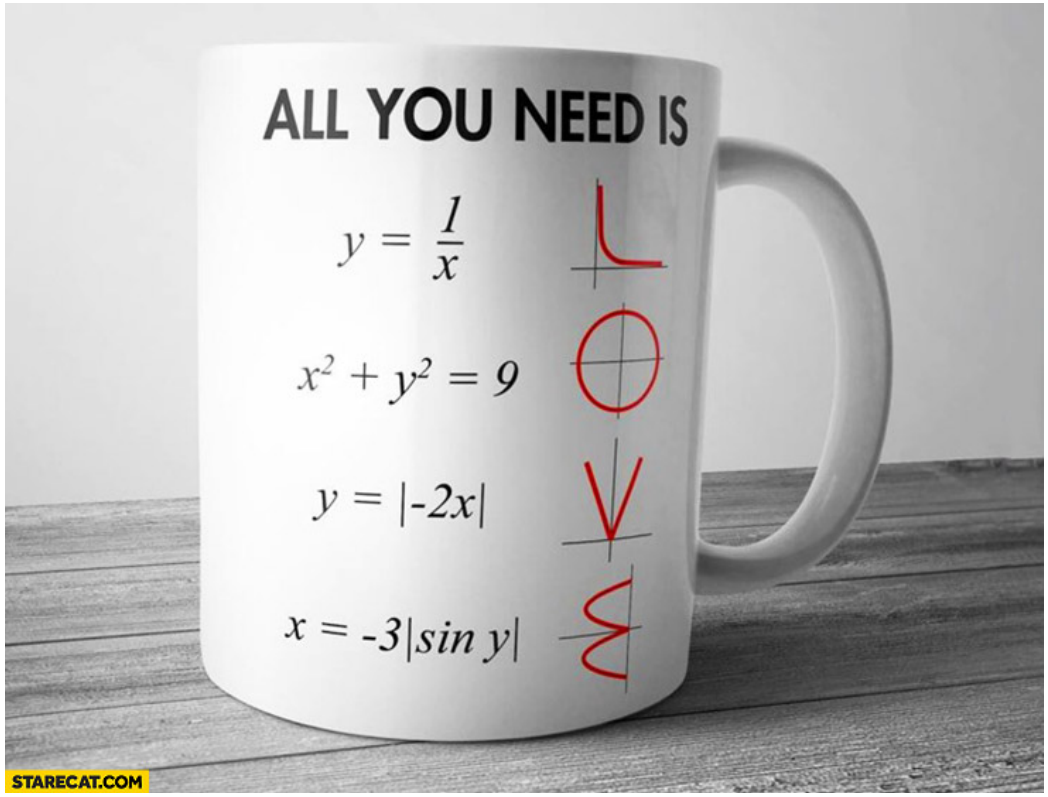
Functions



EECS 203: Discrete Mathematics

Lecture 14

Announcements

- Please fill out your midterm evaluations!
  - We value your feedback
- Midterm 1
  - Grading is coming along
  - We expect to post grades before Thursday's lecture
  - We'll talk about exam stats on Thursday

Upcoming Schedule

	Monday	Tuesday	Wednes.	Thursday	Friday
This week		Today		HW 5 due Discussion 6	
2/28-3/4			*Spring Break!*		
3/7-3/11		Discussion 6		HW 6 due	

- Take a break over the break!
  - Your GSIs and IAs (and faculty) are on break too
  - No Office Hours over break
  - Piazza will not be monitored/answered regularly by staff over the break
    - Feel free to post, but just know not to expect a prompt response

Learning Objectives: Functions & Properties

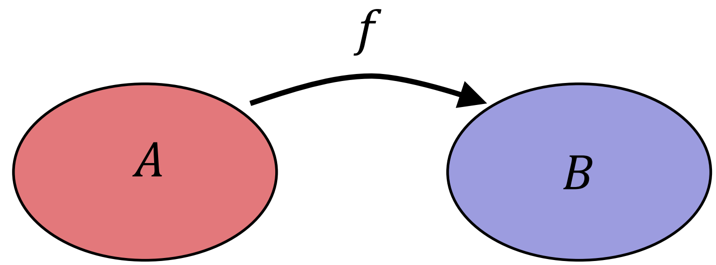
After today's lecture (and this week's readings, discussion & homework), you should be able to:

- **Technical vocab**: mapping, function, domain, codomain, range, onto, one-to-one, bijection, inverse function, function composition
- Function definition
  - Explain the key feature that is required for a mapping to be a function
  - Determine whether a given mapping is a function
- Function properties
  - Determine the properties of a given function: onto, one-to-one, bijection, invertible
  - Explain how the domain and codomain can affect function properties
  - Prove that a given function is or is not onto, and is or is not one-to-one
- Operations with Functions
  - Determine whether a function has an inverse, and if so, find the inverse function.
  - Find the composite function of f with g (mapping, domain & codomain)
  - Explain how "f composed with g" differs from "g composed with f"
  - Determine and prove the properties of a composite function (onto, one-to-one, etc.)

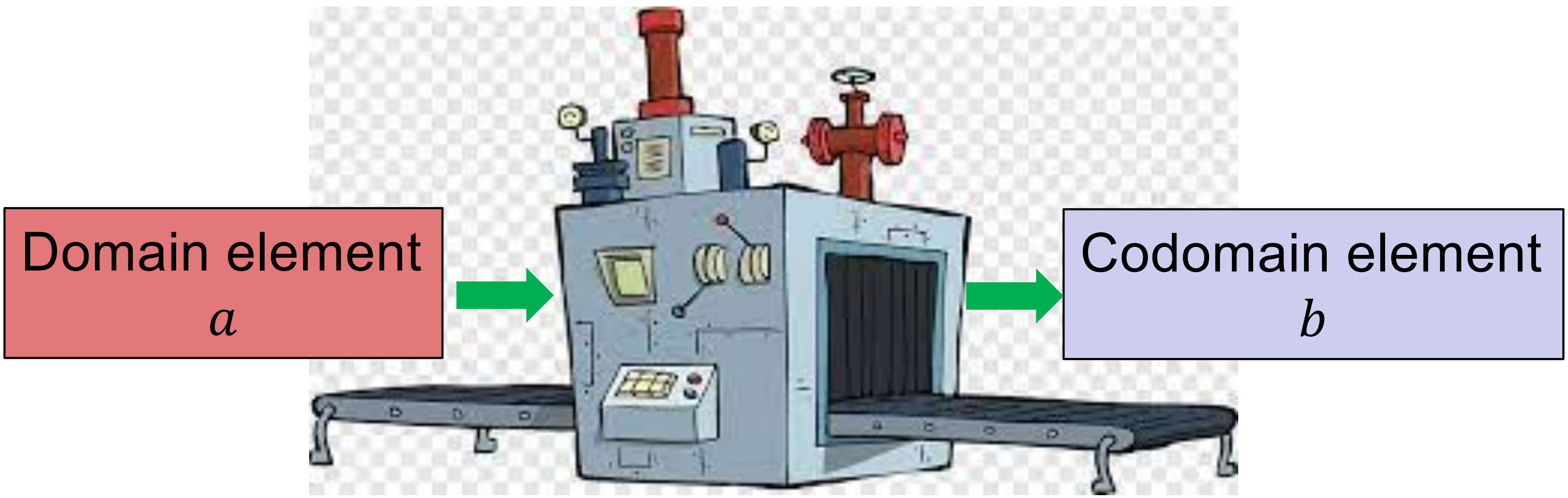
Outline

- **What is a function?**
  - **definition, domain, codomain, range**
- Function properties
  - Onto, One-to-one, bijections
  - Proofs of properties
- Operations with functions
  - Inverse functions
  - Adding and multiplying functions
  - Composition

Functions

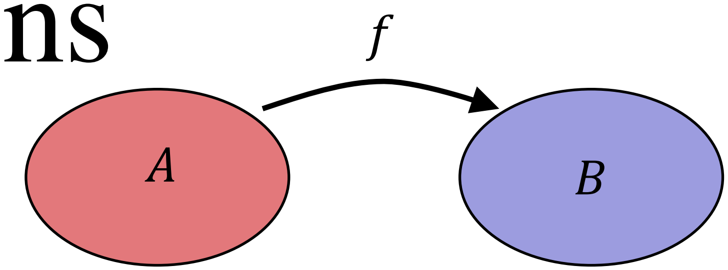


- What is a **function**?
- A **machine** where you can input an element from one given set (the "domain"), and it outputs an element from another given set (the "codomain")



