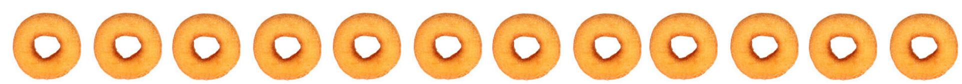
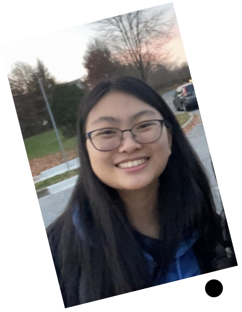
This slide deck posted under Files > Lecture Slides & Handouts > Annotated slides (Diaz)

L23:
Binomial Coefficients
and
Counting with Replacement

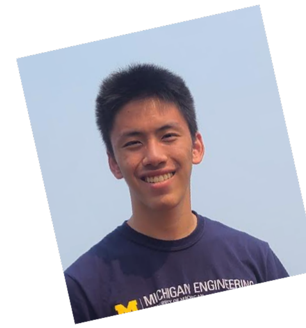
$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$





Apply to be a 203 IA!

- It’s really **fun**!



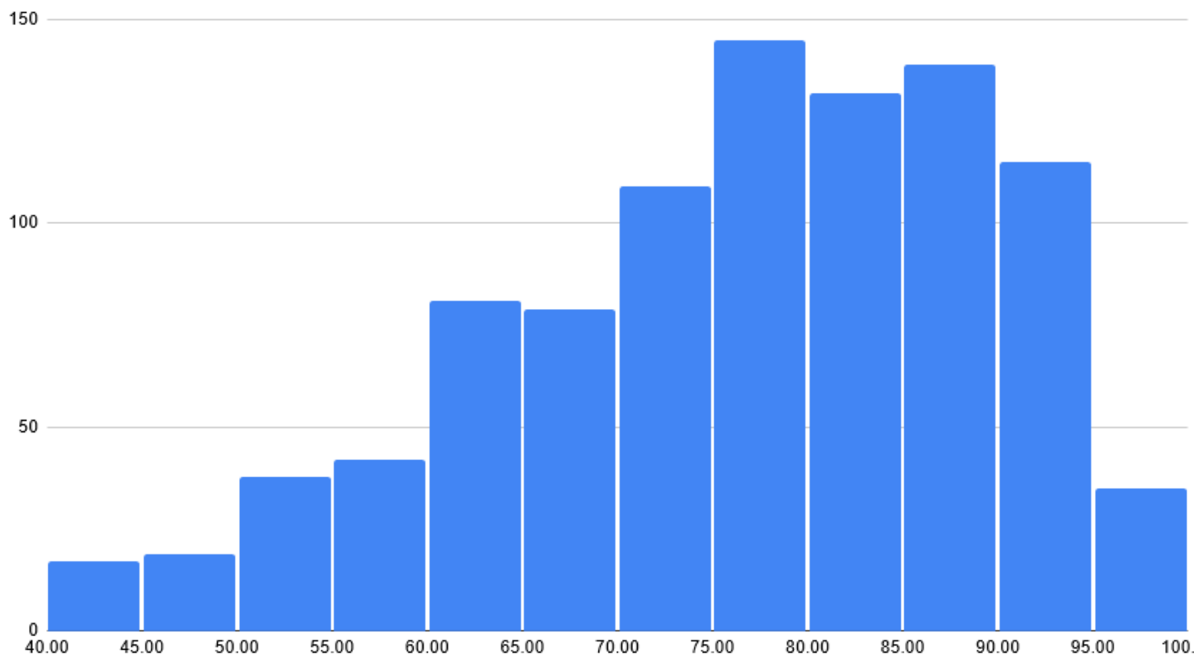
- You get to help students (fun & fulfilling) and you get to do a lot of discrete math (cool!)

- Work ~10 hrs/week; competitive pay
- Full details in Canvas Announcement
- **Application deadline = Friday, April 15**

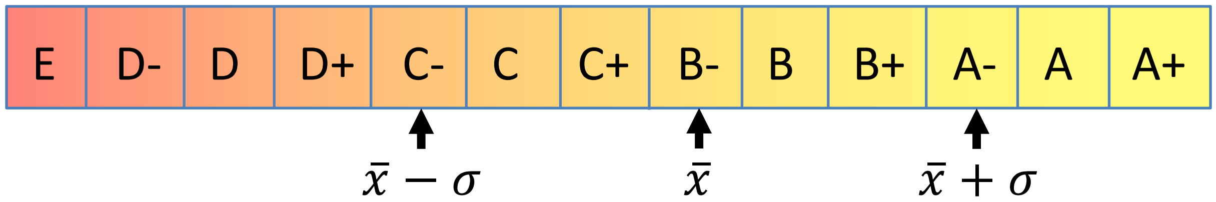
- Have questions about what it’s like to be a 203 IA?
  – Ask any of our IAs or GSIs or faculty, during OH, at discussion, before/after lecture
  – We’d be happy to tell you more about it!

