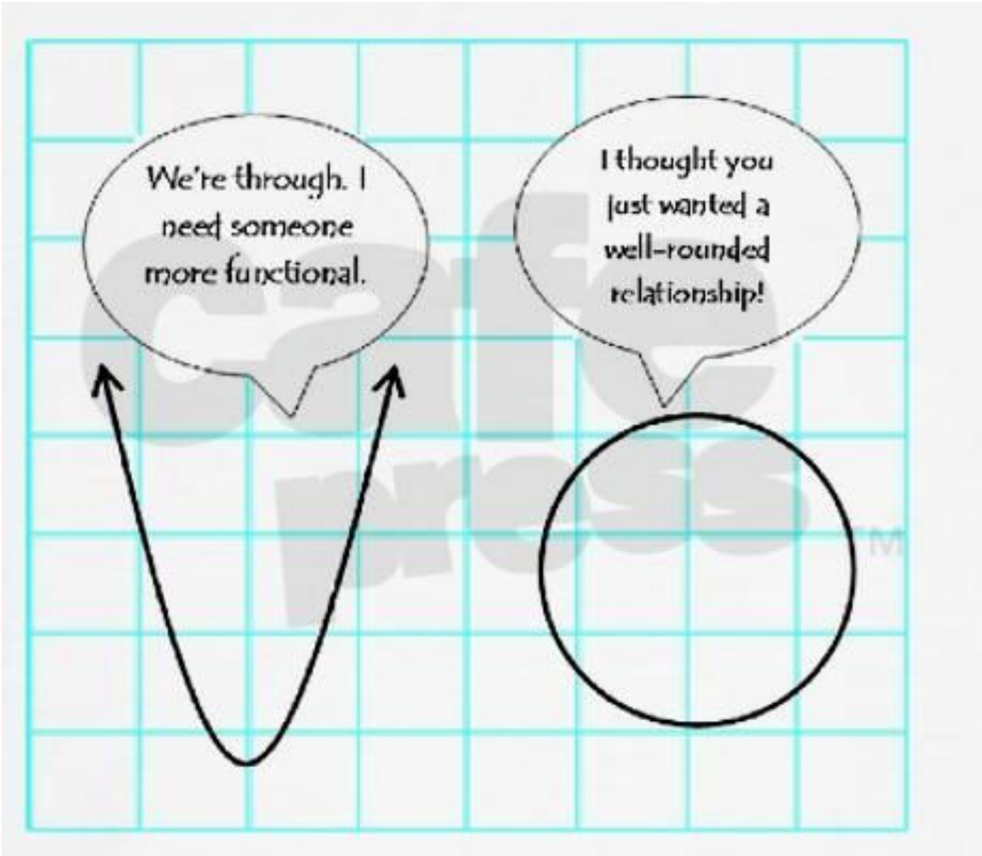
Relations and their Properties (Ch 9)

UM EECS 203 Lecture 16



Welcome Back

- Hope you had a good break!
- Reminders
  - Homework due Thursday
  - Regrade requests for Exam 1 due Thursday
- Individual office-hour appointments available with faculty
  - This week and next
  - If you want to discuss your exam, your progress in the course, or anything else in a 1-on-1 format
  - See 203 Office Hour calendar for times and info
- Exam 2 is Wednesday, March 23
  - 2 weeks from tomorrow
  - More info coming in Thursday's lecture

World Events

- The war in Ukraine is a source of anxiety and stress for many of us.
- We recognize that some students are more deeply affected by these current events than others.
- We want to support you
  - If you and/or your work is being affected, please reach out to us via the Admin Form so that we can look into potential supports and accommodations.

