **truth values** of p, q

p	q	**If** p **then** q
T	T	
T	F	
F	T	
F	F	

If-Then as a joiner

To understand the truth table for **if/then**, think about a statement ***inside a for-all.***

For example:

“For all integers x, if $x > 5$, then x^2 is positive.”

This proposition (as a whole) is **true.**

“if p, then q” sometimes written with a symbol: “p→q”

If-Then as a joiner

To understand the truth table for **if/then**, think about a statement ***inside a for-all.***

For example:

“For all integers x, if $x > 5$, then x^2 is positive.”

So **for any choice of x**, this part should be a **true** if/then proposition.

“If $7 > 5$, then 7^2 is positive”

p: true

q: true

p	q	If p then q
T	T	
T	F	
F	T	
F	F	

“if p, then q” sometimes written with a symbol: “p→q”

If-Then as a joiner

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So **for any choice of x**, this part should be a **true** if/then proposition.

“If $7 > 5$, then 7^2 is positive”

p: true

q: true

p	q	If p then q
T	T	T
T	F	
F	T	
F	F	

“if p, then q” sometimes written with a symbol: “p→q”

If-Then as a joiner

To understand the truth table for **if/then**, think about a statement ***inside a for-all.***

For example:

“For all integers x, if $x > 5$, then x^2 is positive.”

So **for any choice of x**, this part should be a **true** if/then proposition.

“If $3 > 5$, then 3^2 is positive”

p: false

q: true

p	q	**If** p **then** q
T	T	**T**
T	F	
F	T	**T**
F	F	

"if p, then q" sometimes written with a symbol: "p→q"

If-Then as a joiner

To understand the truth table for **if/then**, think about a statement ***inside a for-all.***

For example:

"For all integers x, if $x > 5$, then x^2 is positive."

So **for any choice of x**, this part should be a **true** if/then proposition.

"If $0 > 5$, then 0^2 is positive"

p: false **q: false**

p	q	If p then q
T	T	T
T	F	
F	T	T
F	F	T

"if p, then q" sometimes written with a symbol: "p→q"

If-Then as a joiner

To understand the truth table for **if/then**, think about a statement ***inside a for-all.***

For example:

"For all integers x, if $x > 5$, then x^2 is positive."

So **for any choice of x**, this part should be a **true** if/then proposition.

There is no choice of x where "$x > 5$" is true, but "x^2 is positive" is false...

p	q	If p then q
T	T	T
T	F	??
F	T	T
F	F	T

If-Then as a joiner

To understand the truth table for **if/then**, think about a statement ***inside a for-all.***

For example:

"For all integers x, if $x > -5$, then x^2 is positive."

What do you think?

(a) Proposition is true
(b) Proposition is false
(c) I'm not sure

If-Then as a joiner

To understand the truth table for **if/then**, think about a statement ***inside a for-all.***

For example:

"For all integers x, if $x > -5$, then x^2 is positive."

What do you think?

(a) Proposition is true

(b) Proposition is false

(c) I'm not sure

Consider $x = 0$ …

“if p, then q” sometimes written with a symbol: “p→q”

If-Then as a joiner

To understand the truth table for **if/then**, think about a statement ***inside a for-all.***

For example:

“For all integers x, if $x > -5$, then x^2 is positive.”

if $0 > -5$, then 0^2 is positive

p	q	If p then q
T	T	T
T	F	F
F	T	T
F	F	T

Intuition Pitfalls in If/Then

Caution: “If p, then q” can be unintuitive when **p** is false!

What is the truth value of the proposition
“If I am a walrus, then $0 = 1$”

(A) True
(B) False

p	q	if p then q
T	T	T
T	F	F
F	T	T
F	F	T

Intuition Pitfalls in If/Then

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What is the truth value of the proposition
"If I am a walrus, then $\mathbf{0 = 1}$"

(A) True
(B) False

p	q	if p then q
T	T	T
T	F	F
F	T	T
F	F	T

Intuition Pitfalls in If/Then

If-Then is also unintuitive inside existential quantifiers.

