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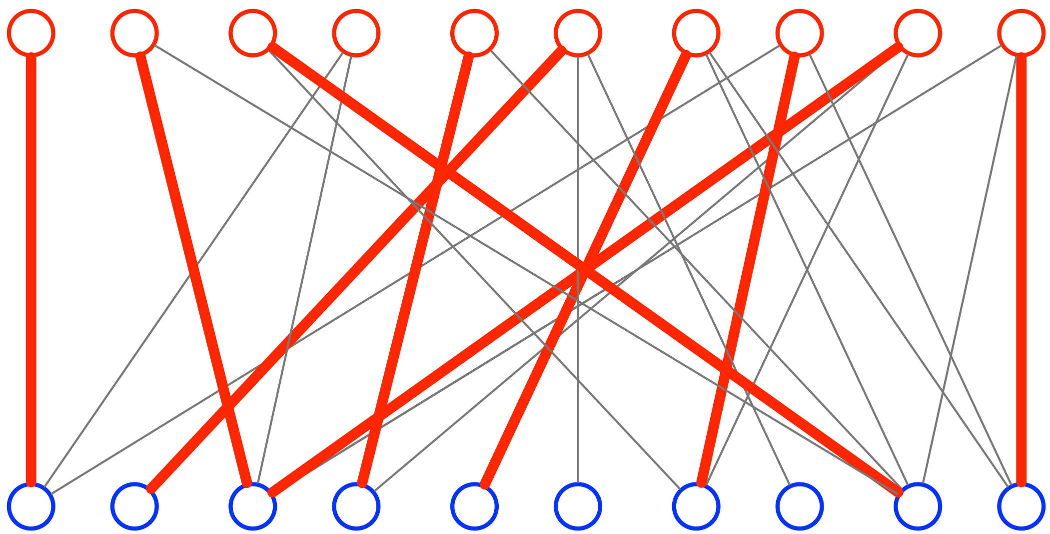
# Lecture 21
# Graphs and Matchings



# Learning Objectives: Lec 13

After today's lecture (and the associated readings, discussion, & homework), you should know:

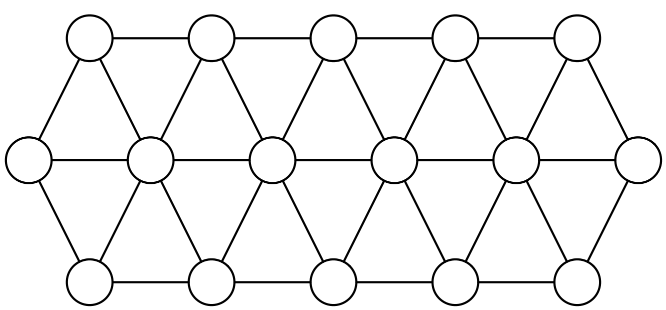
- Bipartite graphs and matchings.
- Deciding whether a bipartite graph has a matching of a certain size.
- Maximum matchings and augmenting paths.
- Hall's Theorem: Constricted sets and perfect matchings.
- Königs Theorem: Vertex Covers and Matchings.

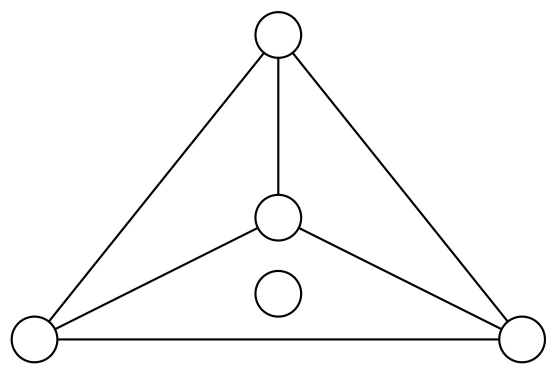
# Outline

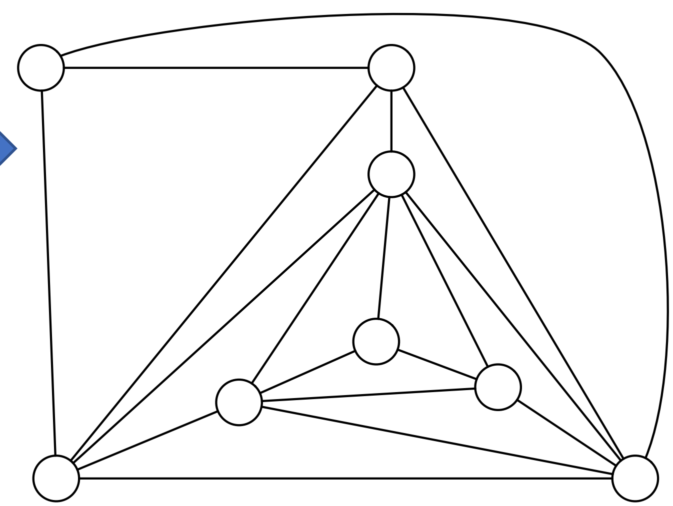
- **Wrap up material from first two graphs lectures**
  - **Applications of Euler's polyhedral formula**
  - Bipartite Graphs (9am lecture only)
- Certificates
- Bipartite graphs and matchings
- Hall's theorem
- König's theorem
- (Bonus: Braess' Paradox)

# Planar Graphs — Triangulation

- ***<u>Defn.</u>*** A plane graph is ***triangulated*** if all faces are bounded by 3 edges and 3 vertices.
- How many of the following three graphs are triangulated?
  - (a) 0
  - (b) 1
  - (c) 2
  - (d) 3





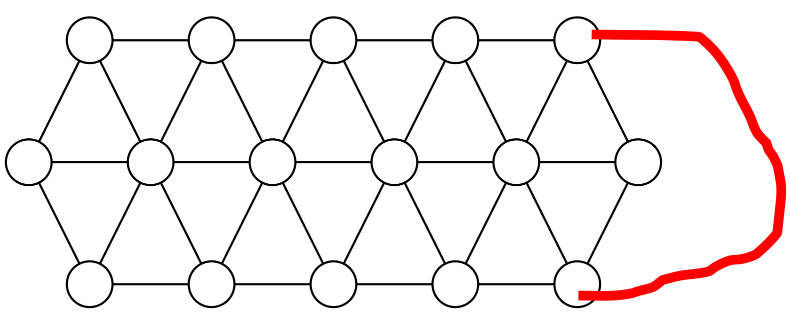


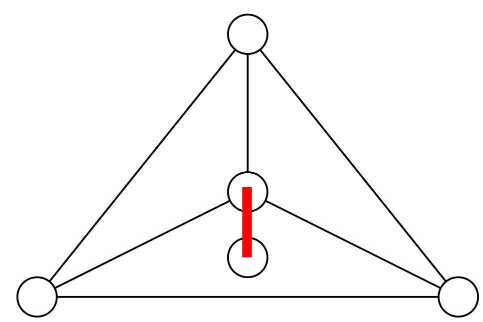
# Number of Edges in a Planar Graph

- What's the ***maximum possible*** number of edges in a ***planar graph*** with $v$ *vertices*?

- What does a maximum-number-of-edges planar graph look like?

- It must be triangulated!
  - If not, we can add an extra edge while keeping it planar

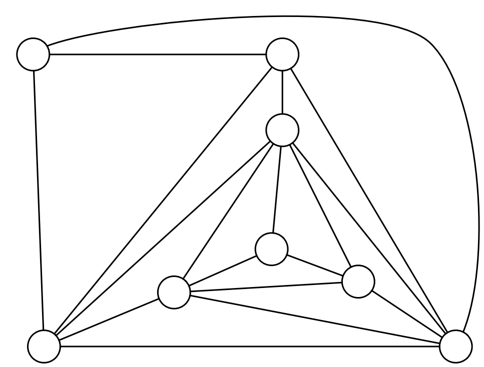




# Number of Edges in a Planar Graph

- What's the ***maximum possible*** number of edges in a ***planar graph*** with $v$ *vertices*?
- How many edges are in a **triangulated** planar graph with $v$ *vertices*?

- $v + f - e - c = 1$ (polyhedral formula, true for **all** planar graphs)
- $c = 1$ (triangulated graph must be connected)
- $3f = \cdots$
  - Each face has 3 edges on it (triangulated)

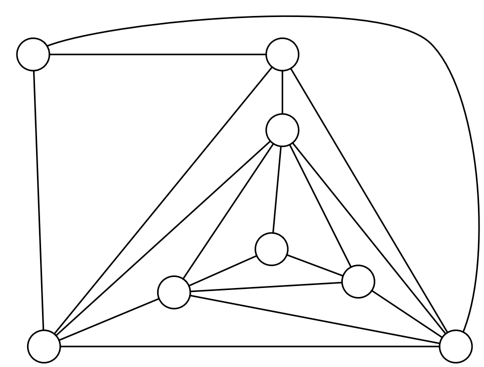


$v = 8,$
$e = 18$

# Number of Edges in a Planar Graph

- What's the ***maximum possible*** number of edges in a ***planar graph*** with $v$ *vertices*?
- How many edges are in a **triangulated** planar graph with $v$ *vertices*?

- $v + f - e - c = 1$ (polyhedral formula, true for **all** planar graphs)
- $c = 1$ (triangulated graph must be connected)
- $3f = 2e$
  - Each face has 3 edges on it (triangulated)
  - Each edge appears on 2 faces
  - So $3f$ counts each edge **twice**



$v = 8$,
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# Number of Edges in a Planar Graph

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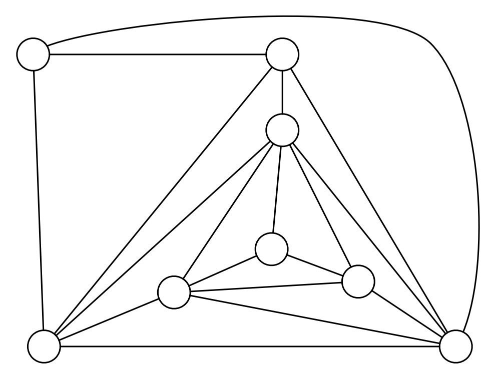
- $v + f - e - c = 1$ (polyhedral formula, true for **all** planar graphs)
- $c = 1$ (triangulated graph must be connected)
- $3f = 2e$ (true of triangulated planar graphs)

Substitution:

$$v + \left(\frac{2e}{3}\right) - e - 1 = 1$$

$$v - \frac{e}{3} = 2$$

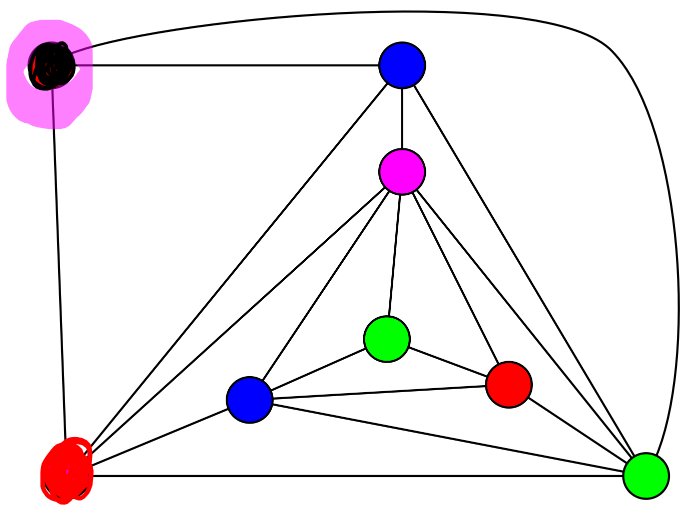
$$\boxed{e = 3v - 6}$$

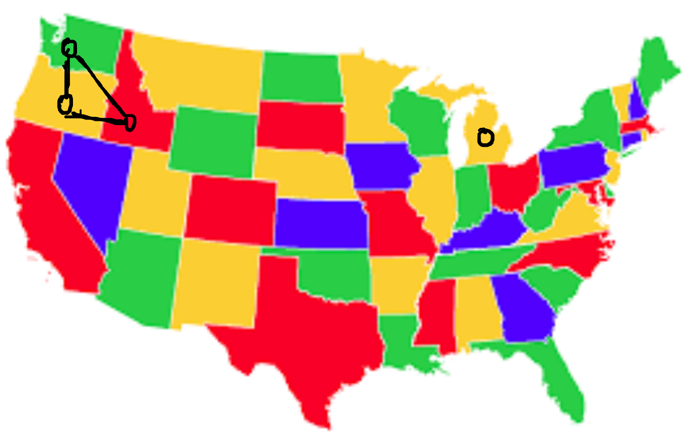


$v = 8$,
$e = 18$

# The 4-Coloring Theorem for Planar Graphs

- **Theorem.** (Appel, Haken 1976) The nodes of any planar graph $G = (V, E)$ can be ***4-colored***, such that all edges go between two differently-colored nodes.
- **The Proof.**
  - … is very very very long. We will not prove it.
  - … but you *can* 4-color any planar graph!





# The ~~4~~ 6-Coloring Theorem for Planar Graphs

- **Theorem.** The vertices of any planar graph $G = (V, E)$ can be ***6-colored***.

- What do the ***Handshake Theorem*** and ***Euler's Polyhedral Formula*** say about the ***average*** vertex degree in a planar graph?

$$\sum_v \deg(v) = 2|E| \qquad \text{and} \qquad |E| \leq 3|V| - 6$$

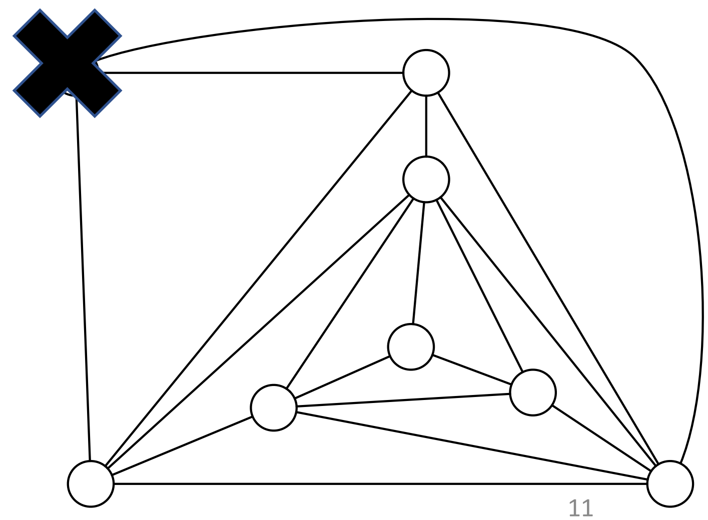
What does this imply about the *minimum* degree?