Midterm 2 Results

Mean: $\overline{x}$ = 74.7
Standard deviation: $\boldsymbol{\sigma \approx 15.7}$



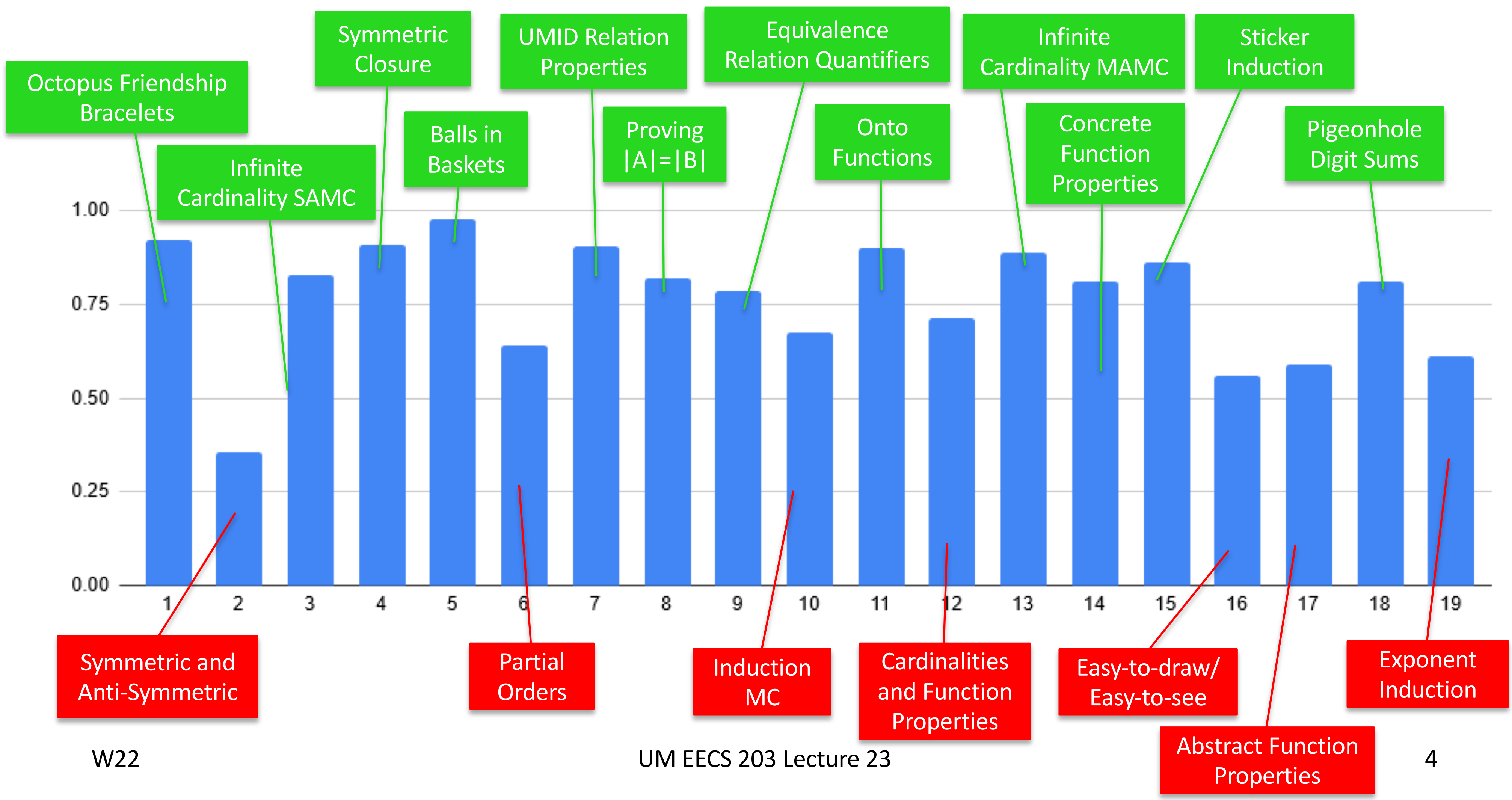
- The curve



Need a C or better for CE/CS major or minor

Question Breakdown

- While exams are not cumulative, the material can be inherently cumulative
- MC order doesn't match our solutions. Here's the labels:



Midterm 2 Results

Mean: $\overline{x}$ = 74.7
Standard deviation: $\sigma \approx 15.7$

- Overall curved exam score on Canvas
  - Canvas grades: Exam 2 (Curved)
- Multiple choice
  - Canvas quiz shows which questions you got correct
  - Look in Canvas GRADES to find your MC score
    - Listed as Exam 2 MC (Raw)
    - See next slide on Grading Details for MAMC
- Free Response
  - Graded work on Gradescope
- Solutions to all on Canvas

Grading Details – MAMC: Partial Credit

Partial Credit:

An incorrect option, A-E, is one that is

- Bubbled when it shouldn't be, or
- Not bubbled when it should be

-1.5 pts for each incorrect option

- All correct: 4 pts
- 1 incorrect : 2.5 pts
- 2 incorrect : 1 pt
- 3 or more incorrect: 0 pts

We have uploaded your MC grade as graded by this system to Canvas under "Exam 2 MC (Raw)". Please let us know if your raw MC doesn't match this grading system, so we can correct it.

Midterm 2: Regrade Requests

- **Open until Thursday, April 7th at 11:59 PM**
- Review Regrade Policy in “Course Policies” document
- First read 3 things
  - The solution (Canvas)
  - Your answer (Gradescope)
  - Rubric (Gradescope)
- “I have read the solution, my answer, and the rubric.”
- Clearly indicate which rubric items you feel you should have been marked for and why

Learning Objectives

After today's lecture (and associated readings, discussion & homework), you should be able to:

- Permutations and Combinations
  - P(n,k) and C(n,k)=$\binom{n}{k}$ notation and what they mean.
- Binomial Coefficients and how they arise in Pascal's triangle.
  - The binomial theorem and how to use it.
- Combinatorial proofs vs. algebraic proofs.
  - Know that combinatorial proofs are a valid proof method.
  - We won't ask you to write combinatorial proofs but you should be able to recognize one when you see it.
- Counting permutations/combinations with and without repetitions (duplicates)
  - "Balls and Bars" method for counting combinations with repetition.

