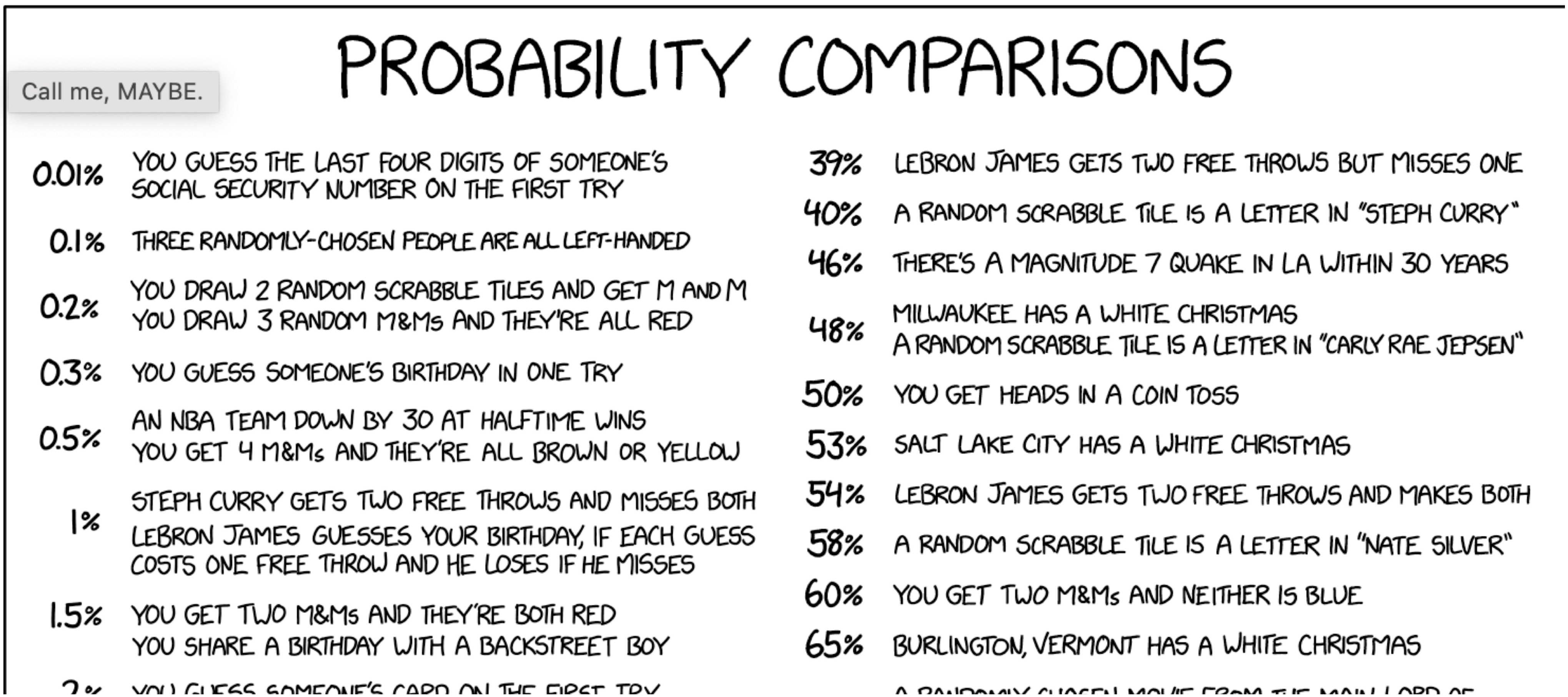
This slide deck posted under Files > Lecture Slides & Handouts > Annotated slides (Diaz)

L24: Generalized Binomial Coefficients + Introduction to Probability



Admin Announcements

more coming soon

- 1-1 meetings with faculty are available.
  - Meetings avail Wednesday morning, Friday afternoon
  - See Office Hours Calendar for times and sign-up link

- Remaining Homeworks
  - Homework 9, due Thursday
  - ***Homework #10*** is due ***TUESDAY April 19***.
    - It covers material from 3 lectures.
  - *Ungraded* Homework 11
    - You won't submit this
    - Problems on material from the final two lectures, so you can get practice before the exam

Outline:

- **Revisit: Donut Shop**
  - **Counting combinations with repetition**
- Combinatorial Proofs (as an fyi)
- Multinomial Coefficients
  - Counting permutations with *some* repetition

- Introduction to discrete probability
  - Experiment, Sample Space, Events
- Conditional Probability
- Independence
- Birthday Problem

Counting with and without replacement (duplicates)

Ways to choose ***sequence*** of k things from a set of n things (***no duplicates***)

$$P(n,k) = \frac{n!}{(n-k)!}$$

Ways to choose ***set*** of k things from a set of n things (***no duplicates***)

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!\ k!}$$

Ways to choose ***sequence*** of k things each one of n ***types*** (***duplicates ok***)

$$n^k$$

Ways to choose ***set*** of k things each one of n ***types*** (***duplicates ok***)

$$\binom{\boldsymbol{balls} + \boldsymbol{bars}}{\boldsymbol{balls}} = \binom{\boldsymbol{balls} + \boldsymbol{bars}}{\boldsymbol{bars}}$$

The "***balls-and-bars***" method

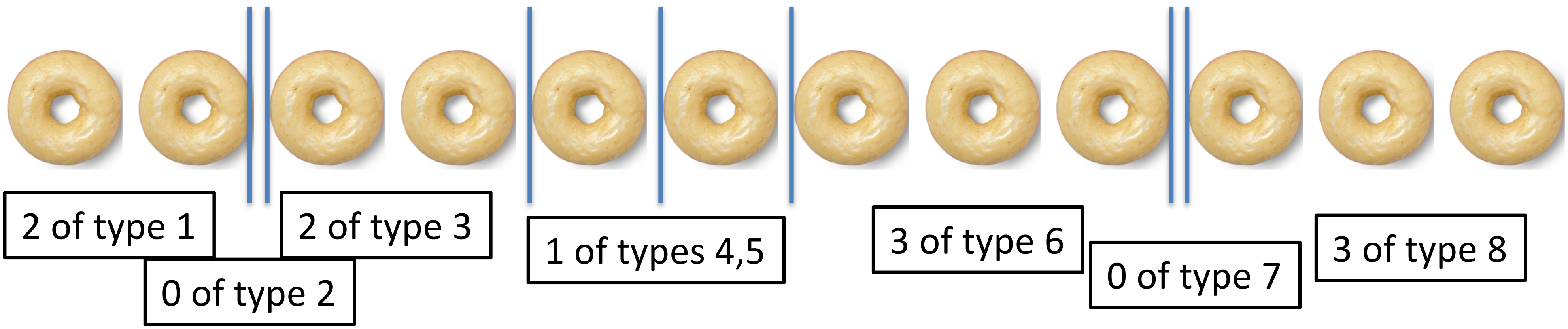
Counting strings of k balls (doughnuts)

$n-1$ bars (dividers)

E.g. o|ooo|oo|oo||oo||oo

Buying Doughnuts

- A ***doughnut*** shop sells 8 types of doughnuts and has an unlimited supply of each type.
- How many ways are there to buy a ***dozen doughnuts***?



