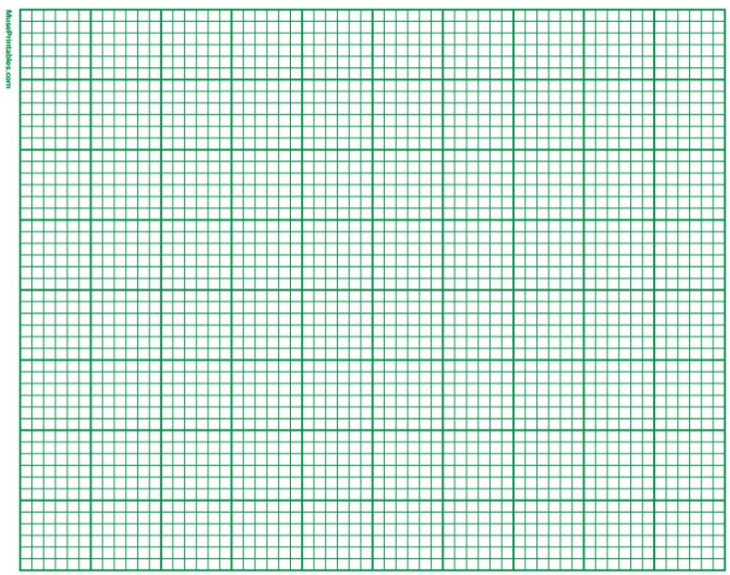
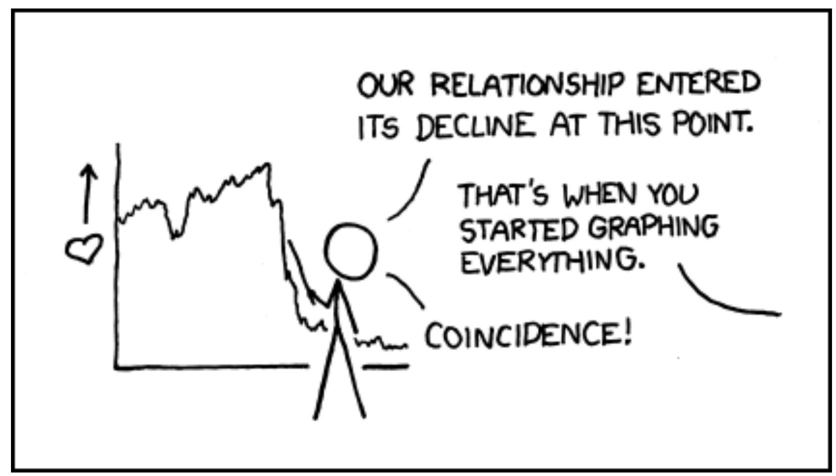
Lecture 18
Introduction to Graphs

$\neq$



$\neq$



Learning Objectives: Lec 18

After today's lecture (and the associated readings, discussion, & homework), you should know:

- **Types of graphs:** undirected, directed
- **Representation of graphs:** Sets of vertices, sets of edges.
- **Terminology and lingo:** bipartite, degree, cycle, tree, spanning tree, connected, connected component.
- **Facts about graphs:** the handshake lemma, characterization of bipartite graphs as 2-colorable, number of edges in a spanning tree.

Outline

- **Definition of a graph**
  - **Variants: undirected, directed**
  - **Variants: simple graphs, multigraphs, and loops.**
  - **A graph vs. a drawing of a graph**
- Graphs and Relations
- Degrees and the Handshake Theorem
- Special graphs: cliques, cycles, hypercubes
- Paths and connected components
- Spanning trees
- Bipartite graphs

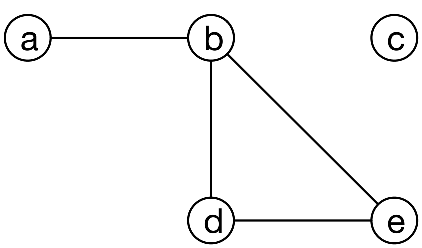
Graphs

The singular form is "vertex" just like index/indices, codex/codices, vortex/vortices, etc. "Vertice" is not a word.