What is the truth value of this proposition?
"There exists an integer x such that if $x = 3$, then x is even."

(A) True
(B) False

p	q	if p then q
T	T	T
T	F	F
F	T	T
F	F	T

Intuition Pitfalls in If/Then

If-Then is also unintuitive inside existential quantifiers.

What is the truth value of this proposition?
"There exists an integer x such that if $x = 3$, then x is even."

(A)True
(B)False

Consider $x = 5$.
"$x = 3$" is false
"x is even" is false
We found an example where the if/then is true

p	q	if p then q
T	T	T
T	F	F
F	T	T
F	F	T

“if p, then q” sometimes written with a symbol: “p↔q”

If and Only If

True Proposition:
For all integers x, x is even **if and only if** $x + 2$ is even.

“p **if and only if** q” means:
(If p, then q) **and** (If q, then p)

Another interpretation:

“p **if and only if** q” means:
p, q have the same truth value

p	q	p if and only if q
T	T	T
T	F	F
F	T	F
F	F	T

Lec 4 Handout: Compound Propositions & Predicates

p	not p
T	
F	

p	q	p and q
T	T	
T	F	
F	T	
F	F	

p	q	p or q
T	T	
T	F	
F	T	
F	F	

The ONLY time that
"if p, then q" is **false** is:

____

____

p	q	if p then q
T	T	
T	F	
F	T	
F	F	

p	q	p if and only if q
T	T	
T	F	
F	T	
F	F	

Exercise: Negation of “if p, then q”

- Complete the two columns given. Then find a compound expression that has the same truth values as “not (if p, then q)” but uses only: “not”, “and”, or “or”

p	q	if p then q	not (if p then q)	________________
T	T			
T	F			
F	T			
F	F			

Exercise: Negation of “if p, then q”

- Complete the two columns given. Then find a compound expression that has the same truth values as “not (if p, then q)” but uses only: “not”, “and”, or “or”

p	q	if p then q	not (if p then q)	**p and not q**
T	T	**T**	**F**	**F**
T	F	**F**	**T**	**T**
F	T	**T**	**F**	**F**
F	F	**T**	**F**	**F**

Handout

Exercises

Complete the truth tables.
(hint: revisit previous exercise)

p	q	________________	________________	if not (p and q), then q and not (p or q)
T	T			
T	F			
F	T			
F	F			

Exercise

- What is the truth table of “if not (p and q), then q and not (p or q)”?
  - Hint: revisit last exercise...

p	q	not (p and q)	q and not (p or q)	if not (p and q), then q and not (p or q)
T	T	**F**	**F**	**T**
T	F	**T**	**F**	**F**
F	T	**T**	**F**	**F**
F	F	**T**	**F**	**F**

Notice: Same truth table as “p and q”! More on this next lecture...

Outline

- Compound Propositions
  - Negation/And/Or
  - Truth Tables
  - If-Then/If and only if

- **Predicates**
  - **Instantiation, For all, There exists**
  - Nested Quantifiers

Predicates

A **proposition** is a declaration about the world that has a **truth value** (either true or false).

A **predicate** is a statement that has **unspecified variable(s)**, and which becomes a proposition once those variables are specified.

- "The integer x is even"
- "The capital of state x is city y"
- "$a^2 + b^2 = c^2$"
- "$2(x + 1) = 2x + 2$" ***True for any x, but not a prop until x is specified***

From Predicates to Propositions

P(x) = "x + 5 = 13"

There are **three main ways** to turn a predicate into a proposition:

Way #1: Instantiation

Specify input(s) for your predicate

P(8) = "$8 + 5 = 13$"

P(1) = "$1 + 5 = 13$"

“There exists x such that p(x)” sometimes written with a symbol: “$\exists$x p(x)”

From Predicates to Propositions

P(x) = “x + 5 = 13”

There are **three main ways** to turn a predicate into a proposition:

1. *Instantiation: pick a value for **x***

Way #2: “Existential Quantification”

Apply “There exists” in front

“There exists an integer **x** for which **x+5=13**”

