模糊控制第二讲：

模糊系统的结构与万能逼近特性

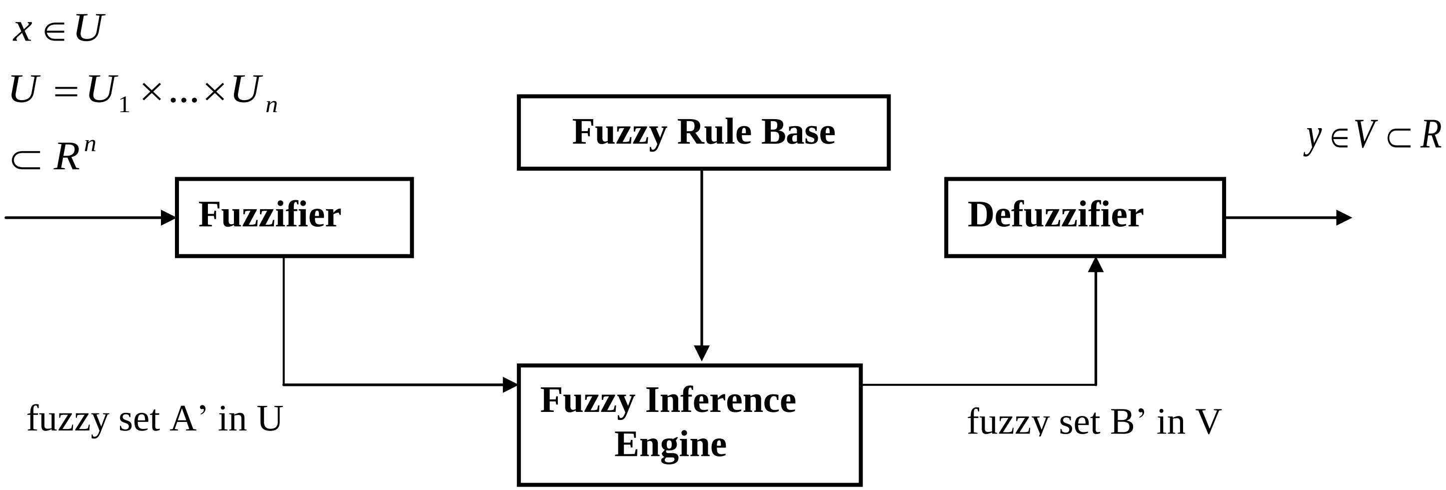
一、　模糊规则库与模糊推理机

二、　模糊器与解模糊器

三、　各种模糊系统的构建

四、　模糊系统的万能逼近特性

一、 模糊规则库与模糊推理机



7.1 Fuzzy Rule Base

A *fuzzy rule base* consists of a collection of fuzzy IF-THEN rules in the following form:

$$Ru^{(l)}: \text{IF } x_1 \text{ is } A_1^l \text{ and } \ldots \text{ and } x_n \text{ is } A_n^l, \text{ THEN } y \text{ is } B^l.$$

where A_i^l and B^l are fuzzy sets in $U_i \subset R$ and $V \subset R$, respectively, and $l = 1,2,\ldots,M$. The rule structure is called canonical form.

Question: Is the rule structure general enough?

Lemma 7.1: The canonical fuzzy IF-THEN rules including the following as special cases:

(a) "Partial rules": IF x_1 is A_1^l and … and x_m is A_m^l, THEN y is B^l, m<n

(b) "or rules": IF x_1 is A_1^l and … and x_m is A_m^l or x_{m+1} is A_{m+1}^l and … and x_n is A_n^l, THEN y is B^l

(c) single fuzzy statement: y is B^l

(d) "Graduate rules": The smaller the x, the bigger the y

(e) Nonfuzzy rules.

Properties of A Set of Rules

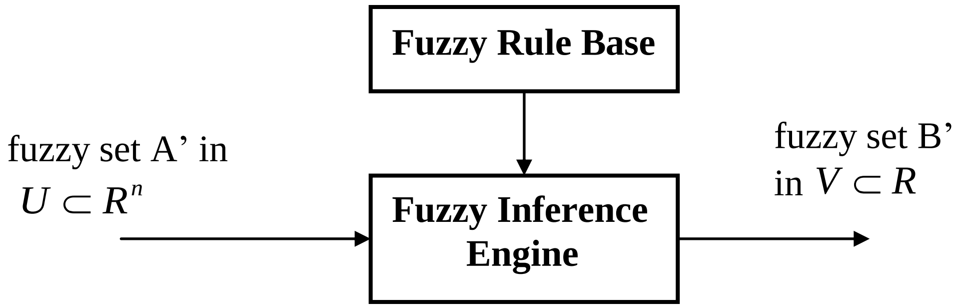
A set of fuzzy IF-THEN rules is *complete* if any x in U, there exists at least one rule, say Ru $^{(l)}$, such that

$$\mu_{A_i^l}(x_i) \neq 0$$

for all i =1,2,…,n.

A set of fuzzy IF-THEN rules is *consistent* if there are no rules with the same IF parts but different THEN parts.

7.2 Fuzzy Inference Engine



In a fuzzy inference engine, fuzzy logic principles are used to combine the fuzzy IF-THEN rules in the fuzzy rule base into a mapping from fuzzy set A' in U to fuzzy set B' in V.

- ***The Single Rule Case***

$$\mu_{B'}(y) = \sup_{x \in U} t[\mu_{A'}(x), \mu_{A \to B}(x, y)]$$

- ***Composition Based Inference***

In composition based inference, all rules in the fuzzy rule base are combined into a single fuzzy relation in $U \times V$, which is then viewed as a single fuzzy IF-THEN rule.

