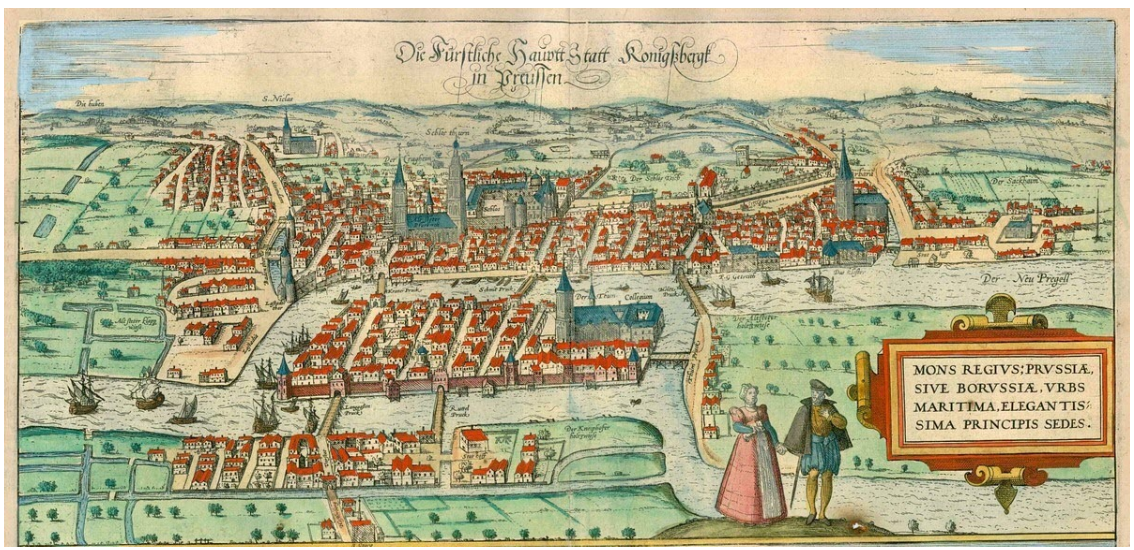
Lecture 19
Euler Cycles and Planar Graphs



Learning Objectives: Lec 19

After today's lecture (and the associated readings, discussion, & homework), you should know:

- Concepts: Euler cycles and Euler paths.
- **Eulerian Graphs** and how to identify them.

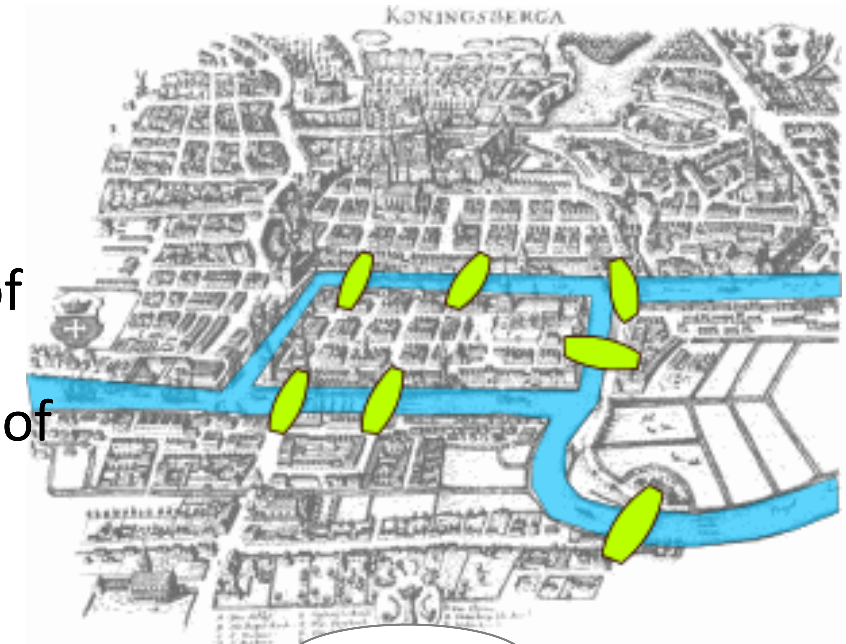
- Planar graphs and planar graph terminology: drawing, embedding, face.
- Euler's polyhedral formula.
- Using Euler's formula: every graph has a vertex of degree ≤ 5. Every planar graph can be 6-colored.

