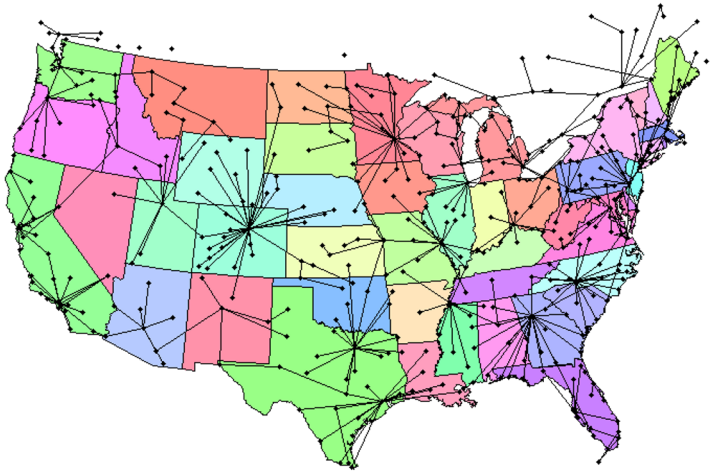
Lecture 20
Minimum Spanning Trees



EECS 281: Data Structures & Algorithms

Minimum Spanning Trees

Data Structures & Algorithms

The Minimum Spanning Tree Problem

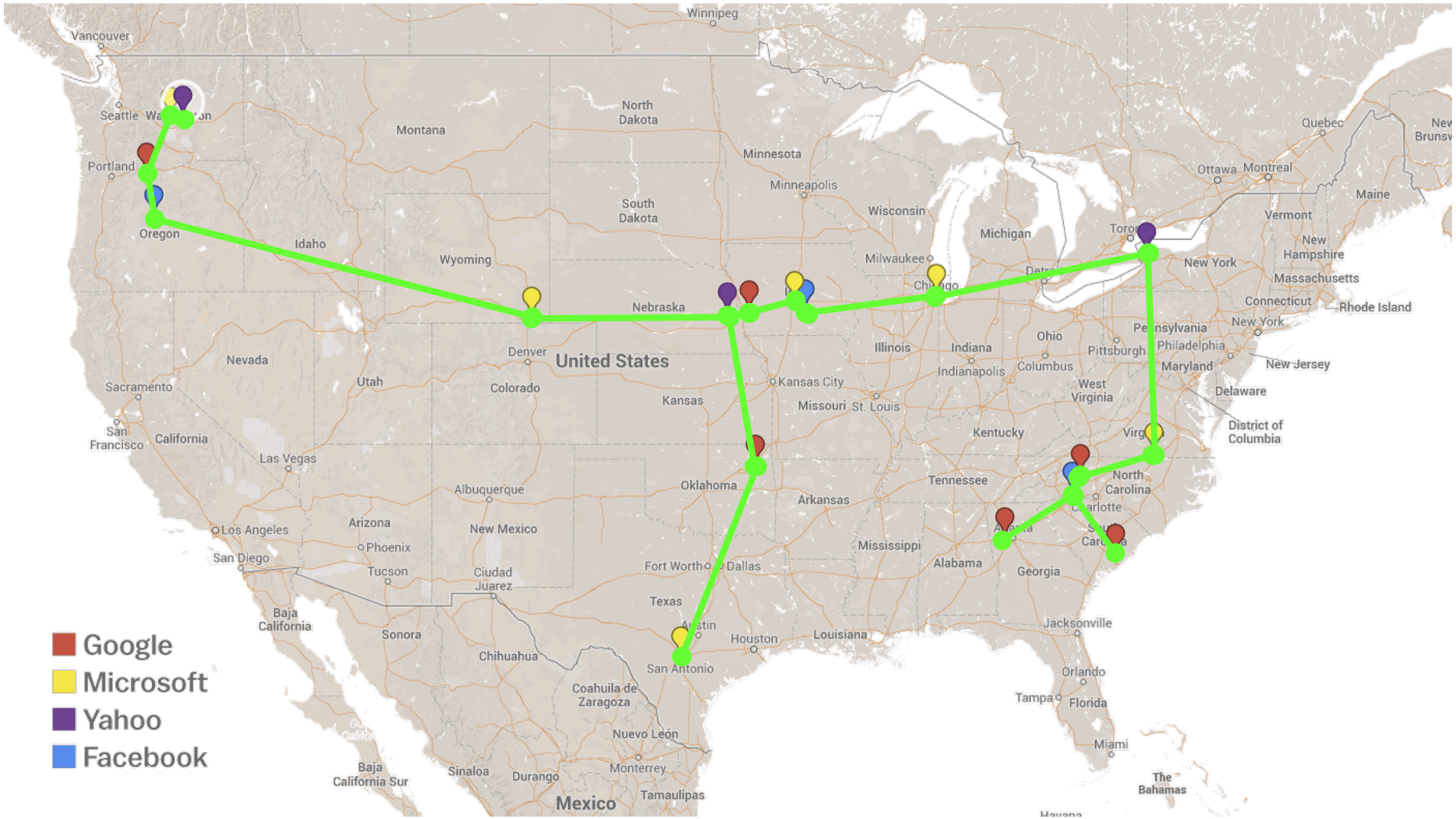
Given: edge-weighted, *undirected* graph $G = (V, E)$

Find: subgraph $T = (V, E')$, $E' \subseteq E$ such that

- All vertices are pair-wise connected
- The sum of all edge weights in T is minimal

- See a cycle in T?
  - Remove edge with highest weight
- Therefore, T must be a tree (no cycles)

Connect All Data-Centers



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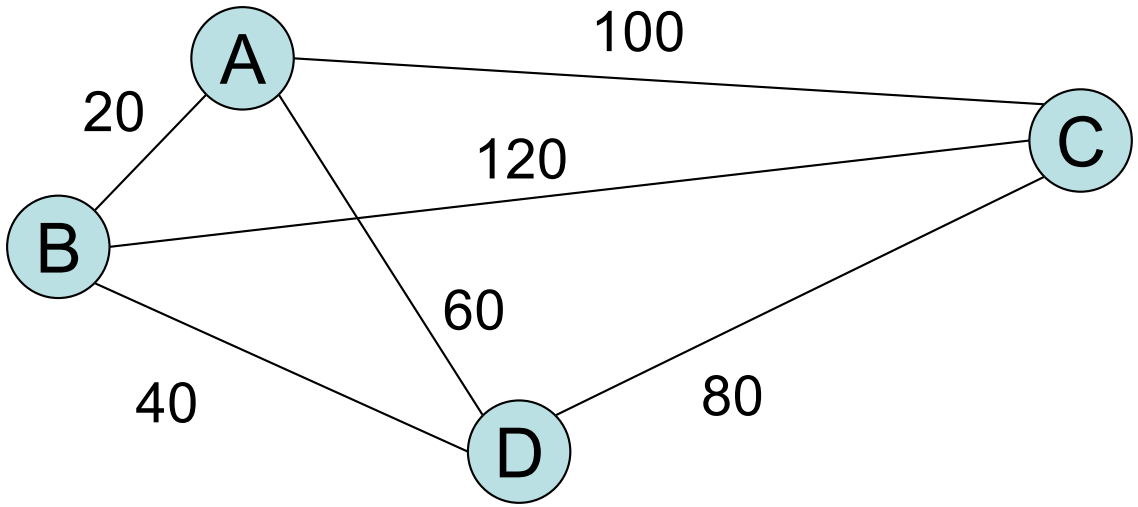
Create an MST

- Minimum Spanning Tree (MST) if:
  - All vertices are pair-wise connected
  - The sum of all edge weights in *T* is minimal

Total Edge Weights:

20+40+60+100+120+80 = 420

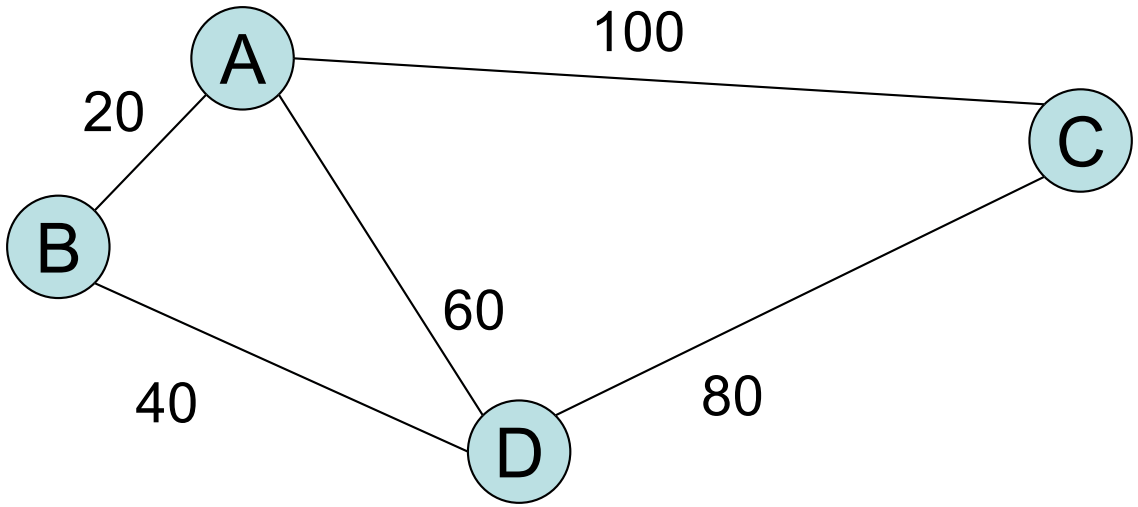


Create an MST

- Minimum Spanning Tree (MST) if:
  - All vertices are pair-wise connected
  - The sum of all edge weights in *T* is minimal

Total Edge Weights:

20+40+60+100+80 = 300

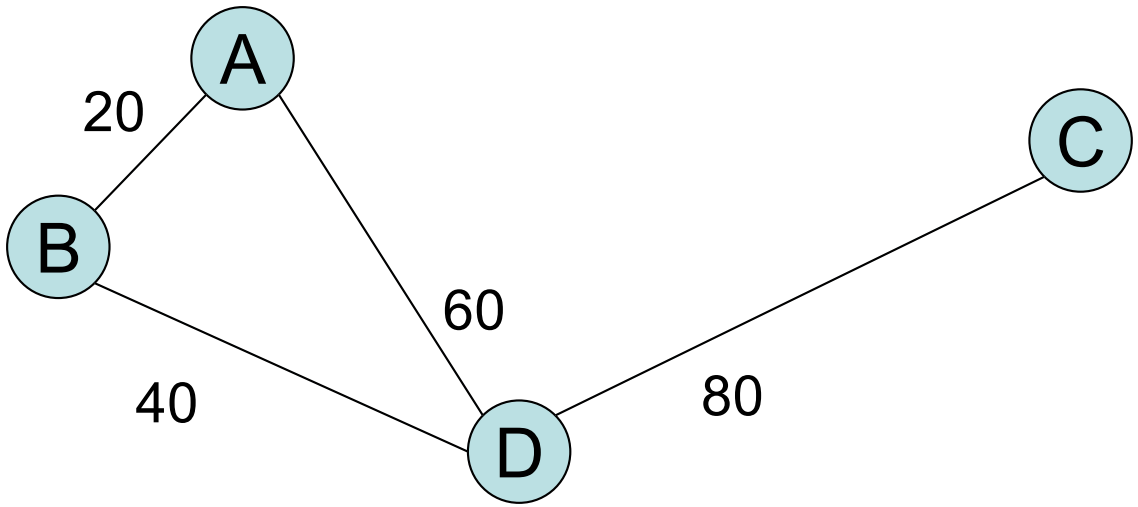


Create an MST

- Minimum Spanning Tree (MST) if:
  - All vertices are pair-wise connected
  - The sum of all edge weights in *T* is minimal

Total Edge Weights:

20+40+60+80 = 200

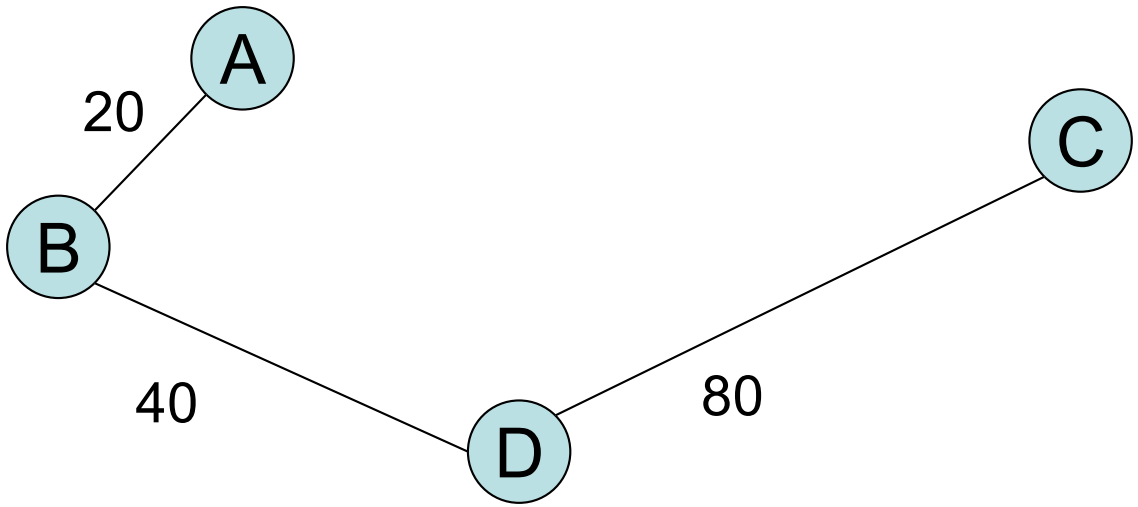


Create an MST

- Minimum Spanning Tree (MST) if:
  - All vertices are pair-wise connected
  - The sum of all edge weights in *T* is minimal

Total Edge Weights:

20+40+80 = 140



MST Quiz

1. Prove that a unique shortest edge must be included in every MST
2. Prove for second shortest edge
3. What about third shortest edge?
4. Show a graph with > 1 MST
5. Show a graph and its MST which avoids *some* shortest edge
6. Show a graph where every longest edge must be in every MST

Minimum Spanning Trees

Data Structures & Algorithms

Prim’s Algorithm

Data Structures & Algorithms

Prim's Algorithm

- Find an MST on edge-weighted, connected, *undirected* graphs
- Greedily select edges one by one and add to a growing sub-graph
- Grows a tree from a single vertex

Prim's Algorithm

1. Initialize a tree with a single vertex, chosen arbitrarily from the graph
2. Grow the tree by one edge: of the edges that connect the tree to vertices not yet in the tree, find the minimum-weight edge, and add that vertex to the tree
3. Repeat step 2 (until all vertices are in the tree)

Prim's Algorithm

- Given graph $G = (V, E)$
- Start with 2 sets of vertices: 'innies' & 'outies'
  - 'Innies' are visited nodes (initially empty)
  - 'Outies' are not yet visited (initially V)
- Select first innie arbitrarily (root of MST)
- Repeat until no more outies
  - Choose outie (v') with smallest distance from *any* innie
  - Move v' from outies to innies
- Implementation issue: use linear search or PQ*?*

Prim: Data structures

- Three vectors
  - Better: a vector of classes or structures!
- For each vertex *v*, record:
  - k_v: Has *v* been visited?
    (initially **false** for all $v \in V$)
  - d_v: What is the minimal edge weight to *v*?
    (initially ∞ for all $v \in V$, except $v_r = 0$)
  - p_v: What vertex precedes (is parent of) *v*?
    (initially **unknown** for all $v \in V$)

Prim's Algorithm

Set starting point d_v to 0.

