第六讲：变分法求解最优控制

最优控制的数学理论之二

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最简变分问题

问题 1 (最简变分问题)

求函数 $x(t):[t_0,t_f]\to\mathbb{R}^n$，在给定的初始和终端时刻 t_0,t_f 满足，$x(t_0)=x_0$, $x(t_f)=x_f$，且最小化性能指标

$$J(x)=\int_{t_0}^{t_f} g(x(t),\dot{x}(t),t)\,\mathrm{d}t. \tag{1}$$

其中 g 取值于 $\mathbb{R}$，二阶连续可微。

函数 $x(t):[t_0,t_f]\to\mathbb{R}^n$ 及其导数 $\dot{x}(t):[t_0,t_f]\to\mathbb{R}^n$ 记为

$$x(t)=\begin{bmatrix} x_1(t)\\ \vdots \\ x_n(t)\end{bmatrix},\qquad \dot{x}(t)=\begin{bmatrix} \dot{x}_1(t)\\ \vdots \\ \dot{x}_n(t)\end{bmatrix},$$

最优控制问题

问题 2 (最优控制问题)

1. 被控对象的状态方程为

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0.$$

2. 容许控制, $u \in \mathcal{U}, \quad x \in \mathcal{X}.$
3. 目标集, $x(t_f) \in \mathcal{S}$

$$\mathcal{S} = [t_0, \infty) \times \{x(t_f) \in \mathbb{R}^n : m(x(t_f), t_f) = 0\}$$

4. 求分段连续的 u, 以最小化性能指标

$$J(u) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t)\, \mathrm{d}t.$$

使用经典变分法求泛函极值的基本过程

- 求泛函变分（根据定义求，或引理 1）
- 求解泛函极值条件（根据定理 1，“导数为零”）

变分问题与最优控制问题：区别

	最简变分	停车例子	导弹例子	时间最短
状态方程	无/$\dot{x}=u$	$\dot{x}=f(x,u,t)$	$\dot{x}=f(x,u,t)$	$\dot{x}=f(x,u,t)$
目标	x_f,t_f fix	x_f,t_f fix	x_f free, t_f fix	$t_f$free
性能指标	$J(x)$	$J(u)$	$J(u)$	$J(u)$

Remark 1 (经典变分求最优控制所需)

- 需处理约束条件【使用拉格朗日乘子法】上节
- 需要处理不同的控制目标【边界条件】本节

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“最简”最优控制问题

问题 3 (“最简”最优控制问题)

状态 $x(t):[t_0,t_f]\to\mathbb{R}$，控制 $u(t):[t_0,t_f]\to\mathbb{R}$，状态方程

$$\dot{x}(t)=f(x(t),u(t),t),\ x(t_0)=x_0$$

固定终端时刻 t_f，固定终端状态 $x(t_f)=x_f$，求最优控制，最小化性能指标

$$J(u)=\int_{t_0}^{t_f} g(x(t),u(t),t)dt.$$

1/4 拉格朗日乘子法

为了处理微分方程约束的泛函极值，引入拉格朗日乘子 $p(t)$

$$\bar{J} = \int_{t_0}^{t_f} \left\{ g(x(t), u(t), t) + p(t)[f(x(t), u(t), t) - \dot{x}(t)] \right\} \mathrm{d}t$$

2/4 求泛函变分

$$
\begin{aligned}
\delta \bar{J} = & \frac{d}{d\alpha} \int_{t_0}^{t_f} \Big\{ g(x + \alpha\delta x, u + \alpha\delta u, t) \\
& + [p + \alpha\delta p][f(x + \alpha\delta x, u + \alpha\delta u, t) - \dot{x} - \alpha\delta\dot{x}] \Big\} \mathrm{d}t \Big|_{\alpha=0} \\
= & \int_{t_0}^{t_f} \Big\{ \frac{\partial g}{\partial x}\delta x + \frac{\partial g}{\partial u}\delta u \\
& + \delta p[f - \dot{x}] + p[\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial u}\delta u - \delta\dot{x}] \Big\} \mathrm{d}t \\
= & \int_{t_0}^{t_f} \Big\{ [\frac{\partial g}{\partial x} + p\frac{\partial f}{\partial x}]\delta x + [\frac{\partial g}{\partial u} + p\frac{\partial f}{\partial u}]\delta u + (f - \dot{x})\delta p - p\delta\dot{x} \Big\} \mathrm{d}t
\end{aligned}
$$

3/4 分部积分去掉变分及其导数的依赖关系

$$
\begin{aligned}
\delta \bar{J} &= \int_{t_0}^{t_f} \left\{ [\frac{\partial g}{\partial x} + p\frac{\partial f}{\partial x}]\delta x + [\frac{\partial g}{\partial u} + p\frac{\partial f}{\partial u}]\delta u + (f - \dot{x})\delta p - p\delta\dot{x} \right\} \mathrm{d}t \\
&= \int_{t_0}^{t_f} \left\{ [\frac{\partial g}{\partial x} + p\frac{\partial f}{\partial x}]\delta x + [\frac{\partial g}{\partial u} + p\frac{\partial f}{\partial u}]\delta u + (f - \dot{x})\delta p + \dot{p}\delta x \right\} \mathrm{d}t \\
&\quad - p\delta x\Big|_{t_0}^{t_f} \\
&= \int_{t_0}^{t_f} \left\{ [\frac{\partial g}{\partial x} + p\frac{\partial f}{\partial x} + \dot{p}]\delta x + [\frac{\partial g}{\partial u} + p\frac{\partial f}{\partial u}]\delta u + (f - \dot{x})\delta p \right\} \mathrm{d}t \\
&\quad - p\delta x\Big|_{t_0}^{t_f}
\end{aligned}
$$

