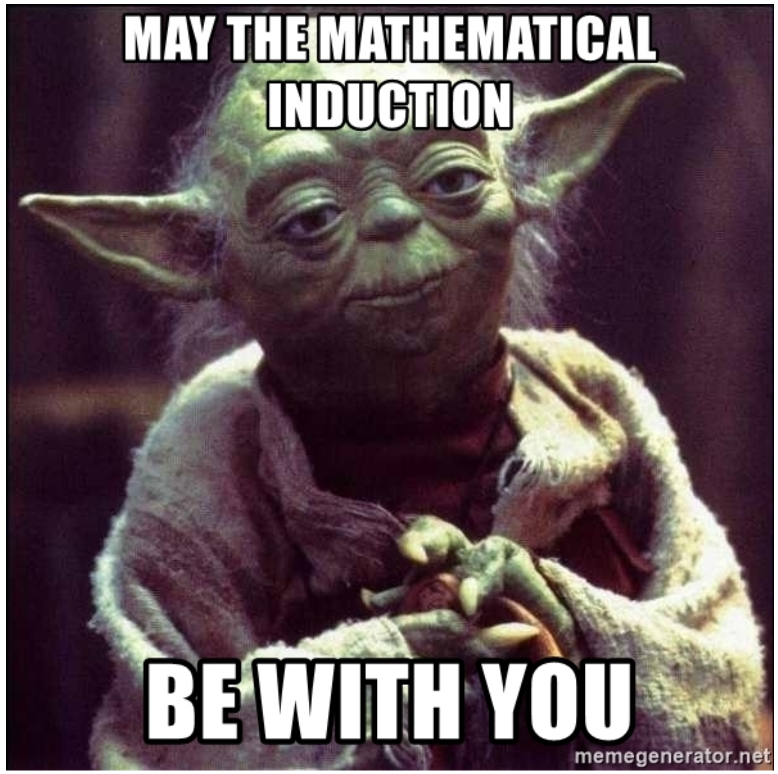
No Handout today

Exam 2 Review



EECS 203: Discrete Mathematics
Lecture 20

Winter 2022

Reminders

Exam 2 is Tomorrow

- 7:00-9:30 pm
- Practice exams and review session recordings are on Canvas

Upcoming Course Schedule

- Discussion 8b starts Thurs
  - More graphs
- Homework 8 due next Thurs 3/31
  - Lecs 18, 19, 21
  - Discussions 8a and 8b

Lec 20 Outline: Exam 2 Review

- Check-in: *How do we feel about the second exam?*
- Session Goals
- Key reminders & logistics
- Top tips & Things to know
- Opportunities to clarify conceptual questions & practice on the topics of the exam

Check-in:
How do we feel about the second exam?

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How *prepared* do you feel for the upcoming exam?

A. I absolutely got this!
B. I believe I will be okay with a bit more study
C. I have a lot to do to be prepared!
D. Yikes! I am worried!

As of today, how much *time and effort* have you invested in preparing for the exam?

A. I have been studying consistently and attended (or watched) the weekend review sessions and test strategies session
B. I have been looking at old exam problems and doing a few when I have time
C. I have been focusing on completing homeworks, not much time for additional study
D. I haven’t even started to think about the exam yet!

Session goals: *How this session will support your exam preparation and confidence*

This session has been designed as complement to the weekend review sessions and test strategies session. Today we will review key information and strategies that will support a successful exam and practice with the topics and problems you feel most in need of review. By the end of the session you should feel ready to:

- Describe the general structure of the exam
- Describe different types of problems you might encounter in the exam
- Plan a set of steps to tackle the upcoming exam, based on a clear understanding of the concepts and procedures assessed in the exam

Reminders & Logistics

What should I expect on the exam?

- 13 Multiple-Choice Questions
  - 4 points each
  - Single-answer multiple choice
  - Multiple-answer multiple choice
    - 0, 1, 2, 3, 4, or 5 correct answers
    - Partial credit awarded
  - On **Canvas**
- 6 Free Response Questions
  - 8 points each
  - Partial credit awarded
  - Show your work!
  - Question prompts on Canvas
  - Upload answers to **Gradescope**
- Covers Lectures 10-11, 13-17
  - Through Relations
  - No graphs

Reminders & Logistics

When/where is the exam?