- *Mamdani Combination*

Let Ru $^{(l)} = A_1^l \times \ldots \times A_n^l \rightarrow B^l$, Q_M be the combination.

$$Q_M = \bigcup_{l-1}^{M} Ru^{(l)}$$

$$\mu_{Q_M}(x,y) = \mu_{Ru^{(1)}}(x,y) \dot{+} \cdots \dot{+} \mu_{Ru^{(M)}}(x,y)$$

where $\dot{+}$ =s-norm

- *Gődel Combination*

$$Q_G = \bigcap_{l=1}^{M} Ru^{(l)}$$

$$\mu_{Q_G}(x,y) = \mu_{Ru^{(1)}}(x,y) * \cdots * \mu_{Ru^{(M)}}(x,y)$$

where $*$ =t-norm

The overall inference engine gives:

$$\mu_{B'}(y) = \sup_{x \in U} t[\mu_{A'}(x), \mu_{Q_M}(x,y)]$$

or

$$\mu_{B'}(y) = \sup_{x \in U} t[\mu_{A'}(x), \mu_{Q_G}(x,y)]$$

- ***Individual-Rule Based Inference***

In individual-rule based inference, each rule determines an output fuzzy set B^{l}' and B' is the combination of B^{l}'

$$\mu_{B'}(y) = \sup_{x \in U} t[\mu_{A'}(x), \mu_{Ru^{(l)}}(x, y)], \quad l = 1,2,\ldots,M$$

$$\mu_{B'}(y) = \mu_{B_1'}(y) \dot{+} \cdots \dot{+} \mu_{B_M'}(y)$$

or

$$\mu_{B'}(y) = \mu_{B_1'}(y) * \cdots * \mu_{B_M'}(y)$$

- ***The Commonly-Used Inference Engines***

- ***Product Inference Engine:***

(1) Individual-rule based inference with $\dot{+}$ combination.
(2) $\rightarrow =$ Mamdani product implication.
(3) All t-norm = product, s-norm = max

$$\mu_{B'}(y) = \max_{l=1}^{M} [\sup_{x \in U} (\mu_{A'}(x) \prod_{i=1}^{n} \mu_{A_l^L}(x_i) \mu_{B_l}(y)]$$

- ***Minimum Inference Engine:***

$$\mu_{B'}(y) = \max_{l=1}^{M}[\sup_{x\in U} \min(\mu_{A'}(x), \mu_{A_1^L}(x_1),\ldots,\mu_{A_n^L}(x_n), \mu_{B_l}(y))]$$

- ***Lukasiewicz Inference Engine:***

(1) Individual-rule based inference with $*$ combination.
(2) $\rightarrow =$ Lukasiewicz implication.
(3) t-norm = min

$$\mu_{B'}(y) = \min_{l=1}^{M}[\sup_{x\in U}(\mu_{A'}(x), 1-\min_{i=1}^{n}\mu_{A_i^l}(x_i)+\mu_{B_l}(y)]$$

- ***Zadeh Inference Engine:***

$$\begin{aligned}\mu_{B'}(y) = \min_{l=1}^{M}\{&\sup_{x\in U}\min[\mu_{A'}(x),\\ &\max(\min(\mu_{A_1^L}(x_1),\ldots,\mu_{A_n^L}(x_n),\mu_{B^L}(y)), 1-\min_{i=1}^{n}\mu_{A_i^l}(x_i)\}\end{aligned}$$

- ***Dienes-Rescher Inference Engine:***

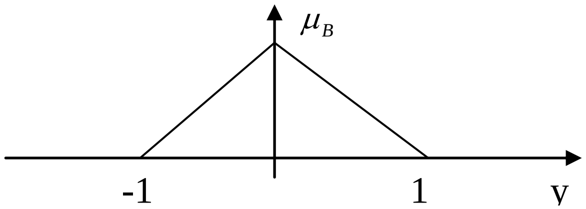
$$\mu_{B'}(y) = \min_{l=1}^{M}\{\sup_{x\in U}\min[\mu_{A'}(x), \max\left(1-\min_{i=1}^{n}\mu_{A_i^l}(x_i), \mu_{B^L}(y)\right)\}$$

Example: Only one rule:

IF x_1 is A_1 and … and x_n is A_n, THEN y is B.

where

$$\mu_B(y) = \begin{cases} 1-|y|, & if \quad -1 \le y \le 1; \\ 0, & otherwise. \end{cases}$$



Let A'= fuzzy singleton, Plot $\mu_B'(y)$ of the five inference engines.

Let $\mu_{A_p}(x_p*) = \min[\mu_{A_1}(x_1*),...,\mu_{An}(x_n*)]$, $\mu_A(x*) = \prod_{i=1}^{n} \mu_{A_i}(x_i*)$

then

$$\mu_{B_p'}(y) = \mu_A(x*)\mu_B(y)$$

$$\mu_{B_M'}(y) = \min[\mu_{A_p}(x_p*), \mu_B(y)]$$

$$\mu_{B_L'}(y) = \min[1,1-\mu_{A_p}(x_p*) + \mu_B(y)]$$

$$\mu_{B_z'}(y) = \max[\min(\mu_{A_p}(x_p*), \mu_B(y)),1-\mu_{A_p}(x_p*)]$$

$$\mu_{B_D'}(y) = \max[1-\mu_{A_p}(x_p*), \mu_B(y)]$$

二、模糊器与解模糊器

8.1 Fuzzifiers

$$x^* \in U \subset R^n \longrightarrow \boxed{\textbf{Fuzzifier}} \longrightarrow \text{A' in U}$$