Outline:

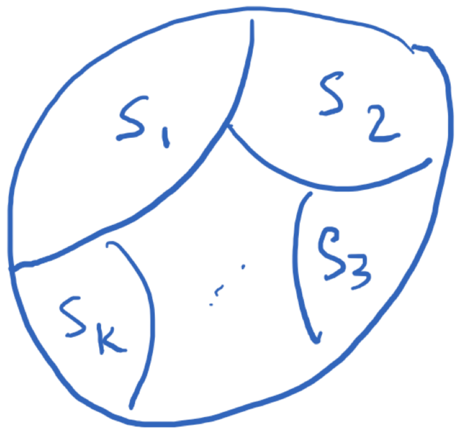
- Recap basic rules of counting, permutations, combinations
- Counting Poker Hands
- Binomial Theorem
  - Binomial coefficients & Pascal’s Triangle
- Combinatorial Proofs
- Balls ‘n’ Bars method
  - Combinations with Replacement

Review: The Basic Rules of Counting

- **Product Rule:** if an object can be chosen in ***k* stages**, with ***exactly n_i choices*** in stage i, and ***no object can be chosen in two different ways***, there are $n_1 \cdot n_2 \cdots n_k$ different objects.

$$\to \boxed{n_1} \to \boxed{n_2} \to \cdots \to \boxed{n_k} \to$$

- **Sum Rule:** If an object is in *exactly one* of the sets $S_1, \dots, S_k$ and these sets are disjoint, then there are $|S_1| + |S_2| \cdots + |S_k|$ different objects.



- **Division Rule:** If there are N ways to choose an object, and each object can be chosen in exactly d ways, there are N/d objects.

Permutations & Combinations

- $\boldsymbol{P(n,k)}$ = the number of ways to pick a ***sequence*** of k things from a set of size n.
  - **(Prod Rule)** $P(n,k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$

- $\boldsymbol{C(n,k) = \binom{n}{k}}$ = the number of ways to pick a ***set*** (unordered) of k things from a set of size n.
  - **(Division Rule)** Start by picking a ***sequence*** a k things, then forget the order. How many ways could you have picked that set?
    - $C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}$

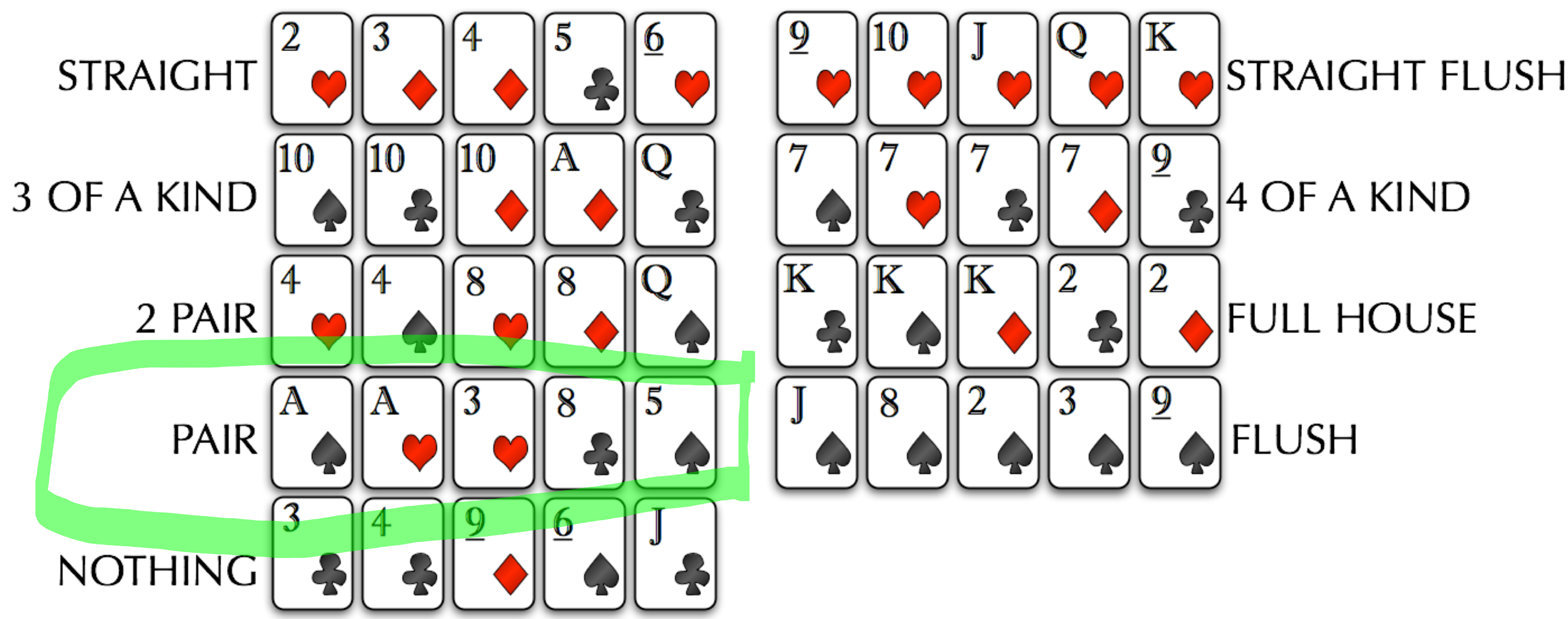
Lecture 23 Handout: Binomials and Counting with Repetition

- $P(n, k)$: number of ways to pick a ***<u>sequence</u>*** of k elements from a set of size n. (No repetitions).
  $P(n, k) =$

- $C(n, k) = \binom{n}{k}$: number of ways to pick a ***<u>set</u>*** of k elements from a set of size n. (No repetitions).
  $\binom{n}{k} =$

Poker Hands

- How many ways are there to make a pair (and nothing better)?