$\delta x(t_0) = 0, \quad \delta x(t_f) = 0$，其余各项都应为零

5/5 最优控制的必要条件

$$\frac{\partial g}{\partial x} + p\frac{\partial f}{\partial x} + \dot{p} = 0 \tag{2}$$

$$\frac{\partial g}{\partial u} + p\frac{\partial f}{\partial u} = 0 \tag{3}$$

$$f - \dot{x} = 0 \tag{4}$$

再加上初值条件和终值条件

$$x(t_0) = x_0 \tag{5}$$

$$x(t_f) = x_f \tag{6}$$

$$\mathcal{H} = g + p \cdot f \Rightarrow \dot{p} = -\frac{\partial \mathcal{H}}{\partial x}, \quad 0 = \frac{\partial \mathcal{H}}{\partial u}, \quad \dot{x} = +\frac{\partial \mathcal{H}}{\partial p}.$$

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处理目标集

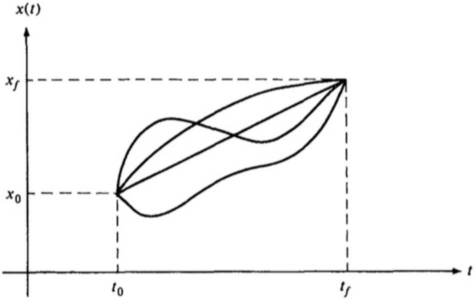
- Case 1: t_f fixed, $x(t_f)$ fixed.
- Case 2: t_f fixed, $x(t_f)$ free.
- Case 3: t_f free, $x(t_f)$ fixed.
- Case 4: t_f free, $x(t_f)$ free 且无关
- 一般目标集: $m(x(t_f), t_f) = 0$.

Case 1: t_f fixed, $x(t_f)$ fixed.

问题 4 (Case 1: t_f fixed, $x(t_f)$ fixed.)

函数 $x(t)$ 初值 $x(t_0) = x_0$, 终值 $x(t_f) = x_f$, 终端时刻固定，求性能指标极值条件

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)dt.$$



Case 1: t_f fixed, $x(t_f)$ fixed.

详见上节欧拉-拉格朗日方程证明，一阶条件如下

$$
\begin{aligned}
0 = \delta J = & \int_{t_0}^{t_f} \{\frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) \cdot \delta x(t) + \frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t) \cdot \delta \dot{x}(t)\} dt \\
= & \int_{t_0}^{t_f} \{\frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) - \frac{\mathrm{d}}{\mathrm{d}t}[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)]\} \cdot \delta x(t) dt \\
& + [\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t) \cdot \delta x(t)]\Big|_{t_0}^{t_f}.
\end{aligned}
$$

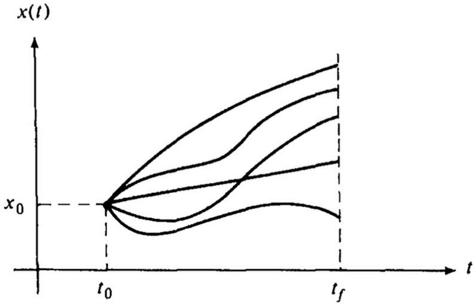
$x(t_0), x(t_f)$ 都不能改变，因此 $\delta x(t_0) = 0, \delta x(t_f) = 0$，边界条件自然为 0，积分号内取 0 即得欧拉-拉格朗日方程

Case 2: t_f fixed, $x(t_f)$ free.

问题 5 (Case 2: t_f fixed, $x(t_f)$ free.)

函数 $x(t)$ 初值 $x(t_0) = x_0$, 终值 $x(t_f)$ 自由, 终端时刻固定, 求性能指标极值条件

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t) dt.$$



1/2 计算泛函变分

$$
\begin{aligned}
0 = \delta J = & \int_{t_0}^{t_f} \{ \frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) \cdot \delta x(t) + \frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t) \cdot \delta \dot{x}(t) \} dt \\
= & \int_{t_0}^{t_f} \{ \frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) - \frac{\mathrm{d}}{\mathrm{d}t} [\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)] \} \cdot \delta x(t) dt \\
& + [\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t) \cdot \delta x(t)] \Big|_{t_0}^{t_f}.
\end{aligned}
$$

“任意的”$\delta x(t)$ 都成立，$\delta x(t_0) = \delta x(t_f) = 0$ 时自然也要成立：

$$
\begin{aligned}
0 = & \frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) - \frac{\mathrm{d}}{\mathrm{d}t} [\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)], \\
0 = & [\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t) \cdot \delta x(t)] \Big|_{t_0}^{t_f}.
\end{aligned} \tag{7}
$$

2/2 泛函极值一阶条件

$$\delta x(t_0) = 0 \quad \Rightarrow \quad \frac{\partial g}{\partial \dot{x}}(x(t_0), \dot{x}(t_0), t_0) \cdot \delta x(t_0) = 0$$

于是，

$$\boxed{\begin{aligned} 0 =& \frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) - \frac{\mathrm{d}}{\mathrm{d}t}[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)], \\ 0 =& \frac{\partial g}{\partial \dot{x}}(x(t_f), \dot{x}(t_f), t_f). \end{aligned} \qquad (8)}$$

例子：终端时刻 fix 终端状态 free 泛函极值

例 1 (终端时刻 fix 终端状态 free 泛函极值)

$x(t_0) = 1$, $t_0 = 0$, $t_f = 5$，终端状态 $x(t_f)$ 自由。最小化性能指标

$$J(x) = \int_0^5 [1 + \dot{x}(t)^2]^{1/2} dt \tag{9}$$

例子：固定终止时间自由终值泛函极值

$$g(\dot{x}) = [1+\dot{x}(t)^2]^{1/2}$$
$$\frac{\partial g}{\partial x} = 0, \quad \frac{\partial g}{\partial \dot{x}} = \frac{\dot{x}(t)}{[1+\dot{x}(t)^2]^{1/2}}$$