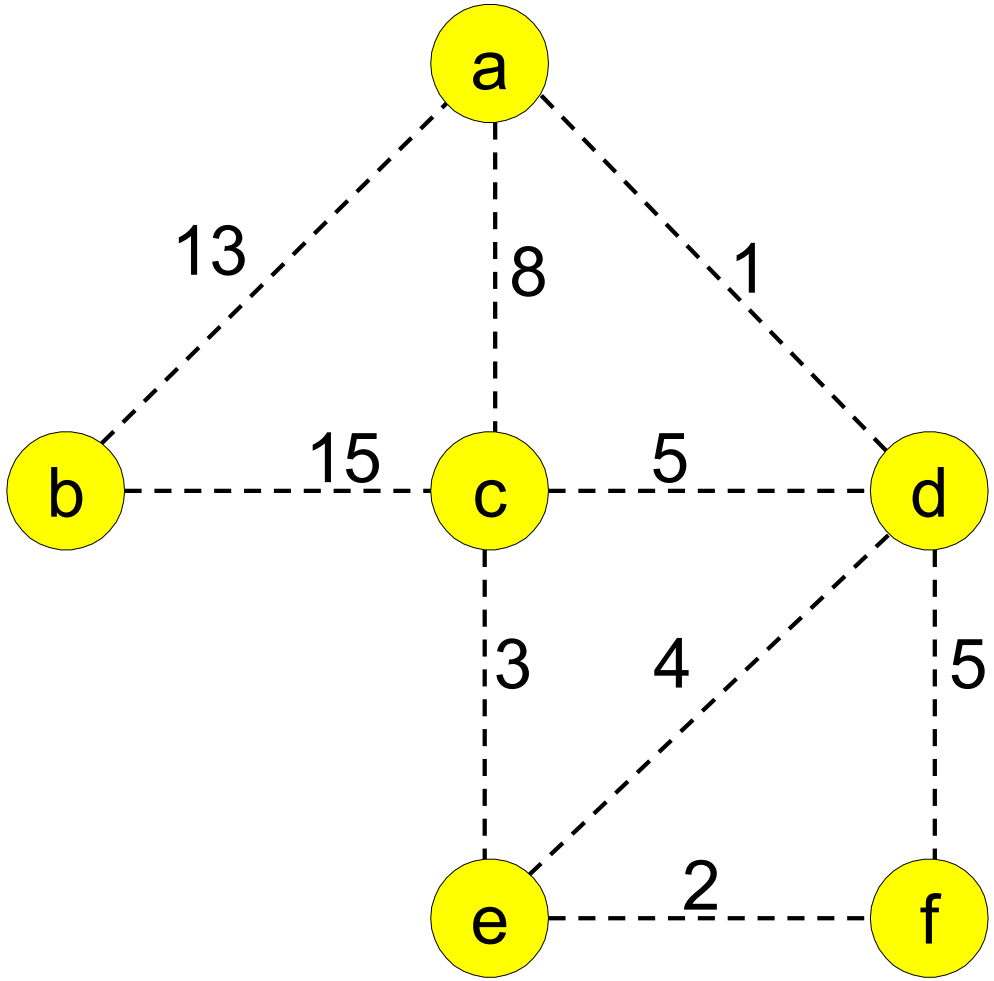
Loop v times (until every k_v is true):

1. **From the set of vertices for which k_v is false, select the vertex v having the smallest tentative distance d_v.**
2. **Set k_v to true.**
3. **For each vertex w adjacent to v for which k_w is false, test whether d_w is greater than distance(v,w). If it is, set d_w to distance(v,w) and set p_w to v.**

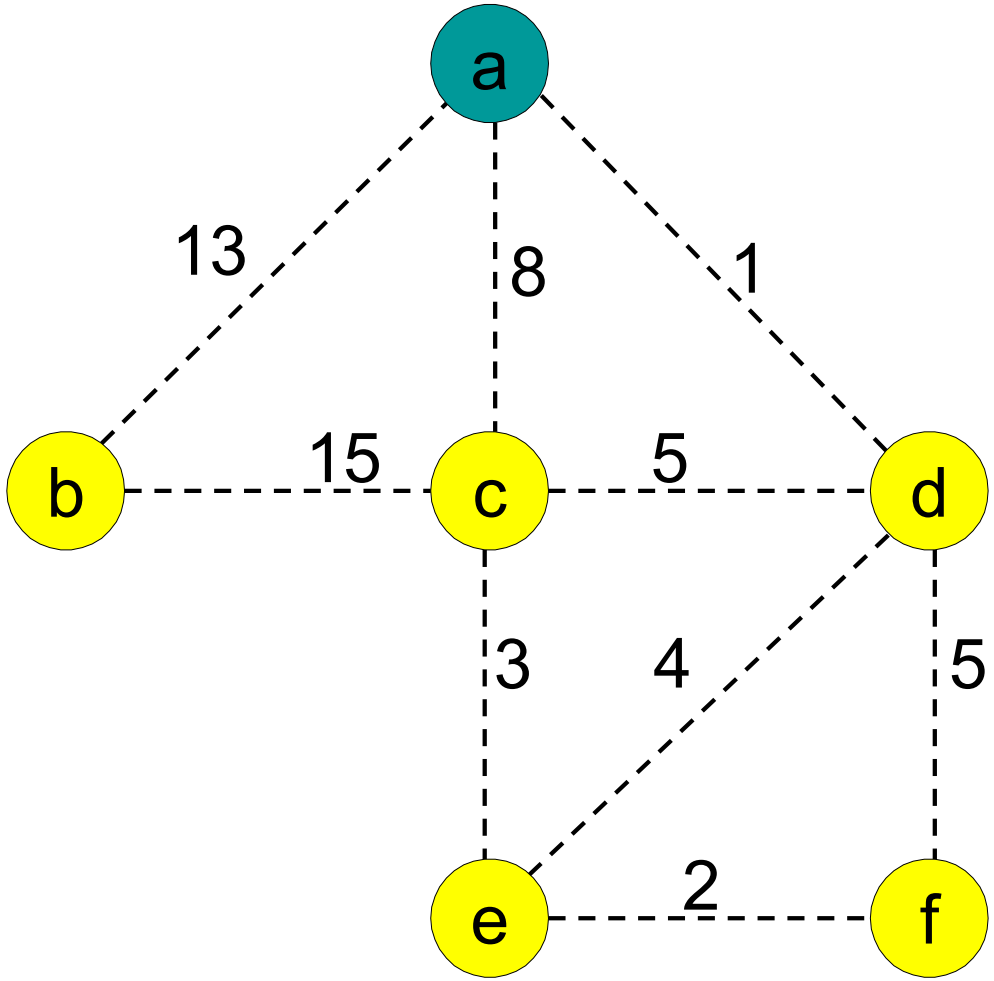
Implementing Prim's

- Implement in the order listed:
  - 1: Loop over all vertices: find smallest `false` k_v
  - 2: Mark k_v as `true`
  - 3: Loop over all vertices: update `false` neighbors of k_v
- Common Mistake: Set the first vertex to `true` outside the loop
- Reordering this can result in a simple algorithm that simply doesn't work

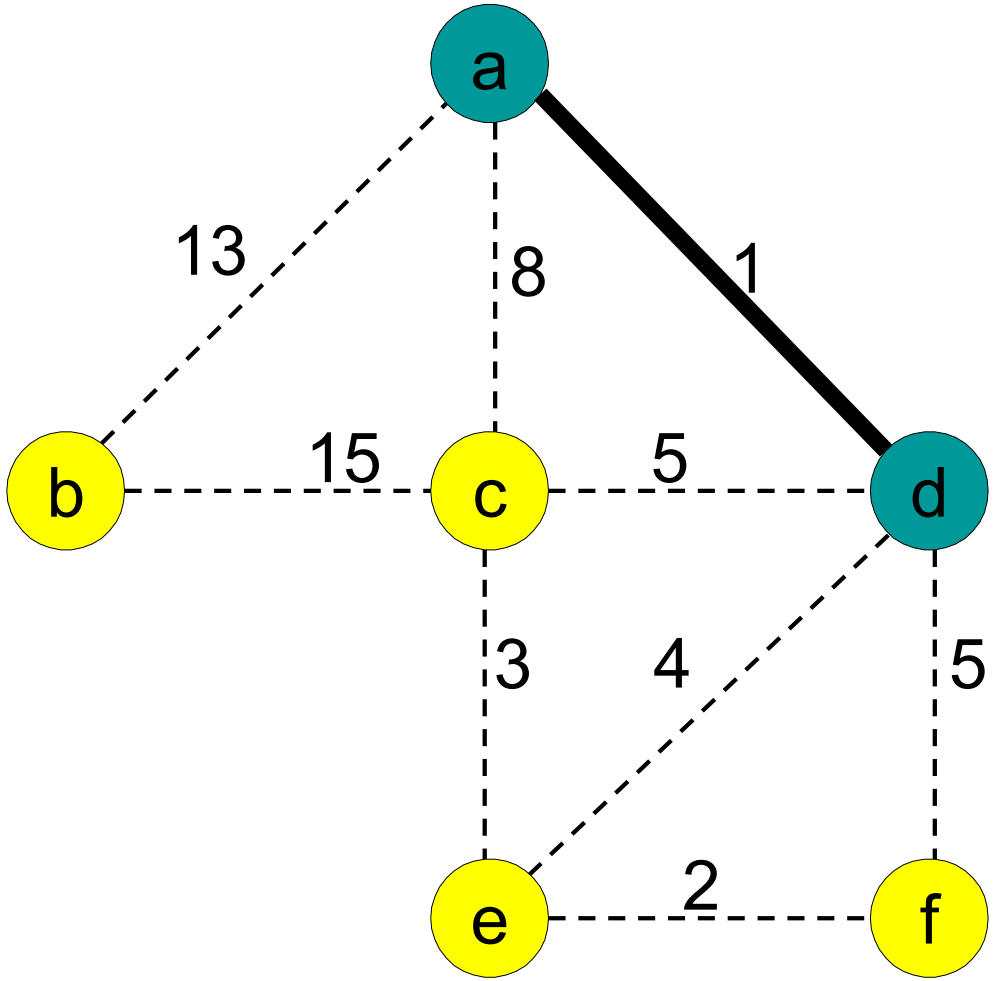
v	k_v	d_v	p_v
a	***F***	0	-
b	***F***	∞	
c	***F***	∞	
d	***F***	∞	
e	***F***	∞	
f	***F***	∞	