Outline

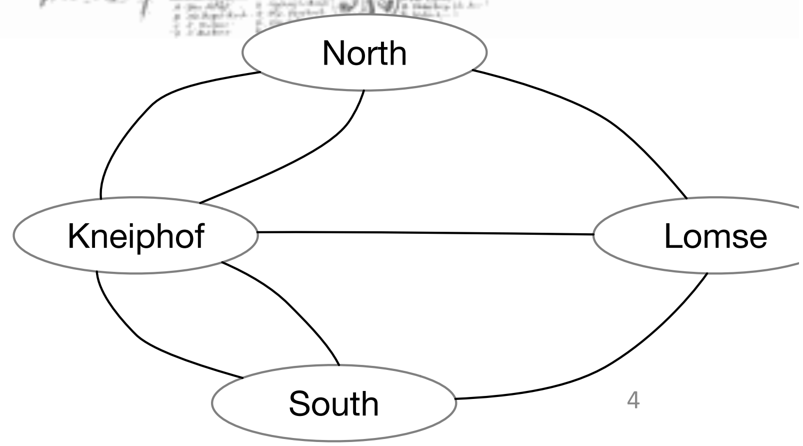
- **The Königsberg bridge puzzle**
- **Euler paths & cycles; Eulerian graphs.**
- Planar graphs
  - Terminology: drawing, partition into vertices, edges, and *faces*. *Triangulated* planar graphs.
- Euler's polyhedral formula
  - Application 1: the maximum number of edges in a planar graph
  - Application 2: every planar graph is 6-colorable.

Königsberg bridge puzzle

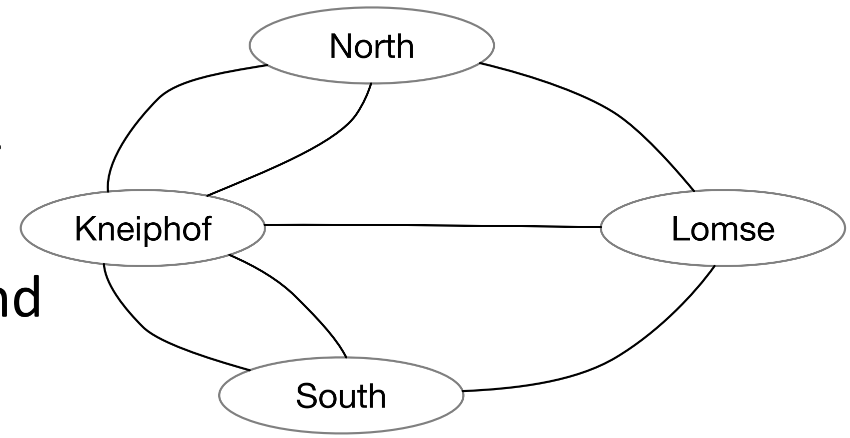
- Graph theory begins with the Königsberg (Kaliningrad) bridge puzzle.
- Q: seven bridges connect the islands of Kneiphof and Lomse to the city north and south of the Pregel river. Is it possible to take a walking tour of the city and cross each bridge ***exactly once***?



- <u>**Euler's observation:**</u> The particular geometry of Königsberg is irrelevant. The only thing that matters is the ***graph*** of landmasses (vertices) and their bridges (edges).



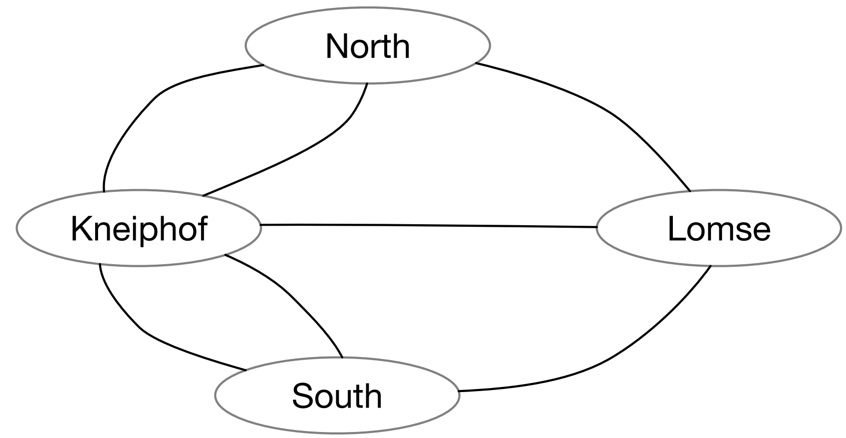
Königsberg bridge puzzle



- **Euler's observation:** The particular geometry of Königsberg is irrelevant. The only thing that matters is the ***graph*** of landmasses (vertices) and their bridges (edges).

- The problem is a little ambiguous. Must the "tour" of the city return to its starting position, or can you start and end in different places?
- **Variant 1:** Is there a ***path*** that uses every edge exactly once? **Euler path**
- **Variant 2:** Is there a ***cycle*** that uses every edge exactly once? **Euler cycle/circuit/tour**
- (Vertices can appear multiple times)

Königsberg bridge puzzle



- **Variant 1:** Is there a path that uses every edge exactly once? [Euler path]
- **Variant 2:** Is there a cycle that uses every edge exactly once? [Euler cycle/circuit/tour]

- **A:** There is no Euler path in the Königsberg graph.
- Proof.
  - Suppose there were such a path that began at s and ended at t.
  - For every other vertex $u \notin \{s, t\}$, the path must enter and exit u an equal number of times, each time using a different edge, so $\deg(u)$ must be ***even***.
  - In other words, there can be at most 2 vertices with odd degree (namely s and t).
  - However, $\deg(north) = \deg(Lomse) = \deg(South) = 3$ are all odd, a contradiction.