Learning Objectives

After today's lecture (and this week's readings, discussion & homework), you should be able to:

- **Technical vocab:** relation, reflexive, symmetric, anti-symmetric, transitive, reflexive closure, symmetric closure, transitive closure

- Explain the difference between a relation and a function
- Determine the properties of a given relation
- Provide examples of relations with specified combinations of properties
- Prove/disprove that a given relation has a particular property

- Compute reflexive closure and symmetric closure
- Compose two relations
- Compose a relation with itself to produce higher powers
- Compute transitive closure (for small relations)

Outline

- **Relations**
  - **Definition**
  - **Representations of relations**
- Properties
  - Reflexive, symmetric, antisymmetric, transitive
  - Asymmetric, irreflexive
- Relations as Sets
- Closure
  - Reflexive closure, symmetric closure
  - (Detour: composing relations & powering relations)
  - Transitive closure

May spill over into next lecture

Motivation

- Relations represent “knowledge of the world.”
- Very relevant to real-world applications (e.g., Relational Database)
- Highly relevant to graphs (e.g., social network)

Functions vs Relations

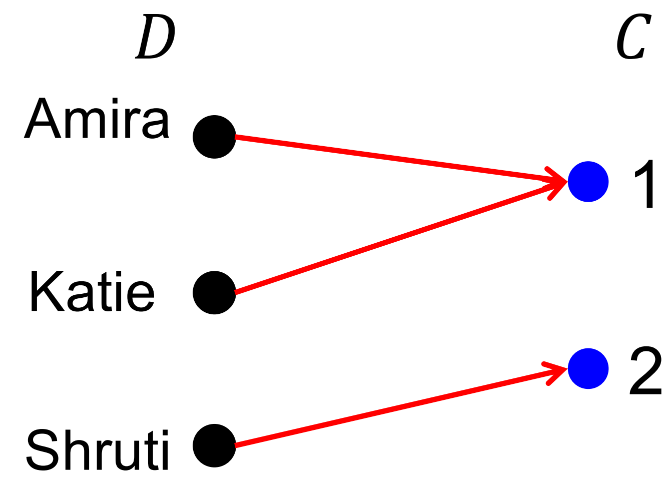
A ***function*** maps each element of the domain to exactly one element in the co-domain.

Domain and co-domain are sets.

Q. why exactly one?

A. because that's what a function is.

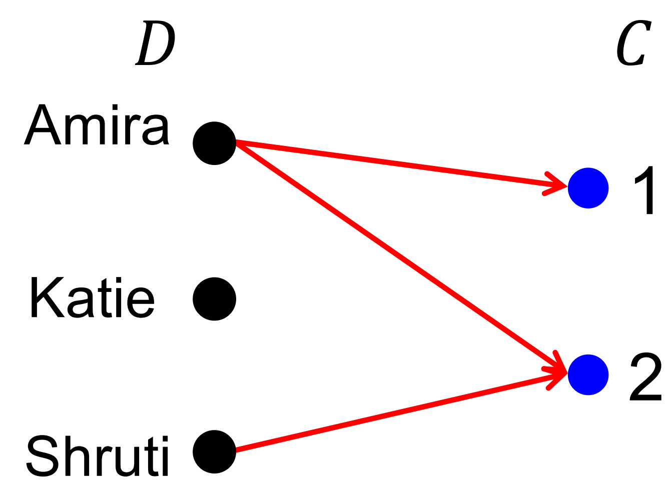
the enrolled-in-discussion-section function



A ***relation*** does not have this restriction.

In a ***relation***, an element of the domain can map to *zero or more* elements of the codomain.

the attends-discussion-section **relation**



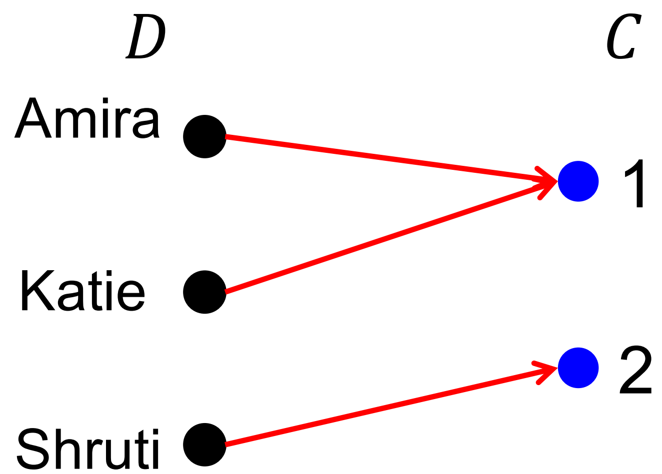
Functions vs Relations

A ***function*** maps each element of the domain to exactly one element in the co-domain.
Domain and co-domain are sets.