- A way to represent things and their pairwise relationships.
- Consists of a pair of sets, one of "vertices" and one of "edges".

- $G = (V, E)$ is an ***undirected graph*** if
  - V is some set (vertices)
  - E is a set of two-element subsets of V (edges)

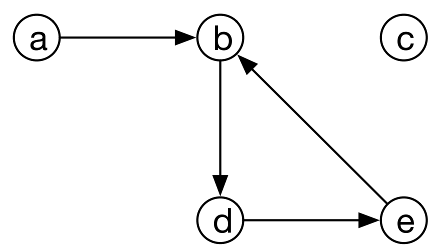
  - $V = \{a, b, c, d, e\}$
  - $E = \{\{a, b\}, \{b, d\}, \{d, e\}, \{b, e\}\}$



- $G = (V, E)$ is a ***directed graph*** if
  - V is some set (vertices)
  - $E \subseteq V \times V$ (subset of ordered pairs of elements from V)

  - $V = \{a, b, c, d, e\}$
  - $E = \{(a, b), (b, d), (d, e), (e, b)\}$

The edge (a, b) is directed FROM a TO b.



Graphs and Social Networks

- Facebook friend graph $G_{FB} = (V, E_{FB})$
  - V : the set of all people on Earth.
  - $\{u, v\} \in E_{FB}$ if u and v are friends (symmetric/mutual relationship)

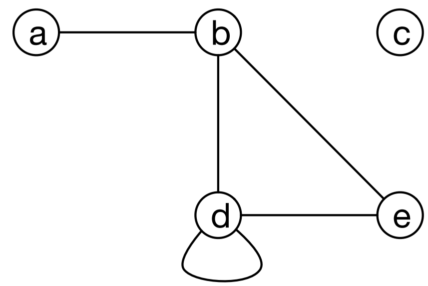
Undirected

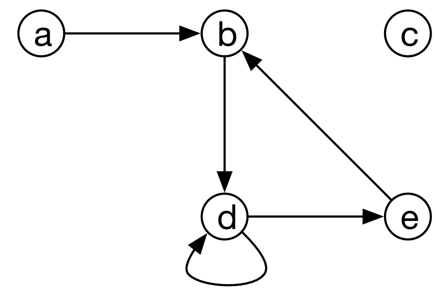
- Twitter network $G_{TW} = (V, E_{TW})$
  - V : the set of all people on Earth.
  - $(u, v) \in E_{TW}$ if u ***follows*** v.
    - Asymmetrical relationship. Not equivalent to v following u!

Directed

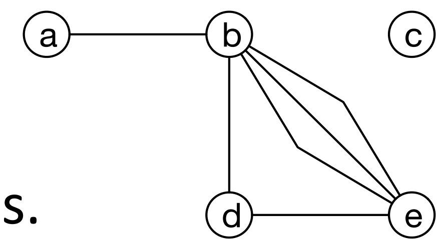
Variants of Graphs

- Some graphs have **"loop"** edges



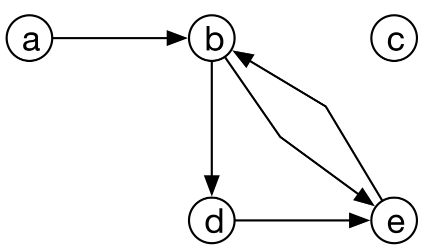


- In ***multigraphs***, edges can have multiplicity:



- ***Simple*** graphs have no loops or multiple edges.

- Is this a ***simple***, ***directed*** graph?