- The average degree is $\frac{\sum_v \deg(v)}{|V|} = \frac{2|E|}{|V|} \leq \frac{2(3|V|-6)}{|V|} < 6.$

Ans: at least one node has degree $\leq 5$

# The ~~4~~ 6-Coloring Theorem for Planar Graphs

- **Theorem.** Every planar graph $G = (V, E)$ can be ***6-colored***.
- **Useful Lemma:** Every planar graph contains a node of degree $\leq 5$.
- **Proof by induction.** P(k) = "Every k-node planar graph $G$ can be 6-colored."
  - Base Case: $k = 0$: empty coloring works. (Or you can think of $k = 1$)
  - Inductive Step:
    - Delete a vertex $v$ with $\deg(v) \leq 5$.



# The ~~4~~ 6-Coloring Theorem for Planar Graphs

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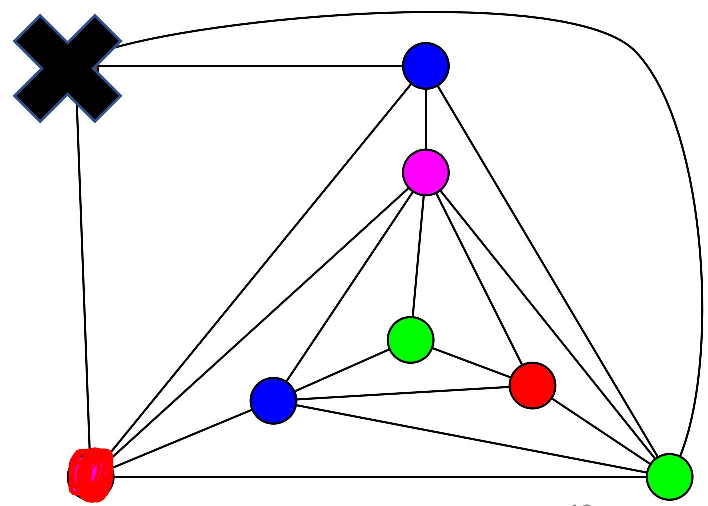
    - **Inductive Hypothesis:** The **remaining** graph can be 6-colored.

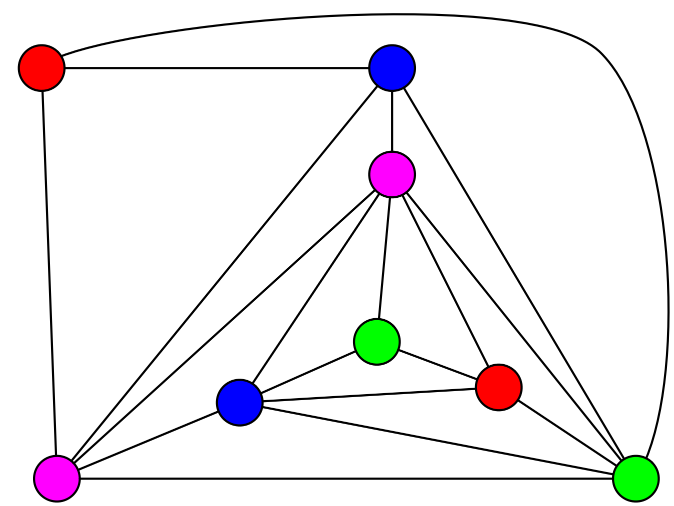


# The ~~4~~ 6-Coloring Theorem for Planar Graphs

P(n): every n-node planar graph can be 6-colored

Claim: P(n) $\forall n \geq 0$

- **Theorem.** Every planar graph $G = (V, E)$ can be ***6-colored***.
- **Useful Lemma:** Every planar graph contains a node of degree $\leq 5$.
- **Proof by induction.** P(k) = "Every k-node planar graph $G$ can be 6-colored."
  - Base Case: $k = 0$: empty coloring works. (Or you can think of $k = 1$)
  - Inductive Step:
    - Delete a vertex $v$ with $\deg(v) \leq 5$.
    - **Inductive Hypothesis:** The **remaining** graph can be 6-colored.
    - Then re-add $v$, and color it differently from all $\leq 5$ neighbors
    - We found a 6-coloring!

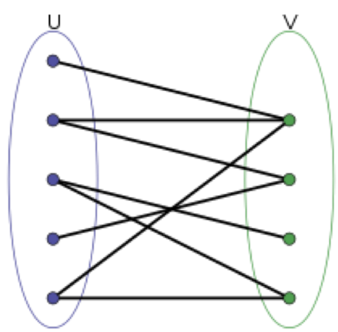


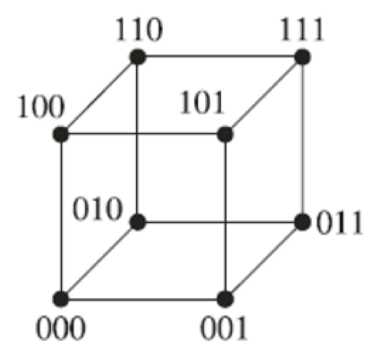
# Outline

- Wrap up material from first two graphs lectures
  - Applications of Euler's polyhedral formula
  - **Bipartite Graphs (9am lecture only)**
- Certificates
- Bipartite graphs and matchings
- Hall's theorem
- König's theorem

# Bipartite Graphs

- A graph $G = (V, E)$ is called ***bipartite*** if
  - You can partition $V = A \cup B$ into two parts, where $A \cap B = \emptyset$.
  - For every edge $\{a, b\} \in E$, $a \in A$ and $b \in B$. (There are no edges between $A$-vertices or $B$-vertices.)



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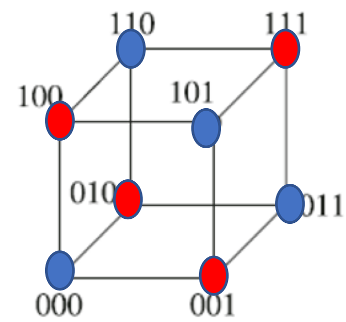
- **Theorem.** The following statements are equivalent:

  (1) $G$ is bipartite.

  (2) $G$ is 2-colorable: we can assign each node “red” or “blue” so that there are no (red, red) or (blue, blue) edges

Proof Sketch:
- Use the vertex subset A as the red nodes
- Use the vertex subset B as the blue nodes



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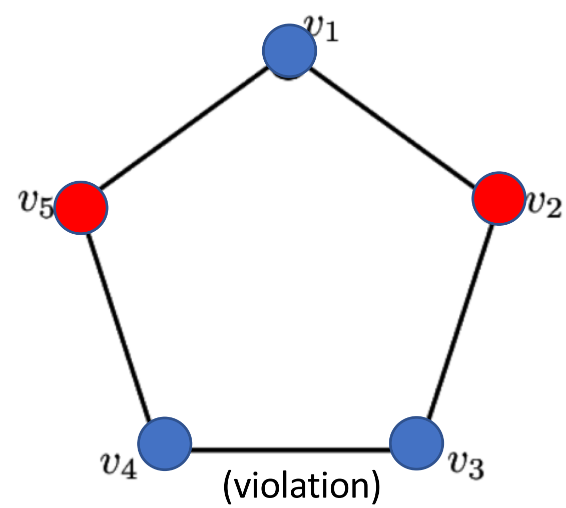
  (3) No odd cycles: For all $k \in \mathbb{Z}^+$, $G$ does not contain $C_{2k+1}$ as a subgraph.

# Bipartite Graphs

- **Theorem.** The following statements are equivalent:
  (2) $G$ is 2-colorable.
  (3) No odd cycles: For all $k \in \mathbb{Z}^+$, $G$ does not contain $C_{2k+1}$ as a subgraph.

(2)→(3) Proof Sketch:

- Seeking contradiction, assume that $G$ is 2-colorable (red+blue) **and** it has $C_{2k+1}$ as a subgraph.
- Then $C_{2k+1}$ itself must be 2-colorable, using the coloring in $G$.
- But we can verify that $C_{2k+1}$ is not 2-colorable.



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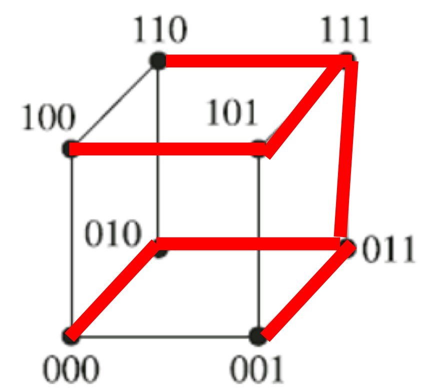
(3)→(2) Proof Sketch:

- Assume no odd cycles, we will build a 2-coloring as follows:
  - Pick any spanning tree $T$ of $G$



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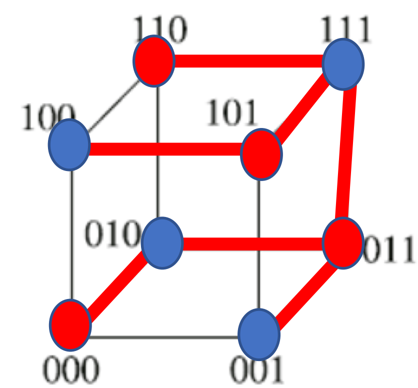

# Bipartite Graphs

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  - Pick any spanning tree $T$ of $G$
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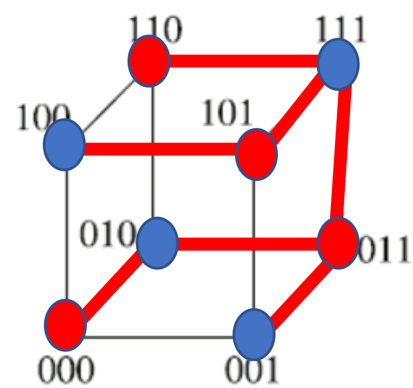
  - **Claim:** This 2-coloring of the spanning tree **also** works for the entire graph! Why?



# Bipartite Graphs

- **Theorem.** The following statements are equivaler
  - (2) $G$ is 2-colorable.
  - (3) No odd cycles: For all $k \in \mathbb{Z}^+$, $G$ does not co

An edge in $G$ between two same-colored nodes would **complete an odd cycle.** We assumed no odd cycles.

(3)→(2) Proof Sketch:

- Assume no odd cycles, we will build a 2-coloring as follows:
  - Pick any spanning tree $T$ of $G$
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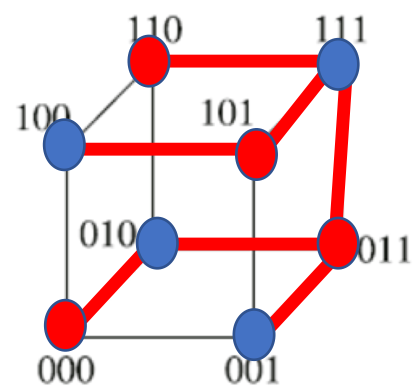
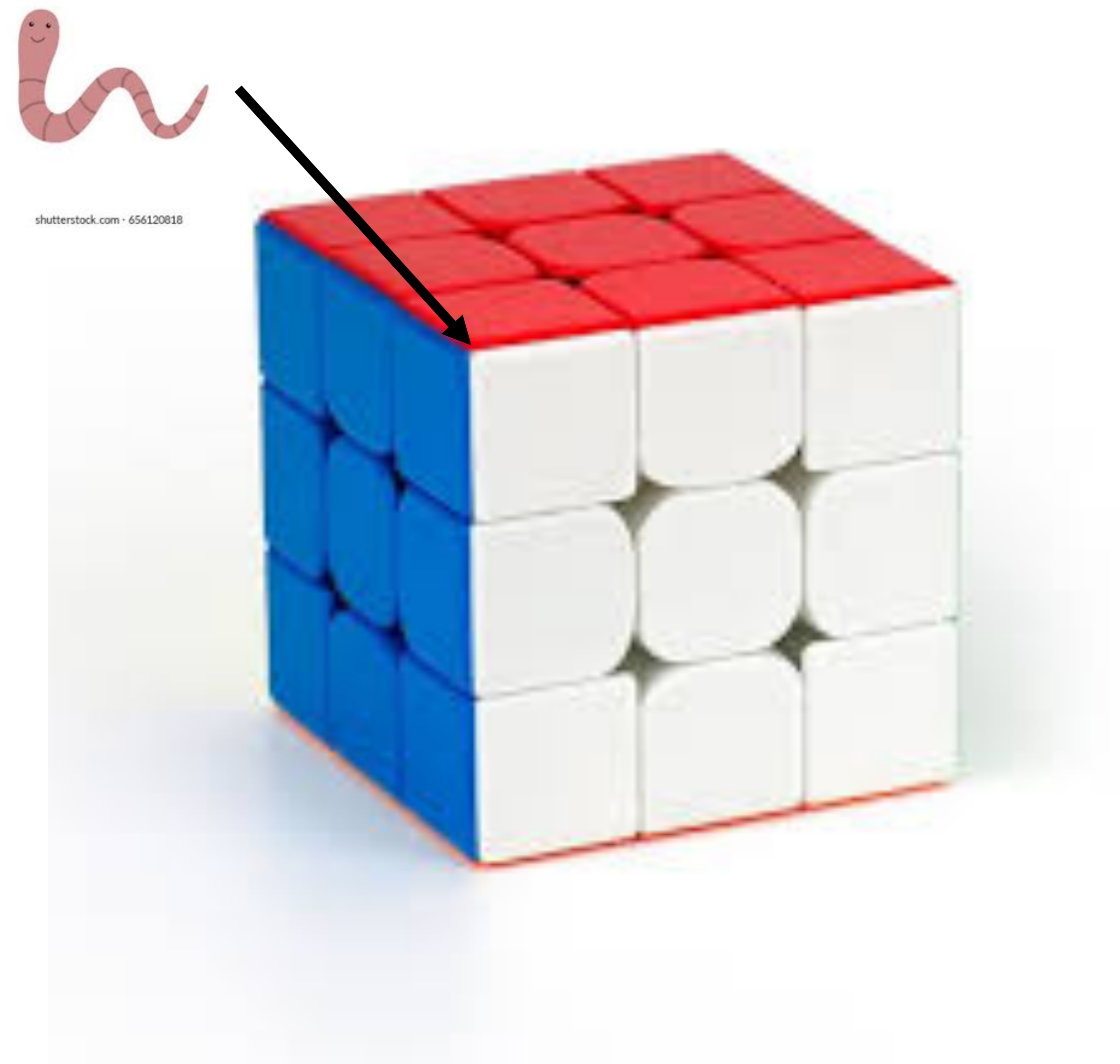
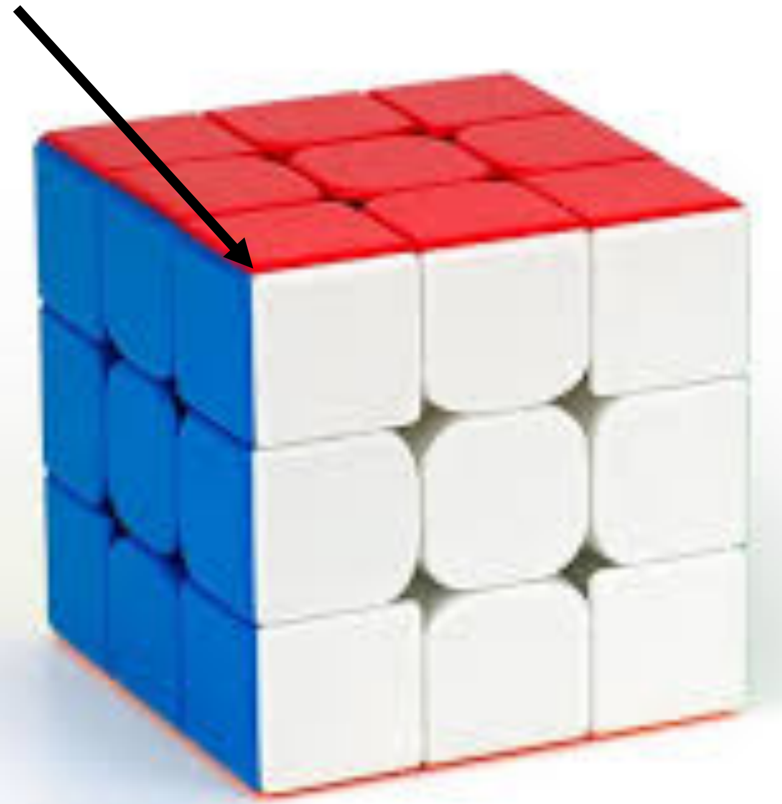


# A Puzzle




- Wanda the worm lives **inside** a corner block of a 3x3x3 cube, subdivided into 1x1x1 subcubes.
- Wanda can tunnel between two subcubes if they **share a face**.
- Wanda wants to see the world before she dies. **Can you help Wanda plan a path to visit each subcube exactly once, and end back in her home cube?**

# A Puzzle

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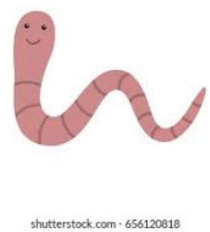
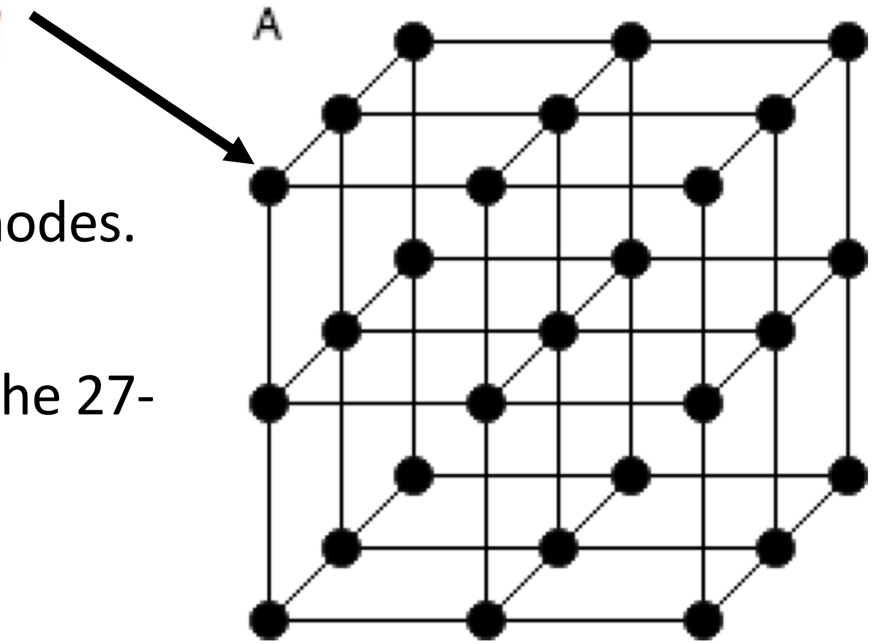
- Wanda can tunnel between two subcubes if they **share a face**.
- Wanda wants to see the world before she dies. **Can you help Wanda plan a path to visit each subcube exactly once, and end back in her home cube?**

No, you can't. ☹

# A Puzzle

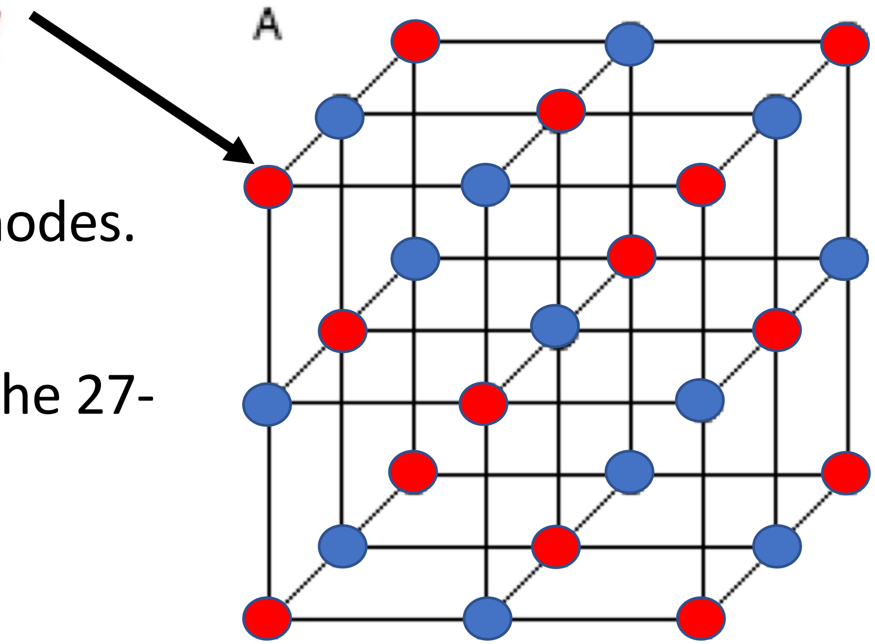



- Model Wanda's cube as a **graph** with 27 nodes.

- Rephased question: **Does this have $C_{27}$**, the 27-cycle, **as a subgraph?**

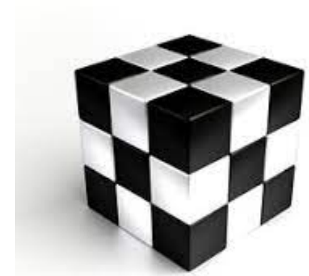
# A Puzzle


- Model Wanda’s cube as a **graph** with 27 nodes.

- Rephased question: **Does this have $C_{27}$**, the 27-cycle, **as a subgraph?**

- The graph is **2-colorable**.



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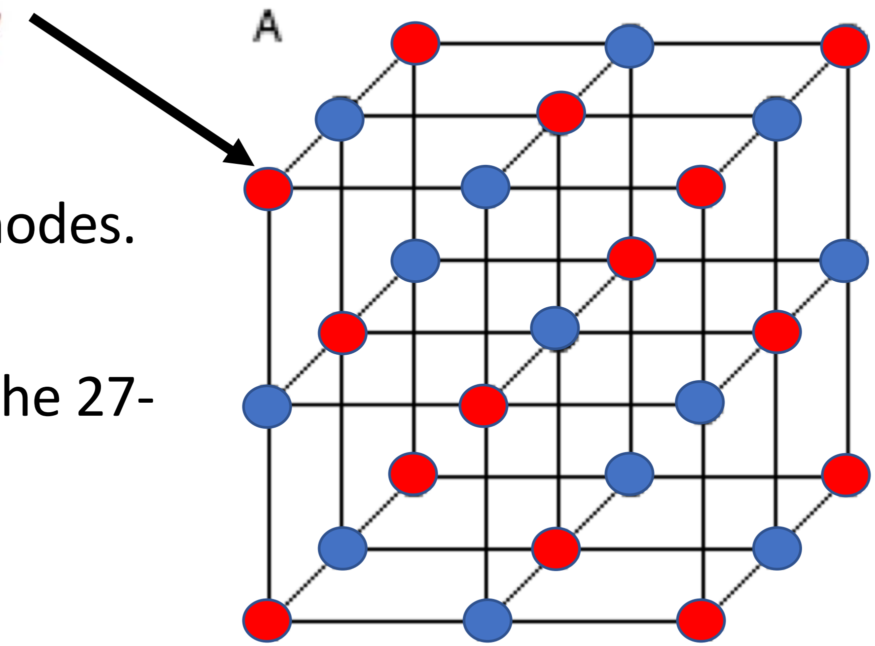
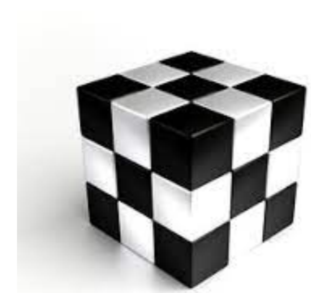



- Model Wanda’s cube as a **graph** with 27 nodes.
- Rephased question: **Does this have $C_{27}$**, the 27-cycle, **as a subgraph?**
- The graph is **2-colorable**.
- So it doesn’t have any odd-length cycles.



# Outline

- **Certificates**
- Bipartite graphs and matchings
- Hall’s theorem
- König’s theorem

# Summary of graph lectures so far

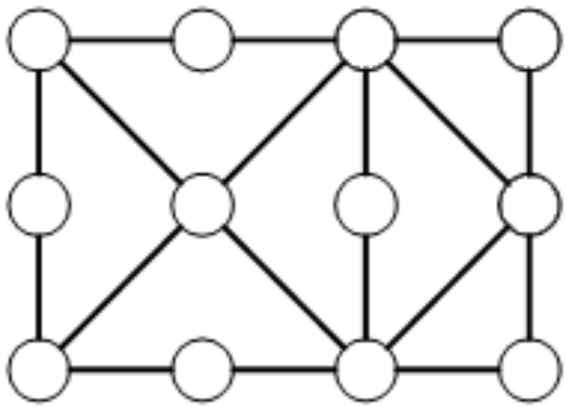
- There are many **properties that a graph might have**
  - Bipartite
  - Planar
  - Has an Euler path
  - (To come in this lecture) has a perfect matching

- It's usually easy from the definition to **prove** that a graph has a property
  - Planar → just show me a planar drawing of your graph
  - Has an Euler path → just show me the Euler path

- But it takes effort to find a way to get **disproofs** that a graph has a property

- **Certificate:** a small/simple part of a graph that **disproves** a graph property

# Certificates



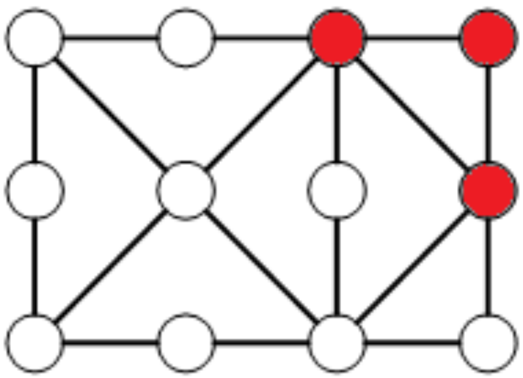
**Reminder**: a **"bipartite"** graph is one whose nodes can be colored {red, blue} so that every edge has different-colored endpoints.

- To **prove** that a graph is bipartite, you can just show a coloring.
- But how can we **disprove** that a graph is bipartite?

One option: try all possible 2-colorings of its nodes

Technically this will work, but it's extremely long and painful.
Computer science is interested in **efficient** checking!

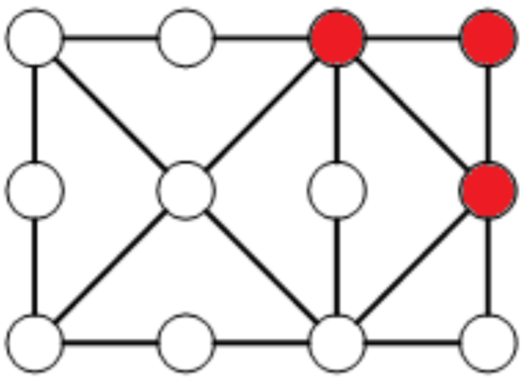
# Exhaustive Certificates



**Triangle subgraphs** are one possible **"certificate"** of non-bipartiteness.

If we can find a triangle subgraph of $G$, this acts as a **disproof** that $G$ is bipartite.

