L22: Counting, Permutations, and Combinations

a "permutation lock"

Computing CARES Panel

RSVP here →

Learning Objectives

After today's lecture (and associated readings, discussion & homework), you should be able to:

- Product rule, sum rule, division rule for counting number of objects.
- Permutations
  - Concept/notation for counting ***sequences*** of things.
- Combinations
  - Concept/notation for counting ***sets*** of things.

Outline:

- **Counting and Probability**
  - **Product and Sum Rules for counting**
- Exercises:
  - Passwords
  - Chessboard configurations
- Permutations: counting bijections
- Notation for k-permutations and k-combinations
- Counting poker hands.

Counting

- The next few lectures are all about ***counting things***, and ***methods*** of counting.
  - Useful inside of proofs using the pigeonhole principle.
  - Used in analysis of algorithms
  - Basis of ***discrete probability*** (coming soon!)

Password Combinatorics

- Suppose a password must be ~~12, 13, or 14~~ 5, 6, 7 characters

  sum rule

  – Each character must be a digit (0-9) or an upper- or lower-case letter (a-z, A-Z)

  – Number of unique characters = $\underbrace{26}_{\#\text{lower}} + \underbrace{26}_{\#\text{upper}} + \underbrace{10}_{\#\text{digits}} = 62$

- How many different passwords?

product rule

Case 1: 5 chars $\Rightarrow$ $\boxed{62 \cdot 62 \cdot 62 \cdot 62 \cdot 62} = \underline{62^5}$ choices

$\underline{62}\ \underline{62}\ \underline{62}\ \underline{62}\ \underline{62}$

Case 2: 6 chars $\Rightarrow$ 62^6 choices

Case 3: 7 chars $\Rightarrow$ 62^7 choices

$$\boxed{\text{Total: } 62^5 + 62^6 + 62^7}$$

sum rule

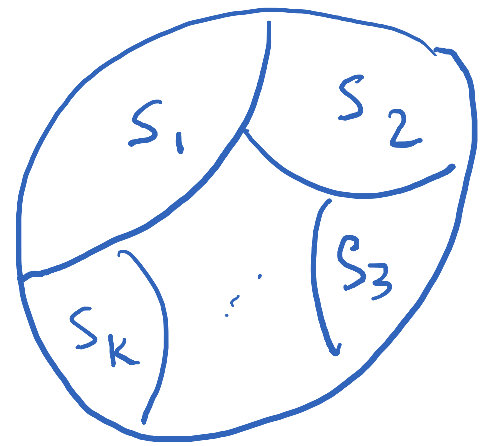
$S_5 \mid S_6 \mid S_7$

The Basic Rules of Counting

- **Product Rule:** if an object can be chosen in <u>k stages</u>, with ***exactly n_i choices*** in stage i, and ***no object can be chosen in two different ways***, there are <u>$n_1 \cdot n_2 \cdot n_3 \cdots n_k$</u> different objects.

$$\rightarrow \boxed{n_1} \rightarrow \boxed{n_2} \rightarrow \cdots \rightarrow \boxed{n_k} \rightarrow$$

- **Sum Rule:** If an object is in exactly <u>one</u> of the sets $S_1, \ldots, S_k$ and these sets are <u>disjoint</u>, then there are <u>$|S_1| + |S_2| + \cdots + |S_k|$</u> different objects.



- **Division Rule:** If there are N ways to choose an object, and each object can be chosen in exactly <u>d</u> ways, there are <u>N/d</u> objects.

used to adjust for overcounting

Password Combinatorics

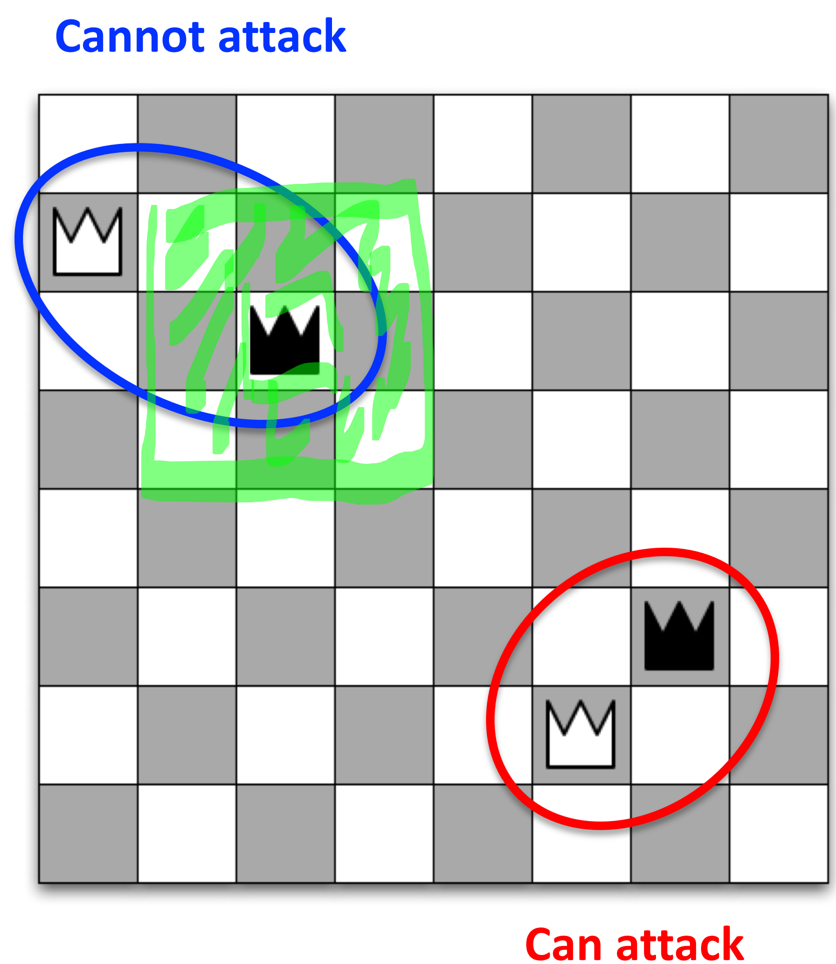
- Suppose a password must be 12, 13, or 14 characters
  - Each character must be a digit (0-9) or an upper- or lower-case letter (a-z, A-Z)
  - Number of unique characters: 10+26+26 = 62
- How many different passwords?
  - **(Product Rule)** A 12-char password can be chosen in 12 stages, 62 options per stage. $(62)^{12}$ different 12-char passwords.
  - Similar calculations for 13- and 14-char passwords.
  - **(Sum Rule)** Passwords partitioned into 3 disjoint sets S_{12}, S_{13}, S_{14}. In total the number of passwords is:

$$|S_{12}| + |S_{13}| + |S_{14}| = (62)^{12} + (62)^{13} + (62)^{14}$$

Chessboard Arrangements

- A king can attack any of the adjacent squares, including the diagonal ones.

- How many ways can two kings be arranged, one white, one black, such that neither is attacking the other?