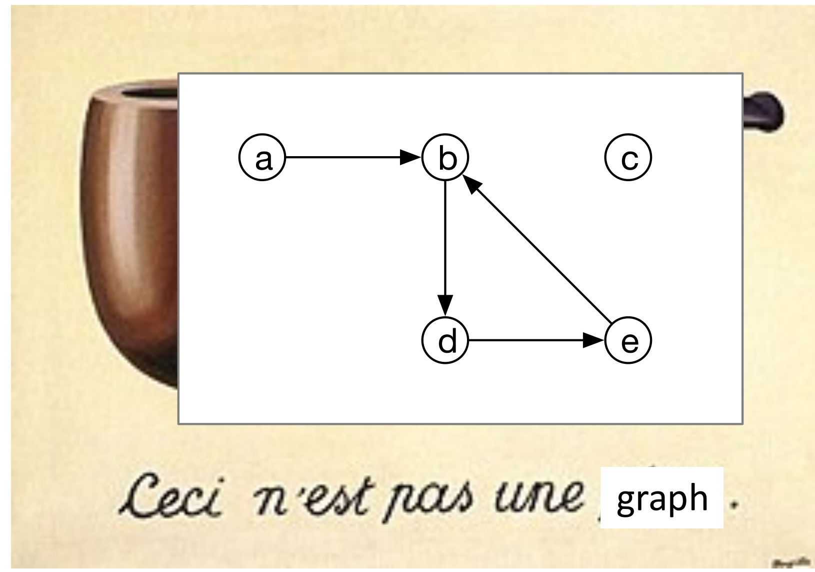


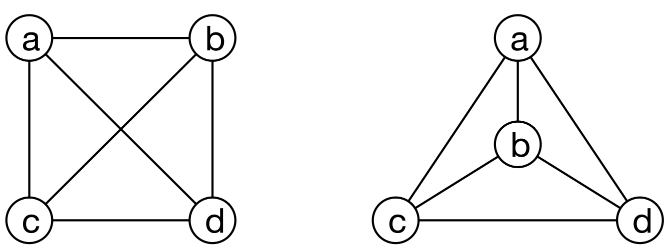
$\boxed{\text{(a) YES! (b) NO!}}$

(b, e) and (e, b) are *distinct edges*, not an edge with multiplicity 2.

Graphs vs. Drawings of Graphs

- A ***graph*** $G = (V, E)$ consists of sets of vertices and edges.
- A ***drawing*** of a graph is a diagram consisting of dots and lines/arrows.
- Every graph can be drawn and every (legal) drawing corresponds to a graph, but they're not the same thing!
  - Two drawings of the same graph $G = (V, E)$,
    $V = \{a, b, c, d\}, \quad E = \big\{\{x, y\} \,\big|\, x, y \in V, x \neq y\big\}$





Outline

- Definition of a graph
    - Variants: undirected, directed
    - Variants: simple graphs, multigraphs, and loops.
    - A graph vs. a drawing of a graph
- **Graphs and Relations**
- Degrees and the Handshake Theorem
- Special graphs: cliques, cycles, hypercubes
- Paths and connected components
- Spanning trees
- Bipartite graphs

Graphs and Relations

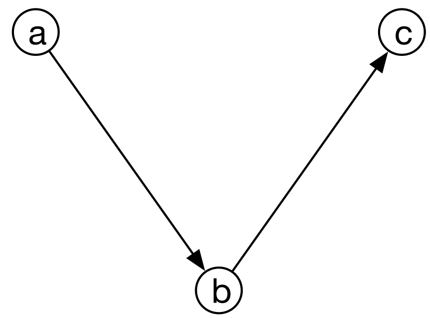
- Let V be a set and $E \subseteq V \times V$ be a relation.
- If E is ***symmetric*** ($\forall a, b \in V.\,(a, b) \in E \leftrightarrow (b, a) \in E$) then you can often regard the graph $G = (V, E)$ as begin *undirected*.
  - Two directed edges $(a, b), (b, a)$ very much like an undirected edge $\{a, b\}$.

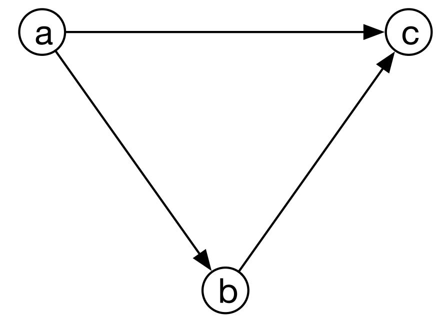
- If E is ***irreflexive*** and ***not symmetric*** then what can you say about $G = (V, E)$?
  - (a) it's an undirected graph.
  - (b) it's a simple undirected graph.
  - (c) it's a directed graph.
  - (d) it's a simple directed graph.
  - (e) it's a directed multigraph.