Which graphs have Euler cycles/paths?

- ***Necessary*** conditions:
  - **Euler path:** there can be at most 2 vertices with odd degree.
  - **Euler cycle:** all vertices must have even degree.
  - **Both Euler path/cycle:** the graph must be connected.
- What are the ***sufficient*** conditions?
- **Euler's Theorem:**

  (1) If G is a connected graph or multigraph and all vertices have even degree, G contains an Euler cycle.

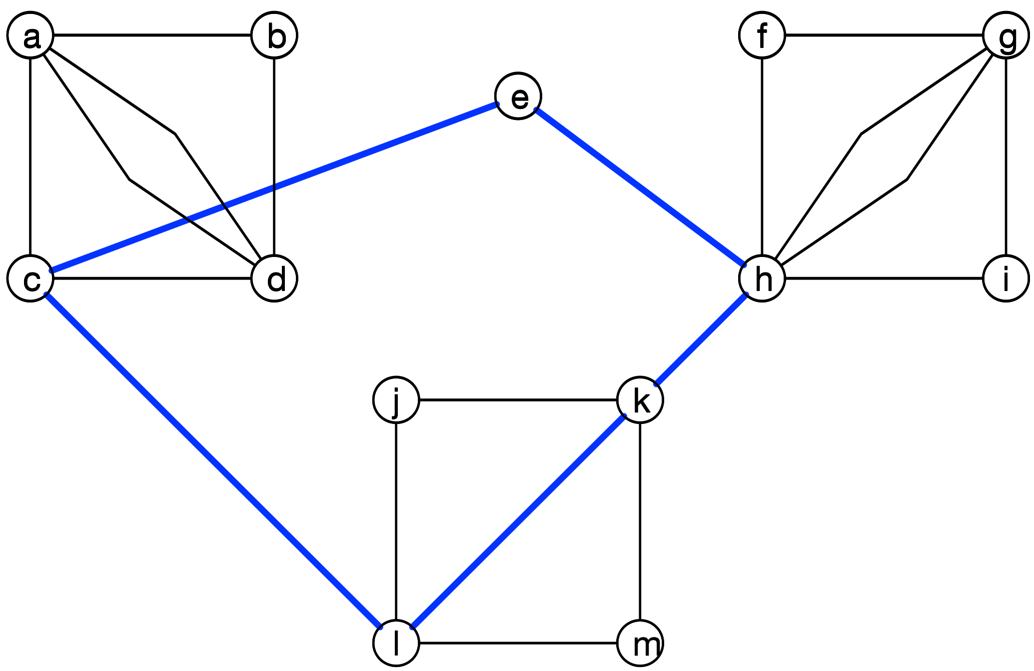
  (2) If G is a connected graph or multigraph and at most 2 vertices have odd degree, G contains an Euler path.
- **Warmup proof:** (1) implies (2).
  - So we only need to prove (1).

The Proof

- Step 1. Find ***<u>some</u>*** cycle (in a graph in which all degrees are even)
  - Generate a sequence $v_0, e_0, v_1, e_1, \dots, e_{k-1}, v_k$ as follows.
    - Let v_0 be an arbitrary starting vertex.
    - Once v_i is known, let $e_i \notin \{e_1, \dots, e_{i-1}\}$ that is incident to v_i and hasn't been used before. Let v_{i+1} be the other endpoint of e_i.
    - Append e_i, v_{i+1} to the sequence and repeat.
  - At some point it will be impossible to find such an edge e_k and the sequence ends at v_k.
  - <u>Claim:</u> $v_0 = v_k$ (the edges form a cycle).
    - Proof: Suppose $v_0 \neq v_k$. All edges incident to v_k appear in $\{e_0, \dots, e_{k-1}\}$, but the number of times the path enters v_k is one more than the number of times it exits v_k, so $\deg(v_k)$ is odd, a contradiction.
- Let $C = (V_C, E_C)$ be the graph where $V_C = \{v_0, \dots, v_k\}$ and $E_C = \{e_0, \dots, e_{k-1}\}$.
- **Q:** What can we say about the degrees in the graph $G' = (V, E - E_C)$?

- **<u>Euler's Theorem:</u>**

  If G is a connected graph or multigraph and all vertices have even degree, G contains an Euler cycle.



- The graph $G = (V, E)$.
- The cycle $C = (V_C, E_C)$.
- The graph $G' = (V, E - E_C)$.

- The graph G' has fewer edges than G.
- Can we apply the inductive hypothesis to G'?