- Suppose we grouped them by type. We need $8 - 1 = 7$ "dividers" to show the grouping.
- This looks like a bit string! 12 "0"s (doughnuts) and 7 "1"s (dividers).
- # of dozens = # of bit strings of length 19 with exactly 12 "0"s

The Balls 'n' Bars Theorem

- The number of ways to choose ***k* objects** each of ***n* different types** ***(with repetition)*** is

(donuts)

(flavors)

balls	= objects
bars	= dividers between types

$$\binom{\text{balls} + \text{bars}}{\text{balls}} = \binom{\text{balls} + \text{bars}}{\text{bars}}$$

$$\binom{k+n-1}{k} = \binom{k+n-1}{n-1}$$

$$\binom{a}{b} = \frac{a!}{(a-b)!\,b!}$$

Use **Balls n' Bars** for any counting problem that has

indistinguishable objects, and

distinguishable boxes

Including problems of the form:

$$x_1 + x_2 + \cdots + x_n = k, \qquad \text{where each } x_i \geq 0$$

Problem: Counting Doughnuts
with lower bounds

- A doughnut shop has 8 kinds of doughnuts. How many ways to buy a dozen doughnuts *with at least 1 of each kind*?
- = Number of solutions to:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12$$

$$\boldsymbol{x_i \geq 1} \text{ for } 1 \leq i \leq 8$$

Problem: Counting Doughnuts

with lower bounds

- A doughnut shop has 8 kinds of doughnuts. How many ways to buy a dozen doughnuts *with at least 1 of each kind*?
- Put 1 doughnut of each type aside, then

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 = \mathbf{4}$$

$$\boldsymbol{y_i} \geq \mathbf{0} \text{ for } 1 \leq i \leq 8$$

(8 doughnuts are already determined so we want $1 + y_i$ of each kind)

- No. of ways:

4 balls, 7 bars $\rightarrow \binom{11}{4}$ $= \binom{11}{7}$

Problem: Counting Doughnuts
with upper bounds

- A doughnut shop has 8 kinds of doughnuts.

  How many ways to buy a dozen doughnuts with at most 4 strawberry iced and at most 2 coconut?

  = Number of solutions to: (natural numbers only)

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12$$

$$x_1 \le 4, x_2 \le 2$$

Problem: Counting Doughnuts

with <u>upper</u> bounds

- A doughnut shop has 8 kinds of doughnuts.

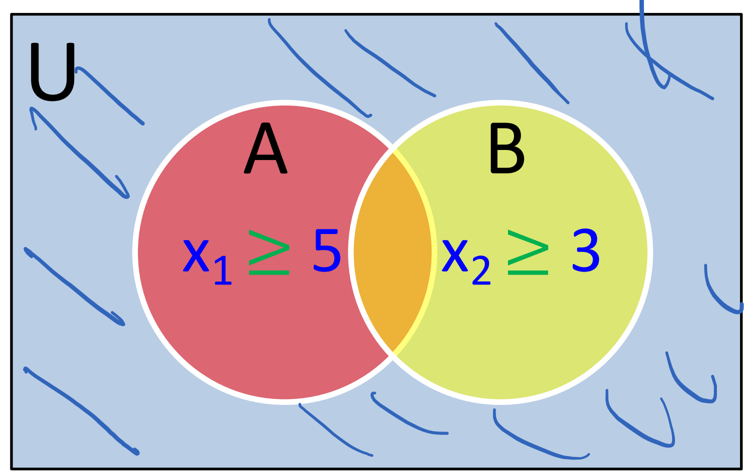
  How many ways to buy a dozen doughnuts with <u>at most</u> 4 strawberry iced and at most 2 coconut?

  = Number of solutions to: (natural numbers only)

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12$$

$$x_1 \leq 4, x_2 \leq 2$$

$\bar{E}$



Solution plan:

- Convert to a "lower bound" problem
- Consider the complement event:

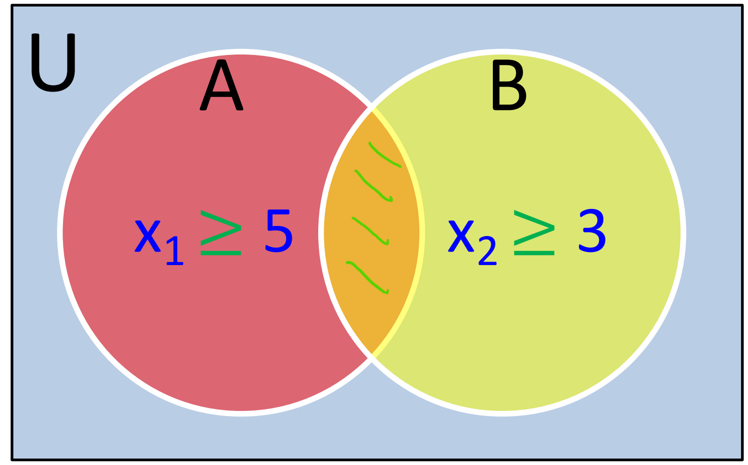
  $\bar{E} = $ #(ways with > 4 strawberry or > 2 coconut)

$$\begin{aligned} |E| &= |U| - |\bar{E}| \\ &= |U| - |A \cup B| \\ &= |U| - (|A| + |B| - |A \cap B|) \end{aligned}$$

<<< inclusion-exclusion principle

Problem: Counting Doughnuts

with upper bounds



Balls = #Objects; Bars = #Types-1

Number of solutions to: (natural numbers only)

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12$$

$$x_1 \leq 4,\ x_2 \leq 2$$

$\binom{\text{balls + bars}}{\text{bars}}$ or $\binom{\text{balls+bars}}{\text{balls}}$

$$|E| = |U| - (|A| + |B| - |A \cap B|)$$

- $|U|$ All Solutions 12 balls, 7 bars $= \binom{19}{7}$
- $|A|$: Solutions with $x_1 \geq 5$ 7 balls, 7 bars $= \binom{14}{7}$
- $|B|$: Solutions with $x_2 \geq 3$ 9 balls, 7 bars $= \binom{16}{9}$
- $|A \cap B|$: Solutions with $x_1 \geq 5$ and $x_2 \geq 3$. 4 balls, 7 bars $= \binom{11}{4}$
- Solutions with $x_1 \leq 4$ and $x_2 \leq 2$: (inclusion-exclusion principle)

$$|E| = |U| - |A| - |B| + |A \cap B| = \binom{19}{7} - \binom{14}{7} - \binom{16}{9} + \binom{11}{4}$$