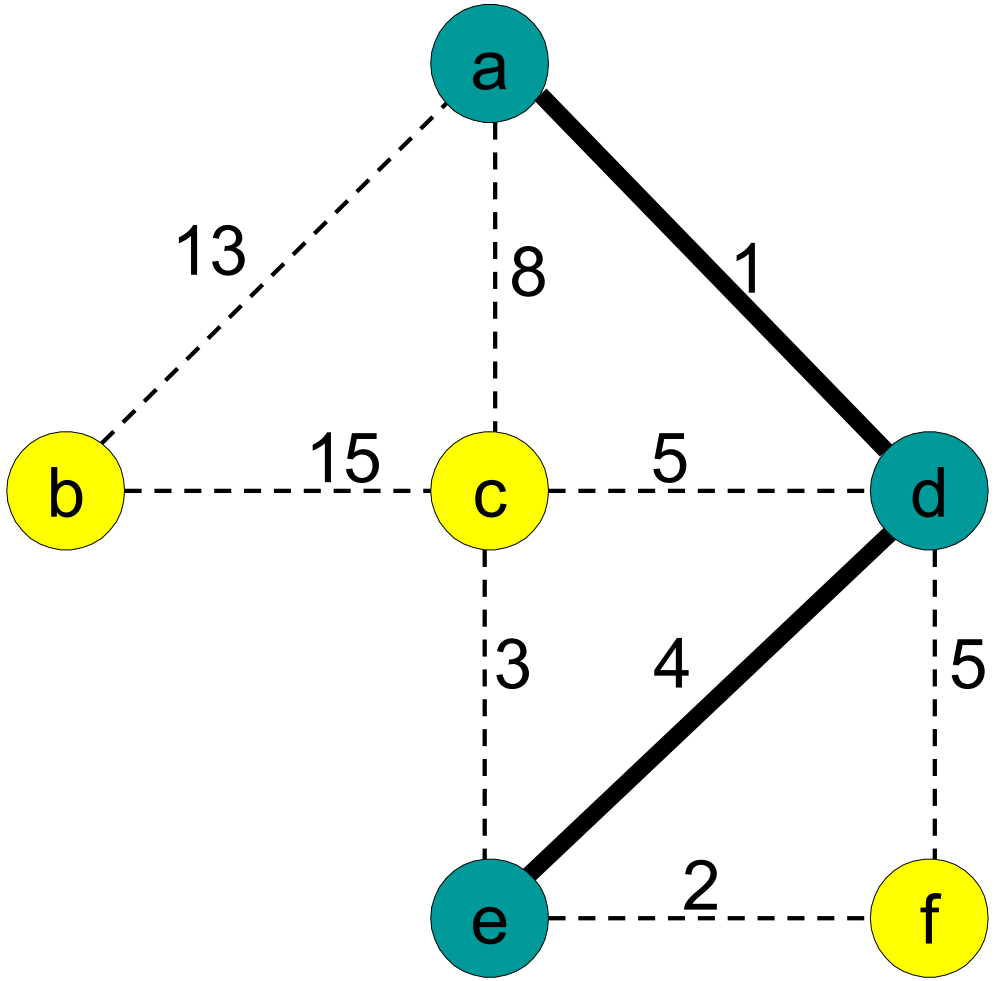
v	k_v	d_v	p_v
a	***T***	0	-
b	***F***	13	a
c	***F***	8	a
d	***F***	1	a
e	***F***	∞	
f	***F***	∞	



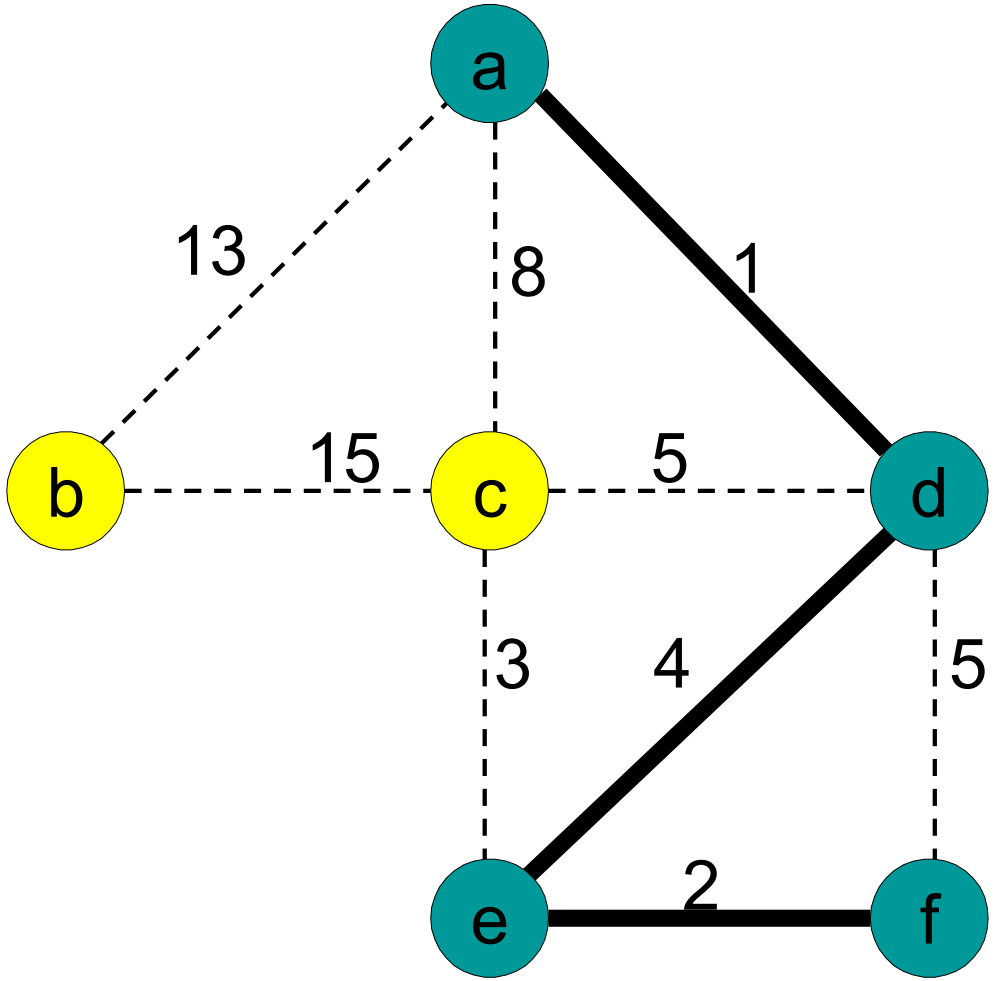
v	k_v	d_v	p_v
a	***T***	0	-
b	***F***	13	a
c	***F***	5	d
d	***T***	1	a
e	***F***	4	d
f	***F***	5	d