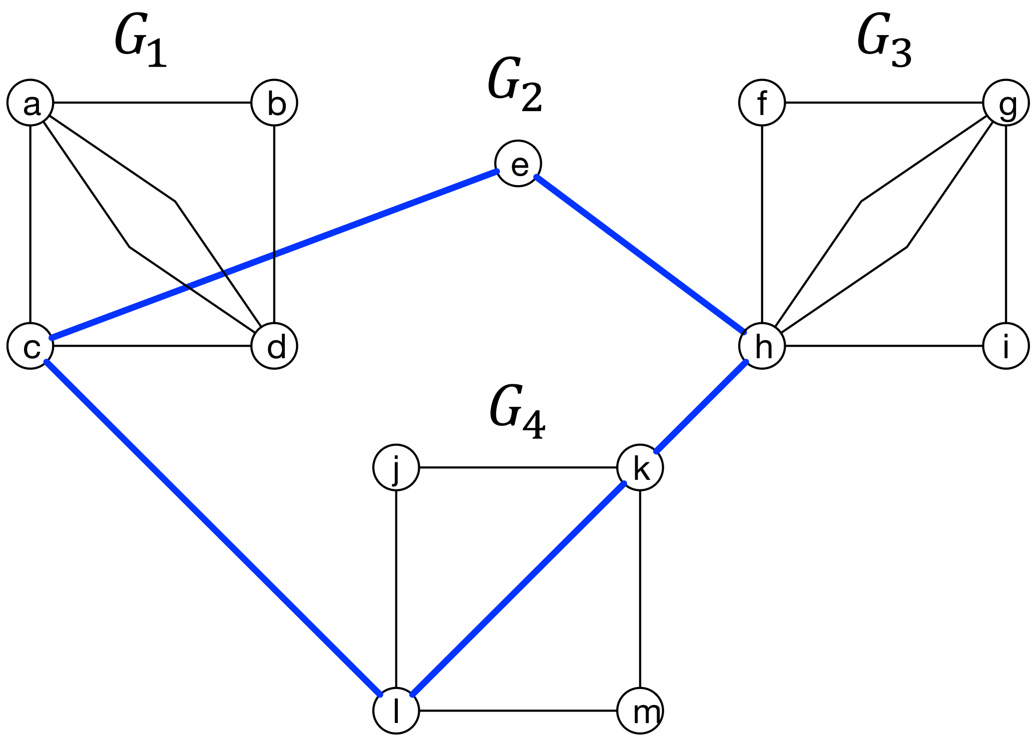
- **<u>Euler's Theorem:</u>**

  If G is a connected graph or multigraph and all vertices have even degree, G contains an Euler cycle.

- <u>Proof, by induction.</u>
  - Let $G = (V, E)$ be a connected graphs whose vertices have even degrees.
  - <u>Base Case:</u> If $|E| = 0$ then $|V| = 1$ and G has an Euler cycle (the empty cycle).
  - <u>Inductive Case:</u> Assume the claim holds for all graphs with less than $|E|$ edges.
  - Step 1: Find any cycle $C = (V_C, E_C)$.
  - Step 2: Form the graph $G' = (V, E - E_C)$. G' may not be connected. Let $G_1, G_2, \ldots, G_k$ be the graphs of the connected components of G'.
  - Step 3: By the inductive hypothesis, each G_i has an Euler cycle C_i.
  - Step 4: Because G is connected, C and each C_i share some vertex, say v_i.
  - Step 5. Form an Euler cycle for G by taking C and "splicing" C_i into it after an occurrence of v_i.

- **Euler's Theorem:**

  If G is a connected graph or multigraph and all vertices have even degree, G contains an Euler cycle.



- Cycles (circularly ordered)
- C: (e, h, k, l, c, e)

- C_1 : (c, a, b, d, a, d, c)
- C_2 : (e)
- C_3 : (h, g, i, h, f, g, h)
- C_4 : (k, m, l, j, k)

- Spliced cycle: $(e, h, g, i, h, f, g, h, k, m, l, j, k, l, c, a, b, d, a, d, c, e)$

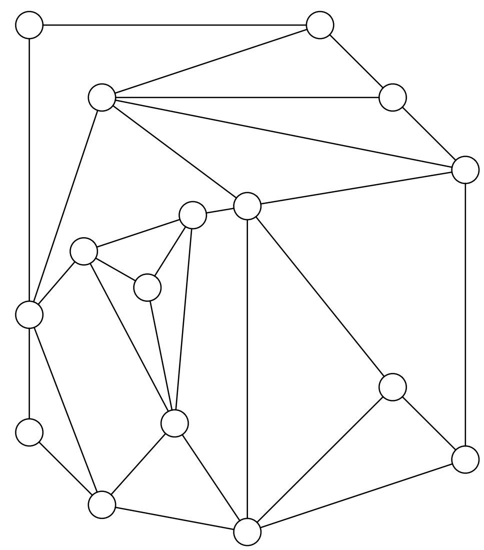
Outline

- The Königsberg bridge puzzle
- Euler paths & cycles; Eulerian graphs
- **Planar graphs**
  - **Terminology: drawing, partition into vertices, edges, and *faces*. *Triangulated* planar graphs.**
- Euler's polyhedral formula
  - Application 1: the maximum number of edges in a planar graph
  - Application 2: every planar graph is 6-colorable.

