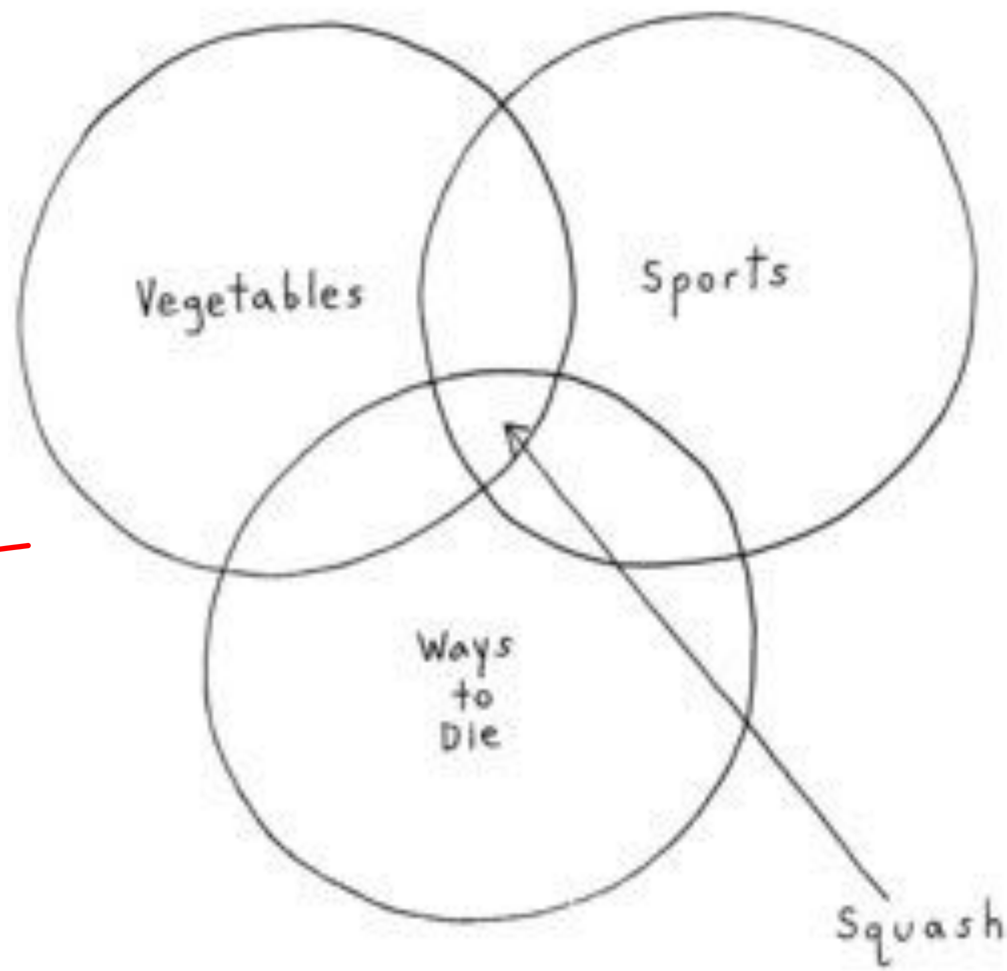
Lecture 7: Sets



Learning Objectives: Lec 7

After today's lecture (and the associated readings, discussion, & homework), you should be able to:

- **Know Technical Vocab:** Set, element, membership predicate (∈), list notation, set builder notation, domain of discourse, complement, union, intersection, set minus, power set, ordered pair, Cartesian product, subset, proper subset, disjoint, cardinality, list of common numerical sets (see slide)

- Know rules of sets (no repeated elements, no ordering, …)
- Prove that one set is a subset of another
- Prove set equalities using set equivalence rules
- Prove set equalities by proving subsets in both directions
- Use inclusion/exclusion to measure the cardinality of a set
- Understand Russel's Paradox

Outline

- **Intro to sets**
  - **Basic definitions**
  - Set-builder notation
- Set operations
- Set relations
  - Set identities
  - Subsets
  - Double-subset equality proofs
  - Other set relations
- Set cardinality
  - Basic definition
  - Inclusion/Exclusion
- Russel's Paradox

Sets

- Sometimes, we want to reason about **collections of objects**.
  - “The integers” {…,-2,-1,0,1,2,…}
  - “The even integers” {…,-4,-2,0,2,4,…}
  - “The truth values” {T,F}

- **Sets** are new discrete structures that formalize “collection of objects”

Sets

- A **“set”** is a collection of items, called **“elements**.”
- **“List notation:”** Define a set by listing its items inside { } brackets.

{a,n,r,b,o} = the set of letters required to spell our fair city

{mon,wed,fri,sat,sun} = days of the week without 203 class

{oct,nov,dec,jan,feb,mar,apr} = snowy months in our fair city

{…,-4,-2,0,2,4,6,8,…} = even integers.

Sets

- A set is ***unordered***

  {mon,wed,fri,sat,sun} = {fri,sat,sun,mon,wed}

- Each item ***appears*** in the set or ***it doesn't***. No concept of multiple appearances.

  {a,n,r,b,o} = {a,n,n,a,r,b,o,r}

- A set can have ***other sets as elements*** (and sets of sets…)

  {{bac.,let.,tom.}, {turk., let., tom., mao.}, {ham,cheese,tom.}}

- A set can have an ***infinitely many*** elements

  {0, 1, 2, 3, 4, 5, …}

- A set can have ***zero elements***

  {} = ∅ (sometimes also ϕ)

Lecture 7 Handout: Sets

- A set is a collection of items
  - Example:
    - "S is the set of all even numbers between 50 and 100, inclusive"
    - S = { $x \in \mathbb{Z} \mid x$ is even $\wedge$ $50 \le x \le 100$ }

- Standard Numerical Sets to know:

Symbol	Elements	Name of Set
∅	{}	Empty set
$\mathbb{N}$	{ 0, 1, 2, 3, ... }	natural numbers
$\mathbb{Z}$	{ ..., -2, -1, 0, 1, 2, ... }	integers
$\mathbb{Z}^+$	{ 1, 2, 3, ... }	positive ints.
$\mathbb{Z}^-$	{ -1, -2, -3, ... }	negative ints
$\mathbb{Q}$	$\{ x \mid \exists a \in \mathbf{Z}\ \exists b \in \mathbf{Z}^+\ \ x = a/b \}$	rationals
$\mathbb{R}$	*(don't try to list the elements)*	reals
$\mathbb{R}^+$	*(don't try to list the elements)*	positive reals

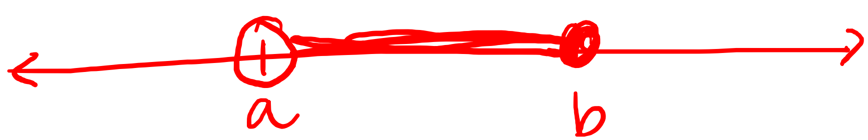
Numerical Sets

Standard Numerical Sets

- $\mathbb{N} = \{0,1,2,3,\dots\}$ *natural numbers*
- $\mathbb{Z} = \{\dots,-2,-1,0,1,2,3,\dots\}$ *integers*
- $\mathbb{Z}^+ = \{1,2,3,\dots\}$ *positive integers*
- $\mathbb{Z}^- = \{-1,-2,-3,\dots\}$ *negative integers*
- $\mathbb{Q}$ *rational numbers*
- $\mathbb{R}$ *real numbers*
- $\mathbb{R}^+$ *positive real numbers*
- Intervals: $[a,b], (a,b], [a,b), (a,b)$

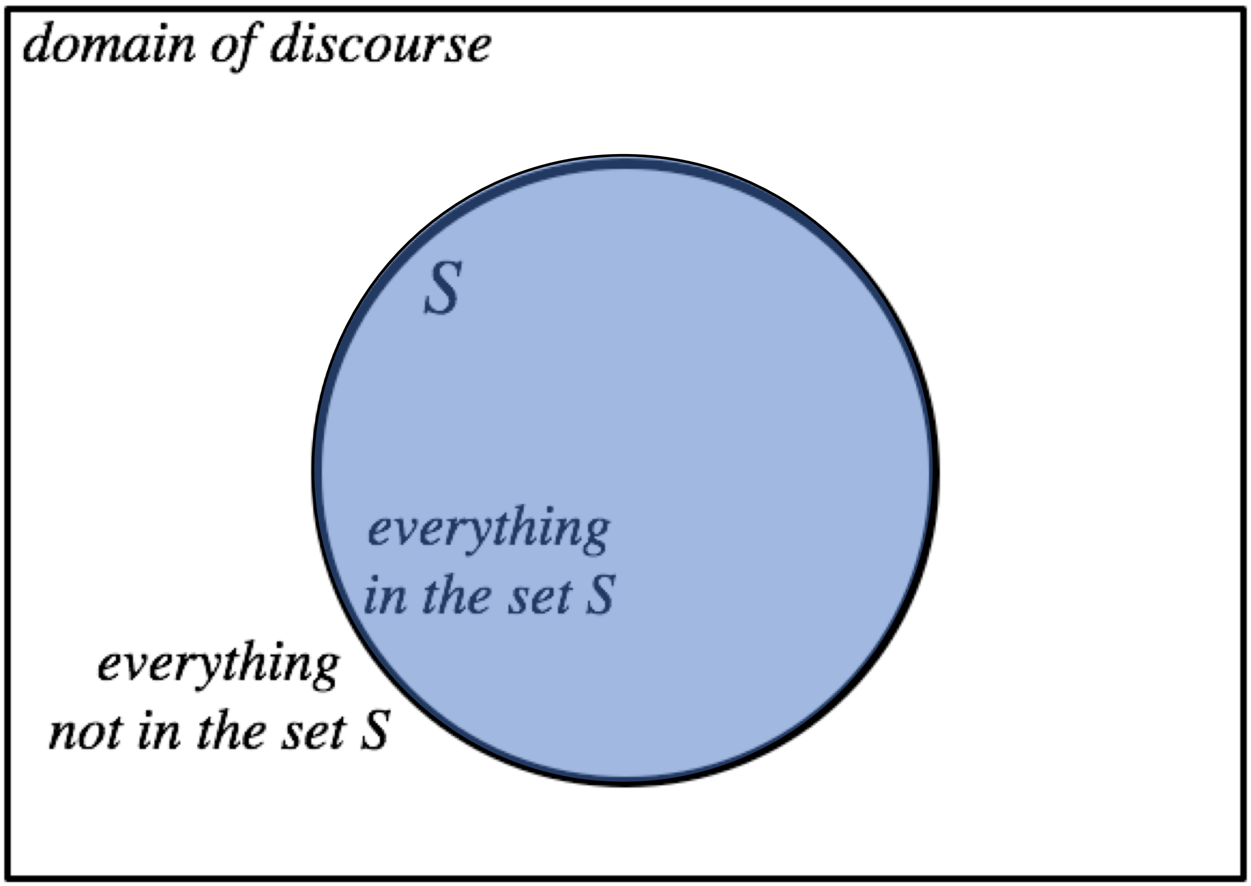
  all real numbers between a and b, [] brackets include endpoint, () brackets omit endpoint

$(a, b]$



Sets

Sets are defined within a "**domain of discourse"**
(all possible elements that *could* be in the set)
Sometimes just **"domain"** for short
Often left implicit if it's clear in from context, or it doesn't matter for the problem we're trying to solve



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Membership Predicate

- **Notation:**
  - $x \in S$ means "x is an element of S"
  - $x \notin S$ means "x is not an element of S"

- For $S = \{0,1,3,4\}$ we have $1 \in S, 2 \notin S$.