Outline:

- Revisit: Donut Shop
  - Counting combinations with repetition
- **Combinatorial Proofs (as an fyi)**
- Multinomial Coefficients
  - Counting permutations with *some* repetition

- Introduction to discrete probability
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Combinatorial Proofs

- Oooohh… another proof method!
- Based on COUNTING
  - Specifically counting the same thing in different ways

- **Goal = *exposure* to combinatorial proofs**
  - Know what combinatorial proofs are
  - Understand why they are a valid proof method
  - We will **<u>not</u>** ask you to write your own combinatorial proofs in this class

Combinatorial Proofs vs. Algebraic Proofs

- ***Algebraic proof*** of identity: $\binom{n}{k} = \binom{n}{n-k}$
  - $$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$
- ***Combinatorial proof***. Count the same thing in two different ways.
  - LHS: There are n faculty members and you must form a committee containing k of them. There are $\binom{n}{k}$ ways to choose the k people on the committee.
  - RHS: On the other hand, this is the same as picking $n-k$ faculty members to ***not*** serve on the committee. There are $\binom{n}{n-k}$ ways to choose the $n-k$ non-members, hence $\binom{n}{k} = \binom{n}{n-k}$.

Combinatorial Proofs vs. Algebraic Proofs

- **Pascal's Identity**: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
- **Combinatorial proof.**

Combinatorial Proofs vs. Algebraic Proofs

- **Pascal's Identity**: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
- **Combinatorial proof.**
- LHS: There are $n+1$ people, n freshmen and 1 sophomore, and you are forming a team of k people. There are $\binom{n+1}{k}$ ways to do this.
- RHS: Consider separately the cases where the sophomore is or is not on the team:
  - Case 1: sophomore ***is not*** the team.
    - Pick the k team members from the n freshmen: $\binom{n}{k}$ ways to do do this.
  - Case 2: sophomore ***is*** the team.
    - Put the sophomore on the team: $\binom{1}{1} = 1$ way to do this
    - Then, pick the other $k-1$ team members from the n freshmen: $\binom{n}{k-1}$ ways
  - **(Sum Rule)** Total ways for RHS $= \binom{n}{k} + \binom{n}{k-1}$
- Since both sides were counting the same thing, we have proven

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Learning Objectives

After today's lecture (and associated readings, discussion & homework), you should be able to:

- Know how to use multinomial coefficients.
- ~~Using the inclusion-exclusion formula to count.~~

- Basics of discrete probability: *experiments* and *outcomes*, *sample spaces*, and *events*.

Outline:

- Revisit: Donut Shop
  - Counting combinations with repetition
- Combinatorial Proofs (as an fyi)
- **Multinomial Coefficients**
  - **Counting permutations with *some* repetition**

    i.e, sequences
- Introduction to discrete probability
  - Experiment, Sample Space, Events
- Conditional Probability
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Sequences with *some* repetition

- Example: How many license plates are there if each license plate must be 3 letters followed by 3 digits?
  - **L** = {A,B,…,Z}   **D** = {0,1,…,9}
  - Any license plate ∈ **L**×**L**×**L**×**D**×**D**×**D**

$\underline{26}\ \underline{26}\ \underline{26}\ \underline{10}\ \underline{10}\ \underline{10}$

$\Rightarrow 26^3\, 10^3$ ways

product rule

Sequences with *some* repetition

- Example: How many license plates are there if each license plate must be 3 letters followed by 3 digits?
  - **L** = {A,B,…,Z}   **D** = {0,1,…,9}
  - Any license plate ∈ **L**×**L**×**L**×**D**×**D**×**D**

Choose a plate number in 6 stages

26 choices in each of first 3 stages

10 choices in each last 3 stages

Product rule: $(26)^3(10)^3$

In general, # ways (n choices, r stages) = n^r

Problem: Vanity Plates

- Wouldn't it be nice to have your name on your license plate? (or if that's taken, an ***anagram*** of your name)
  - ***Q.*** How many different anagrams could you have?

    (WRONG) A. If your name has *n letters* it must be *n!*

~~RAMSEY~~

M Y R E A S

$\binom{6}{1}\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} = 6!$

R A M S E Y

~~DEKANEYD~~

E D D Y A L E N

2 Ds

2 Es

$\binom{8}{1}\binom{7}{1}\binom{6}{1}\binom{5}{2}\binom{3}{2}\binom{1}{1}$

C A C Y C L C D E N

~~DEKANEYD~~

E D D Y A L E N

$\binom{8}{2}\binom{6}{1}\binom{5}{2}\binom{3}{1}\binom{2}{1}\binom{1}{1}$

D L E A N Y

Problem: Vanity Plates

- Wouldn't it be nice to have your name on your license plate? (or if that's taken, an ***anagram*** of your name)
  - ***Q.*** How many different anagrams could you have?

    (WRONG) A. If your name has *n letters* it must be *n!*
  - ***A.*** It depends on your name!

E.g., "ZZZZZZ" only has one choice

"AZZZZZ" has 6 choices

"PETTIE" has 180 choices (TETEPI, etc.)

"GRAETZ" has 720 choices

"KIMDIAZ" has 2520 choices

Variation on repetition: Vanity Plates

- Wouldn't it be nice to have your name on your license plate? (or if that's taken, an ***anagram*** of your name)
  - ***Q.*** How many different anagrams could you have?
- Choose locations for each letter in stages: "PETTIE"
  - E's: $\binom{6}{2} = 15$ choices
  - T's: $\binom{4}{2} = 6$ choices
  - P: $\binom{2}{1} = 2$ choices
  - I: $\binom{1}{1} = 1$ choice
  - # choices: $15 \cdot 6 \cdot 2 \cdot 1 = 180$

I P E E T T

$\binom{6}{2}\binom{4}{2}\ 2 \cdot 1$

place Es T I P

I P E E T T

$\binom{6}{1}\binom{5}{1}\binom{4}{2}\binom{2}{2}$

Variation on repetition: Unique orderings

- In general
  - N-length sequences
  - k different objects
  - n_i repetitions of the ith object $\quad N = n_1 + n_2 + \cdots + n_k$
- Choosing the sequence in stages