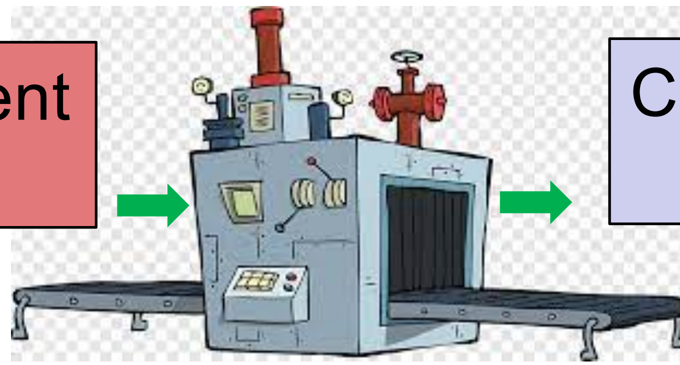
- Output is **exactly one** codomain element! Never 0, never ≥ 2
- It **is** possible that two domain elems give the same codomain elem

Writing Functions



- **Notation**: $f : A \to B$
  - Means that f is a function with **domain A** and **codomain B**

Domain element
a



Codomain element
b

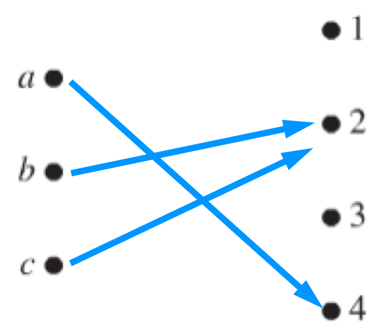
Two ways to write a function:

(1) Defining the transformation

$$f : \color{red}{\mathbb{Z}} \to \color{blue}{\mathbb{Z}}, \boldsymbol{f(x) = x + 1}$$

(2) Give the mapping by hand

$$f : \color{red}{\{a, b, c\}} \to \color{red}{\{1, 2, 3, 4\}}$$



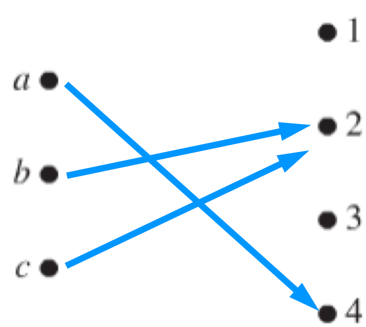
Note: exactly one arrow leaving each point on the left

Range vs. Codomain

$f : D \rightarrow C$

- Every element of the **domain** maps to exactly one element of the **codomain**.
- **Not** every element of the **codomain** must get mapped to.
- **Range:** The set of elements in the **codomain** that **do** get mapped to.

f:

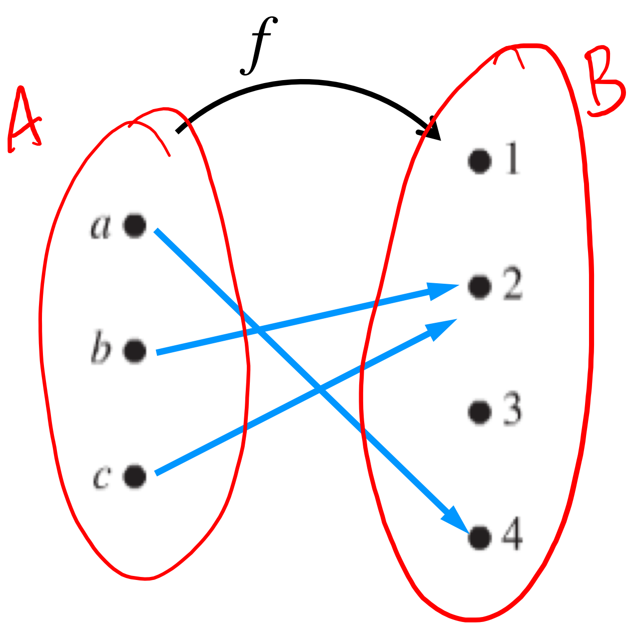


domain: {a, b, c}

codomain: {1, 2, 3, 4}
range: {2, 4}

Lec 14 Handout: Functions & Properties

- A **function** f : A $\rightarrow$ B is a machine where you input an element from a given set (the "domain") and it outputs an element from another given set (the "codomain").
- **Key Requirements** (to be a function):
  - Every element of the domain is valid input.
  - For every input, the function produces exactly one output
  - A mapping not necessarily satisfying these requirements is a **relation**.



Domain = {a, b, c}

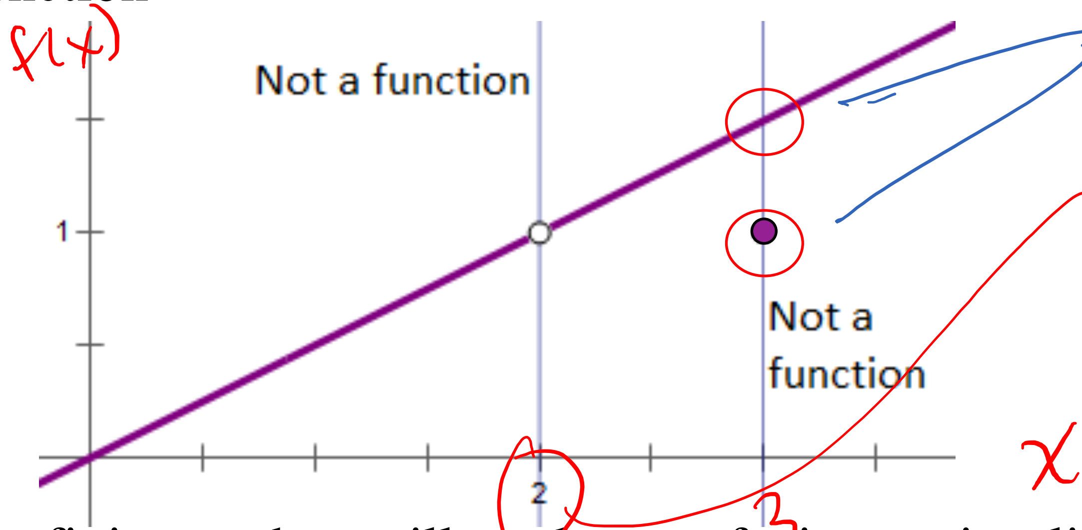
Codomain = {1, 2, 3, 4}

Range = {2, 4}

Vertical Line Test (not a proof)

For number-valued functions

- To visually check if something is a function, you can do the vertical line test
  - Imagine a vertical line at each point in the domain
  - If any have 0 or 2 (or more) intersections with the "function", then it is not a function



f(3) → 1, 1.5 !

f(2) DNE!

  - Graphs are finite, so they will not be proofs, just a visualization tool

Floor and Ceiling Functions

- The floor function of x, denoted $\lfloor x \rfloor$
  - The largest integer smaller than or equal to x
  - Examples:

    $\lfloor 1.2 \rfloor = 1$, $\lfloor 3 \rfloor =$?, $\lfloor -2.5 \rfloor =$?
- The ceiling function of x, denoted $\lceil x \rceil$
  - The smallest integer greater than or equal to x

    $\lceil 1.2 \rceil = 2$, $\lceil 3 \rceil =$?, $\lceil -2.5 \rceil =$?
- Some useful properties in Rosen Section 2.3

Floor and Ceiling Functions

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- With domain and codomain $\mathbb{R}$:
- Define $f: \mathbb{R} \to \boxed{\mathbb{R}}$ such that $f(x) = y$ if and only if $y = \lfloor x \rfloor$.
- Q: What is the **range** of f?
  (i.e., the set of values that f can take)

A. $\mathbb{R}$ ← this is the codomain

B. $\mathbb{R}^+$

C. $\mathbb{N}$

D. $\mathbb{Z}$

E. $\mathbb{Z}^+$

Floor and Ceiling Functions

- With domain and codomain $\mathbb{R}$:
- Define $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = y$ if and only if $y = \lceil x \rceil$.
- Q: What is the **range** of f?
  (i.e., the set of values that f can take)

  A. $\mathbb{R}$

  B. $\mathbb{R}^+$

  C. $\mathbb{N}$

  D. $\mathbb{Z}$ <= Correct answer

  E. $\mathbb{Z}^+$

Handout

Examples (or not!) of Functions

1. **Domain** and **codomain** are both the set of people **P**

   getchild(x) = y iff y is the child of x

   – Is this a function? No.

   - because 1 person can have more than 1 child
   - also, some people don't have kids ⇒ f(Joe) DNE

A. Yes
B. No

2. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = y$ iff $x = y^2$

   – Is this a function? No.