- The predicate $P(x) =$ "$x \in S$" is the **"membership predicate"** of S.

Set Builder Notation

- Instead of listing elements, we can **define** a set by giving its membership predicate.
- Called **"set builder notation:"**

Examples:

"S is the set of all odd prime integers"

$$S = \{x \in \mathbb{Z} \mid x \text{ is odd and } x \text{ is prime}\}$$

Read the middle bar as "such that"

"all integers x"
(the left part specifies the **domain)**

Set Builder Notation

- Instead of listing elements, we can **define** a set by giving its membership predicate.
- Called **“set builder notation:”**

Examples:

“T is the set of sophomores enrolled in EECS203”

T = $\{x \mid x \text{ is a sophomore and } x \text{ is in EECS203}\}$

“such that”

“all [people] x”

Set Builder Notation

- Instead of listing elements, we can **define** a set by giving its membership predicate.
- Called **"set builder notation:"**

Examples:

"R is the set of all integers that are the sum of two primes"

$$R = \{x \in \mathbb{Z} \mid \exists y, z \in \mathbb{Z}\ (\mathbf{Prime}(y) \wedge \mathbf{Prime}(z) \wedge x = y + z)\}$$

"such that"

"all integers x"

"there are two integers y and z"

"such that y and z are both prime and $x = y + z$"

Set Builder Notation

- Instead of listing elements, we can **define** a set by giving its membership predicate.
- Called **"set builder notation"**

General form of set builder notation:

"R is the set of all elements in domain D satisfying membership predicate P"

$$R = \{x \in D \mid P(x)\}$$

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Exercise: Set Operations

Handout

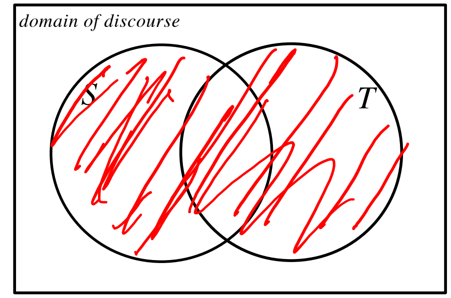
- **S ∪ T** : "the ***union*** of S and T"

  or

  $\rightarrow$ $x \in (S \cup T) \equiv \underbrace{x \in S} \vee \underbrace{x \in T}$

  Example: $\{1,2\} \cup \{2,3\} =$ {1, 2, 3}

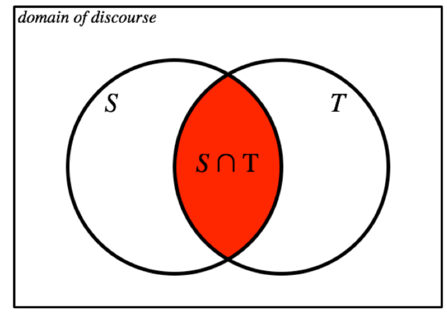
S ∪ T



- **S ∩ T** : "the ***intersection*** of S and T"

  $x \in (S \cap T) \equiv$ $x \in S \wedge x \in T$

  Example: $\{1,2\} \cap \{2,3\} =$ {2}



- $\bar{S}$: "the ***complement*** of S"

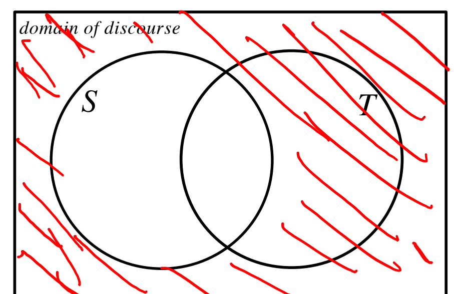
  $x \in \bar{S} \equiv x \notin S$

  Example: $\overline{\{1,2\}} =$ depends on the domain

  If $D = \{1, 2, 3\}$
  $\bar{S} = \{3\}$

  If $D = \mathbb{R}$
  $\bar{S} = (-\infty, 1) \cup (1, 2) \cup (2, \infty)$

$\bar{S}$



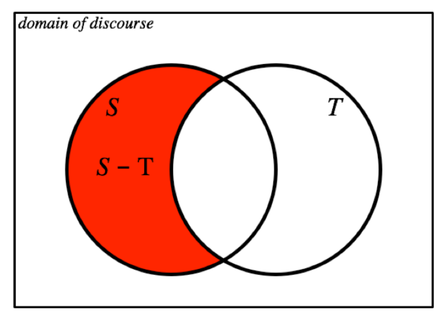
- **S – T** : "S ***minus*** T"

  $x \in (S - T) \equiv$ $x \in S \wedge x \notin T$ $\leftarrow$ main defn.

  Example: $\{1,2\} - \{2,3\} =$ ______

also $x \in S \oplus T \wedge x \notin T$

~~$x \in (S \wedge \neg T)$~~



Set Complements

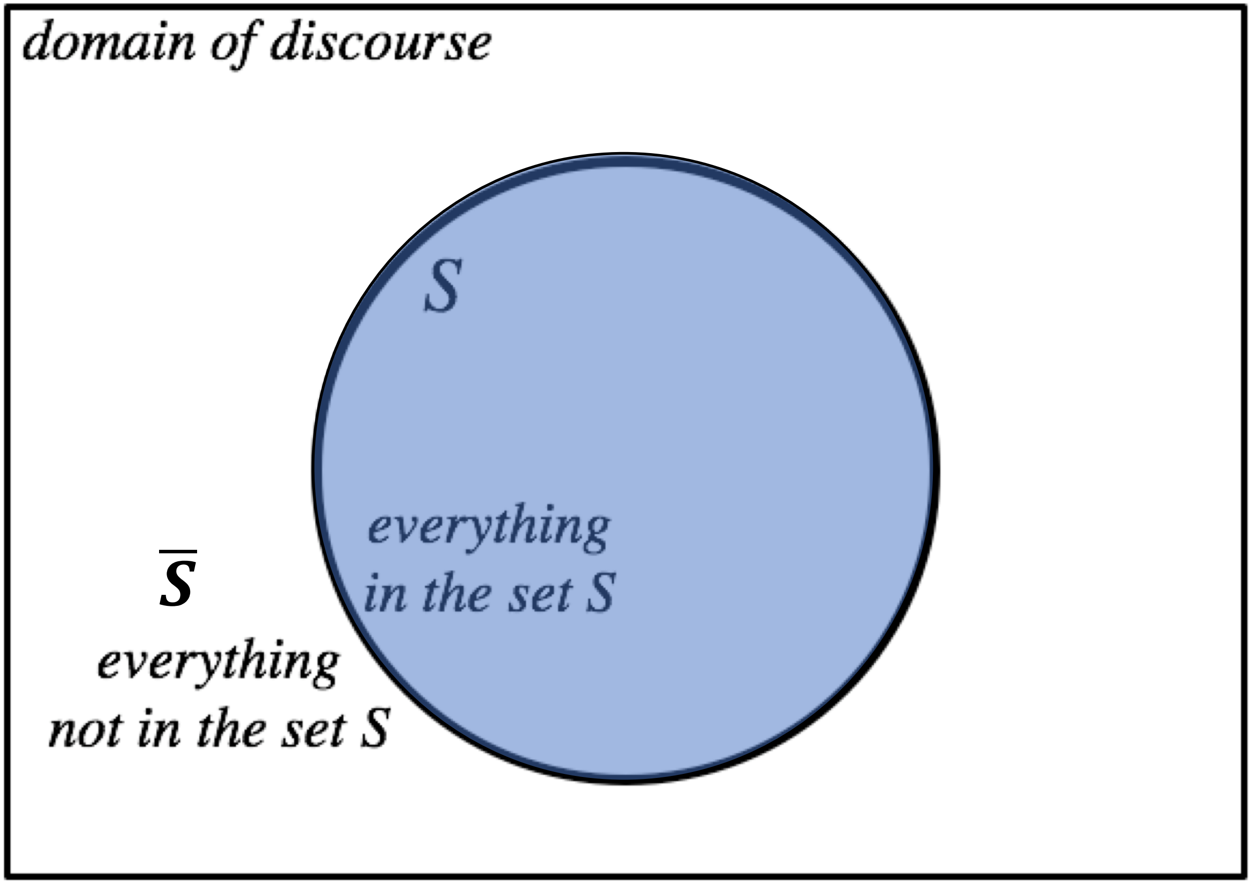
- Every set S has a **complement** $\bar{S}$
- $\bar{S}$ is the set of elements in the domain **not** in S. ***You must know the domain in order to find the complement.***

Example: $S = \{1,2,3\}$

If domain = $\mathbb{Z}$ (integers): $\bar{S} = \{\dots, -3, -2, -1, 0, 4, 5, 6, \dots\}$