$\binom{N}{n_1}$ ways to place repetitions of 1st object

$\binom{N-n_1}{n_2}$ ways to place repetitions of 2nd object

$\binom{N-n_1-n_2}{n_3}$ ways to place repetitions of 3rd object

$\vdots$

$\binom{N-n_1-n_2-\cdots-n_{k-1}}{n_k}$ ways to place repetitions of last object

- Therefore, total number of unique orderings

$$\binom{N}{n_1}\binom{N-n_1}{n_2}\binom{N-n_1-n_2}{n_3}\ldots\binom{N-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

Variation on repetition: Unique orderings

$$\binom{N}{n_1}\binom{N-n_1}{n_2}\binom{N-n_1-n_2}{n_3}\dots\binom{N-n_1-n_2-\cdots-n_{k-1}}{n_k} =$$

$$\frac{N!}{(N-n_1)!n_1!}$$

Variation on repetition: Unique orderings

$$\binom{N}{n_1}\binom{N-n_1}{n_2}\binom{N-n_1-n_2}{n_3}\ldots\binom{N-n_1-n_2-\cdots-n_{k-1}}{n_k}=$$

$$\frac{N!}{(N-n_1)!n_1!}\frac{(N-n_1)!}{(N-n_1-n_2)!n_2!}\frac{(N-n_1-n_2)!}{(N-n_1-n_2-n_3)!n_3!}\cdots\frac{(N-n_1-n_2-\cdots-n_{k-1})!}{(N-n_1-n_2-\cdots-n_k)!n_k!}=$$

$$\frac{N!}{\cancel{(N-n_1)!}n_1!}\frac{\cancel{(N-n_1)!}}{\cancel{(N-n_1-n_2)!}n_2!}\frac{\cancel{(N-n_1-n_2)!}}{\cancel{(N-n_1-n_2-n_3)!}n_3!}\cdots\frac{\cancel{(N-n_1-n_2-\cdots-n_{k-1})!}}{\cancelto{0}{(N-n_1-n_2-\cdots-n_k)}!n_k!}=$$

$$\frac{N!}{n_1!\,n_2!\,n_3!\ldots n_k!}$$

Variation on repetition: Multinomial

- In general
  - N-length sequences
  - k different objects
  - n_i repetitions of the ith object $\quad N = n_1 + n_2 + \cdots + n_k$
- Therefore, total number of unique orderings

$$\text{Multinomial Coefficient: } \frac{N!}{n_1!n_2!n_3!\ldots n_k!}$$

- Denoted by

$$\binom{N}{n_1, n_2, n_3, \ldots, n_k}$$

More Counting Examples in Main L24 Slide Deck

- Bigger/messier examples
- Inclusion-exclusion with > 2 sets

- How many ***onto*** functions are there of the form $f\colon \{1, \ldots, 10\} \to \{1, \ldots, 5\}$?

- How many bijections $f\colon \{1, \cdots, n\} \to \{1, \ldots, n\}$ such that $\forall x \;\; f(x) \neq x$?

Studying Recommendations

- The **product rule, sum rule,** and **division rule** are your friends, and will never fail you. Master these.

- Permutation, combination, balls-and-bars and multinomial formulas are **shortcuts that only work in certain situations.**
  - Permutations: pick **k** distinct items in order (derived from product rule)
  - Combination formula: pick **k** distinct items, order doesn't matter (derived from product + division rule)

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- **Introduction to discrete probability**
  - **Experiment, Sample Space, Events**
- Conditional Probability
- Independence
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Discrete Probability: Terminology

- Experiment: Procedure that yields an outcome
  - E.g., Tossing a coin three times
    - Outcome: HHH in one trial, HTH in another trial, etc.

- Sample space: Set of all possible outcomes in the experiment
  - E.g., Set of all possible outcomes
    - S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

☆ $E \subseteq S$ ☆

- Event: subset of the sample space (i.e., event is a set consisting of individual outcomes)
  - E.g., Event that # of heads is an even number.
    - E = {HHT, HTH, THH, TTT}

Discrete Probability: Terminology

- Experiment: Procedure that yields an outcome
- Sample space: Set of all possible outcomes
- Event: subset of the sample space (i.e., event is a set consisting of individual outcomes)

- If S is a sample space of **equally likely** outcomes, the probability of an event E is

$$\boxed{p(E) = \frac{|E|}{|S|}}$$

  - Counting comes into play here!
  - E.g., probability of an event E of having even # of heads when tossing a coin three times = 4/8 = 0.5

Example: 1 Die

- We roll a single die, what are the possible outcomes (sample space)?

  S = {1,2,3,4,5,6} $|S| = 6$

- What is an event that the outcome (number on the die) is a perfect square?

  E = {1,4} $|E| = 2$

- What is the probability of getting a perfect square number when you roll a die? (Assume it is a fair die)

  p(E) = |E|/|S| = 2/6 = 0.33

Example: 2 Dice

- We roll a pair of dice, what is the **sample space**?

- What is the **event** that the sum of two dice is 7?

- What is the probability of E?

Example: 2 Dice

- We roll a pair of dice, what is the **sample space**?

  S = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
  (2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
  (3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
  (4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
  (5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
  (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

  |S| = 36

- What is the **event** that the sum of two dice is 7?

  E = {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}

  |E| = 6

- What is the probability of E?

  p(E) = |E|/|S| = 6/36 = 1/6

Example: 2 Dice

- Could we also solve this using the set of *unordered* pairs as the sample space?
- **Sample space** $|S| = 21$

  S = {{1,1}, {1,2}, {1,3}, {1,4}, {1,5}, {1,6},
  {2,2}, {2,3}, {2,4}, {2,5}, {2,6},
  {3,3}, {3,4}, {3,5}, {3,6},
  {4,4}, {4,5}, {4,6}, {5,5}, {5,6}, {6,6}}

These outcomes are **not** equally likely.
$p(\{1,1\}) \neq p(\{1,2\})$

So we can't use $p(E) = \frac{|E|}{|S|}$

- **Event** that the sum of two dice is 7

  E = { {1,6}, {2,5}, {3,4} }

$|E| = 3$

- What is the probability of E?

  p(E) = |E|/|S| = 3/21 = ~~1/7~~

This answer is different than our previous calculation of $p(E) = 1/6$, and is in fact **incorrect**.
Where did we go wrong?