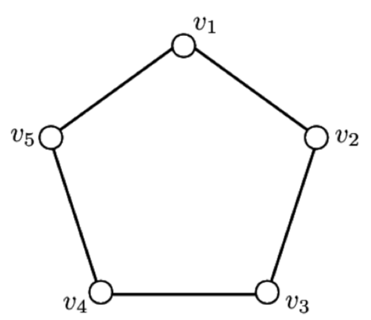
# Exhaustive Certificates



**Triangle subgraphs** are one possible **"certificate"** of non-bipartiteness.

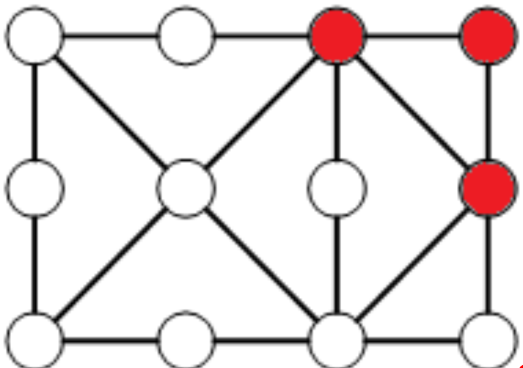
Are they **exhaustive certificates**?

*(Can **every** non-bipartite graph be proved non-bipartite using a triangle subgraph?)*




**No**: there are some graphs that are non-bipartite, even though you can't prove this via a triangle subgraph.

# Exhaustive Certificates



**Triangle subgraphs** are one possible **"certificate"** of non-bipartiteness.

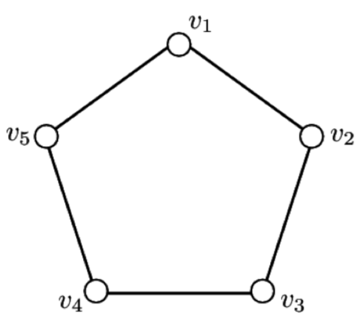
**Proved in first graph lecture:** Odd cycle subgraphs are **exhaustive certificates** of non-bipartiteness.

- *(Certificate)* If $G$ has an odd cycle as a subgraph, then $G$ is not bipartite

$\neg q \to \neg p$

- *(Exhaustive)* and if $G$ does **not** have any odd cycles as subgraphs, then $G$ **is** bipartite.

$q \to p$




Theorem (from first graph lecture):
$\underbrace{G \text{ is bipartite}}_{p}$ if and only if $\underbrace{G \text{ does not have any odd cycles as subgraphs}}_{q}$.

$\longleftrightarrow$

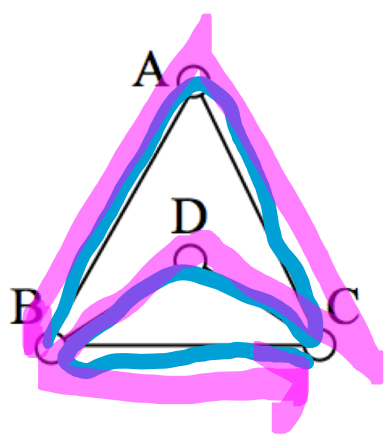
$p \to q \equiv \neg q \to \neg p$

$p \leftarrow q$

# Exhaustive Certificates

**Reminder**: an **“Euler path”** in a (multi)graph is one that uses every edge exactly once.

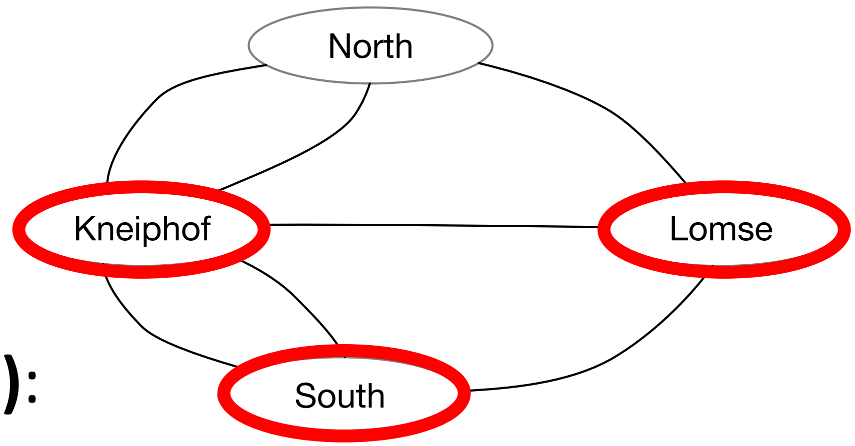
- To **prove** that a graph has an Euler path: just show the Euler path.
- But how can we **disprove** that a graph has an Euler path?



# Exhaustive Certificates

**Reminder**: an **"Euler path"** in a (multi)graph is one that uses every edge exactly once.

**Observation**: Any three nodes with odd degree form a **certificate** that $G$ does not have an Euler path.

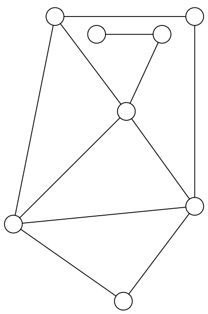



**Euler's Theorem (proved last lecture)**:
These are **exhaustive certificates.**

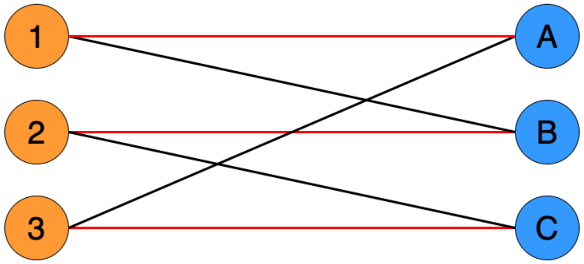
*(If there are **not** three odd-degree nodes, then $G$ **does** have an Euler path.)*

# Exhaustive Certificates

- **Planarity** also has a nice exhaustive certificate theorem
  - To prove that $G$ is planar: just show me a plane drawing
  - **Kuratowski's Theorem (not covered)** helps you disprove planarity



- Today's lecture:
  - A new graph property: **"Has a perfect matching"**
  - An exhaustive certificate theorem for this property in bipartite graphs



# Outline

- Certificates
- **Bipartite graphs and matchings**
- Hall’s theorem
- König’s theorem

# Bipartite Graphs and *Matchings*

- We have 30 IAs and 30 discussion sections to cover.
- Before the semester we asked all IAs which discussion sections they could/couldn't teach.
- Then, we need to **assign each IA** to a discussion.

Discussion Sections

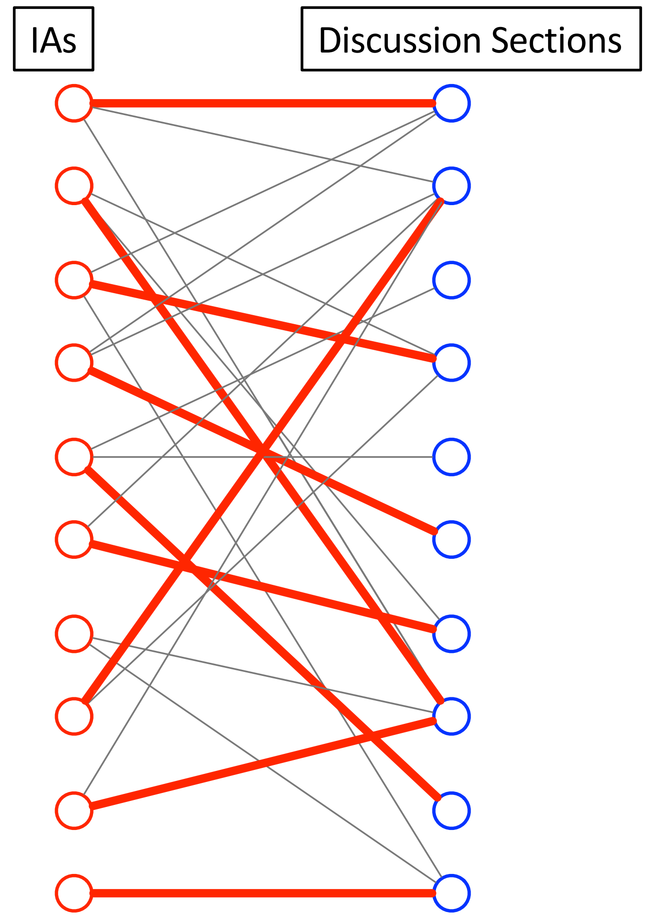
IAs

| Email Address | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt | Unt |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ashwinak@umich.edu | 5 | 5 | 5 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 1 | 5 | 5 | 5 | 3 | 1 | 1 | 1 | 3 | 1 |
| inwang@umich.edu | 1 | 1 | 1 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 5 | 3 | 3 | 4 | 5 | 1 | 1 | 1 | 1 | 1 |
| mitchang@umich.edu | 3 | 5 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 5 | 3 | 1 | 1 | 1 | 1 | 5 | 5 | 1 |
| rcgutman@umich.edu | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 5 |
| qinx@umich.edu | 1 | 4 | 4 | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 1 | 1 | 4 | 2 | 1 | 2 | 4 | 3 | 2 | 1 | 1 | 1 | 5 | 5 | 1 |
| lxhao@umich.edu | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 5 | 5 | 5 | 1 | 4 | 3 | 1 | 1 | 1 | 2 | 4 | 4 | 4 | 3 | 3 |
| niharj@umich.edu | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 5 |
| shbhgrwl@umich.edu | 1 | 1 | 1 | 2 | 3 | 3 | 1 | 1 | 1 | 1 | 5 | 4 | 1 | 4 | 3 | 5 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 3 | 1 |
| iansm@umich.edu | 2 | 5 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 5 | 5 | 1 | 1 | 5 | 1 | 4 | 4 | 1 | 1 | 1 | 1 | 1 | 5 | 1 |
| bait@umich.edu | 2 | 1 | 1 | 4 | 4 | 5 | 5 | 5 | 5 | 5 | 3 | 1 | 4 | 2 | 1 | 4 | 1 | 1 | 1 | 3 | 3 | 3 | 3 | 3 | 2 |
| angzh@umich.edu | 1 | 1 | 1 | 5 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 1 | 1 | 1 | 5 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 4 | 2 | 1 |
| calebqiu@umich.edu | 1 | 1 | 1 | 4 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 3 | 1 | 5 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| dsinghal@umich.edu | 4 | 5 | 5 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 5 | 1 |
| benxzh@umich.edu | 2 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 5 | 5 | 1 | 5 | 5 | 2 | 2 | 1 | 1 | 1 | 1 | 4 | 4 | 2 |
| jcantre@umich.edu | 1 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 4 | 1 |
| tchiang@umich.edu | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 5 | 4 | 4 | 1 | 1 | 5 |
| rpolly@umich.edu | 1 | 3 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 5 | 1 | 1 | 1 | 4 | 5 | 5 | 5 | 4 | 1 |
| feyrer@umich.edu | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| rschled@umich.edu | 2 | 3 | 3 | 1 | 4 | 4 | 4 | 4 | 4 | 4 | 1 | 1 | 5 | 5 | 1 | 5 | 2 | 2 | 2 | 3 | 1 | 1 | 1 | 3 | 1 |
| floxic@umich.edu | 1 | 1 | 1 | 2 | 4 | 4 | 2 | 2 | 2 | 2 | 4 | 1 | 1 | 4 | 1 | 1 | 1 | 1 | 1 | 2 | 4 | 4 | 4 | 4 | 2 |
| brianoo@umich.edu | 1 | 1 | 1 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 2 | 1 | 4 | 5 | 1 | 3 | 1 | 1 | 1 | 2 | 3 | 3 | 3 | 3 | 2 |
| kkwanliu@umich.edu | 2 | 2 | 2 | 3 | 3 | 3 | 1 | 1 | 1 | 1 | 3 | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 4 | 5 | 4 | 4 | 5 | 4 | 3 |
| emshedde@umich.edu | 1 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 2 | 5 | 5 | 2 | 5 | 1 | 1 | 1 | 2 | 4 | 5 | 5 | 5 | 5 | 2 |
| rqkang@umich.edu | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 3 | 4 | 1 | 3 | 1 | 2 | 2 | 5 | 5 | 3 | 3 | 1 | 1 | 3 |
| asbhown@umich.edu | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 5 | 5 | 4 | 1 | 5 | 1 |

# Bipartite Graphs and *Matchings*

- We have 30 IAs and 30 discussion sections to cover.
- Before the semester we asked all IAs which discussion sections they could/couldn't teach.
- Then, we need to **assign each IA** to a discussion.
- ***Matching***: any set of edges that do not share any endpoints.

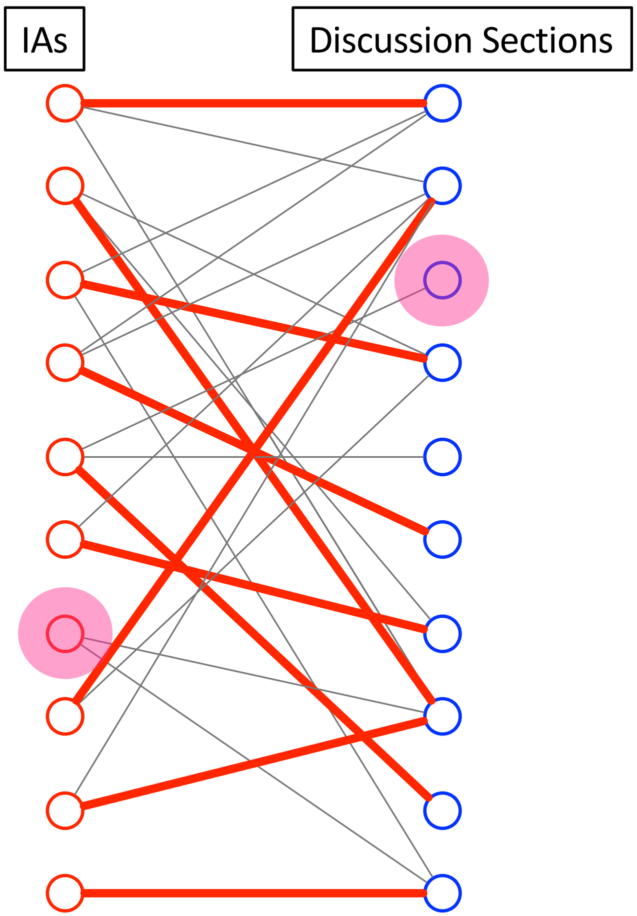
The red edges form a ***matching***



# Bipartite Graphs and *Matchings*

- We have 30 IAs and 30 discussion sections to cover.
- Before the semester we asked all IAs which discussion sections they could/couldn't teach.
- Then, we need to **assign each IA** to a discussion.
- ***Matching***: any set of edges that do not share any endpoints.
- ***Perfect matching***: all vertices are adjacent to a matched edge.