Planar Graphs

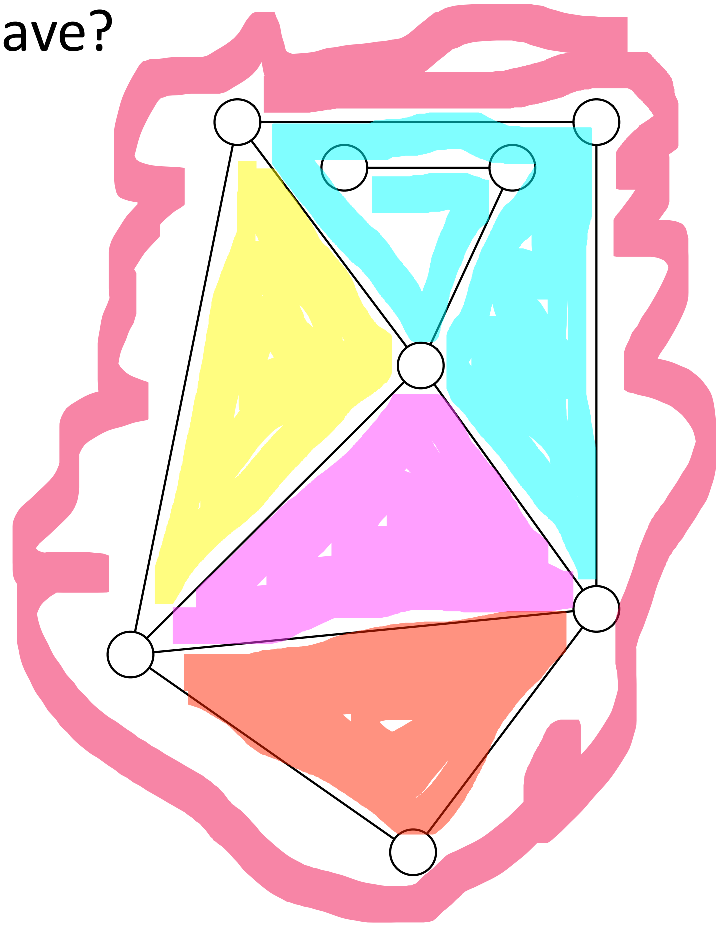
- ***Breaking news!*** You live on the surface of a sphere.
- …and you may be interested in graphs that can be drawn "nicely" on a sphere.

- ***Defn.*** A ***planar*** graph is one that can be drawn ~~*in the plane*~~ (or ***on a sphere***) so that edges do not intersect (except at vertices).
- ***Defn.*** A ***plane*** graph is one that is drawn in the plane.
  - The points of the plane are partitioned into three types:
  - ***Vertices*** (points)
  - ***Edges*** (curved lines connecting two vertices)
  - ***Faces*** (connected regions of the plane after removing vertices & edges).



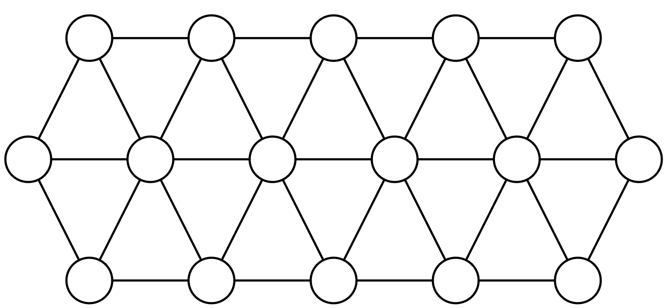
Planar Graphs

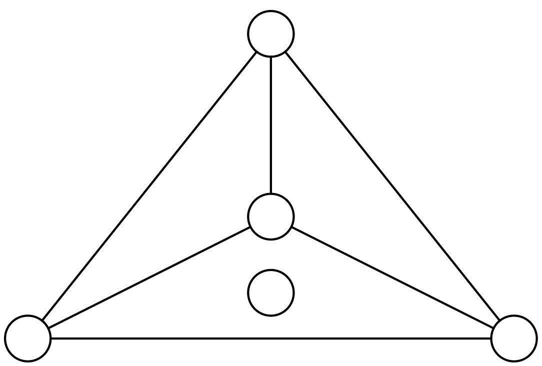
- How many faces does this plane graph have?
  - (a) 3
  - (b) 4
  - (c) 5
  - (d) 6
  - (e) faces?

