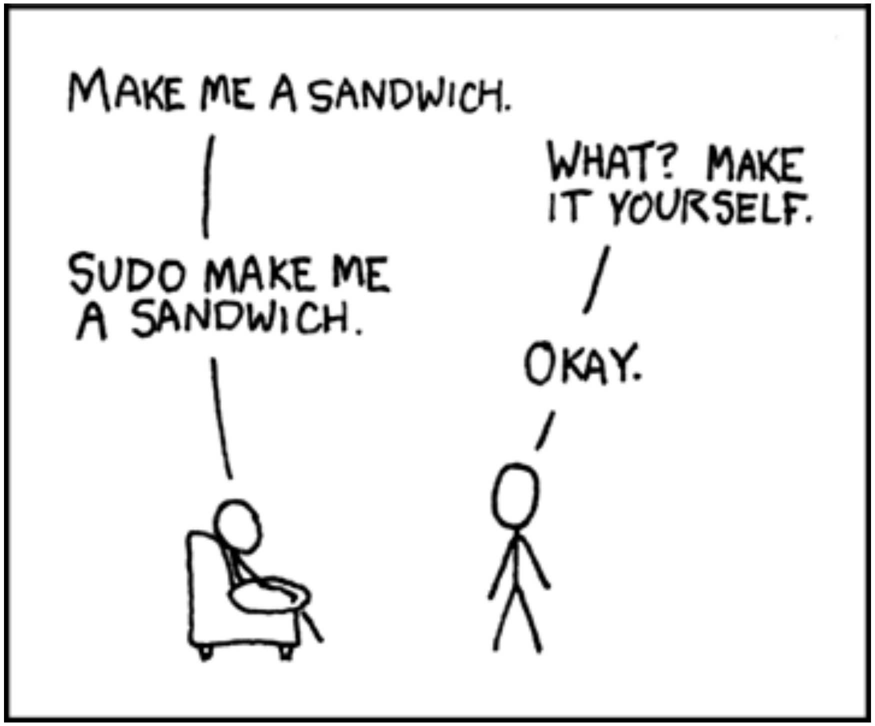
Closure, Equivalence Relations & Partial Orders

UM EECS 203 Winter 2022

Lecture 17



Exam 2 Planning

	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
This week				Practice Exams released	Today HW 6 due		
3/13-3/19				Practice Exam Solutions released Special Topics Review 7-9pm	HW 7 due	Study Session/ Test Strategies Session 1-3pm	Review session #1: 1-4 pm
	Review session #2: 1-4 pm			**3/23: Exam 2: 7-9pm**			

- Exam 2 covers: Induction, Pigeonhole Principle, Functions, Relations (HW 5 – 7)
- Please note: the material is inherently cumulative
- See “Exam 2 Information” doc on Canvas *(coming soon!)* and related Piazza post

Learning Objectives

After today's lecture (and this week's readings, discussion & homework), you should be able to:

- **Technical vocab:** reflexive closure, symmetric closure, transitive closure, equivalence relation, equivalence class, partial order

- Compute reflexive closure and symmetric closure
- Compose relations & compose a relation with itself to produce higher powers
- Compute transitive closure (for small relations)

- Determine whether a relation is an equivalence relation
- Find the equivalence classes for an equivalence relation
- Determine whether a relation is a partial order
- Determine whether a relation is a total order

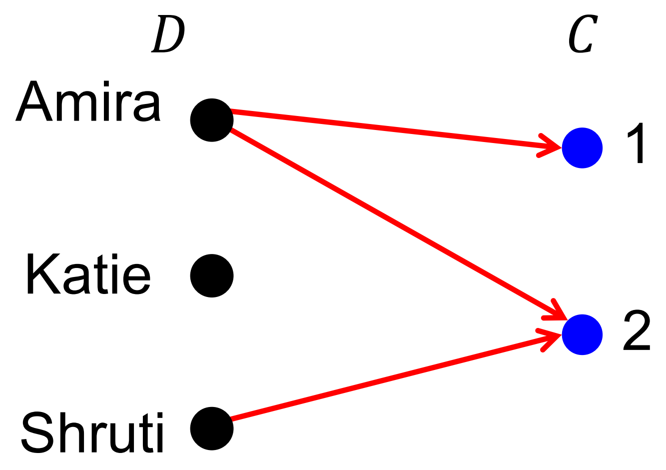
Outline

- Closure
  - Reflexive closure, symmetric closure
  - (Detour: composing relations & powering relations)
  - Transitive closure
- Equivalence relations
  - Definition and examples
  - Equivalence Classes
  - Partitions
- Partial Orderings
  - Definition and examples
- Bonus/Optional
  - Hasse Diagrams

Review: Relations

A ***binary relation*** R between sets D and C is a subset of $D \times C$.
* An element of D can be related to *zero or more* elements of C.

the attends-discussion-section **relation**



Ways to represent a relation R:

1. **Graph** representation
2. **Set** representation $R = \{$ (Amira,1), (Amira,2), (Shruti,2) $\}$
3. **Matrix** representation

R	1	2
Amira	1	1
Katie	0	0
Shruti	0	1

can be used interchangeably:
- "x relates to y (by R)"
- "xRy"
- "(x,y) ∈ R
- "R(x,y)"

Review: Properties of Relations

- $R \subseteq V \times V$ is a binary relation (over the set V)
  - $(x, y) \in R$, xRy, $R(x, y)$ all mean the same thing

- Some properties
  - ***REFLEXIVE*** $\forall x \quad xRx$
  - ***SYMMETRIC*** $\forall x,y \; xRy \leftrightarrow yRx$
  - ***ANTISYMMETRIC*** $\forall x,y \; (xRy \wedge yRx) \rightarrow x=y$
  - ***TRANSITIVE*** $\forall x,y,z \; (xRy \wedge yRz) \rightarrow xRz$

- Other properties
  - ***IRREFLEXIVE*** $\forall x \; \neg(xRx)$
    - Q. Is "irreflexive" the same as "not reflexive"? No!
  - ***ASYMMETRIC*** $\forall x,y \; xRy \rightarrow \neg(yRx)$
    - Q. Is "asymmetric" the same as "not symmetric"? No!

Review: Which properties do these satisfy?

Relation	Domain	Ref.	Sym.	Antisym	Trans.
$<$	$\mathbf{R}$	✗	✗	✓	✓
$\subseteq$	sets	✓	✗	✓	✓
$=$	$\mathbf{Z}$	✓	✓	✓	✓
"has a non-empty intersection with"	sets	✗	✓	✗	✗
"is a sister of"	people	✗	✗	✗	✗
"is a sibling of"	people	✗	✓	✗	✗
"is a descendant of"	people	✗	✗	F → ✓	✓
"is divisible by"	$\mathbf{Z}$	✓	✗	✗	✓

Outline

- **Closure**
  - **Reflexive closure, symmetric closure**
  - (Detour: composing relations & powering relations)
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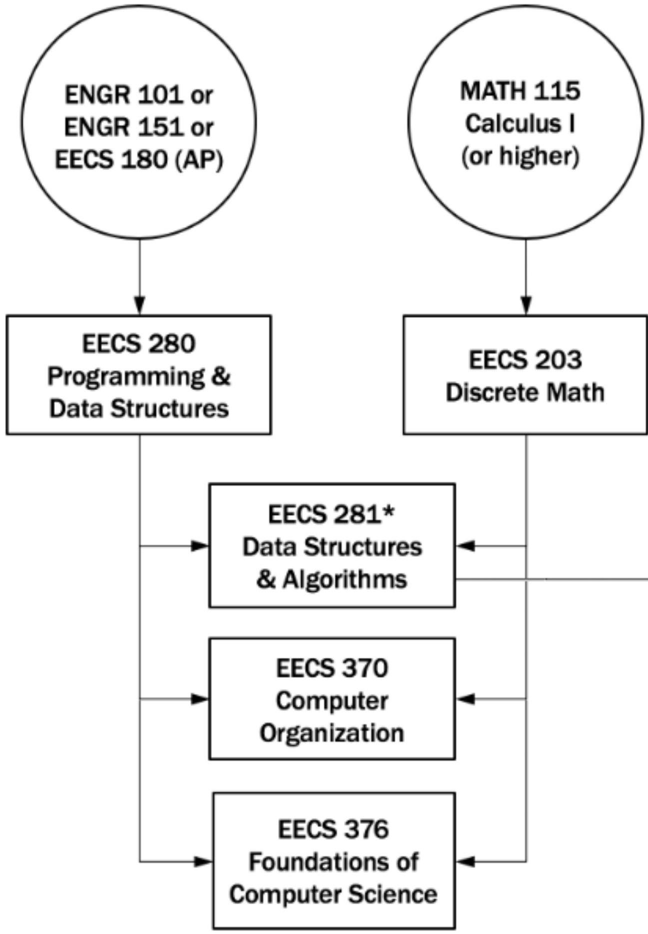
Example Application: Prereq Structure



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What courses must a student take before EECS481?