The red edges form a ***matching***, but not a ***perfect matching***.

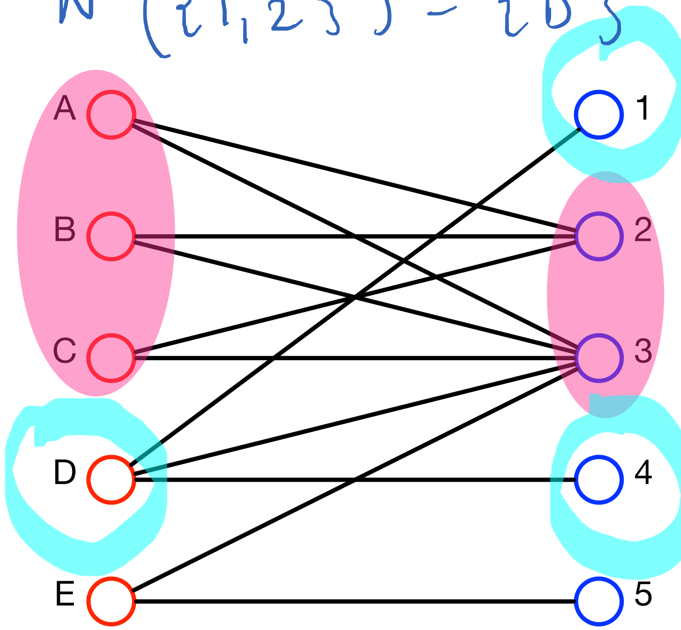


# Matching Certificates

$\boldsymbol{N(\{A, B, C\}) = \{2, 3\}}$

$N(\{1,2\}) = \{D\}$

- Does this graph have a ***perfect matching***?
  - (a) Yes.
  - (b) No.




This part of the graph is a **certificate** that no perfect matching exists!

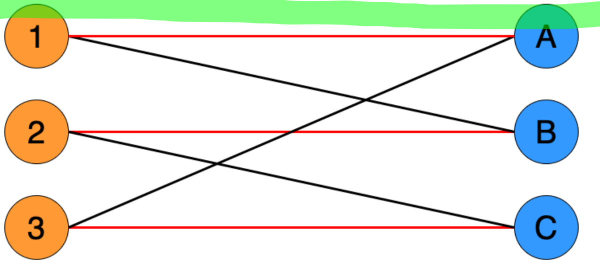
S is a "constricted set"

- For a **set** of nodes $S$, $N(S) = \{v \mid \{s, v\} \in E, s \in S\}$ is the "**neighborhood**" of $S$.
- A node set $S$ with $\boldsymbol{|N(S)| < |S|}$ works as a **certificate** that $G$ has no perfect matching.
  - *By pigeonhole, not possible for **every** node in $S$ to match with a **unique** node in $N(S)$.*

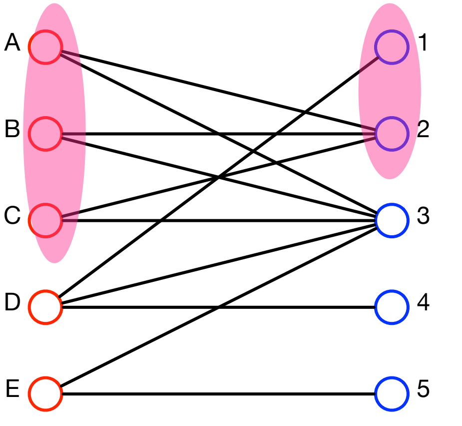
# Matching Certificates

Hall's Thm: For a bipartite graph G, there is a perfect matching iff there are no constricted sets

- Given a graph $G$, how do we **prove** that it has a perfect matching?
  - Just show me the matching.




- How do we **disprove** that $G$ has a perfect matching?
  - Show me a **certificate** of the form: node set $S$ with $|N(S)| < |S|$




**Hall's Theorem (main proof for today):**

For **bipartite** graphs $G$, these are **exhaustive certificates**.

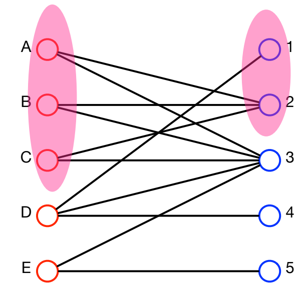
*When $G$ is bipartite, if **no** node set $S$ has $|N(S)| < |S|$, then $G$ **has** a perfect matching.*

# Outline

- Certificates
- Bipartite graphs and matchings
- **Hall's theorem**
- König's theorem

# Hall’s Theorem

**Certificate** that $G$ has no perfect matching: node set $S$ with $|N(S)| < |S|$



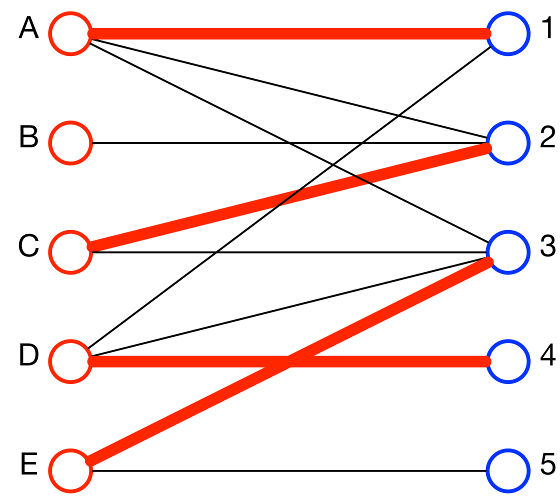

**Hall’s Theorem:** For **bipartite** graphs $G$, these certificates are exhaustive.

*When $G$ is bipartite, if no node set $S$ has $|N(S)| < |S|$, then $G$ has a perfect matching.*

**High-Level Proof Strategy:**

<
- Let $G$ be a bipartite graph in which no set $S$ has $|N(S)|\ \cancel{\geq}\ |S|$
- Let $M$ be **any matching** in $G$.

- We will prove: if $M$ is not yet a perfect matching, then we can modify it to get a bigger matching $M'$.
- *(Then repeat until we have a perfect matching)*



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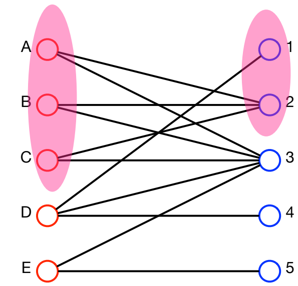
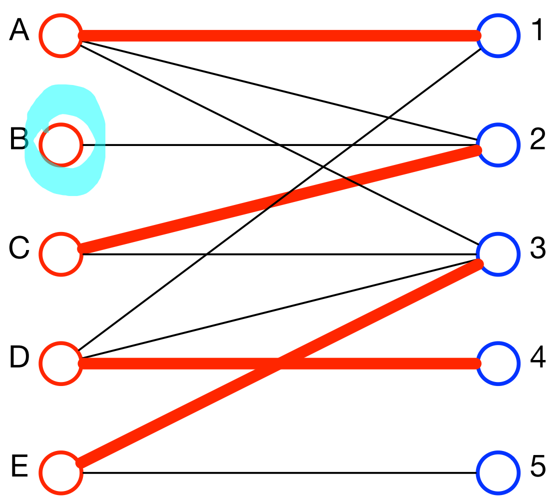



**Hall’s Theorem:** For **bipartite** graphs $G$, these certificates are exhaustive.
*When $G$ is bipartite, if no node set $S$ has $|N(S)| < |S|$, then $G$ has a perfect matching.*

**How could we make this matching bigger?**

- We **can’t** simply add a new edge
  - *$\{b, 5\}$ is not an edge in the input graph*
- We **can** more carefully toggle edges in/out of the matching
  - *Need to keep a valid matching*
  - *Goal: toggle in 1 more edge than we toggle out*



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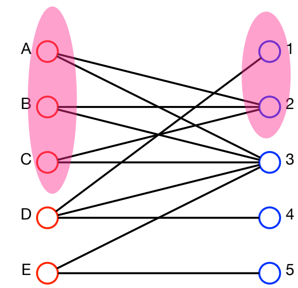


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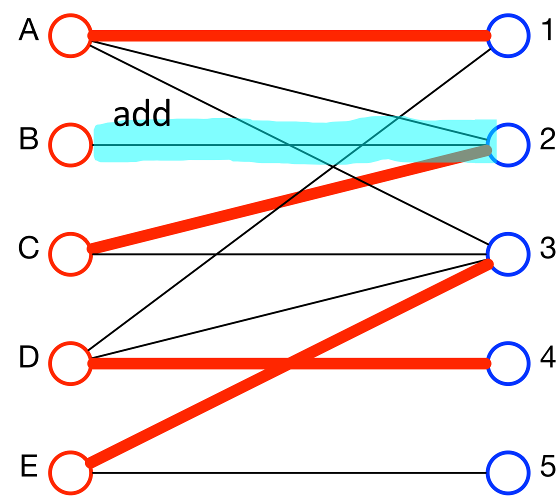

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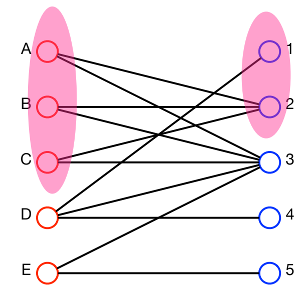


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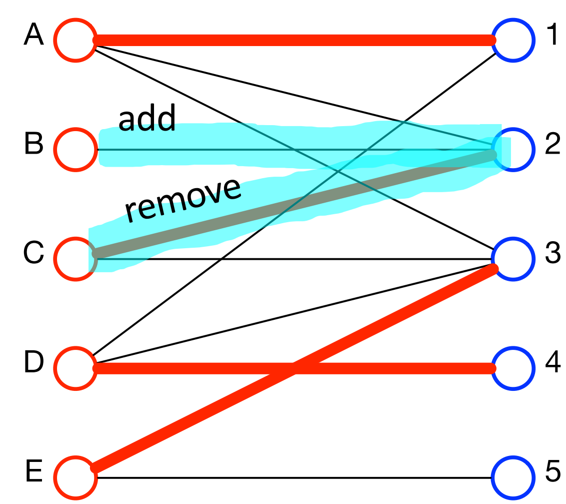

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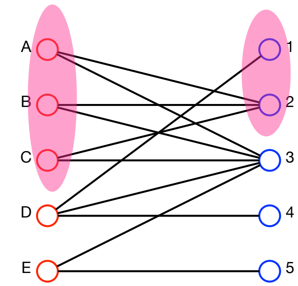


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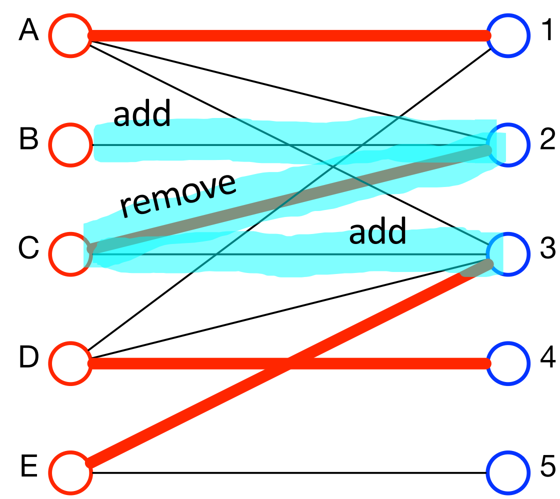

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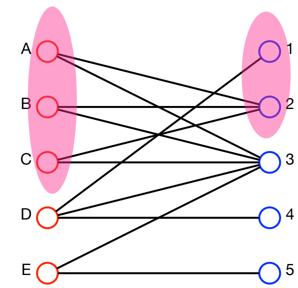


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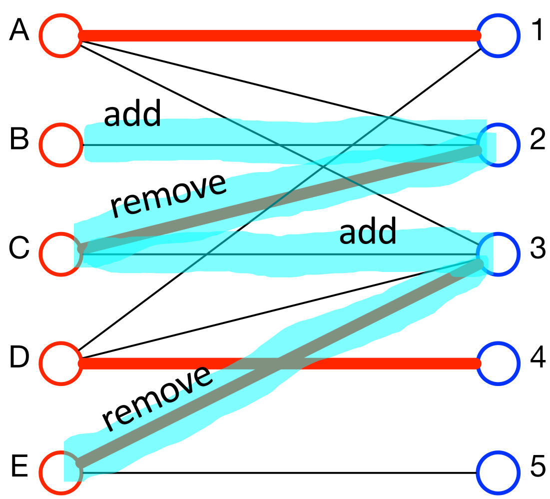

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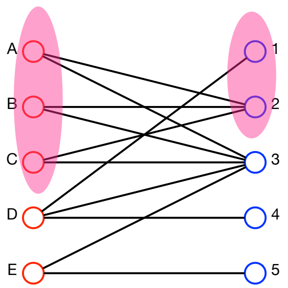