Exercise: How many ways to make a pair

(and nothing better)?

A♣   A♠   2♠   7♦   3♥

Total:

$$\frac{13 \cdot 48 \cdot 44 \cdot 40 \binom{4}{2}}{3!}$$

Stage 1: pick rank for the pair : 13 choices   A

Stage 2: pick 1st nonpair card : $52-4 = 48$ choices   2♠

≠ different rank than the pair

stage 3: pick 2nd nonpair card : $52-8 = 44$ choices   7♦

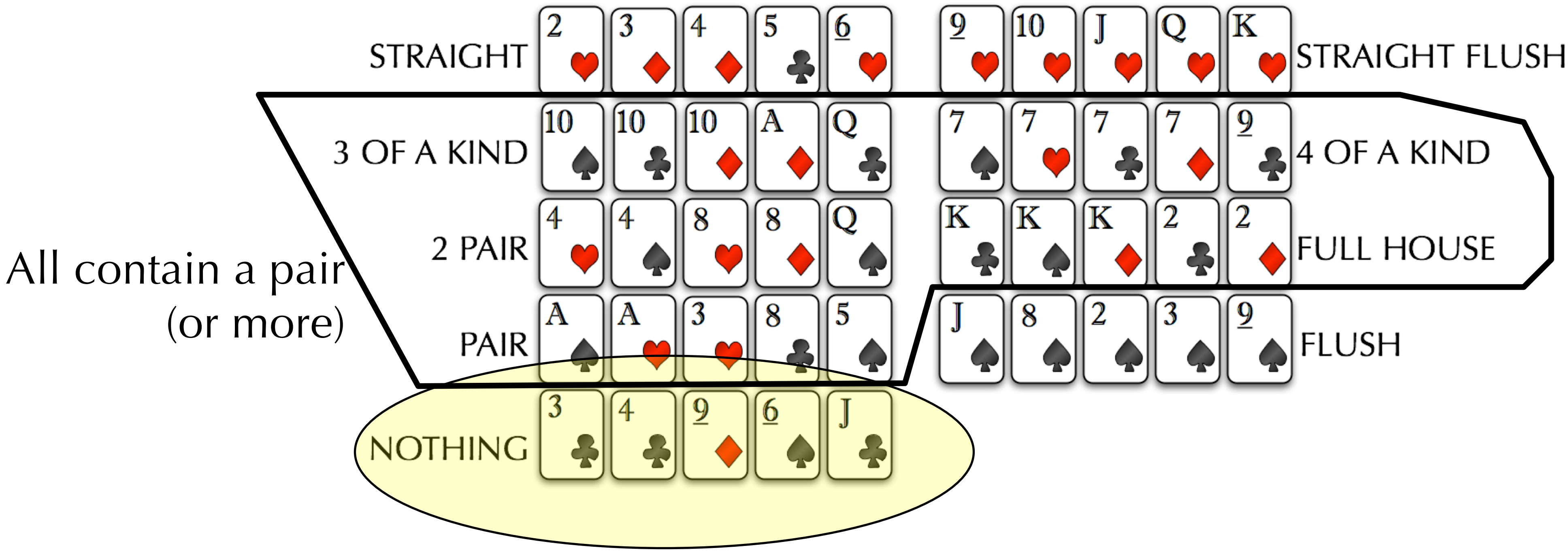
different rank

stage 4: pick 3rd nonpair card $52-12 = 40$ choices   3♥

different rank

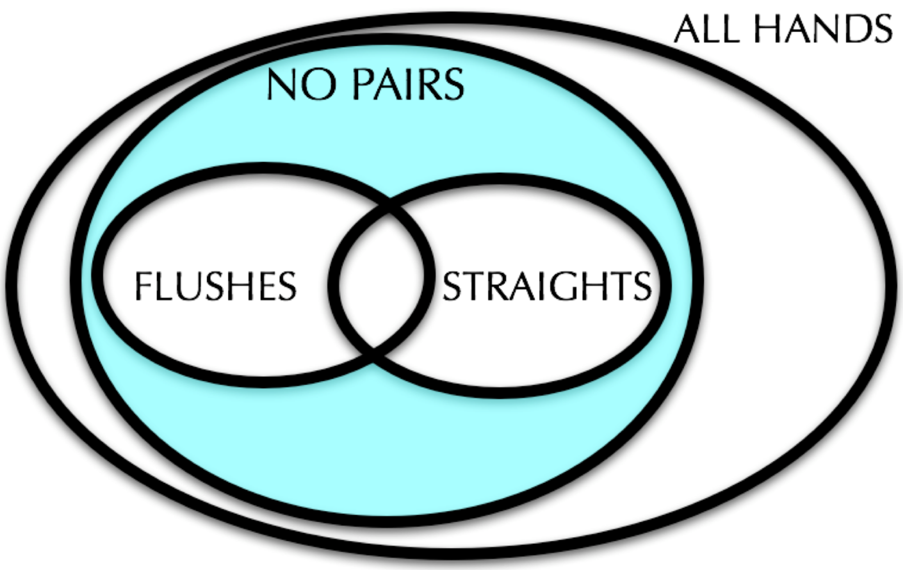
Stage 5: pick suits for the pair $\binom{4}{2}$ {♥, ♦}

Exercise: How many ways to make nothing?