任意时刻满足欧拉-拉格朗日方程，

$$0 = \frac{\partial g}{\partial x} - \frac{d}{dt}[\frac{\partial g}{\partial \dot{x}}] = \frac{d}{dt}\left[\frac{\dot{x}(t)}{[1+\dot{x}(t)^2]^{1/2}}\right] \Rightarrow \frac{\ddot{x}^2}{(1+\dot{x}^2)^{3/2}} = 0 \Rightarrow \ddot{x} = 0.$$

终端时刻满足自由终端状态的边界条件

$$\frac{\dot{x}(5)}{[1+\dot{x}(5)^2]^{1/2}} = 0 \Rightarrow \dot{x}(5) = 0$$

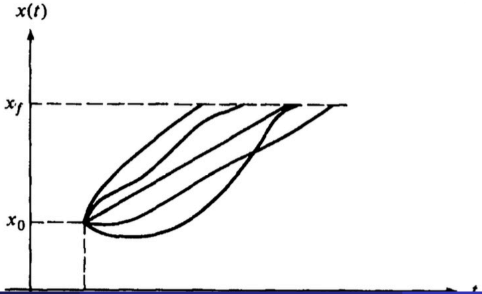
以及 $x(0)=1$ 得到 $x(t) = c_1 t + c_2, c_1 = 0, c_2 = 1$

Case 3: t_f free, $x(t_f)$ fixed.

问题 6 (Case 3: t_f free, $x(t_f)$ fixed.)

函数 $x(t)$ 初值 $x(t_0)=x_0$, 终端状态 $x(t_f)=x_f$ 固定, 终端时刻 t_f 自由，求性能指标极值条件

$$J(x)=\int_{t_0}^{t_f} g(x(t),\dot{x}(t),t)dt.$$



1/5 计算泛函增量的线性部分

计算泛函增量，允许状态变分 $\delta x(t)$ 和终端时刻变分 δt_f

$$
\begin{aligned}
\Delta J &= \int_{t_0}^{t_f+\delta t_f} g(x+\delta x, \dot{x}+\delta \dot{x}, t)dt - \int_{t_0}^{t_f} g(x, \dot{x}, t)dt \\
&= \int_{t_0}^{t_f} \{g(x+\delta x, \dot{x}+\delta \dot{x}, t) - g(x, \dot{x}, t)\} dt \\
&\quad + \int_{t_f}^{t_f+\delta t_f} g(x+\delta x, \dot{x}+\delta \dot{x}, t)dt \\
&= \int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x}(x, \dot{x}, t) \cdot \delta x(t) + \frac{\partial g}{\partial \dot{x}}(x, \dot{x}, t) \cdot \delta \dot{x}(t) \right\} dt \\
&\quad + g(x(t_f)+\delta x(t_f), \dot{x}(t_f)+\delta \dot{x}(t_f), t_f)\delta t_f + o(\| \cdot \|).
\end{aligned}
$$

2/5 终端时刻变分项泰勒展开

把 δt_f 一项泰勒展开

$$\begin{aligned}
&g(x(t_f)+\delta x(t_f),\dot{x}(t_f)+\delta\dot{x}(t_f),t_f)\delta t_f\\
=&g(x(t_f),\dot{x}(t_f),t_f)\delta t_f+\frac{\partial g}{\partial x}(x(t_f),\dot{x}(t_f),t_f)\cdot\delta x(t_f)\delta t_f\\
&+\frac{\partial g}{\partial \dot{x}}(x(t_f),\dot{x}(t_f),t_f)\cdot\delta\dot{x}(t_f)\delta t_f+o(\|\cdot\|)\\
=&g(x(t_f),\dot{x}(t_f),t_f)\delta t_f+o(\|\cdot\|)
\end{aligned}$$

$$\begin{aligned}
\delta J=&\int_{t_0}^{t_f}\left\{\frac{\partial g}{\partial x}(x,\dot{x},t)\cdot\delta x(t)+\frac{\partial g}{\partial \dot{x}}(x,\dot{x},t)\cdot\delta\dot{x}(t)\right\}dt\\
&+g(x(t_f),\dot{x}(t_f),t_f)\delta t_f.
\end{aligned}$$

3/5 分部积分化简

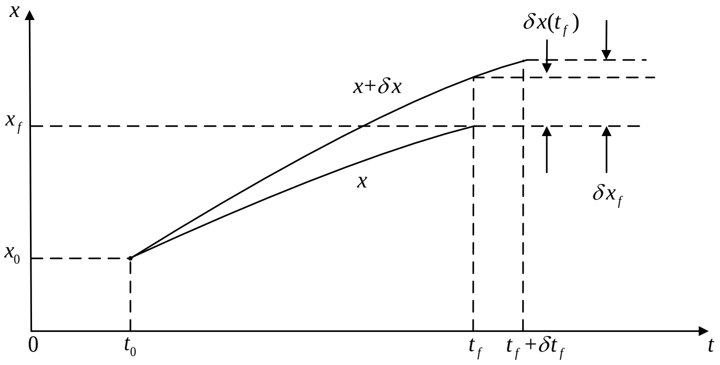
$$
\begin{aligned}
\delta J = &\int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\right] \right\} \cdot \delta x(t)\, \mathrm{d}t \\
&+ \frac{\partial g}{\partial \dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \delta x(t_f) \\
&+ g(x(t_f), \dot{x}(t_f), t_f)\delta t_f.
\end{aligned}
$$