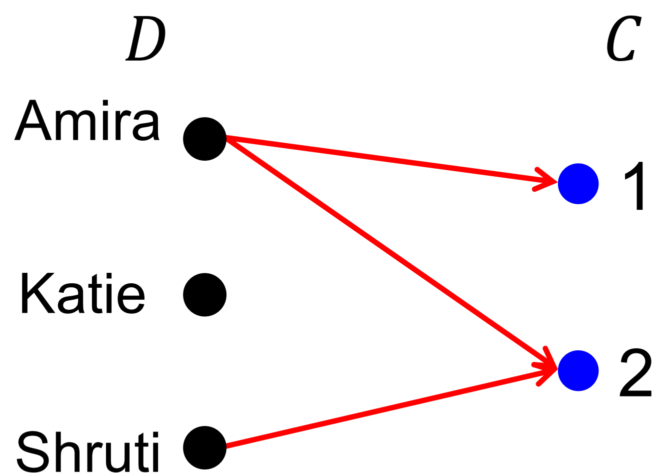
Definition:
A ***binary relation*** R between sets D and C is a subset of $D \times C$.

Key feature:
An element of D can be related to *zero or more* elements of C.

the enrolled-in-discussion-section function



the attends-discussion-section **relation**

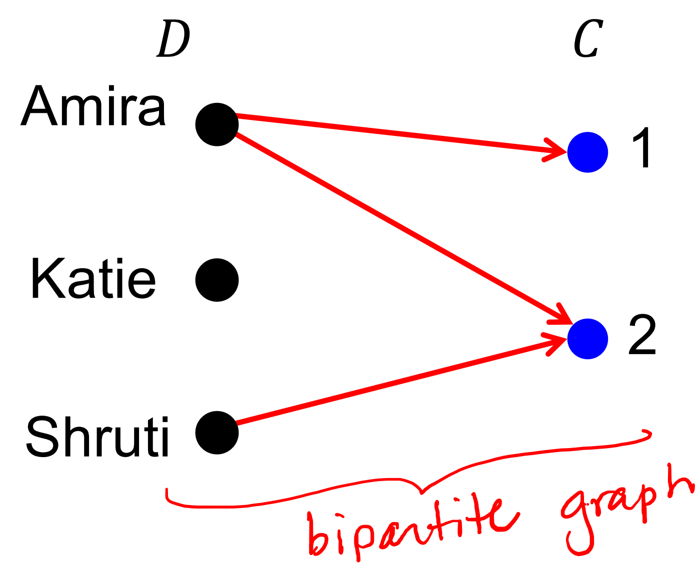


Representations of Relations

A ***binary relation*** R between sets D and C is a subset of $D \times C$.

* An element of D can be related to *zero or more* elements of C.

the attends-discussion-section **relation**



Ways to represent a relation R:

1. **Graph** representation
2. **Set** representation $R = \{ (\text{Amira},1), (\text{Amira},2), (\text{Shruti},2) \}$
3. **0-1 Matrix** representation

elements of codomain

elements of domain

R	1	2
Amira	1	1
Katie	0	0
Shruti	0	1

can be used interchangeably:

"x relates to y (by R)"

"xRy"

"(x,y) ∈ R

"R(x,y)"

Relations on the same set

Very common: Domain = Co-Domain

Example:

pMq ≡ "p and q have been in a movie together"

