# Mathematical Induction

EECS 203: Discrete Mathematics

Lecture 10

# FRAMING MATTERS!

A positive perspective can help you succeed in engineering.

## ENGINEERING EDUCATION RESEARCH FOUND THAT

Viewing situations as opportunities instead of threats (positive framing) is strongly correlated with student success outcomes in engineering.

This relationship increased during COVID.

(Finelli, Millunchick, Zhou, Zhao and Reeping, in progress)

## ORGANIZATIONAL STUDIES AND PSYCHOLOGICAL RESEARCH SHOW THAT

You can shift your framing to be more positive.

Doing so is an important wellness strategy.

## UNLOCK MORE OF YOUR POTENTIAL AS A STUDENT

Practice reframing strategies using a 30-minute Canvas Module:

- Reframe negative thinking
- Set goals & savor success
- Leverage your resources

## SCAN THE QR CODE TO GET STARTED!

Complete the Module and get free Michigan Engineering winter gear (hat, gloves or scarf)! ...while supplies last.

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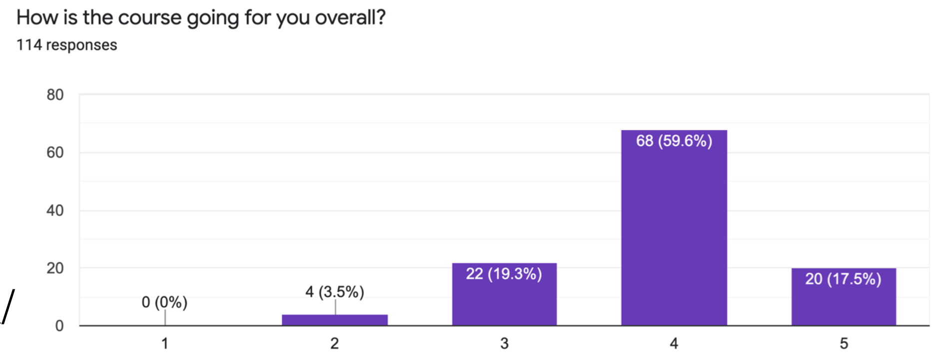
# Thanks for your Feedback!




- Overall, the course is going well (avg: 3.9 / 5)
- What is working?
  - Collaboration (in-lecture and on homework/ groupwork)
  - Helpful: Office hours, Piazza, discussion
  - Handouts, lecture format & pace
- What is challenging?
  - All-new material, a lot to digest
  - Big step from lecture to hw problems
  - Slides: there are a lot of them, sometimes we skip around
- How can I improve your experience?
  - More practice problems – done!
  - Post handout answers – done!
  - Cover harder questions, include review of harder homework/groupwork questions
    - I agree this would be helpful. We may be able to add it to discussion and/or create concept videos on these topics

# Exam 1: Weds. Feb 16

### Info & Resources

- Exam 1 Information Document
  - On Canvas home page
- STUDENT RESOURCES Piazza post
  - Additional practice problems
  - Concept videos
  - Reference sheet suggestions for Exam 1

### Exam Reviews

- Review Sessions this weekend
  - See Canvas announcement
- Special Topics Reviews
  - “Crash course” on specific topics
  - Strictly reviewing old problems (no new problems will be covered)
- Discussion this week (starting Thurs)
  - Half induction (new material)
  - Half exam review
- Lecture next Tues will be exam review

# Lecture 10 Outline

- **What is Mathematical Induction?**
  - **New proof method!**
  - **What it is & why it works**
    - **Mathematical Induction $\approx$ Dominoes**
    - **Example 1**
  - Guide for Inductive Proofs
- Examples
  - Another equality
  - *Aside*: Warning: Common pitfalls in proving LHS = RHS
  - An inequality
  - Tiling a checkerboard

# Proof by Induction

- New proof method!
- One of THE most important things you'll take away from this class
- Chapter 5
  - Read 5.1 and 5.2
- Today: What is induction + lots of examples
- Next time: More examples + strong induction

# Why do we care about induction?

- Mathematical Induction = Recursion
  - E.g., a recursive function (see: EECS 280)
  - A recursive function is a function that calls itself
- How do you know your recursive function works?
  - Use mathematical induction to prove that it does
- How do you prove your program is correct?
  - Often with mathematical induction

# “Logic” vs “Counting” Proof Styles

**“Logical Equivalence” proof styles**

proof by contradiction

proof by contrapositive

proof by cases

...

**“Counting” proof styles**

proof by induction (today)

pigeonhole principle (soon)

diagonalization (soon)

...