Chessboard Arrangements

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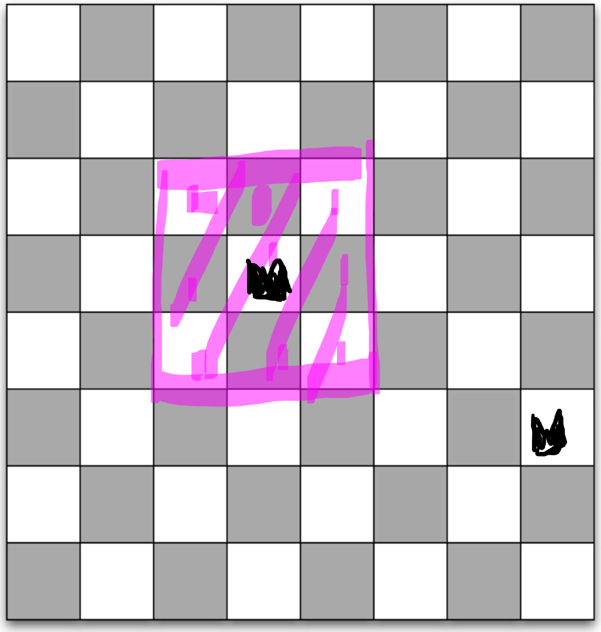


Attempt #1: (Incorrect)

Stage 1: place Black King

$\Rightarrow$ 64 choices

Stage 2: place White King

$\Rightarrow 64 - 9 = 55$ choices

Total: $64 \cdot 55$

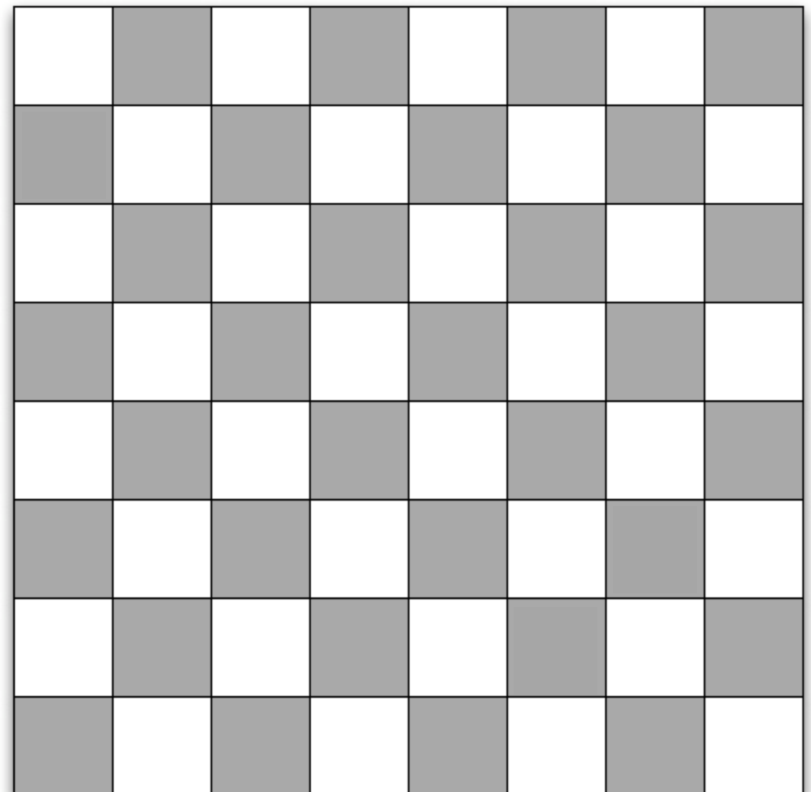
55 choices if Black K was in the interior, but if Black K on edge or in corner, not 55 choices b/c not 8 neighbors.

Chessboard Arrangements

- Try choosing the arrangement in ***two stages*** (product rule).

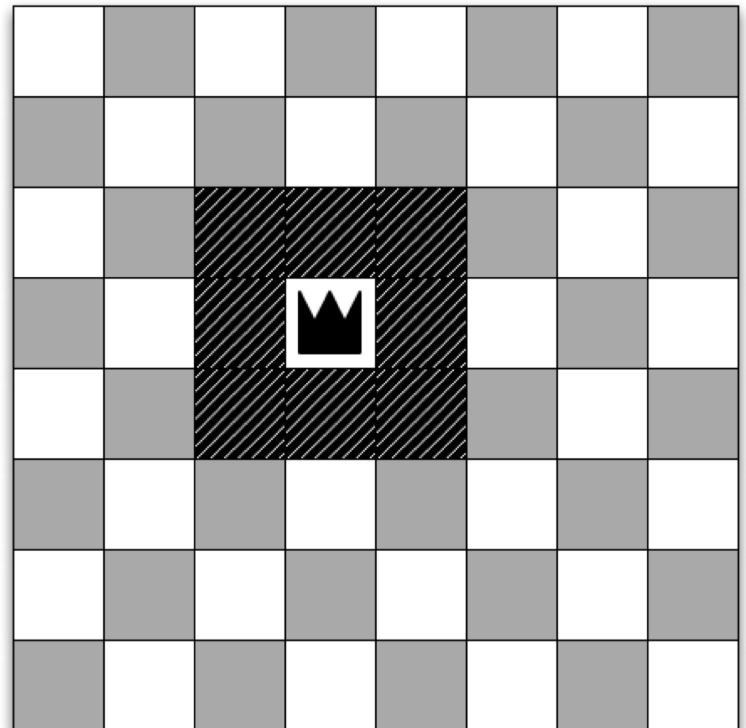
- <u>Stage 1:</u> place black king.

- <u>Stage 2:</u> place white king.



Chessboard Arrangements

- Try choosing the arrangement in ***two stages*** (product rule).

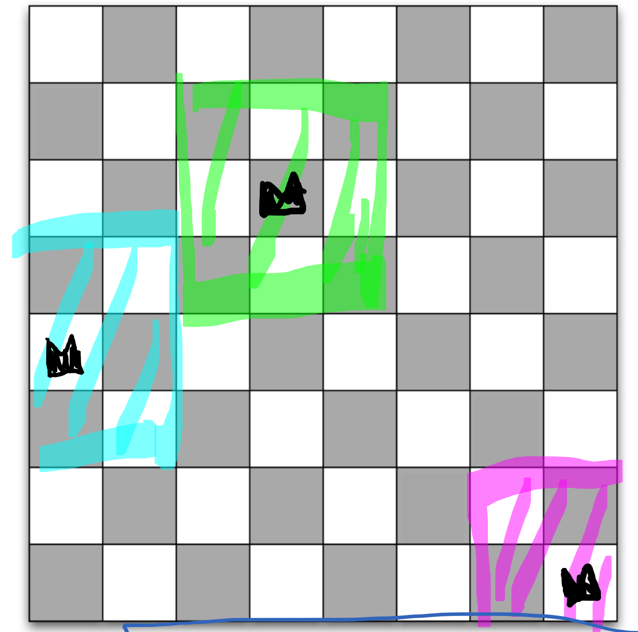


- Stage 1: place black king.
  - $8^2 = 64$ choices.
- Stage 2: place white king.
  - $64 - 9 = 55$ options.
- So $(64)(55)$ is the correct answer?

No. So where did we go wrong?