If domain = $\mathbb{N}$ (natural numbers): $\bar{S} = \{0, 4, 5, 6, \dots\}$

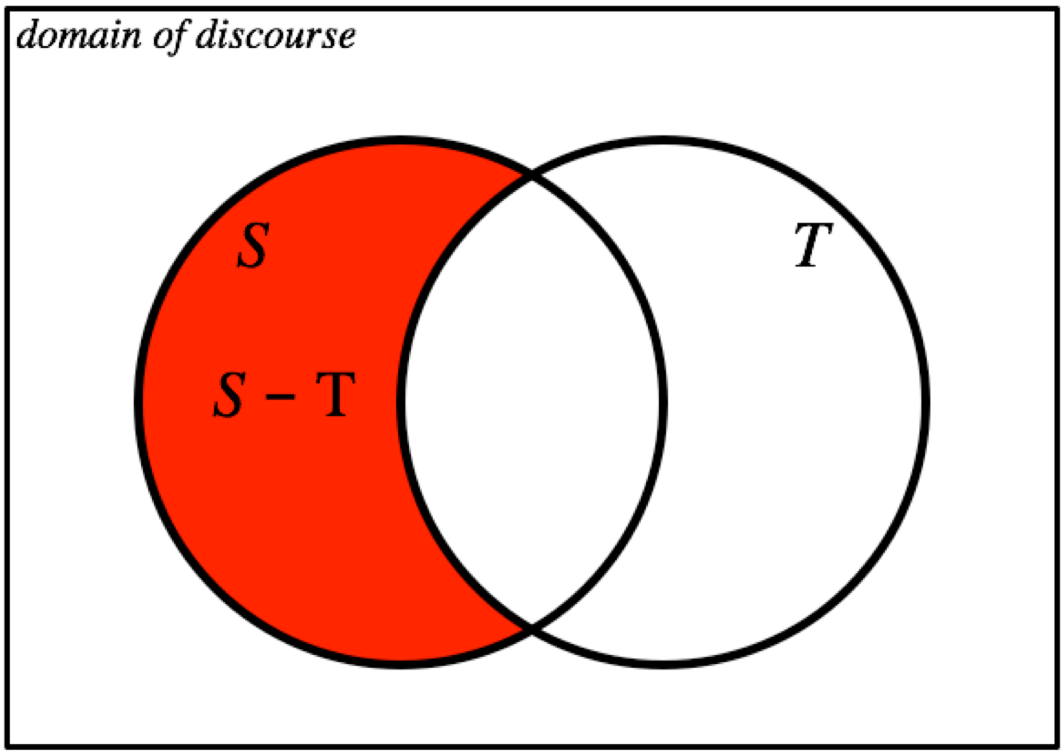
Set Complements



- **Alt definition:** the membership predicate of $\bar{S}$ is the **negation** of the membership predicate of S.
- $S = \{x \mid P(x)\}$
- $\bar{S} = \{x \mid \neg P(x)\}$

Set Difference

- Given sets S and T over the same domain, their **difference** $S - T$ is a third set that contains all the elements in S, **except** those in T.



Set Difference Exercise

How does the membership predicate of $S - T$ relate to the membership predicates of S, T?

(Answer will be a compound proposition made from ors, ands, and nots)

- $S = \{x \mid P(x)\}$
- $T = \{x \mid Q(x)\}$

- $S - T = \{x \mid$ ______________________________ $\}$

Set Difference Exercise

How does the membership predicate of $S - T$ relate to the membership predicates of S, T?

(Answer will be a compound proposition made from ors, ands, and nots)

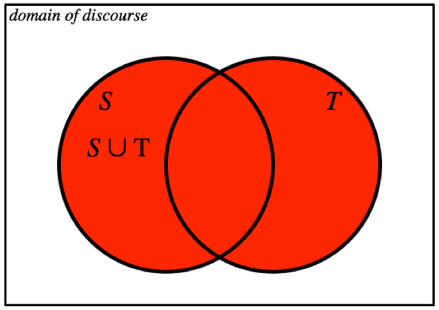
- $S = \{x \mid P(x)\}$
- $T = \{x \mid Q(x)\}$

- $S - T = \{x \mid \mathbf{P(x) \land \neg Q(x)}\}$

- **S ∪ T** : "the *union* of S and T" (a new set)

  $x \in (S \cup T) \equiv x \in S \vee x \in T$

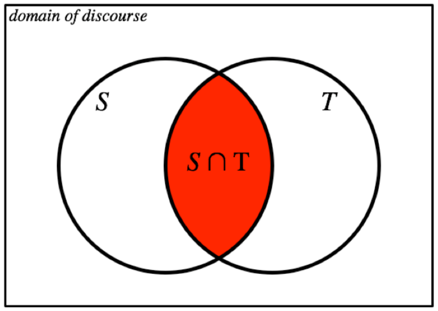
  $\{1,2\} \cup \{2,3\} = \{1,2,3\}$



- **S ∩ T** : "the *intersection* of S and T" (a new set)

  $x \in (S \cap T) \equiv x \in S \wedge x \in T$

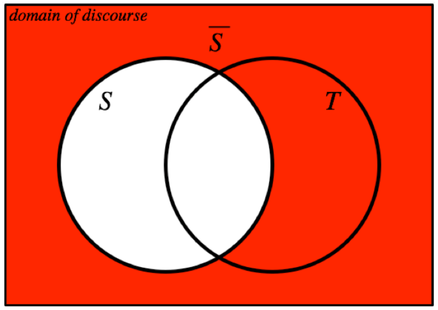
  $\{1,2\} \cap \{2,3\} = \{2\}$



- $\bar{S}$: "the *complement* of S" (a new set)

  $x \in \bar{S} \equiv x \notin S$

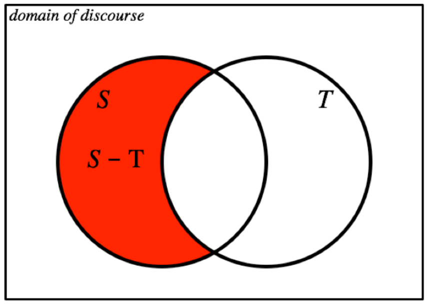
  $\overline{\{1,2\}} = ?$ $\boxed{\{3\} \text{ if our domain is } \{1,2,3\}}$



- **S – T** : "S *minus* T" (a new set)

  $x \in (S - T) \equiv x \in S \wedge x \notin T$

  $\{1,2\} - \{2,3\} = \{1\}$



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Set Equality

$\boldsymbol{S = T}$ means that sets S and T have exactly the same elements.

Alt definition: $S = T$ means for all x, $x \in S$ if and only if $x \in T$.

There are two ways to prove a set identity $S = T$

1. Use "set identity rules" to get from S to T (or from T to S)
   very similar to logical equivalences

2. Prove that an arbitrary element of S is also in T, and that an arbitrary element of T is also in S.
   similar to how we prove "p if and only if q" by proving both "if p, then q" and "if q, then p"

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Set Identity

DeMorgan for Sets:

$$\overline{S \cup T} = \overline{S} \cap \overline{T}$$

$$\overline{S \cup T}$$

Set Identity

DeMorgan for Sets:

$$\overline{S \cup T} = \overline{S} \cap \overline{T}$$

$\overline{S \cup T}$	
$= \{x \mid x \notin (S \cup T)\}$	definition of complement
$= \{x \mid \neg(x \in (S \cup T))\}$	definition of $\notin$
$= \{x \mid \neg(x \in S \vee x \in T)\}$	definition of $\cup$
$= \{x \mid x \notin S \wedge x \notin T\}$	DeMorgan's law
$= \{x \mid x \in \overline{S} \wedge x \in \overline{T}\}$	definition of complement
$= \{x \mid x \in \overline{S} \cap \overline{T}\}$	definition of $\cap$
$= \overline{S} \cap \overline{T}$	definition of $\in$

Most logical equivalence laws have a corresponding set identity law.

Set Identities

- **Identity** laws

$$A \cap U = A$$
$$A \cup \emptyset = A$$

- **Domination** laws

$$A \cup U = U$$
$$A \cap \emptyset = \emptyset$$

- **Idempotent** laws

$$A \cup A = A$$
$$A \cap A = A$$

- **Complementation** law

$$\overline{(\overline{A})} = A$$

- **Commutative** laws

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- **Definition of Set Minus (not in book)**

$$A - B = A \cap \bar{B}$$

- **Associative** laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- **Distributive** laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- **De Morgan's** laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

- **Absorption** laws

$$A \cap (A \cup B) = A$$
$$A \cup (A \cap B) = A$$

- **Complement** laws

$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \emptyset$$

Exercise: Set Identities

Prove: $A - (B \cap C) = (A - B) \cup (A - C)$

Exercise: Set Identities

Prove: $A - (B \cap C) = (A - B) \cup (A - C)$

$A - (B \cap C)$

$= A \cap \overline{(B \cap C)}$ Def. of set difference

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Prove: $A - (B \cap C) = (A - B) \cup (A - C)$

$A - (B \cap C)$

$= A \cap \overline{(B \cap C)}$ Def. of set difference

$= A \cap \left(\overline{B} \cup \overline{C}\right)$ De Morgan's

Exercise: Set Identities

Prove: $A - (B \cap C) = (A - B) \cup (A - C)$

$$
\begin{aligned}
& A - (B \cap C) & \\
&= A \cap \overline{(B \cap C)} & \text{Def. of set difference} \\
&= A \cap \left(\overline{B} \cup \overline{C}\right) & \text{De Morgan's} \\
&= \left(A \cap \overline{B}\right) \cup \left(A \cap \overline{C}\right) & \text{Distributive Law}
\end{aligned}
$$

Exercise: Set Identities

Prove: $A - (B \cap C) = (A - B) \cup (A - C)$

$A - (B \cap C)$	
$= A \cap \overline{(B \cap C)}$	Def. of set difference
$= A \cap \left(\overline{B} \cup \overline{C}\right)$	De Morgan's
$= \left(A \cap \overline{B}\right) \cup \left(A \cap \overline{C}\right)$	Distributive Law
$= (A - B) \cup (A - C)$	Def. of set complement

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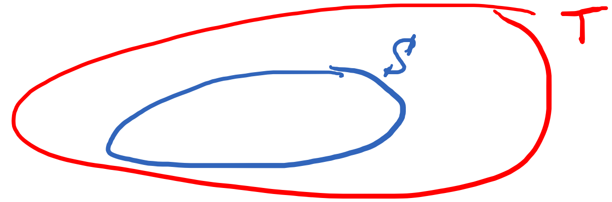
There are two ways to prove a set identity $S = T$