Valid because of **logical equivalences**

**Proof by Contradiction:**

$(\neg p \rightarrow F) \rightarrow p$

Valid because of **counting ideas ...**

# Induction

- **Induction** is a new proof style

- It's useful when you have a **predicate $\boldsymbol{P(n)}$ over the natural numbers $\mathbb{N}$**
  - Or positive integers $\mathbb{Z}^+$ or similar

$P(n) \ \forall n \geq n_0$

- And you want to prove $\forall n\, P(n)$

# Example 1

Prove, for all $n \in \mathbb{N}$,

$P(n)$: $0 + 1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

Try a few values of $n$ …

But this is not a proof!

- $P(0): \ 0 = \frac{0(0+1)}{2}$
- $P(1): \ 0 + 1 = 1 = \frac{1(1+1)}{2}$
- $P(2): \ 0 + 1 + 2 = 3 = \frac{6}{2} = \frac{2(2+1)}{2}$
- $P(3): \ 0 + 1 + 2 + 3 = 6 = \frac{12}{2} = \frac{3(3+1)}{2}$
- …..

# Example 1

Prove, for all $n \in \mathbf{N}$,

$P(n)$: $0 + 1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

Try a few values of

- $P(0)$ : $0 = \frac{0(0+1)}{2}$
- $P(1)$ : $0 + 1 = 1 = \frac{1}{}$
- $P(2)$ : $0 + 1 + 2 = 3$
- $P(3)$ : $0 + 1 + 2 + 3$
- …..

**Idea:**

What if we could prove

$P(0) \rightarrow P(1)$

$P(1) \rightarrow P(2)$

$P(2) \rightarrow P(3)$

$\vdots$

i.e.,

$P(k) \rightarrow P(k+1)$

for an arbitrary $k \in \mathbf{N}$

# Mathematical Induction ≈ Dominos

Let $P(n)$ be a predicate.

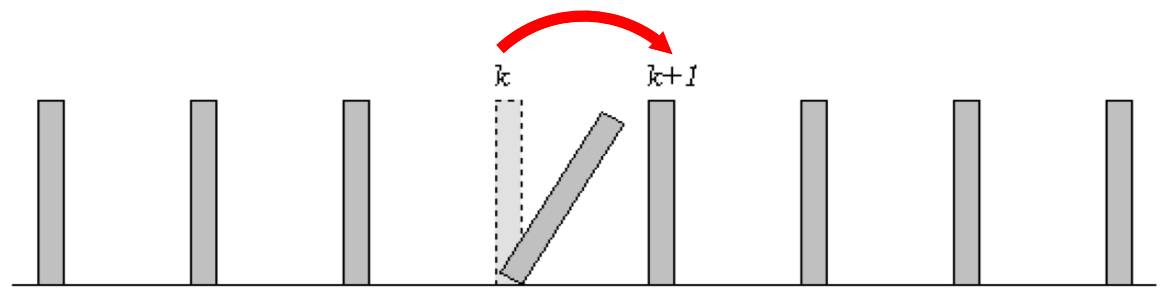
$P(n)$: $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

**Goal:** Prove that $\boldsymbol{P(n)}$ is true for all $\boldsymbol{n} \in \mathbf{N}$

**Step 1: "Inductive Step"**

If you can knock down one domino, then you can knock down the next one.

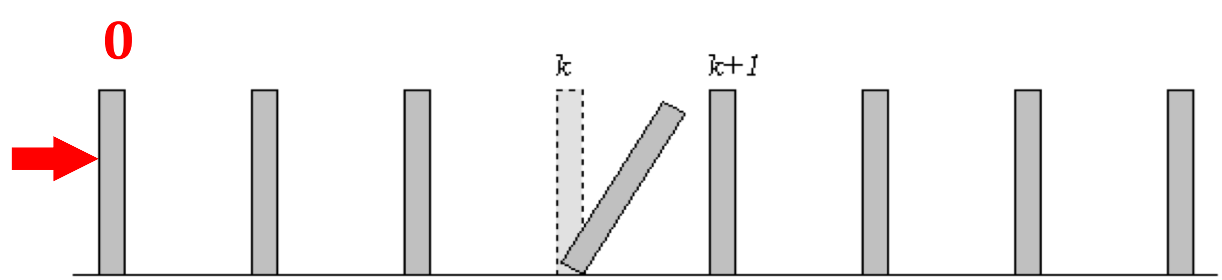
**For any $\boldsymbol{k} \in \mathbf{N}$,**
$\boldsymbol{P(k) \rightarrow P(k+1)}$




**Step 2: "Base Case"**

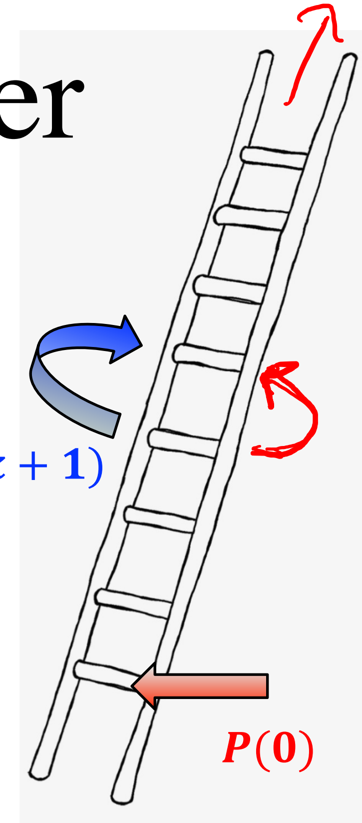
You can knock down the first domino

$\boldsymbol{P(0)}$




Therefore, you can knock down all dominos.