Irreflexive only means that $\forall a \in V.\,(a, a) \notin E$, i.e., no loops are allowed. I.e., it is a ***simple graph***.

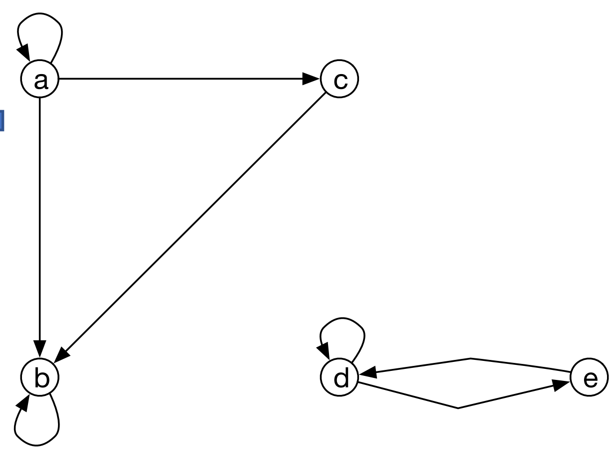
Graphs and Relations

- Let V be a set and $E \subseteq V \times V$ be a relation.
- If E is ***transitive***, what does that look like in terms of the graph?
  - $\forall a, b, c. (a, b), (b, c) \in E \rightarrow (a, c) \in E$





- Is this directed graph transitive?
  - (a) YES!
  - (b) NO!



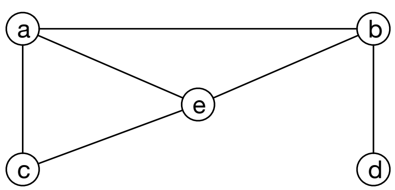
$(e, d), (d, e) \in E \rightarrow (e, e) \in E$.
However, the loops $(a, a), (b, b) \in E$ don't "need" to be there.

Outline

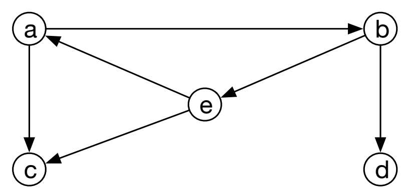
- Definition of a graph
  - Variants: undirected, directed
  - Variants: simple graphs, multigraphs, and loops.
  - A graph vs. a drawing of a graph
- Graphs and Relations
- **Degrees and the Handshake Theorem**
- **Special graphs: cliques, cycles, hypercubes**
- Paths and connected components
- Spanning trees
- Bipartite graphs

Degrees, Indegrees, and Outdegrees

- In a simple graph, vertices u, v are "adjacent" or "neighbors" if
  - $\{v, u\} \in E$ (undirected graph) or $(v, u) \in E$ (directed).
- Here is some useful notation for talking about neighborhoods:
  - $N(u) = \{v \mid \{u, v\} \in E\}$ (the "neighborhood of u" in an undirected graph)
  - $N^+(u) = \{v \mid (u, v) \in E\}$ (the "out-neighborhood of u" in a directed graph)
  - $N^-(u) = \{v \mid (v, u) \in E\}$ (the "in-neighborhood of u" in a directed graph)



$N(e) = \{a, b, c\}$



$N^+(e) = \{a, c\}$

$N^-(e) = \{b\}$

- The ***degree*** of a vertex is the number of adjacent edges.
  - $\deg(u) = |N(u)|$ (undirected graphs)
  - $\deg^+(u) = |N^+(u)|$ (out-degree in directed graphs)
  - $\deg^-(u) = |N^-(u)|$ (in-degree in directed graphs)

The Handshake Theorem

In an undirected graph $G = (V, E)$, what is $\sum_{v \in V} \deg(v)$?

The Handshake Theorem

- **Theorem.** In a simple undirected graph $G = (V, E)$,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

- **Proof.** Every edge $\{u, v\} \in E$ contributes 1 to $\deg(u)$ and 1 to $\deg(v)$.

Does this Theorem hold for ***non-simple*** graphs?
How should we define $\deg(v)$?