上式已经利用了 $\delta x(t_0) = 0$

4/5 变分之间的依赖关系

δx_f 由两部分组成 $\delta x(t_f)$ 与 δt_f，可采用下式估计

$$0 = \delta x_f \approx \delta x(t_f) + \dot{x}(t_f)\delta t_f \Rightarrow \delta x(t_f) \approx -\dot{x}(t_f)\delta t_f$$



5/5 泛函极值一阶条件

$$\delta J = \Big\{ g(x(t_f), \dot{x}(t_f), t_f) - \frac{\partial g}{\partial \dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \dot{x}(t_f) \Big\} \delta t_f + \int_{t_0}^{t_f} \Big\{ \frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) - \frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\Big] \Big\} \cdot \delta x(t)\, \mathrm{d}t.$$

$$0 = \frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) - \frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\Big], \tag{10}$$

$$0 = g(x(t_f), \dot{x}(t_f), t_f) - \frac{\partial g}{\partial \dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \dot{x}(t_f). \tag{11}$$

例子： t_f free, $x(t_f)$ fixed.

例 2 (t_f free, $x(t_f)$ fixed.)

$x(1) = 4$, $x(t_f) = 4$, t_f free，最小化性能指标

$$J(x) = \int_1^{t_f} [2x(t) + \frac{1}{2}\dot{x}(t)^2] dt \tag{12}$$

例子：t_f free, $x(t_f)$ fixed.

$$g = 2x(t) + \frac{1}{2}\dot{x}(t)^2$$

欧拉方程

$$0 = 2 - \frac{d}{dt}\dot{x} \Rightarrow x(t) = t^2 + c_1 t + c_2 \tag{13}$$

边界条件

$$0 = [g - \frac{\partial g}{\partial \dot{x}} \cdot \dot{x}]\Big|_{t_f} = 2x(t_f) + \frac{1}{2}\dot{x}(t_f)^2 - \dot{x}(t_f)^2$$

$$x(1) = 4,\ x(t_f) = 4 \Rightarrow$$

$$1 + c_1 + c_2 = 4,\ t_f^2 + c_1 t_f + c_2 = 4,\ 2 * 4 - \frac{1}{2}(2t_f + c_1)^2 = 0 \Rightarrow$$

$$t_f = 5,\ c_1 = -6,\ c_2 = 9$$

Case 4: t_f free, $x(t_f)$ free 且无关.

问题 7 (Case 4: t_f free, $x(t_f)$ free 且无关.)

函数 $x(t)$ 初值 $x(t_0) = x_0$, t_f free, $x(t_f)$ free 且二者无关. 求性能指标极值条件

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t) dt.$$

1/3 求泛函变分

由 Case3 分析可得

$$\delta J = \int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\right] \right\} \cdot \delta x(t)\,\mathrm{d}t + \frac{\partial g}{\partial \dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \delta x(t_f) + g(x(t_f), \dot{x}(t_f), t_f)\delta t_f.$$

同样由 Case3 分析，δx_f 由两部分组成 $\delta x(t_f)$ 与 δt_f

$$\delta x_f \approx \delta x(t_f) + \dot{x}(t_f)\delta t_f$$

2/3 处理边界条件

$$
\begin{aligned}
\delta J =& \frac{\partial g}{\partial \dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot [\delta x_f - \dot{x}(t_f)\delta t_f] \\
& + \int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\right] \right\} \cdot \delta x(t) dt \\
& + g(x(t_f), \dot{x}(t_f), t_f)\delta t_f \\
=& \frac{\partial g}{\partial \dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \delta x_f \\
& + \left[g(x(t_f), \dot{x}(t_f), t_f) - \frac{\partial g}{\partial \dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \dot{x}(t_f) \right] \delta t_f \\
& + \int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\right] \right\} \cdot \delta x(t) dt
\end{aligned}
$$

3/3 泛函极值一阶条件

由泛函极值的一阶条件可得

$$\frac{\partial g}{\partial x}(x(t),\dot{x}(t),t)-\frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x(t),\dot{x}(t),t)\right]=0$$

$$\frac{\partial g}{\partial \dot{x}}(x(t_f),\dot{x}(t_f),t_f)=0$$

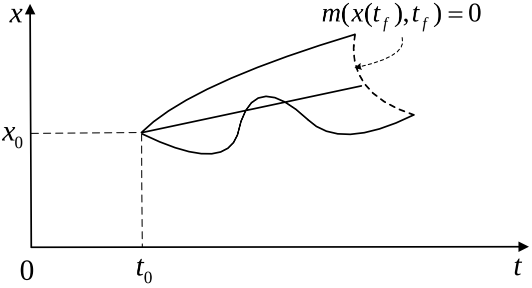
$$g(x(t_f),\dot{x}(t_f),t_f)-\frac{\partial g}{\partial \dot{x}}(x(t_f),\dot{x}(t_f),t_f)\cdot\dot{x}(t_f)=0$$

一般目标集：$m(x(t_f), t_f) = 0.$

问题 8 (一般目标集：$m(x(t_f), t_f) = 0.$)

$x(t)$ 初值 $x(t_0) = x_0$, $m(x(t_f), t_f) = 0$. 求性能指标极值条件

$$J(x) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)dt.$$



三种形式的性能指标

例 3 (Lagrange 形式)

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)dt$$

例 4 (Mayer 形式)

$$J(x) = h(x(t_f), t_f)$$

例 5 (Bolza 形式)

$$J(x) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)dt$$

其他形式转化为 Lagrange 形式

$$
\begin{aligned}
J(u) &= h(x(t_f), t_f) \\
&= h(x(t_0), t_0) + \int_{t_0}^{t_f} \frac{d}{dt}[h(x(t), t)]dt
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
J(u) &= h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)dt \\
&= h(x(t_0), t_0) + \int_{t_0}^{t_f} \{g(x(t), \dot{x}(t), t) + \frac{d}{dt}[h(x(t), t)]\}dt
\end{aligned} \tag{15}
$$