Also specify a **domain** of **possible variables**

Domain: integers

"For all x, p(x)" sometimes written with a symbol: "$\forall$x p(x)"

From Predicates to Propositions

P(x) = "x + 5 = 13"

There are **three main ways** to turn a predicate into a proposition:

1. *Instantiation: pick a value for* ***x***
2. *Existential Quantification: "there exists* ***x*** *such that* ***P(x)*** *is true"*

Way #3: "Universal Quantification"

Apply "For all" in front

Again specify a **domain** of **possible variables**

"For all integers **x, x + 5 = 13**"

(false proposition)

Domain: integers

From Predicates to Propositions

P(x) = "x + 5 = 13"

There are **three main ways** to turn a predicate into a proposition:

1. *Instantiation: pick a value for **x***
2. *Existential Quantification: "there exists **x** such that **P(x)** is true"*
3. *Universal Quantification: "for all **x**, **P(x)** is true"*

Another View:

- **Existential Quantification =** all possible instantiations, joined with OR
  - "... **(-1 + 5 = 13) or (0 + 5 = 13) or (1 + 5 = 13) or (2 + 5 = 13) or ...**"

- **Universal Quantification =** all possible instantiations, joined with AND
  - "... **(-1 + 5 = 13) and (0 + 5 = 13) and (1 + 5 = 13) and (2 + 5 = 13) and ...**"

From Predicates to Propositions

Handout

3 ways to turn a predicate, ***P(x)***, into a proposition:

1. *Instantiation: pick a _____________________________*
2. *Existential Quantification: say __________________________* ***P(x)*** *is true*
3. *Universal Quantification: say ___________________________* ***P(x)*** *is true*

Domain Specification

$$P(x) = \text{“}2x = 7\text{”}$$

What is the truth value of **“there exists x such that P(x)?”**

(A) True
(B) False
(C) Not enough information
(D) Not sure

Domain Specification

$$P(x) = \text{“}2x = 7\text{”}$$

What is the truth value of **“there exists x such that P(x)?”**

(A) True
(B) False
(C) Not enough information
(D) Not sure

What is the **domain**?
What values of **x** are we allowed to consider here?

Domain Specification

P(x) = “2x = 7”

There exists an **integer** x such that P(x)

False

There exists a **real number** x such that P(x)

True (consider $x = 3.5$…)

Outline

- Compound Propositions
  - Negation/And/Or
  - Truth Tables
  - If-Then/If and only if

- Predicates
  - Instantiation, For all, There exists
  - **Nested Quantifiers**

Multivariable Predicates

- Predicates can also have **several variables**.
  - $P(x, y) = "x + y = 2"$
  - $C(x, y) = "city\ x\ is\ the\ capital\ of\ state\ y"$
  - $T(a, b, c) = "a^2 + b^2 = c^2"$

- You can apply **different quantifiers** to these variables if you want.

Exercise

$$P(x, y) = "x + y = 2"$$

A

There exists an integer $\boldsymbol{x}$ such that **for all** integers $\boldsymbol{y}$, $P(x, y)$

B

For all integers $\boldsymbol{y}$, **there exists** an integer $\boldsymbol{x}$ such that $P(x, y)$

1. Both are true
2. A is true, B is false
3. A is false, B is true
4. Both are false
5. Not sure

Exercise

$$P(x,y) = "x + y = 2"$$

A

There exists an integer x such that **for all** integers y, $P(x,y)$

B

For all integers y, **there exists** an integer x such that $P(x,y)$

1. Both are true
2. A is true, B is false
3. **A is false, B is true**
4. Both are false
5. Not sure

Exercise

$$P(x, y) = "x + y = 2"$$

A

There exists an integer $\boldsymbol{x}$ such that **for all** integers $\boldsymbol{y}$, $P(x, y)$

There is no specific example of an integer x

(like $x = 1, x = 3.5, x = 12$...)

where, for any integer y you add to this x,

you get 2.