2. Prove that an arbitrary element of S is also in T, and that an arbitrary element of T is also in S.

similar to how we prove "p if and only if q" by proving both "if p, then q" and "if q, then p"

Subsets



"Set S is a **subset** of set T" means that every element of S is also an element of T, ***and T might contain some other elements, too.***
Written: $S \subseteq T$

In logic: $S \subseteq T$ means $\forall x\,(x \in S \rightarrow x \in T)$

For all S
$S \subseteq S$

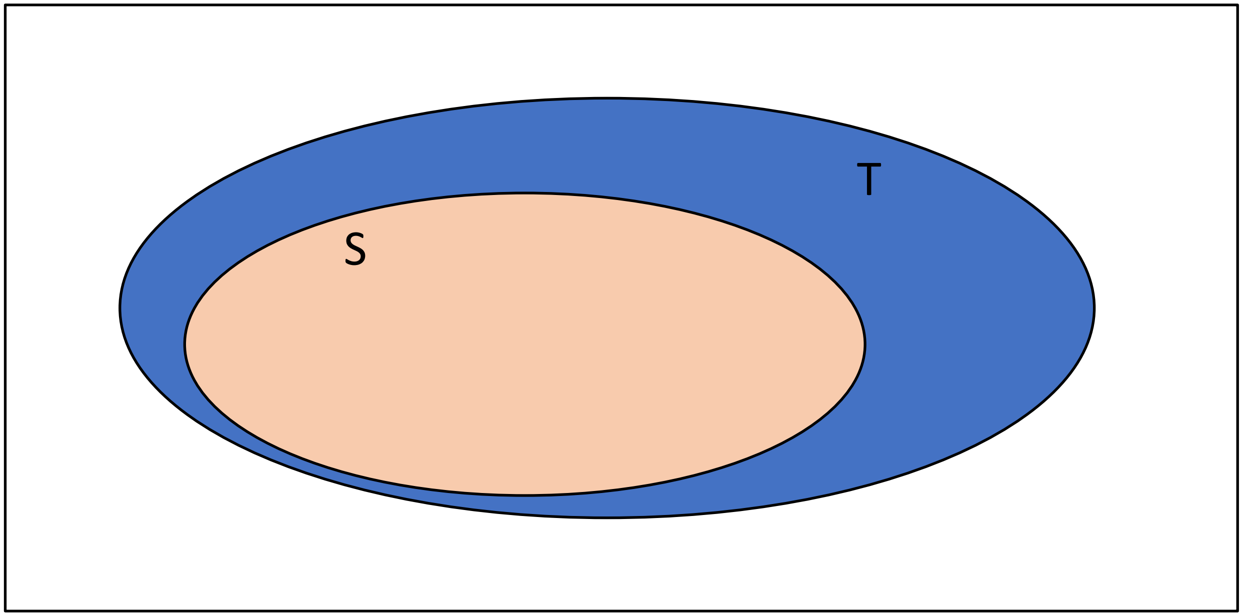
Example 1:
$$S = \{2,3\}$$
$$T = \{1,2,3,4,5\}$$
So $S \subseteq T$ ***is a true proposition***

Example 2:
$$S = \{1,2,3,4,5\}$$
$$T = \{1,2,3,4,5\}$$
So $S \subseteq T$ ***is a true proposition***

Example 3:
$$S = \{\,\}$$
$$T = \{1,2,3,4,5\}$$
So $S \subseteq T$ ***is a true proposition***

For all S
$\emptyset \subseteq S$

Subsets

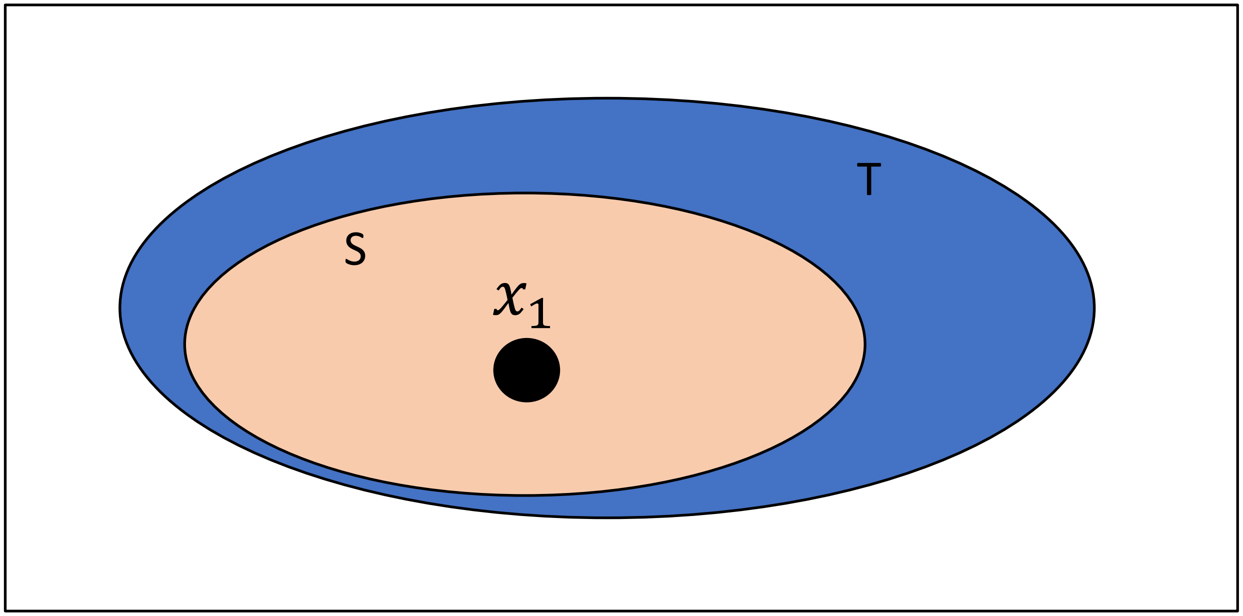


Alt Definition: $S \subseteq T$ means that, for all x, *if* $x \in S$, *then* $x \in T$.

- $S = \{x \mid P(x)\}$
- $T = \{x \mid Q(x)\}$
- $S \subseteq T$ means for all x, $P(x) \rightarrow Q(x)$

$x \in S$	$x \in T$	if $\boldsymbol{x} \in \boldsymbol{S}$ then $\boldsymbol{x} \in \boldsymbol{T}$
T	T	T
T	F	F
F	T	T
F	F	T

Subsets



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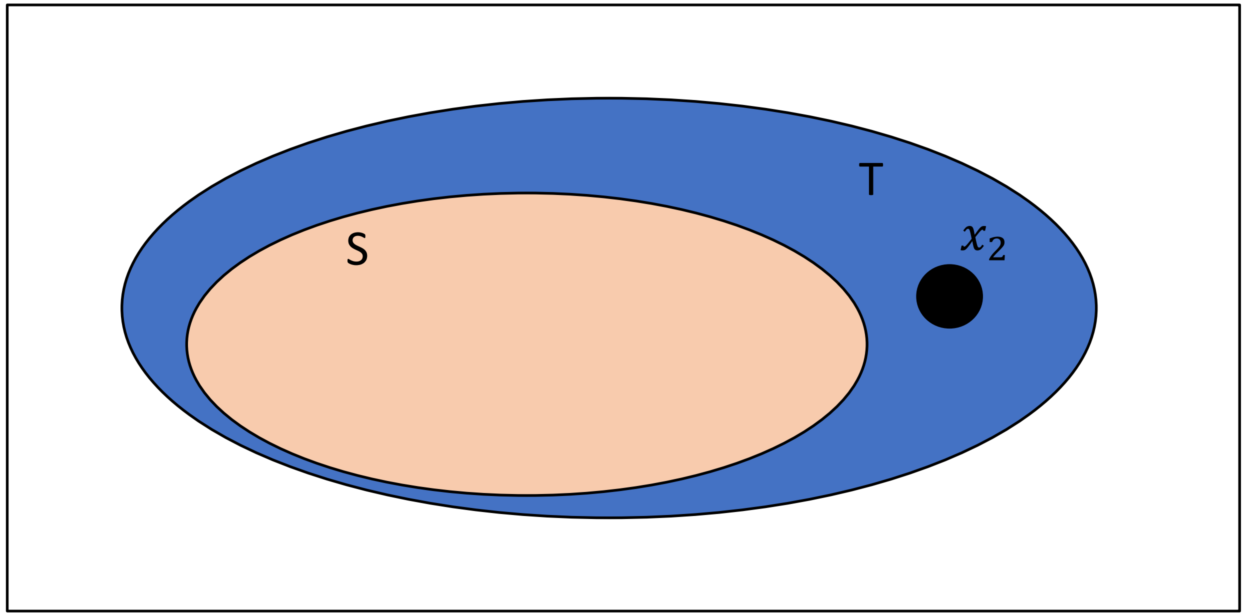
Subsets



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T	F	F
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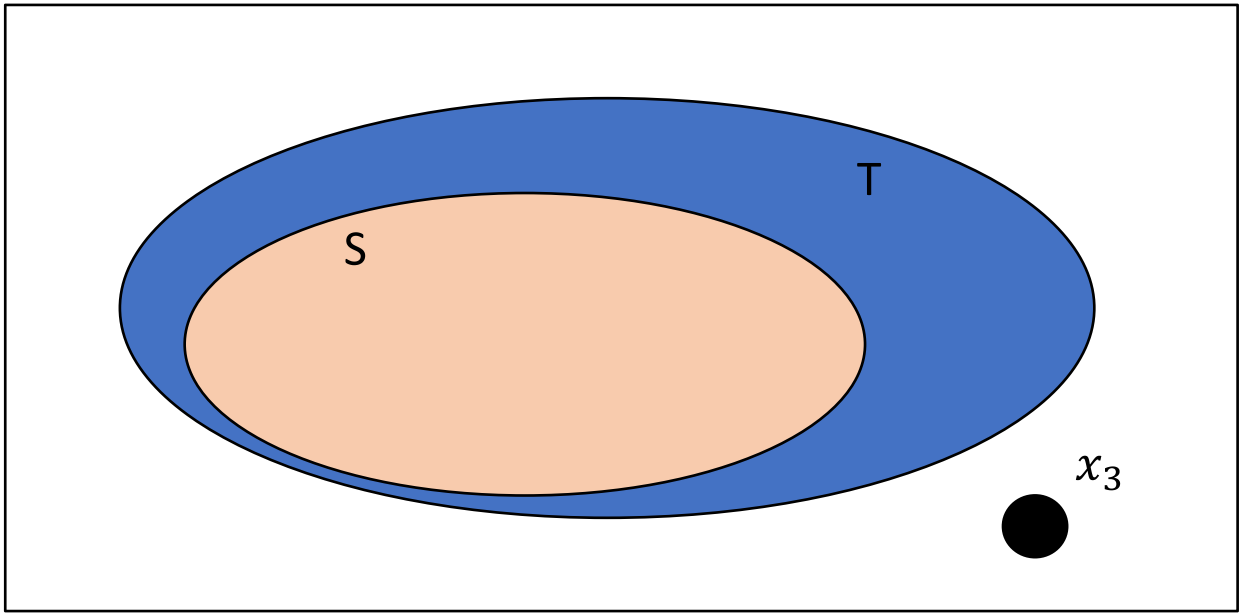
Subsets