$h(x(t_0), t_0)$ 与控制无关，上述问题可转化为 Lagrange 问题

Mayer 形式性能指标的变分

$$\bar{g} = \frac{d}{dt}[h(x(t), t)] = \frac{\partial h}{\partial x}(x(t), t) \cdot \dot{x}(t) + \frac{\partial h}{\partial t}(x(t), t)$$

$$\begin{aligned}\delta h = & \frac{\partial \bar{g}}{\partial \dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \delta x_f \\ & + [\bar{g}(x(t_f), \dot{x}(t_f), t_f) - \frac{\partial \bar{g}}{\partial \dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \dot{x}(t_f)]\delta t_f \\ & + \int_{t_0}^{t_f} \{\frac{\partial \bar{g}}{\partial x}(x(t), \dot{x}(t), t) - \frac{d}{dt}[\frac{\partial \bar{g}}{\partial \dot{x}}(x(t), \dot{x}(t), t)]\} \cdot \delta x(t) dt.\end{aligned}$$

$$\begin{aligned}\frac{\partial \bar{g}}{\partial x}(x(t), \dot{x}(t), t) &= \frac{\partial^2 h}{\partial x^2}(x(t), t)\dot{x}(t) + \frac{\partial^2 h}{\partial x \partial t}(x(t), t), \\ \frac{\partial \bar{g}}{\partial \dot{x}}(x(t), \dot{x}(t), t) &= \frac{\partial h}{\partial x}(x(t), t), \\ \frac{\mathrm{d}}{\mathrm{d}t}[\frac{\partial \bar{g}}{\partial \dot{x}}(x(t), \dot{x}(t), t)] &= \frac{\partial^2 h}{\partial x^2}(x(t), t)\dot{x}(t) + \frac{\partial^2 h}{\partial x \partial t}(x(t), t).\end{aligned}$$

$$\delta h = \frac{\partial h}{\partial x}(x(t_f), t_f) \cdot \delta x_f + \frac{\partial h}{\partial t}(x(t_f), t_f)\delta t_f$$

1/3 拉格朗日乘子法，终端变分

对终端约束 m，取拉格朗日乘子

$$\bar{J}(x,t_f,\lambda) := h(x(t_f),t_f) + \lambda^T m(x(t_f),t_f) + \int_{t_0}^{t_f} g(x(t),\dot{x}(t),t)dt.$$

其中 λ 是向量（而非函数）
前两项为 Mayer 形式性能指标，利用上页结果可得变分

$$\frac{\partial h}{\partial x}(x(t_f),t_f)\cdot\delta x_f + \frac{\partial h}{\partial t}(x(t_f),t_f)\delta t_f \\ + m(x(t_f),t_f)\cdot\delta\lambda + \lambda\cdot\left[\frac{\partial m}{\partial x}(x(t_f),t_f)\delta x_f + \frac{\partial m}{\partial t}(x(t_f),t_f)\delta t_f\right]$$

2/3 运行代价变分

后项为 Lagrange 形式性能指标，终端时刻自由，终端状态自由，变分与 Case4 相同

$$
\begin{aligned}
&\frac{\partial g}{\partial \dot{x}}(x(t_f),\dot{x}(t_f),t_f)\cdot\delta x_f \\
&\quad+\left[g(x(t_f),\dot{x}(t_f),t_f)-\frac{\partial g}{\partial \dot{x}}(x(t_f),\dot{x}(t_f),t_f)\cdot\dot{x}(t_f)\right]\delta t_f \\
&\quad+\int_{t_0}^{t_f}\left\{\frac{\partial g}{\partial x}(x(t),\dot{x}(t),t)-\frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x(t),\dot{x}(t),t)\right]\right\}\cdot\delta x(t)dt
\end{aligned}
$$

3/3 泛函变分

性能指标泛函变分为上述二者加和，并令

$$\bar{h}(x(t_f),t_f,\lambda)=h(x(t_f),t_f)+\lambda\cdot m(x(t_f),t_f).$$

得到增广的性能指标泛函的变分：

$$\begin{aligned}\delta\bar{J}=&\Big[\frac{\partial\bar{h}}{\partial t}(x(t_f),t_f)+g(x(t_f),\dot{x}(t_f),t_f)-\frac{\partial g}{\partial\dot{x}}(x(t_f),\dot{x}(t_f),t_f)\cdot\dot{x}(t_f)\Big]\delta t_f\\&+\Big[\frac{\partial\bar{h}}{\partial x}(x(t_f),t_f)+\frac{\partial g}{\partial\dot{x}}(x(t_f),\dot{x}(t_f),t_f)\Big]\cdot\delta x_f\\&+m(x(t_f),t_f)\cdot\delta\lambda\\&+\int_{t_0}^{t_f}\Big\{\frac{\partial g}{\partial x}(x(t),\dot{x}(t),t)-\frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{\partial g}{\partial\dot{x}}(x(t),\dot{x}(t),t)\Big]\Big\}\cdot\delta x(t)\,\mathrm{d}t.\end{aligned}\tag{16}$$

例子：终值约束 t_f, $x(t_f)$ free, $m(x(t_f), t_f) = 0$

例 6 (终值约束 t_f, $x(t_f)$ free, $m(x(t_f), t_f) = 0$)