- *Singleton fuzzifier*

$$\mu_{A'}(x) = \begin{cases} 1, & \text{if} \quad x = x^*; \\ 0, & \text{otherwise.} \end{cases}$$

- *Gaussian fuzzifier*

$$\mu_{A'}(x) = \exp\left(-\left(\frac{x_1 - x_1^*}{a_1}\right)^2\right) * \cdots * \exp\left(-\left(\frac{x_n - x_n^*}{a_n}\right)^2\right)$$

- *Triangular fuzzifier*

$$\mu_{A'}(x) = \begin{cases} \left(1 - \frac{|x_1 - x_1^*|}{b_1}\right) * \cdots * \left(1 - \frac{|x_n - x_n^*|}{b_n}\right), & \text{if } |x_i - x_i^*| \le \mathrm{b_i}; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 8.1 Consider the canonical fuzzy rules

IF x_1 is A_1^l and … and x_n is A_n^l, THEN y is B^l

with l=1,2,…,M and

$$\mu_{A_i^l}(x_i) = \exp(-(\frac{x_i - \bar{x}_i^l}{\sigma_i^l})^2)$$

where $\bar{x}_i^l$ and σ_i^l are constants. If we use the Gaussian fuzzifier with * = product, then the product inference engine gives

$$\mu_{B'}(y) = \max_{l=1}^{M}\left\{\prod_{i=1}^{n}\exp(-(\frac{x_{ip}^l - \bar{x}_i^l}{\sigma_i^l})^2)\exp(-(\frac{x_{ip}^l - x_i^*}{a_i})^2)\mu_{B^l}(y))\right\}$$

where

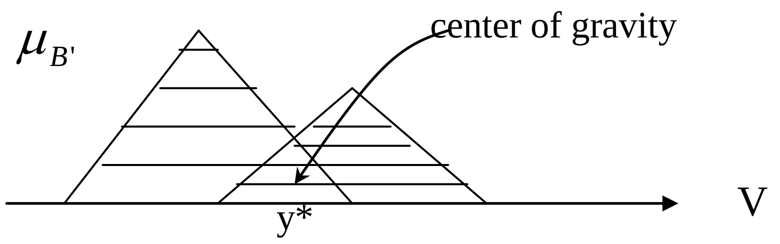
$$x_{ip}^l = \frac{a_i^2\bar{x}_i^l + \left(\sigma_i^l\right)^2 x_i^*}{a_i^2 + \left(\sigma_i^l\right)^2}$$

8.2 Defuzzifiers

e.g.

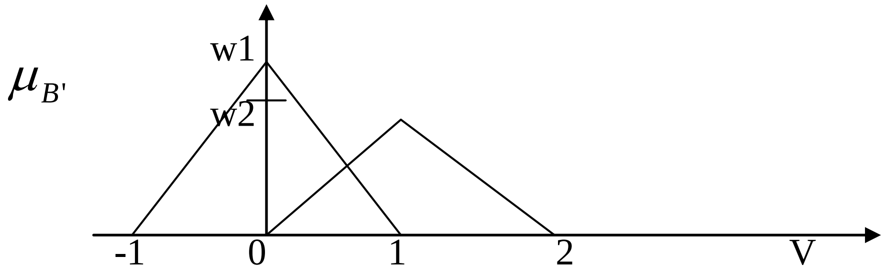
$$\mu_{B'}(y) = \max_{l=1}^{M} \left\{ \sup_{x \in U} \left(\mu_{A'}(x) \prod_{i=1}^{n} \mu_{A_i^l}(x_i) \mu_{B^l}(y) \right) \right\}$$

- *Center of Gravity Defuzzifier*

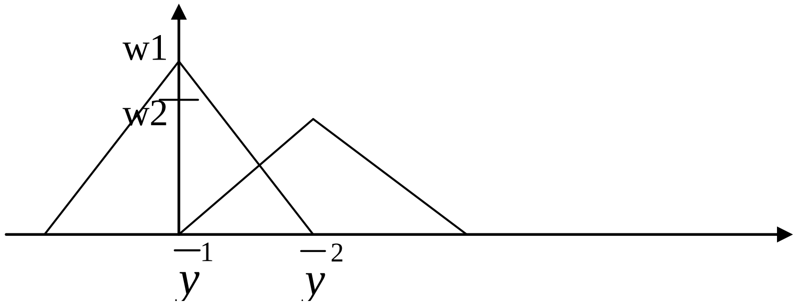


$$y^* = \frac{\int_V y \mu_{B'}(y) dy}{\int_V \mu_{B'}(y) dy}$$

Example*:*



• *Center Average Defuzzifier*



$$y^* = \frac{\sum_{l=1}^{M} \bar{y}^l \omega_l}{\sum_{l=1}^{M} \omega_l}$$

where $\bar{y}^l$: center of l'th fuzzy set and ω_l : height of l'th fuzzy set.

Example: If $\bar{y}^1$ =0, $\bar{y}^2$ =1, then $y^* = \dfrac{\omega_2}{\omega_1 + \omega_2}$.