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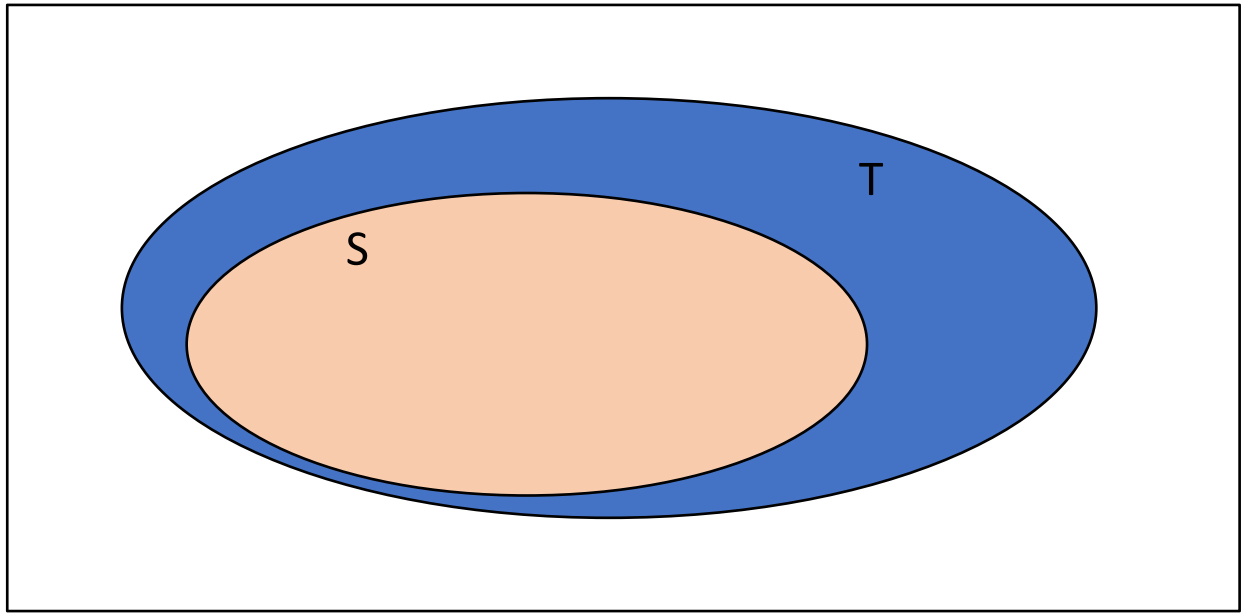
Subsets



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- $T = \{x \mid Q(x)\}$
- $S \subseteq T$ means for all x, $P(x) \rightarrow Q(x)$

$x \in S$	$x \in T$	if $\boldsymbol{x} \in \boldsymbol{S}$ then $\boldsymbol{x} \in \boldsymbol{T}$
T	T	T
T	F	F
F	T	T
F	F	T

For all points in the domain, $x \in S \rightarrow x \in T$ is true

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- Russel’s Paradox

Set Equality

$\boldsymbol{S = T}$ means that sets S and T have exactly the same elements.

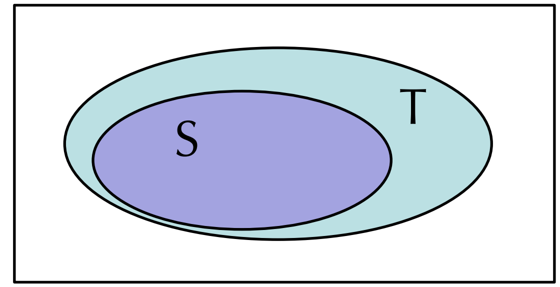
Alt definition: $S = T$ means for all x, $x \in S$ if and only if $x \in T$.

There are two ways to prove a set identity $S = T$

2. Prove that $\boldsymbol{S \subseteq T}$, and also $\boldsymbol{T \subseteq S}$.

Subsets ; How to prove A = B

$S \subseteq T$



"S is a ***subset*** of T"

- $\mathbf{S \subseteq T}$: $\forall x(x \in S \to$ _x ∈ T_ $)$
- "Every element of S is also in T"

T or F $\{a, b\} \subseteq \{a, b, c\}$

T or F $\emptyset \subseteq \{a, b, c\}$

T or F $b \subseteq \{a, b, c\}$

b is not a set

T or F $\{a, b, c\} \subseteq \{a, b, c\}$

How to Prove A = B

Two ways to prove set equality, e.g., A = B

1. Use set identities
2. Show each side is a _subset_ of the other:
   - A _$\subseteq$_ B, and B _$\subseteq$_ A

Equality Proof

$R = \{r \in \mathbb{Z} \mid r \equiv 0 \ (mod\ 2)\}$
$S = \{s \in \mathbb{Z} \mid s \equiv 0 \ (mod\ 3)\}$
$T = \{t \in \mathbb{Z} \mid t \equiv 0 \ (mod\ 6)\}$

Proposition: $R \cap S = T$

Proof: ***goal is to prove*** $(\boldsymbol{R} \cap \boldsymbol{S}) \subseteq \boldsymbol{T}$***, and*** $\boldsymbol{T} \subseteq (\boldsymbol{R} \cap \boldsymbol{S})$

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$R = \{r \in \mathbb{Z} \mid r \equiv 0\ (mod\ 2)\}$
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- ***To prove*** $\boldsymbol{T} \subseteq (\boldsymbol{R} \cap \boldsymbol{S})$***, we need to prove that the definition of subset holds.***
  - **Definition:** $A \subseteq B$ means that, for all x, *if* $x \in A$, *then* $x \in B$.
  - Here, $A = T$ and $B = R \cap S$, so we need to show that:
    for all x, *if* $x \in T$, *then* $x \in R \cap S$.

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- ***To prove*** $\boldsymbol{T} \subseteq (\boldsymbol{R \cap S})$***, we need to prove that the definition of subset holds.***
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- Let x be an arbitrary element of the domain.

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Proposition: $R \cap S = T$

Proof: ***goal is to prove*** $(R \cap S) \subseteq T$***, and*** $T \subseteq (R \cap S)$

- ***To prove*** $T \subseteq (R \cap S)$***, we need to prove that the definition of subset holds.***
  - **Definition:** $A \subseteq B$ means that, for all x, *if* $x \in A$, *then* $x \in B$.
  - Here, $A = T$ and $B = R \cap S$, so we need to show that:
    for all x, *if* $x \in T$, *then* $x \in R \cap S$.

- Let x be an arbitrary element of the domain.
- Assume $x \in T$.

Equality Proof

$R = \{r \in \mathbb{Z} \mid r \equiv 0 \ (mod\ 2)\}$
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  - Here, $A = T$ and $B = R \cap S$, so we need to show that:
    for all x, *if* $x \in T$, *then* $x \in R \cap S$.

- Let x be an arbitrary element of the domain.
- Assume $x \in T$.
- So $x \equiv 0 \ (mod\ 6)$.

Equality Proof

$R = \{r \in \mathbb{Z} \mid r \equiv 0 \ (mod\ 2)\}$
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  - Here, $A = T$ and $B = R \cap S$, so we need to show that:
    for all x, *if* $x \in T$, *then* $x \in R \cap S$.

- Let x be an arbitrary element of the domain.
- Assume $x \in T$.
- So $x \equiv 0 \ (mod\ 6)$.
- So there exists an integer k with $x = 6k = (2)(3)k$

Equality Proof

$R = \{r \in \mathbb{Z} \mid r \equiv 0 \ (mod\ 2)\}$
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- ***To prove*** $\boldsymbol{T} \subseteq (\boldsymbol{R} \cap \boldsymbol{S})$***, we need to prove that the definition of subset holds.***
  - **Definition:** $A \subseteq B$ means that, for all x, *if* $x \in A$, *then* $x \in B$.
  - Here, $A = T$ and $B = R \cap S$, so we need to show that:
    for all x, *if* $x \in T$, *then* $x \in R \cap S$.

- Let x be an arbitrary element of the domain.
- Assume $x \in T$.
- So $x \equiv 0 \ (mod\ 6)$.
- So there exists an integer k with $x = 6k = (2)(3)k$
- In mod 2, x reduces to: $x \equiv (0)(3)k \equiv 0 \ (mod\ 2)$, **so $\boldsymbol{x} \in \boldsymbol{R}$**

Equality Proof

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Proof: *goal is to prove* $(R \cap S) \subseteq T$*, and* $T \subseteq (R \cap S)$

- ***To prove*** $T \subseteq (R \cap S)$***, we need to prove that the definition of subset holds.***
  - **Definition:** $A \subseteq B$ means that, for all x, *if* $x \in A$, *then* $x \in B$.
  - Here, $A = T$ and $B = R \cap S$, so we need to show that:
    for all x, *if* $x \in T$, *then* $x \in R \cap S$.