Chessboard Arrangements

- A king can attack any of the adjacent squares, including the diagonal ones.
- How many ways can two kings be arranged, one white, one black, such that neither is attacking the other?



Case 1: Place Black K in interior
$\Rightarrow$ 36 choices
↳ $36 \cdot 55$ ways
place white K $\Rightarrow 64 - 9 = 55$

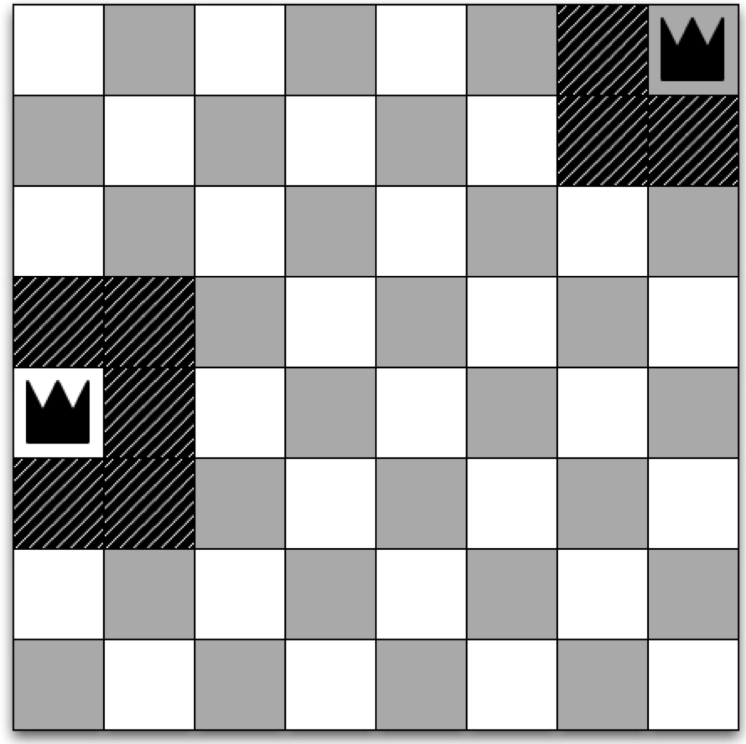
Case 2: Black K in corner
$\Rightarrow$ 4 choices
↳ $4 \cdot 60$
place White King: $64 - 4 = 60$

Case 3: Black K on edge but not corner
$\Rightarrow$ 24 choices
↳ $24 \cdot 58$
place white K: $64 - 6 = 58$

Total:
$36 \cdot 55 + 4 \cdot 60$
$+ 24 \cdot 58$

Chessboard Arrangements

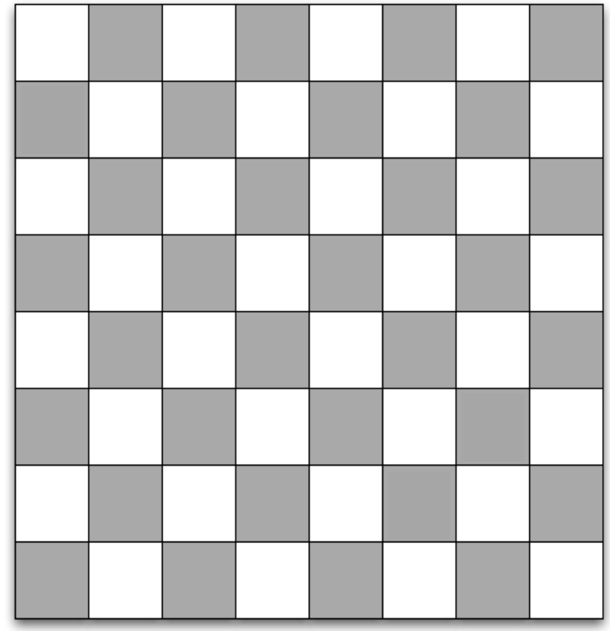
- S_1: black king in interior
  - Stage 1: 36 options
  - Stage 2: $64 - 9 = 55$ options
- S_2: black king in corner
  - Stage 1: 4 options
  - Stage 2: $64 - 4 = 60$ options
- S_3: black king on side
  - Stage 1: 24 options
  - Stage 2: $64 - 6 = 58$ options
- **(Sum Rule)**
  - $|S_1| + |S_2| + |S_3| = \boxed{36 \cdot 55 + 4 \cdot 60 + 24 \cdot 58} = 3612$



Leave YOUR answers **unsimplified** (unless otherwise prompted)

Chessboard Arrangements #2

- Same rules, but what if both kings are white (instead of one white, one black).
- How many arrangements are there now?



Chessboard Arrangements #2

- Same rules, but what if both kings are white (instead of one white, one black).
- How many arrangements are there now?

- **(Division Rule)** There are 3612 ways to place two kings ***in order*** (first one, then the other). When order doesn't matter (because both kings are white), each arrangement can be selected in exactly 2 ways.
- Therefore there are $\frac{3612}{2} = 1806$ arrangements of two white kings.

Outline:

- Counting and Probability
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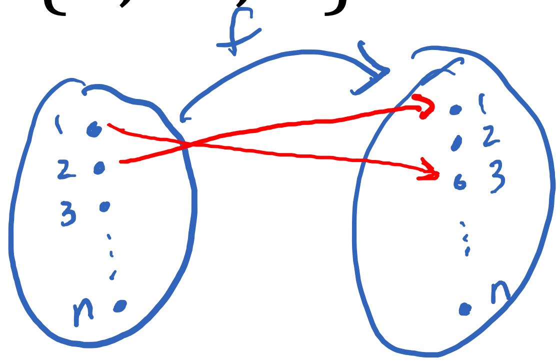
- **Permutations: counting bijections**
- Notation for k-permutations and k-combinations
- Counting poker hands.

Permutations & Bijections

- How many bijections are there of the form $f:\{1,\dots,n\} \to \{1,\dots,n\}$?

$n(n-1)(n-2)\cdots(1)$

$= n!$



$f(1) = \underline{3}$ ← n choices

$f(2) = \underline{1}$ ← $n-1$ choices

can't map to $f(1)$ because f is a bij. (f is 1-1)

$f(3) = \underline{\qquad}$ ← $n-2$ choices

⋮

$f(n) = \underline{\qquad}$ ← 1 choice

Permutations & Bijections

- How many bijections are there of the form $f: \{1, \dots, n\} \to \{1, \dots, n\}$?
- **(Product Rule)**
  - Stage 1: choose $f(1) = ?$ (n choices)
  - Stage 2: choose $f(2) = ?$ ($n - 1$ choices left)
  - …
  - Stage $n - 1$: choose $f(n-1) = ?$ (2 choices left)
  - Stage n: choose $f(n) = ?$ (1 choice left)
- Answer: $n! = \prod_{1 \le i \le n} i$.

Permutations & Combinations

- $P(n,k)$ = the number of ways to pick a ***sequence*** of k things from a set of size n.
  - **(Prod Rule)** $P(n,k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$.
- $C(n,k) = \binom{n}{k}$ = the number of ways to pick a ***set*** (unordered) of k things from a set of size n.
  - **(Division Rule)** Start by picking a ***sequence*** of k things, then forget the order. How many ways could you have picked that set?
  - $C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!k!}$.