- **Theorem.** Similarly, in a directed graph $G = (V, E)$,

$$\sum_{v \in V} \deg^{+}(v) = \sum_{v \in V} \deg^{-}(v) = |\mathrm{E}|$$

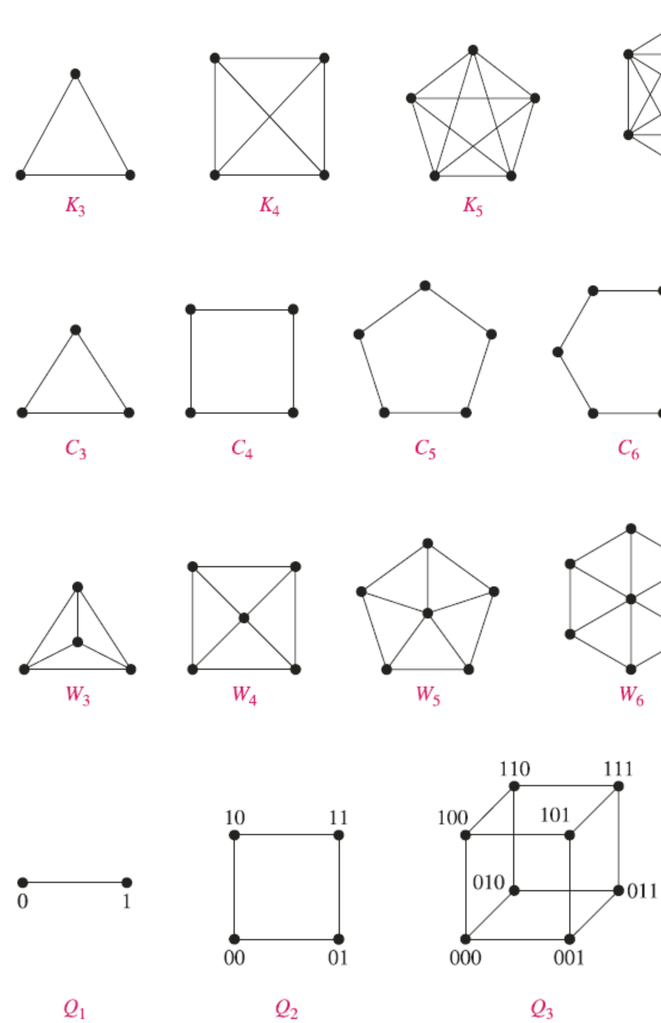
Degree sequences

- I'm thinking of a simple undirected graph with 6 vertices whose degrees are $1,2,2,3,3,4$. How many edges does it have?
  - (a) 13
  - (b) 15
  - (c) 26
  - (d) 30
  - (e) there is no such graph

- **Corollary** (of the Handshake Theorem). Every graph has an ***even number*** of vertices with ***odd degree***.
- **Proof by contradiction.** If one had an odd number of vertices with odd degree, then $\sum_{v\in V} \deg(v)$ would be odd, but $2|E|$ is clearly even, a contradiction.

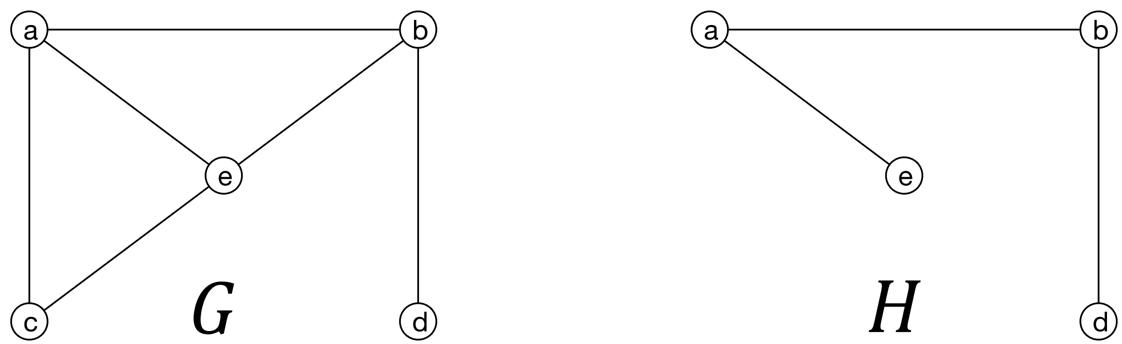
Special Undirected Graphs

- K_n : the complete graph on n vertices.
- C_n : the cycle on n vertices.
- ”Wheels”
- ”Hypercubes”