How can we make sure a course isn’t accidentally a prereq for itself?

Closures

The **closure** of a relation **R** with respect to a property (reflexive, transitive, symmetric, etc.) is the smallest relation S containing R that has that property.

THINK:

- Start with **R**.
- Try to find a **violation** of the property in question.
- Fix the violation by **increasing** the relation.
- Once there are no more violations, the final relation is **S**.

Closures

Handout

The **closure** of a relation **R** with respect to a property (reflexive, transitive, symmetric, etc.) is:

the smallest relation containing R that has that property.

THINK:

- Start with **R**.
- Try to find a **violation** of the property in question.
- Fix the violation by increasing the relation.

add edges

- Once there are no more violations, the final relation is **S**.

Reflexive Closure

The **closure** of a relation **R** with respect to a property (reflexive, transitive, symmetric, etc.) is the smallest relation S containing R that has that property.

THINK:

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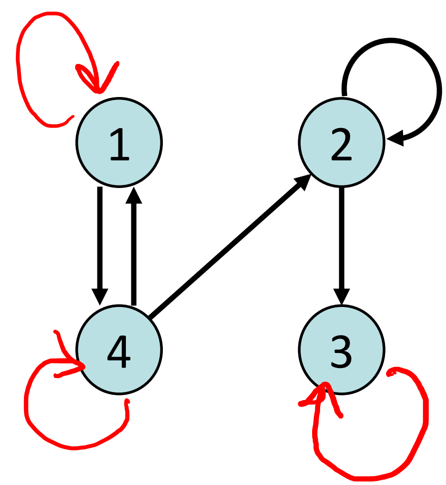
- Fix the violation by **increasing** the relation.
- Once there are no more violations, the final relation is **S**.

R represented by:

find the **reflexive** closure



Reflexive Closure

The **closure** of a relation **R** with respect to a property (reflexive, transitive, symmetric, etc.) is the smallest relation S containing R that has that property.

THINK:

- Start with **R**.
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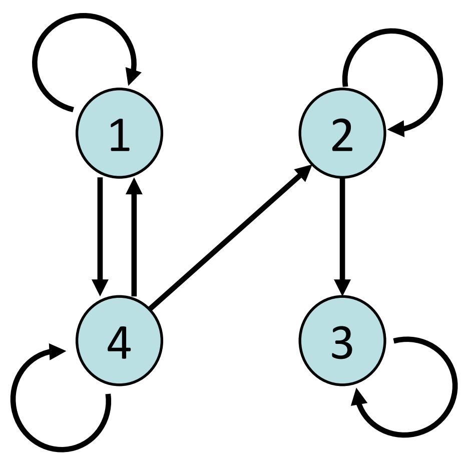
R represented by:

find the **reflexive** closure



= Representation of the reflexive closure of **R**.

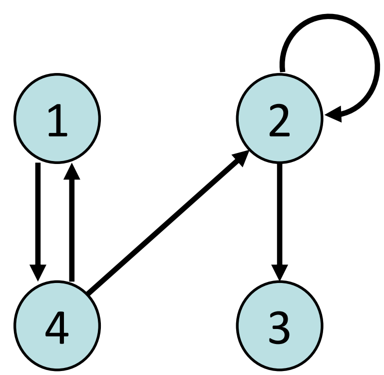
Symmetric Closure

The **closure** of a relation **R** with respect to a property (reflexive, transitive, symmetric, etc.) is the smallest relation S containing R that has that property.

THINK:

- Start with **R**.
- Try to find a **violation** of the property in question.
- Fix the violation by **increasing** the relation.
- Once there are no more violations, the final relation is **S**.

R represented by:

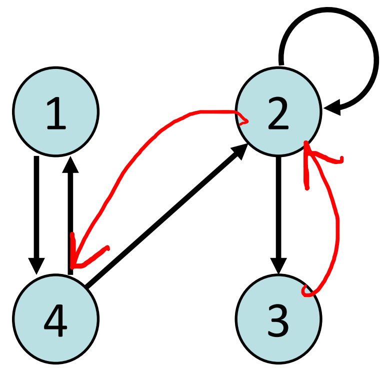


find the **symmetric** closure

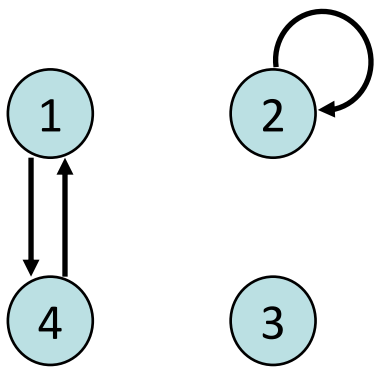
Symmetric closure

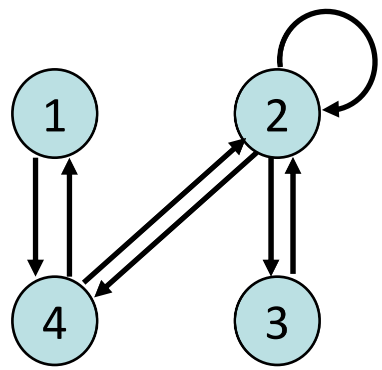
Slido.com #203203

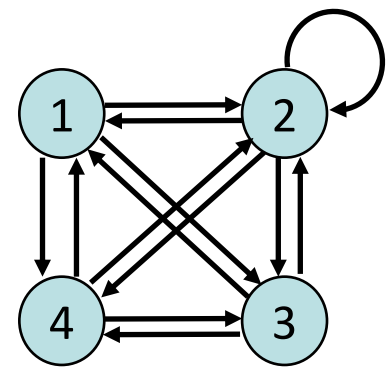
Which of these is the matrix representation of the **symmetric** closure of **R**?



R is ***SYMMETRIC***
For all x,y:
if xRy, then yRx

A:


B:


C:


(D) More than one of these represents a symmetric closure of R.

(E) Not sure

Outline

- Closure
  - Reflexive closure, symmetric closure
  - **(Detour: composing relations & powering relations)**
  - **Transitive closure**
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Transitive Closures

Goal: Given a relation **R**, compute the transitive closure $\boxed{\mathbf{R}^*}$:

$$\mathbf{R} \cup \mathbf{R}^2 \cup \mathbf{R}^3 \cup \ldots = \mathbf{R}^*$$

First, what are $\mathbf{R}^2$, $\mathbf{R}^3$, etc., and how do we find them?

Answer:

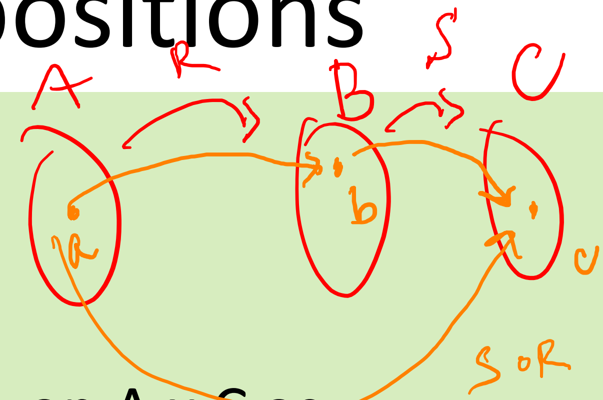
Relation Compositions:

$\mathbf{R}^2 = \mathbf{R} \circ \mathbf{R}$

$\mathbf{R}^3 = \mathbf{R} \circ \mathbf{R} \circ \mathbf{R}$

…

Relation Compositions



Given sets A, B, C, and

1) relation **R** on A x B
2) relation **S** on B x C

We can define ***composite relation*** $S \circ R$ on A x C as:

For a∈A, c∈C: a($S \circ R$)c iff ∃b∈B (a**R**b ∧ b**S**c)

Example: A = UM students, B = courses, C = profs

1) R defined as: aRb means “student a taking class b“
2) S defined as: bSc means ”class b taught by prof c“

Then:

3) a(S∘R)c means “student a taking a class taught by prof c”