Permutations, and Combinations

(A, B, C) is different (B, A, C)   {A, B, C} same as {B, A, C}

Permutations

$$P(n,k) = \frac{n!}{(n-k)!}$$

- # of __sequences__ of length k, selected from a set of n things

order matters

- Repetition allowed? __No__

Ex: Have group of n ppl
Take k of them and
line them up

7 ppl, line up 3 for a pic: $\underbrace{\underline{7}\ \underline{6}\ \underline{5}} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1}$

Combinations

"n choose k"

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

- # of __sets__ of size k, selected from a set of n things

order doesn't matter

- Repetition allowed? __No__

Ex: Group of n ppl.
Choose k to get ice cream

Have 7 ppl, choose 3 to get ice cream: $7\cdot6\cdot5/3!$

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- **Counting poker hands.**

Poker Hands

- A deck consists of 52 cards (13 ***ranks*** in 4 ***suits***).
- How many ***5-card hands*** are there?

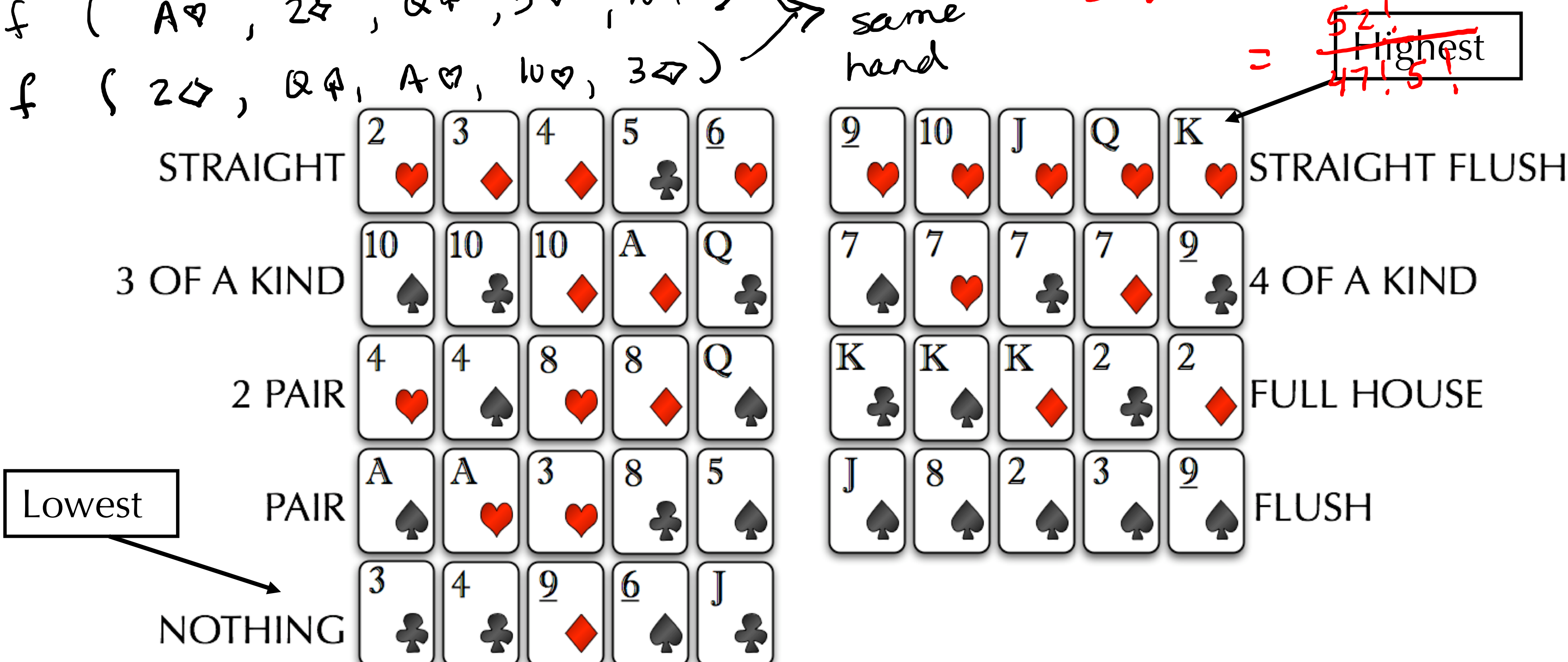
$\underline{52}\quad \underline{51}\quad \underline{50}\quad \underline{49}\quad \underline{48}$

f (A♡ , 2♢ , Q♠ , 3♢ , 10♡)

f (2♢ , Q♠ , A♡ , 10♡ , 3♢)

same hand

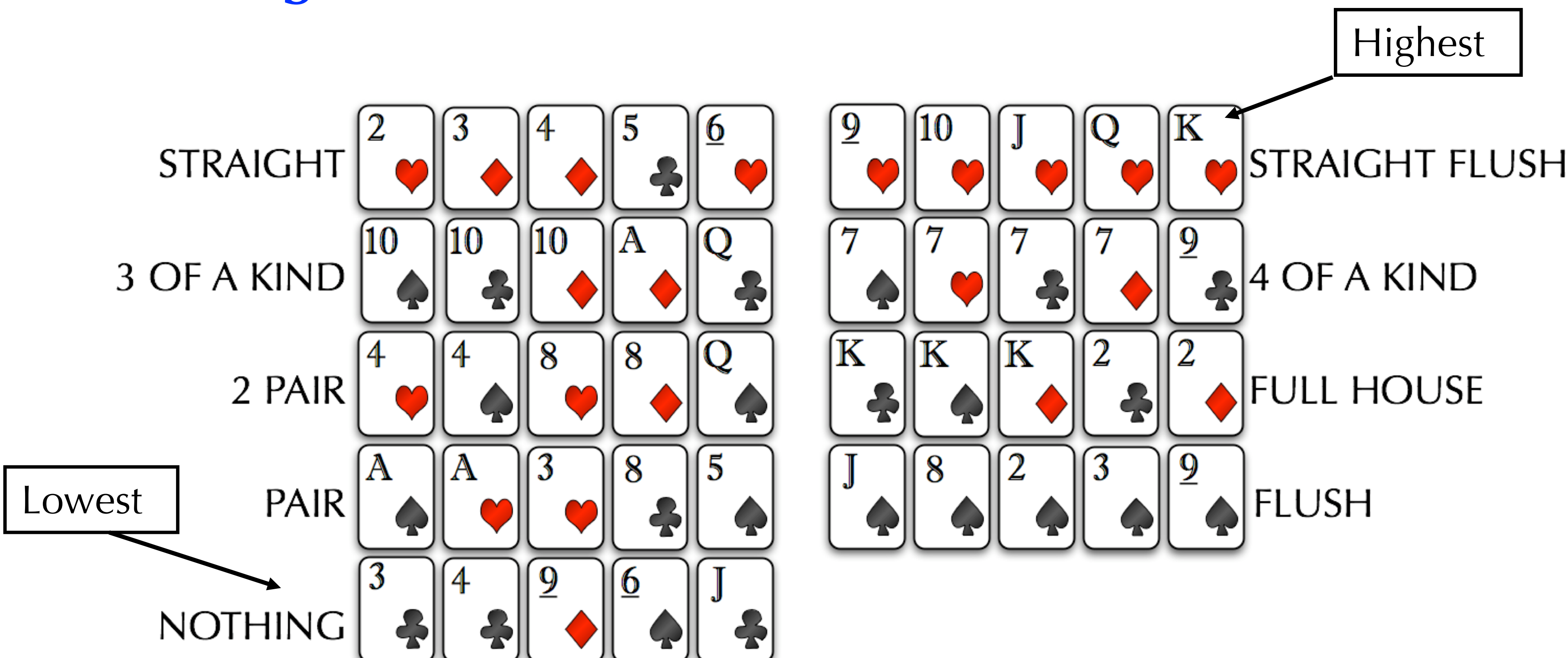
$$\frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5!} = \binom{52}{5} = \frac{52!}{47!\,5!}$$