False
(this is not a proof, just intuition)

B

For all integers $\boldsymbol{y}$, **there exists** an integer $\boldsymbol{x}$ such that $P(x, y)$

Exercise

$$P(x, y) = "x + y = 2"$$

A

There exists an integer $\boldsymbol{x}$ such that **for all** integers $\boldsymbol{y}$, $P(x, y)$

There is no specific example of an integer x
(like $x = 1, x = 3.5, x = 12 \ldots$)
where, for any integer y you add to this x, you get 2.

False
(this is not a proof, just intuition)

B

For all integers $\boldsymbol{y}$, **there exists** an integer $\boldsymbol{x}$ such that $P(x, y)$

Proof:

- Let y be an arbitrary integer

Exercise

$$P(x,y) = "x + y = 2"$$

A

There exists an integer x such that **for all** integers y, $P(x,y)$

There is no specific example of an integer x

(like $x = 1, x = 3.5, x = 12\ldots$)

where, for any integer y you add to this x, you get 2.

False
(this is not a proof, just intuition)

B

For all integers y, **there exists** an integer x such that $P(x,y)$

Proof:

- Let y be an arbitrary integer
- Consider the specific integer $x = 2 - y$

Exercise

$$P(x,y) = "x + y = 2"$$

A

There exists an integer x such that **for all** integers y, $P(x,y)$

There is no specific example of an integer x
(like $x = 1, x = 3.5, x = 12 \ldots$)
where, for any integer y you add to this x, you get 2.

False
(this is not a proof, just intuition)

B

For all integers y, **there exists** an integer x such that $P(x,y)$

Proof:

- Let y be an arbitrary integer
- Consider the specific integer $x = 2 - y$
- We have $x + y = (2 - y) + y = 2$.

Exercise

$$P(x, y) = "x + y = 2"$$

A

There exists an integer x such that for all integers y, $P(x, y)$

There is no specific example of an integer x

(like $x = 1, x = 3.5, x = 12$...)

where, for any integer y you add to this x, you get 2.

False

(this is not a proof, just intuition)

B

For all integers y, there exists an integer x such that $P(x, y)$

Proof:

- Let y be an arbitrary integer
- Consider the specific integer $x = 2 - y$
- We have $x + y = (2 - y) + y = 2$.

True

Nested Quantifiers

How could we quantify a **2-variable** predicate to a proposition?

1. There exists **x** such that there exists **y** such that **P(x, y).**
2. There exists **y** such that there exists **x** such that **P(x, y).**

3. There exists **x** such that for all **y**, **P(x, y).**
4. For all **y**, there exists **x** such that **P(x, y).**

5. For all **x**, there exists **y** such that **P(x, y).**
6. There exists **x** such that for all **y**, **P(x, y).**

7. For all **x**, for all **y**, **P(x, y)**.
8. For all **y**, for all **x**, **P(x, y)**.

Does switching the order actually matter?

Nested Quantifiers

How could we quantify a **2-variable** predicate to a proposition?

1. There exists **x** such that there exists **y** such that **P(x, y).**
2. There exists **y** such that there exists **x** such that **P(x, y).**

THE SAME

3. There exists **x** such that for all **y**, **P(x, y).**
4. For all **y**, there exists **x** such that **P(x, y).**

5. For all **x**, there exists **y** such tha
6. There exists **x** such that for all **y**

7. For all **x**, for all **y**, **P(x, y)**.
8. For all **y**, for all **x**, **P(x, y)**.

Same quantifier (both for-all or both there-exists):
You can swap the order without changing meaning.

Different quantifiers (one for-all, other there-exists):
Swapping the order changes the meaning!

Nested Quantifiers – additional exercises

Handout

1. Let **P(x,y) = “4x – y = 0”,** domain = integers

Determine whether each of the following propositions is true or false:

1. There exists **x** such that there exists **y** such that **P(x, y).** True / False
2. There exists **x** such that for all **y**, **P(x, y).** True / False
3. For all **y**, there exists **x** such that **P(x, y).** True / False
4. For all **x**, there exists **y** such that **P(x, y).** True / False
5. There exists **y** such that for all **x**, **P(x, y).** True / False
6. For all **x**, for all **y**, **P(x, y)**. True / False