Binary Relation

- ↔ Predicate
- ↔ Directed Graph
- ↔ Set of ordered pairs
- ↔ 0–1 Matrix

Lecture 16 Handout: Relations & Properties

A **binary relation** R between D and C is a **subset** of $D \times C$

- Unlike a function, in a **relation** an element of the domain can relate to zero or more elements of the codomain.
  - Very common: $D = C$

Example: Consider the relation $\leq$ over {-5, 1, 2} $\leftrightarrow$ xRy iff $x \leq y$

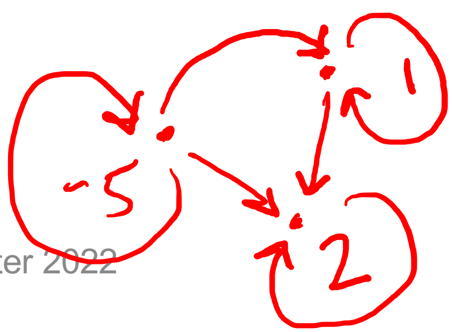
a) Domain $D = \{-5, 1, 2\}$ Codomain $C = \{-5, 1, 2\}$

b) $D \times C = \{(-5,-5), (-5,1), (-5,2), (1,-5), (1,1), (1,2), (2,-5), (2,1), (2,2)\}$

c) Set representation: $R = \{(-5,-5), (-5,1), (-5,2), (1,1), (1,2), (2,2)\}$

d) Graph representation

directed graph
"digraph"



e) Matrix representation

$$\begin{array}{c|ccc} & -5 & 1 & 2 \leftarrow \text{codomain} \\ \hline \text{domain}\ -5 & & & \\ 1 & & & \\ 2 & & & \end{array} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Outline

- Relations
  - Definition
  - Representations of relations
- **Properties**
  - **Reflexive, symmetric, antisymmetric, transitive**
  - **Asymmetric, irreflexive**
- Relations as Sets
- Closure
  - Reflexive closure, symmetric closure
  - (Detour: composing relations & powering relations)
  - Transitive closure

May spill over into next lecture

Properties of Relations [on the same set]