# Induction ≈ Climbing the Ladder




We want to show that $\forall n \geq 0$ $P(n)$ is true.

- Think of the natural numbers as a ladder.
- 0, 1, 2, 3, 4, 5, 6, . . .

- From *each* ladder step, you can reach the *next*.
  **"Inductive step"**

  $P(0) \rightarrow P(1)$, $P(1) \rightarrow P(2)$, $P(2) \rightarrow P(3)$, . . .

  $$\forall k \geq 0 \quad P(k) \rightarrow P(k+1)$$

- You can *get on* the ladder (at the bottom): **"Base case"**

  $P(0)$

- Then, by mathematical induction, you can climb up the whole ladder:

  $$\forall n \geq 0 \;\; P(n)$$

# Example 1

Prove, $\forall n \in \mathbb{N},\ P(n): 0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Handout

**Inductive Step**:

Goal is to show $\underbrace{P(k) \rightarrow P(k+1)}$ $\underbrace{\text{for all } k \in \mathbf{N}}$

$p \rightarrow q$

assume p

q

Let k be an arbitrary natural numer

**Base case**: Goal is to show $\boldsymbol{P(0)}$ is true.

# Example 1

Prove, $\forall n \in \mathbb{N}$, $\boxed{P(n)}$: $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Handout

$p \rightarrow q$

**Inductive Step**: Let $k$ be an arbitrary natural number.

$p$

- Assume $\boldsymbol{P(k)}$: $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ ← **Inductive Hypothesis**

$q$

- Want to show $\boldsymbol{P(k+1)}$: $0 + 1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

**Base case**: Goal is to show $\boldsymbol{P(0)}$ is true.

# Example 1

Prove, $\forall n \in \mathbb{N},\ P(n): 0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Handout

**Inductive Step**: Let $k$ be an arbitrary natural number.

- Assume $\boldsymbol{P(k)}$: $\underbrace{0 + 1 + 2 + \cdots + k} = \frac{k(k+1)}{2}$ ← **Inductive Hypothesis**
- Want to show $\boldsymbol{P(k+1)}$: $0 + 1 + 2 + \cdots + k + (k+1) = \frac{\boxed{(k+1)}(k+2)}{2}$

$$\underbrace{0 + 1 + 2 + \cdots + k} + (k+1)$$

$$= (0 + 1 + 2 + \cdots + k) + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \qquad \text{Inductive Hypothesis}$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k\boxed{(k+1)} + 2\boxed{(k+1)}}{2}$$

$$= \boxed{\frac{(k+1)(k+2)}{2}}$$

$\therefore \forall k \in \mathbb{N} \quad P(k) \to P(k+1)$

# Example 1

Prove, $\forall n \in \mathbb{N},\ P(n): 0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Handout

**Inductive Step**: Let $k$ be an arbitrary natural number.

- Assume $\boldsymbol{P(k)}$: $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ ← **Inductive Hypothesis**
- Want to show $\boldsymbol{P(k+1)}$: $0 + 1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

$$
\begin{aligned}
0 + 1 + 2 + \cdots + k + (k+1) \\
&= (0 + 1 + 2 + \cdots + k) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{Inductive Hypothesis} \\
&= \frac{(k+1)(k+2)}{2} && \text{Algebra}
\end{aligned}
$$

- Thus $P(k) \rightarrow P(k+1)$ for any natural number $k$

**Base case**: $P(0)$: $0 = \frac{0(0+1)}{2}$.

# Example 1

Prove, $\forall n \in \mathbb{N},\ P(n): 0+1+2+\cdots+n = \frac{n(n+1)}{2}$

Handout

**Inductive Step**: Let $k$ be an arbitrary natural number.

Inductive Hypothesis

- Assume $\boldsymbol{P(k)}$: $0+1+2+\cdots+k = \frac{k(k+1)}{2}$
- Want to show $\boldsymbol{P(k+1)}$: $0+1+2+\cdots+k+(k+1) = \frac{(k+1)(k+2)}{2}$

$$
\begin{aligned}
0+1+2+\cdots+k+(k+1) \\
&= (0+1+2+\cdots+k)+(k+1) \\
&= \frac{k(k+1)}{2}+(k+1) && \text{Inductive Hypothesis} \\
&= \frac{(k+1)(k+2)}{2} && \text{Algebra}
\end{aligned}
$$