   - $f(-1)$ DNE ← in the codomain
   - $f(4) \to 2$, $f(4) \to -2$

A. Yes
B. No

3. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = y$ iff $y = 1/x^2$

   – Is this a function?

   $f(0)$ DNE

A. Yes
B. No

Examples (or not!) of Functions

1. **Domain** and **codomain** are both the set of people **P**
   getMother(x) = y iff y is the mother of x
   – Is this a function?

A. Yes
B. No

Examples (or not!) of Functions

1. **Domain** and **codomain** are both the set of people **P**
   getMother(x) = y iff y is the mother of x
   – Is this a function?
   – **Not a function:** not everyone has exactly 1 mother.

A. Yes
B. No

2. **f : R → R (R =** all real numbers), $\mathbf{f(x) = y}$ iff $\mathbf{x = y^2}$
   – Is this a function?

A. Yes
B. No

Examples (or not!) of Functions

1. **Domain** and **codomain** are both the set of people **P**
   getMother(x) = y iff y is the mother of x
   – Is this a function?
   – **Not a function:** not everyone has exactly 1 mother.

A. Yes
B. No

2. **f : R → R (R =** all real numbers), **f(x) = y** iff **$x = y^2$**
   – Is this a function?
   – **Not a function:**
     – some values of **x** map to 2 y's: **f(4) = 2** and **f(4) = -2**
     – some values of **x** don't map to any $y \in R$: **f(-1)**

A. Yes
B. No

3. **f : R → R (R =** all real numbers), **f(x) = y** iff **$y = 1/x^2$**
   – Is this a function?

A. Yes
B. No

Examples (or not!) of Functions

1. **Domain** and **codomain** are both the set of people **P**
   getMother(x) = y iff y is the mother of x
   – Is this a function?
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A. Yes
B. No

2. **f :** **R** $\rightarrow$ **R** (**R =** all real numbers), **f(x) = y** iff **x = y**^{2}
   – Is this a function?
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     – some values of **x** don't map to any y ∈ R: **f(-1)**

A. Yes
B. No

3. **f :** **R** $\rightarrow$ **R** (**R =** all real numbers), **f(x) = y** iff **y = 1/x**^{2}
   – Is this a function?
   – **Not a function: f(0) has no corresponding y**

A. Yes
B. No

Outline

- What is a function?
  - definition, domain, codomain, range
- **Function properties**
  - **Onto, One-to-one, bijections**
  - Proofs of properties
- Operations with functions
  - Inverse functions
  - Adding and multiplying functions
  - Composition

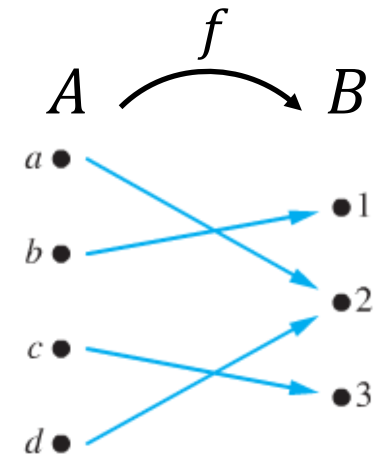
Onto

Given a function $f: A \to B$

- f is **onto** (or "**surjective**") iff

$$\forall b \in B \ \ \exists a \in A \ \ [\, f(a) = b \,]$$

Example: Mapping of guests to rooms in a full hotel



onto

"f is onto" means the same thing as "range(f) = codomain(f)"

for every element of the codomain

there is an elem. of the domain

that maps to it.

every element in the codomain is mapped to

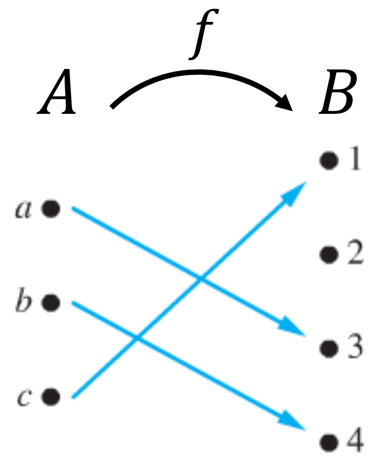
One-to-one

Given a function $f: A \to B$

- f is **one-to-one** (or "**injective**") iff

$$\forall a, b \in A \;\; [\,(f(a) = f(b)) \to (a = b)]$$

$\equiv \forall a, b \in A \;\; [\, a \neq b \to f(a) \neq f(b) \,]$



one-to-one

Example: Mapping of students to seats in a [possibly non-full] classroom

if I told you the output, you could tell me the input

Onto, One-to-one, Bijections, Inverse Functions

Handout

Given a function $f: A \to B$

- f is **onto** iff

  $\forall b \in B \; \exists a \in A \; [f(a) = b]$

- f is **one-to-one** iff

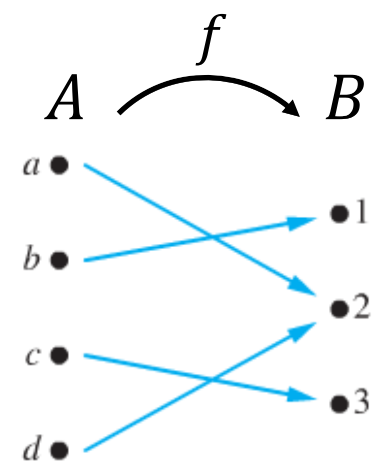
  $\forall a, b \in A \; [f(a) = f(b) \to (a = b)]$

  $\equiv \; \forall a, b \in A \; [a \neq b \to f(a) \neq f(b)]$

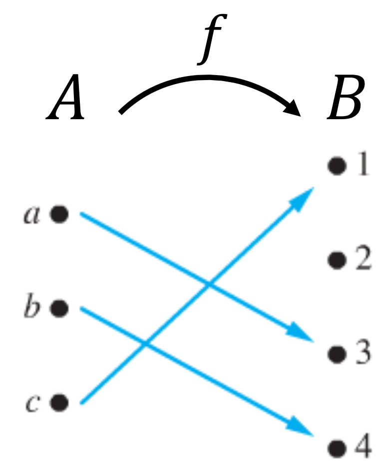
- f is a **bijection** iff

- f is **invertible** iff f is __________________

  - $f^{-1}(b) = a \;\; \leftrightarrow \;\; f(\_\_\_) = \_\_\_$



onto

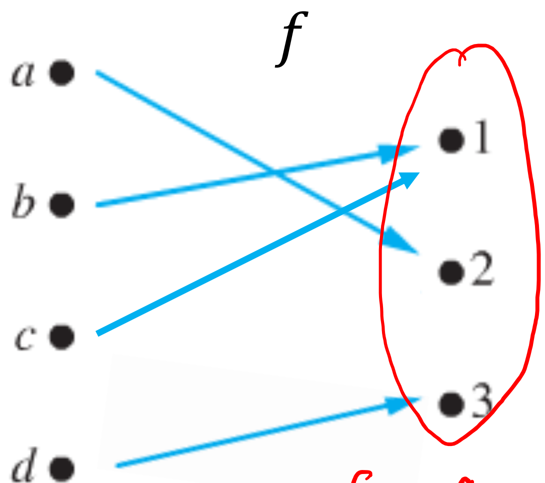


one-to-one

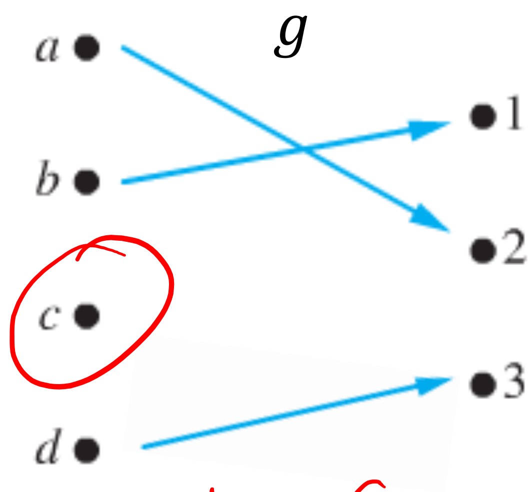
Exercise: One-to-one and Onto

Handout

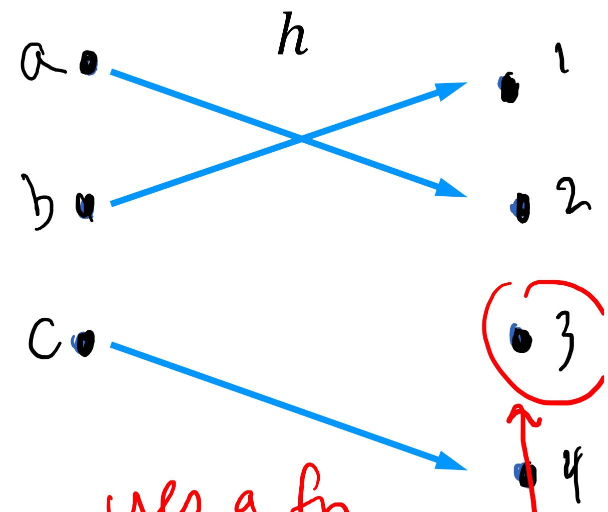
1. **For each of f, g, and h: is it a function? If so, which properties does it have?**