- Let x be an arbitrary element of the domain.
- Assume $x \in T$.
- So $x \equiv 0 \ (mod\ 6)$.
- So there exists an integer k with $x = 6k = (2)(3)k$
- In mod 2, x reduces to: $x \equiv (0)(3)k \equiv 0 \ (mod\ 2)$, **so $x \in R$**
- In mod 3, x reduces to: $x \equiv (2)(0)k \equiv 0 \ (mod\ 3)$, **so $x \in S$**

Equality Proof

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- ***To prove*** $\boldsymbol{T} \subseteq (\boldsymbol{R} \cap \boldsymbol{S})$***, we need to prove that the definition of subset holds.***
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- Let x be an arbitrary element of the domain.
- Assume $x \in T$.
- So $x \equiv 0 \ (mod\ 6)$.
- So there exists an integer k with $x = 6k = (2)(3)k$
- In mod 2, x reduces to: $x \equiv (0)(3)k \equiv 0 \ (mod\ 2)$, **so** $\boldsymbol{x} \in \boldsymbol{R}$
- In mod 3, x reduces to: $x \equiv (2)(0)k \equiv 0 \ (mod\ 3)$, **so** $\boldsymbol{x} \in \boldsymbol{S}$
- Since $x \in R$ and $x \in S$, we have $\boldsymbol{x} \in \boldsymbol{R} \cap \boldsymbol{S}$

Equality Proof

$R = \{r \in \mathbb{Z} \mid r \equiv 0 \ (mod\ 2)\}$
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- ***To prove*** $\boldsymbol{T} \subseteq (\boldsymbol{R} \cap \boldsymbol{S})$***, we need to prove that the definition of subset holds.***
  - **Definition:** $A \subseteq B$ means that, for all x, *if* $x \in A$, *then* $x \in B$.
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- Let x be an arbitrary element of the domain.
- Assume $x \in T$.
- So $x \equiv 0 \ (mod\ 6)$.
- So there exists an integer k with $x = 6k = (2)(3)k$
- In mod 2, x reduces to: $x \equiv (0)(3)k \equiv 0 \ (mod\ 2)$, **so** $\boldsymbol{x} \in \boldsymbol{R}$
- In mod 3, x reduces to: $x \equiv (2)(0)k \equiv 0 \ (mod\ 3)$, **so** $\boldsymbol{x} \in \boldsymbol{S}$
- Since $x \in R$ and $x \in S$, we have $\boldsymbol{x} \in \boldsymbol{R} \cap \boldsymbol{S}$
- **So** $\boldsymbol{T} \subseteq \boldsymbol{R} \cap \boldsymbol{S}$

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- Let x be an arbitrary element of the domain.

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- Let x be an arbitrary element of the domain.
- Assume $x \in R \cap S$.
- So $x \equiv 0 \ (mod\ 2)$ and $x \equiv 0 \ (mod\ 3)$.

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- So there exist integers k_1, k_2 with $x = 2k_1$ and $x = 3k_2$.

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- For any integer k_1, we have $2\mathrm{k}_1$ equivalent to one of $\{0,2,4\}$ (mod 6)

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$R = \{r \in \mathbb{Z} \mid r \equiv 0 \ (mod\ 2)\}$
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- So there exist integers k_1, k_2 with $x = 2k_1$ and $x = 3k_2$.
- For any integer k_1, we have $2\mathrm{k}_1$ equivalent to one of $\{0,2,4\}$ (mod 6)
- For any integer k_2, we have $3k_2$ equivalent to one of $\{0,3\}$ (mod 6)
- Since $x = 2k_1 = 3k_2$, then $2k_1$ and $3k_2$ must have the same remainder in mod 6

Equality Proof

$R = \{r \in \mathbb{Z} \mid r \equiv 0\ (mod\ 2)\}$
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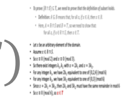

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  - **Definition:** $A \subseteq B$ means that, for all x, *if* $x \in A$, *then* $x \in B$.
  - Here, $A = R \cap S$ and $B = T$, so we need to show that:
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- Assume $x \in R \cap S$.
- So $x \equiv 0\ (mod\ 2)$ and $x \equiv 0\ (mod\ 3)$.
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- Since $x = 2k_1 = 3k_2$, then $2k_1$ and $3k_2$ must have the same remainder in mod 6
- So $x \equiv 0\ (mod\ 6)$, so $\boldsymbol{x \in T}$

Equality Proof

$R = \{r \in \mathbb{Z} \mid r \equiv 0\ (mod\ 2)\}$
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- Let x be an arbitrary element of the domain.
- Assume $x \in R \cap S$.
- So $x \equiv 0\ (mod\ 2)$ and $x \equiv 0\ (mod\ 3)$.
- So there exist integers k_1, k_2 with $x = 2k_1$ and $x = 3k_2$.
- For any integer k_1, we have $2\mathrm{k}_1$ equivalent to one of $\{0,2,4\}$ (mod 6)
- For any integer k_2, we have $3k_2$ equivalent to one of $\{0,3\}$ (mod 6)
- Since $x = 2k_1 = 3k_2$, then $2k_1$ and $3k_2$ must have the same remainder in mod 6
- So $x \equiv 0\ (mod\ 6)$, so $\boldsymbol{x} \in \boldsymbol{T}$
- **So** $(\boldsymbol{R} \cap \boldsymbol{S}) \subseteq \boldsymbol{T}$

Equality Proof

$R = \{r \in \mathbb{Z} \mid r \equiv 0 \ (mod\ 2)\}$
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The full proof = **both** of the previous parts, plus concluding sentence:

"We have proved $(\mathrm{R} \cap \mathrm{S}) \subseteq \mathrm{T}$ and $\mathrm{T} \subseteq (\mathrm{R} \cap \mathrm{S})$, so $(\mathrm{R} \cap \mathrm{S}) = \mathrm{T}$"

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$T = \{t \in \mathbb{Z} \mid t \equiv 0 \ (mod\ 6)\}$

Proposition: $R \cap S = T$

Proof: ***goal is to prove*** $\boldsymbol{R \cap S \subseteq T}$***, and*** $\boldsymbol{T \subseteq R \cap S}$

The full proof = **both** of the proofs on the previous two sets of slides.

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Other set operations

$S = \{ \overbrace{\{\text{siamese}, \text{tabby}\}}^{a}, \underbrace{\{\text{german shepherd}, \text{pitbull}, \text{beagle}\}}_{a} \}$

$P(S) = \{ \emptyset, \overbrace{\{ \underbrace{\{s,t\}}_{a}, \underbrace{\{g,p,b\}}_{b} \}}^{S}, \{ \underbrace{\{s,t\}}_{a} \}, \{ \underbrace{\{g,p,b\}}_{b} \} \}$

- The "**Power Set**" of a set S is the set of all subsets of S.
- Written $P(S)$.

$S = \{a, b\}$

$P(S) = \{ \emptyset, \{a,b\}, \{a\}, \{b\} \}$

- Example: Find the power set of {cat, dog}

$P(\{cat, dog\}) = \{ \emptyset, \{cat\}, \{dog\}, \{cat, dog\} \}$

  - There are 4 possible subsets
  - { }, {cat}, {dog}, {cat, dog}
  - So P({cat, dog}) = {{ }, {cat}, {dog}, {cat, dog}}

$P(\{cat, dog, bird\})$ has 8 elements.

★ If S has n elements, then $P(S)$ has 2^n elements

Other set operations

- An **ordered pair** (a, b) is two elements from a domain
  - Order matters: $(a, b) \neq (b, a)$
  - Repeats allowed: (a, a) is valid

- The **Cartesian product** of two sets A, B is the set of all ordered pairs (a, b) with a taken from A, and b taken from B
  - Notation: $A \times B$
  - Formally: $A \times B = \{(a, b) \mid a \in A, b \in B\}$

Example: B = {white, wheat, rye}, M = {turkey, tuna}. Find $B \times M$.

$B \times M$ is the set of the 6 possible ordered pairs:

$$B \times M = \{ (white, turkey), (white, tuna), (wheat, turkey), (wheat, tuna), (rye, turkey), (rye, tuna) \}$$

These are the 6 elements of $B \times M$

B × M = { (white, turkey), (white, tuna), (wheat, turkey), (wheat, tuna), (rye, turkey), (rye, tuna) }

Other Set Operations

Handout

The **cardinality** of a set S means the number of

____

$|\{a, 22, dog\}| =$

$|\{a, \{e\}, \{i, o\}, u\}| =$

$|\{x \mid x \in \mathbf{N} \text{ and } x^2 = x\}| =$

$|\{x \mid x \in \mathbf{N} \text{ and Prime}(x)\}| =$

$|\mathbf{N}| =$

$|\mathbf{R}| =$

For a set S, the **power set** of S is:

____

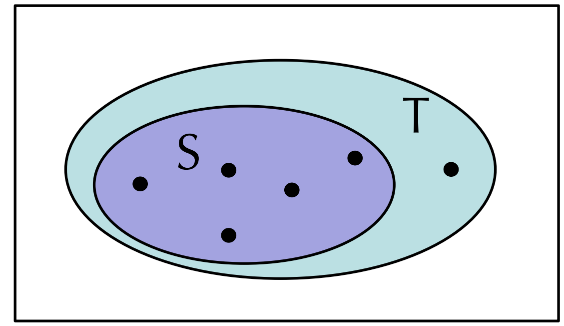
The **Cartesian product** of sets A and B is

$$A \times B = \{ \_\_\_\_\_\_\_\_\_\_ \mid a \in A \wedge b \in B \}$$

Other Relationships Between Sets

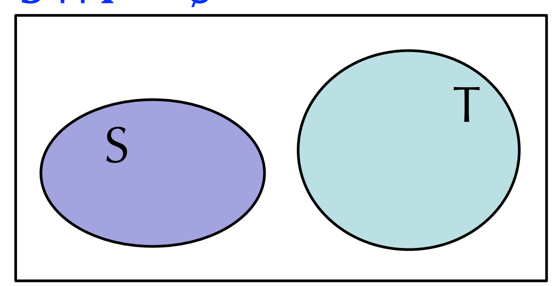
Handout

- "S is a *proper subset* of T"
  - **S $\subsetneq$ T** : $S \subseteq T \wedge S \neq T$
  - $\{1,2\} \subsetneq \{1,2,3\}$