Discrete Probability: Terminology

- Experiment: Procedure that yields an outcome
- Sample space: Set of all possible outcomes
- Event: subset of the sample space (i.e., event is a set consisting of individual outcomes)

- If S is a sample space of **equally likely** outcomes, the probability of an event E is

$$p(E) = \frac{|E|}{|S|}$$

  - Counting comes into play here!
  - E.g., probability of an event E of having even # of heads when tossing a coin three times = 4/8 = 0.5

Discrete Probability: Nonuniform Probabilities

- **Sample space:** set S (finite or countably infinite set)

- $\boldsymbol{p}: S \to [0,1]$ assigns probability to each point in S.
  - Rephrased: for each $s \in S,\ 0 \le p(s) \le 1$.
  - $\sum_{s \in S} p(s) = 1$

- **Probability of Events:**

$$p(E) = \sum_{s \in E} p(s)$$

Example: 2 Dice

- Could we also solve this using the set of *unordered* pairs as the sample space?
- **Sample space**

  S = {{1,1}, {1,2}, {1,3}, {1,4}, {1,5}, {1,6},
  {2,2}, {2,3}, {2,4}, {2,5}, {2,6},
  {3,3}, {3,4}, {3,5}, {3,6},
  {4,4}, {4,5}, {4,6}, {5,5}, {5,6}, {6,6}}

These outcomes are **not** equally likely.
$p(\{1,1\}) \neq p(\{1,2\})$

So we can't use $p(E) = \frac{|E|}{|S|}$

- **Event** that the sum of two dice is 7

  E = { {1,6}, {2,5}, {3,4} }

Instead we'd need

$$p(E) = \sum_{s \in E} p(s)$$
$$= p(\{1,6\}) + p(\{2,5\}) + p(\{3,4\})$$
$$= \cdots$$ (This is more work than I want to do!)

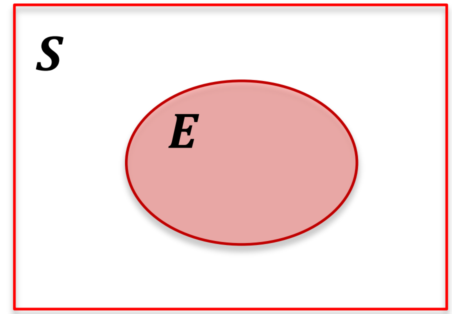
- What is the probability of E?

Key Takeaways:

- Use a sample space of **equally likely** outcomes whenever possible
- **Sometimes this means *imposing* an order**, to make the counting easier
- **Ordered dice rolls is a sample space of equally likely ~~events~~** outcomes

Discrete Probability

- **Experiment**: Procedure that yields an outcome
- **Sample space**: Set of all possible outcomes
- **Event**: subset of the sample space



If S is a sample space of **equally likely** outcomes, the **probability** of an event E is

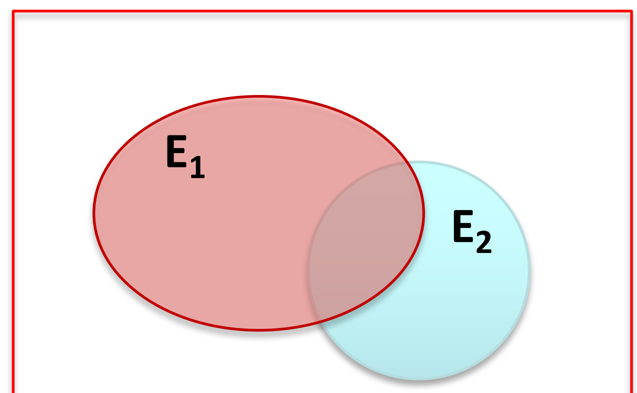
$$p(E) = \frac{|E|}{|S|}$$

If S consists of **unequally likely** outcomes, then

$$p(E) = \sum_{s \in E} p(s)$$

Fun facts:

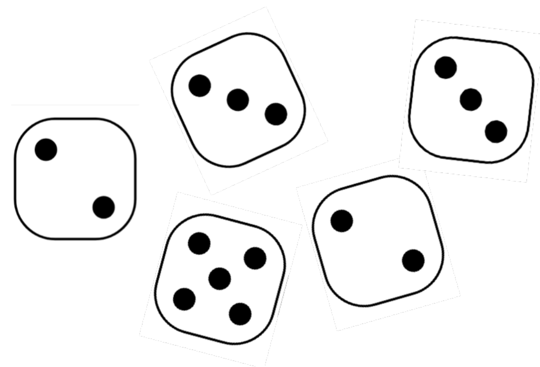
- $0 \leq p(E) \leq 1$
- $p(E) + p(\bar{E}) = 1$
- $p(\bar{E}) = 1 - p(E)$
- $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

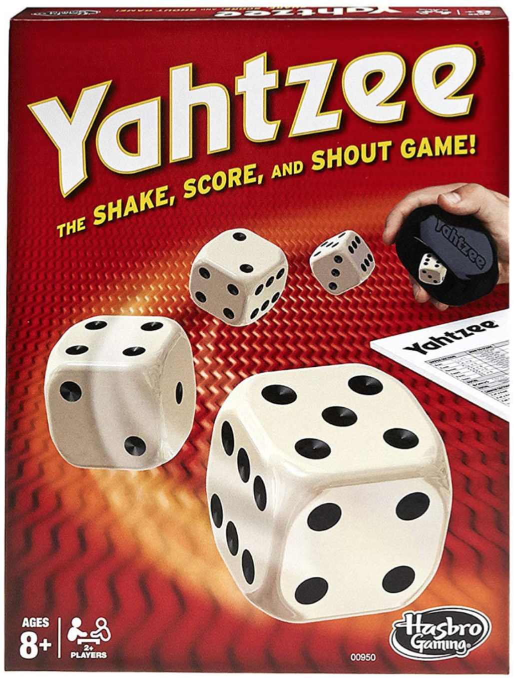


Yahtzee Exercise 1

- In Yahtzee, a player rolls five 6-sided dice
- How many **ordered** dice rolls are possible?

One possible
Yahtzee roll



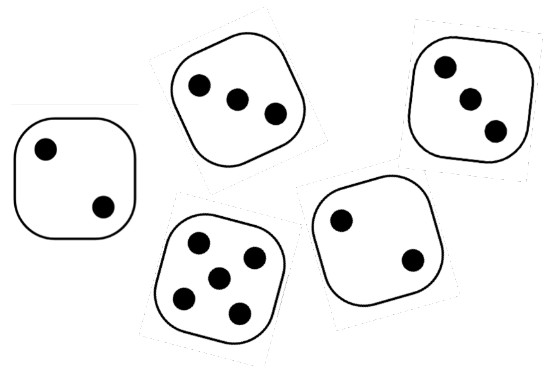


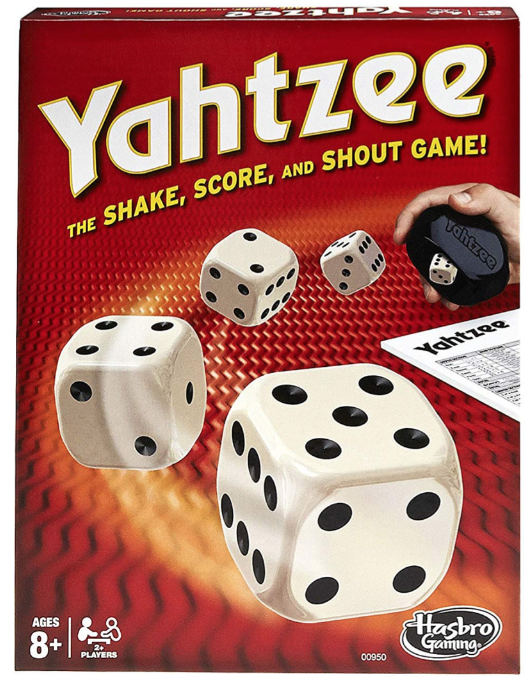
A: C(10,5)
B: C(6,5)
C: 6^5
D: None of these
E: Not sure