yes a fn
yes onto
Not 1-1



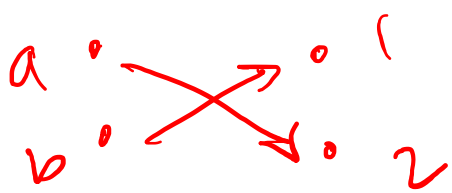
Not a fn



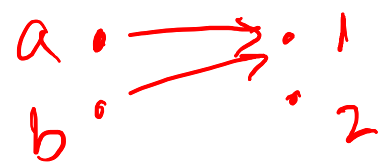
yes a fn.
Not onto
yes 1-1

2. **Draw a function that is**

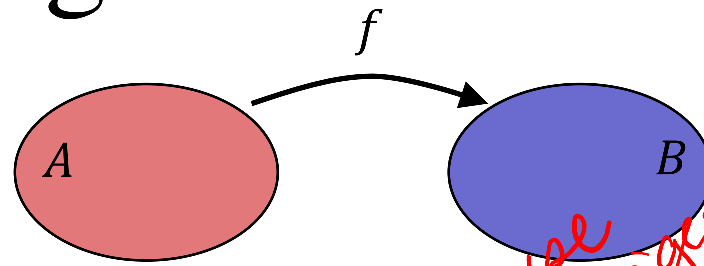
   a) **Onto and one-to-one**



   b) **Neither onto nor one-to-one**



Functions and Pigeons

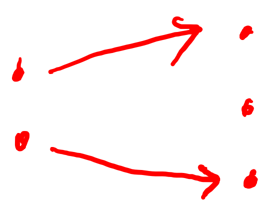


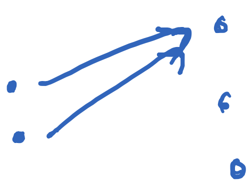
Let $f: A \to B$ be a function

because more pigeons than holes

- If $|A| < |B|$
  - can f be onto? No
  - can f be 1-1? yes

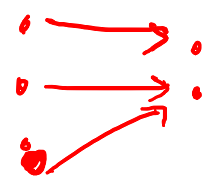
  must f be 1-1? No





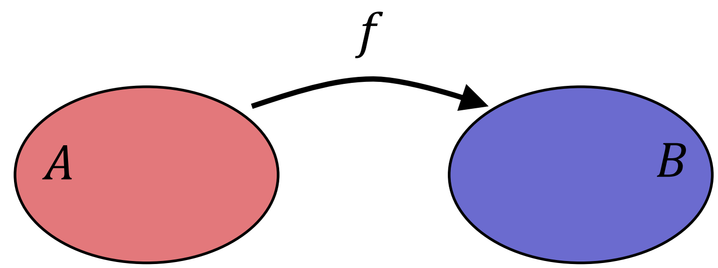
- If $|A| > |B|$
  - can f be onto? yes
  - can f be 1-1? No,

  must f be onto?



Functions and Pigeons

Let $f: A \rightarrow B$ be a function



- If $|A| < |B|$
  - can f be onto? No
  - can f be 1-1? Yes
  - must f be 1-1? No
- If $|A| > |B|$
  - can f be onto? Yes
  - can f be 1-1? No
  - must f be onto? No

Properties Practice

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- Let $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = 2x + 4$.

Which describes the function f?

A. Onto and one-to-one

B. Onto but not one-to-one

C. Not onto, but one-to-one

D. Neither onto nor one-to-one

Bijections

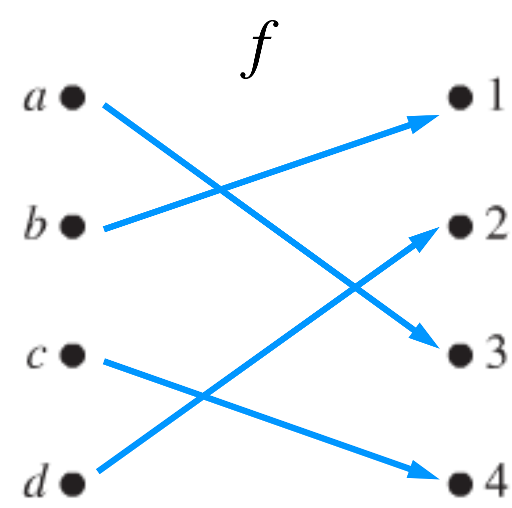
- If $f: A \to B$ is both one-to-one and onto then we say that f is a bijection.

> - Confusingly, a bijection is also called a *one-to-one correspondence*.
> - So be aware: "f is a one-to-one correspondence" actually means "f is a bijection", NOT "f is one-to-one".

- Examples

$f: \mathbb{R} \to \mathbb{R}$,
$f(x) = 2x + 4$

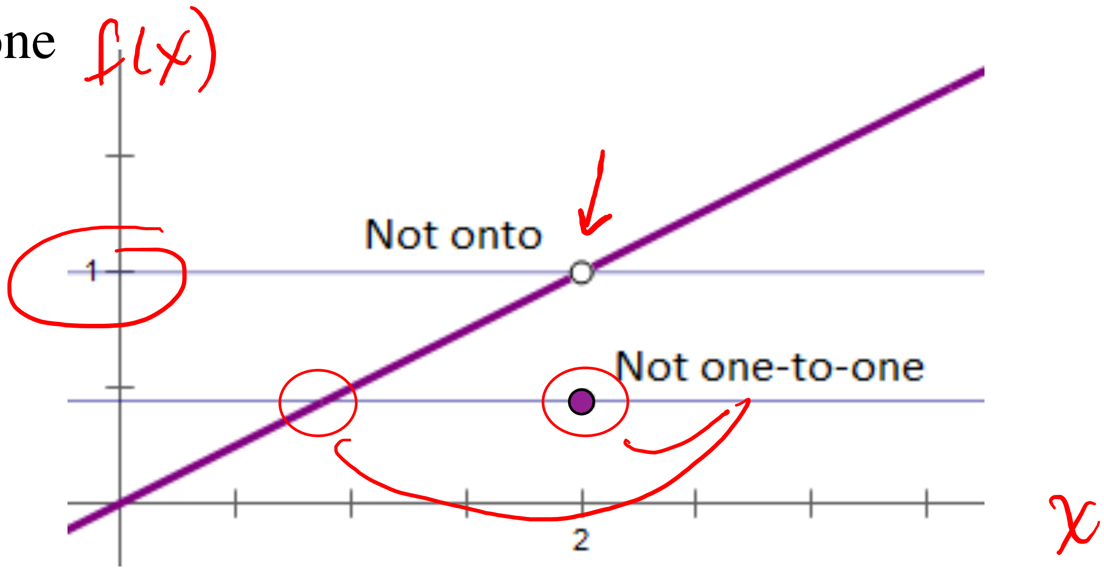
$f: \mathbb{R} \to \mathbb{R}$,
$f(x) = x^3$



Horizontal Line Test (not a proof)

For number-valued functions

- To visually check if something is one-to-one or onto, you can do the horizontal line test
  - Imagine a horizontal line at each point in the co-domain
  - If any have 0 intersections with the function, then it is not onto
  - If any have 2 or more intersections with the function, then it is not one-to-one



  - Again, graphs are finite, so they will not be proofs, just a visualization tool

Outline

- What is a function?
  - definition, domain, codomain, range
- Function properties
  - Onto, One-to-one, bijections
  - **Proofs of properties**
- Operations with functions
  - Inverse functions
  - Adding and multiplying functions
  - Composition

Proof: $f(x)$ is one-to-one

Handout

Let $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2x + 4$ **Claim:** f is **one-to-one**

Proof: $f(x)$ is one-to-one

Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 4$ **Claim:** f is **one-to-one**

In Logic: $\forall a, b \in A\ [\boxed{(f(b) = f(a))} \to \boxed{(b = a)}]$

for all intro

$p \to q$

Proof style: Direct proof (prove a forall, prove p → q)

Proof:

Proof: Let a, b be arbitrary real numbers.