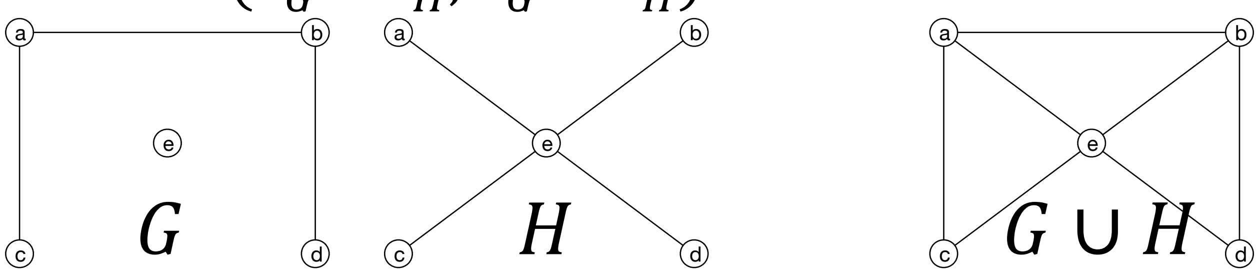


Subgraphs and Disjoint Unions

- $H = (V_H, E_H)$ is a ***subgraph*** of $G = (V_G, E_G)$ if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.



- The ***union*** of $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is the graph
  - $G \cup H = (V_G \cup V_H, E_G \cup E_H)$.



Outline

- Definition of a graph
  - Variants: undirected, directed
  - Variants: simple graphs, multigraphs, and loops.
  - A graph vs. a drawing of a graph
- Graphs and Relations
- Degrees and the Handshake Theorem
- Special graphs: cliques, cycles, hypercubes
- **Paths and connected components**
- **Spanning trees**
- Bipartite graphs

Connectivity

- Let $G = (V, E)$ be a simple, undirected graph.
- **Definition.** A ***path*** $(u_0, u_1, \dots, u_k)$ is a sequence of vertices in which consecutive vertices are connected by an edge, i.e., $\forall i \in [0, k). \{u_i, u_{i+1}\} \in E$. A ***simple path*** does not repeat any vertex.
- **Definition.** Two vertices u, v are ***connected*** if there is a path $(u, \dots, v)$.

Connectivity

- Define the relation $(u, v) \in Conn$ iff u is connected to v in $G = (V, E)$.
- Which properties does $Conn$ have?
  - Asymmetric? No
  - Antisymmetric? No
  - Symmetric? Yes
  - Transitive? Yes
  - Reflexive? Yes
  - Irreflexive? No

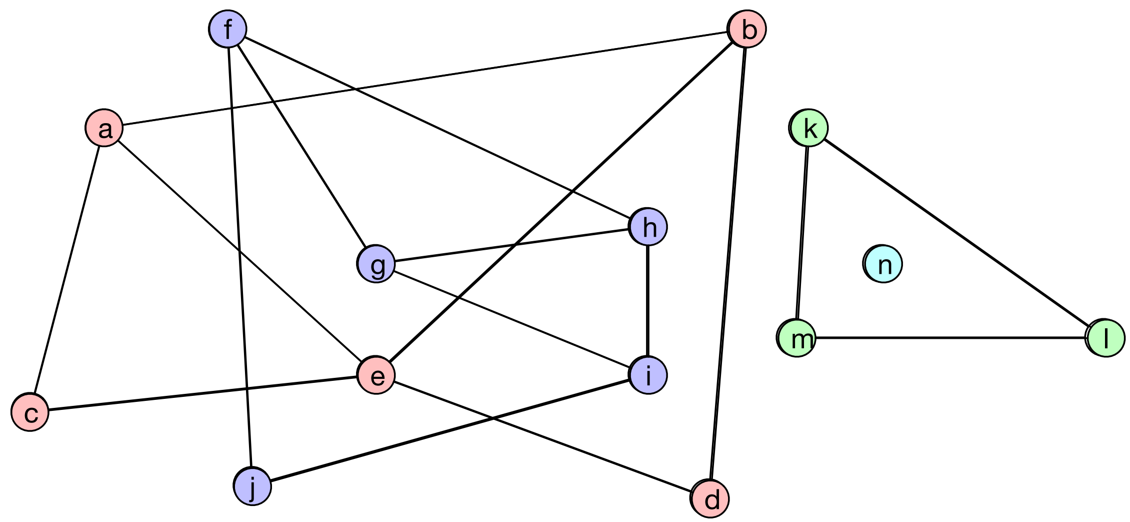
A relation that is symmetric, transitive, and reflexive is called an ***equivalence relation***