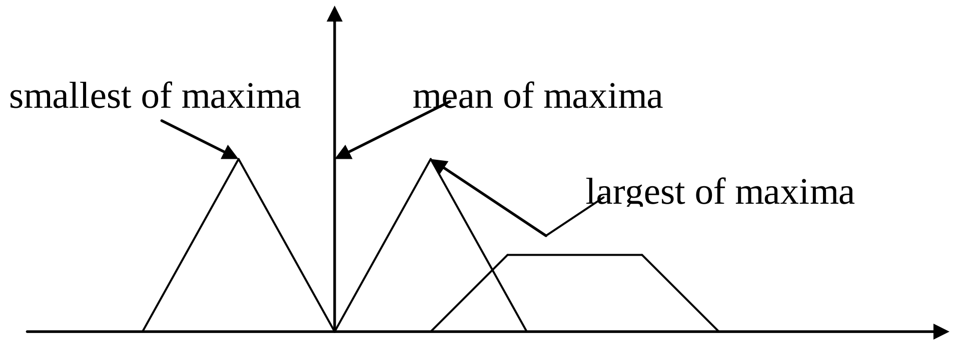
• *Maximum Defuzzifier*

Define the height set of B’ as

$$hgh(B') = \{y \in V \mid \mu_{B'}(y) = \sup_{y\in V} \mu_{B'}(y)\}$$

then

$$y^* = \begin{cases} \text{any point in hgh(B')} \\ \inf\{y \in \text{hgh(B')}\} & \text{(smallest of maxima defuzzifier)} \\ \sup\{y \in \text{hgh(B')}\} & \text{(largest of maxima defuzzifier)} \\ \dfrac{\int_{hgh(B')} y\,dy}{\int_{hgh(B')} dy} & \text{(mean of maxima defuzzifier).} \end{cases}$$

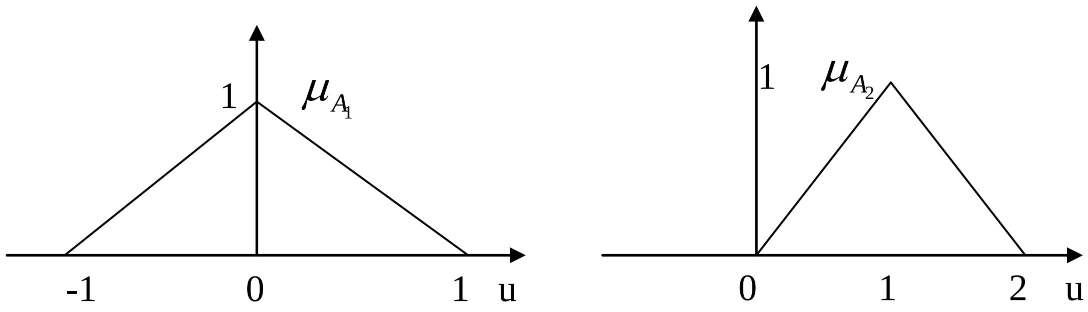


Example: Consider a fuzzy system with the following two rules:

IF x_1 is A_1 and x_2 is A_2, THEN y is A_1;

IF x_1 is A_2 and x_2 is A_1, THEN y is A_2.

where



Suppose the input to fuzzy system is $(x_1^*, x_2^*) = (0.3, 0.6)$ and we use the singleton fuzzifier. Determine the output y* if the fuzzy system in the following situations:

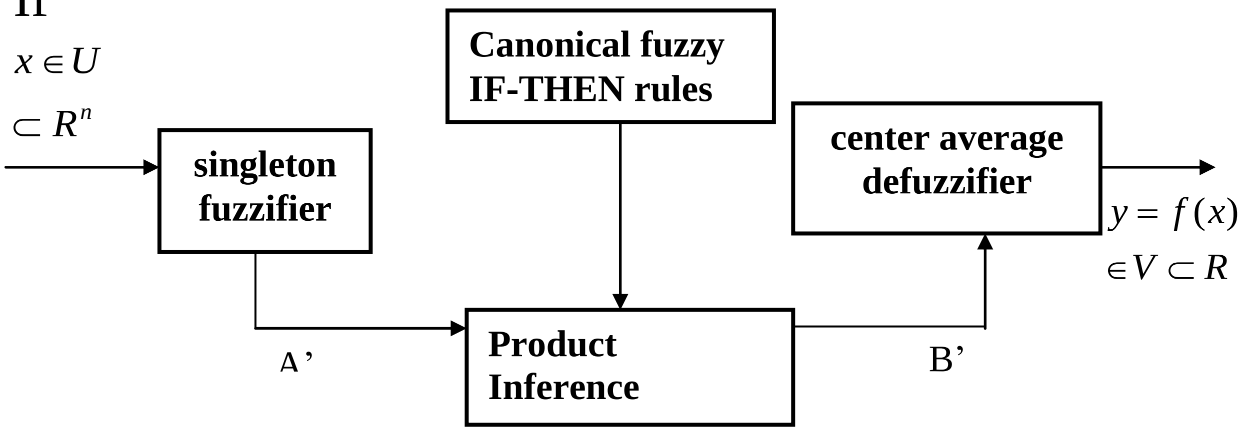
(a) product inference engine (7.23) and center average defuzzifier.
(b) Product inference engine and center of gravity defuzzifier.
(c) Lukasiewicz inference engine (7.30) and mean of maxima defuzzifier.
(d) Lukasiewicz inference engine and certer average defuzzifier.