When **A=B=C**, we can have **R∘R = R^2.**

Relation Compositions

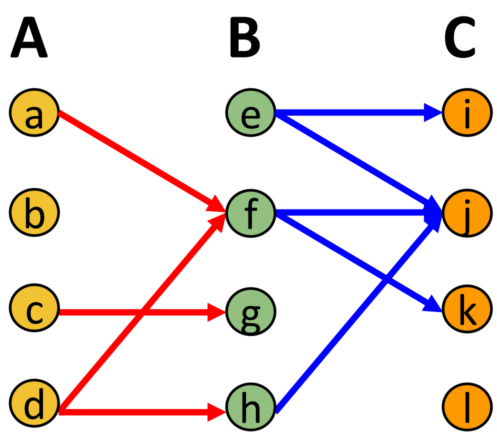
Given sets A, B, C, and

1) relation **R** on A x B
2) relation **S** on B x C

We can define ***composite relation*** **S∘R** on A x C as:

For $a \in A$, $c \in C$: $a(S \circ R)c$ iff $\exists b \in B\ (aRb \land bSc)$

Bipartite graph representation



Relation Compositions

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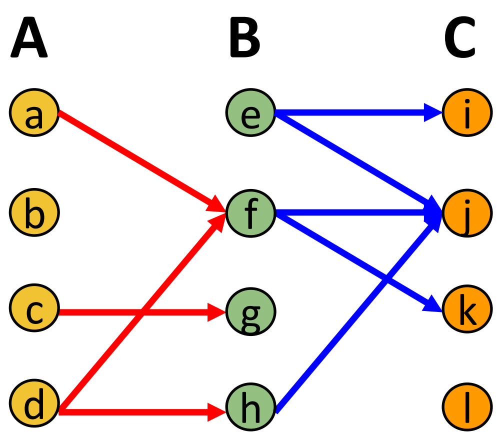
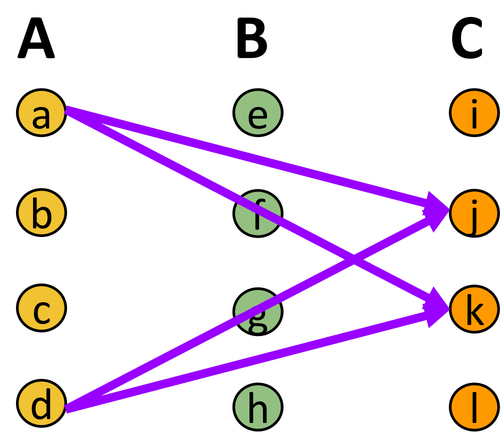
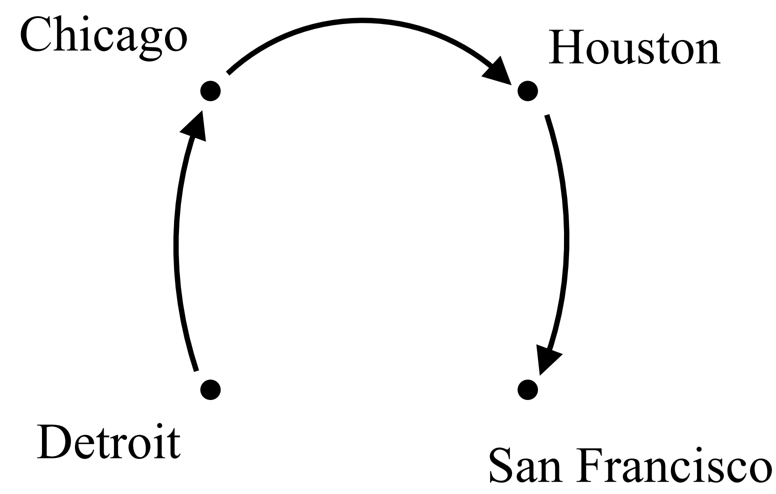
Bipartite graph representation





Powering Relations



Domain = Co-domain = Cities

$x\boldsymbol{F}y$ if possible to fly from x to y

Define $F^2 = F \circ F$

$x\boldsymbol{F}^2 z \equiv \exists y \, (x\boldsymbol{F}y \wedge \, y\boldsymbol{F}z)$

Q. What does it mean if $x\boldsymbol{F}^2 z$?

You can fly from x to z in exactly 2 flights

Q. What would it mean if $\boldsymbol{F}^2 \subseteq \boldsymbol{F}$?

$$\forall x, z \, [(\exists y \, xFy \wedge yFz) \rightarrow xFz]$$

$$\forall x, z, y \, [(xFy \wedge yFz) \rightarrow xFz]$$

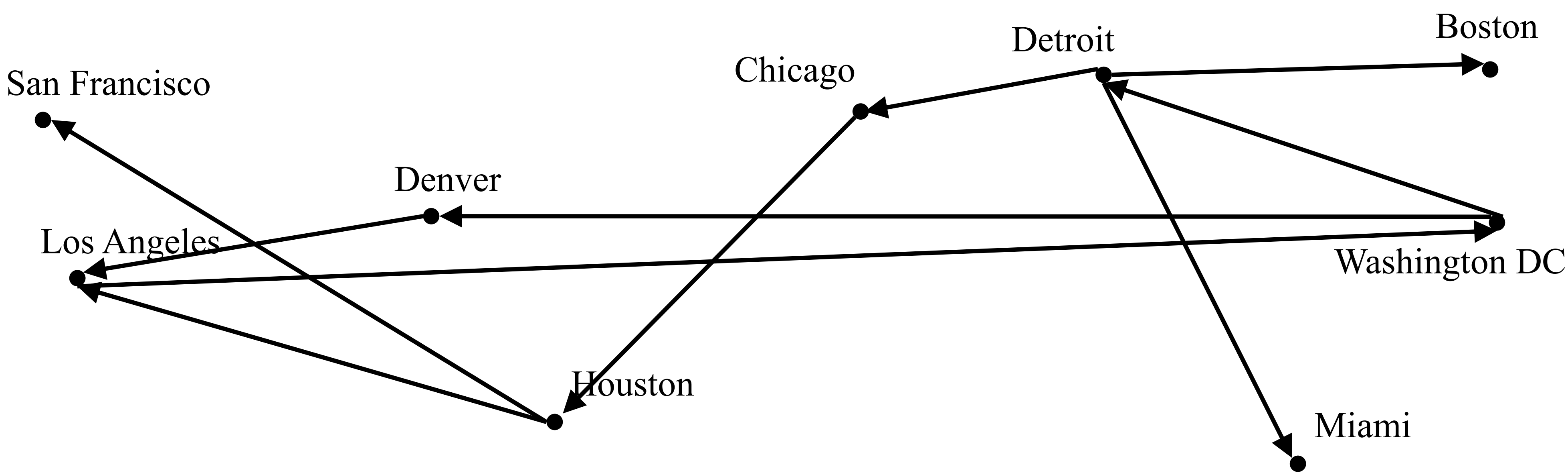
This means $\boldsymbol{F}$ is transitive!

Transitive Closures

Goal: Given a relation **R**, compute the **transitive closure** $\mathbf{R}^*$:

$$\mathbf{R}^* = \mathbf{R} \cup \mathbf{R}^2 \cup \mathbf{R}^3 \cup \ldots$$

$aR^k b \equiv$ there is a path of length k from a to b

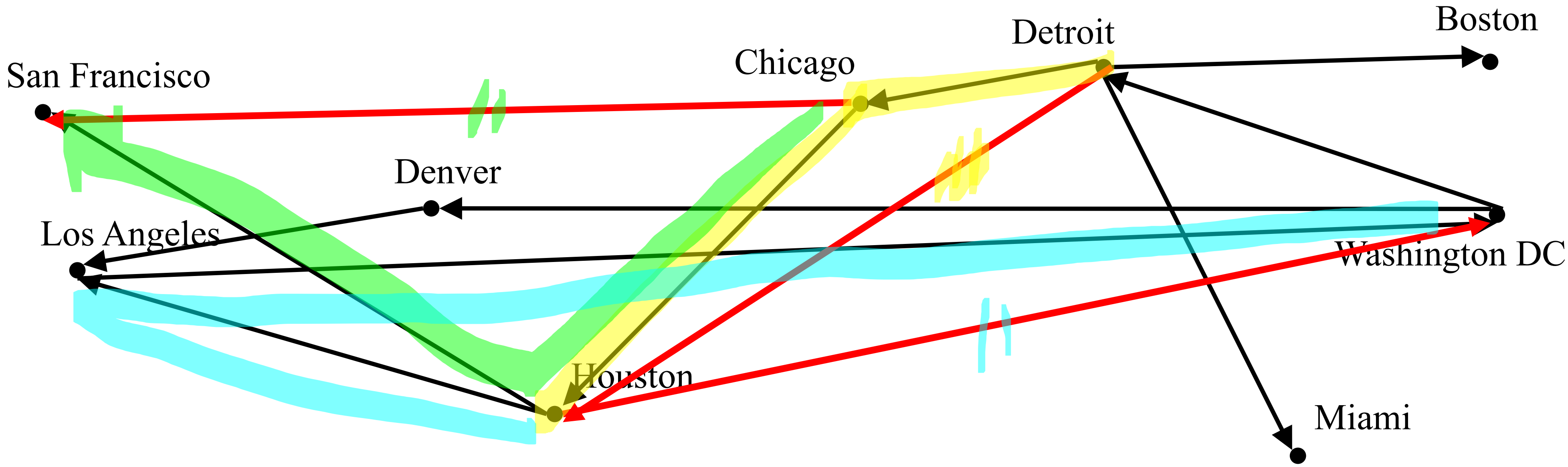


Transitive Closures

Goal: Given a relation **R**, compute the **transitive closure** $\mathbf{R}^*$:

$$\mathbf{R}^* = \mathbf{R} \cup \mathbf{R}^2 \cup \mathbf{R}^3 \cup \ldots$$

Violations of transitivity = directed paths of length 2 w/o edge from start to finish. Fixing these violations = **unioning** $\boldsymbol{R^2}$ to the relation



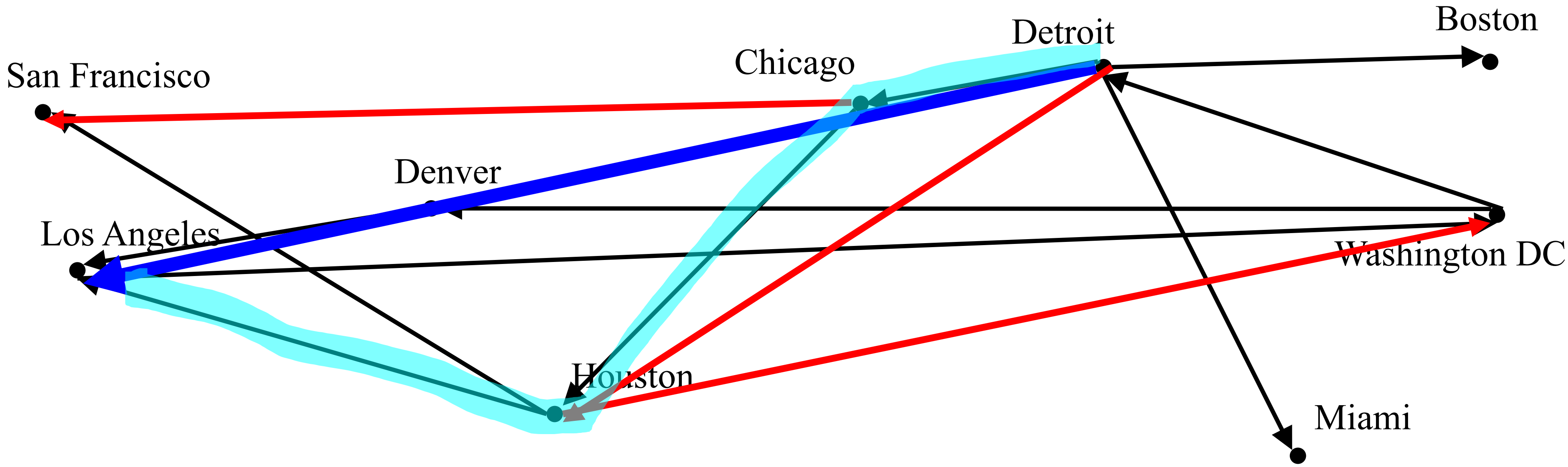
(not all arrows in R^2 drawn here)

Transitive Closures

Goal: Given a relation **R**, compute the **transitive closure** $\mathbf{R}^*$:

$$\mathbf{R}^* = \mathbf{R} \cup \mathbf{R}^2 \cup \mathbf{R}^3 \cup \ldots$$

New violations of transitivity = directed paths of length 3 w/o edge from start to finish. Now fix these violations = **unioning** $\boldsymbol{R^3}$ to the relation

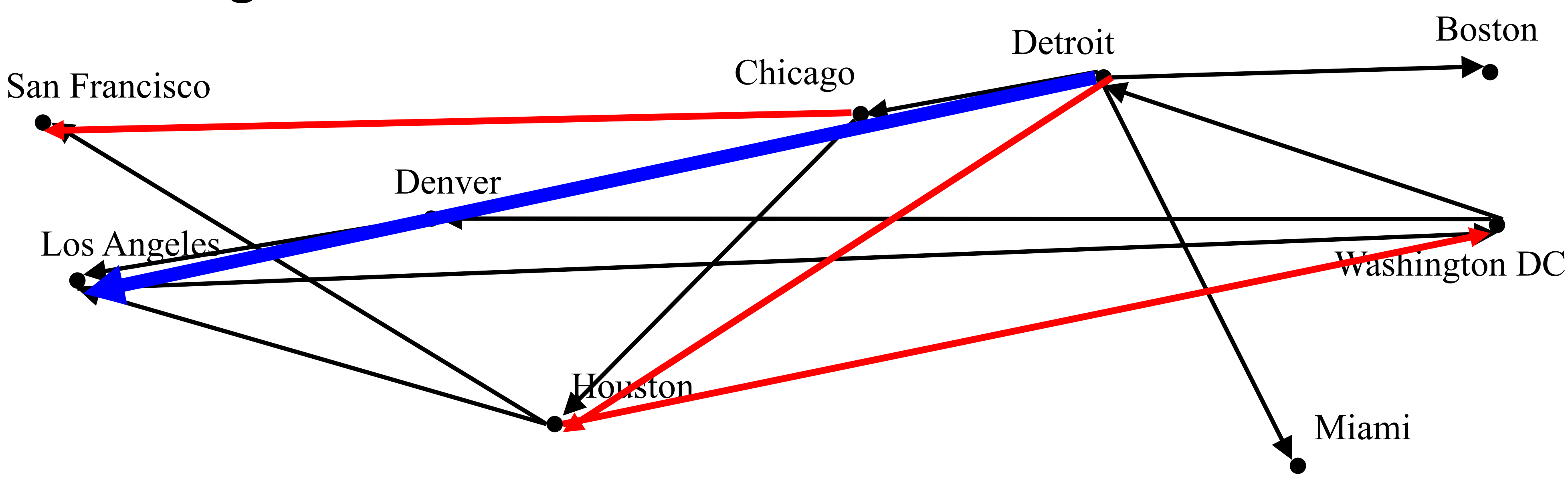


Transitive Closures

Goal: Given a relation **R**, compute the **transitive closure** R^*:

$$R^* = R \cup R^2 \cup R^3 \cup \ldots$$

New violations of transitivity = directed paths of length 3 w/o edge from start to finish. Now fix these violations = **unioning** R^3 to the relation



Then R^4, etc.

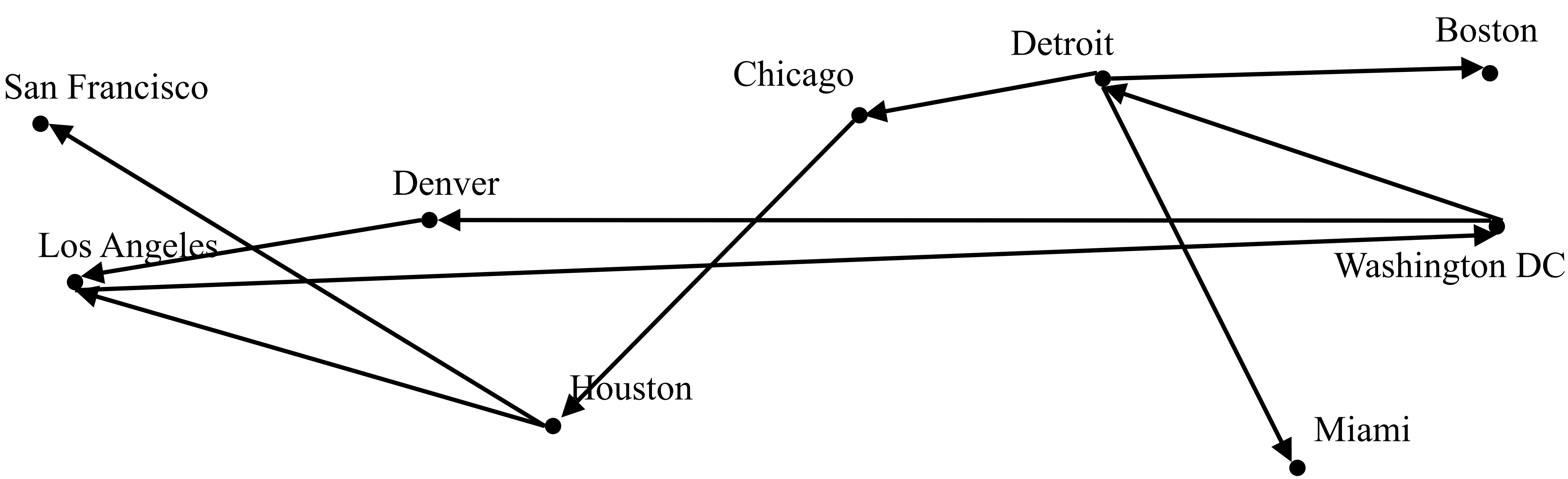
Transitive Closures

Goal: Given a relation **R**, compute the **transitive closure** $\mathbf{R}^*$:

$$R^* = R \cup R^2 \cup R^3 \cup \ldots$$

In a graph representation, $\mathbf{R}^*$ is the **reachability** relation:

a $\mathbf{R}^*$ **b** means there is a path from a to b (of any length ≥ 1).



Closures

The **closure** of a relation $R \subseteq A \times A$
with respect to a property (reflexive, transitive, symmetric, etc.)
is the smallest relation S containing R (i.e., $R \subseteq S$)
that has that property.

all self loops.

The **reflexive closure** of R is $R \cup \Delta$, where

$$\Delta = \{(a,a) \mid a \in A\}$$

The **symmetric closure** of R is $R \cup R^{-1}$ where

$$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$

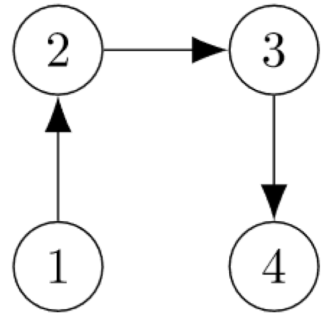
The **transitive closure** of R is R*, where

$$R^* = R^1 \cup R^2 \cup R^3 \cup \cdots R^n \cdots = \bigcup_{i=1}^{\infty} R^i$$

Transitive Closure

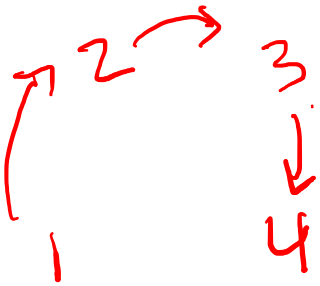
Handout

Given a relation R, the transitive closure R^* is
$$R^* = R \cup R^2 \cup R^3 \cup R^4$$

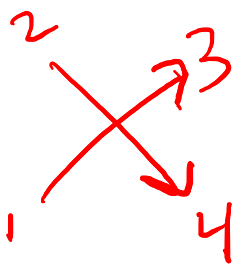


R $\qquad$ R^*

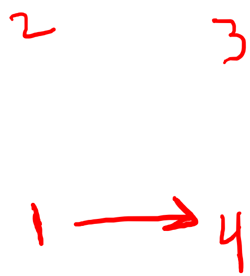
- $aR^kb \equiv$ there is a path of length k from a to b:



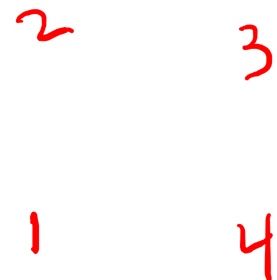
$R = R^1$



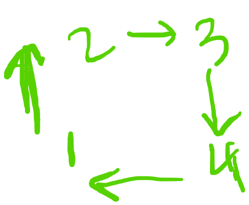
R^2



R^3

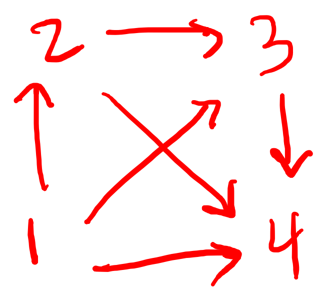


R^4 is empty

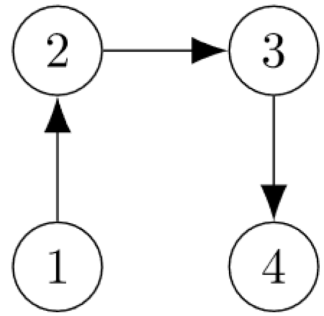


$$R^* = R^1 \cup R^2 \cup R^3 \cup R^4$$

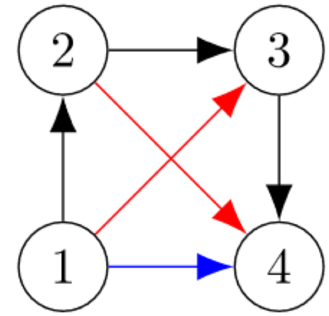
$R^* \Rightarrow$



Transitive Closure

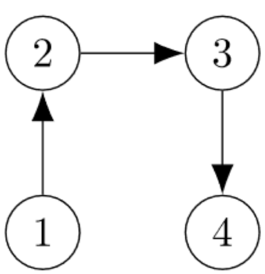


R

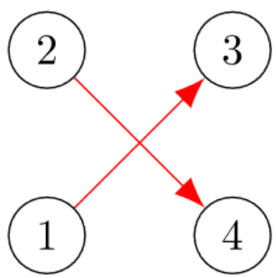


$R^* =$
$R^1 \cup R^2 \cup R^3 \cup R^4$

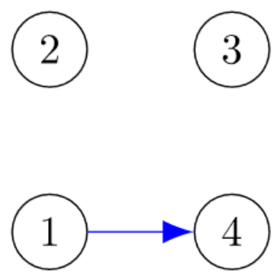
- To get the transitive closure, we need to add aR^*b if there is a path from a to b of *any* length, not just 2
- Let aR^kb if and only if there is a path of length k from a to b:



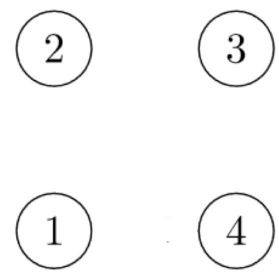
$R = R^1$



R^2

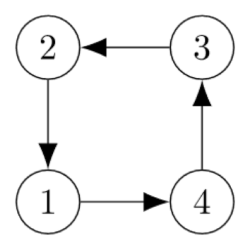


R^3



R^4

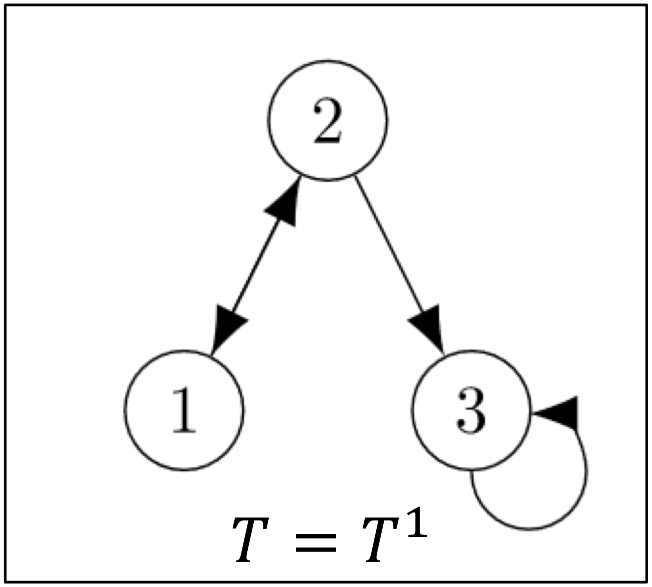
- Note we went up to R^4 because the longest path (without repeats) might be 4 steps long. For example, S=



- $1S^41$, so we need $1S^*1$

Exercise: Closure

Handout



a) Reflexive closure of T:

b) Symmetric closure of T:

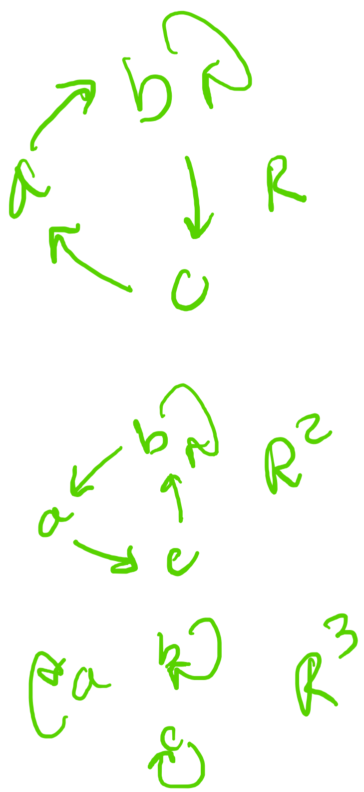
c) T^*, Transitive closure of T:

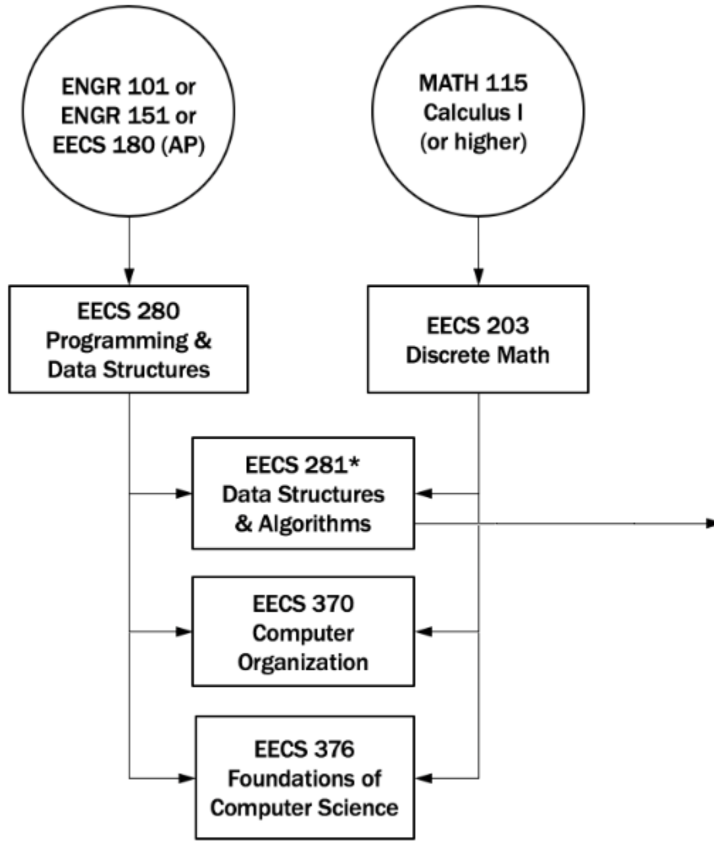
$\boldsymbol{T^2}$

$\boldsymbol{T^3}$

$\boldsymbol{T^*} =$ ________________

Example Application: Prereq Structure





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EECS 483	•
EECS 484	•
EECS 485	•
EECS 486	•
EECS 487	•
EECS 489	•
EECS 490	•
EECS 491	•

What courses must a student take before EECS481?

- Compute transitive closure, and find all courses c where Prereq(c,481)

How can we make sure a course isn't accidentally a prereq for itself?

- Transitive closure had better be irreflexive

Outline

- Closure
  - Reflexive closure, symmetric closure
  - (Detour: composing relations & powering relations)
  - Transitive closure
- **Equivalence relations** = R S T
  - **Definition and examples**
  - Equivalence Classes
  - Partitions
- Partial Orderings = R (AS) T
  - Definition and examples
- Bonus/Optional
  - Hasse Diagrams

Equivalence Relations

Equivalence Relation = RST

- R is an ***equivalence relation*** if it is
  - ***Reflexive*** Edges (1,1), (2,2), etc.
  - ***Symmetric*** (v,w) implies (w,v)
  - ***Transitive*** (v,w) and (w,x) imply (v,x)

Equivalence Relations

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  - ***Symmetric*** (v,w) implies (w,v)
  - ***Transitive*** (v,w) and (w,x) imply (v,x)
- Examples
  - x=y (equality over reals, naturals, etc.)
  - $a \equiv b\ (mod\ m)$ (modular equivalence)
  - $p \equiv q$ (logical equivalence)
    - E.g., $x \rightarrow (y \vee z) \equiv (\neg z) \rightarrow ((\neg y) \rightarrow x)$
  - $|A|=|B|$ (equality over the size of sets)
  - (a,b) “=“ (c,d) iff a=c and b=d (equality of pairs)

Outline

- Closure
  - Reflexive closure, symmetric closure
  - (Detour: composing relations & powering relations)
  - Transitive closure
- Equivalence relations
  - Definition and examples
  - **Equivalence Classes**
  - **Partitions**
- Partial Orderings
  - Definition and examples
- Bonus/Optional
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Equivalence Classes

Let R be an equivalence relation on set S

[R is reflexive, symmetric, transitive]

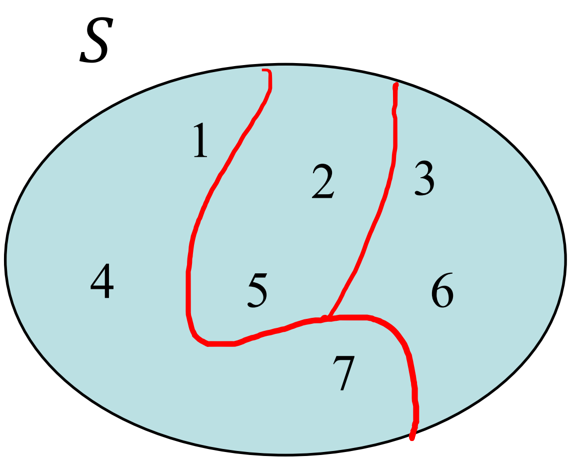
The ***equivalence class*** of $a \in S$ is denoted as $[a]_R$ and defined as

$$[a]_R = \{b\colon aRb\}$$

set of all elements that a relates to.

Equivalence Classes: $[a]_R = \{b: aRb\}$

- Let $R = \{(x, y) \mid x \equiv y \ (mod\ 3)\}$
- Domain: $S = \{1,2,3,4,5,6,7\}$

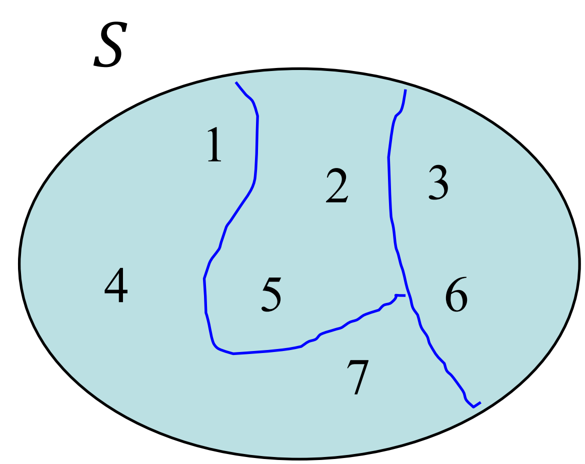


- $[1]_R =$ $\{1, 4, 7\}$
- $[2]_R =$ $\{2, 5\}$
- $[3]_R =$ $\{3, 6\}$
- $[4]_R =$ $\{1, 4, 7\} = [1]_R = [7]_R$
- $[5]_R =$ $\{2, 5\}$
- $[6]_R =$ $\{3, 6\}$
- $[7]_R =$ $\{1, 4, 7\}$

R partitions S into 3 equiv. classes.

Equivalence Classes: $[a]_R = \{b : aRb\}$

- Let $R = \{(x, y) |\ x \equiv y\ (mod\ 3)\ \}$
- Domain: $S = \{1,2,3,4,5,6,7\}$



  - $[1]_R = \{1, 4, 7\}$
  - $[2]_R = \{2, 5\}$
  - $[3]_R = \{3, 6\}$
  - $[4]_R = \{1, 4, 7\} = [1]_R$
  - $[5]_R = \{2, 5\} = [2]_R$
  - $[6]_R = \{3, 6\} = [3]_R$
  - $[7]_R = \{1, 4, 7\} = [1]_R$

R partitions S into 3 equivalence classes

Equivalence Relations

Handout

Consider a relation R over the set S.

R is an **equivalence relation** iff — RST

it is reflexive, symmetric, transitive

For equivalence relation R on set S, the **equivalence class** of $a \in S$, is

$$[a]_R = \{ b : aR_b \}$$

The equivalence classes of R form a partition of the set S.