Yahtzee Exercise 1

- In Yahtzee, a player rolls five 6-sided dice
- How many **ordered** dice rolls are possible?

One possible Yahtzee roll



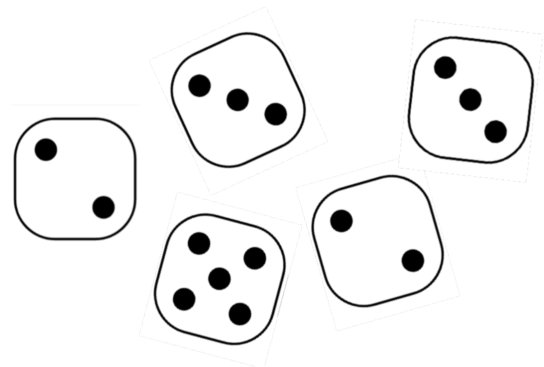


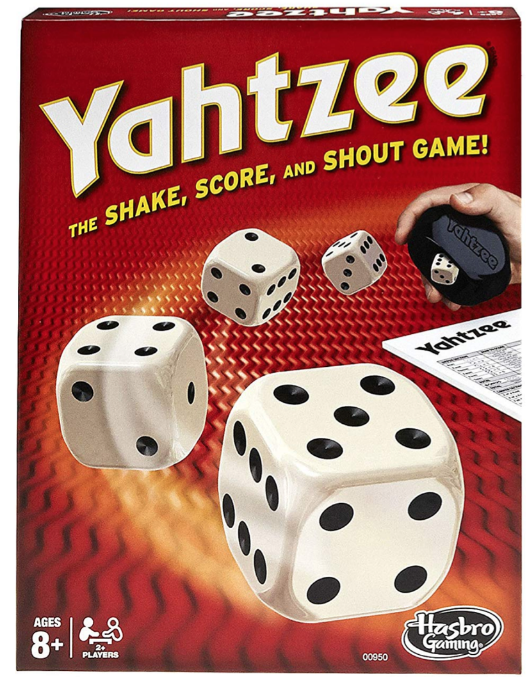
A: C(10,5)
B: C(6,5)
C: 6^5 (product rule)
D: None of these
E: Not sure

Yahtzee Exercise 2

- In Yahtzee, a player rolls five 6-sided dice
- How many **unordered** dice rolls are possible?

One possible
Yahtzee roll





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Yahtzee Exercise 2

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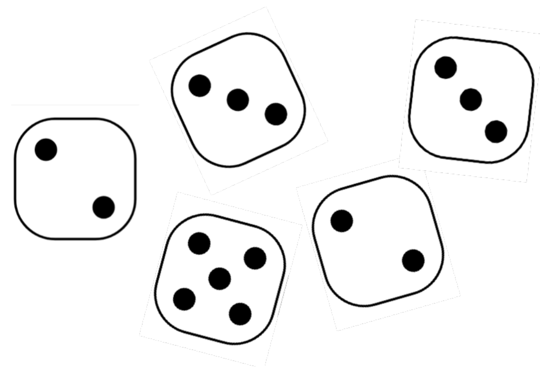
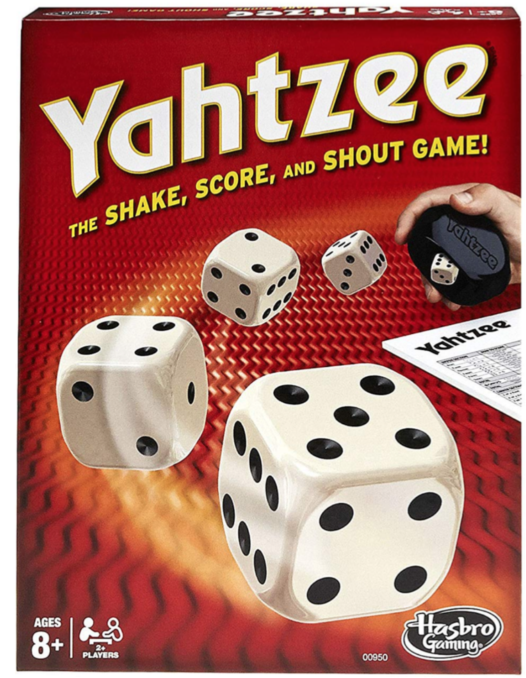




A: C(10,5)
B: C(6,5)
C: 6^5
D: None of these
E: Not sure

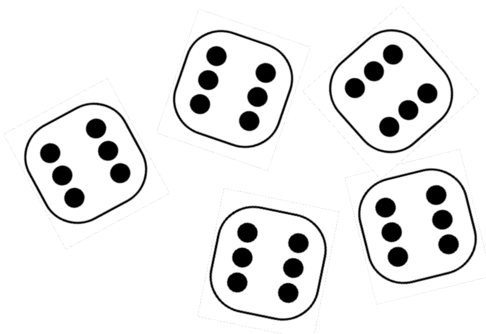
Choose **5** times from among **6** possibilities.

Example:
| ○○ | ○○ | | ○ |
(two 2's, two 3's, one 5)

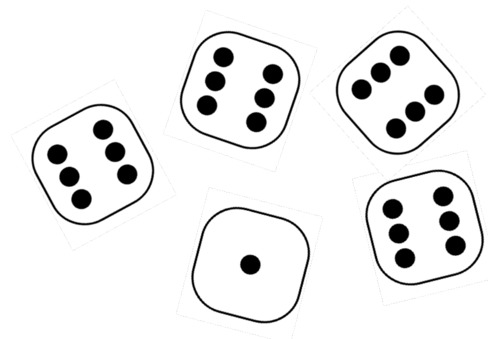
5 balls, 5 bars → C(10, 5)

Yahtzee Exercise 3

- In Yahtzee, a player rolls five 6-sided dice
- Is is more/less/equally likely that your roll five 6s or four 6s and a 1?



vs.