Poker Hands

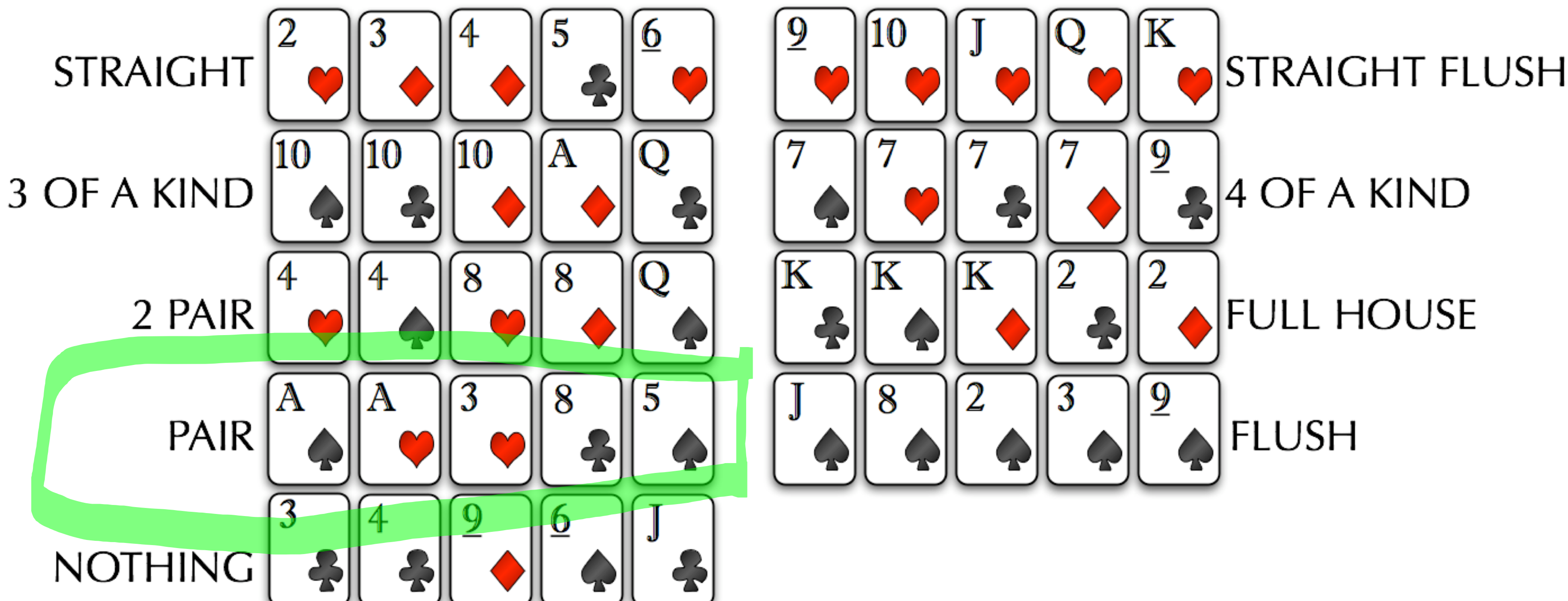
- A deck consists of 52 cards (13 ***ranks*** in 4 ***suits***).
- How many ***5-card hands*** are there?

$$\binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960$$



Poker Hands

- How many ways are there to make a pair (and nothing better)?



Exercise: How many ways to make a pair

(and nothing better)?

A♣ A♦ 2♠ 7♦ 3♥

Total:

$$\frac{13 \cdot 48 \cdot 44 \cdot 40 \binom{4}{2}}{3!}$$

Stage 1: pick rank for the pair : 13 choices   A

Stage 2: pick 1st nonpair card : $52-4 = 48$ choices   2♠

≥ different rank than the pair

stage 3: pick 2nd nonpair card : $52-8 = 44$ choices   7♦

different rank

stage 4: pick 3rd nonpair card $52-12 = 40$ choices   3♥

different rank

Stage 5: pick suits for the pair $\binom{4}{2}$ {♥, ♦}

Poker Hands

- How many ways are there to make a pair (and nothing better)?
- Stage 1: pick the rank of the pair
  - 13 options.
- Stage 2: pick the suits of the pair
  - $\binom{4}{2} = 6$ ways to pick 2 suits.
- Stage 3: pick the 3rd card of a ***different*** rank : 48 options.
- Stage 4: pick the 4^{th} card of a ***different*** rank : 44 options.
- Stage 5: pick the 5^{th} card of a ***different*** rank : 40 options.