三、各种模糊系统的构建

9.1 Formulas of some classes of fuzzy systems

- *Fuzzy System with Center Average Defuzzifier*

Lemma 9.1: If

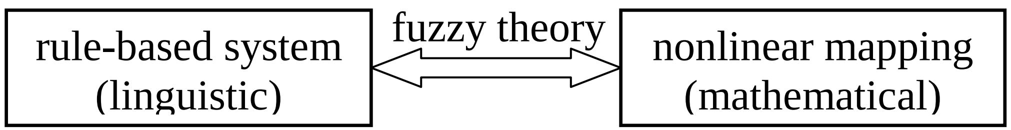


then

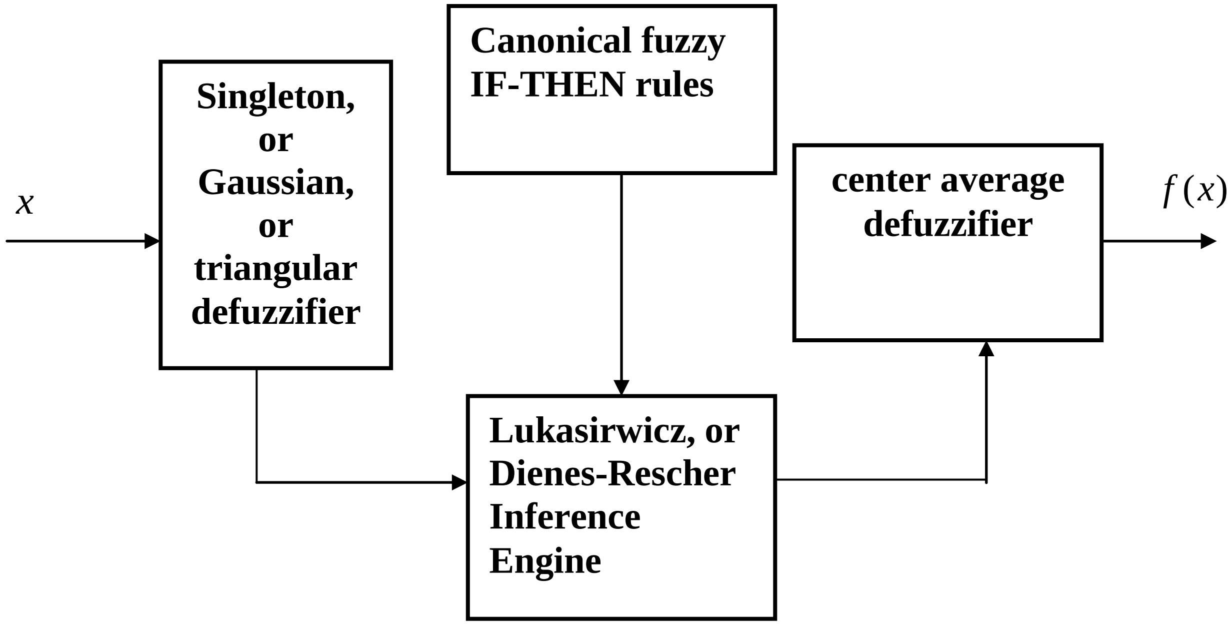
$$f(x) = \frac{\sum_{l=1}^{M} \bar{y}^l \left(\prod_{i=1}^{n} \mu_{A_i^l}(x_i)\right)}{\sum_{l=1}^{M} \left(\prod_{i=1}^{n} \mu_{A_i^l}(x_i)\right)}$$

where $\bar{y}^l$ is the center of $\mu_B{}^l$.

The dual rule of fuzzy systems



Lemma 9.3 If



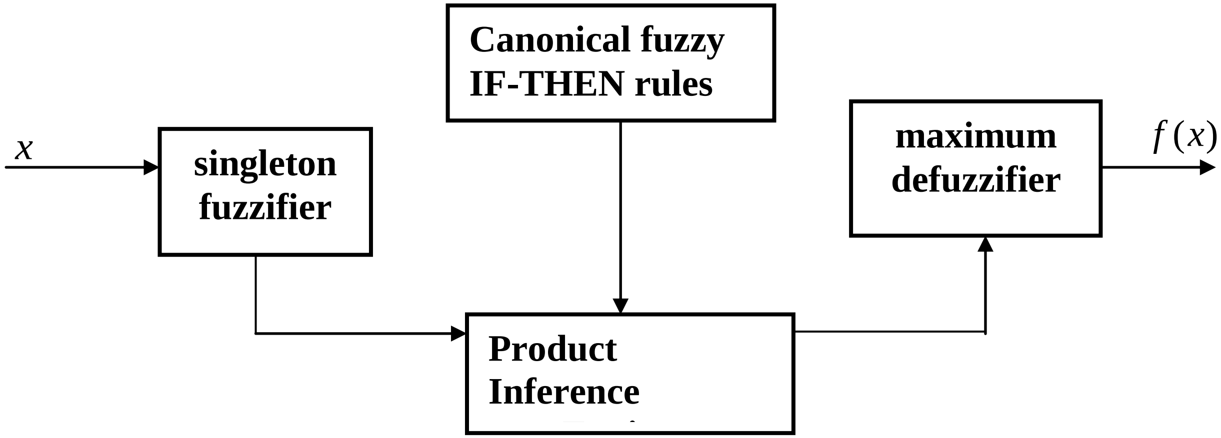
then:

$$f(x) = \frac{1}{M}\sum_{l=1}^{M}\bar{y}^l$$

where $\bar{y}^l$ is the center of B^l.