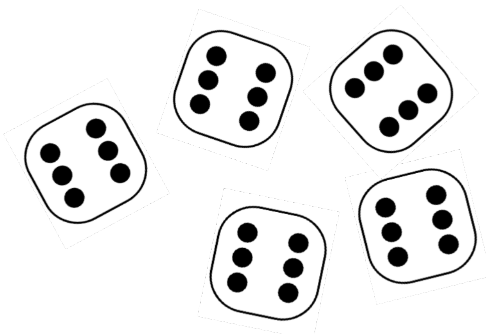
A: The rolls are equally likely

B: You are **less** likely to roll five 6s

C: You are **more** likely to roll five 6s

Yahtzee Exercise 3

- In Yahtzee, a player rolls five 6-sided dice
- Is is more/less/equally likely that your roll five 6s or four 6s and a 1?



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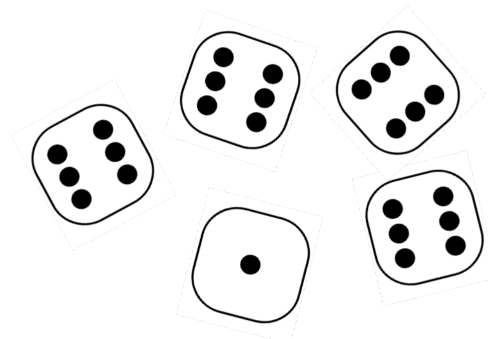


A: The rolls are equally likely

B: You are **less** likely to roll five 6s

C: You are **more** likely to roll five 6s

5 times <u>less</u> likely to roll five 6's

Outline:

- Revisit: Donut Shop
  - Counting combinations with repetition
- Combinatorial Proofs (as an fyi)
- Multinomial Coefficients
  - Counting permutations with *some* repetition

- Introduction to discrete probability
  - Experiment, Sample Space, Events
- **Conditional Probability**
- **Independence**
- Birthday Problem

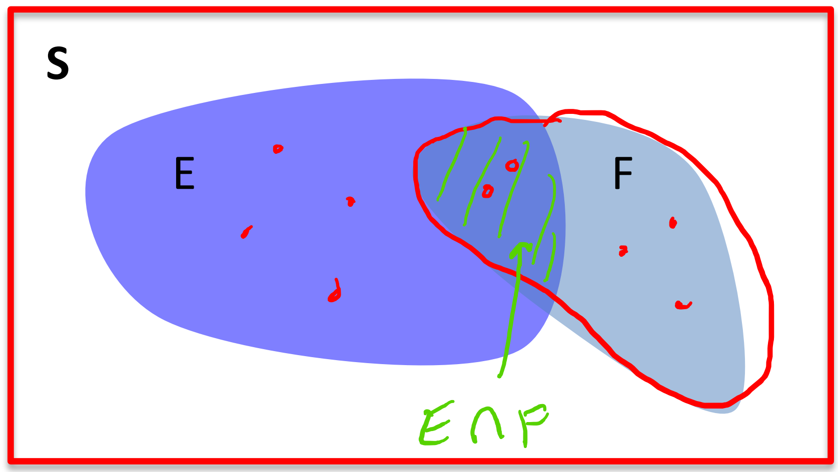
Conditional Probability

Let E and F be events with $p(F) > 0$. The **conditional probability** of E given F, represented as $p(E \mid F)$, is

$$p(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{|E \cap F|}{|F|}$$

Read as "probability of E given F"

For a sample space S with *equally likely* outcomes



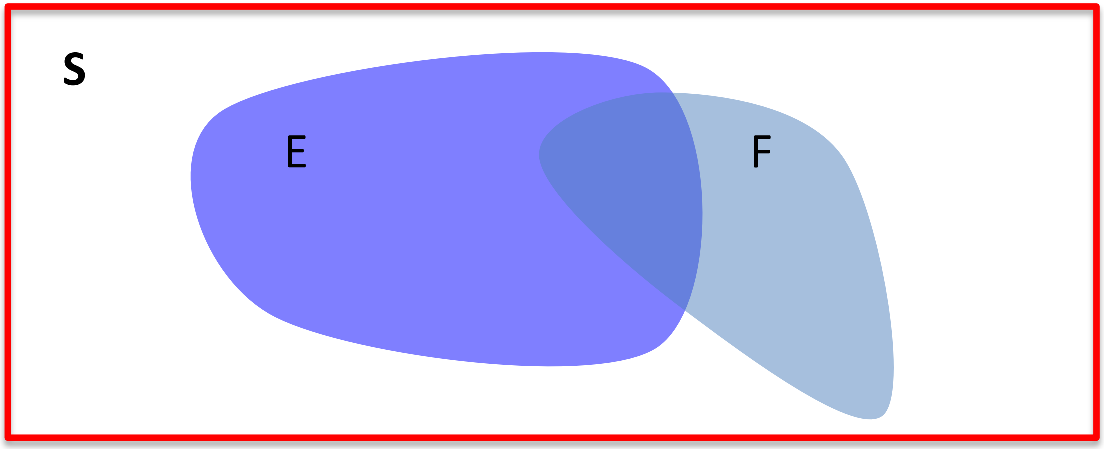
Meaning of conditional probability:

- Given that F happened, what is the probability that E happened?
- Essentially: ***F* becomes our new sample space** (because we are given that F happened)
- The only part of event E that is in our new sample space F is $E \cap F$
- So **our new event is $\boldsymbol{E \cap F}$**

Conditional Probability

The **conditional probability** of event E given event F is

$$p(\_\_\_\_\_\_) =$$



Example: Flip a coin 3 times.

- E = total of two Heads
- F = first flip was Tails

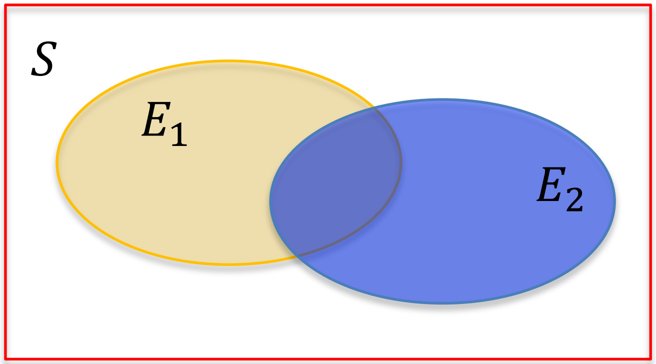
Find p(E), p(F), p(E ∩ F), p(E|F), p(F|E)

Independent Events

- **Independence.** Events E and F are ***independent*** iff

$$p(E|F) = p(E)$$

In words: knowing that F happened doesn't change the probability that E happened.