- The equivalence classes of $Conn$ are called the ***connected components*** of G.

Connectivity

- Define the relation $(u, v) \in Conn$ iff u is connected to v in $G = (V, E)$.
- The equivalence classes of $Conn$ are called the ***connected components*** of G.

- How many connected components are there in this graph G?
- (a) 2
- (b) 3
- (c) 4
- (d) 5
- (e) 6

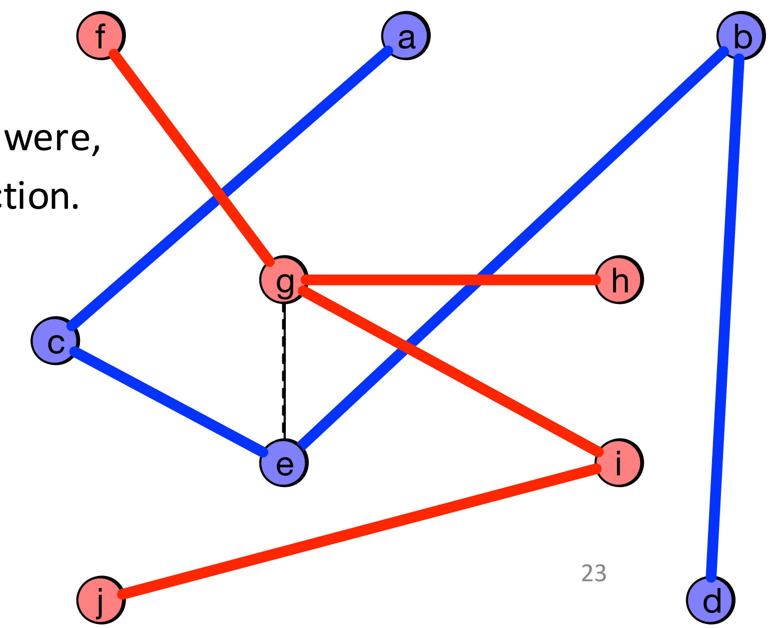


Trees

- **Definition.** A ***tree*** is a connected ***acyclic*** graph.
  - ***Acyclic*** means no subgraph is a cycle.
- If $G = (V, E)$ is a graph, *a* ***spanning tree*** is a subgraph $T = (V, E_T)$ that is a tree. (I.e., it "spans" all of V.)
- **Theorem.** If $T = (V, E)$ is a tree and $u, v \in V$, there is a *unique* simple path from u to v.
  - **Proof:** if there were two simple paths from u to v, T would contain a cycle.
- **Theorem.** Every tree on n vertices contains $n - 1$ edges.

Trees

- **Theorem.** Every tree on n vertices contains $n-1$ edges.
- **Proof by induction.**
  - Base case: $n = 1$. The only tree is a graph with 1 vertex and 0 edges.
  - General case: Assume the claim holds for all $n' < n$. Let $T = (V, E)$ be any tree with $|V| = n$.
    - Pick any edge in E, say it is $\{e, g\}$.
    - There is no path from e to g in $T' = (V, E - \{\{e, g\}\})$; if there were, that path and $\{e, g\}$ would form a cycle in T, a contradiction.
    - Let T_e, T_g be the connected components of T' containing e, g.
    - $T_e = (V_e, E_e), T_g = (V_g, E_g)$ are acyclic and therefore trees.
    - By the inductive hypothesis, T_e contains $|V_e| - 1$ edges and T_g contains $|V_g| - 1$ edges. With $\{e, g\}$, T contains $1 + (|V_e| - 1) + (|V_g| - 1) = |V| - 1 = n - 1$ edges.