- Thus $P(k) \rightarrow P(k+1)$ for any natural number $k$

**Base case**: $P(0)$: $0 = \frac{0(0+1)}{2}$.

**Conclusion:** By induction, $P(n)$ is true for all natural numbers $n$.

# Lecture 10 Outline

- What is Mathematical Induction?
  - New proof method!
  - What it is & why it works
    - Mathematical Induction ≈ Dominoes
    - Example 1
  - **Guide for Inductive Proofs**
- Examples
  - Another equality
  - *Aside*: Warning: Common pitfalls in proving LHS = RHS
  - An inequality
  - Tiling a checkerboard

# Guide for Induction Proofs

- Restate the claim you are trying to prove : $P(n)$ $\forall n \geq n_0$

- **Base case**: Prove the claim holds for the "first" value of $n$
  - Prove $P(n_0)$ is true

- **Inductive Step**: Prove that $P(k) \rightarrow P(k+1)$ for an arbitrary integer $k$ in the desired range.
  - Let $k$ be an arbitrary integer with $k \geq n_0$
  - Assume $P(k)$
  - Show that $P(k+1)$ holds

Equivalently: Show $P(k-1) \rightarrow P(k)$

- Conclusion: explain that you've proven the desired claim.

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Handout

# Induction Proof: Another Equality

P(n):

- Claim: $\boxed{\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^n} = 1 - \frac{1}{3^n}}$ $\quad \forall n \geq 1$

# Guide for Induction Proofs

- Restate the claim you are trying to prove
- **Base case**: Prove the claim holds for the "first" value of $n$
  - Prove $P(n_0)$ is true
- **Inductive Step**: Prove that $P(k) \rightarrow P(k+1)$ for an arbitrary integer $k$ in the desired range.
  - Let $k$ be an arbitrary integer with $k \geq n_0$
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# Induction Proof: Another Equality

P(n):

- Claim: $\boxed{\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^n}=1-\frac{1}{3^n}}$ $\forall n \geq 1$

**Inductive Step**: Let $k$ be an arbitrary integer with $k \geq 1$. Assume **P(k)**, i.e., assume

$$\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^k}=1-\frac{1}{3^k}$$

Want to show **P(k+1)**: $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^k}+\frac{2}{3^{k+1}}=\boxed{1-\frac{1}{3^{k+1}}}$

$$\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^k}+\frac{2}{3^{k+1}}=1-\frac{1}{3^k}+\frac{2}{3^{k+1}}$$

(assuming IH **P(k)**)

$$=1-\frac{3}{3}\cdot\frac{1}{3^k}+\frac{2}{3^{k+1}}$$

$$=1-\frac{3}{3^{k+1}}+\frac{2}{3^{k+1}}$$

$$=\boxed{1-\frac{1}{3^{k+1}}}$$

**Base case: P(1):** Prove $\frac{2}{3^1}=1-\frac{1}{3^1}$

$$\frac{2}{3^1}=\frac{2}{3}=1-\frac{1}{3}=1-\frac{1}{3^1}$$

**Conclusion:** By mathematical induction, the claim holds for all $n \geq 1$.

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# Warning: Always work in **One Direction**

**P(k+1), P($n_0$)** often look like **LHS = RHS**, or **LHS > RHS**, etc.

Proper mathematical proofs:

- To prove LHS = RHS, start with one side and work your way ***in one direction*** until you get the other side
  - Work in one direction from LHS to RHS – or –
  - Work in one direction from RHS to LHS

**Example:** Say we want to prove: $2x^2 - 1^x = x^2 - 2 + (x^2 + 1)$

**Correct:** ✓

$$\begin{aligned} 2x^2 - 1^x &= 2x^2 - 1 \\ &= x^2 + x^2 - 1 \\ &= x^2 + x^2 - 2 + 1 \\ &= x^2 - 2 + (x^2 + 1) \end{aligned}$$

**Incorrect:** X

$$2x^2 - 1^x = x^2 - 2 + (x^2 + 1)$$
$$2x^2 - 1 = 2x^2 - 2 + 1$$
$$2x^2 - 1 = 2x^2 - 1$$

# Warning: Always work in **One Direction**

**P(k+1), P($n_0$)** often look like **L**

Proper mathematical proofs:

- To prove LHS = RHS, start
  ***one direction*** until you get t
  - Work in one direction from L
  - Work in one direction from R

**Example:** Say we want to prove:

*Incorrect* approach:

1. Start by setting LHS = RHS
2. Work both sides of the equation until you have the same expression on both sides

Logically, doing this is equivalent to proving:
"if LHS = RHS, then true"

which does NOT prove "LHS = RHS"

**Correct:** ✓

$2x^2$

$$\begin{aligned} LHS &= \ldots \\ &= \ldots \\ &\vdots \\ &= RHS \end{aligned}$$

1

- 1)

**Incorrect:** X

$$LHS = RHS$$
$$\ldots = \ldots$$
$$\vdots$$
$$expression_1 = expression_1$$

# Warning: Always work in **One Direction**

## Why only in one direction?

- Because not every mathematical operation is reversible

**Examples:**

$$a = b \ \rightarrow \ a^2 = b^2$$

$$a^2 = b^2 \ \not\rightarrow \ a = b$$

For any constant $c \in \mathbf{R}$:

$$a = b \ \rightarrow \ ac = bc$$

$$ac = bc \ \not\rightarrow \ a = b$$

With the wrong approach, we can "prove" things that are not true!

**Wrong approach:** Start with LHS = RHS, then work both sides until you get the same expression on both sides.

**"Proof" that 0 = 2:**

$$0 = 2$$
$$0 - 1 = 2 - 1$$
$$-1 = 1$$
$$(-1)^2 = 1^2$$
$$1 = 1$$

**"Proof" that 1024 = -57:**

$$1024 = -57$$
$$0 \cdot 1024 = 0 \cdot 57$$
$$0 = 0$$

# Warning: Always work in **One Direction**

Handout

## **Summary:** To prove LHS = RHS

***Correct*** approach:

1. Start with ___one side___
2. Work your way ___one direction___ until you get the other side

***Incorrect*** approach:

1. Start by setting ___LHS = RHS___
2. ___Work both sides.___ of the equation until you have the same expression on both sides

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Handout

# Induction Proof: An Inequality

- Claim: $\overset{P(n)}{\boxed{2n + 3 \leq 2^n}}$ $\quad\quad\quad\quad \forall n \geq 4$

# Guide for Induction Proofs

- Restate the claim you are trying to prove
- **Base case**: Prove the claim holds for the "first" value of $n$
  - Prove $P(n_0)$ is true
- **Inductive Step**: Prove that $P(k) \rightarrow P(k+1)$ for an arbitrary integer $k$ in the desired range.
  - Let $k$ be an arbitrary integer with $k \geq n_0$
  - Assume $P(k)$
  - Show that $P(k+1)$ holds

Equivalently: Show $P(k-1) \rightarrow P(k)$

- Conclusion: explain that you've proven the desired claim.

# Induction Proof: An Inequality

- Claim: $2n + 3 \leq 2^n$ $\forall n \geq 4$

**Inductive Step**: For an arbitrary $k \geq 4$, assume **P(k)**: $2k + 3 \leq 2^k$

Want to show **P(k+1)**: $2(k + 1) + 3 \leq 2^{k+1}$

$$2(k+1) + 3 = 2k + 2 + 3$$

$$= 2k + 3 + 2$$

$$\leq 2^k + \boxed{2}$$

another $\leq$

$$\leq 2^k + \boxed{2^k}$$

because $2 < 2^k$ $\forall k \geq 4$

$$= 2^k \cdot 2^1$$

$$= \boxed{2^{k+1}}$$

# Induction Proof: An Inequality

- Claim: $2n + 3 \leq 2^n \qquad \forall n \geq 4$

**Inductive Step**: For an arbitrary $k \geq 4$, assume **P(k)**: $2k + 3 \leq 2^k$

Want to show **P(k+1)**: $2(k+1) + 3 \leq 2^{k+1}$

# Induction Proof: An Inequality

- Claim: $2n + 3 \leq 2^n$ $\quad\quad\quad\quad \forall n \geq 4$

**Inductive Step**: For an arbitrary $k \geq 4$, assume **P(k)**: $2k + 3 \leq 2^k$

Want to show **P(k+1)**: $2(k + 1) + 3 \leq 2^{k+1}$

$$
\begin{aligned}
2(k + 1) + 3 &= 2k + 2 + 3 \\
&= 2k + 3 + 2 \\
&\leq 2^k + 2 && \text{(by I.H., i.e., apply } \mathbf{P(k)}\text{)} \\
&\leq 2^k + 2^k && \text{(because } 2 \leq 2^k \quad \forall k \geq 4\text{)} \\
&= 2^k \cdot 2 \\
&= 2^{k+1}
\end{aligned}
$$

**Base Case**: **P(4):** Prove $2(4) + 3 \leq 2^4$ :

$$2(4) + 3 \;=\; 11 \;\leq\; 16 \;=\; 2^4$$

**Conclusion:** By mathematical induction, $2n + 3 \leq 2^n$ holds for all $n \geq 4$.

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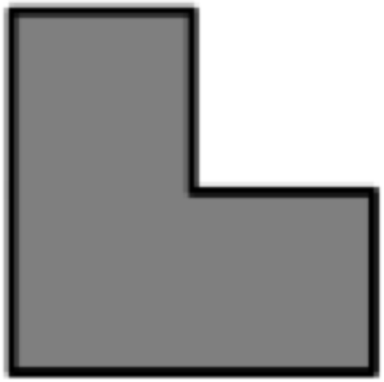
# Tiling a Checkerboard

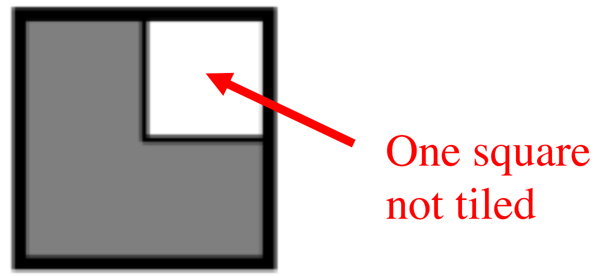
- For $n \geq 1$, consider a $2^n \times 2^n$ checkerboard.
- Can we cover it with right triominos? No!



A right triomino

***Counterexample***:

- A $2^1 \times 2^1$ checkerboard cannot be completely tiled with triominos




- Let $P(n)$ be the proposition:
  - $P(n)$: A $2^n \times 2^n$ checkerboard *with any one square removed* can be tiled using right triominos.
- Use mathematical induction to prove $\forall n \geq 1\ P(n)$.

# Tiling a Checkerboard

- $\boldsymbol{P(n)}$: A $2^n \times 2^n$ checkerboard with any one square removed can be tiled using right triominos.
- **Prove:** $\forall n \geq 1\ P(n)$
- We'll do the **inductive step** first.
- Then figure out the **base case**.

# Guide for Induction Proofs

- Restate the claim you are trying to prove
- **Base case**: Prove the claim holds for the "first" value of $n$
  - Prove $P(n_0)$ is true
- **Inductive Step**: Prove that $P(k) \rightarrow P(k+1)$ for an arbitrary integer $k$ in the desired range.
  - Let $k$ be an arbitrary integer with $k \geq n_0$
  - Assume $P(k)$
  - Show that $P(k+1)$ holds

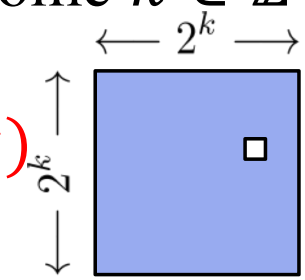
Equivalently: Show $P(k-1) \rightarrow P(k)$

- Conclusion: explain that you've proven the desired claim.

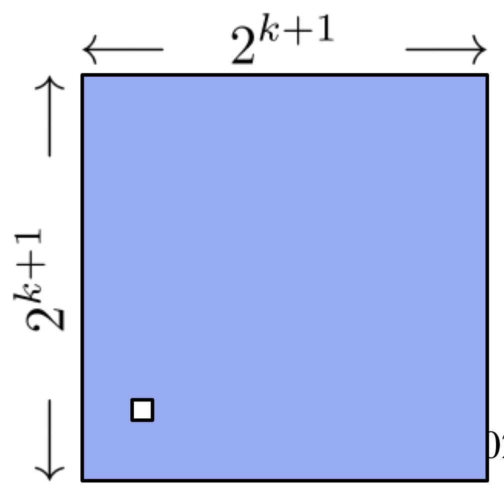
# Tiling a Checkerboard

- $\boldsymbol{P(n)}$: A $2^n \times 2^n$ checkerboard with any one square removed can be tiled using right triominos.
- **Inductive Step**: Prove $\forall k \in \mathbb{Z}^+ \quad P(k) \to P(k+1)$
  - Inductive Hypothesis: Assume $P(k)$ holds, for some $k \in \mathbb{Z}^+$.
    - A $2^k \times 2^k$ checkerboard with any one square removed can be tiled using right triominos.

$P(k)$



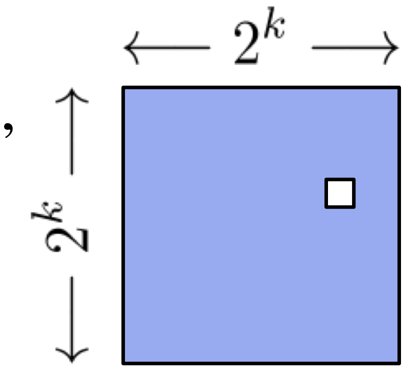

  - *Want to show*: A $2^{k+1} \times 2^{k+1}$ checkerboard with any one square removed can be tiled using right triominos.