- **Alternative Definition:** E and F are are ***independent*** iff

$$= p(E \cap F) = p(E) \cdot p(F)$$

Independence of Events

Events E and F are **independent** if and only if

$$p(E|F) = \_\_\_\_\_\_\_\_\_\_$$

(alternatively)

E and F are **independent** if and only if

$$p(E \cap F) = \_\_\_\_\_\_\_\_\_\_\_\_\_$$

Are these independent?

- Roll two dice
  - E: the sum of the two dice is 5
  - F: the first die is a 1
  -

- Roll two dice
  - E: the sum of the two dice is 7
  - F: the first die is a 1

Are these independent?

- Roll two dice
  - E: the sum of the two dice is 5
  - F: the first die is a 1

E = {(1,4), (2,3), (3,2), (4,1)},
F = {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)},

p(E) = 4/36 = 1/9,   p(F) = 1/6
*p(E|F) = 1/6 $\neq$ p(E)   **Not Independent**
*Or via: $p(E \cap F)$ = 1/36 $\neq$ 1/9 * 1/6 = p(E)p(F)

- Roll two dice
  - E: the sum of the two dice is 7
  - F: the first die is a 1

p(E) = 6/36 = 1/6
p(F) = 1/6
* p(E|F) = 1/6 = p(E)   **Independent**
*Or via: $p(E \cap F)$ = 1/36 = 1/6 * 1/6 = p(E)p(F)

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- Independence
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Birthday Problem

How many people have to be in a room to assure that the probability that at least two of them have the same birthday is greater than 1/2?

a) 23
b) 183
c) 365
d) 730

Birthday Problem

How many people have to be in a room to assure that the probability that at least two of them have the same birthday is greater than 1/2?

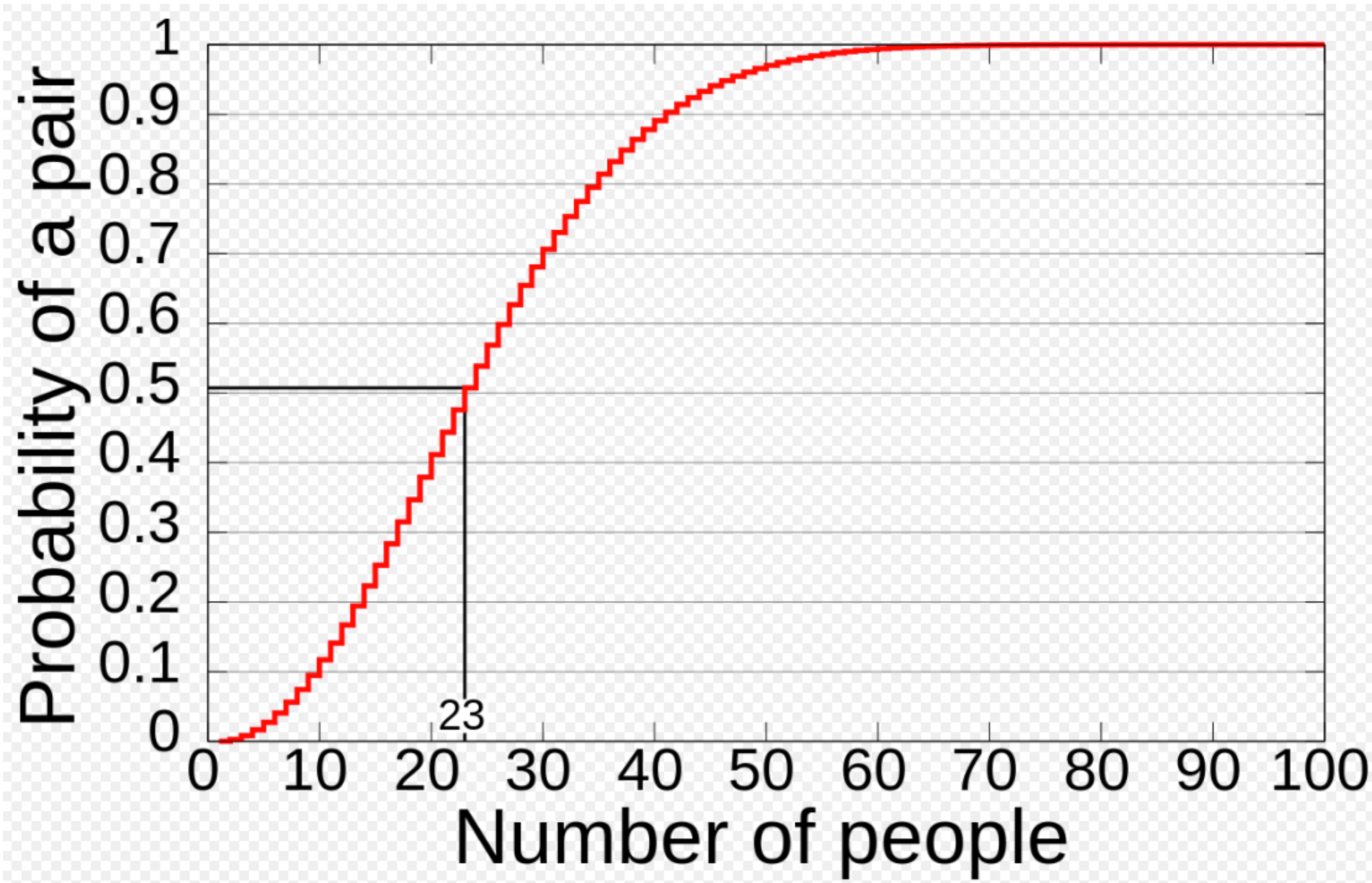
Assume 366 possible birthdays, each equally likely

a) 23
b) 183
c) 365
d) 730

- Let p_n be the probability that no people share a birthday among n people in a room.
  - Then 1 - p_n is the probability that 2 or more share a birthday.
  - We want the smallest n so that $1 - p_n > 1/2$.
- Sample space S: given n people (in some order), #ways of assigning a birthday to each person. $|S| = 366^n$
- E = "no people share a birthday among n people" $|E| = P(366,n)$

$$p_n = p(E) = \frac{P(366,n)}{366^n}$$ when $\boldsymbol{n} = \mathbf{23},\ 1 - p_n \approx 0.506$

Birthday Problem



More Yahtzee Exercises in Main L24 Slides

- You roll 5 indistinguishable dice.

- What is the probability that all 5 show the same number? (Yahtzee)

- What is the probability that you roll a straight? ($\{1,2,3,4,5\}$ or $\{2,3,4,5,6\}$)