Very common: Domain = Co-Domain

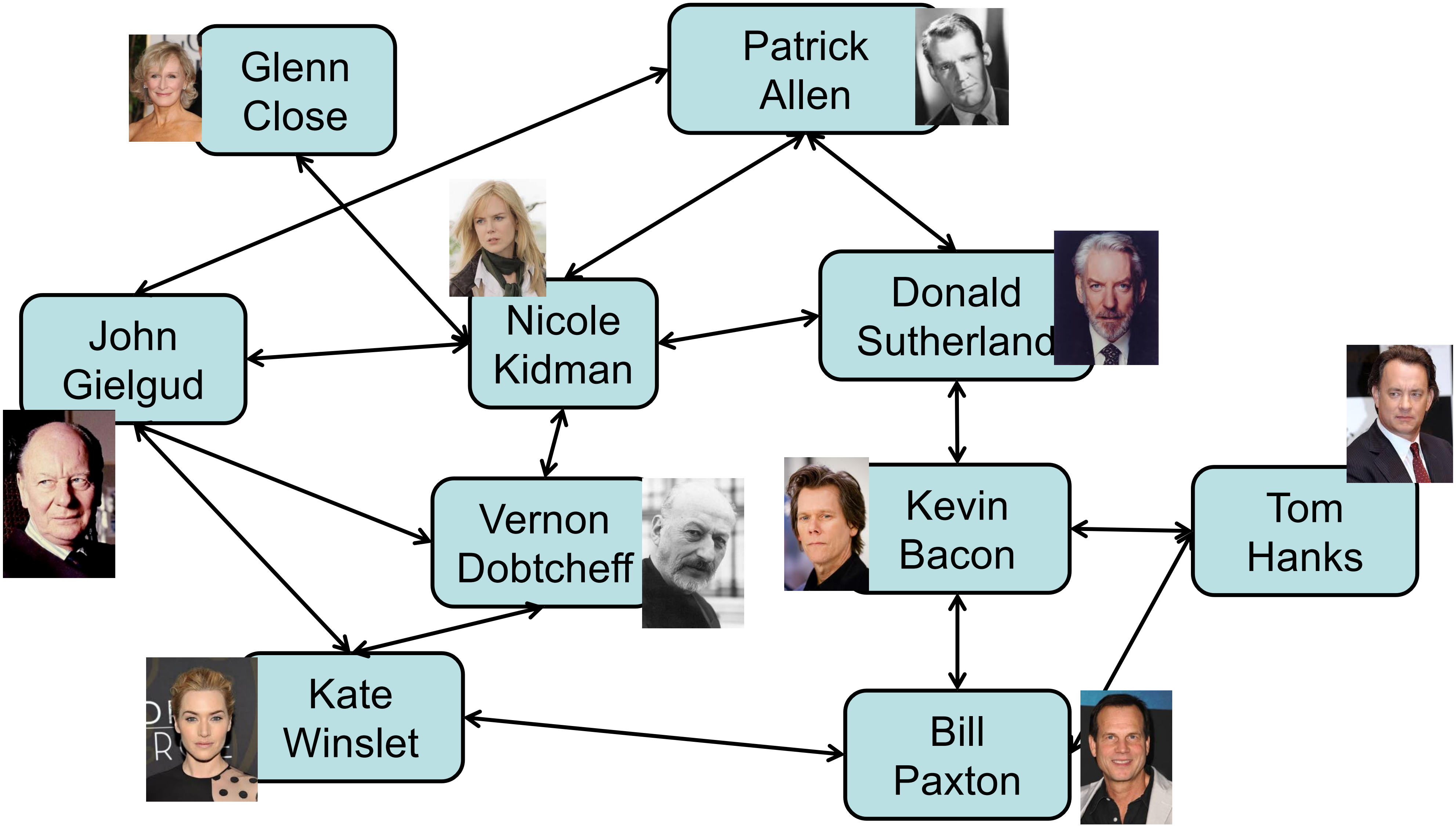
$\leq$ on $\{-5, 1, 2\}$

R is ***REFLEXIVE***	For all x:	xRx	yes
R is ***SYMMETRIC***	For all x,y:	$xRy \to yRx$	no
R is ***ANTISYMMETRIC***	For all x,y:	$(xRy \wedge yRx) \to x=y$	yes
R is ***TRANSITIVE***	For all x,y,z:	$(xRy \wedge yRz) \to xRz$	yes
R is **asymmetric**	For all x,y:	$xRy \to y \not{R} x$	no
R is **irreflexive**	For all x:	$x \not{R} x$	no

Relations on the same set

pMq ≡ p&q in a movie together

(representation of M is shown below, but not all edges are shown to avoid clutter)



Properties Example

Very common: Domain = Co-Domain

pMq ≡ p&q in a movie together

Is M ***REFLEXIVE?*** Defn.: For all p: pMp

Is M ***SYMMETRIC?*** Defn.: For all p,q: pMq $\leftrightarrow$ qMp

Is M ***ANTISYMMETRIC?*** Defn.: For all p,q: (pMq $\wedge$ qMp) $\rightarrow$ p=q

Is M ***TRANSITIVE?*** Defn.: For all p,q,r: (pMq $\wedge$ qMr)$\rightarrow$pMr

Properties Example

Very common: Domain = Co-Domain

$pMq \equiv$ p&q in a movie together

Is M ***REFLEXIVE?*** Defn.: For all p: pMp

Yes... If our domain is people who have been in a movie

Not if our domain is all people

Is M ***SYMMETRIC?*** Defn.: For all p,q: $pMq \leftrightarrow qMp$

Yes

Is M ***ANTISYMMETRIC?*** Defn.: For all p,q: $(pMq \wedge qMp) \rightarrow p=q$

No

Is M ***TRANSITIVE?*** Defn.: For all p,q,r: $(pMq \wedge qMr) \rightarrow pMr$

No

Properties Example

Very common: Domain = Co-Domain (in this case, positive integers)