• *Fuzzy system with Maximum Defuzzifier*

Lemma 9.4: If



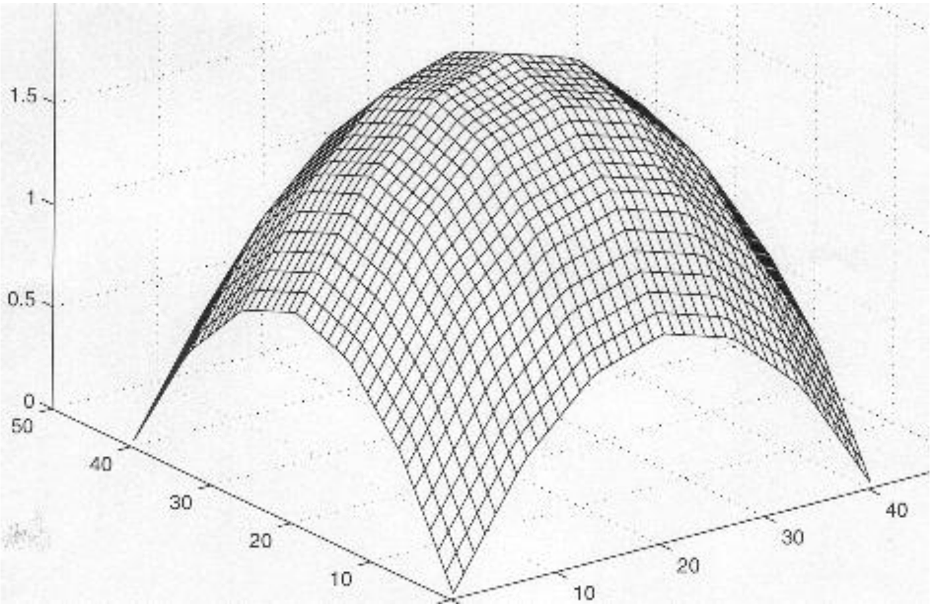
then:

$$f(x) = \bar{y}^{l_*}$$

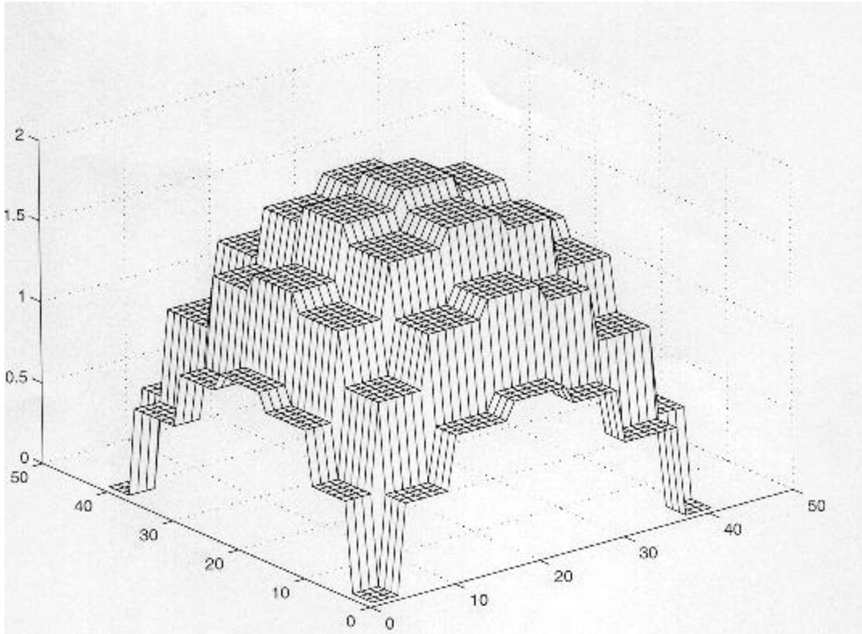
where $l_* \in \{1,2,...M\}$ such that

$$\prod_{i=1}^{n} \mu_{A_i^{l_*}}(x_i) \geq \prod_{i=1}^{n} \mu_{A_i^{l}}(x_i)$$

for all $l = 1,2,...,M$.



Fuzzy system with center average defuzzifier



Fuzzy system with maximum defuzzifier

四、模糊系统的万能逼近特性

Question: Given any nonlinear function g(x) over $x \in U \subset R^n$, can we find a fuzzy system f(x) such that the difference between g(x) and f(x) over U can be arbitrarily small?

Theorem 9.1: (Universal Approximation Theorem) Fuzzy system in the form of:

$$f(x) = \frac{\sum_{l=1}^{M} \bar{y}^l [\prod_{i=1}^{n} a_i{}^l \exp(-(\frac{x_i - \bar{x}_i{}^l}{\sigma_i{}^l})^2)]}{\sum_{l=1}^{M} [\prod_{i=1}^{n} a_i{}^l \exp(-(\frac{x_i - \bar{x}_i{}^l}{\sigma_i{}^l})^2)]} \qquad (9.6)$$

are universal approximators. That is, for any continuous function g(x) on the compact set $U \subset R^n$, there exists a fuzzy system f(x) in the form of (9.6) such that

$$\sup_{x \in U} | f(x) - g(x) | < \varepsilon$$

where $\varepsilon > 0$ can be arbitrarily small.

Stone-Weierstrass Theorem: Let Z be a set of continuous functions on a compact set U. If

(i) Z is an algebra, i.e., Z is closed under addition, multiplication, and scalar multiplication.
(ii) Z seperates points on U, i.e., $\forall x, y \in U, x \neq y, \exists f \in Z$ s.t. $f(x) \neq 0$.
(iii) Z vanishes at no point of U, i.e., $\forall x \in U, \exists f \in Z$ s.t. $f(x) \neq 0$.

Then for any continuous g(x), there exists $f \in Z$ s.t.

$$\sup_{x \in U} | f(x) - g(x) | < \varepsilon$$

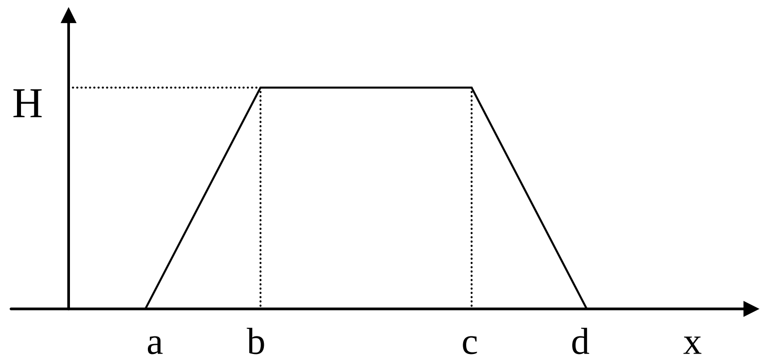
• **The Basic Problem**: Find a fuzzy system f(x) to approximate nonlinear mapping g(x): $U \subset R^n \to R$. Three cases:

(1) The analytic formula of g(x) is known.
(2) The formula of g(x) is unknown, but for any $x \in U$, we know the value g(x).
(3) We only know a limited number of input-output pairs $(x^j, g(x^j))$,where $x^j \in U$ cannot be arbitrarily chosen.