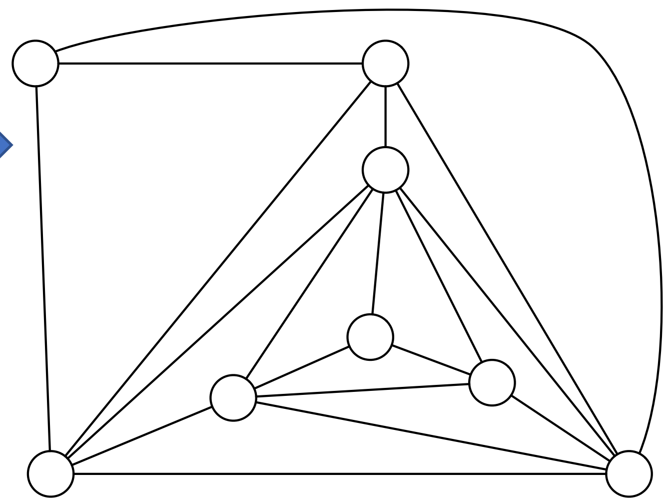


Planar Graphs — Triangulation

- ***Defn.*** A plane graph is ***triangulated*** if all faces are bounded by 3 edges and 3 vertices.
- How many of the following three graphs are triangulated?
  - (a) 0
  - (b) 1
  - (c) 2
  - (d) 3







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- **Euler's polyhedral formula**
  - Application 1: the maximum number of edges in a planar graph
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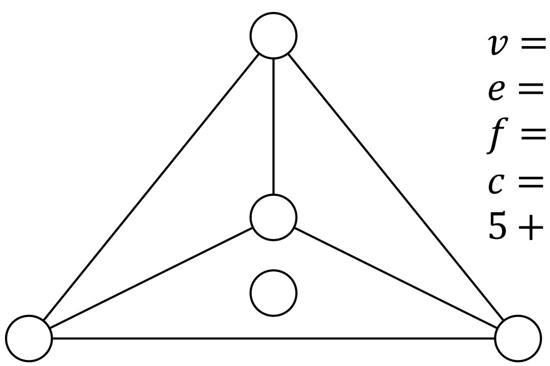
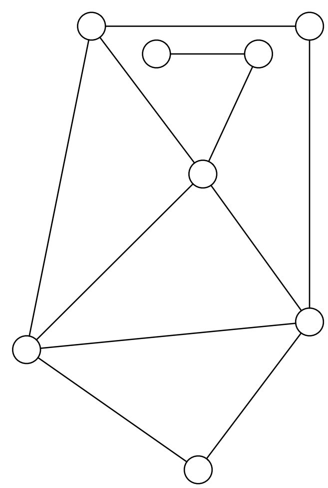
Euler’s Polyhedral Formula

- **<u>Theorem.</u>** (Euler) Suppose G is a plane graph with
  - v vertices,
  - e edges,
  - f faces, and
  - c connected components.
  - Then $v + f - e - c = 1$.



$v = 5$,
$e = 6$,
$f = 4$,
$c = 2$,
$5 + 4 - 6 - 2 = 1$.



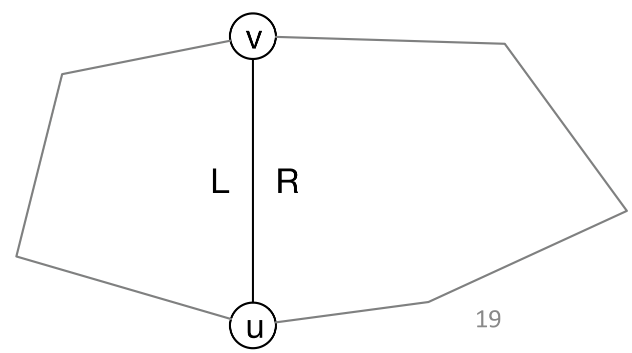
$v = 8$,
$e = 11$,
$f = 5$,
$c = 1$,
$8 + 5 - 11 - 1 = 1$.

Euler's Polyhedral Formula

- **<u>Theorem.</u>** (Euler) Suppose $G = (V, E)$ is a plane graph with v vertices, e edges, f faces, and, c connected components. Then $v + f - e - c = 1$.
- **<u>Proof by induction.</u>** (What should we do induction over?)
  - Let $G = (V, E)$ be an arbitrary plane graph.
  - <u>Base Case:</u> If $|E| = 0$ then each vertex is in a separate connected component.
    - $v = |V|$
    - $c = |V|$
    - $f = 1$
    - $v + f - e - c = |V| + 1 - 0 - |V| = 1.$

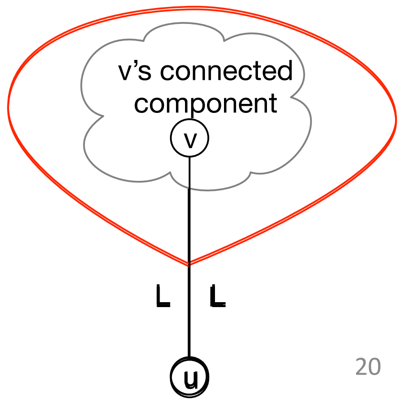
Euler's Polyhedral Formula

- **Theorem.** (Euler) Suppose $G = (V, E)$ is a plane graph with v vertices, e edges, f faces, and c connected components. Then $v + f - e - c = 1$.
- **Proof by induction.** (What should we do induction over?)
  - Let $G = (V, E)$ be an arbitrary plane graph.
  - Base Case: If $|E| = 0$ then each vertex is in a separate connected component.
  - Inductive Case: Assume the claim holds for all graphs with less than $|E|$ edges.
    - Pick an arbitrary edge $\{u, v\}$. Let L and R be the faces to the left and right of $\{u, v\}$.
    - Let $G' = (V, E - \{\{u, v\}\})$ be the plane graph, with parameters v', e', f', c'.
    - By the inductive hypothesis, $v' + f' - e' - c' = 1$.
    - Case 1. $L \neq R$
      - In G', $L \cup R$ is part of one face. $f' = f - 1, c' = c$.
      - $v + f - e - c = v' + (f' + 1) - (e' + 1) - c' = 1$.