Assume $f(b) = f(a)$ assume p.

$2b + 4 = 2a + 4$ apply defn of f(x)

$2b = 2a$

$b = a$ q

So $\forall a, b \in \mathbb{R}\ [f(b) = f(a) \to b = a]$

$\therefore$ f is one-to-one.

p … q

$\therefore p \to q$

Proof: $f(x)$ is one-to-one

Let $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2x + 4$ **Claim:** f is **one-to-one**

In Logic: $\forall a, b \in A\ [\ \big(f(b) = f(a)\big) \to (b = a)\ ]$

Proof style: Direct proof (prove a forall, prove p → q)

<u>Proof</u>: Let a, b be *arbitrary* real numbers (the domain of f).

Suppose $f(a) = f(b)$. Then we have:

$$\begin{aligned} f(b) &= f(a) && \text{(Assumption)} \\ 2b + 4 &= 2a + 4 && \text{(Definition of } f\text{)} \\ 2b &= 2a && \text{(Algebra…)} \\ b &= a \end{aligned}$$

So, $\big(f(b) = f(a)\big) \to (b = a)$.

Since a and b were arbitrary, we can say

$$\forall a, b \in \mathbb{R}\ [\ \big(f(b) = f(a)\big) \to (b = a)]$$

Thus, $f(x)$ is one-to-one (by the definition of 1-1)

Proof: $f(x)$ is onto

Handout

Let $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2x + 4$ **Claim:** f is **onto**

Proof: $f(x)$ is onto

Let $f: \mathbb{R} \to \mathbb{R}$, $f(\boxed{x}) = 2\boxed{x} + 4$ **Claim:** f is **onto**

In Logic: $\forall b \in B \; \exists a \in A \, [f(a) = b]$

Proof style: Direct proof

(For an arbitrary b, we will find an a that maps to it)

2 **Proof:**

1 **Side work:**

Side work:

$f(a) = b$

$2a + 4 = b$

$2a = b - 4$

$a = \frac{b-4}{2}$

Proof: Let b be an arbitrary real number (the codomain).

Define $a = \frac{b-4}{2}$

$$f(a) = f\left(\boxed{\frac{b-4}{2}}\right) = 2\left(\frac{b-4}{2}\right) + 4 = b - 4 + 4 = b$$

$\therefore f$ is onto.

Proof: $f(x)$ is onto

Let $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2x + 4$ **Claim:** f is **onto**

In Logic: $\forall b \in B\ \exists a \in A\ [\, f(a) = b]$

Proof style: Direct proof

(For an arbitrary b, we will find an a that maps to it)

2

Proof: Let b be an *arbitrary* real number (the *co*domain of f). Define $a = \ ?$

1

Side work:

$$\begin{aligned} f(a) &= b \\ 2a + 4 &= b \\ 2a &= b - 4 \\ a &= \frac{b-4}{2} \end{aligned}$$

Proof: $f(x)$ is onto

Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 4$ **Claim:** f is **onto**

In Logic: $\forall b \in B \ \exists a \in A \, [\, f(a) = b]$

Proof style: Direct proof

(For an arbitrary b, we will find an a that maps to it)

1

Side work:

$$
\begin{aligned}
f(a) &= b \\
2a + 4 &= b \\
2a &= b - 4 \\
a &= \frac{b-4}{2}
\end{aligned}
$$

2

Proof: Let b be an *arbitrary* real number (the *co*domain of f). Define $a = \frac{b-4}{2}$. Note that $a \in \mathbb{R}$, the domain of f. Then,

$$
\begin{aligned}
f(a) &= 2a + 4 \\
&= 2\left(\frac{b-4}{2}\right) + 4 \\
&= b - 4 + 4 \\
&= b
\end{aligned}
$$

So, $\exists a \in \mathbb{R} \ [f(a) = b]$. (We found an a in the domain that maps to b.)

Since b was arbitrary, we can say $\forall b \in \mathbb{R} \ \exists a \in \mathbb{R} \, [\, f(a) = b]$

Thus, $f(x)$ is onto (by the definition of onto).

Proof: $f(x)$ is **not** onto

Handout

Let $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^3 + 1$ **Claim:** f is **not onto**

Proof: $f(x)$ is **not** onto

Let $f: \mathbb{Z} \to \mathbb{Z},\ f(x) = x^3 + 1$ **Claim:** f is **not onto**

In Logic: $\neg(\forall b \in B\ \exists a \in A\ [\ f(a) = b\ ])$

$\equiv\ \exists b \in B\ \forall a \in A\ \neg[\ f(a) = b\ ]$

$\equiv\ \exists b \in B\ \forall a \in A\ [\ f(a) \neq b\ ]$

In words: there is an element of the codomain that is not mapped to by any element of the domain

Proof approach: Find an element of the codomain that is not mapped to.

Thoughts: $f(-1) = 0,\ f(0) = 1,\ f(1) = 2,\ f(2) = 9,\ f(3) = 28, \ldots$

Proof: $f(x)$ is **not** onto

Let $f: \mathbb{Z} \rightarrow \mathbb{Z},\ f(x) = x^3 + 1$ **Claim:** f is **not onto**

<u>Proof</u>:

- Consider $b = 3$ (note that 3 is in the codomain of f).
- Setting $f(a) = b$, we have $f(a) = a^3 + 1 = 3$.
- Solving for a, we get $a = 2^{1/3}$.
- Note that the domain of f is integers, so $a = 2^{1/3}$ is not in the domain.
- Thus no element of the domain maps to 3, so f cannot be onto.

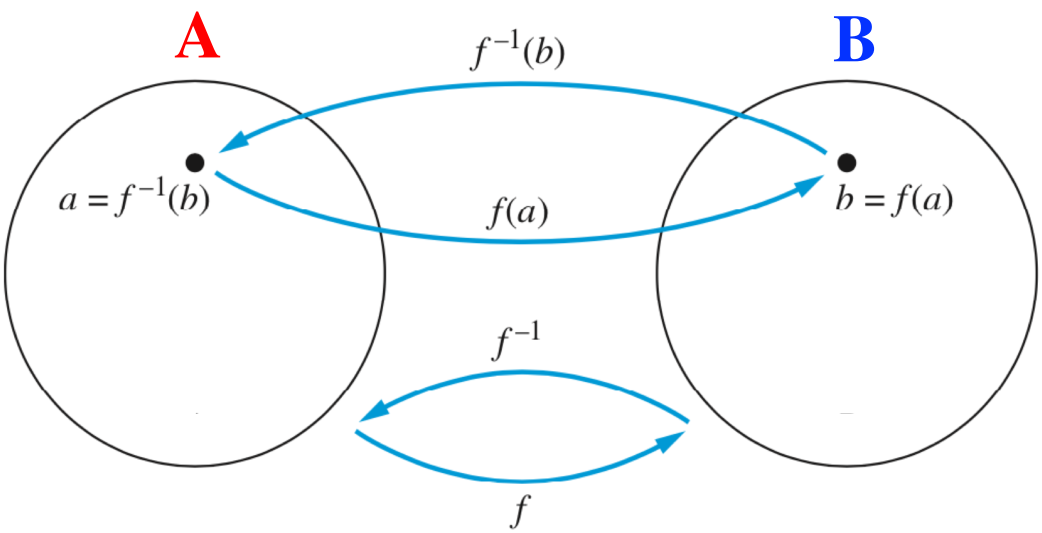
Outline

- What is a function?
  - definition, domain, codomain, range
- Function properties
  - Onto, One-to-one, bijections
  - Proofs of properties
- Operations with functions
  - **Inverse functions**
  - **Adding and multiplying functions**
  - Composition

Inverse Functions

- If f: $A \to B$ is a bijection, then f has an **inverse function** $f^{-1}: B \to A$, where

$$f^{-1}(b) = a \text{ iff } f(a) = b.$$



For any a in the domain of f
(i.e., the codomain of f^{-1}):
$f^{-1}(f(a)) = a$

For any b in the codomain of f
(i.e., the domain of f^{-1}):
$f(f^{-1}(b)) = b$

- Example:
  - $f: \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x + 4$
  - $f^{-1}: \mathbb{R} \to \mathbb{R}, \; f^{-1}(x) = (x - 4)/2$