10.1 Preliminary Concepts

• Pseudo-Trapzoid Membership Function:

$$\mu_A(x;a,b,c,d,H) = \begin{cases} I(x), & \mathrm{x} \in [\mathrm{a},\mathrm{b}) \\ H, & \mathrm{x} \in [\mathrm{b},\mathrm{c}] \\ D(x), & \mathrm{x} \in (\mathrm{c},\mathrm{d}] \\ 0, & \mathrm{x} \in R-(\mathrm{a},\mathrm{d}) \end{cases}$$



where I(x): nondecreasing with I(a)=0, I(b)=H.
D(x): nondecreasing with D(c)=H, D(d)=0.

• *Completeness of Fuzzy Sets*: Fuzzy sets $A^1, A^2, \ldots, A^N$ in $W \subset R$ are said to be complete on W if for any $x \in W$, there exists A^j such that $\mu_{A^j}(x) > 0$.

• *Consistency of Fuzzy Sets*: Fuzzy sets $A^1, A^2, \ldots, A^N$ in $W \subset R$ are said to be consistent on W if $\mu_{A^j}(x) = 1$ for some $x \in W$ implies that $\mu_{A^i}(x) = 0$ for all $i \neq j$.

• *High Set of Fuzzy Set*:

$$hgh(A) = \{x \in W \mid \mu_A(x) = \sup_{x' \in W} \mu_A(x')\}$$

• *Order between Fuzzy Sets*:

For any fuzzy sets A, B in $W \subset R$, we say A>B if hgh(A)>hgh(B) (i.e., for any $x \in hgh(A), x' \in hgh(B)$, we have x>x').

• **Lemma 10.1**: If $A^1, A^2, \ldots, A^N$ are normal and consistent fuzzy sets in $W \subset R$ with pseudo-trapzoid membership functions $\mu_{A^i}(x; a_i, b_i, c_i, d_i)$, then there exists a rearrangement $\{i_1, i_2, \ldots, i_N\}$ of {1,2,…N} such that

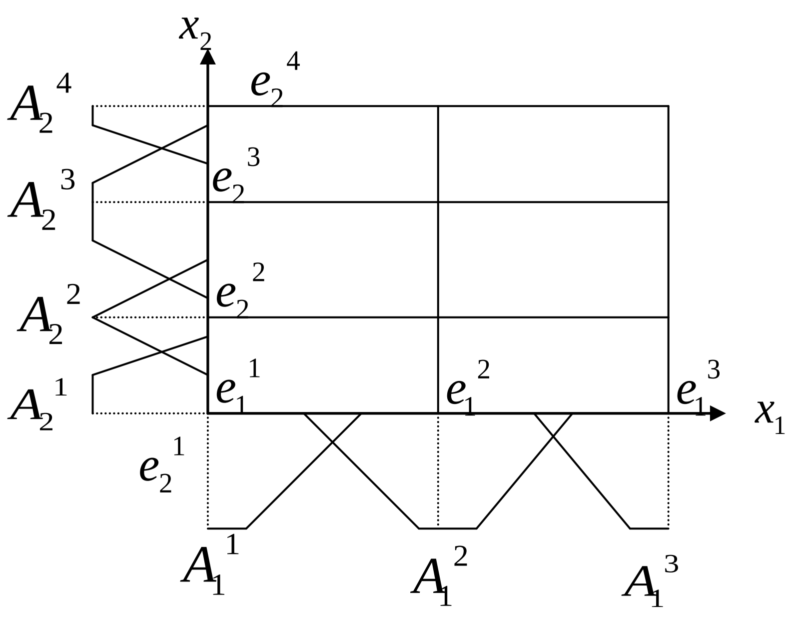
$$A^{i_1} < A^{i_2} < \cdots < A^{i_N}$$

10.2 Design of the Fuzzy System

• **The Problem**: Let g(x) be a function on $U=[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]\subset R^2$ and we know the value g(x) for any $x\in U$. Design a fuzzy system f(x) to approximate g(x) over U.

Design of Fuzzy System

• *Step 1:* Define Ni fuzzy sets A_i^1, …, A_i^{Ni} in $[\alpha_i,\beta_i]$ which are normal, consistent, complete with pseudo-trapzoid membership functions, and $A_i^1 < A_i^2 < \cdots < A_i^{N_i}$. $i=1,2.$



• *Step 2:* Construct M= $N_1 \times N_2$ rules:

$$Ru^{i_1 i_2} : IF \quad x_1 \text{ is } A_1^{i_1} \text{ and } x_2 \text{ is } A_2^{i_2}, \quad \text{THEN } y \text{ is } B^{i_1 i_2}.$$

where $i_1 = 1,\cdots,N_1, i_2 = 1,\cdots,N_2$, and the center of $B^{i_1 i_2}$ is

$$\bar{y}^{i_1 i_2} = g(e_1^{i_1}, e_2^{i_2})$$

• *Step 3:* Design the fuzzy system as

$$f(x) = \frac{\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} \bar{y}^{i_1 i_2}(\mu_{A_1^{i_1}}(x_1)\mu_{A_2^{i_2}}(x_2))}{\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}(\mu_{A_1^{i_1}}(x_1)\mu_{A_2^{i_2}}(x_2))} \qquad (10.10)$$

Theorem 10.1 The designed fuzzy system satisfies

$$\sup_{x\in U} |g(x) - f(x)| \le \left\| \frac{\partial g}{\partial x_1} \right\|_\infty h_1 + \left\| \frac{\partial g}{\partial x_2} \right\|_\infty h_2$$

where $h_i = \max_{1\le j\le N_I - 1} |e_i^{j+1} - e_i^{j}|$ and $\| d(x) \|_\infty = \sup_{x\in U} | d(x) |$.