Exercise: How many ways to make nothing?

- We're counting hands:
  (1) without pairs
  (2) that also do not contain straights or flushes

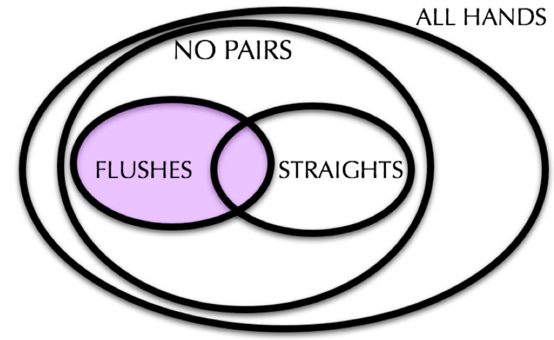


- These L23 "Diaz" slides only have #straights and #flushes.
- Full problem/solution in L22 slides.

(# nothing) = (#no pairs) – (#straights) – (#flushes) +(#straight flushes)

Part of Bigger Exercise: How many ways to make nothing?

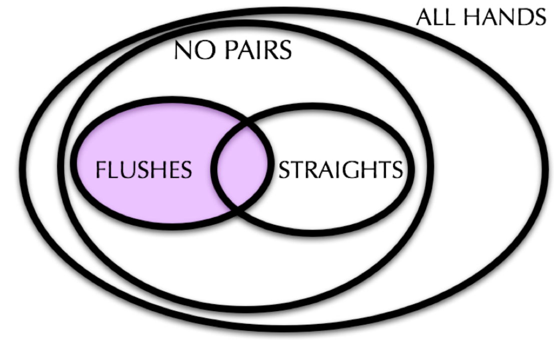
Flushes



- Exercise: how many ways are there to make ***a flush*** (all 5 cards the same suit).

Part of Bigger Exercise: How many ways to make nothing?

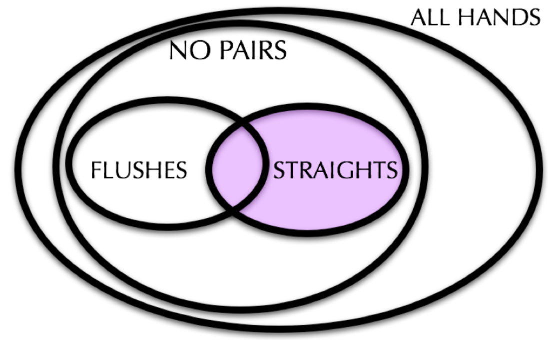
Flushes



- <u>Exercise:</u> how many ways are there to make ***a flush*** (all 5 cards the same suit). Try 2 stages.
- <u>Solution:</u>
  - Stage 1: pick a suit. 4 choices.
  - Stage 2: pick a set of 5 cards in that suit. $\binom{13}{5} = \frac{13!}{8!5!}$
  - Every hand is picked in exactly one way: $4\binom{13}{5}$

Part of Bigger Exercise: How many ways to make nothing?

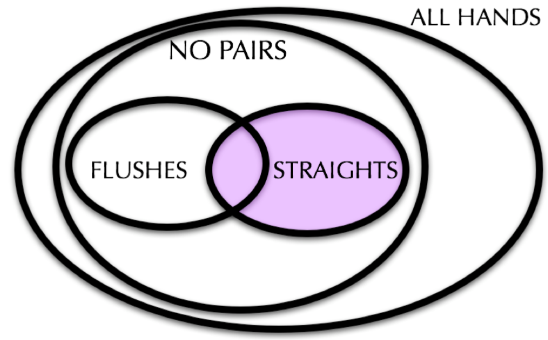
Straights



- Exercise: how many ways are there to make ***straight*** (5 consecutive ranks; aces can be low or high).

Part of Bigger Exercise: How many ways to make nothing?

Straights



- Exercise: how many ways are there to make ***straight*** (5 consecutive ranks; aces can be low or high).
- Solution:
  - Stage 1: pick the lowest rank. 10 choices {A,2,3,…,10}.
  - Stage 2: pick the suit of the lowest card. 4 choices.
  - …
  - Stage 6: pick the suit of the highest card. 4 choices.
  - Each hand picked in exactly one way. $10 \cdot 4^5$

Outline:

- Recap basic rules of counting, permutations, combinations
- Counting Poker Hands
- Binomial Theorem
  - Binomial coefficients & Pascal's Triangle
- Combinatorial Proofs
- Balls 'n' Bars method
  - Combinations with Replacement

Combinations and $(x + y)^n$

- What is the coefficient of x^2 in the expansion of $(x + 1)^5$?
- *Hint: Don't solve by brute force. Look for a counting problem.*

Expand as the product of five factors:

$$(x + y)^5 = (x + y)(x + y)(x + y)(x + y)(x + y)$$

The Binomial Theorem

- $\binom{n}{k}$ "n choose k" is called the *binomial coefficient*

- Binomial Theorem: For any x and y and non-negative integer n

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}}$$

$\binom{n}{k}$ is the number of ways to choose k xs out of the n "$(x+y)$" factors

Proving things with the binomial theorem

- Binomial Theorem: for any x and y

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Theorem: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$

- – In other words ?