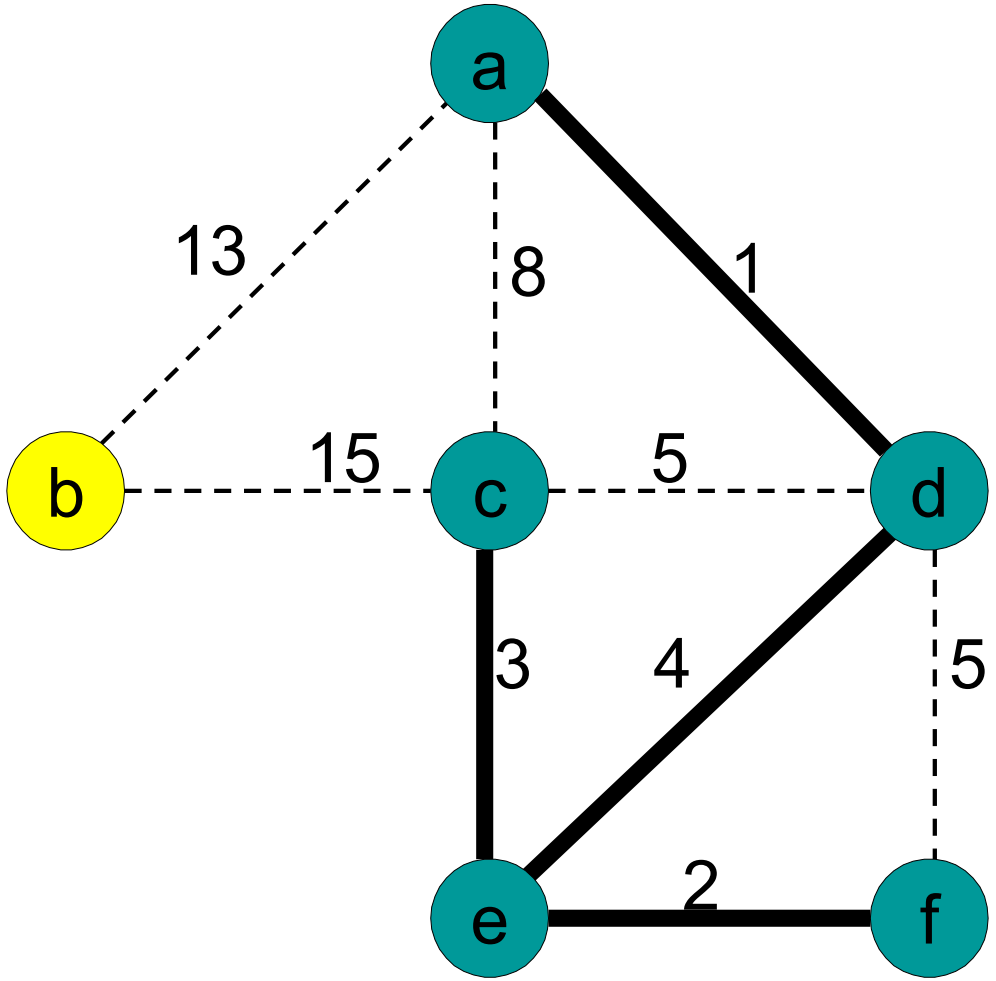
v	k_v	d_v	p_v
a	***T***	0	-
b	***F***	13	a
c	***F***	3	e
d	***T***	1	a
e	***T***	4	d
f	***F***	2	e



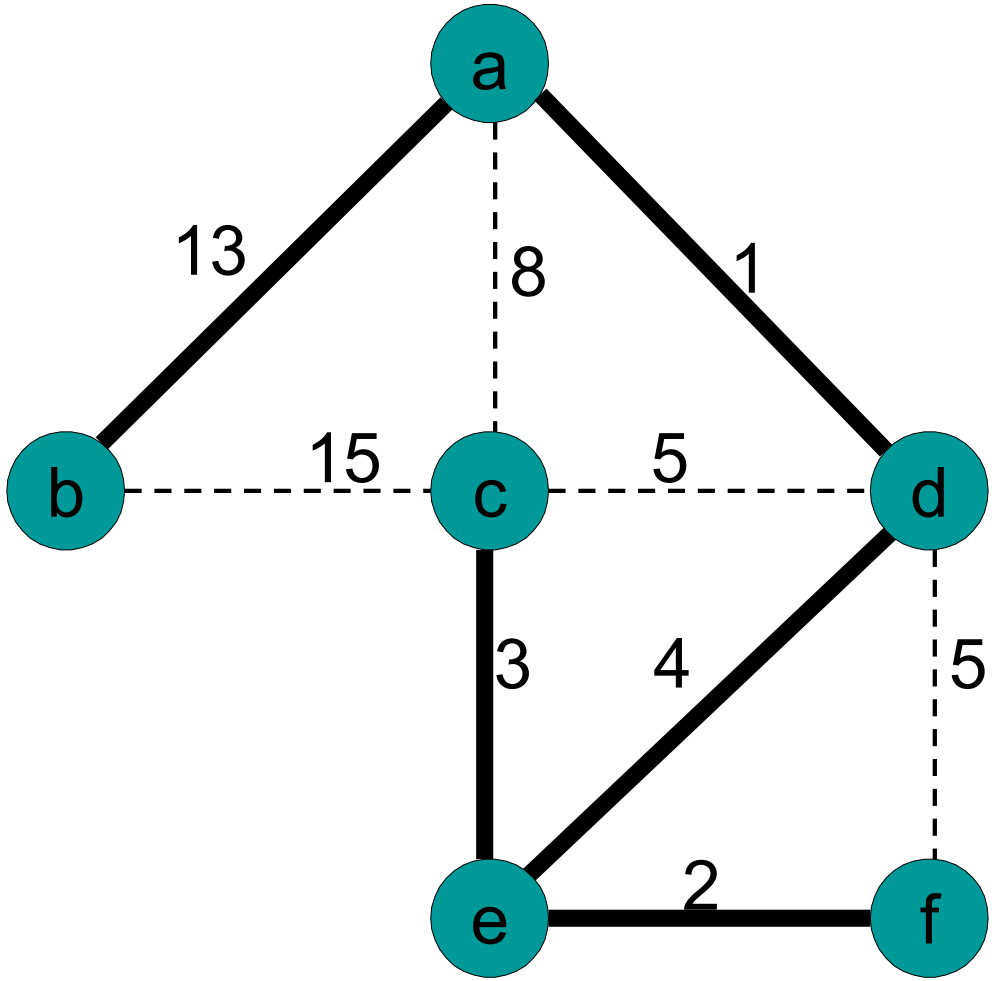
v	k_v	d_v	p_v
a	***T***	0	-
b	***F***	13	a
c	***F***	3	e
d	***T***	1	a
e	***T***	4	d
f	***T***	2	e



v	k_v	d_v	p_v
a	***T***	0	-
b	***F***	13	a
c	***T***	3	e
d	***T***	1	a
e	***T***	4	d
f	***T***	2	e

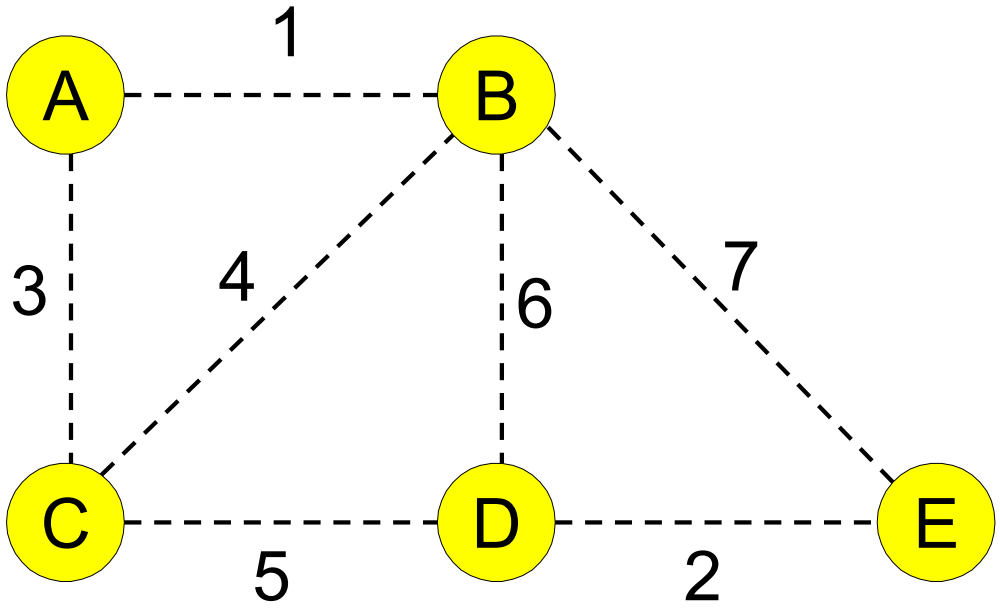


v	k_v	d_v	p_v
a	***T***	0	
b	***T***	13	a
c	***T***	3	e
d	***T***	1	a
e	***T***	4	d
f	***T***	2	e



MST This! (Prim's)

Using Prim's; start at node A



v	k_v	d_v	p_v
A			
B			
C			
D			
E			

Complexity – Linear Search

Repeat until every k_v is true: $|V|$ times

1. **From the set of vertices for which k_v is false, select the vertex v having the smallest tentative distance d_v.** $O(|V|)$
2. **Set k_v to true.** $O(1)$
3. **For each vertex w adjacent to v for which k_w is false, test whether d_w is greater than distance(v,w). If it is, set d_w to distance(v,w) and set p_w to v.**

Most at this vertex: $O(|V|)$. Cost of each: $O(1)$.