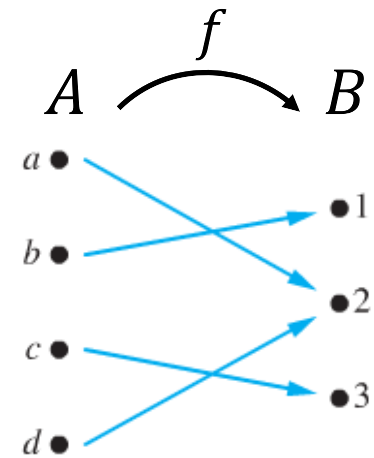
Adding and Multiplying Functions

- If functions f_1 and f_2 map from an arbitrary set to **R**, then:
  - $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
  - $(f_1 f_2)(x) = f_1(x)\, f_2(x)$

Onto, One-to-one, Bijections, Inverse Functions

Handout

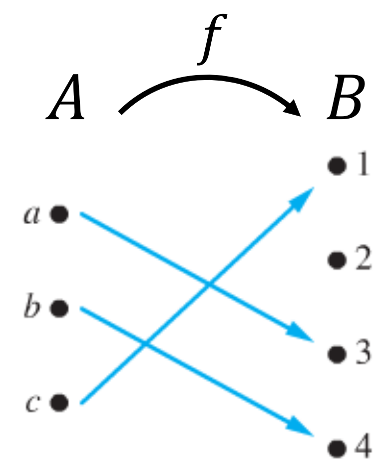
Given a function $f: A \to B$



onto

- f is **onto** iff

- f is **one-to-one** iff



one-to-one

- f is a **bijection** iff

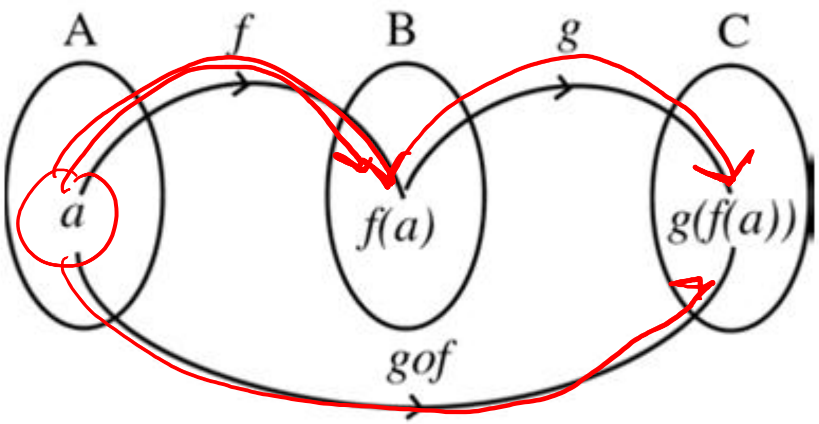
- f is **invertible** iff f is ________________
  - $f^{-1}(b) = a \;\leftrightarrow\; f(\_\_\_) = \_\_\_$

Outline

- What is a function?
  - definition, domain, codomain, range
- Function properties
  - Onto, One-to-one, bijections
  - Proofs of properties
- Operations with functions
  - Inverse functions
  - Adding and multiplying functions
  - **Composition**

Composition of Functions

- Suppose $f: A \to B$ and $g: B \to C$
- The **composition** $\boldsymbol{h = g \circ f}$ is a function $h: A \to C$,



$$\boldsymbol{h(a) = (g \circ f)(a) = g\big(f(a)\big)}$$

  – For $f(a) = b$ and $g(b) = c$,
  $$h(a) = (g \circ f)(a) = g\big(f(a)\big) = g(b) = c$$

- Note that $f \circ g$ is not defined (unless $C \subseteq A$)
  – $(f \circ g)(b) = f(g(b))$, so $g(b)$ needs to be in the domain of f
  – Output of the first function you apply must be legal input to the second function you apply

Exercise: $f \circ g$

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Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be defined as:

- $f(x) = 2\boxed{x} + 3$
- $g(x) = \underbrace{3x + 1}$

Which of the following is $(f \circ g)(x)$?

- A. 5x+4
- B. 6x+3
- C. 6x+4
- D. 6x+5
- E. 6x+10

$(f \circ g)(x) = f(g(x))$

$= f(\boxed{3x+1})$

$= 2(3x+1) + 3$

$= 6x + 2 + 3$

$= 6x + 5$

Exercise: $f \circ g$

Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be defined as:

- $f(x) = 2x + 3$
- $g(x) = 3x + 1$

Which of the following is $(f \circ g)(x)$?

- A. 5x+4
- B. 6x+3
- C. 6x+4
- D. 6x+5 <= Correct answer
- E. 6x+10

Function Properties with Composition

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- Let $f: A \to B$ and $g: B \to C$
  - Q_1: If f and g are 1-1, is $g \circ f$ also 1-1?
  - Q_2: If f and g are onto, is $g \circ f$ also onto?
  - Q_3: If g and $g \circ f$ are onto, is f also onto?

- The answers to $[Q_1, Q_2, Q_3]$ are:

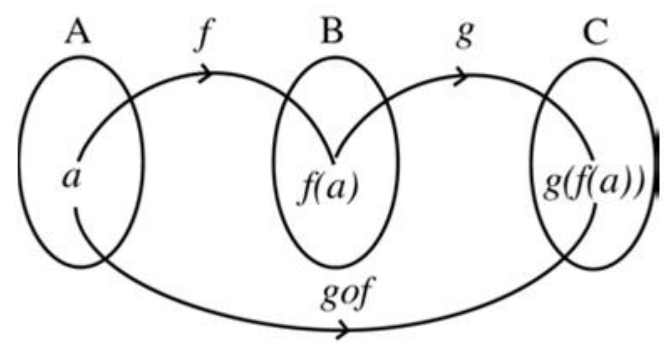
  A. [yes, yes, yes]

  B. [no, yes, yes]

  C. [yes, no, yes]

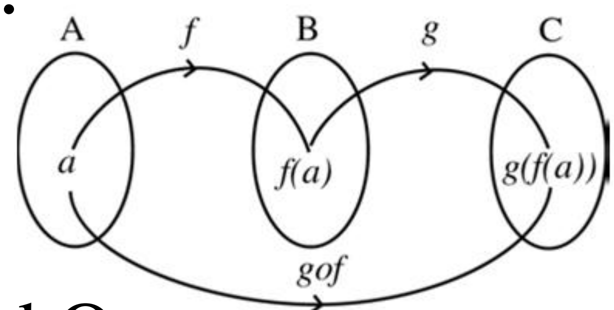
  D. [yes, yes, no]

  E. [no, no, no]



Function Properties with Composition

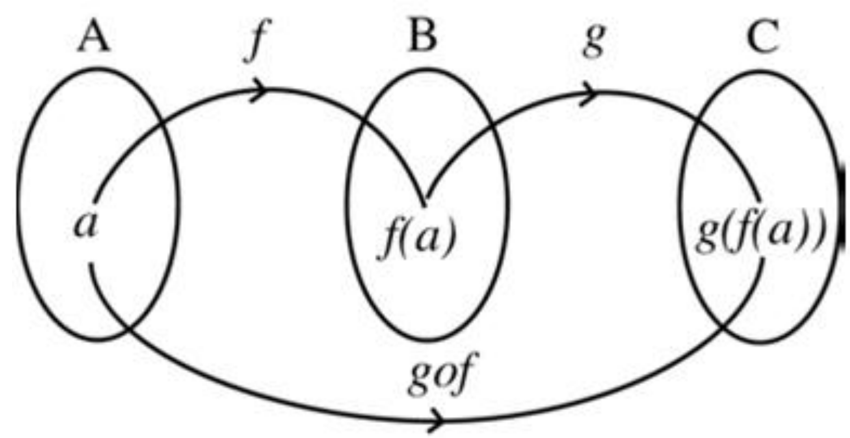
- Let $f : A \rightarrow B$ and $g : B \rightarrow C$
  - Q_1: If f and g are 1-1, must $g \circ f$ be 1-1?
  - Q_2: If f and g are onto, must $g \circ f$ be onto?
  - Q_3: If g and $g \circ f$ are onto, must f be onto?
- The answers to [Q_1, Q_2, Q_3] are:
  - D. [yes, yes, no]



We will prove the answer for Q_2 and Q_3.