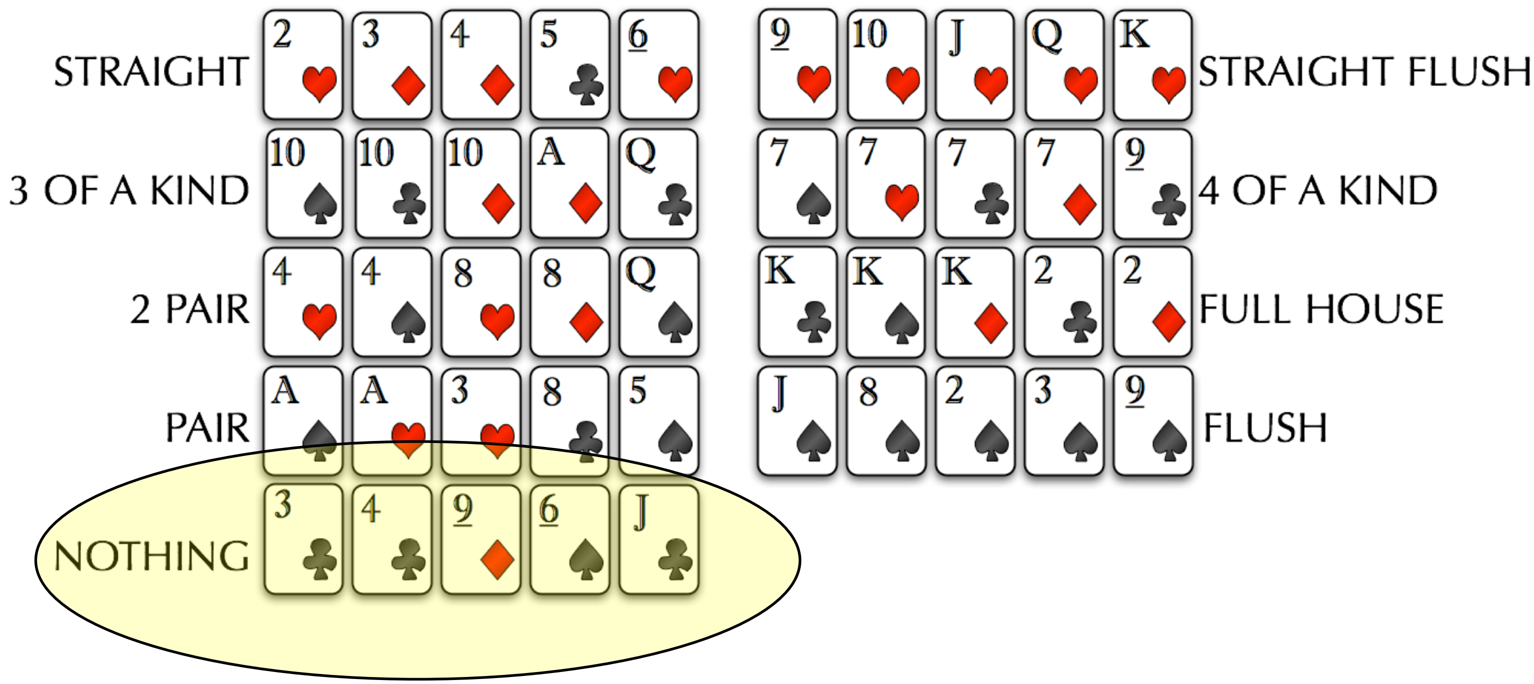
$$\boxed{\text{Is the number of pairs } 13 \cdot 6 \cdot 48 \cdot 44 \cdot 40 \text{ ?}}$$

Poker Hands

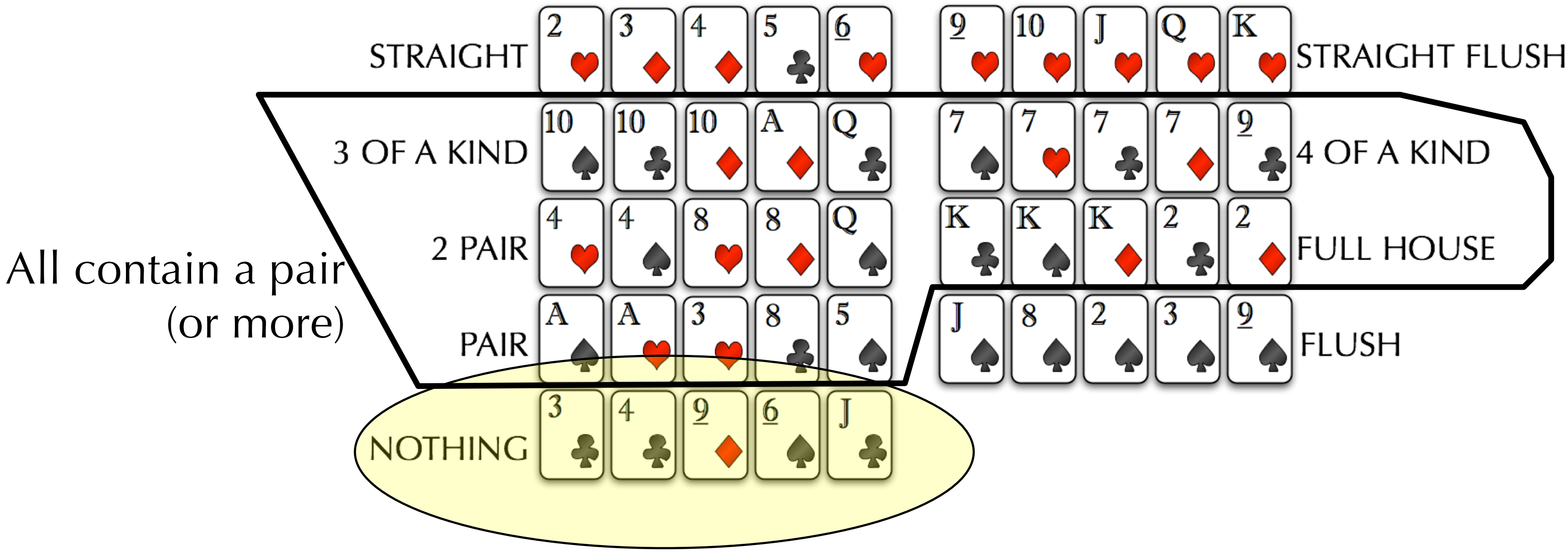
- How many ways are there to make a pair (and nothing better)?
- Stage 1: pick the rank of the pair
  - 13 options.
- Stage 2: pick the suits of the pair
  - $\binom{4}{2} = 6$ ways to pick 2 suits.
- Stage 3: pick the 3rd card of a ***different*** rank : 48 options.
- Stage 4: pick the 4th card of a ***different*** rank : 44 options.
- Stage 5: pick the 5th card of a ***different*** rank : 40 options.
- How many ways can the same hand be picked?
- **(Prod., Div. Rule)** $\frac{13 \cdot 6 \cdot 48 \cdot 44 \cdot 40}{3!} = 1{,}098{,}240$ pa

These 3 cards could have been picked in any order!

Exercise: How many ways to make nothing?