"the Divides relation": **aDb ≡ a divides b** $\leftrightarrow$ b is a multiple of a

Is D ***REFLEXIVE?*** Defn.: For all a: aDa

Is D ***SYMMETRIC?*** Defn.: For all a,b: aDb $\leftrightarrow$ bDa

Is D ***ANTISYMMETRIC?*** Defn.: For all a,b: (aDb $\wedge$ bDa) $\rightarrow$ a=b

Is D ***TRANSITIVE?*** Defn.: For all a,b,c: (aDb $\wedge$ bDc)$\rightarrow$aDc

Properties Example

Very common: Domain = Co-Domain (in this case, positive integers)

"the Divides relation": **aDb ≡ a divides b** ⟷ b is a multiple of a

Is D ***REFLEXIVE?*** Defn.: For all a: aDa

Yes: $a = 1 \cdot a$

Is D ***SYMMETRIC?*** Defn.: For all a,b: aDb ↔ bDa

No: $2D4$, but $4\overline{D}2$

Is D ***ANTISYMMETRIC?*** Defn.: For all a,b: (aDb ∧ bDa) → a=b

Yes: if aDb and bDa, then $a \leq b$ and $b \leq a$, so a= b

Is D ***TRANSITIVE?*** Defn.: For all a,b,c: (aDb ∧ bDc)→aDc

Yes: if aDb and bDc, then $b = k_1 a$ and $c = k_2 b$, so $c = k_2 k_1 a$ and thus aDc

Which properties do these satisfy?

Relation	Domain	Ref.	Sym.	Antisym	Trans.
$<$	$\mathbb{R}$	✗	✗	✓ F → _ ≡ T	✓
$\subseteq$	sets	✓	✗	✓	✓
$=$	$\mathbb{Z}$	✓	✓	✓	✓
"has a non-empty intersection with"	sets	✗ False for $\emptyset$	✓	✗	✗
"is a sister of"	people	✗	✗	✗	✗ when $x = z$
"is a sibling of"	people	✗	✓	✗	✗
"is a descendant of"	people	✗	✗	✓ F → _ ≡ T	✓
"is divisible by"	$\mathbb{Z}$	✓	✗	✗ Ex: $-4, 4$	✓

a is divisible by b $\longleftrightarrow$ a is a multiple of b

Warnings: Properties of Binary Relations

The names for properties of relations can get confusing!

- ***antisymmetric*** : *not* equivalent to "not symmetric". Meaning: it's never the case for $a \neq b$ that both *aRb* and *bRa* hold.

- ***asymmetric*** : also *not* equivalent to "not symmetric". Meaning: it's never the case that both *aRb* and *bRa* hold.

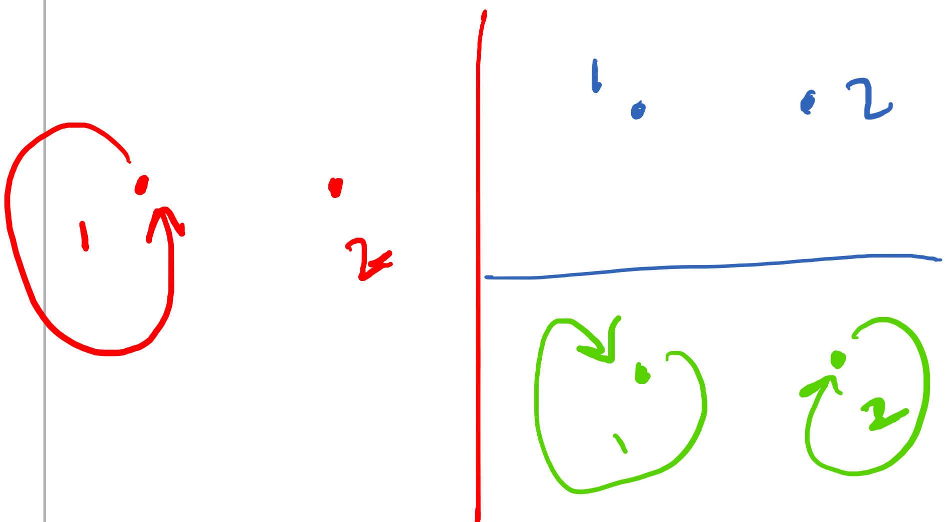
  (an asymmetric relation is also antisymmetric)