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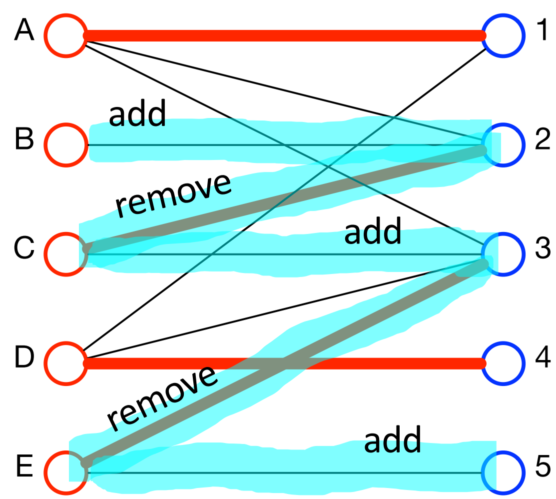

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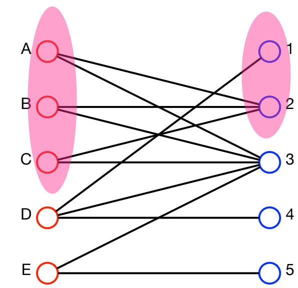


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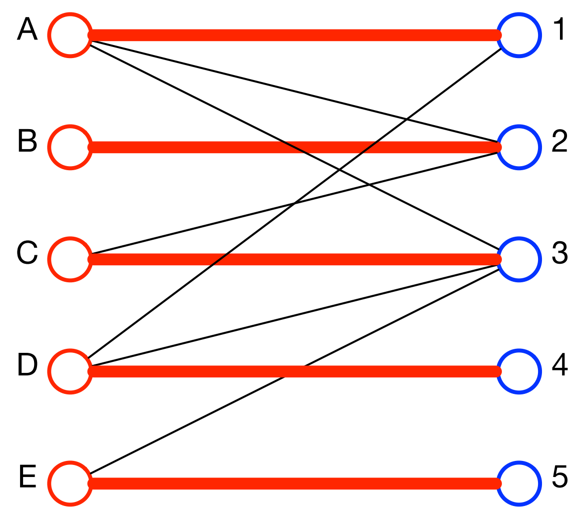


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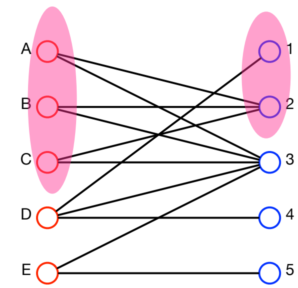
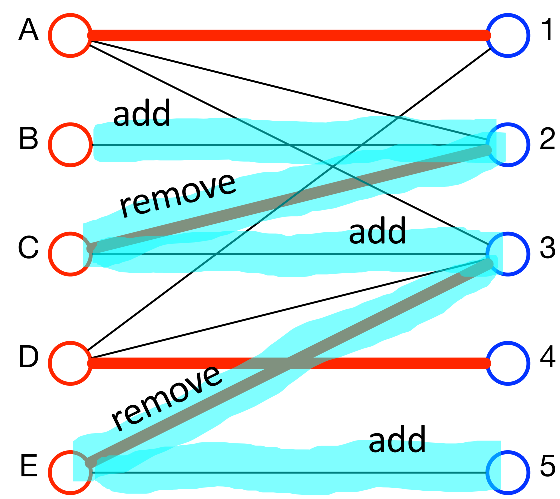



**Hall's Theorem:** For **bipartite** graphs $G$, these certificates are exhaustive.

*When $G$ is bipartite, if no node set $S$ has $|N(S)| < |S|$, then $G$ has a perfect matching.*

An ***augmenting path*** for a matching $M$ is a path $P = (v_1, v_2, v_3, \ldots, v_{2k})$, where:

- The **endpoint nodes** $v_1, v_{2k}$ are both unmatched
- The **edges** alternate between out/in of **M**




**The Point of Augmenting Paths:**

If we can find an augmenting path, then we can safely toggle its edges in/out of the matching

- Matching stays valid
- Size of matching increases by 1

# Hall's Theorem

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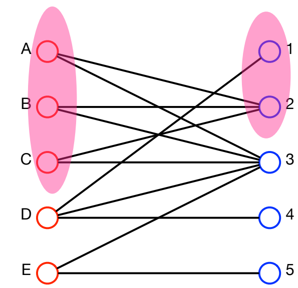


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An ***augmenting path*** for a matching $M$ is a path $P = (v_1, v_2, v_3, \ldots, v_{2k})$, where:

- The **endpoint nodes** $v_1, v_{2k}$ are both unmatched

Still to do: show that an augmenting path **always exists**

**The Point of Augmenting Paths:**

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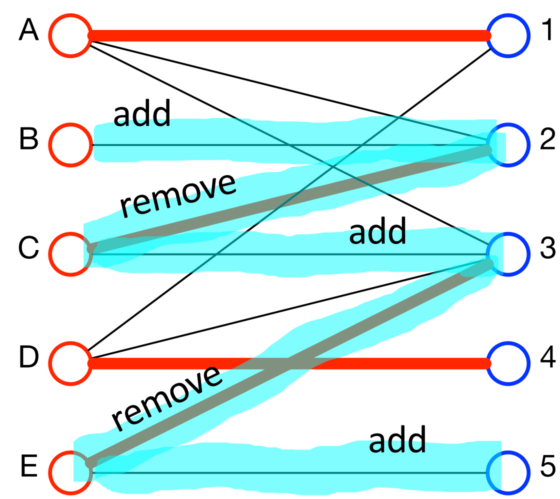
- Matching stays valid
- Size of matching increases by 1



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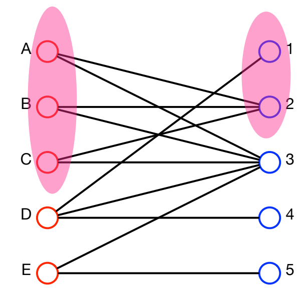


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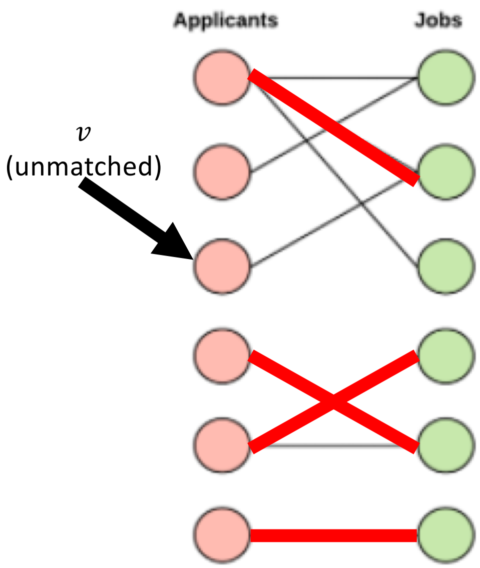
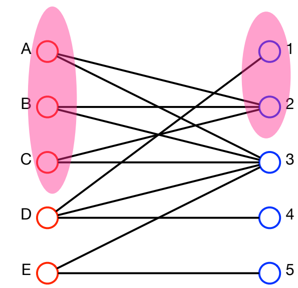



Strategy for finding an augmenting path:

- **Start:** A bipartite graph with a **partial matching**
- Pick your favorite **unmatched node** on the left (v)

# Hall's Theorem

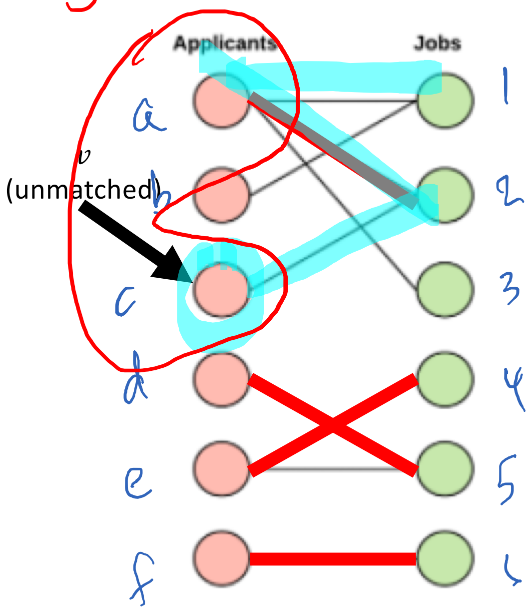
**Certificate** that $G$ has no perfect matching: node set $S$ with $|N(S)| < |S|$




**Hall's Theorem:** For **bipartite** graphs $G$, these certificates are exhaustive.

*When $G$ is bipartite, if no node set $S$ has $|N(S)| < |S|$, then $G$ has a perfect matching.*

$S = \{a, c\}$

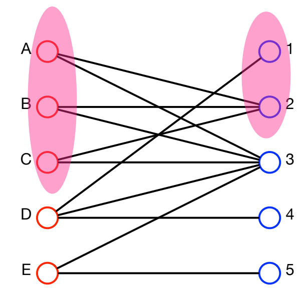



## Strategy for finding an augmenting path:

- **Start:** A bipartite graph with a **partial matching**
- Pick your favorite **unmatched node** on the left (v)
- Let $S$ be all the nodes on the **left side** that we can reach using a path that:
  - Starts at v
  - Alternates edges out/in the matching
  - *(including $v \in S$)*

# Hall's Theorem

**Certificate** that $G$ has no perfect matching: node set $S$ with $|N(S)| < |S|$




**Hall's Theorem:** For **bipartite** graphs $G$, these certificates are exhaustive.

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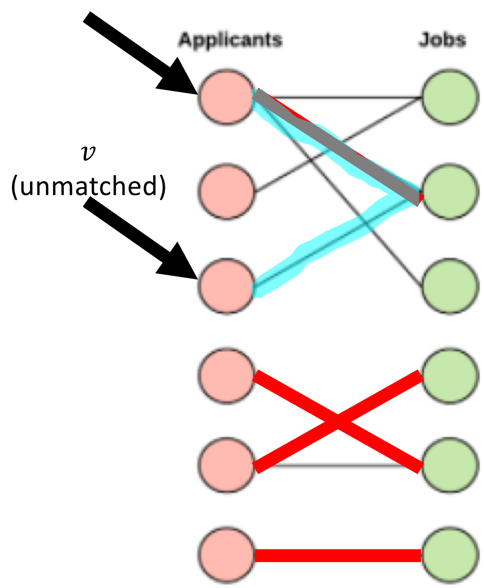


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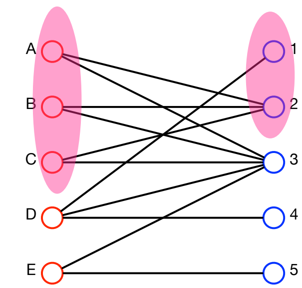


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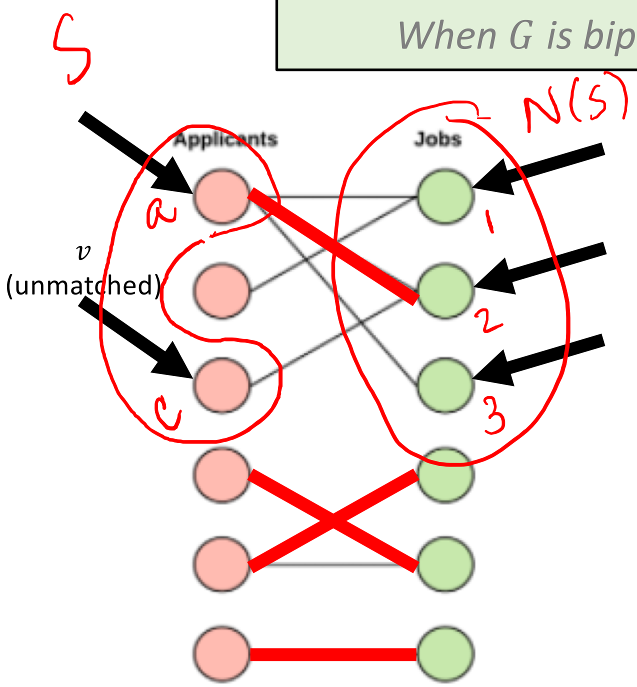


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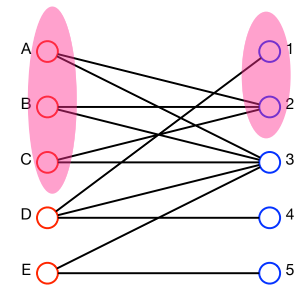


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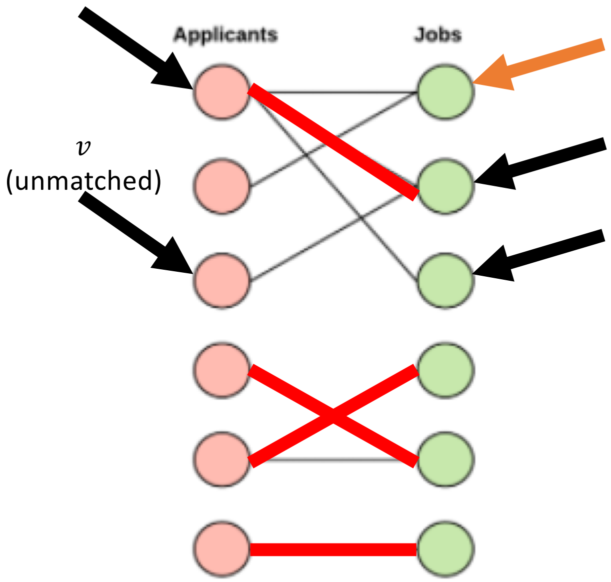


Strategy for finding an augmenting path:

- **Start:** A bipartite graph with a **partial matching**
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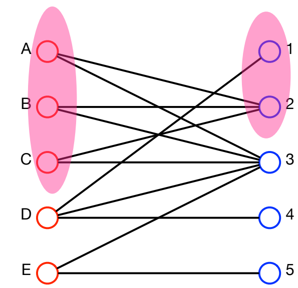
- Look at $N(S)$ on the right. Notice that $|N(S)| \geq |S|$