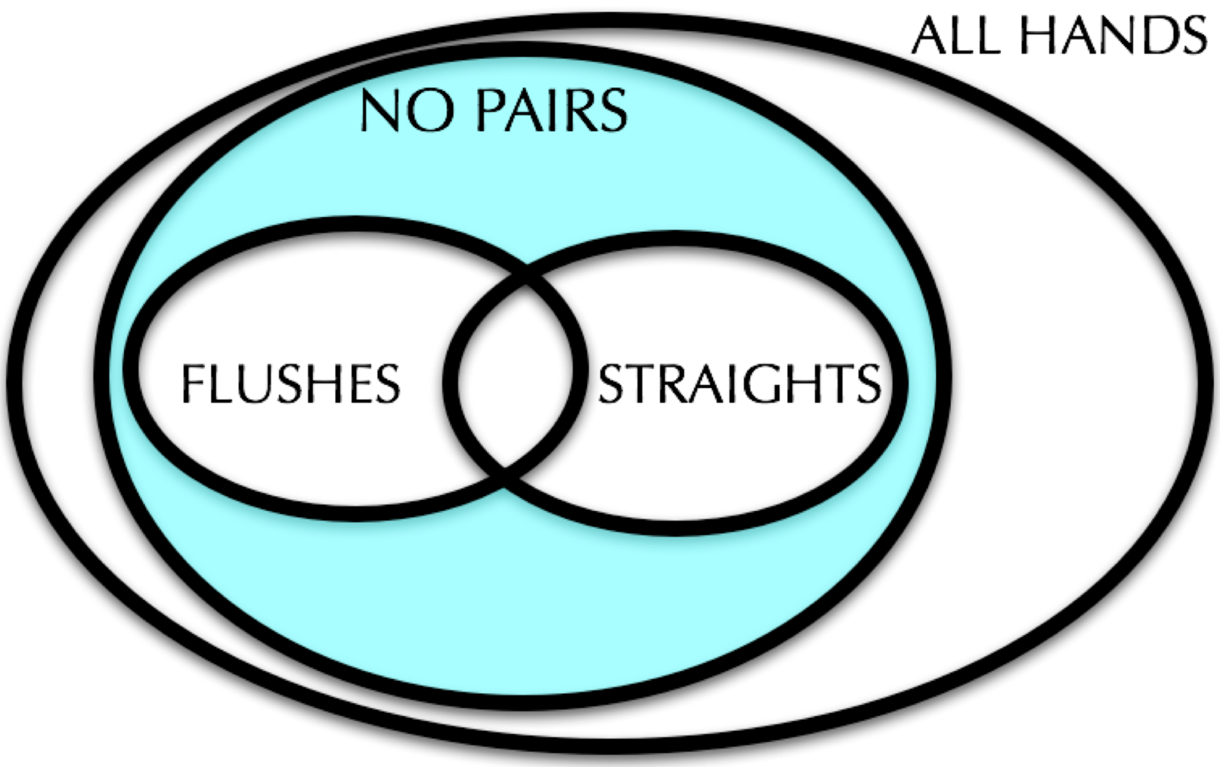


Exercise: How many ways to make nothing?



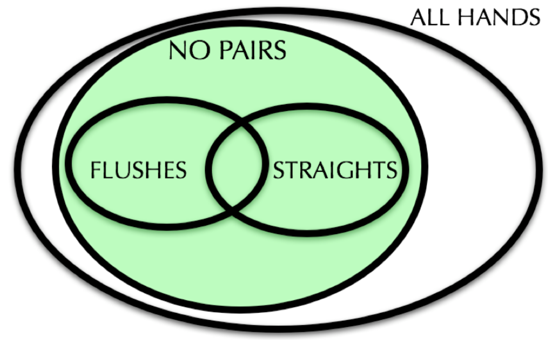
Exercise: How many ways to make nothing?

- We're counting hands:
  (1) without pairs
  (2) that also do not contain straights or flushes



Part of Bigger Exercise: How many ways to make nothing?

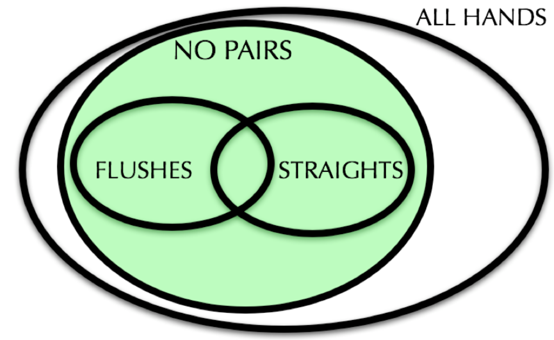
No Pairs



- <u>Exercise:</u> how many ways are there to make ***no pairs*** (all 5 cards different rank). (Pick it in stages!)

Part of Bigger Exercise: How many ways to make nothing?

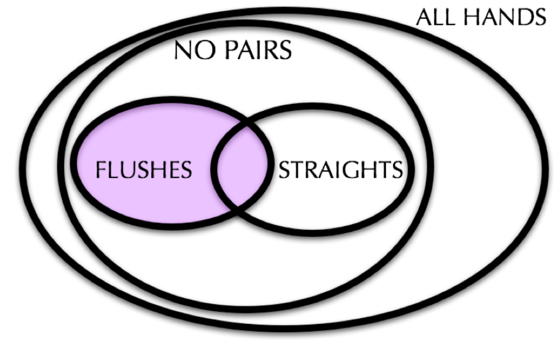
No Pairs



- Exercise: how many ways are there to make ***no pairs*** (all 5 cards different rank). (Pick it in stages!)
- Solution:
  - Stage 1: pick any card. 52 choices.
  - Stage 2: pick any card of a new rank. 48 choices.
  - Stage 3: pick any card of a new rank. 44 choices.
  - Stage 4: pick any card of a new rank. 40 choices.
  - Stage 5: pick any card of a new rank. 36 choices.
  - Observation: every hand could have been picked in 5! ways.
  - $\frac{52\cdot48\cdot44\cdot40\cdot36}{5!} = 1{,}317{,}888$ hands with no pairs.

Part of Bigger Exercise: How many ways to make nothing?

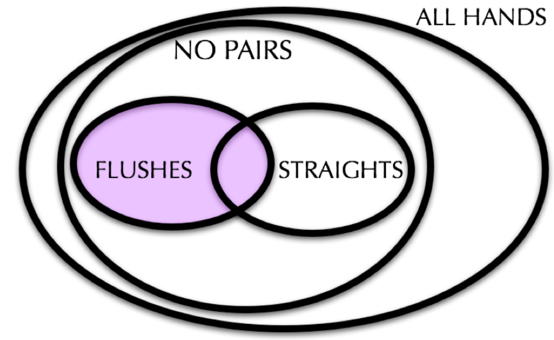
Flushes



- Exercise: how many ways are there to make ***a flush*** (all 5 cards the same suit). Try 2 stages.

Part of Bigger Exercise: How many ways to make nothing?

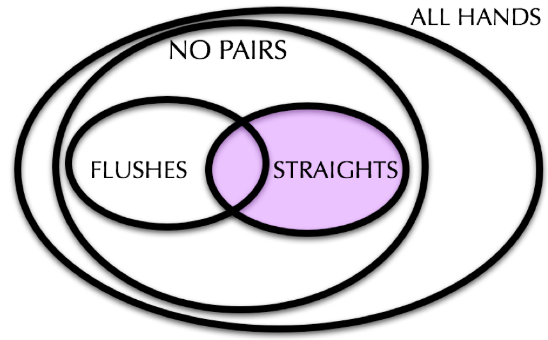
Flushes



- Exercise: how many ways are there to make ***a flush*** (all 5 cards the same suit). Try 2 stages.
- Solution:
  - Stage 1: pick a suit. 4 choices.
  - Stage 2: pick a set of 5 cards in that suit. $\binom{13}{5} = \frac{13!}{8!5!}$
  - Every hand is picked in exactly one way: $4\binom{13}{5} = 5{,}148$ flushes.

Part of Bigger Exercise: How many ways to make nothing?

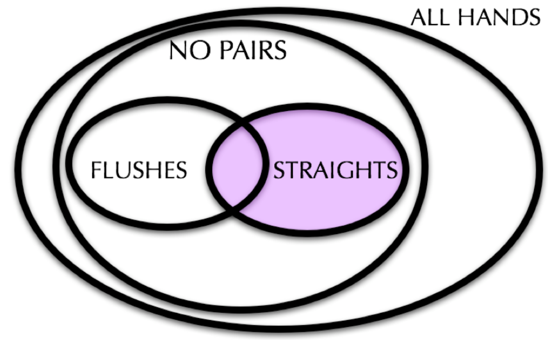
Straights



- Exercise: how many ways are there to make ***straight*** (5 consecutive ranks; aces can be low or high).