$x(0) = 0$, t_f 和 $x(t_f)$ 自由，但需满足

$$x(t_f) + 5t_f - 15 = 0$$

最小化性能指标

$$J(x) = \int_0^{t_f} [1 + \dot{x}(t)^2]^{1/2} dt \tag{17}$$

例子：终值约束 t_f, $x(t_f)$ free, $m(x(t_f), t_f) = 0$

令 $\bar{h}(x(t_f), t_f, \lambda) = \lambda(x(t_f) + 5t - 15)$ 有

$$x(t) = c_1 t + c_2, \quad g = [1 + \dot{x}(t)^2]^{1/2}$$

$$0 = \left[\frac{\partial \bar{h}}{\partial t} + g - \frac{\partial g}{\partial \dot{x}}\dot{x}\right]\Big|_{t_f} = \left[5\lambda + g - \frac{\partial g}{\partial \dot{x}}\dot{x}\right]\Big|_{t_f}$$

$$0 = \left[\frac{\partial \bar{h}}{\partial x} + \frac{\partial g}{\partial \dot{x}}\right]\Big|_{t_f} = \left[\lambda + \frac{\partial g}{\partial \dot{x}}\right]\Big|_{t_f} \Rightarrow \lambda = -\frac{\partial g}{\partial \dot{x}}\Big|_{t_f},$$

$$\text{上两式得}: 0 = \left[-5\frac{\partial g}{\partial \dot{x}} + g - \dot{x}\frac{\partial g}{\partial \dot{x}}\right]\Big|_{t_f} = \left[g - \frac{\partial g}{\partial \dot{x}}[5 + \dot{x}]\right]\Big|_{t_f} \Rightarrow$$

$$[1 + \dot{x}(t_f)^2]^{1/2} - \frac{\dot{x}(t_f)}{[1 + \dot{x}(t_f)^2]^{1/2}}[5 + \dot{x}(t_f)] = 0 \Rightarrow \dot{x}(t_f) = \frac{1}{5}$$

$$x(t) = \frac{1}{5}t, \quad \frac{1}{5}t_f + 5t_f - 15 = 0, \Rightarrow t_f = \frac{75}{26}$$

t_f 固定, $x(t_f)$ 自由, $m(x(t_f), t_f) = 0$.

$$x(t_0) = x_0,\, t_f \text{ fix},\, x(t_f) \text{ free},\, m(x(t_f), t_f) = 0$$

$$J(x) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)dt.$$

变分法求泛函极值的过程

- 初步计算泛函变分
- 使用分部积分公式处理变分之间的导数依赖
- 将终止时刻状态变分 δx_f 分为由状态自身变分 $\delta x(t_f)$ 和终止时刻变分 δt_f 组成的两部分
- 泛函极值的一阶条件

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变分法求解最优控制问题

问题 9 (变分法求解最优控制问题)

$x(t) : [t_0, t_f] \to \mathbb{R}^n$, $u(t) : [t_0, t_f] \to \mathbb{R}^m$ 状态方程

$$\dot{x}(t) = f(x(t), u(t), t),\ x(t_0) = x_0$$

最小化性能指标

$$J(u) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t)dt.$$

终止时刻 t_f 和终止状态 $x(t_f)$ 待定

1/5 引入拉格朗日乘子

为了处理微分方程约束的泛函极值，引入拉格朗日乘子

$$p(t) = [p_1(t), p_2(t), \ldots, p_n(t)]^T.$$

最小化性能指标

$$\bar{J}(u,p) = h(x(t_f), t_f) + \int_{t_0}^{t_f} \{g(x(t), u(t), t) + p^T(t)[f(x(t), u(t), t) - \dot{x}(t)]\}dt$$

使用 Hamiltonian 函数表示，则

$$\bar{J} = h(x(t_f), t_f) + \int_{t_0}^{t_f} \left\{\mathcal{H}(x(t), u(t), p(t), t) - p(t) \cdot \dot{x}(t)\right\} \mathrm{d}t.$$

2/5 泛函变分

$$
\begin{aligned}
\delta \bar{J} =& \frac{\partial h}{\partial x}(x(t_f), t_f) \cdot \delta x_f + \frac{\partial h}{\partial t}(x(t_f), t_f)\delta t_f \\
&+ \int_{t_0}^{t_f} \Big\{ \frac{\partial \mathcal{H}}{\partial x}(x(t), u(t), p(t), t) \cdot \delta x(t) + \frac{\partial \mathcal{H}}{\partial u}(x(t), u(t), p(t), t) \cdot \delta u(t) \\
&\qquad + \frac{\partial \mathcal{H}}{\partial p}(x(t), u(t), p(t), t) \cdot \delta p(t) - \dot{x}(t) \cdot \delta p(t) - p(t) \cdot \delta \dot{x}(t) \Big\} \, \mathrm{d}t \\
&+ \Big[\mathcal{H}(x(t), u(t), p(t), t) - p(t) \cdot \dot{x}(t) \Big] \Big|_{t_0}^{t_f} \delta t_f \\
=& \frac{\partial h}{\partial x}(x(t_f), t_f) \cdot \delta x_f \\
&+ \Big[\frac{\partial h}{\partial t}(x(t_f), t_f) + \mathcal{H}(x(t_f), u(t_f), t_f) - p(t_f) \cdot \dot{x}(t_f) \Big] \delta t_f \\
&+ \int_{t_0}^{t_f} \Big\{ \frac{\partial \mathcal{H}}{\partial x}(x(t), u(t), t) \cdot \delta x(t) + \frac{\partial \mathcal{H}}{\partial u}(x(t), u(t), t) \cdot \delta u(t) \\
&\qquad + \Big[\frac{\partial \mathcal{H}}{\partial p}(x(t), u(t), t) - \dot{x}(t) \Big] \cdot \delta p(t) - p(t) \cdot \delta \dot{x}(t) \Big\} \, \mathrm{d}t.
\end{aligned}
$$