- Tomorrow: Weds, March 23
- Exam Timing: Starts 7pm, 2 hours
  - +15m for breaks/interruptions
  - Your Canvas quiz will auto-submit at 9:15pm
  - +15 minutes to upload to Gradescope
  - Hard deadline for gradescope submission: 9:30. YOU WILL LOSE POINTS AT 9:31
- You can take the test from anywhere
  - Optional: take exam from a North Campus classroom, if you filled out the request form

Exam Logistics

- Zoom Helpline
  - https://umich.zoom.us/j/93850208854
  - Meeting ID: **938 5020 8854**
  - Joining #: +1 312 626 6799. Write this down!
  - Not in AA? Look up your joining #: https://umich.zoom.us/u/ads3iwNVIv
- Use Helpline for:
  - Clarifying questions about the test
  - Logistical issues
  - IF YOU HAVE GRADESCOPE UPLOAD ISSUES, GET ON THE HELPLINE TO LET US KNOW **BEFORE** 9:30

Reminders & Logistics

Submitting:

- You can submit to Gradescope as many times as you want. Only the last one will count. Don't submit after 9:30!
- Your Canvas quiz will auto-submit itself at 9:15 (if you haven't submitted it previously).
- If your have trouble with your Canvas quiz, disable your adblocker.

What Resources Can I Use?

- Note Sheet
  - But you must make it yourself
  - Can include screenshots, pictures, can be typed or handwritten
  - Not required, but *highly recommended*
- Any class materials
  - Textbook
  - Lecture slides
  - Notes you took

What Resources **Can't** I Use?

- No other humans
  - Either in-person or on the internet
- No googling the question prompt
- No posting answers/hints anywhere yourself

Top Tips

- Do the practice exams
  - And read the Common Mistakes listed in the solutions
- Manage your time during the exam
  - Do the questions you know first
  - Come back to questions you’re unsure of later
- Show all your work for Free Response questions
  - Partial credit is available
- You can ask us if you have questions during the exam
  - Use the Zoom Helpline: **https://umich.zoom.us/j/93850208854**

Things to know

- Review the posted learning objectives
- Know the definitions of
  - A function, onto, 1-1, bijection, function composition
  - Countable, uncountable, etc.
  - Relation properties, equivalence relation, partial order
  - (^ this list is not exhaustive)
- Proof methods, tips, etc
  - Most proofs can be broken down into applying definitions and properties
    - Ex: proving R is transitive $\equiv$ proving the definition of transitive holds $\equiv$ proving an "implies" inside a "for all"
    - How do you prove a "for all…" statement?
    - How do you prove "if …., then …."
  - How to do inductive proofs

Guide for [weak] Induction Proofs

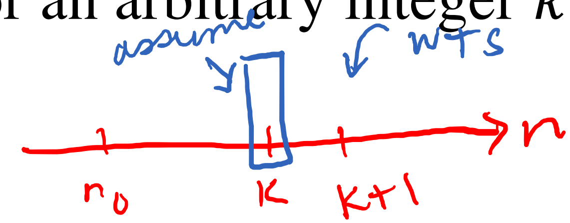
$P(n)$: "with n staff members, $n-1$ gifts are exchanged."

- Restate the claim you are trying to prove

  Claim: $P(n)$ $\forall n \geq n_0$

- **Base case**: Prove the claim holds for the "first" value of n
  - Prove $P(n_0)$ is true

- **Inductive Step**: Prove that $P(k) \rightarrow P(k+1)$ for an arbitrary integer k in the desired range.
  - Let k be an arbitrary integer with $k \geq n_0$
  - Assume $P(k)$
  - Show that $P(k+1)$ holds



Equivalently: Show $P(k-1) \rightarrow P(k)$

- Conclusion: explain that you've proven the desired claim.