Nested Quantifiers – additional exercises

Handout

P(x,y) = “the square at row **x**, column **y** is shaded”. Shade squares so that …

(a)

“*There exists* **x**, such that *there exists* **y**, such that **P(x,y)**” is **TRUE**

(b)

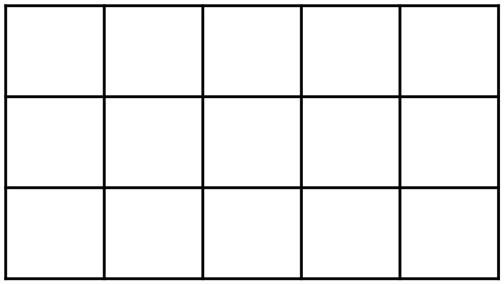
“*There exists* **x**, such that *there exists* **y**, such that **P(x,y)**” is **FALSE**

Nested Quantifiers – additional exercises

Handout

P(x,y) = "the square at row **x**, column **y** is shaded". Shade squares so that …

(c)

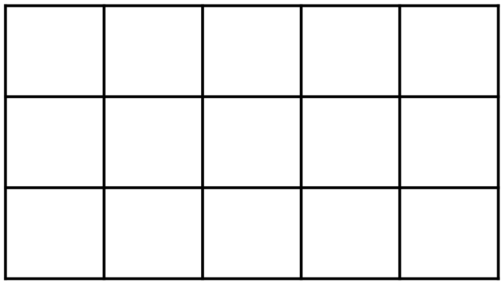


"*For all* **x**, *for all* **y**, **P(x,y)**"

is **TRUE**

Question: Does P(x,y) have to be **true** for *every* pair of (x,y)?

(d)



"*For all* **x**, *for all* **y**, **P(x,y)**"

is **FALSE**

Question: Does P(x,y) have to be **false** for *every* pair of (x,y)?

Nested Quantifiers – additional exercises

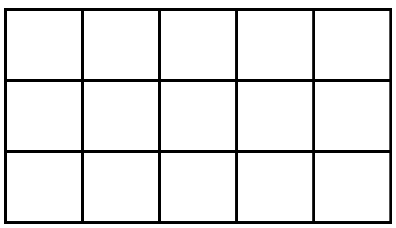
Handout

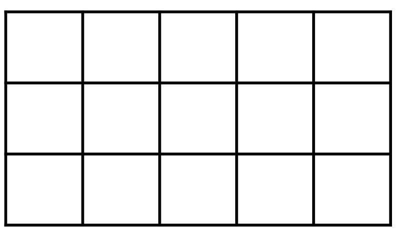
Select the logical expression that matches the English statement.

Then provide *two different shadings* that satisfy the statement.

(e) *English*: **The grid has an entire column that is shaded.**

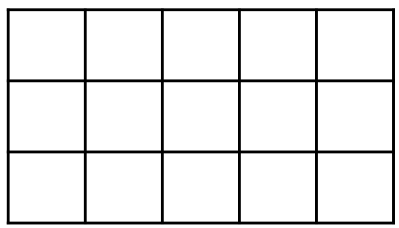
Logic: ______________

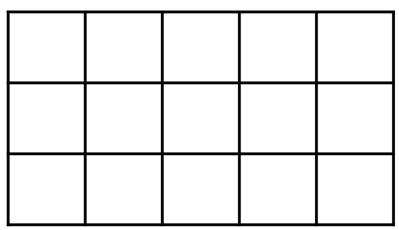




(f) *English*: **Every column has at least one shaded square.**

Logic: _____________





Answer options:

A. There exists an **x**, such that for all **y**, **P(x, y).**

B. For all **y**, there exists an **x**, such that **P(x, y).**

C. For all **x**, there exists a **y**, such that **P(x, y).**

D. There exists a **y**, such that for all **x**, **P(x, y).**

Wrapup

- **Today:** Basic ways to create propositions and analyze their truth value
  - not, and, or, if-then, if-and-only-if, quantified predicates

- These are not the only ways!
  - Other examples not covered today: xor, nor, nand, …

- **Next Time:**
  - Translating English sentences into logic