Prim's (Heap) Algorithm

```
Algorithm Prims_Heaps(G, s0)
  //Initialize
  n = |V|                                   O(1)
  create_table(n) //stores k,d,p            O(V)
  create_pq() //empty heap                  O(1)
  table[s0].d = 0                           O(1)
  insert_pq(0, s0)                          O(1)
```

Prim's (Heap) Algorithm

```
while (!pq.isempty)                                  O(E)
  v0 = getMin() //heap top() & pop()                 O(log E)
  if (!table[v0].k) //not known                      O(1)
    table[v0].k = true                               O(1)
    for each vi ∈ Adj[v0]                            O(1 + E/V)
      distance = weight(v0, vi)                      O(1)
      if (distance < table[vi].d)                    O(1)
        table[vi].d = distance                       O(1)
        table[vi].p = v0                             O(1)
        insert_pq(distance, vi)                      O(log E)
```

Complexity – Heaps

Repeat until every k_v is true: — $|V|$ times

1. **From the set of vertices for which k_v is false, select the vertex v having the smallest tentative distance d_v.** — $O(\log |V|)$
2. **Set k_v to true.** — $O(1)$
3. **For each vertex w adjacent to v for which k_w is false, test whether d_w is greater than distance(v,w). If it is, set d_w to distance(v,w) and set p_w to v.**

Most at this vertex: $O(|V|)$. Cost of each: $O(\log(|V|)$.
Note: Visits every edge once (over all iterations) = $O(|E|)$.

Prim's: Complexity Summary

- ***O(V²)*** for the simplest nested-loop implementation

- ***O(E* log *E)*** with heaps
  - Is this always faster?
  - Think about the complexity of the PQ version for dense versus sparse graphs

Prim’s Algorithm

Data Structures & Algorithms

Kruskal's Algorithm

Data Structures & Algorithms

Kruskal's Algorithm

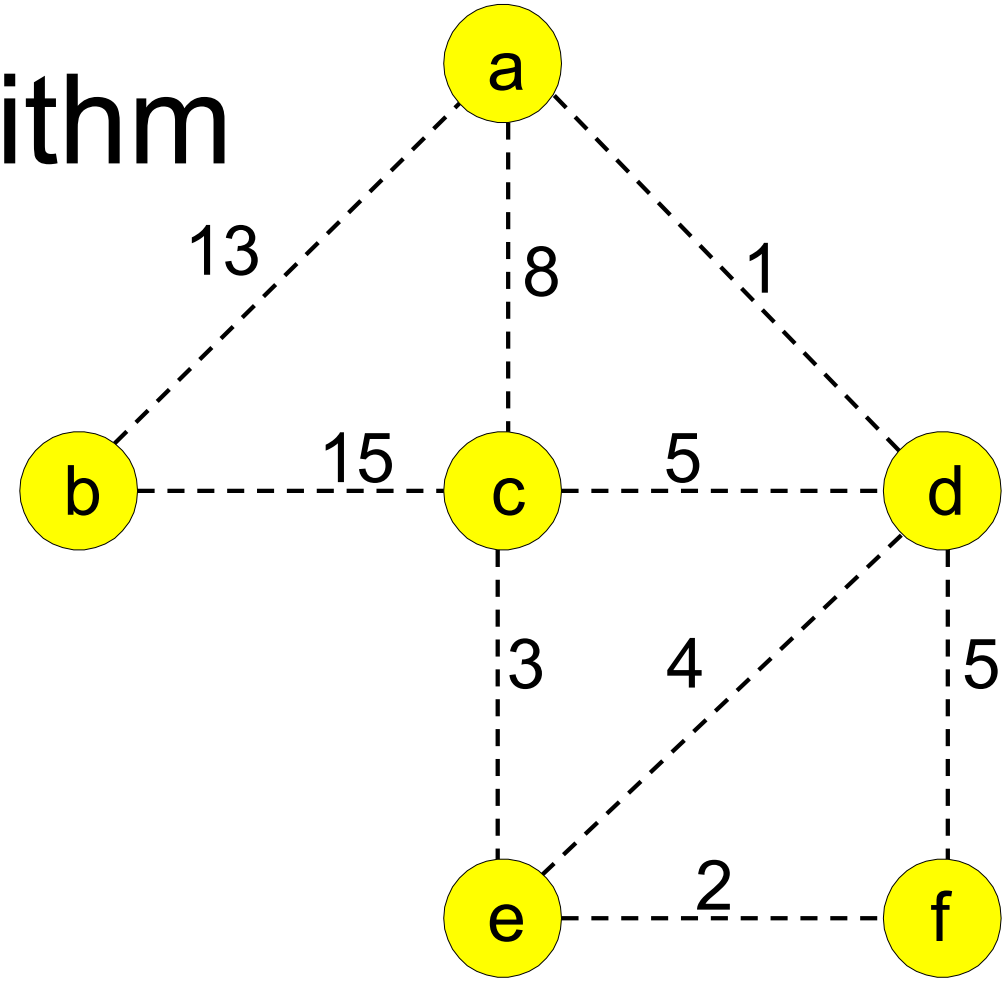
- Find an MST on edge-weighted, connected, *undirected* graphs
- Greedily select edges one by one and add to a growing sub-graph
- Grows a forest of trees that eventually merges into a single tree

Kruskal's Algorithm

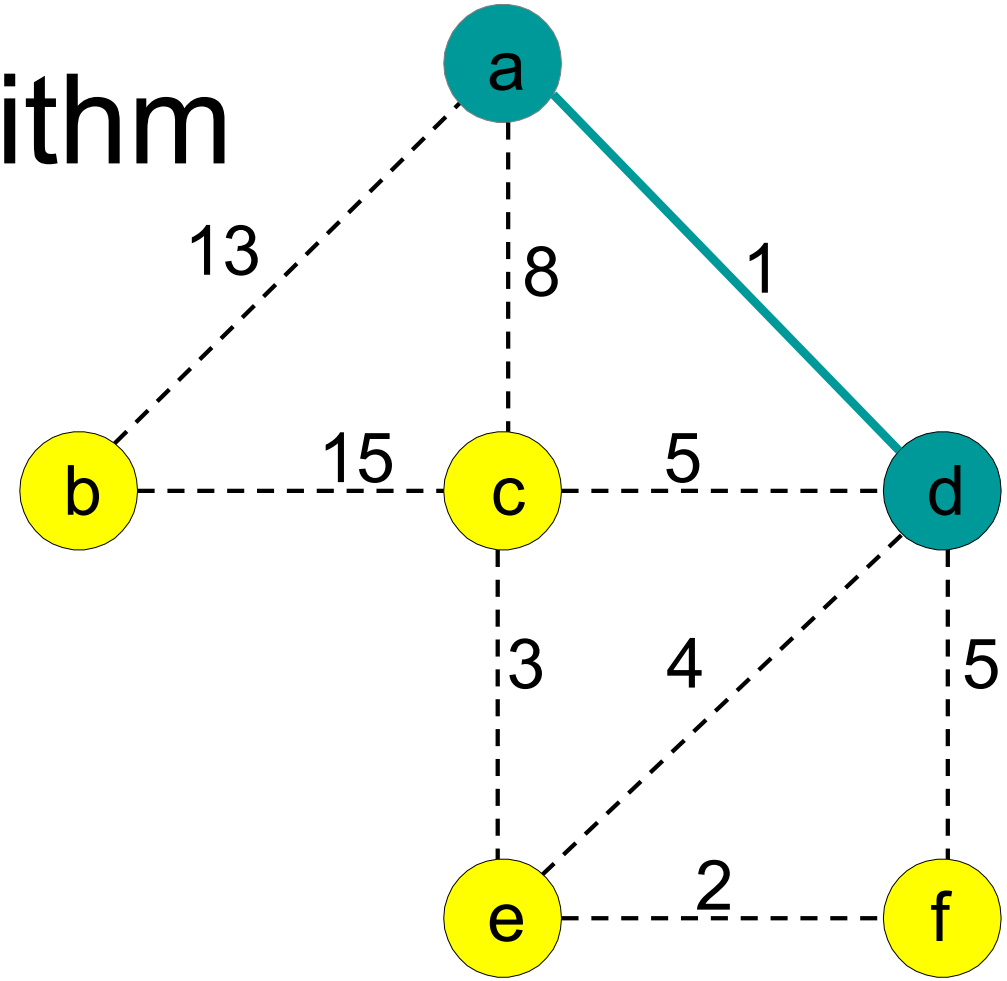
1. Presort all edges: $O(E \log E) \approx O(E \log V)$ time
2. Try inserting in order of increasing weight
3. Some edges will be discarded so as not to create cycles