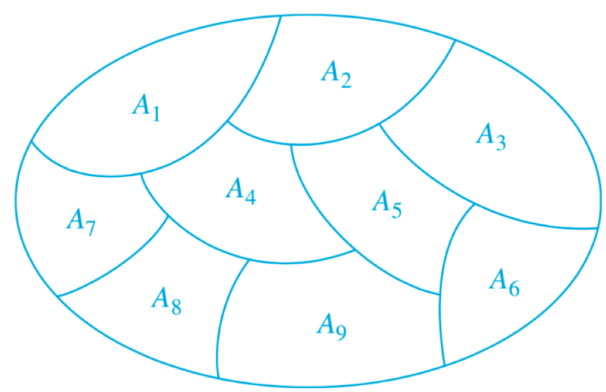
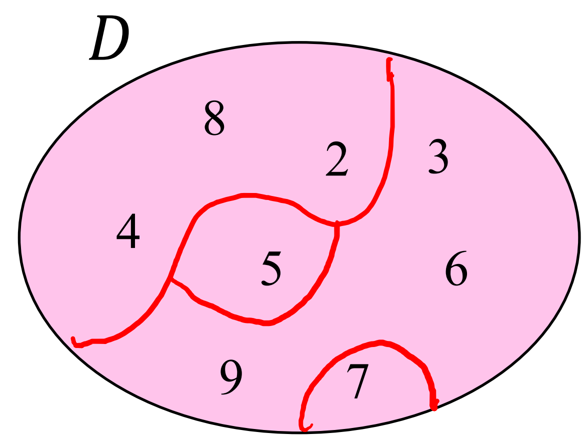


FIGURE 1 **A Partition of a Set.**

Equivalence Classes: $[a]_R = \{b : aRb\}$

Handout

- $R = \{(x, y) \mid x \text{ and } y \text{ have the same largest prime divisor}\}$
- Domain: $D = \{2,3,4,5,6,7,8,9\}$



1. Find the equivalence classes of R.
2. Draw the partition they create.

$[2]_R = \{2, 4, 8\}$

$[3]_R = \{3, 6, 9\}$

$[5]_R = \{5\}$

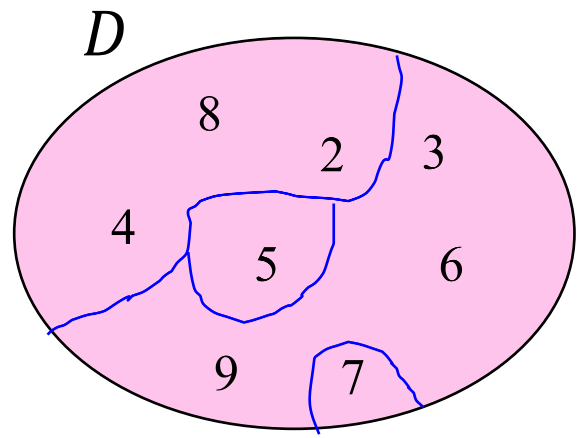
$[7]_R = \{7\}$

4 Equiv. classes.

Equivalence Classes: $[a]_R = \{b: aRb\}$

Handout

- $R = \{(x, y) |\ x \text{ and } y \text{ have the same largest prime divisor}\}$
- Domain: $D = \{2,3,4,5,6,7,8,9\}$



1. Find the equivalence classes of R.
2. Draw the partition they create.

R partitions D into 4 equivalence classes

- $[2]_R = \{2,4,8\} = [4]_R = [8]_R$
- $[3]_R = \{3,6,9\} = [3]_R = [9]_R$
- $[5]_R = \{5\}$
- $[7]_R = \{7\}$

Equivalence Classes

Handout

Lemma: Let R be an equivalence relation on S.
Then

R S T

1. If aRb, then $[a]_R$ = $[b]_R$
2. If not aRb, then $[a]_R \cap [b]_R$ = $\emptyset$

Proof of 1: $aRb \rightarrow ([a]_R = [b]_R)$

Suppose aRb

Let $x \in [a]_R$

$\Rightarrow aRx$ by defn of $[a]_R$

$\Rightarrow xRa$ by symmetry

$\Rightarrow xRb$ by transitivity: $xRa \wedge aR$

$\Rightarrow bRx$ by symmetry

So $x \in [b]_R$ by defn of $[b]_R$

Equivalence Classes

Lemma: Let R be an equivalence relation on S. Then

1. If aRb, then $[a]_R = [b]_R$
2. If not aRb, then $[a]_R \cap [b]_R = \emptyset$

Proof of 1: $aRb \rightarrow [a]_R = [b]_R$

Suppose aRb, and consider $x \in S$.

$x \in [a]_R \Rightarrow aRx$	Defn of $[a]_R$
$\Rightarrow xRa$	symmetry
$\Rightarrow xRb$	transitivity
$\Rightarrow bRx$	symmetry
$\Rightarrow x \in [b]_R$	Defn of $[b]_R$

Equivalence Classes

Lemma: Let R be an equivalence relation on S. Then

1. If aRb, then $[a]_R = [b]_R$
2. If not aRb, then $[a]_R \cap [b]_R = \emptyset$

Proof of 2: $\neg(aRb) \rightarrow ([a]_R \cap [b]_R = \emptyset)$

Suppose to the contrary that $\exists\, x \in [a]_R \cap [b]_R$

$$x \in [a]_R \wedge x \in [b]_R \Rightarrow aRx \text{ and } bRx$$
$$\Rightarrow aRx \text{ and } xRb$$
$$\Rightarrow aRb, \text{ contradicting } \neg(aRb)$$

Thus, $[a]_R$ and $[b]_R$ are either identical or disjoint.

and

$$\bigcup_{a \in S} [a]_R = S$$

Partitions

Given a set S, a **partition** of S is a collection of nonempty subsets of S, such that each element in S belongs to exactly one subset.

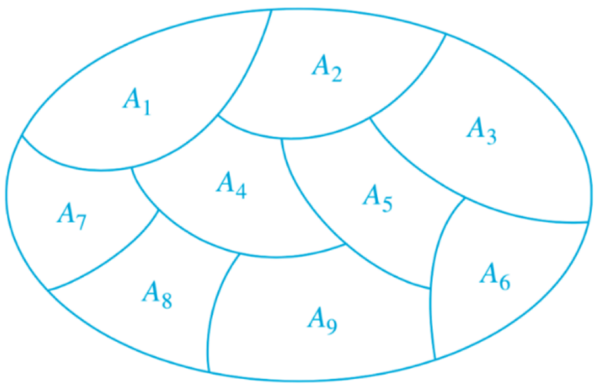


FIGURE 1 A Partition of a Set.

Examples

- $\boldsymbol{S} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$
  - One partition: $\{\{\mathbf{1}, \mathbf{2}\}, \{\mathbf{3}\}, \{\mathbf{4}, \mathbf{5}\}\}$
  - Another partition: $\{\{\mathbf{1}\}, \{\mathbf{2}\}, \{\mathbf{345}\}\}$

- $\boldsymbol{S}$ = **Integers**
  - One partition: {Even Integers, Odd Integers}
  - Another partition: {positive integers, 0, negative integers}
  - (Many more partitions exist)

Partitions

Given a set S, a **partition** of S is a collection of nonempty subsets of S, such that each element in S belongs to exactly one subset.

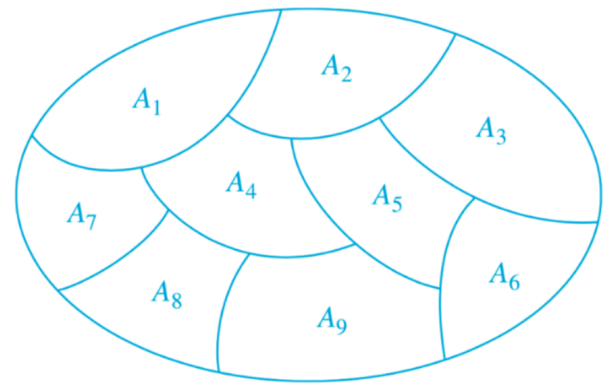


FIGURE 1 **A Partition of a Set.**

- The individual subsets in a partition are called **blocks** or **parts**.
- Each block, A_i, is an equivalence class.
  - And thus each A_i is nonempty.
- So S is the **union** of **disjoint** equivalence classes of R.
  - $S = \bigcup A_i$
  - $A_i \cap A_j = \emptyset$ for all $i \neq j$

Equivalence Relation/Classes Example

- Domain=Codomain= S = set of pairs of numbers on two dice.
  - $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
    $(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
    $(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$
    $(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$
    $(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$
    $(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$|S| = 36$

- Relation="SameTotal": $(a,b) SameTotal (c,d)$ iff $a + b = c + d$
  - For example, $(1,4) SameTotal (3,2)$ because $1 + 4 = 3 + 2$
- Q: Is "SameTotal" an Equivalence Relation?
  - Reflexive? Symmetric? Transitive?