Euler's Polyhedral Formula

- **Theorem.** (Euler) Suppose $G = (V, E)$ is a plane graph with v vertices, e edges, f faces, and, c connected components. Then $v + f - e - c = 1$.
- **Proof by induction.** (What should we do induction over?)
  - Let $G = (V, E)$ be an arbitrary plane graph.
  - Base Case: If $|E| = 0$ then each vertex is in a separate connected component.
  - Inductive Case: Assume the claim holds for all graphs with less than $|E|$ edges.
    - Pick an arbitrary edge $\{u, v\}$. Let L and R be the faces to the left and right of $\{u, v\}$.
    - Let $G' = (V, E - \{\{u, v\}\})$ be the plane graph, with parameters v', e', f', c'.
    - Case 2. $L = R$
      - Then there is a ***cycle*** that intersects only $\{u, v\}$.
      - u and v are in distinct connected components.
        - $f' = f, c' = c + 1$.
      - $v + f - e - c = v' + f' - (e' + 1) - (c' + 1) = 1$.



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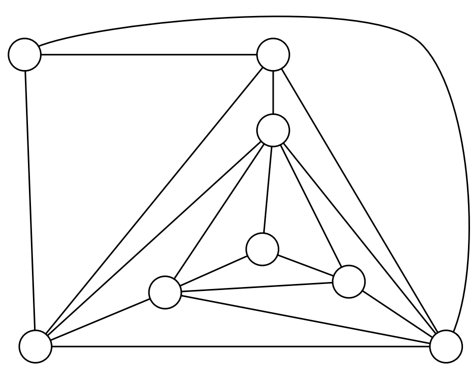
Number of Edges in a Planar Graph

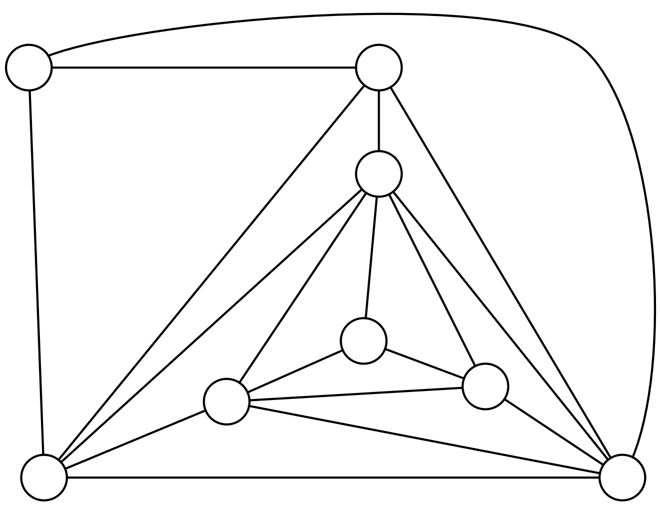
- What's the ***maximum*** number of edges in a ***planar graph*** with v *vertices*?
- Use Euler's polyhedral formula!
  - If $G = (V, E)$ has the maximum number of edges, it must be ***triangulated***.
  - In a triangulated graph, every face is bounded by 3 edges, and every edge bounds 2 faces.
  - $3f = 2e$ and $c = 1$.
  - $v + f - e - c = 1$ (Euler's polyhedral formula)
  - $v + \left(\frac{2e}{3}\right) - e = 2$
  - $v - 2 = \frac{e}{3}$
  - $e = 3v - 6$