3/5 处理变分依赖关系

依然利用变分法技巧去掉变分之间依赖。由：

$$\delta x_f \approx \delta x(t_f) + \dot{x}(t_f)\delta t_f,$$

再结合分部积分公式，可将上式中函数变分的导数项化为：

$$\begin{aligned}\int_{t_0}^{t_f} -p(t)\cdot\delta\dot{x}(t)\,\mathrm{d}t &= -p(t_f)\cdot\delta x(t_f) + \int_{t_0}^{t_f} \dot{p}(t)\cdot\delta x(t)\,\mathrm{d}t \\ &= -p(t_f)\cdot\left[\delta x_f - \dot{x}(t_f)\delta t_f\right] + \int_{t_0}^{t_f} \dot{p}(t)\cdot\delta x(t)\,\mathrm{d}t.\end{aligned}$$

4/5 一阶条件

$$
\begin{aligned}
\delta \bar{J} = & \Big[\frac{\partial h}{\partial x}(x(t_f), t_f) - p(t_f)\Big] \cdot \delta x_f \\
& + \Big[\frac{\partial h}{\partial t}(x(t_f), t_f) + \mathcal{H}(x(t_f), u(t_f), t_f)\Big]\delta t_f \\
& + \int_{t_0}^{t_f} \Big\{ \Big[\frac{\partial \mathcal{H}}{\partial x}(x(t), u(t), t) + \dot{p}(t)\Big] \cdot \delta x(t) + \frac{\partial \mathcal{H}}{\partial u}(x(t), u(t), t) \cdot \delta u(t) \\
& \qquad + \Big[\frac{\partial \mathcal{H}}{\partial p}(x(t), u(t), t) - \dot{x}(t)\Big] \cdot \delta p(t) \Big\} \, \mathrm{d}t.
\end{aligned}
$$

5/5 极小值原理

$$\text{极值条件}:0 = \frac{\partial \mathcal{H}}{\partial u}(x(t), u(t), t). \tag{18}$$

$$\text{状态方程}:\dot{p}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x(t), u(t), t). \tag{19}$$

$$\text{协态方程}:\dot{x}(t) = +\frac{\partial \mathcal{H}}{\partial p}(x(t), u(t), t). \tag{20}$$

以及边界条件：

$$0 = \left[\frac{\partial h}{\partial x}(x(t_f), t_f) - p(t_f)\right] \cdot \delta x_f. \tag{21}$$

$$0 = \left[\frac{\partial h}{\partial t}(x(t_f), t_f) + \mathcal{H}(x(t_f), u(t_f), t_f)\right] \delta t_f. \tag{22}$$

例子：终端时刻固定、终端状态自由

例 7 (终端时刻固定、终端状态自由)

状态变量 $x(t):[t_0,t_f]\to\mathbb{R}^2$，控制变量 $u(t):[t_0,t_f]\to\mathbb{R}$。状态初值 $x(t_0)=x_0$，状态方程为

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= -x_2(t)+u(t).\end{aligned}$$

最小化性能指标

$$J(u)=\int_{t_0}^{t_f}\frac{1}{2}[x_1^2(t)+u^2(t)]\mathrm{d}t.$$

终端时刻固定、终端状态自由

1/3 极值条件

$$\mathcal{H}(x(t),u(t),p(t),t)=\frac{1}{2}\left[x_1^2(t)+u^2(t)\right]+p_1(t)x_2(t)+p_2(t)\left[-x_2(t)+u(t)\right.$$

$$0=\frac{\partial\mathcal{H}}{\partial u}=u(t)+p_2(t).$$

即，

$$u(t)=-p_2(t). \tag{23}$$

2/3 状态方程协态方程

$$\dot{x}_1(t) = +\frac{\partial \mathcal{H}}{\partial p_1} = x_2(t) \tag{24}$$

$$\dot{x}_2(t) = +\frac{\partial \mathcal{H}}{\partial p_2} = -x_2(t) + u(t) = -x_2(t) - p_2(t) \tag{25}$$

$$\dot{p}_1(t) = -\frac{\partial \mathcal{H}}{\partial x_1} = -x_1(t) \tag{26}$$

$$\dot{p}_2(t) = -\frac{\partial \mathcal{H}}{\partial x_2} = -p_1(t) + p_2(t). \tag{27}$$

3/3 边界条件

$$0 = \left[\frac{\partial h}{\partial x}(x(t_f), t_f) - p(t_f)\right] \cdot \delta x_f.$$

$$0 = \left[\frac{\partial h}{\partial t}(x(t_f), t_f) + \mathcal{H}(x(t_f), u(t_f), t_f)\right] \delta t_f.$$

终端时刻固定、所有终端状态都自由。$\delta t_f = 0$，δx_f 可变

$$0 = \frac{\partial h}{\partial x}(x(t_f), t_f) - p(t_f) = -p(t_f).$$

此外，状态变量应满足初值

$$x(t_0) = x_0.$$

例子：终端时刻自由、终端状态自由

例 8 (终端时刻自由、终端状态自由)

状态变量 $x(t):[t_0,t_f]\rightarrow\mathbb{R}^2$，控制变量 $u(t):[t_0,t_f]\rightarrow\mathbb{R}$。状态初值 $x(t_0)=x_0$，状态方程为

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= -x_2(t)+u(t).\end{aligned}$$

最小化性能指标

$$J(u)=\int_{t_0}^{t_f}\frac{1}{2}[x_1^2(t)+u^2(t)]\mathrm{d}t.$$

终端时刻自由、终端状态自由

边界条件

极值条件、状态方程、协态方程相同

$$
\begin{aligned}
0 &= \Big[\frac{\partial h}{\partial x}(x(t_f), t_f) - p(t_f)\Big] \cdot \delta x_f. \\
0 &= \Big[\frac{\partial h}{\partial t}(x(t_f), t_f) + \mathcal{H}(x(t_f), u(t_f), t_f)\Big] \delta t_f.
\end{aligned}
$$