Proof: f, g onto $\rightarrow g \circ f$ onto

- Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be functions
- Let $h = g \circ f$



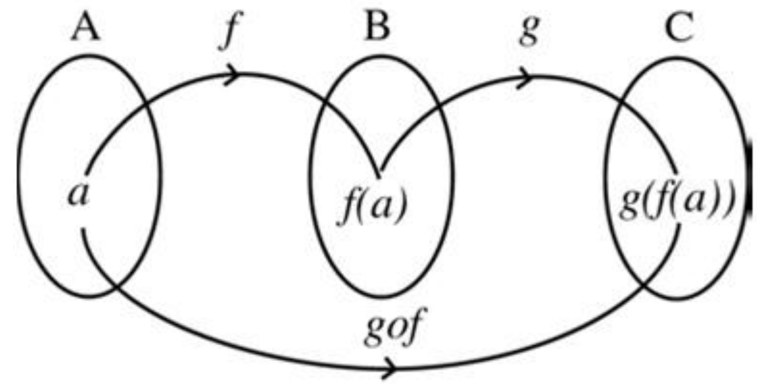
- We want to prove that if f and g are onto, then h is onto, i.e.,

$$\forall c \in C \; \exists a \in A \;\; [h(a) = c]$$

Proof: f, g onto $\rightarrow g \circ f$ onto

Handout

- $f: A \rightarrow B$ is onto

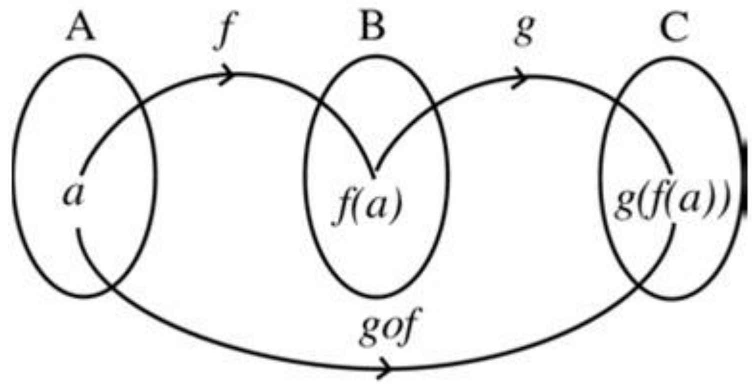


- $g: B \rightarrow C$ is onto

- So $g \circ f$ is onto

Proof: f, g onto $\rightarrow g \circ f$ onto

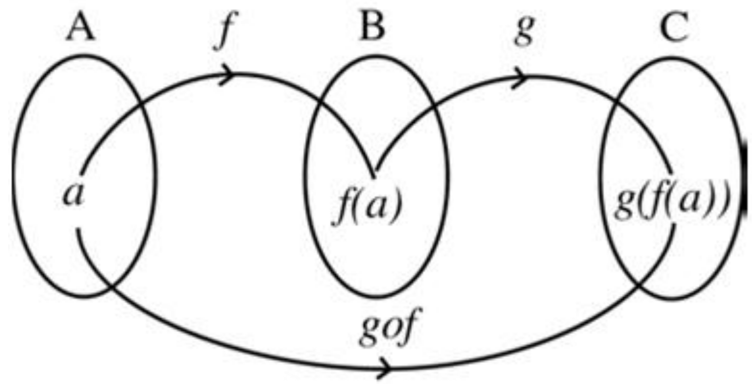
- $f: A \rightarrow B$ is onto, then by definition
  - $\forall b \in B\ \exists a \in A\ [f(a) = b]$
- $g: B \rightarrow C$ is onto, then by defn
  - $\forall c \in C\ \exists b \in B\ [g(b) = c]$



- $\forall c \in C\ \exists a \in A\ [(g \circ f)(a) = c]$
- So $g \circ f$ is onto (by definition of onto)

Proof: f, g onto $\rightarrow g \circ f$ onto

- $f: A \rightarrow B$ is onto, then by definition
  - $\forall b \in B \ \exists a \in A \ [f(a) = b]$
- $g: B \rightarrow C$ is onto, then by defn
  - $\forall c \in C \ \exists b \in B \ [g(b) = c]$
- Consider any element $c_0 \in C$

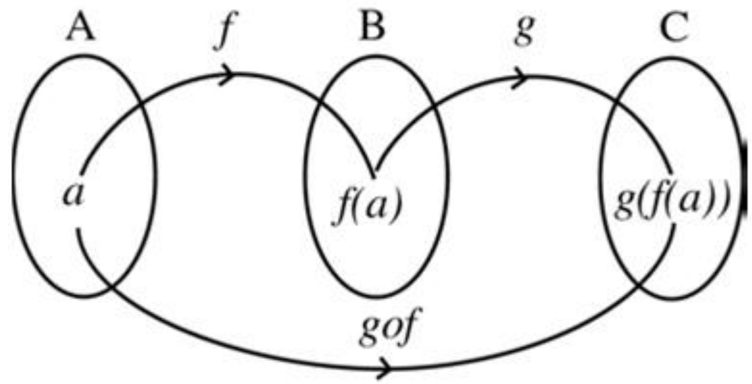


$$(g \circ f)(a_0) = g(f(a_0)) = c_0$$

  - So $\exists a \in A$ such that $(g \circ f)(a) = c_0$.
- $\forall c \in C \ \exists a \in A \ [(g \circ f)(a) = c]$
- So $g \circ f$ is onto (by definition of onto)

Proof: f, g onto $\rightarrow g \circ f$ onto

- $f: A \rightarrow B$ is onto, then by definition
  - $\forall b \in B \;\; \exists a \in A \;\; [f(a) = b]$



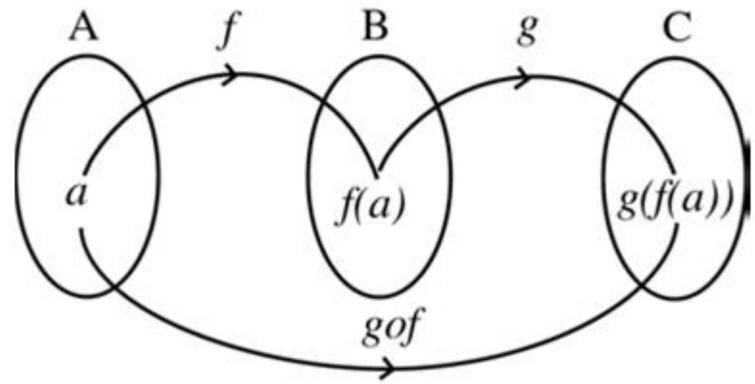
- $g: B \rightarrow C$ is onto, then by defn
  - $\forall c \in C \;\; \exists b \in B \;\; [g(b) = c]$
- Consider any element $c_0 \in C$
  - g is onto so there must be some element b_0 with $g(b_0) = c_0$.
  - f is onto so there must be some element a_0 with $f(a_0) = b_0$.

$$(g \circ f)(a_0) = g(f(a_0)) = c_0$$

  - So $\exists a \in A$ such that $(g \circ f)(a) = c_0$.
- $\forall c \in C \; \exists a \in A \;\; [(g \circ f)(a) = c]$
- So $g \circ f$ is onto (by definition of onto)

Proof: f, g onto $\rightarrow g \circ f$ onto

- $f: A \rightarrow B$ is onto, then by definition
  - $\forall b \in B \ \exists a \in A \ [f(a) = b]$
- $g: B \rightarrow C$ is onto, then by defn
  - $\forall c \in C \ \exists b \in B \ [g(b) = c]$

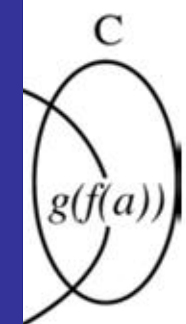


- Consider any element $c_0 \in C$
  - g is onto so there must be some element b_0 with $g(b_0) = c_0$.
  - f is onto so there must be some element a_0 with $f(a_0) = b_0$.
  - So $(g \circ f)(a_0) = g\big(f(a_0)\big) = g(b_0) = c_0$.
  - So $\exists a \in A$ such that $(g \circ f)(a) = c_0$.
- $\forall c \in C \ \exists a \in A \ [(g \circ f)(a) = c]$
- So $g \circ f$ is onto (by definition of onto)

Proof: f, g onto $\rightarrow g \circ f$ onto

Key takeaways from this proof:

- This is a proof like any other “logic” proof we’ve ever written
- Write down the premises and conclusion
- Work from the back *and* the front as you see fit
- The whole proof boils down to:
  - Applying rules of logical proofs (aka natural deduction rules)
  - Applying definitions



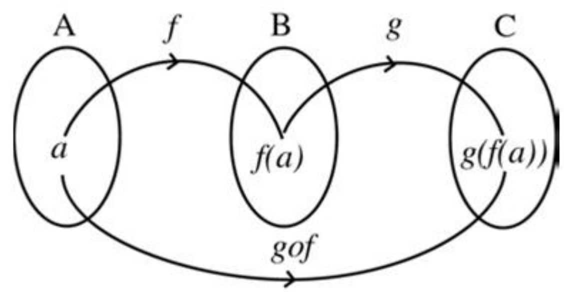
$= c_0.$

$= b_0.$

- So $g \circ f$ is onto (by definition of onto)