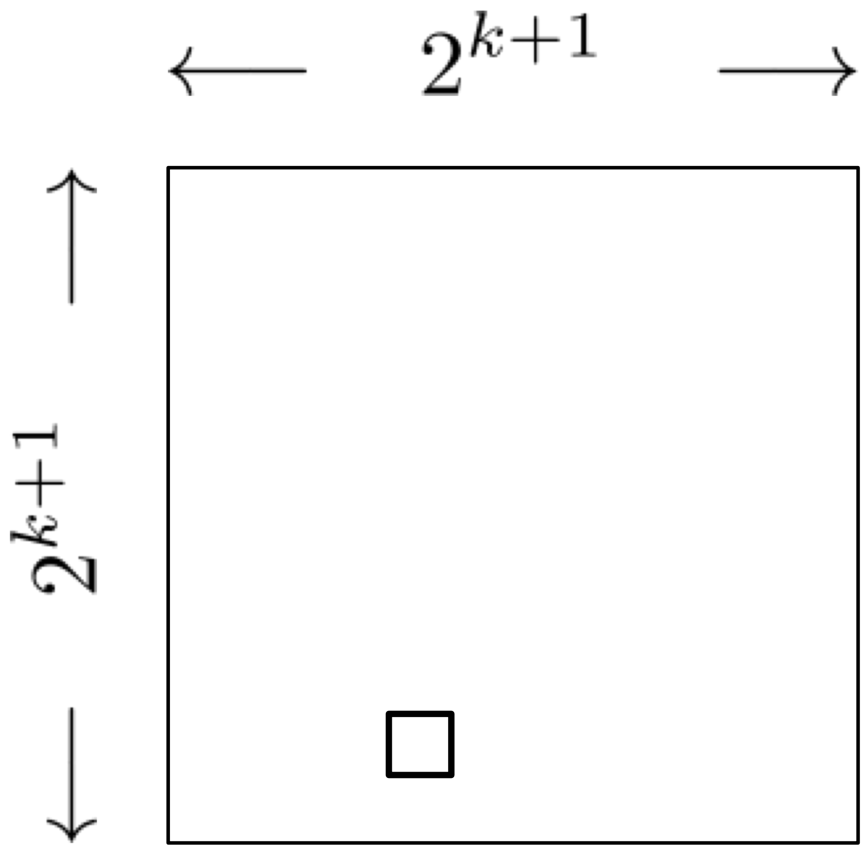



$P(k+1)$

# Tiling a Checkerboard

– Inductive Hypothesis: For an arbitrary positive integer $k$, a $2^k \times 2^k$ checkerboard with any one square removed can be tiled using right triominos.



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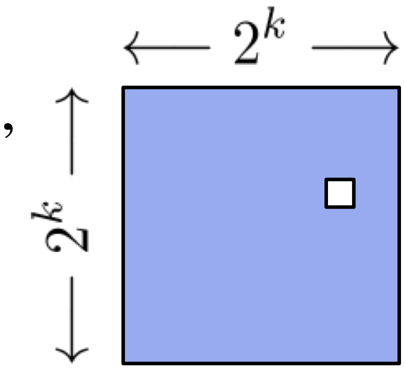