- ***irreflexive*** : *not* equivalent to "not reflexive". Meaning: it's never the case that *aRa* holds.

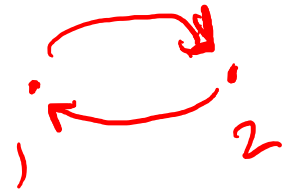
See textbook

Exercise: Draw a graph for a relation that is

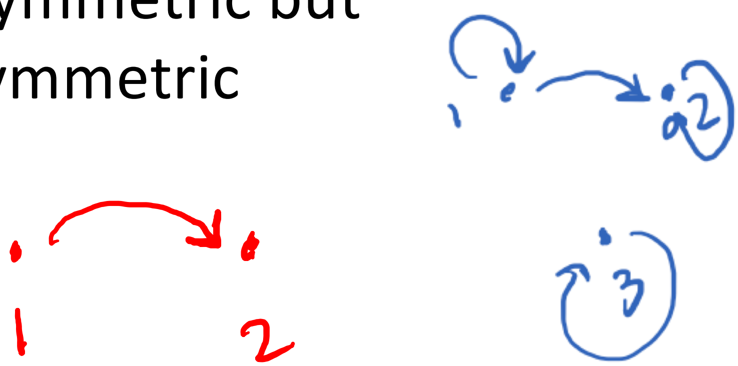
Symmetric and antisymmetric



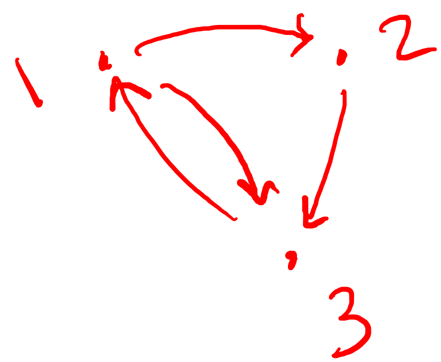
Symmetric but
not antisymmetric



Not symmetric but
antisymmetric



Not symmetric and
not antisymmetric



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May spill over into next lecture

Relations are Sets: $\cap, \cup, \oplus, -, \bar{\ }$

Because relations are just sets, all the usual set theoretic operations are defined between relations that are subsets of the same Cartesian product.

Q: Suppose we have relations on {1,2} given by $R = \{(1,1), (2,2)\}$, $S = \{(1,1),(1,2)\}$. Find:

1. The union $R \cup S$
2. The intersection $R \cap S$
3. The symmetric difference $R \oplus S$
4. The difference R-S
5. The complement $\bar{R}$

Relations are Sets: ∩,∪, ⊕, - , ‾

Suppose we have relations on {1,2} given by

R = {(1,1), (2,2)}, S = {(1,1),(1,2)}. Find:

1. The union $R \cup S$ $= \{(1,1), (1,2), (2,2)\}$

2. The intersection $R \cap S$ $= \{(1,1)\}$

3. The symmetric difference $R \oplus S$ $= \{(1,2), (2,2)\}$

4. The difference $R - S$ $= \{(2,2)\}$

5. The complement of R: $\bar{R}$ $= \mathcal{U} - R = \{(1,2), (2,1)\}$

$\mathcal{U} = \{1,2\} \times \{1,2\}$