Part of Bigger Exercise: How many ways to make nothing?

Straights



- Exercise: how many ways are there to make ***straight*** (5 consecutive ranks; aces can be low or high).
- Solution:
  - Stage 1: pick the lowest rank. 10 choices {A,2,3,…,10}.
  - Stage 2: pick the suit of the lowest card. 4 choices.
  - …
  - Stage 6: pick the suit of the highest card. 4 choices.
  - Each hand picked in exactly one way. $10 \cdot 4^5 = 10{,}240$ straights.

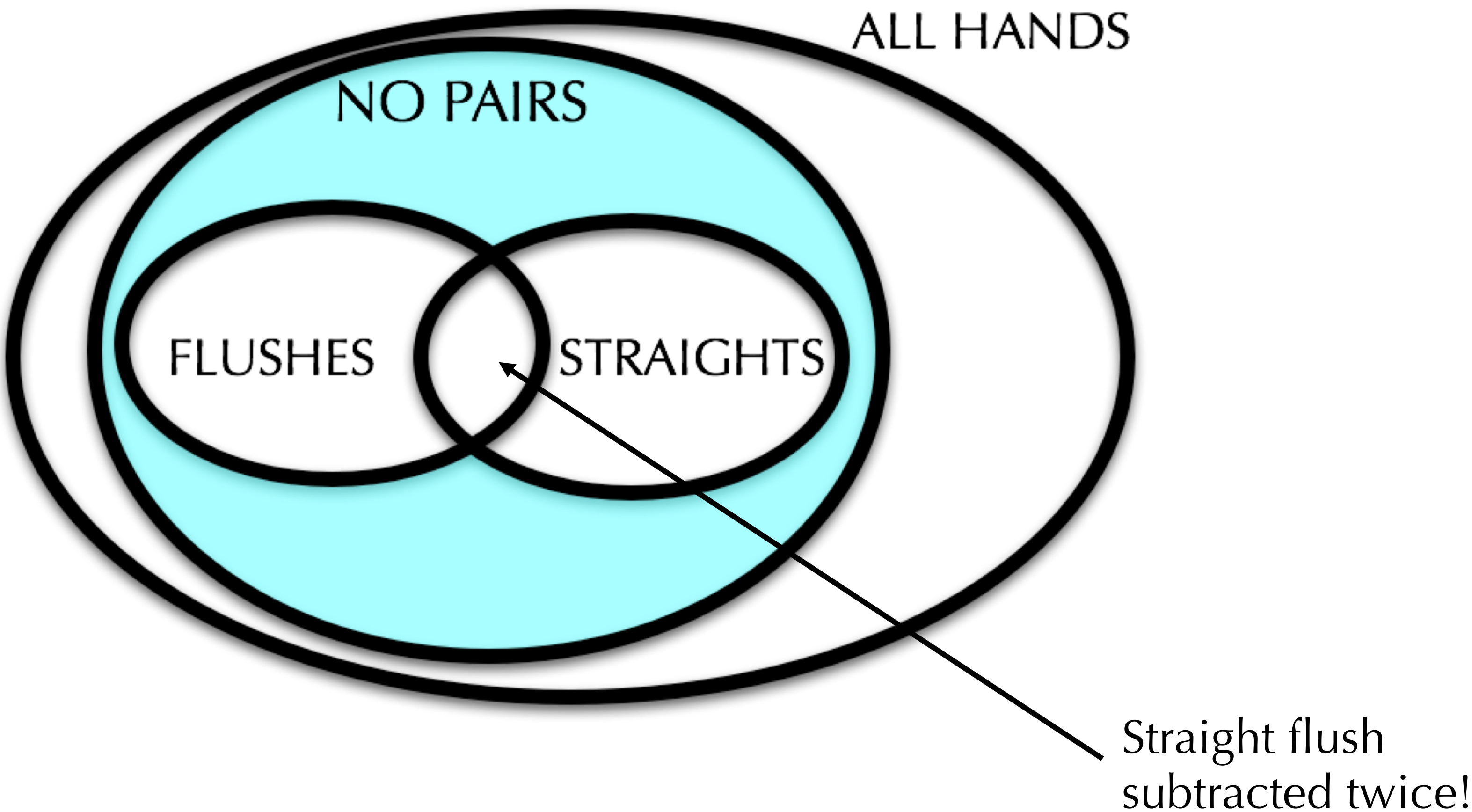
Are we done? What is the next exercise?

Part of Bigger Exercise: How many ways to make nothing?

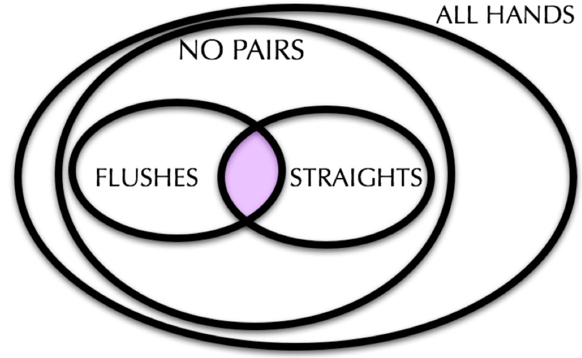
Back to the Venn diagram

Ways to make "nothing"

= (#no pairs) - (#flushes) - (#straights) **+ (#straightflushes)**



Part of Bigger Exercise: How many ways to make nothing?

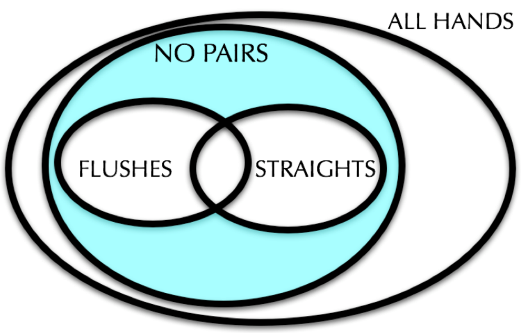


Straight Flushes

- Exercise: how many ways are there to make a ***straight flush*** (5 consecutive rank, all the same suit).
- Solution:
  - Stage 1: pick the suit. 4 choices.
  - Stage 2: pick the lowest rank. 10 choices.
  - (Once stages 1 and 2 are done all 5 cards are determined. There is nothing more to decide.)
  - $4 \cdot 10 = 40$ straight flushes.

Exercise: How many ways to make nothing?

- We're counting hands:
  (1) without pairs
  (2) that also do not contain straights or flushes



(# nothing) = (#no pairs) – (#straights) – (#flushes) +(#straight flushes)

$$= 4\binom{13}{5} - 10 \cdot 4^5 - 4\binom{13}{5} + 4 \cdot 10$$

$$= 1{,}302{,}540.$$

Recap

- **Product**, **Sum**, **Division** rules are applicable to all kinds of counting problems.
- Permutations and Combinations are particularly useful concepts in counting:
  - $P(n,k) = \frac{n!}{(n-k)!}$: the number of ways to pick a ***sequence*** of k things, out of a set of n possible things.

    Order matters

  - $C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$: the number of ways to pick a ***set*** of k things, out of a set of n possible things.

    Order doesn't matter