- At least one node $u \in N(S)$ must be unmatched!
  - *There are only $|S| - 1$ matched nodes in $S$*
  - *For each **matched** node in $N(S)$, we can discover its matched pair in $S$ by extending the alternating path*
  - *So there are only $|S| - 1$ matched nodes in $N(S)$*

# Hall’s Theorem

**Certificate** that $G$ has no perfect matching: node set $S$ with $|N(S)| < |S|$

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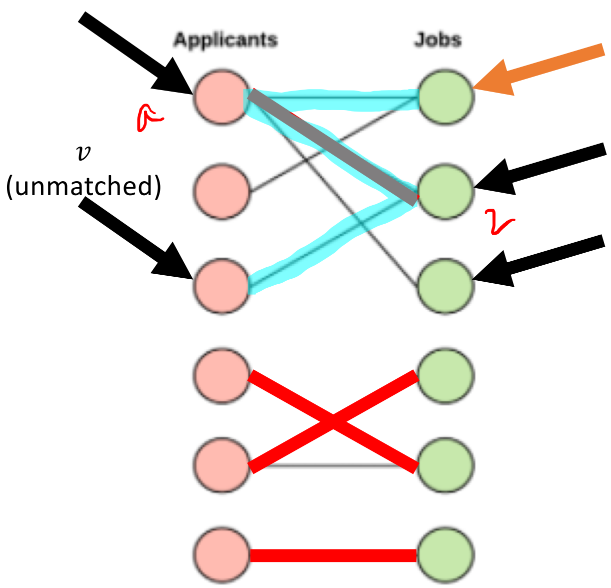


Strategy for finding an augmenting path:

- **Start:** A bipartite graph with a **partial matching**
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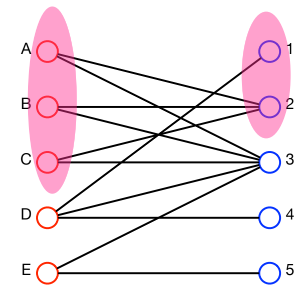
- Look at $N(S)$ on the right. Notice that $|N(S)| \geq |S|$
- At least one node $u \in N(S)$ must be unmatched!
- So we have a $v \to u$ augmenting path

# Hall’s Theorem

**Certificate** that $G$ has no perfect matching: node set $S$ with $|N(S)| < |S|$

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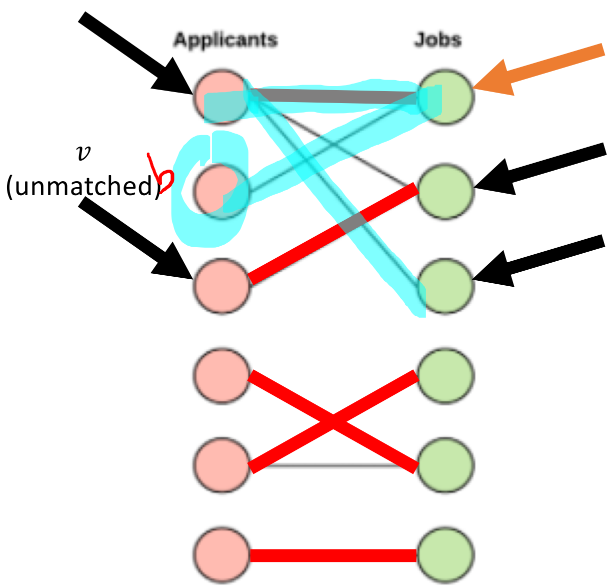
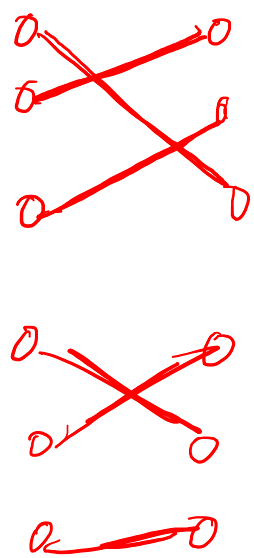


Strategy for finding an augmenting path:

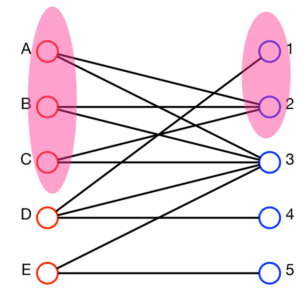
- **Start:** A bipartite graph with a **partial matching**
- Pick your favorite **unmatched node** on the left (v)
- Let $S$ be all the nodes on the **left side** that we can reach using an **alternating path** starting at **v**.
- Look at $N(S)$ on the right. Notice that $|N(S)| \geq |S|$
- At least one node $u \in N(S)$ must be unmatched!
- So we have a $v \rightarrow u$ augmenting path
- *(Toggle augmenting path to increase the matching)*

# Outline

- Certificates
- Bipartite graphs and matchings
- Hall’s theorem
- **König’s theorem**

# Hall vs. König

**Certificate** that $G$ has no perfect matching: node set $S$ with $|N(S)| < |S|$




**Hall's Theorem:** For **bipartite** graphs $G$, these certificates are exhaustive.

*When $G$ is bipartite, if no node set $S$ has $|N(S)| < |S|$, then $G$ has a perfect matching.*

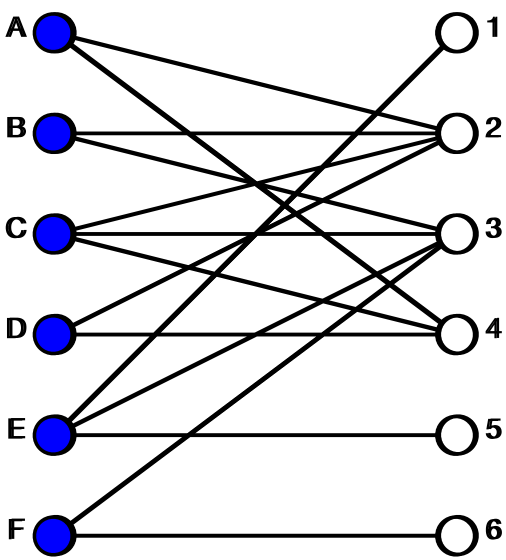
- Even if we don't have a **perfect** matching, we still might hope to have a **large** matching.
  - *Even if we can't hire applicants to cover all jobs, we want to cover as many jobs as possible.*

- What might be a **certificate** that there is no matching of size $\geq k$?

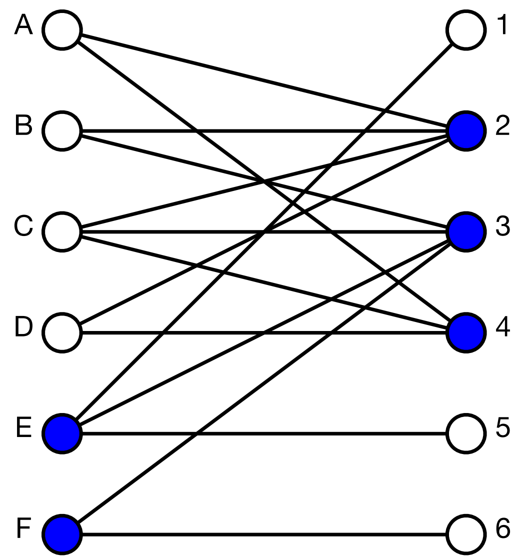
# Matchings and Vertex Covers

- A ***vertex cover*** in $G = (V, E)$ is a subset of vertices $\boldsymbol{C} \subseteq V$ that "covers" all the edges. For every $\{u, v\} \in E$, we have $(\boldsymbol{u} \in \boldsymbol{C})$ ***or*** $(\boldsymbol{v} \in \boldsymbol{C})$.

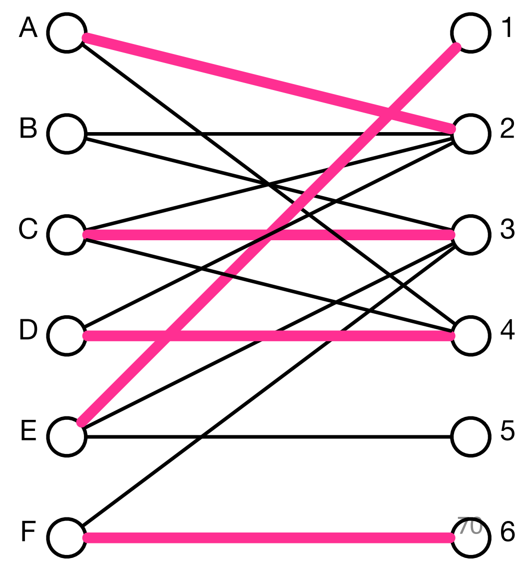
$C$ = set of left vertices is a V.C.
size: 6




This graph has a smaller V.C.
size: 5




Coincidentally the same size as a maximum size matching. Size: 5.



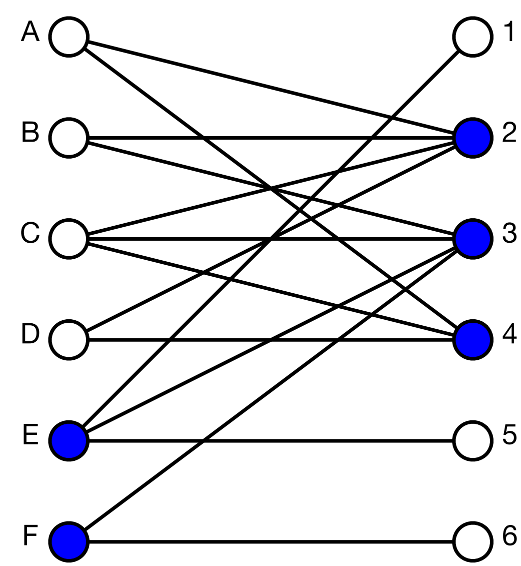
# Matchings and Vertex Covers

König's Thm: The minimum size of a vertex cover = maximum sized matching.

- A ***vertex cover*** in $G = (V, E)$ is a subset of vertices $\boldsymbol{C} \subseteq V$ that "covers" all the edges. For every $\{u, v\} \in E$, we have $(\boldsymbol{u} \in \boldsymbol{C})$ ***or*** $(\boldsymbol{v} \in \boldsymbol{C})$.

A vertex cover of size $k$ works as a **certificate** that there is no matching of size $> k$.

- In any matching $M$, every edge touches one of the k nodes in the vertex cover.
- But no two edges in $M$ can touch the same node.




## Theorem (König-Egerváry):

In bipartite graphs, these certificates are **exhaustive**.

*(min vertex cover size k → exists matching of size k)*

# Recap

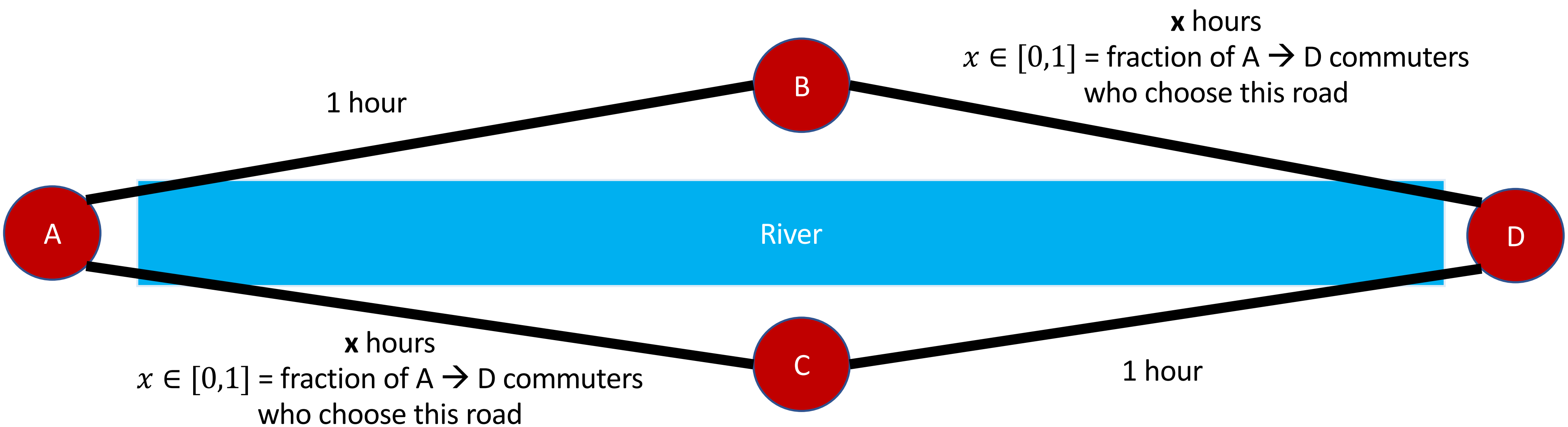
- Many graph properties are easy to **prove** but not as obvious to **disprove** *(or sometimes the other way around)*
  - Bipartiteness, planarity, existence of Euler path …
- A **certificate** is a small/simple feature of your graph that lets you **disprove** the property *(or sometimes prove)*
- Certificates are particularly cool/useful when they are **exhaustive**
  - Bipartiteness → odd cycle subgraphs
  - Euler path → three odd-degree nodes
  - Perfect matching → node subset with $|N(S)| < |S|$
  - Size k matching → size k vertex cover

# Bonus: Braess' Paradox

# Designing Road Networks

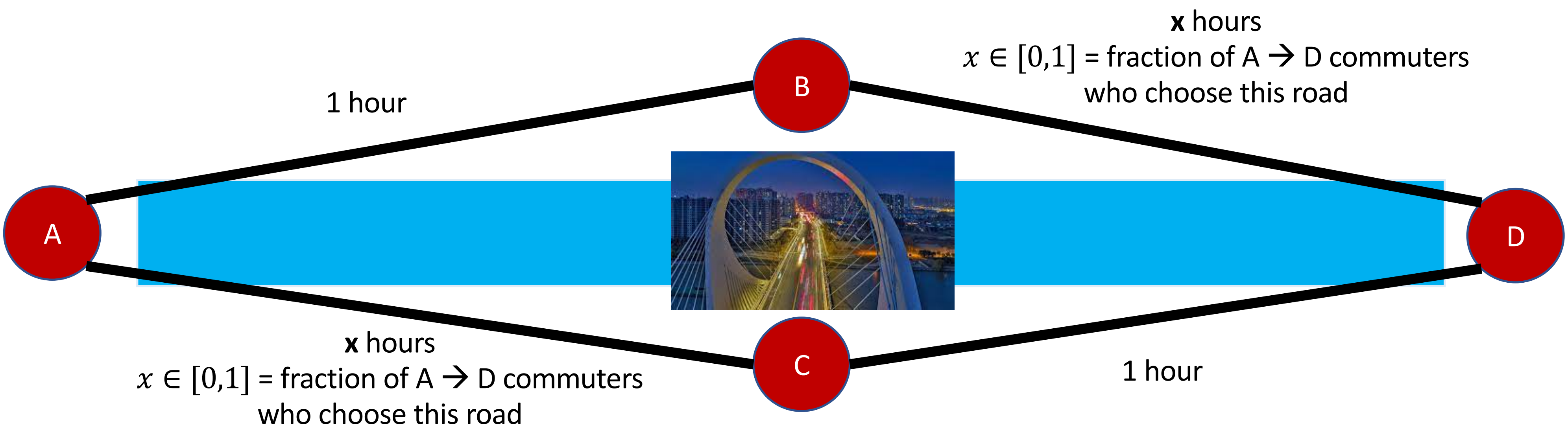
- Common use of (planar?) graph theory: design good road networks
  - “Good” = people can drive between points quickly
- Time to travel a road might be **traffic sensitive**
  - Different roads might depend differently on traffic (1-lane vs 6-lane highway)
- Commuters are **rational** and **selfish**
  - They will always choose the fastest route, given current traffic

# Commuting Network



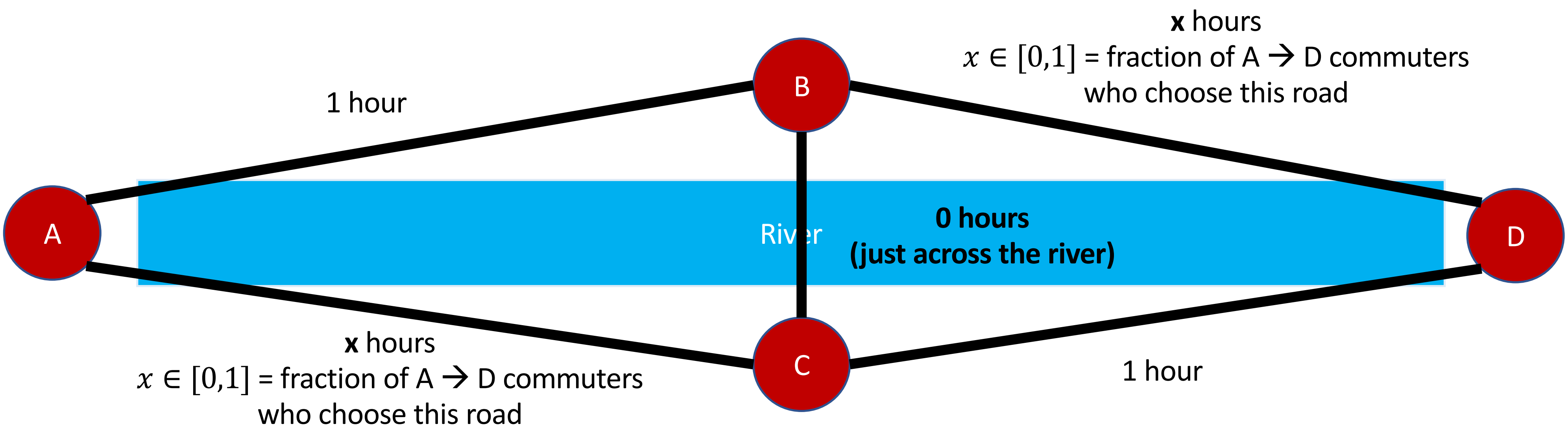

Commuters travel between A and D.

# Commuting Network



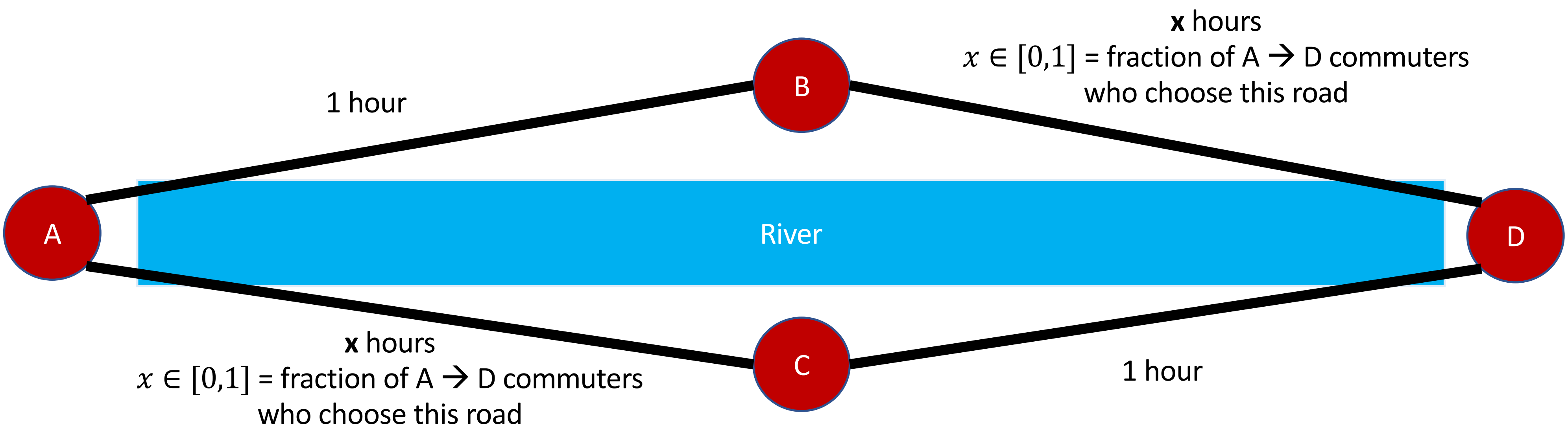

Good news!
The city has agreed to allocate $1 million dollars for new bridge construction projects to improve your network!

# Commuting Network




Good news!
The city has agreed to allocate $1 million dollars for new bridge construction projects to improve your network!
**How much does this improve average commute time?**

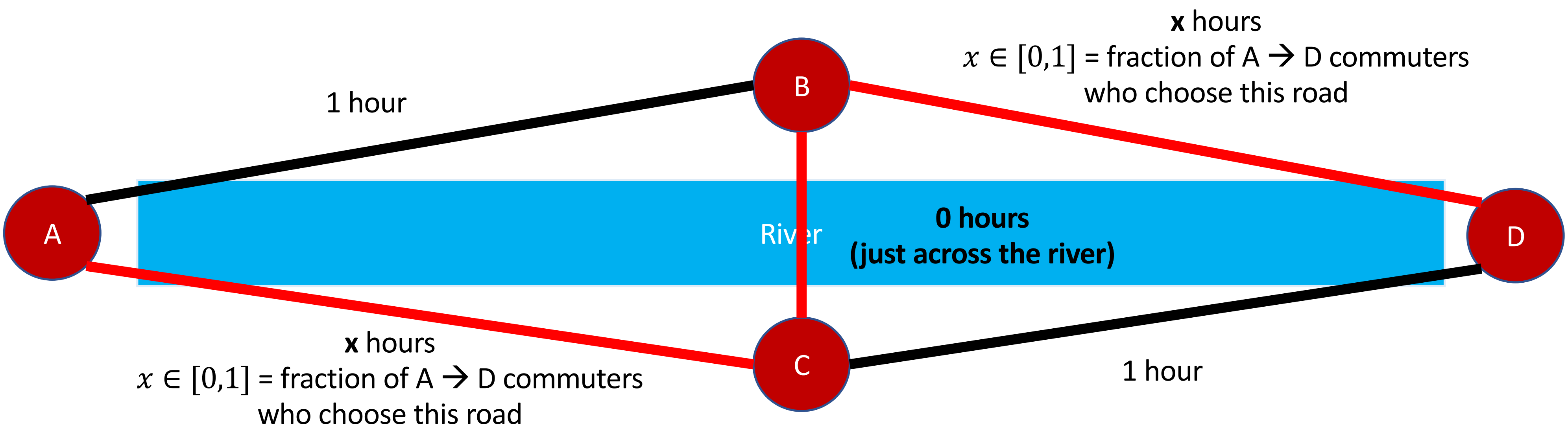
# Commuting Network




Before: average commute time is **1.5 hours**

- Half the commuters will go one way, half will go the other way
- Both $x = \frac{1}{2}$

# Commuting Network

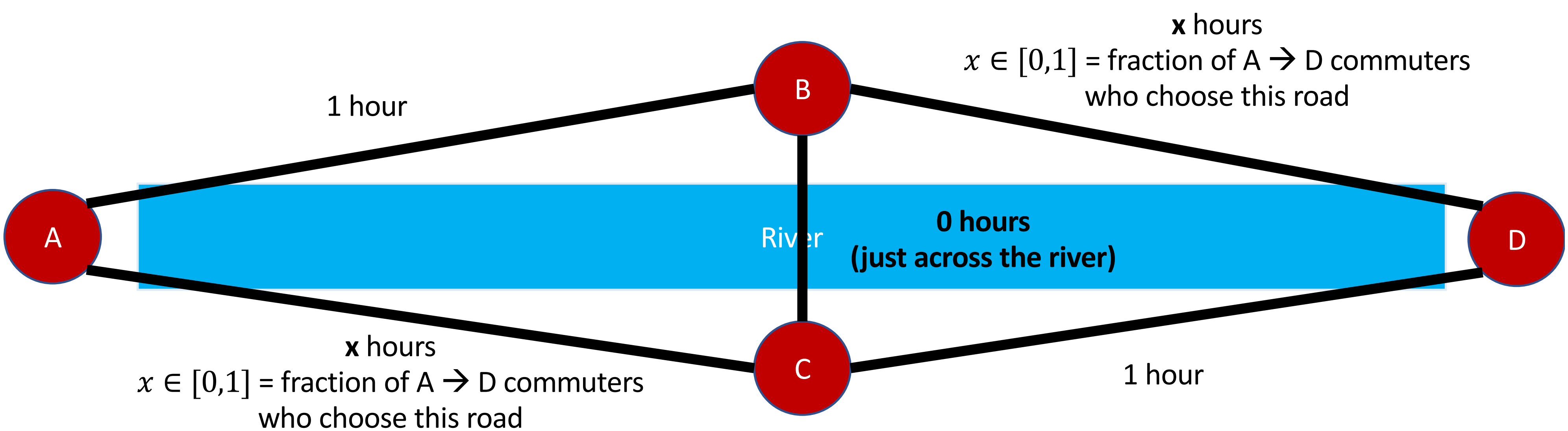



After: average commute time is **2 hours**

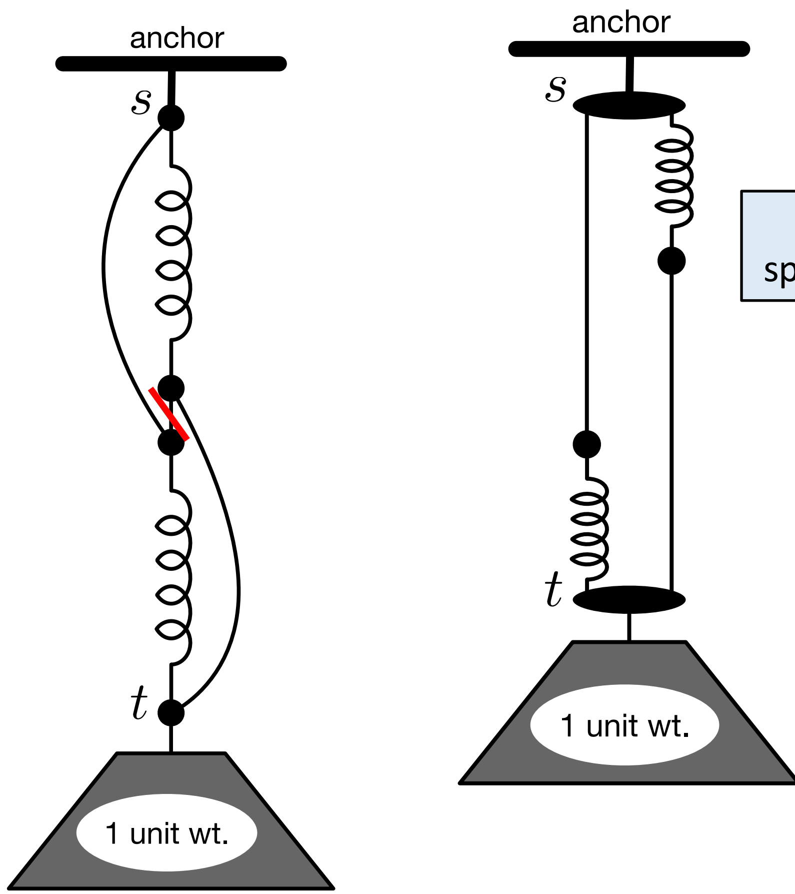
- Everyone uses both variable-length roads + the bridge

# Braess's Paradox

- Adding a new road **harmed travel time!** 1.5 hours → 2 hours



# Another Version of Braess's Paradox




**Cutting** a string in a system of strings and springs can **improve** its ability to lift a weight!