Guide for **Strong** Induction Proofs

- Restate the claim you are trying to prove

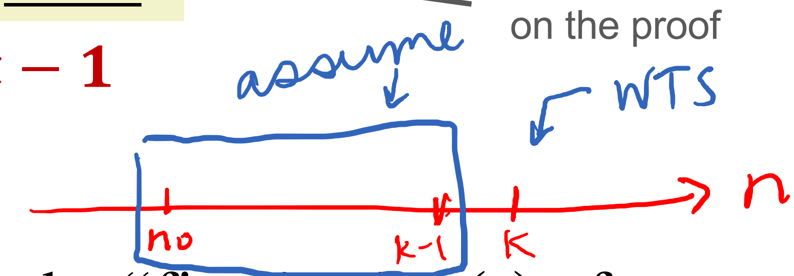
> Equivalently: Show
> $[\, P(n_0) \wedge P(n_0+1) \wedge \cdots \wedge P(\boldsymbol{k}) \,] \rightarrow P(\boldsymbol{k}+\mathbf{1})$

- **Inductive Step**: Prove that for an arbitrary integer k in the desired range,

$$[\, P(\boldsymbol{n_0}) \wedge P(n_0+1) \wedge \cdots \wedge P(\boldsymbol{k}-\mathbf{1}) \,] \rightarrow P(\boldsymbol{k})$$

  - Let k be an arbitrary integer with $k \geq$ _____
  - Assume $P(j)$ is true for all $\boldsymbol{n_o} \leq j \leq \boldsymbol{k}-\mathbf{1}$
  - Show that $P(\boldsymbol{k})$ holds

value depends on the proof



- **Base case(s)**: Prove the claim holds for the "first" **value(s)** of n
  - Prove $P(n_0)$ is true
  - May also need to prove $P(n_0+1)$ and more, depending on the inductive step
- Conclusion: explain that you've proven the desired claim.

Idea: *Create your own* template for other proofs

To prove a relation R is symmetric, I need to prove …

the definition of symmetric holds:

$$\forall \boldsymbol{a}, \boldsymbol{b} \quad \boldsymbol{aRb} \rightarrow \boldsymbol{bRa}$$

What structure do you need to prove this definition?
What are some key phrases you can use along the way?

You’ve got this!

- Good luck!
- We have confidence in you
- Take care of yourself
  - Sleep, eat, take a walk, recharge…

- Reminder: this is just one test
  - … in one course
  - … in your whole life
- You are a multidimensional person with your own set of interests, circumstances, and challenges
  - you ≠ your exam score

Opportunities to Practice

Opportunities to Practice: You choose the topic(s)!

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1. Weak Induction
2. Strong Induction
3. Pigeonhole Principle
4. Proving a function is/is not onto and 1-1
5. Onto and 1-1 with $f \circ g$
6. Countability
7. Relations properties proof

Weak Induction Proof

Prove the following identity by mathematical induction:

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}, \quad \text{for all } n \geq 2$$

Weak Induction Proof

Prove the following identity by mathematical induction:

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\dots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n},\quad \text{for all } n\geq 2$$

Solution: Define $P(n)$ to be $(1-\frac{1}{4})(1-\frac{1}{9})\dots(1-\frac{1}{n^2})=\frac{n+1}{2n}$.

Inductive Step:

Inductive hypothesis: Assume $P(k)$ is true, i.e., $(1-\frac{1}{4})(1-\frac{1}{9})\dots(1-\frac{1}{k^2})=\frac{k+1}{2k}$, for some integer $k\geq 2$.

We want to show $P(k+1)$, i.e., $(1-\frac{1}{4})\left(1-\frac{1}{9}\right)\dots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)=\frac{(k+1)+1}{2(k+1)}$:

$$\begin{aligned}
\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\dots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right) &= \left((1-\frac{1}{4})(1-\frac{1}{9})\dots(1-\frac{1}{k^2})\right)\left(1-\frac{1}{(k+1)^2}\right)\\
&= \left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right)\\
&= \left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)\\
&= \left(\frac{1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)}\right)\\
&= \frac{(k+1)^2-1}{2k(k+1)}\\
&= \frac{k^2+2k+1-1}{2k(k+1)}\\
&= \frac{k^2+2k}{2k(k+1)}\\
&= \frac{k+2}{2(k+1)}\\
&= \frac{(k+1)+1}{2(k+1)}
\end{aligned}$$

Base case: $n=2$.

$$\left(1-\frac{1}{n^2}\right)=\left(1-\frac{1}{4}\right)=\frac{3}{4}=\frac{2+1}{2(2)}=\frac{n+1}{2n}$$

where we applied the Inductive Hypothesis on Line 1 to get Line 2.
Thus $P(k+1)$ holds, when $P(k)$ is true.