- Initial edges may be disjoint
  - Kruskal's grows a forest (union of disjoint trees)

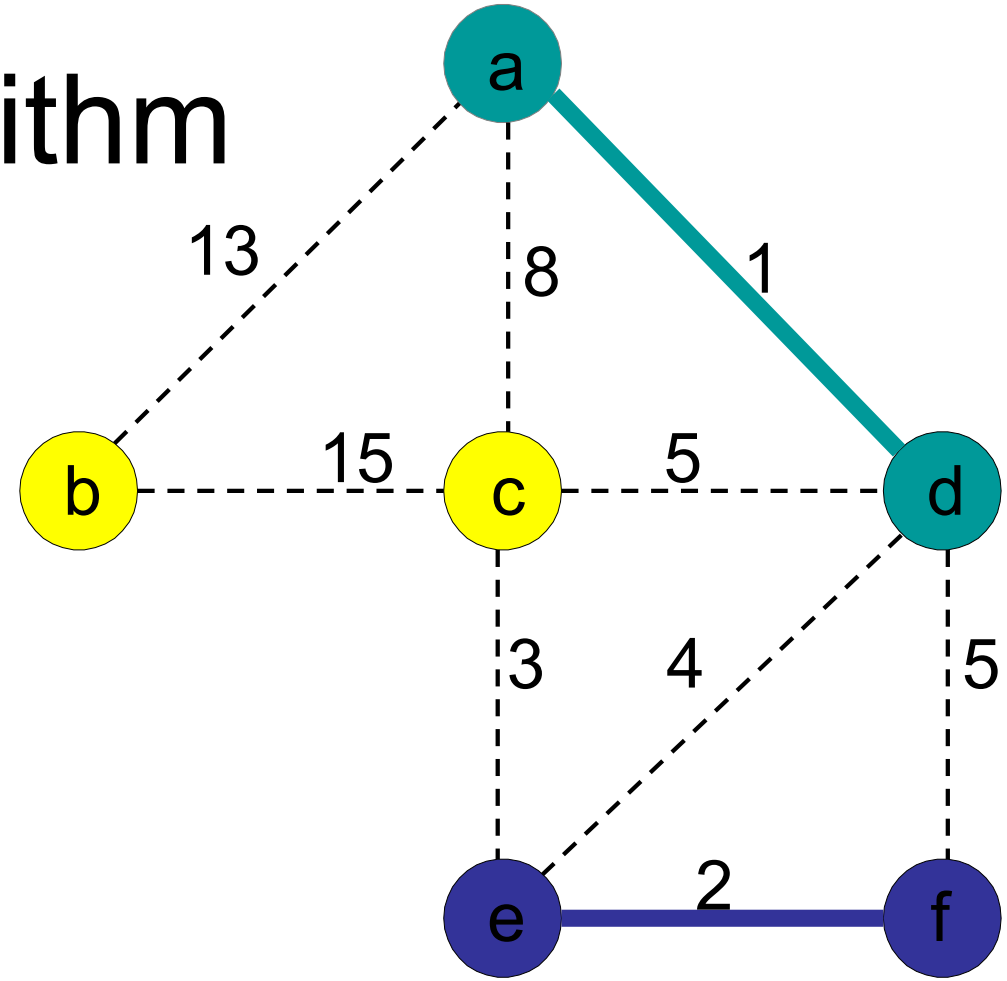
Kruskal's Algorithm



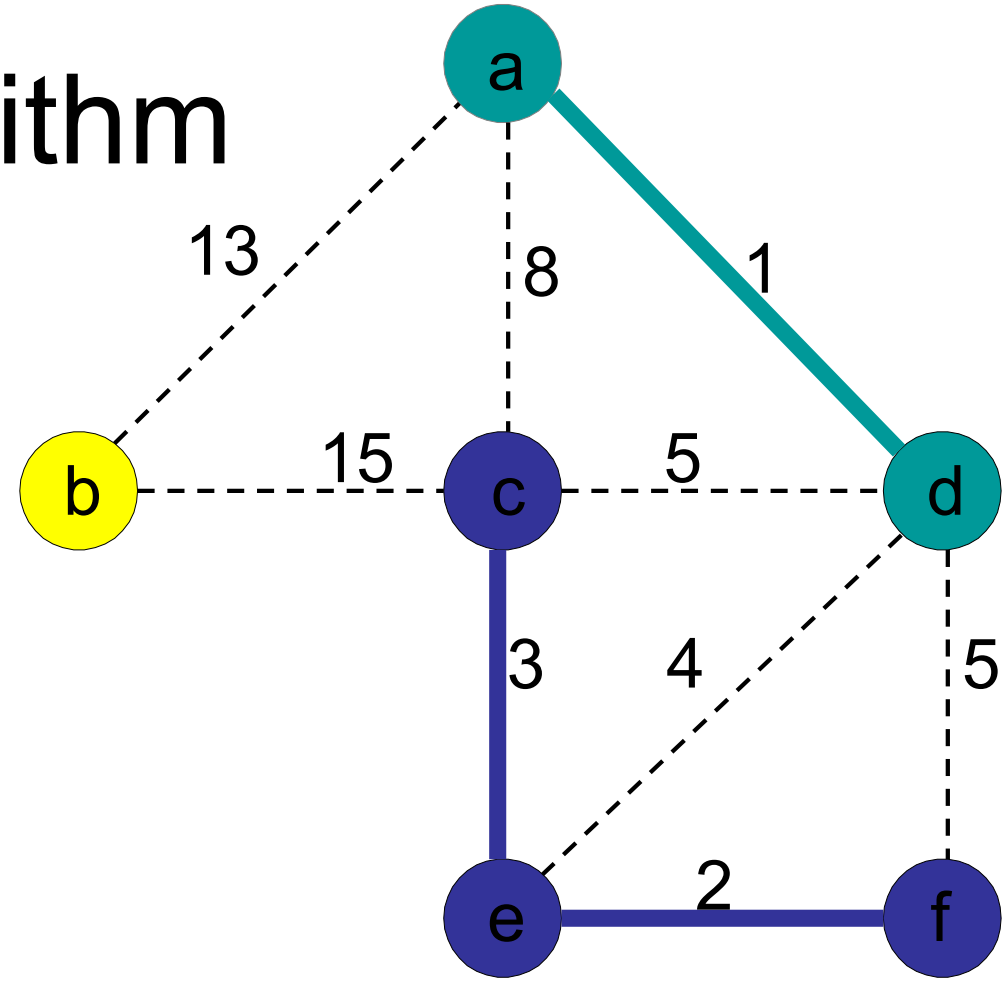
Kruskal's Algorithm



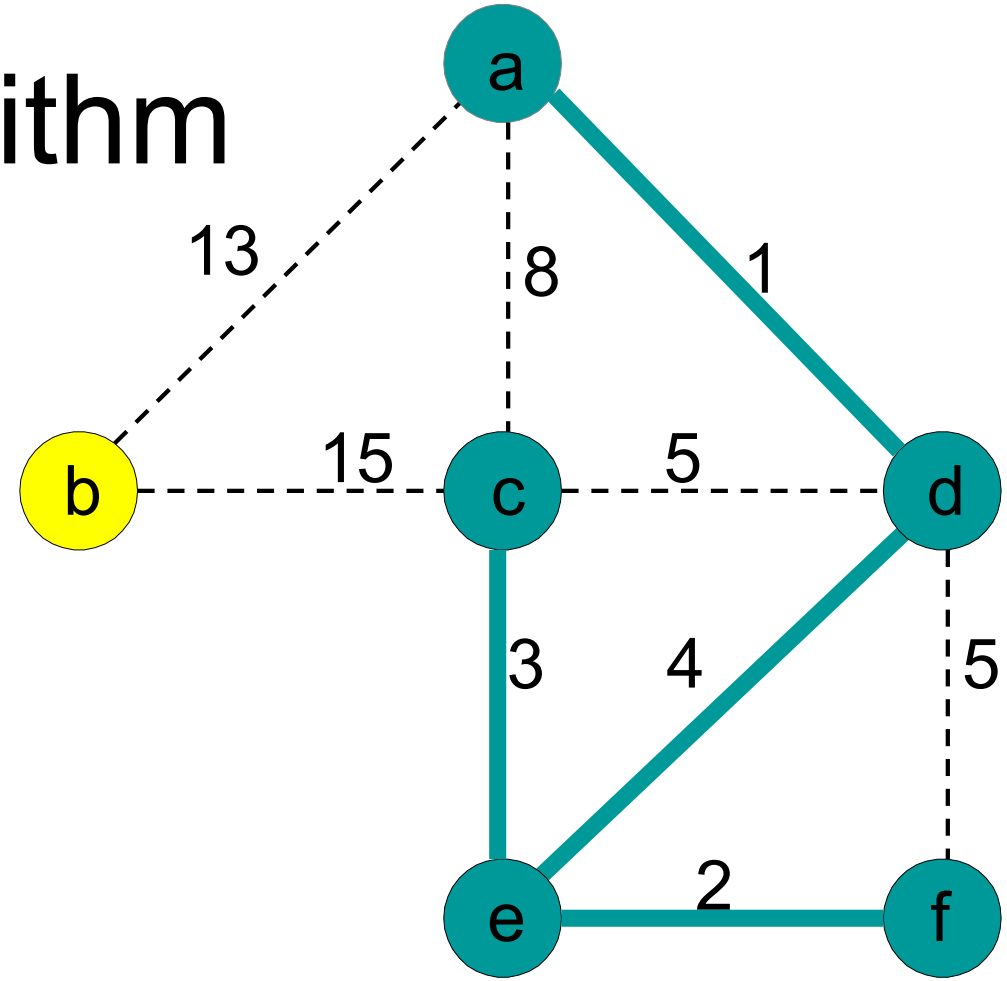
Kruskal's Algorithm



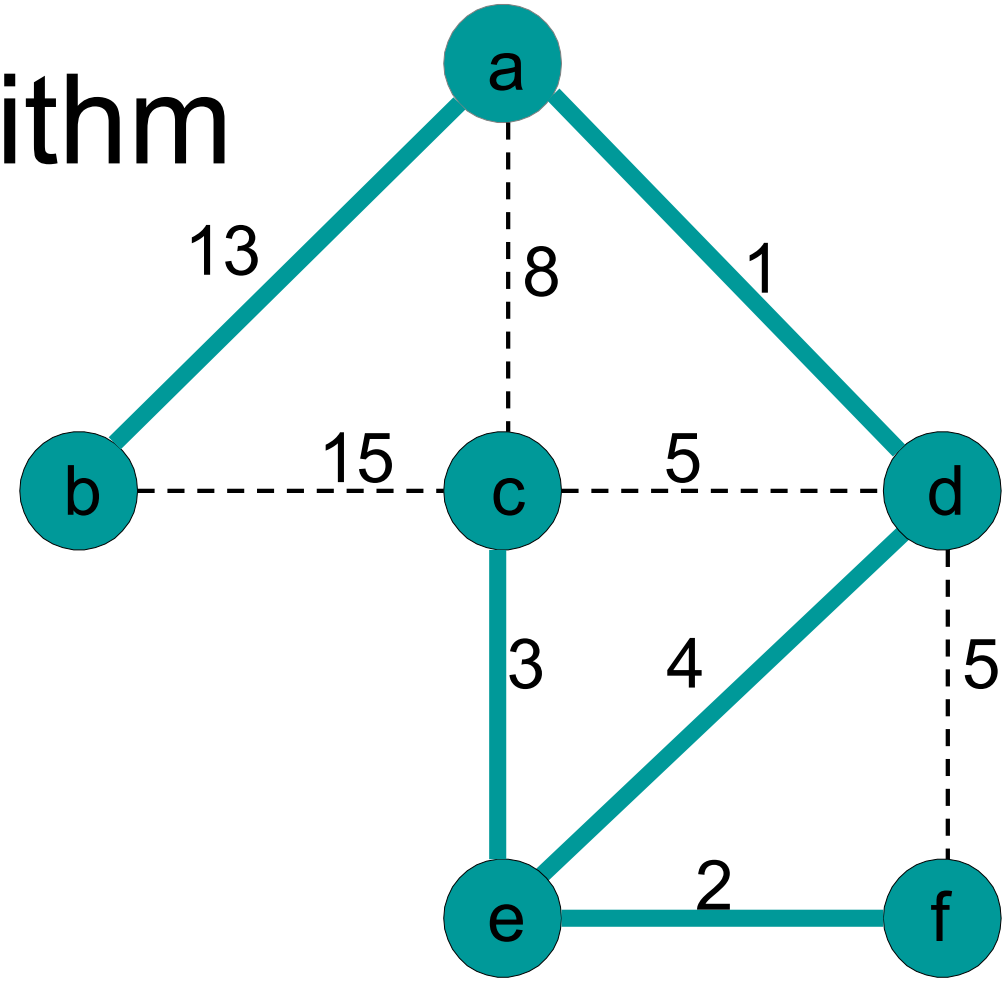
Kruskal's Algorithm



Kruskal's Algorithm



Kruskal's Algorithm



Kruskal: Complexity Analysis

- Sorting takes $E \log V$
  - Happens to be the bottleneck of entire algorithm
- Remaining work: a loop over E edges
  - Discarding an edge is trivial $O(1)$
  - Adding an edge is easy $O(1)$
  - Most time spent testing for cycles $O(?)$
  - Good news: takes less than $\log E \approx \log V$
- Key idea: if vertices v_i and v_j are connected, then a new edge would create a cycle
  - Only need to maintain disjoint sets

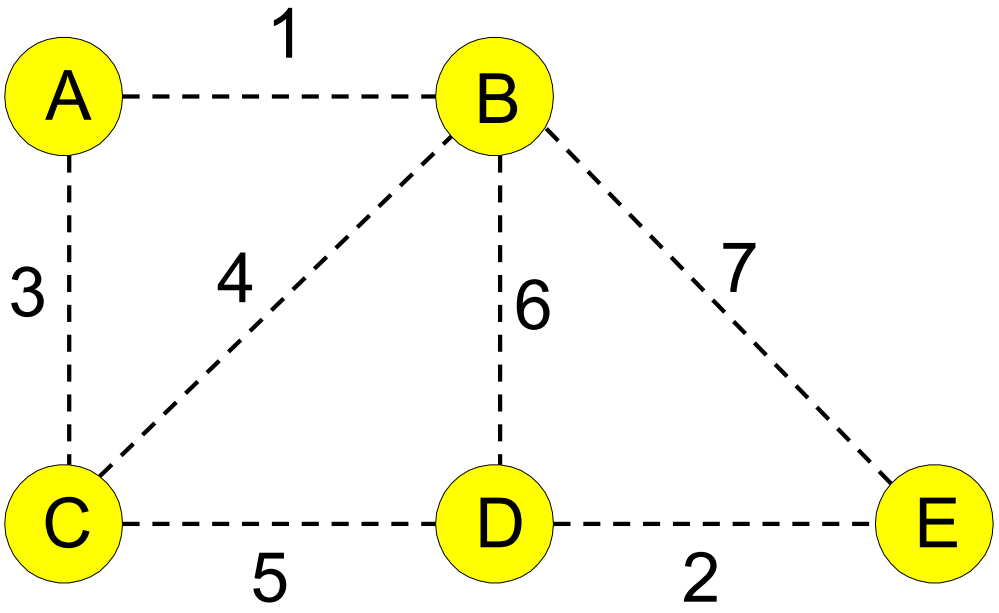
Maintaining Disjoint Sets

- *N* locations with no connecting roads
- Roads are added one by one
  - Distances are unimportant (for now)
  - Connectivity is important
- Want to connect cities ASAP
  - Redundant roads would slow us down

Q: For two cities *k* and *j*, would road *(k, j)* be redundant?

A: Use a Union-Find data structure.

MST this! (Kruskal's)



Weight	Edge
1	
2	
3	
4	
5	
6	
7	

Vertex	A	B	C	D	E
Representative					

Kruskal's Algorithm

Data Structures & Algorithms

MST Summary

- MST is lowest-cost sub-graph that
  - Includes all nodes in a graph
  - Keeps all nodes connected
- Two algorithms to find MST
  - Prim: iteratively adds closest node to current tree – very similar to Dijkstra, $O(V^2)$ or $O(E \log V)$
  - Kruskal: iteratively builds forest by adding minimal edges, $O(E \log V)$
- For dense *G*, use the nested-loop Prim variant
- For sparse *G*, Kruskal is faster
  - Relies on the efficiency of sorting algorithms
  - Relies on the efficiency of union-find

Take-Home MST Quiz

- Prove that Kruskal always finds an MST
- Prove that Prim always finds an MST
- Prove that Prim can start at any vertex
- Hint: revisit in-class MST quiz