Proof: $g, g \circ f$ onto $\not\Rightarrow$ f onto

- Let $f: A \to B$ and $g: B \to C$ be functions
- Let $h = g \circ f$



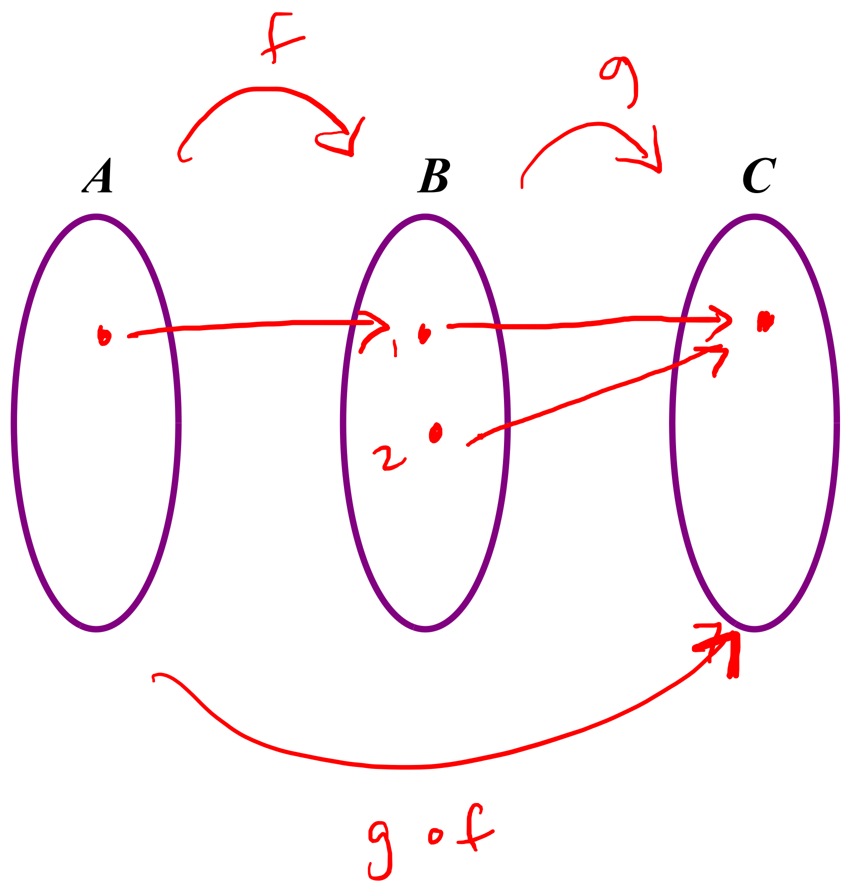
- We wish to prove that when g and h are onto, then f might not be onto.
  - We need to find an example of functions f and g such that g and h are onto, but ***f* is not onto**

$$\neg(\forall b \in B \ \exists a \in A \ [f(a) = b])$$

$$\equiv \quad \exists b \in B \ \forall a \in A \ [f(a) \neq b]$$

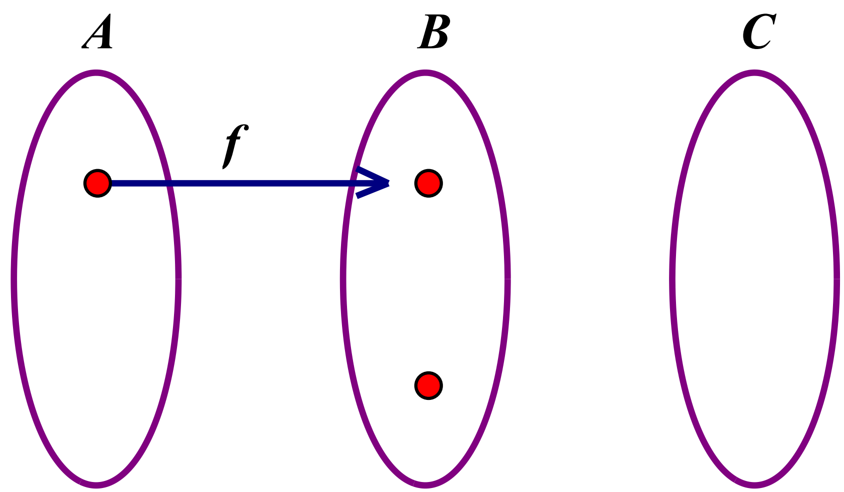
Proof: $g, g \circ f$ onto $\not\to$ f onto

- Let's start with a function f that is not onto



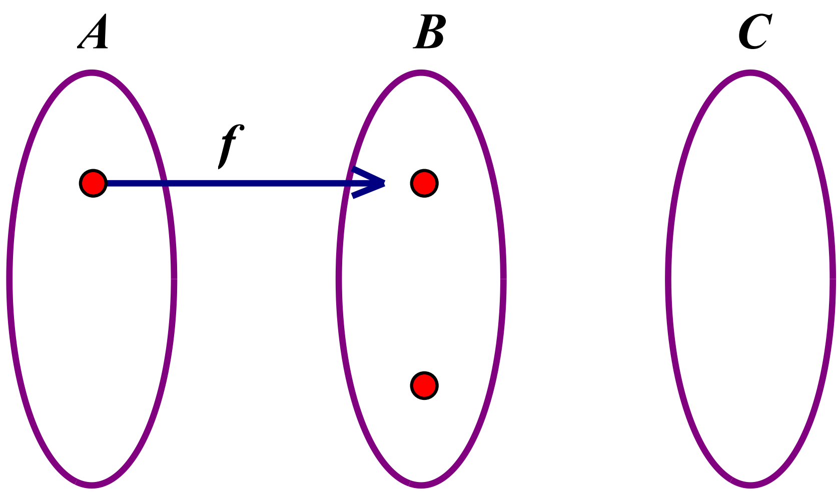
Proof: $g, g \circ f$ onto $\not\Rightarrow f$ onto

- Let's start with a function *f* that is not onto



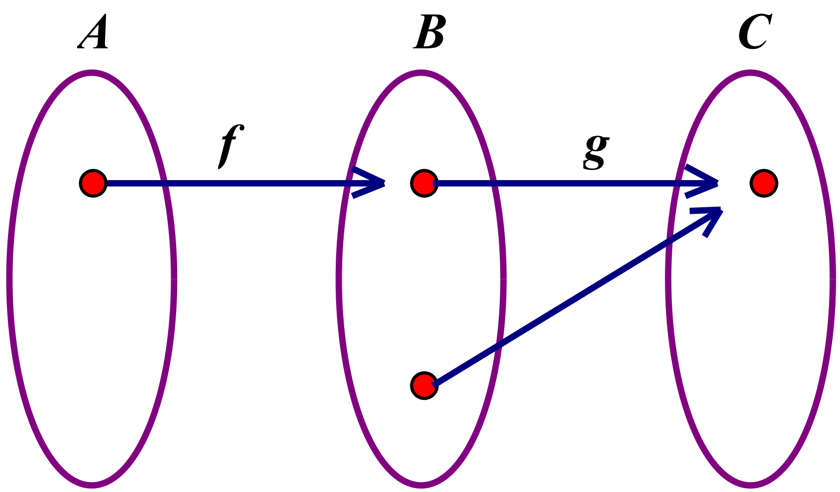
Proof: $g, g \circ f$ onto $\not\Rightarrow f$ onto

- Now, we need to make g onto in such a way that the composition is onto



Proof: $g, g \circ f$ onto $\not\Rightarrow f$ onto

- Now, we need to make g onto in such a way that the composition is onto



- Every element in C can be reached from B (g is onto)
- Every element in C can be reached from A ($g \circ f$ is onto)
- There is an element in B that cannot be reached from A (f is not onto)

Wrapup

- We'll see functions come up throughout this course
- The properties of **onto**, **one-to-one**, and **invertibility** are important in:
  - Counting (later this term)
  - Hashing functions (also uses modular arithmetic)
    - Many-to-one mappings
    - Finding the pre-image of a hashed value (what values map to the hashed value)
  - Cryptography and error-correcting codes
    - One-to-one and onto functions
  - Decryption of an encrypted message
    - Finding the pre-image of a point in the range (what value maps to that point)
  - 3-D translations/rotations
    - One-to-one correspondences, i.e., bijections