Outline

- Definition of a graph
  - Variants: undirected, directed
  - Variants: simple graphs, multigraphs, and loops.
  - A graph vs. a drawing of a graph
- Graphs and Relations
- Degrees and the Handshake Theorem
- Special graphs: cliques, cycles, hypercubes
- Paths and connected components
- Spanning trees
- **Bipartite graphs**

Bipartite Graphs

- A graph $G = (V, E)$ is called ***bipartite*** if
  - You can partition $V = A \cup B$ into two parts, where $A \cap B = \emptyset$.
  - For every edge $\{a, b\} \in E$, $a \in A$ and $b \in B$. (There are no edges between A-vertices or B-vertices.)
- **Theorem.** The following statements are equivalent:

  (1) G is bipartite.

  (2) G is 2-colorable.
  - There is a function $f: V \rightarrow \{\text{red}, \text{blue}\}$ s.t. $\{u, v\} \in E \rightarrow f(u) \neq f(v)$.

  (3) G does not contain any C_{2k+1} (an odd cycle) as a subgraph.

- (1) $\leftrightarrow$ (2)
  - If $G = (V, E)$ is bipartite we can write it as $G = (A \cup B, E)$ such that $\{u, v\} \in E \rightarrow u \in A, v \in B$.
  - Color every vertex in A "red" and every vertex in B "blue".
    - $f: V \rightarrow \{\text{red}, \text{blue}\}$ be such that $f(u) = \begin{cases} \text{red} & u \in A \\ \text{blue} & u \in B \end{cases}$.
    - By definition, $\{u, v\} \in E \rightarrow f(u) \neq f(v)$, so G is 2-colorable.
  - In the reverse direction, set $A = f^{-1}(\text{red})$ and $B = f^{-1}(\text{blue})$.
    - Then $\{u, v\} \in E \rightarrow u \in A, v \in B$, so G is bipartite.

- $(2) \to (3)$
- Proof by contradiction.
  - Suppose G is 2-colorable and contains C_{2k+1} (odd cycle) as a subgraph.
  - Call the vertices of the cycle $(v_0, v_1, v_2, v_3, \dots, v_{2k}, v_0)$.
  - Wlog $f(v_0) = \text{blue}$.
    - Then $f(v_1) = f(v_3) = f(v_5) = \cdots = f(v_{2k-1}) = \text{red}$,
    - And $f(v_0) = f(v_2) = f(v_4) = \cdots = f(v_{2k}) = \text{blue}$.
  - But then $f(v_0) = f(v_{2k})$, so f is not a 2-coloring, a contradiction.

- $(3) \rightarrow (2)$
- If the claim holds for every connected component of G then it holds for G as well. (Combine the 2-colorings of each component.) Wlog we can assume G is connected.
- Let T be any spanning tree of G and $v_0 \in V$ any vertex.
- $f(u) = \begin{cases} \text{blue} & \text{if the path in } T \text{ from } v_0 \text{ to } u \text{ has } \textbf{even} \text{ length.} \\ \text{red} & \text{if the path in } T \text{ from } v_0 \text{ to } u \text{ has } \textbf{odd} \text{ length.} \end{cases}$
- If f were not a 2-coloring then some edge $\{x, y\}$ has $f(x) = f(y)$.
  - Let $(v_0, v_1, v_2, v_3, \dots, v_k)$ be the T-path from v_0 to $v_k = x$, and
  - $(v_0, v_1, \dots, v_i, v'_{i+1}, v'_{i+2}, \dots, v'_j)$ be the T-path from v_0 to $v'_j = y$.
- Then $(v_i, v_{i+1}, \dots, v_k, v'_j, v'_{j-1}, \dots, v'_{i+1}, v_i)$ is a cycle with length $k - i + 1 + j - i$, but since $f(v_k) = f(v'_j)$, $(k - i) \equiv (j - i) \pmod 2$, it follows that
- $k - i + 1 + j - i \equiv 1 \pmod 2$, meaning the cycle has odd length, contradicting (3).