So by mathematical induction $P(n)$ is true for all $n\geq 2$.

Strong Induction Proof

$P(n) =$ "Yunsoo can buy n pencils with only packs of 3 and 8"

Yunsoo is buying pencils for his upcoming exam. Pencils come in packs of 3 pencils and 8 pencils. Let $P(n)$ be the predicate that Yunsoo can purchase a total of n pencils buying only packs of 3 pencils and/or 8 pencils. $P(n)$ is true for all $n \geq n_0$.

Claim: $P(n)$ $\forall n \geq n_0$

(a) What is the smallest possible value of n_0? No justification necessary. $n_0 = 14$

(b) For your n_0 from Part (a), prove $\forall n \geq n_0$ $P(n)$ **using induction**.

(c) For what numbers of pencils less than n_0 is it possible for Yunsoo to purchase buying only packs of the provided sizes? Prove $P(n)$ for those values.

Strong Induction Proof

$14 \leq j \leq k-1$

Solution:

(a) $n_0 =$ _____14_____

(b) Claim: $P(n)$ for all $n \geq 14$.

Induction Step

Let $k \geq 17$

Let $k \geq$ ____

Assume $P(j)$ is true for all j when $\boxed{14 \leq j < k}$. (Inductive Hypothesis)

Show $P(k)$ is true:

$k \geq 17$

- $14 \leq k - 3 < k$ (because $k \geq 17$)
- $P(k) = P(k - 3 + 3)$
- $P(k - 3) \to P(k)$ by adding a pack of 3 pencils, and $P(k - 3)$ is true by our inductive hypothesis
- Therefore, $P(k)$.

or Assume:

$14 \leq j \leq k$

WTS:

$P(k+1)$

Base Cases

- $P(14)$ is true, because $14 = 3 + 3 + 8$
- $P(15)$ is true, because $15 = 3 + 3 + 3 + 3 + 3$
- $P(16)$ is true, because $16 = 8 + 8$

Thus, by strong induction we have shown that $\forall n \geq n_0 \ P(n)$ is true.

(c) Yunsoo can purchase 3, 6, 8, 9, 11, and 12 pencils with the given pack sizes.

- $3 = 3$
- $6 = 3 \cdot 2$
- $8 = 8$
- $9 = 3 \cdot 3$
- $11 = 3 + 8$
- $12 = 3 \cdot 4$

Pigeonhole Principle

Grace has 3 red balls, 3 blue balls, and 4 green balls. She wants to put the balls into x boxes that can each hold any number of balls. What's the largest value of x such that Grace can guarantee that at least two of the same colored balls are in some box?

a) 3

b) 4

c) 7

d) 9

e) 10

Pigeonhole Principle

Grace has 3 red balls, 3 blue balls, and 4 green balls. She wants to put the balls into x boxes that can each hold any number of balls. What's the largest value of x such that Grace can guarantee that at least two of the same colored balls are in some box?

a) 3

b) 4

c) 7

d) 9

e) 10

Solution: (a)

This is because for red balls and blue balls, it's possible that they are distributed evenly among the three boxes. However, for green balls, there are 4 balls but only 3 boxes, so using the pigeonhole principle, $\lceil \frac{4}{3} \rceil = 2$.

For any value of $x > 3$ (ex. 4), we can't apply pigeonhole principle because the number of boxes are greater or equal to the number of balls of each color.

Onto and 1-1 Proofs

Problem 17. (8 points)

(a) Define $f : \mathbb{R} - \{-\frac{5}{4}\} \longrightarrow \mathbb{R}$: $f(x) = \frac{3}{4x+5}$. Prove or disprove that f is one-to-one.

(b) Define $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$: $g(x) = 3x^5 + 9$. Prove or disprove that g is onto.

Onto and 1-1 Proofs

(a) Define $f : \mathbb{R} - \{-\frac{5}{4}\} \longrightarrow \mathbb{R}$: $f(x) = \frac{3}{4x+5}$. Prove or disprove that f is one-to-one.