$\forall (a,b)$

$(a,b) \, R \, (a,b)$?

- A: Yes

Equivalence Relation/Classes Example

- Domain=Codomain= S = set of pairs of numbers on dice. $|S| = 36$
- Relation="SameTotal": $(a,b) SameTotal (c,d)$ iff $a+b=c+d$
  - Q: Is it an Equivalence Relation? Reflexive? Symmetric? Transitive?
  - A: Yes
- The Equivalence Classes: → 11 of them
  - $[(1,1)]_{\text{SameTotal}}$ = {(1,1)}
  - $[(2,1)]_{\text{SameTotal}}$ = {(1,2), (2,1)}
  - $[(2,2)]_{\text{SameTotal}}$ = {(1,3), (2,2), (3,1)}
  - $[(1,4)]_{\text{SameTotal}}$ = {(1,4), (2,3), (3,2), (4,1)}
  - $[(3,3)]_{\text{SameTotal}}$ = {(1,5), (2,4), (3,3), (4,2), (5,1)}
  - $[(4,3)]_{\text{SameTotal}}$ = {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}
  - $[(6,2)]_{\text{SameTotal}}$ = {(2,6), (3,5), (4,4), (5,3), (6,2)}
  - $[(5,4)]_{\text{SameTotal}}$ = {(3,6), (4,5), (5,4), (6,3)}
  - $[(4,6)]_{\text{SameTotal}}$ = {(4,6), (5,5), (6,4)}
  - $[(6,5)]_{\text{SameTotal}}$ = {(5,6), (6,5)}
  - $[(6,6)]_{\text{SameTotal}}$ = {(6,6)}
- They partition S into 11 blocks.

- Are these equivalence classes meaningful in craps?
  - Lose on total 2,3,12
  - Win on total 7
  - Else, roll same total before 7

Partitions, Equivalence Classes

Theorem:

If $\{Ai\}$ is *any* partition of S, then there exists an equivalence relation R, whose equivalence classes are exactly the blocks A_i.

Proof: If $\{A_i\}$ partitions S then define relation R on S to be

$R = \{(a,b) : \exists\, i,\ a \in A_i \text{ and } b \in A_i\}$

Next show that R is an equivalence relation.

Reflexive and symmetric. Transitive?

Suppose aRb and bRc. Then a and b are in A_i, and b and c are in A_j.

But $b \in A_i \cap A_j$, so $A_i = A_j$.

So, a, b, c $\in A_i$, thus aRc.

Outline

- Closure
  - Reflexive closure, symmetric closure
  - (Detour: composing relations & powering relations)
  - Transitive closure
- Equivalence relations = RST
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  - Equivalence Classes
  - Partitions
- **Partial Orderings** = R (AS) T
  - **Definition and examples**
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Partial Orders

- A ***partial order*** ranks some pairs of elements in a consistent & transitive manner

Consistency:

A team can't be both better than & worse than another

Transitivity:

Portugal > England,

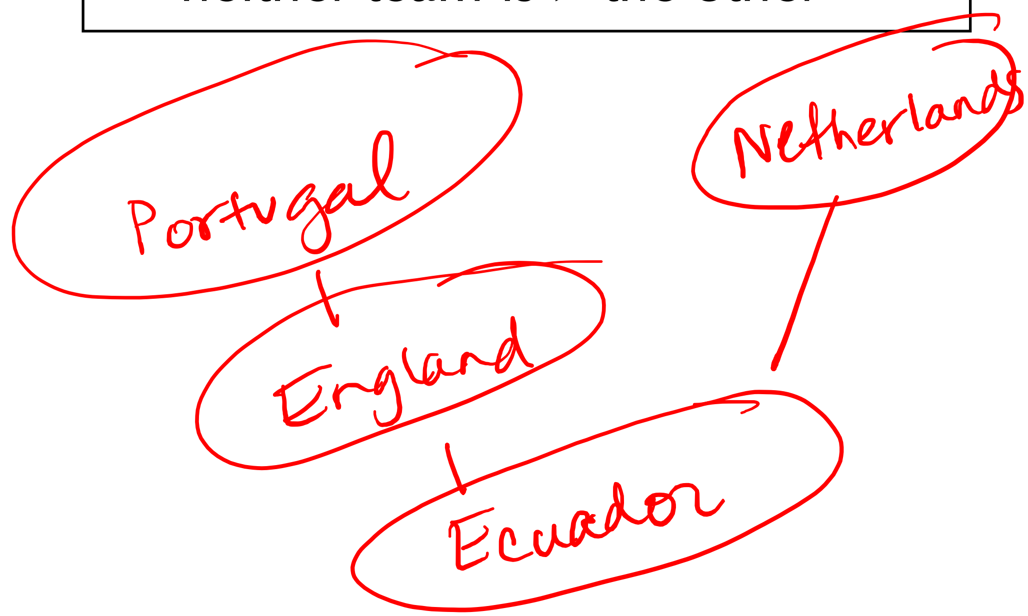
England > Ecuador,

therefore Portugal > Ecuador

For partial ordering, some elements could be incomparable:

e.g., England & Netherlands

- neither team is > the other



Partial Orders

Partial Order = R(AS)T

- A relation R on a set S is called ***partial order*** when R is:
  - ***Reflexive*** Edges (a,a), (b,b), etc.
  - ***Antisymmetric*** aRb and bRa → a=b
  - ***Transitive*** aRb and bRc → aRc

Example:
the "divides" partial order over $\{1,2,3,4,5,6\}$
e.g., $2 \mid 4$

- **Poset**: a set S together with partial ordering R is called a partially ordered set, or poset, denoted by (S, R)
  - E.g., ({1,2,3,4,5,6}, |) is a poset.
  - Other examples: ({1,2,3,4}, ≤)

Partial Orders vs Total Orders

- A ***partial order*** is:
  - ***Reflexive*** Edges (1,1), (2,2), etc.
  - ***Antisymmetric*** (v,w) and (w,v) imply v=w
  - ***Transitive*** (v,w) and (w,x) imply (v,x)

Example:
the "divides" partial order over {1,2,3,4,5,6}
e.g., $2 \mid 4$

- and a ***total order*** if it's a partial order and :
  - ***Total*** either (v,w) or (w,v) (except the case v=w)
    - **Every** pair of elements must be "comparable"
  - **Example:** the "less than or equal" total order over {1,2,3,4}:
    $1 \leq 2 \leq 3 \leq 4$

Partial Orders

- A ***partial order*** is:
  - ***Reflexive*** Edges (1,1), (2,2), etc.
  - ***Antisymmetric*** (v,w) and (w,v) imply v=w
  - ***Transitive*** (v,w) and (w,x) imply (v,x)
- and a ***total order*** if it's a partial order and :
  - ***Total*** either (v,w) or (w,v) except the case v=w
- Examples: Is each a partial order? Total order?
  - $x \leq y$ (reals) Partial and Total
  - $x \subseteq y$ (sets) Partial, not Total. Ex: $A = \{1,2\}$ and $B = \{3,4\}$ are not comparable
  - "x ancestral to (above on family tree) y" (people)

    Partial, not Total. Ex: x = me, y = you

Partial Orders & Total Orders

Handout

Partial Order: A relation R on a set S is called *partial order* iff

R is: ____

Total Order: A relation R on a set S is called *total order* when it is a partial order and every pair of elements is _______________.

Poset = "partially-ordered set" = a set S together with a partial ordering R, denoted (S, R)

Example: Are these posets? If so, are they totally ordered?

(a) $(\mathbb{Z}, \geq)$

(b) $(\mathbb{Z}^+, \mid)$

Partial & Total Orders

Definition: A relation R on a set S is called a ***partial ordering*** if it is **reflexive, antisymmetric**, and **transitive**. A set S together with a partial ordering R is called a ***partially ordered set***, or ***poset***.

Are these posets? If so, are they totally ordered?

- $(\mathbb{Z}, \geq)$
  - R, AS, T: poset.
  - Total order

- $(\mathbb{Z}^+, |)$
  - R, AS, T: poset
  - Not a total order
    - Consider 3 and 5. Does 3|5 or does 5|3?
    - Neither (3,5) nor (5,3) is in the relation, so it's not a total order