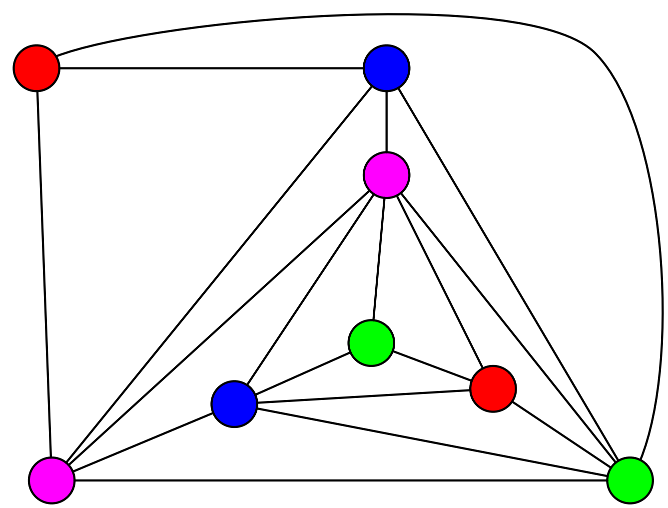


$v = 8,$
$e = 18$

The 4-Coloring Theorem for Planar Graphs

- **<u>Theorem.</u>** (Appel, Haken 1976) Every planar graph $G = (V, E)$ can be ***4-colored***. There exists a function $f: V \to \{1,2,3,4\}$ such that $\{u, v\} \in E \to f(u) \neq f(v)$.
- **<u>The Proof.</u>**
  - … is very very very long. We will not prove it.
  - … but you <u>*can*</u> 4-color any planar graph!





The ~~4~~ 6-Coloring Theorem for Planar Graphs

- **Theorem.** Every planar graph $G = (V, E)$ can be ***6-colored***. There exists a function $f : V \rightarrow \{1,2,3,4,5,6\}$ such that $\{u, v\} \in E \rightarrow f(u) \neq f(v)$.

- What do the ***Handshake Theorem*** and ***Euler's Polyhedral Formula*** say about the ***average*** vertex degree in a planar graph?

$$\sum_{v} \deg(v) = 2|E| \qquad \text{and} \qquad |E| \leq 3|V| - 6$$

What does this imply about the *minimum* degree?

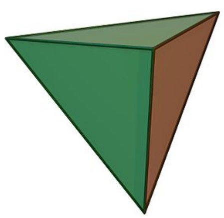
- The average degree is $\frac{\sum_v \deg(v)}{|V|} = \frac{2|E|}{|V|} \leq \frac{2(3|V|-6)}{|V|} < 6.$

The ~~4~~ 6-Coloring Theorem for Planar Graphs

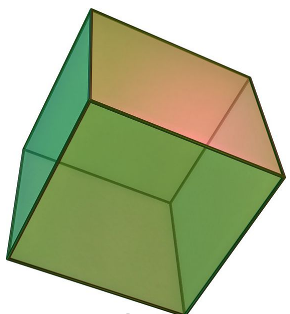
- **Theorem.** Every planar graph $G = (V, E)$ can be ***6-colored***. There exists a function $f: V \to \{1,2,3,4,5,6\}$ such that $\{u, v\} \in E \to f(u) \neq f(v)$.
- **Proof by induction.**
  - Let $G = (V, E)$ be an arbitrary planar graph.
  - Base Cases: If $|V| \leq 6$ then the claim is clearly true (give every vertex a different color).
  - Inductive Case: Assume the claim is true for all planar graphs with less than $|V|$ vertices.
  - Pick a vertex v with $\deg(v) \leq 5$. (**Handshake** + **Polyhedral** formula imply avg degree < 6.)
  - $G' = (V', E')$ is G with v and all incident edges removed. By inductive hypothesis, G' has a 6-coloring $f': V' \to \{1, \ldots, 6\}$.
  - By pigeonhole principle, $\{1,2,3,4,5,6\} - \{f(u) \mid \{u, v\} \in E\} \neq \emptyset$.
  - Create the coloring $f: V \to \{1,2,3,4,5,6\}$
    Cardinality 6 Cardinality ≤ 5
    - Set $f(u) = f'(u)$ for $u \neq v$,
    - and $f(v)$ to be any color left in $\{1,2,3,4,5,6\} - \{f(u) \mid \{u, v\} \in E\}$.

Euler's polyhedral formula for dice.

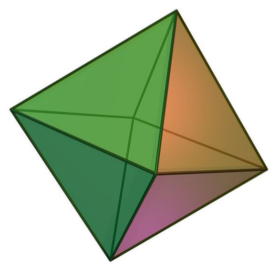
- ***(3D) Polyhedra*** graphs can be drawn on spheres. Some examples:



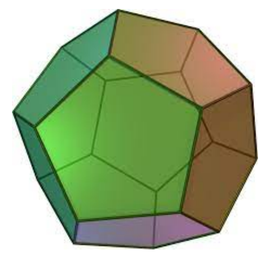
$v = 4,$
$e = 6,$
$f = 4.$
$4 + 4 - 6 - 1 = 1$



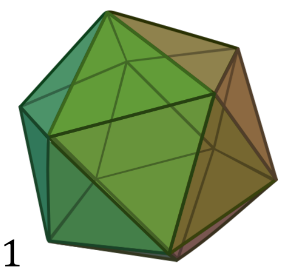
$v = 8,$
$e = 12,$
$f = 6.$
$8 + 6 - 12 - 1 = 1$



$v = 6,$
$e = 12,$
$f = 8.$
$6 + 8 - 12 - 1 = 1$



$v = 20,$
$e = 30,$
$f = 12.$
$20 + 12 - 30 - 1 = 1$



$v = 12,$
$e = 30,$
$f = 20.$
$12 + 20 - 30 - 1 = 1$