(b) Define $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$: $g(x) = 3x^5 + 9$. Prove or disprove that g is onto.

Solution:

(a) f is one-to-one. To prove this we need to show that for any $x_1, x_2 \in \mathbb{R} - \{-\frac{5}{4}\}$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
Let $x_1, x_2 \in \mathbb{R} - \{-\frac{5}{4}\}$, and let $f(x_1) = f(x_2)$. Then

$$\begin{aligned} f(x_1) &= f(x_2) && \text{Assumption} \\ \frac{3}{4x_1+5} &= \frac{3}{4x_2+5} && \text{Definition of } f(x) \\ 3(4x_2+5) &= 3(4x_1+5) && \text{Multiply both sides by } (4x_1+5)(4x_2+5) \\ 4x_2+5 &= 4x_1+5 && \text{Divide by 3} \\ 4x_2 &= 4x_1 && \text{Minus 5} \\ x_2 &= x_1 && \text{Divide by 4} \end{aligned}$$

Therefore, f is one-to-one, by the definition of one-to-one.

(b) g is not onto, because g maps from the positive reals to the reals. A counterexample is 6, which is in the codomain, but is not mapped to by any number in the domain.

$$\begin{aligned} g(x) = 3x^5 + 9 &= 6 \\ 3x^5 &= -3 \\ x^5 &= -1 \\ x &= -1 \end{aligned}$$

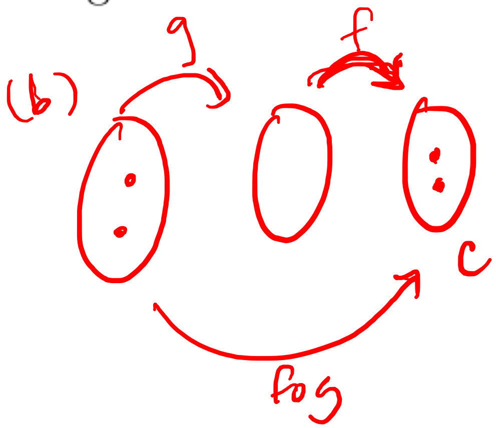
6 is only mapped to by -1 which is not in the domain.

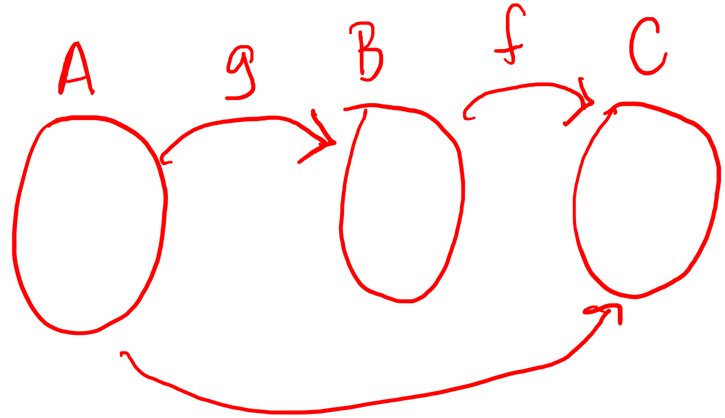
Onto and 1-1 with Function Composition

Suppose $g : A \to B$ and $f : B \to C$ are two functions such that $f \circ g$ is bijective. Which of the following statements are guaranteed to be true?

- ~~(a)~~ g is onto.
- (b) f is onto.
- (c) g is one-to-one
- ~~(d)~~ f is one-to-one

(b)

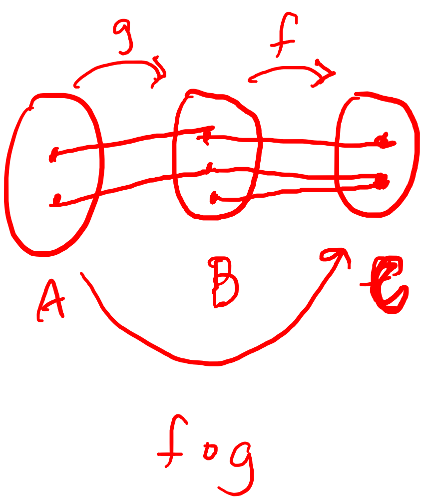


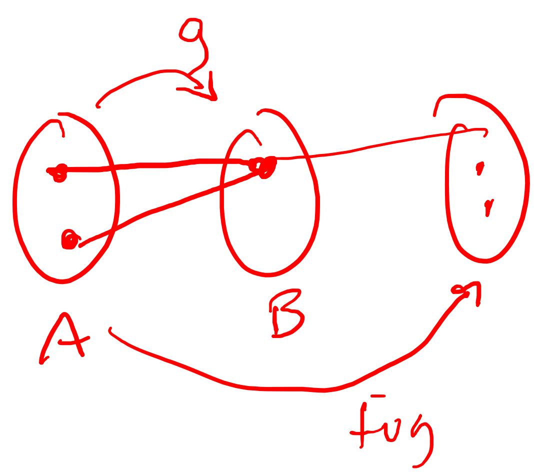


fog is a bijection

★ fog is onto → f is onto

(a) and (d)





★ fog is 1-1 → g is 1-1

Onto, 1-1 with Function Composition

Suppose $g : A \to B$ and $f : B \to C$ are two functions such that $f \circ g$ is bijective. Which of the following statements are guaranteed to be true?

(a) g is onto.

(b) f is onto.

(c) g is one-to-one

(d) f is one-to-one

Solution: (b), (c)
(a) and (d) may not be true. For example, let $C = \{c\}$, $B = \{b_1, b_2\}$, $A = \{a\}$. Let $g(a) = b_1$, $f(b_1) = f(b_2) = c$. Then $f \circ g$ is bijective but g is not onto and f is not one-to-one.

Countability

Which of the following sets are "countably infinite"?

↳ same size as $\mathbb{Z}^+$

" countable" means finite or countably infinite

~~(a)~~ $\mathbb{R} - \mathbb{Z}$

~~(b)~~ $\mathbb{R} - \{x \mid x \in \mathbb{R} \wedge (|x| > 10)\}$

(c) $\mathbb{Q} \cup \mathbb{Z}^3$ $\qquad \mathbb{Z}^3 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

(d) $\mathbb{Q}^+ - \mathbb{Z}$

~~(e)~~ $\mathbb{Z} \cap (\mathbb{Z} \times \mathbb{Z}) = \emptyset$

$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\{\ldots, (1,1), (1,2), (-5,-27), \ldots\}$

Some countable sets: $\mathbb{Z}^+$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N}$, $\mathbb{Z} \times \mathbb{Z}$, any finite set

Some uncountable sets: $\mathbb{R}$, irrationals

Countability

Which of the following sets are countably infinite?

(a) $\mathbb{R} - \mathbb{Z}$

(b) $\mathbb{R} - \{x \mid x \in \mathbb{R} \wedge (|x| > 10)\}$

(c) $\mathbb{Q} \cup \mathbb{Z}^3$

(d) $\mathbb{Q}^+ - \mathbb{Z}$

(e) $\mathbb{Z} \cap (\mathbb{Z} \times \mathbb{Z})$

Solution: (c), (d)

(a) This is the set of all reals excluding integers, which is uncountably infinite.

(b) This is the set of all real numbers whose absolute value is less than or equal to 10, which is uncountably infinite.

(c) This is the union of two countably infinite sets, which is countably infinite.

(d) This is the set of all positive rational numbers excluding integers, which is countably infinite.

(e) These sets are disjoint, so their intersection is the empty set, which is finite.

Relations Properties Proof

Let R be a relation defined on a set A. Prove or disprove:

$$\text{If } R \text{ is irreflexive and transitive, then } R \text{ is asymmetric}$$

Relations Properties Proof

Let R be a relation defined on a set A. Prove or disprove:

If R is irreflexive and transitive, then R is asymmetric

Solution: We prove this by contradiction.
Suppose that R is irreflexive and transitive but **not** asymmetric. Then there exist $a, b \in A$ such that aRb and bRa. Because R is transitive, this implies that aRa and bRb which contradicts that R is irreflexive.

Good Luck!

- You’ve got this