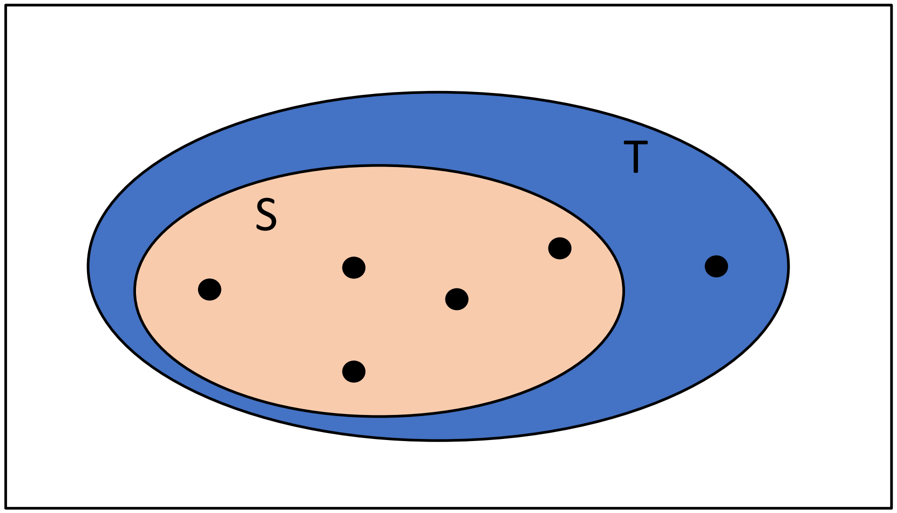
- "S and T are *disjoint*"
  - **Means:** $S \cap T = \emptyset$
  - {1,2} and {3,4} are disjoint

$S \cap T = \emptyset$



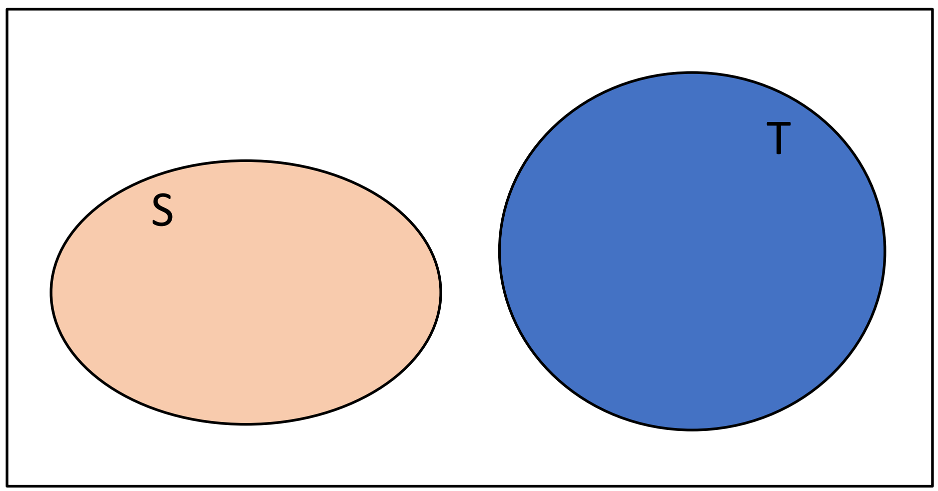
Other set relations

- Subsets allow for equality: $S \subseteq T$ includes the possibility of $S = T$.
- We say S is a **"proper subset"** of T if $S \subseteq T$ but they are not equal
  - There is at least one element in T but not S.
- Proper subsets are written $S \subsetneq T$ or sometimes $S \subset T$



Other set relations

- Two sets with empty intersection are called “**disjoint**” ***(from each other)***.
  - **Means: S ∩ T = ∅**
  - **Example:** $\{1,2\}$ and $\{3,4\}$ are disjoint



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A question we can’t answer yet

- How many integers between 1 and 300 (inclusive) are divisible by 5?

$$\frac{300}{5} = 60$$

- How many integers between 1 and 300 (inclusive) are divisible by at least one of {2,3,5}?

We’ll solve this by the end of this section.

Set Cardinality

- The **cardinality** of a set is its number of elements.
- **Notation: “cardinality of S”** is written $|S|$ (vertical bar on either side)

Examples:

$$|\{0,1,2\}| = 3$$
$$|\{\text{apple}, \text{banana}, \text{carrot}, \text{dragonfruit}\}| = 4$$
$$|\{\quad\}| = 0$$
$$|\mathbb{Z}| = \infty$$

Set Cardinality

$|S|$ = *number* of elements in S

Examples

$|\{1,2,3\}| =$ 3

$|\underbrace{\{\{1,2,3\}\}}| =$ 1

$|\{a, \{e\}, \{i, o\}, u\}| =$

$|\{x \in \mathbb{N} \mid \mathrm{x}^2 = \mathrm{x}\}| =$

$|\{x \in \mathbb{N} \mid x \text{ is prime}\}| =$

$|\mathbb{N}| =$

$|\mathbb{R}| =$

Other Set Operations

Handout

The **cardinality** of a set S means the number of

elements in the set.

$|\{a, 22, dog\}| =$ 3

$|\{a, \underbrace{\{e\}}, \underbrace{\{i, o\}}, u\}| =$ 4

$|\{x \mid x \in \mathbf{N} \text{ and } x^2 = x\}| =$ 2

$|\{x \mid x \in \mathbf{N} \text{ and Prime}(x)\}| =$ ∞

$|\mathbf{N}| =$ ∞

$|\mathbf{R}| =$ ∞

For a set S, the **power set** of S is:

____

The **Cartesian product** of sets A and B is

$$A \times B = \{ \_\_\_\_\_\_\_\_\_\_ \mid a \in A \wedge b \in B \}$$

Set Cardinality

$|S|$ = *number* of elements in S

Examples

$|\{1,2,3\}| = 3$

$|\{\{1,2,3\}\}| = 1$ ***only one element (the set {1,2,3})***

$|\{a, \{e\}, \{i, o\}, u\}| = 4$ $\{i, o\}$ ***only counts as one element***

$|\{x \in \mathbb{N} \mid \mathrm{x}^2 = \mathrm{x}\}| = 2$ ***This describes the set*** $\{0, 1\}$

$|\{x \in \mathbb{N} \mid x \text{ is prime}\}| = \infty$ ***Infinitely many primes***

$|\mathbb{N}| = \infty$

$|\mathbb{R}| = \infty$

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Cardinality vs. Unions

$$S_m = \{x \in \mathbb{Z} \mid 1 \leq x \leq 30 \text{ and } x \equiv 0 \ (mod\ m)\}$$

What is $|S_2 \cup S_3|$?

One strategy: "brute force"

Count from 1 to 30

Check whether each individual number is a multiple of 2 or 3

Slow to implement, and too hard for bigger numbers (what if $30 \rightarrow 3000$?)

Another strategy:

Calculate the number of multiples of 2: $\frac{30}{2} = 15$

Calculate the number of multiples of 3: $\frac{30}{3} = 10$

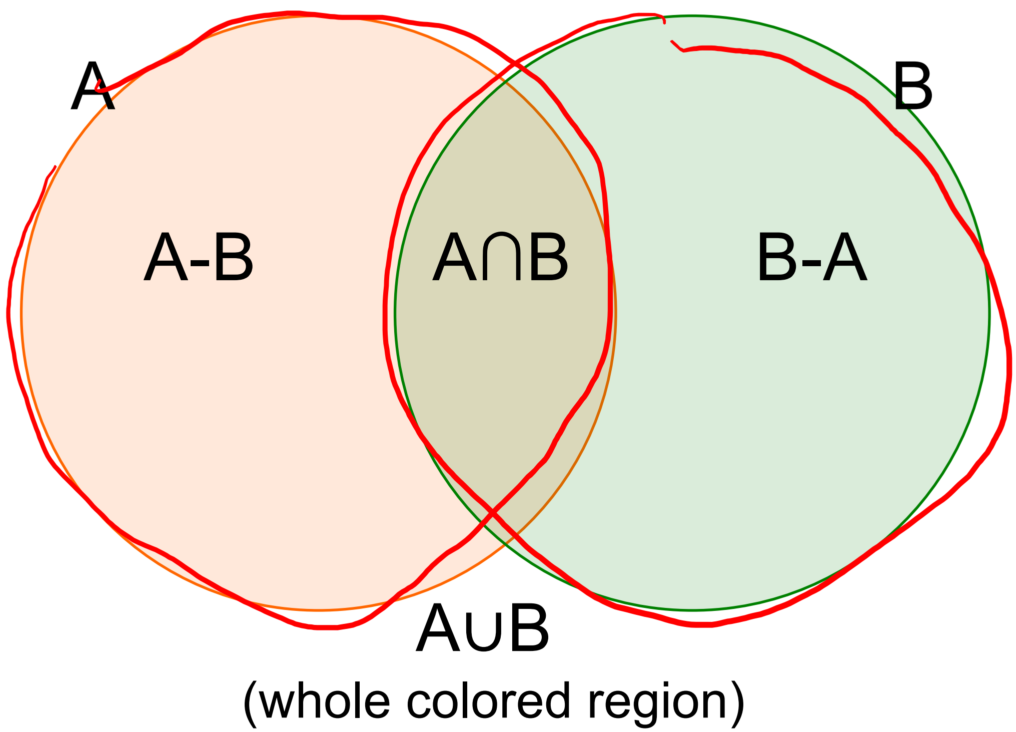
How many elements got counted twice?

Cardinality of unions and intersections

"Inclusion-Exclusion Rules"

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cap B| = |A| + |B| - |A \cup B|$$



"Inclusion-Exclusion"

$|A \cup B| = |A| + |B| - |A \cap B|$

Cardinality vs. Unions

How many numbers between 1 and 30 are multiples of 2 or 3?

$$S_m = \{x \in \mathbb{Z} \mid 1 \le x \le 30 \text{ and } x \equiv 0 \ (mod\ m)\}$$

S_m = Multiples of m from 1 to 30

What is $|S_2 \cup S_3|$?

$$|S_2 \cup S_3| = |S_2| + |S_3| - |S_2 \cap S_3|$$

$R = \{r \in \mathbb{Z} \mid r \equiv 0 \ (mod\ 2)\}$
$S = \{s \in \mathbb{Z} \mid s \equiv 0 \ (mod\ 3)\}$
$T = \{t \in \mathbb{Z} \mid t \equiv 0 \ (mod\ 6)\}$

Theorem: $R \cap S = T$

"Inclusion-Exclusion"

$|A \cup B| = |A| + |B| - |A \cap B|$

Cardinality vs. Unions

How many numbers between 1 and 30 are multiples of 2 or 3?

$$S_m = \{x \in \mathbb{Z} \mid 1 \leq x \leq 30 \text{ and } x \equiv 0 \ (mod\ m)\}$$

S_m = Multiples of m from 1 to 30

What is $|S_2 \cup S_3|$?

$$|S_2 \cup S_3| = |S_2| + |S_3| - |\boldsymbol{S_6}|$$

$$|S_2| = \frac{30}{2} = 15$$

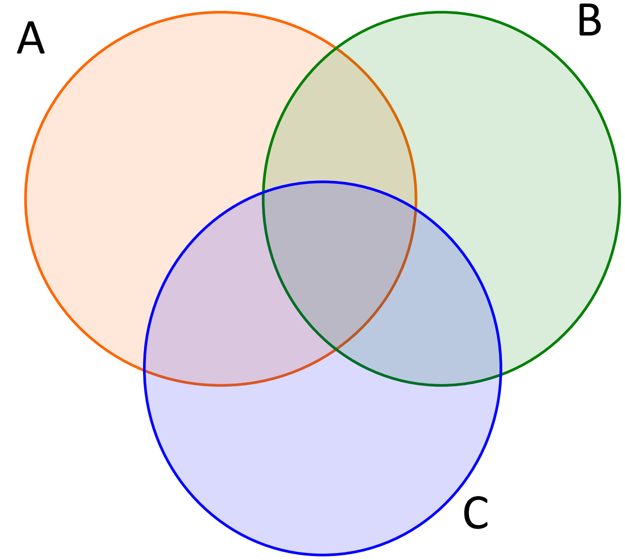
$$|S_3| = \frac{30}{3} = 10$$

$$|S_6| = \frac{30}{6} = 5$$

$$|S_2 \cup S_3| = 20$$

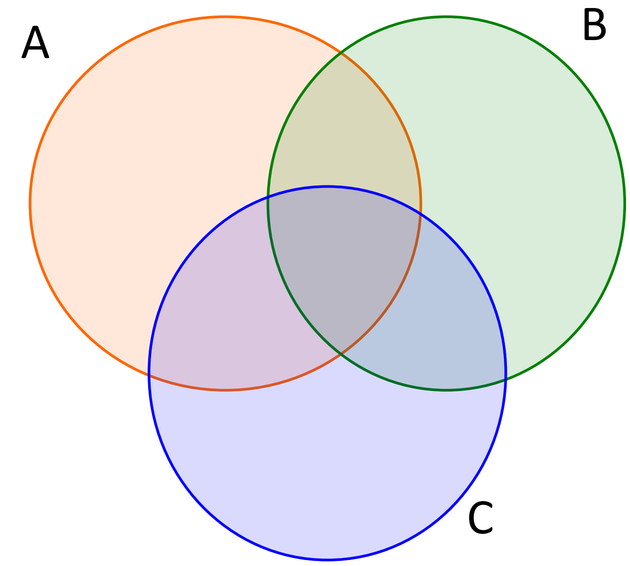
Inclusion-Exclusion for 3 sets

- What is |AUBUC|?



Inclusion-Exclusion for 3 sets

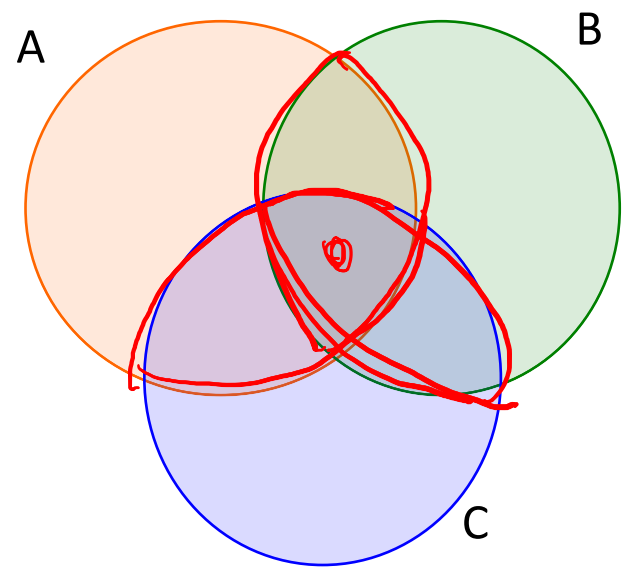
- What is |AUBUC|?



$$= \underbrace{|A|+|B|+|C|}_{\text{Add individual sizes}}$$

Inclusion-Exclusion for 3 sets

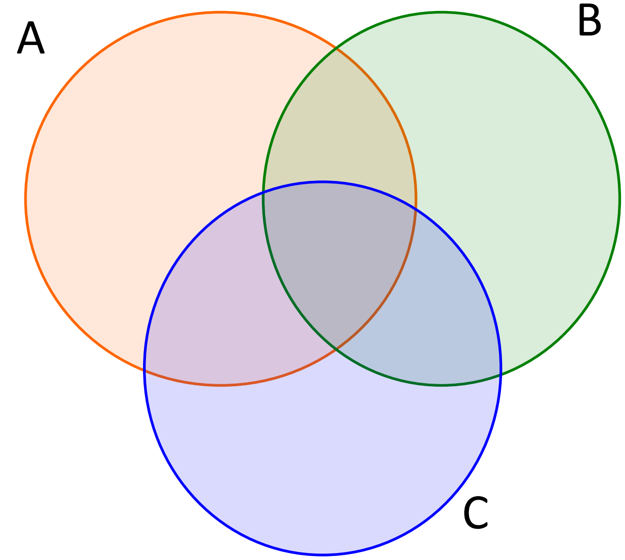
- What is |AUBUC|?



$$= \underbrace{|A|+|B|+|C|}_{\text{Add individual sizes}} \underbrace{-|A\cap B| - |A\cap C| - |B\cap C|}_{\text{Subtract pairwise } \cap\text{'s}}$$

Inclusion-Exclusion for 3 sets

- What is $|A \cup B \cup C|$?



$$= \underbrace{|A|+|B|+|C|}_{\text{Add individual sizes}} \underbrace{- |A \cap B| - |A \cap C| - |B \cap C|}_{\text{Subtract pairwise } \cap\text{'s}} \underbrace{+ |A \cap B \cap C|}_{\text{Add 3-wise } \cap}$$

Original Question: How many multiples of 2, 3, or 5 from 1 to 300?

Now $S_m = \{x \in \mathbb{Z} \mid 1 \leq x \leq \mathbf{300} \text{ and } x \equiv 0 \ (mod\ m)\}$

What is $|S_2 \cup S_3 \cup S_5|$?

Lemma: If a, b are relatively prime, then $S_a \cap S_b = S_{a \cdot b}$

Original Question: How many multiples of 2, 3, or 5 from 1 to 300?

Now $S_m = \{x \in \mathbb{Z} \mid 1 \leq x \leq \mathbf{300} \text{ and } x \equiv 0 \ (mod\ m)\}$

What is $|S_2 \cup S_3 \cup S_5|$?

Lemma: If a, b are relatively prime, then $S_a \cap S_b = S_{a \cdot b}$

First Inclusion:

$$|S_2| + |S_3| + |S_5| = \frac{300}{2} + \frac{300}{3} + \frac{300}{5} = 150 + 100 + 60 = 310$$

Second Exclusion:

$$-|S_{2\cdot 3}| - |S_{2\cdot 5}| - |S_{3\cdot 5}| = -\frac{300}{6} - \frac{300}{10} - \frac{300}{15} = -50 - 30 - 20 = -100$$

Third Inclusion:

$$|S_{2\cdot 3\cdot 5}| = \frac{300}{30} = 10$$

$$|S_2 \cup S_3 \cup S_5| = 310 - 100 + 10 = \mathbf{220}$$

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Russell's Paradox

In a small village there is a barber who shaves exactly those people who do not shave themselves

Does the barber shave herself?

- Suppose she **does** shave herself.
  - Then she's one of the people who do shave themselves, which means the barber doesn't shave her. But she's the barber, so we have a contradiction: she does shave herself and she doesn't shave herself.

- Suppose she **doesn't** shave herself.
  - Then she's one of the people who don't shave themselves, so the barber shaves her. But she's the barber, so again we have a contradiction: she doesn't shave herself and she does shave herself.

- Neither option is possible! **It's not actually possible to have a barber with this property.**

What does this have to do with sets?

Back to membership predicates

- Every set S has a membership predicate: $P(x) = "x \in S“$
- But does every membership predicate have a set S?

- **That is:** let D be a domain, and let $P(x)$ be an arbitrary predicate over D. Is there always a set S whose membership predicate is P?

Of course there is … just take the set of domain elements that satisfy P(x).

Right?

Russell’s Paradox

the domain $D =$ **all sets** set has (as an element)

In ~~a small village~~ there is a ~~barber~~ who ~~shaves~~ exactly those ~~people~~ who do not ~~shave~~ themselves

sets has (as an element)

- **In general, sets can have themselves as elements!**

$S = \{1,2,S\}$ is allowed.

- So $P(S) =$ "$S \in S$“ is a valid predicate (over the domain of all sets)
- And so is $Q(s) =$ "$S \notin S$"

Russell's Paradox

*In the **set of all sets**, is there a **set** that **has as an element** exactly those **sets** that do not **have themselves as elements**?*

- **Sets can contain themselves!** $\boldsymbol{Q(S)} = "\boldsymbol{S \notin S}"$ is valid.
- Is there a set $\mathbf{R} = \{S \in D \mid Q(S)\}$ ***(with membership pred.*** $Q(S)$***)***?
  - We can't have $\mathbf{R} \in \boldsymbol{R}$, because that means $Q(\boldsymbol{R})$ is true, which means $\mathbf{R} \notin \boldsymbol{R}$.
  - We can't have $\mathbf{R} \notin \boldsymbol{R}$, because that means $Q(\boldsymbol{R})$ is false, which means $\mathbf{R} \in \boldsymbol{R}$.
  - **Conclusion: no such set** $\boldsymbol{R}$ **exists.**

Russell's Paradox

- **Russel's Paradox:** "Let R the set of all sets that do not have themselves as elements. Does R have itself as an element?"

- **Conclusion:** Actually, R can't exist! So there exists a membership predicate that doesn't correspond to any set.

- These are **extremely rare**. Outside of these special examples, all membership predicates correspond to sets.

Wrapup

Sets are:

- **Like logic:** Combining/comparing sets is like combining/comparing their membership functions
- **Unlike logic:** Important concepts of ***counting elements***
- Russel's paradox was our first example of a "diagonalization proof." We'll see this again later!