Proving things with the binomial theorem

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Theorem: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$

– In other words ?

- Proof: Set $x = 1, y = 1$

The Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \qquad x \quad y$$

- Exercise: What is the coefficient of "x" in $\left(x^2 + \frac{1}{x}\right)^{20}$?

The Binomial Theorem

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- Exercise: What is the coefficient of "x" in $\left(x^2 + \frac{1}{x}\right)^{20}$?

The Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- Exercise: What is the coefficient of "x" in $\left(x^2 + \frac{1}{x}\right)^{20}$?
  - Need to find a where $\binom{20}{k} (x^2)^k (\frac{1}{x})^{20-k} = ax$
  - $a = \binom{20}{k}$, where k is the value that solves
  $$(x^2)^k (x^{-1})^{20-k} = x^1$$
  - $x^{2k} x^{-(20-k)} = x^{2k-20+k} = x^{3k-20}$
  - So $3k - 20 = 1$, yielding $k = 7$.
  - So coef. of x in binomial expansion of $\left(x^2 + \frac{1}{x}\right)^{20}$ is $\binom{20}{7}$

Binomial Coefficients and Pascal's Triangle

- $(x+y)^0 = \mathbf{1}$ (when $x + y \neq 0$)
- $(x+y)^1 = \mathbf{1} \cdot x + \mathbf{1} \cdot y$
- $(x+y)^2 = \mathbf{1} \cdot x^2 + \mathbf{2} \cdot xy + \mathbf{1} \cdot y^2$
- $(x+y)^3 = \mathbf{1} \cdot x^3 + \mathbf{3} \cdot xy^2 + \mathbf{3} \cdot x^2y + \mathbf{1} \cdot y^3$

How do you get the coefficients of $(x+y)^n$ from the coefficients of $(x+y)^{n-1}$?

Coefficient list of $(x+y)^0$	1
Coefficient list of $(x+y)^1$	1 1
Coefficient list of $(x+y)^2$	1 2 1
Coefficient list of $(x+y)^3$	1 3 3 1
of $(x+y)^4$	1 4 6 4 1
of $(x+y)^5$	1 5 10 10 5 1

Binomial Coefficients and Pascal's Triangle

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- $(x+y)^1 = \mathbf{1} \cdot x + \mathbf{1} \cdot y$
- $(x+y)^2 = \mathbf{1} \cdot x^2 + \mathbf{2} \cdot xy + \mathbf{1} \cdot y^2$
- $(x+y)^3 = \mathbf{1} \cdot x^3 + \mathbf{3} \cdot xy^2 + \mathbf{3} \cdot x^2y + \mathbf{1} \cdot y^3$

How do you get the coefficients of $(x+y)^n$ from the coefficients of $(x+y)^{n-1}$?

Expressed in "choose" notation:

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Outline:

- Recap basic rules of counting, permutations, combinations
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- **Combinatorial Proofs**
- Balls 'n' Bars method
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Combinatorial Proofs

- Oooohh… another proof method!
- Based on COUNTING
  - Specifically counting the same thing in different ways

- **Goal = *exposure* to combinatorial proofs**
  - Know what combinatorial proofs are
  - Understand why they are a valid proof method
  - We will <u>**not**</u> ask you to write your own combinatorial proofs in this class

Combinatorial Proofs vs. Algebraic Proofs

- ***Algebraic proof*** of identity: $\binom{n}{k} = \binom{n}{n-k}$
  - $\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$

- ***Combinatorial proof***. Count the same thing in two different ways.
  - LHS: There are n faculty members and you must form a committee containing k of them. There are $\binom{n}{k}$ ways to choose the k people on the committee.
  - RHS: On the other hand, this is the same as picking $n-k$ faculty members to ***not*** serve on the committee. There are $\binom{n}{n-k}$ ways to choose the $n-k$ non-members, hence $\binom{n}{k} = \binom{n}{n-k}$.

Combinatorial Proofs vs. Algebraic Proofs

- **Pascal's Identity**: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
- **Combinatorial proof.**

Combinatorial Proofs vs. Algebraic Proofs

- **Pascal's Identity**: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
- **Combinatorial proof.**
- LHS: There are $n+1$ people, n freshmen and 1 sophomore, and you are forming a team of k people. There are $\binom{n+1}{k}$ ways to do this.
- RHS: Consider separately the cases where the sophomore is or is not on the team:
  - Case 1: sophomore ***is not*** the team.
    - Pick the k team members from the n freshmen: $\binom{n}{k}$ ways to do do this.
  - Case 2: sophomore ***is*** the team.
    - Put the sophomore on the team: $\binom{1}{1} = 1$ way to do this
    - Then, pick the other $k-1$ team members from the n freshmen: $\binom{n}{k-1}$ ways
  - **(Sum Rule)** Total ways for RHS $= \binom{n}{k} + \binom{n}{k-1}$
- Since both sides were counting the same thing, we have proven

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

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Counting with and without replacement (duplicates)

Ways to choose ***sequence*** of k things from a set of n things (***no duplicates***)

Ways to choose ***set*** of k things from a set of n things (***no duplicates***)

Ways to choose ***sequence*** of k things each one of n ***types*** (***duplicates ok***)

Ways to choose ***set*** of k things each one of n ***types*** (***duplicates ok***)

Example: Counting Bitstrings

- How many 9-bit strings contain exactly five “1”s?

- Note: a “bit” is either 0 or 1
  - One 9-bit string with five “1”s: `0 0 1 1 1 0 1 1 0`

Problem: Counting Doughnuts

- A doughnut shop has 8 kinds of doughnuts.
  How many ways to buy a dozen doughnuts?

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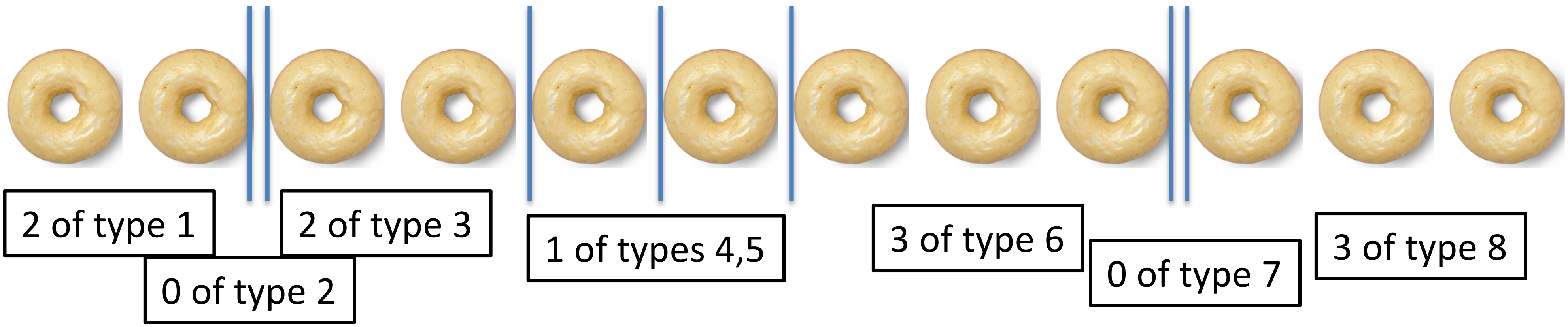
Buying Doughnuts

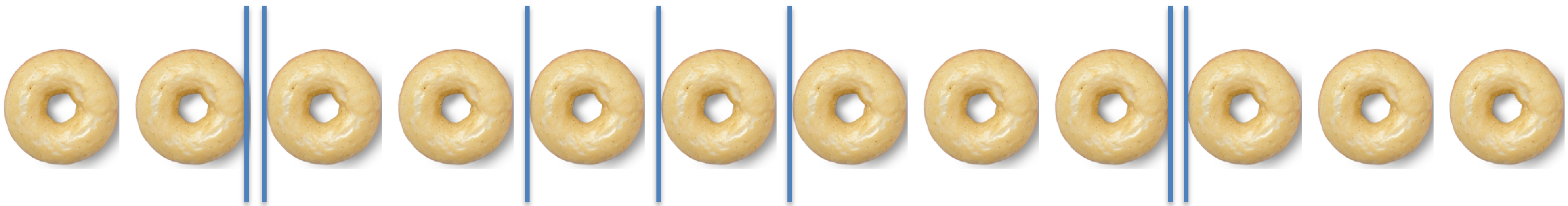
- A ***doughnut*** shop sells 8 types of doughnuts and has an unlimited supply of each type.
- How many ways are there to buy a ***dozen doughnuts***?



- Suppose we grouped them by type. We need $8 - 1 = 7$ "dividers" to show the grouping.
- This looks like a bit string! 12 "0"s (doughnuts) and 7 "1"s (dividers).

Buying Doughnuts

- A ***doughnut*** shop sells 8 types of doughnuts and has an unlimited supply of each type.
- How many ways are there to buy a ***dozen doughnuts***?



- We have a ***bijection*** between two sets:
  - "ways to pick dozen doughtnuts of eight types" and
  - "ways to choose a bit string with length 19 having 7 "1"s.

Tricky to think about

Much easier! $\binom{19}{7}$

Problem: Counting Doughnuts

- A doughnut shop has 8 kinds of doughnuts.
  How many ways to buy a dozen doughnuts?

Another view:

- Say you buy x_1 donuts of type 1, x_2 of type 2, etc.
- Then the number of ways to buy a dozen donuts = the number of solutions to the following equation, where each $x_i \geq 0$:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12$$

The Balls 'n' Bars Theorem

- The number of ways to choose ***k* objects** each of ***n* different types** ***(with repetition)*** is

$balls$ = ____________

$bars$ = ____________

Use **Balls n' Bars** for any counting problem that has

__________________________ objects, and

__________________________ boxes

Including problems of the form:

$$x_1 + x_2 + \cdots + x_n = k, \qquad \text{where each } x_i \geq 0$$

Counting with and without replacement (duplicates)

Ways to choose ***sequence*** of k things from a set of n things (***no duplicates***)

$$P(n,k) = \frac{n!}{(n-k)!}$$

Ways to choose ***set*** of k things from a set of n things (***no duplicates***)

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Ways to choose ***sequence*** of k things each one of n ***types*** (***duplicates ok***)

$$n^k$$

Ways to choose ***set*** of k things each one of n ***types*** (***duplicates ok***)

$$\binom{\boldsymbol{balls} + \boldsymbol{bars}}{\boldsymbol{balls}} = \binom{\boldsymbol{balls} + \boldsymbol{bars}}{\boldsymbol{bars}}$$

The "***balls-and-bars***" method

Counting strings of k balls (doughnuts)

$n-1$ bars (dividers)

E.g. o|ooo|oo|oo||oo||oo

Problem: Counting Doughnuts with **lower bounds**

- A doughnut shop has 8 kinds of doughnuts. How many ways to buy a dozen doughnuts

  with at least 1 of each kind?