终端时刻固定、终端状态自由，$\delta t_f = 0$，δx_f 可变。得到边界条件

$$
0 = \frac{\partial h}{\partial x}(x(t_f), t_f) - p(t_f) = -p(t_f).
$$

此外，状态变量应满足初值

$$
x(t_0) = x_0.
$$

例子：终端时刻固定、终端状态部分自由部分固定

例 9 (终端时刻固定、终端状态部分自由部分固定)

状态变量 $x(t):[t_0,t_f]\to\mathbb{R}^2$，控制变量 $u(t):[t_0,t_f]\to\mathbb{R}$。状态初值 $x(t_0)=x_0$，状态方程为

$$\begin{aligned}\dot{x}_1(t)&=x_2(t),\\ \dot{x}_2(t)&=-x_2(t)+u(t).\end{aligned}$$

最小化性能指标

$$J(u)=\int_{t_0}^{t_f}\frac{1}{2}[x_1^2(t)+u^2(t)]\mathrm{d}t.$$

终端时刻固定、x_1 自由，$x_2(t_f)=b$

边界条件

极值条件、状态方程、协态方程相同

$$0 = [\frac{\partial h}{\partial x}(x(t_f), t_f) - p(t_f)] \cdot \delta x_f.$$

$$0 = [\frac{\partial h}{\partial t}(x(t_f), t_f) + \mathcal{H}(x(t_f), u(t_f), t_f)]\delta t_f.$$

终端时刻固定、终端时刻的状态 x_1 自由，$x_2(t_f) = b$，边界条件，

$$0 = \frac{\partial h}{\partial x_1}(x(t_f), t_f) - p_1(t_f) = -p_1(t_f).$$

此外，状态变量应满足初值

$$x(t_0) = x_0,$$

和目标集的约束

$$x_2(t_f) = b.$$

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稳态 Hamiltonian

定理 1 (稳态 Hamiltonian)

Hamiltonian 不显式依赖于时间，则最优控制的 Hamiltonian 满足

$$\mathcal{H}(x(t), u(t), p(t)) = c_1, \ \forall t \in [t_0, t_f]. \tag{28}$$

其中 c_1 为常数

稳态 Hamiltonian 1/1

Proof.

无论边界条件如何，

$$0 = \frac{\partial \mathcal{H}}{\partial u} \tag{29}$$

$$\dot{x} = +\frac{\partial \mathcal{H}}{\partial p} \tag{30}$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial x} \tag{31}$$

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(x(t), u(t), p(t)) &= \frac{\partial \mathcal{H}}{\partial x}\dot{x} + \frac{\partial \mathcal{H}}{\partial u}\dot{u} + \frac{\partial \mathcal{H}}{\partial p}\dot{p} \\ &= -\dot{p}\cdot\dot{x} + 0\cdot\dot{u} + \dot{x}\cdot\dot{p} \\ &= 0\end{aligned}$$

终端时刻自由，稳态 Hamiltonian

定理 2 (终端时刻自由，稳态 Hamiltonian)

t_f free, 且 Hamiltonian 和终端代价都不显式依赖于时间，则最优控制的 Hamiltonian 满足

$$\mathcal{H}(x(t), u(t), p(t)) = 0, \ \forall t \in [t_0, t_f]. \tag{32}$$

终端时刻自由，稳态 Hamiltonian 1/1

Proof.

继续上一定理的证明。若 t_f 自由，由边界条件-(22)，有

$$\frac{\partial h}{\partial t}(x(t_f)) + \mathcal{H}(x(t_f), u(t_f), p(t_f)) = 0.$$

而终端代价 h 并不显含 t，因此，

$$\mathcal{H}(x(t_f), u(t_f), p(t_f)) = 0.$$

而由上一定理，已知哈密尔顿函数沿着最优控制总是常数，于是当终端时刻 t_f 自由时，这个常数 $c_1 = 0$，即，

$$\mathcal{H}(x(t), u(t), p(t)) = 0, \ \forall t \in [t_0, t_f].$$

□

变分法求最优控制

$\delta J = 0$ 的必要条件

Remark 2 (变分法求最优控制的过程)

- 使用拉格朗日乘子法处理各类等式约束
- 求增广形式性能指标的泛函变分
- 使用分部积分公式处理变分之间的导数依赖
- 将终止时刻状态变分 δx_f 分为由状态自身变分 $\delta x(t_f)$ 和终止时刻变分 δt_f 组成的两部分
- 得到泛函极值的一阶条件

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5 稳态系统的极小值原理

6 附录：Matlab 求解常微分方程

ode45

```
[t,k]=ode45( @krhs, T:-dt:0, [b 0. b] );

      function dkdt=krhs(t,k)
            dkdt=[k(2)^2;
                  -k(1) + k(2)*k(3);
                  -2.*k(2) + k(3)^2];
      end

K(:,:)=k(N+1:-1:1,:);
```

bvp4c

```
solinit = bvpinit(linspace(0,T,N), [0 0 0.5 0.5]);
options = bvpset('Stats','on','RelTol',1e-1);

sol = bvp4c(@BVP_ode, @BVP_bc, solinit, options);

t = sol.x
y = sol.y
```

define ODE

```
% ODE:
% dx1/dt= x2, dx2/dt = -p2, dp1/dt=0, dp2/dt = -p1;
function dydt = BVP_ode(t,y)
dydt = [ y(2)
         -y(4)
         0
         -y(3) ];
% The boundary conditions:
% x1(0) = -2, x2(0) = 1, x1(tf) = 0, x2(tf) = 0;
function res = BVP_bc(ya,yb)
res = [ ya(1) + 2
        ya(2) - 1
        yb(1)
        yb(2) ];
```