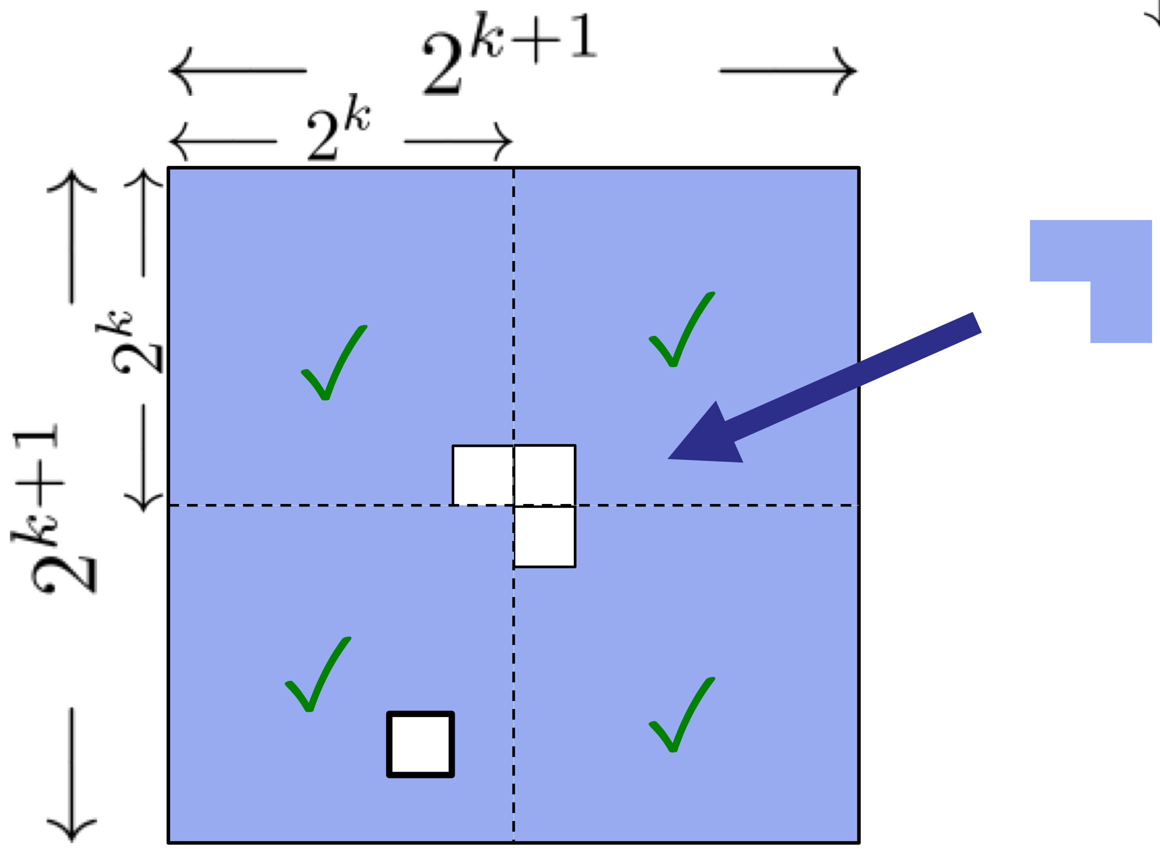

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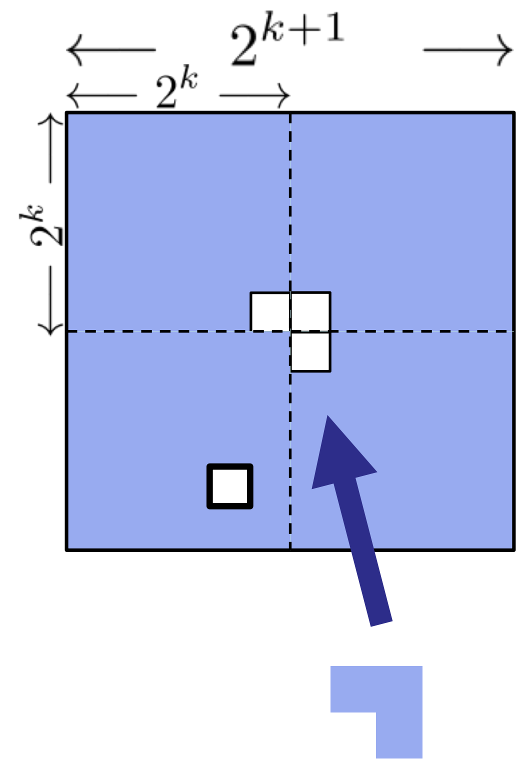
**Inductive Step:** for an arbitrary $k \in \mathbb{Z}^+$, assume a $2^k \times 2^k$ checkerboard with any one square removed can be tiled using right triominos. (Inductive Hypothesis)

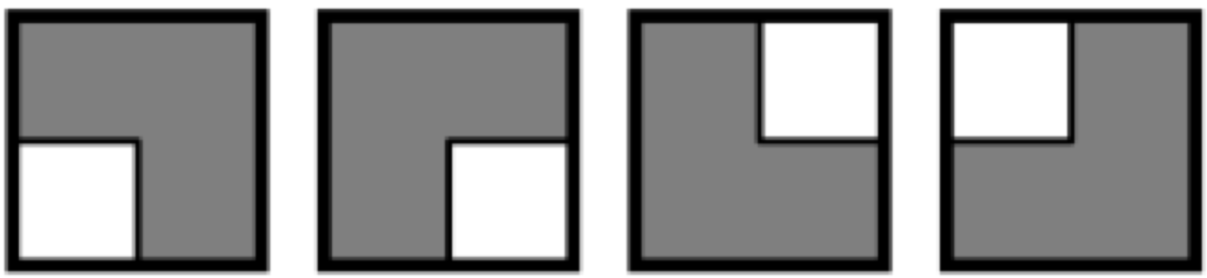
- Consider a $2^{k+1} \times 2^{k+1}$ checkerboard with any one square removed. We can tile it as follows:
- Divide checkerboard into four $2^k \times 2^k$ quadrants.
  - The missing square is in one quadrant. From the other three quadrants, remove the square closest to the center of the checkerboard.
  - Assuming IH, each of the four quadrants missing one square can be tiled
  - Place one last triomino to cover the removed squares at the center of the checkerboard.

– Thus, $P(k) \rightarrow P(k+1)$ for any positive integer $k$.



# Tiling a Checkerboard

- $\boldsymbol{P(n)}$: A $2^n \times 2^n$ checkerboard with any one square removed can be tiled using right triominos.
- **Prove:** $\forall n \geq 1 \ P(n)$
- **Base case**:
  - $P(1)$ says that a $2 \times 2$ checkerboard with any one square removed can be tiled using right triominos.
    - True, by inspection of all cases:



- By mathematical induction, $P(n)$ holds $\forall n \geq 1$.

# Wrap up

- Proof by Induction is a key concept of this course

- Induction is often how you prove that your program is correct
  - Recursive functions
  - Loops / loop invariants
  - See Sections 5.4 and 5.5 if you're interested in more details

- You'll continue to use induction in 281 and 376

- Next time: more induction examples + strong induction