= number of solutions to the following equation, where x_i is the number of doughnuts of kind i.

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12, \quad \text{each } x_i \geq \underline{\hspace{3em}}$$

Problem: Counting Doughnuts
with lower bounds

- A doughnut shop has 8 kinds of doughnuts. How many ways to buy a dozen doughnuts *with at least 1 of each kind*?
- = Number of solutions to:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12$$
$$\boldsymbol{x_i \geq 1} \text{ for } 1 \leq i \leq 8$$

Problem: Counting Doughnuts
with lower bounds

- A doughnut shop has 8 kinds of doughnuts. How many ways to buy a dozen doughnuts *with at least 1 of each kind*?
- Put 1 doughnut of each type aside, then

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 = \mathbf{4}$$

$$\boldsymbol{y_i \geq 0} \text{ for } 1 \leq i \leq 8$$

(8 doughnuts are already determined so we want $1 + y_i$ of each kind)

- No. of ways:

4 balls, 7 bars $\Rightarrow$ $\binom{11}{4}$

Problem: Counting Doughnuts
with upper bounds

- A doughnut shop has 8 kinds of doughnuts.

  How many ways to buy a dozen doughnuts with at most 4 strawberry iced and at most 2 coconut?

  = Number of solutions to: (natural numbers only)

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12$$

$$x_1 \leq 4, x_2 \leq 2$$

Problem: Counting Doughnuts
with upper bounds

- A doughnut shop has 8 kinds of doughnuts.

  How many ways to buy a dozen doughnuts with at most 4 strawberry iced and at most 2 coconut?

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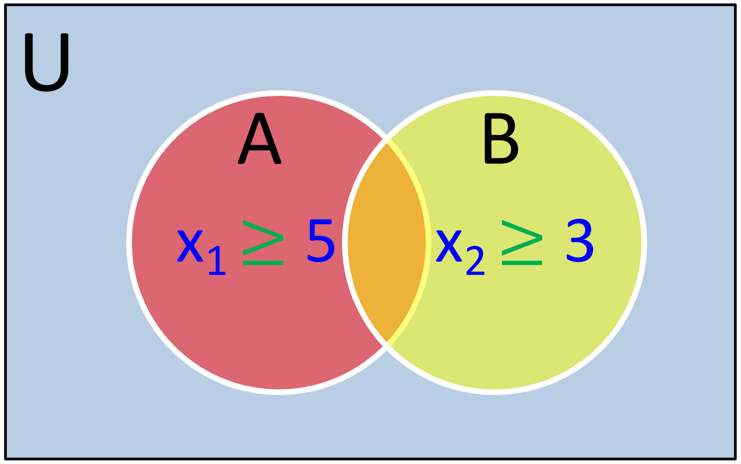
$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12$$

$$x_1 \leq 4, x_2 \leq 2$$

Solution plan:

- Convert to a "lower bound" problem
- Consider the complement event:

  $\bar{E}$ = #(ways with > 4 strawberry or > 2 coconut)

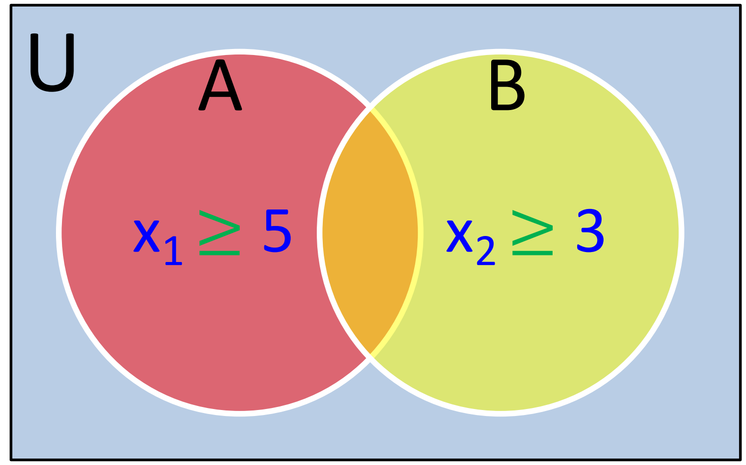


$$\begin{aligned} |E| &= |U| - |\bar{E}| \\ &= |U| - |A \cup B| \\ &= |U| - (|A| + |B| - |A \cap B|) \end{aligned}$$

<<< inclusion-exclusion principle

Problem: Counting Doughnuts

with upper bounds



Balls = #Objects; Bars = #Types-1

Number of solutions to: (natural numbers only)

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12$$

$$x_1 \leq 4,\ x_2 \leq 2$$

$$|E| = |U| - (|A| + |B| - |A \cap B|)$$

- $|U|$ All Solutions 12 balls, 7 bars $= \binom{19}{12}$
- $|A|$: Solutions with $x_1 \geq 5$ 7 balls, 7 bars $= \binom{14}{7}$
- $|B|$: Solutions with $x_2 \geq 3$ 9 balls, 7 bars $= \binom{16}{9}$
- $|A \cap B|$: Solutions with $x_1 \geq 5$ and $x_2 \geq 3$. 4 balls, 7 bars $= \binom{11}{4}$
- Solutions with $x_1 \leq 4$ and $x_2 \leq 2$: (inclusion-exclusion principle)

$$|E| = |U| - |A| - |B| + |A \cap B|) = \binom{19}{12} - \binom{14}{7} - \binom{16}{9} + \binom{11}{4}$$