Proof:

Remark: For continuously differentiable g(x), $\|\frac{\partial g}{\partial x_1}\|_\infty$ and $\|\frac{\partial g}{\partial x_2}\|_\infty$ are finite, so by making h_1, h_2 sufficiently small, we can make $\| f - g \|_\infty$ arbitrarily small.

Example: Design a fuzzy system f(x) to approximate g(x)=sinx over U=[-3,3] with accuracy $\varepsilon = 0.2$, that is,

$$\sup_{x\in U} | g(x) - f(x) | \leq | \varepsilon .$$

Example: Design a fuzzy system f(x) to approximate $g(x)=0.52+0.1x_1+0.28x_2-0.06 x_1x_2$ over U=$[-1,1]\times[-1,1]$ with accuracy $\varepsilon = 0.1$.

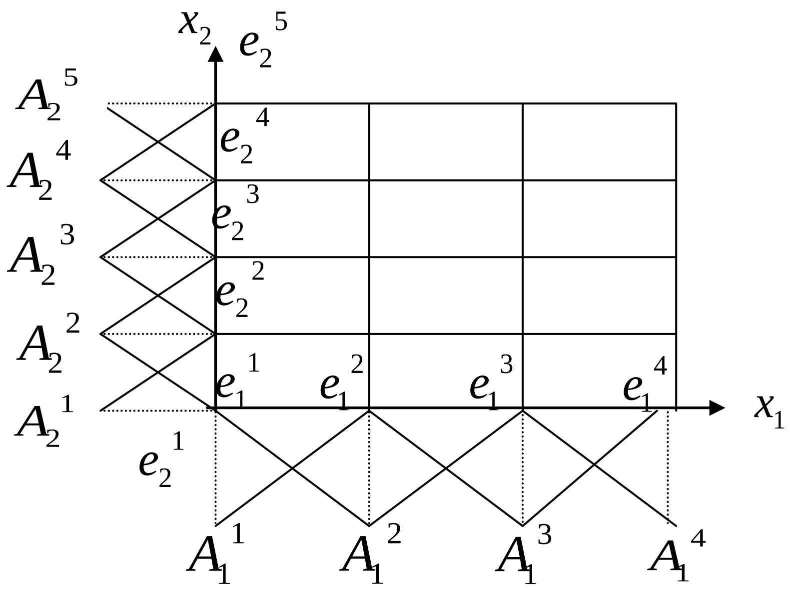
Lemma 10.3: Let f(x) be the designed fuzzy system (10.10) and $e_1^{i_1}, e_2^{i_2}$ be the points defined in the design procedure. Then

$$f(e_1^{i_1}, e_2^{i_2}) = g(e_1^{i_1}, e_2^{i_2})$$

Proof:

11.1 Fuzzy Systems with Second-Order Approximation Accuracy

• **Theorem 11.1**: If we use the following triangular membership functions in designing the fuzzy system f(x) in Chapter 10:



then

$$\sup_{x \in U} | f(x) - g(x) | \le \frac{1}{8} [\| \frac{\partial^2 g}{\partial x_1^2} \|_\infty h_1^2 + \| \frac{\partial^2 g}{\partial x_2^2} \|_\infty h_2^2]$$

where $h_i = \max_{1 \le j \le N_i - 1} | e_i^{j+1} - e_i^j |$.

Example: Design a fuzzy system f(x) to approximate g(x)=sin(x) with $\varepsilon = 0.2$.

Corrolary: Let f(x) be the fuzzy system designed above. If

$$g(x) = \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} a_{k_1 k_2} x_1^{k_1} x_2^{k_2}$$

where $a_{k_1 k_2}$ are constants, then f(x)=g(x) for all $x \in U$.

11.2 Design of the Fuzzy Systems with Maximum Defuzzifier

- Step 1: Same as in Section 11.1;
- Step 2: Same as in Section 10.2;
- Step 3: The designed fuzzy system is

$$f(x) = \bar{y}^{i_1^* i_2^*} = g(e_1^{i_1^*}, e_2^{i_2^*})$$

where $i_1^* i_2^*$ is such that

$$\mu_{A_1^{i_1^*}}(x_1)\mu_{A_2^{i_2^*}}(x_2) \geq \mu_{A_1^{i_1}}(x_1)\mu_{A_2^{i_2}}(x_2)$$

for all $i_1 = 1,2,...,N_1$, $i_2 = 1,2,...,N_2$.

• **Theorem 11.2**: The fuzzy system designed above satisfies

$$\sup_{x\in U} | g(x) - f(x) | \le \| \frac{\partial g}{\partial x_1} \|_\infty h_1 + \| \frac{\partial g}{\partial x_2} \|_\infty h_2$$

Example: Design a fuzzy system f(x) to approximate g(x)=sinx over U=[-3,3] .

Lemma: Fuzzy system with maximum defuzzifier cannot be second-order approximators.
Proof: