SIT 787 - MATHEMATICS FOR AI

ASSIGNMENT-3

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Question 1, Graph Theory:

Consider the following incidence matrix of a graph $G=(V,E)$ with $V=\{a,b,c,d\}$ and $\{e_1,e_2,e_3,e_4,e_5,e_6\}$

$$M = \begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline a & 1 & -1 & 0 & 1 & -1 & 0 \\ b & -1 & 1 & -1 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 1 & 1 \\ d & 0 & 0 & 1 & -1 & 0 & -1 \end{array}$$

a, What type of graph does M represent?

$G=(V,E)$.

Vertices $\Rightarrow V=\{a,b,c,d\}$

Edges $\Rightarrow E=\{e_1,e_2,e_3,e_4,e_5,e_6\}$

$$\text{Incidence Matrix } M \Rightarrow \begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline a & 1 & -1 & 0 & 1 & -1 & 0 \\ b & -1 & 1 & -1 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 1 & 1 \\ d & 0 & 0 & 1 & -1 & 0 & -1 \end{array}$$

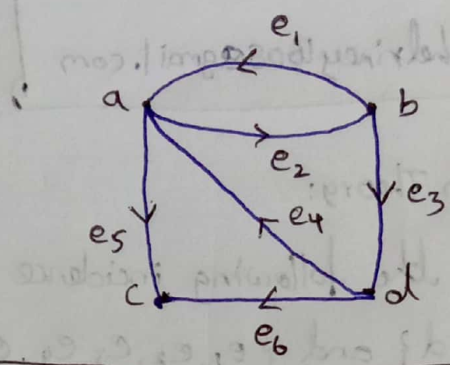
The rows of the matrix represent vertex and columns of the matrix represent Edges, the values of M shows the relation between Vertices and Edges.

In the incidence Matrix 'M' the values represent

For (i) -1 the edge is directed away from vertex

(ii) 0 there is no path

(iii) 1 the edge is directed towards the vertex



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b) Find the Adjacency Matrix A for their graph.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

① e_1 connects to the vertex of b and a

② e_2 connects to the vertex of a and b

③ e_3 connects to the vertex of b and d

④ e_4 connects to the vertex of d and a

⑤ e_5 connects to the vertex of a and c

⑥ e_6 connects to the vertex of d and c

Now we write Adjacency Matrix as

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

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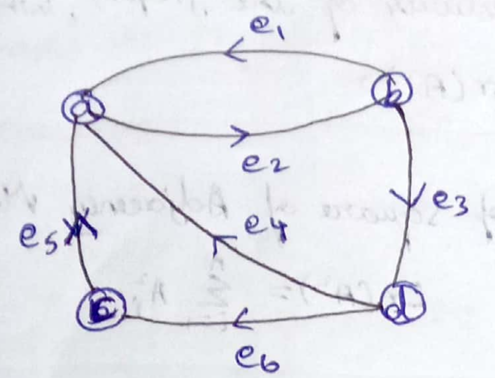
c, Draw the graph



d, How many paths of length 2 are there between nodes b & c

To find the path, we need to multiply the A with times of length given. The length given is 2,

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

length = 2

Multiply $A \Rightarrow A * A$

$$A \cdot A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+0+0 & 0+0+0+0 & 0+0+0+0 & 0+1+0+0 \\ 0+0+0+1 & 1+0+0+0 & 1+0+0+1 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 1+0+0+0 & 1+0+0+0 & 0+0+0+0 \end{bmatrix}$$

$$= \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$\Rightarrow$ $\boxed{\text{The paths between nodes b \& c is 2}}$

e) In terms of connectivity of the graph, what is your interpretation of $tr(A^2)$?

Trace of square of Adjacency Matrix

$$tr(A^2) = \sum_{i=1}^{n} A^2_{i,i}$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

For Interpretation of $tr(A^2)$ it represents the sum of number of paths of length 2 that starts and ends at the same vertex.

So,

For Vertex 1: $A^2[0][0] = 1 \Rightarrow$ No. of paths = 1

For vertex 2: $A^2[1][1] = 1 \Rightarrow$ No. of paths = 1

For vertex 3: $A^2[2][2] = 0 \Rightarrow$ No paths present

For vertex 4: $A^2[3][3] = 0 \Rightarrow$ No paths present

This gives insights into local connectivity of the graph,

the higher the value of $tr(A^2)$, the more closer loops of length 2 indicates denser local connectivity around the vertices involved.

In vertices 3 and 4, it does not have any path of length 2 to return itself.

f, Without direct calculation, find one of the eigenvalues of A based on the information you can get from A, then calculate its corresponding eigen vector.

$tr(A) = A[0][0] + A[1][1] + A[2][2] + A[3][3]$

$= 0 + 0 + 0 + 0$

$= 0$

Sum of the eigen values of A is 0

Determinant of A =

In A 3rd row is all of 0

$\Rightarrow det(A) = 0$

Eigen value is also 0.

Eigen vector:

Finding the eigenvector for the eigenvalue 0.

Eigen vector for $\lambda = 0$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

swap R_1 & R_2

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Subtracting R_4 with R_1 at R_4

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Swap R_3 and R_4

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_1 + v_4 = 0$

$v_2 + v_3 = 0$

$v_3 - v_4 = 0$

$v_3 = v_4$

$v_1 + v_4 = 0 \Rightarrow v_1 = -v_4 \Rightarrow v_4 = -v_2$

$v_2 + v_3 = 0 \Rightarrow v_3 = -v_2$

$v_2 = -v_3 = -(-v_2) = v_2$

$$v = \begin{bmatrix} v_2 \\ v_2 \\ -v_2 \\ -v_2 \end{bmatrix}$$

taking v_2 common

$$\Rightarrow v_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

The EigenVector corresponding to eigenvalue

$$\boxed{v = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}$$

Question 2) Probability, Bayes' Theorem,

A data scientist is interested in studying the relationship between having a social media account and obesity, He collected some data represented in the following table.

Individual	Having a social media account	obesity
Individual 1	Yes	Yes
Individual 2	Yes	Yes
Individual 3	Yes	Yes
Individual 4	No	No
Individual 5	No	Yes

a, Find the probability distribution of having a social media account and obesity

Lets create contigency table

	Obesity (yes)	obesity (No)	Total
social media (yes)	3	0	3
social media (No)	1	1	2
	4	1	5

① Probability of having a social media account distribution

Lets take x,

$$P(x = yes) = \frac{\text{No. of Individuals has social media}}{\text{Total No. of Individuals}}$$

$$= \frac{3}{5}$$

$$\boxed{P(x = yes) = 0.6}$$

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Probability of Individuals who has No Social media ③

$$P(X=No) = \frac{\text{Prob of Individuals has no social media}}{\text{Total no. of Individuals}}$$

$$= \frac{2}{5}$$

$$\boxed{P(X=No) = 0.4}$$

② Probability distribution of having obesity

Let's take Y.

$$P(Y=yes) = \frac{\text{Prob of Individuals has obesity}}{\text{Total no. of Individuals}}$$

$$= \frac{4}{5}$$

$$\boxed{P(Y=yes) = 0.8}$$

probability of Individuals who has no obsity

$$P(Y=No) = \frac{\text{Prob of Individuals with no obesity}}{\text{Total. No of Individual}}$$

$$= \frac{1}{5}$$

$$\boxed{P(Y=No) = 0.2}$$

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b, Find the joint probability distribution of having a social media account and obesity.

Joint probability distribution,

let define P(x, y)

$$P(x=yes, y=yes) = \frac{3}{5} = 0.6$$

$$P(x=yes, y=No) = \frac{0}{5} = 0$$

$$P(x=No, y=yes) = \frac{1}{5} = 0.2$$

$P(x = No, y = No) = \frac{1}{5} = 0.2$

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c) Are these two distributions independents?

For two distributions to be independent, by checking the joint distribution is equal to the marginal distribution for all the combinations.

$$P(x,y) = P(x)\,P(y)$$

Probability distribution of having social media and not.

$P(x = yes) = 0.6$
$P(x = No) = 0.4$

Probability distribution of having obesity and not.

$P(y = yes) = 0.8$
$P(y = No) = 0.2$

Joint probability distribution,

$P(x = yes, y = yes) = \frac{3}{5} = 0.6$

$P(x = yes, y = No) = \frac{0}{5} = 0$

$P(x = No, y = yes) = \frac{1}{5} = 0.2$

$P(x = No, y = No) = \frac{1}{5} = 0.2$

Check the Independent,

$$P(x,y) = P(x)\,P(y)$$

$P(x = yes, y = yes) = 0.6$

$P(x = yes) = 0.6$
$P(y = yes) = 0.8$

$0.6 = 0.6 \times 0.8 = 0.48.$

$\Rightarrow 0.6 \neq 0.48 \Rightarrow P(x,y) \neq P(x).P(y)$
The varitables are not independent.

d) Compute the mutual information between having a social media account and obesity

$$\text{mutual information} = \sum_x \sum_y P_{x,y}(x,y) \ln \frac{P_{x,y}(x,y)}{P_x(x)\, P_y(y)}$$
$I(X;Y)$

i) $P(X = \text{Yes}, Y = \text{Yes}) \Rightarrow$

$P(X = \text{yes}, Y = \text{yes}) = 0.6$

$P(X = \text{yes}) = 0.6$

$P(Y = \text{yes}) = 0.8$

$P(X = \text{yes}) \cdot P(Y = \text{yes}) = 0.6 \times 0.8 = 0.48.$

$\Rightarrow \log\left(\frac{0.6}{0.48}\right) = \log(1.25)$

$\simeq 0.2231$

$0.6 \times 0.2231 = 0.1338$

ii) $P(X = \text{Yes}, Y = \text{No}) \Rightarrow$

$P(X = \text{Yes}, Y = \text{No}) = 0$

$P(X = \text{Yes}) = 0.6$

$P(Y = \text{No}) = 0.2$

$\Rightarrow 0.6 \times 0.2 = 0.12$

$\Rightarrow \log\left(\frac{0}{0.12}\right) = \log(0).$

$\Rightarrow 0 \times \log(0) = 0$

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iii, for $P(X = No,\ Y = yes)$

$P(X = No, Y = yes) = 0.2$

$P(X = No) = 0.4$

$P(Y = yes) = 0.8$

$P(X, Y) = P(X) \cdot P(Y)$

$= 0.4 \times 0.8$

$= 0.32$

$\Rightarrow \log\left(\frac{0.2}{0.32}\right) = \log(0.625) =$

$\simeq -0.470$

$I(X:Y) \Rightarrow 0.2 \times -0.470$

$= -0.094$

iv, for $P(X = No, Y = No)$

$P(X = No, Y = No) = 0.2$

$P(X = No) = 0.4$

$P(Y = No) = 0.2$

$P(X) \cdot P(Y) = 0.4 \times 0.2 = 0.08$

$\Rightarrow \log\left(\frac{0.2}{0.08}\right) = \log(2.5)$

$\simeq 0.916$

$I(X:Y) = 0.2 \times 0.916 = 0.1832$

$I(X, Y) = 0.1338 + 0 - 0.094 + 0.1832$

$\boxed{I(X, Y) = 0.22306}$

Mutual Information between having a Social media acco[unt] and obesity is 0.22306

This value quantifies the information about the individual has social media acount who has obesity

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Q, Theoretically, what is the mutual Information of two independent random variables?

Mutual Information of two independent random variables is theoritically zero.

Because it measures the amount of information gained about one variable through observing other. If x and y are random variables, where x provides no information about y.

for independent variables =>

$P_{x,y}(x,y) = P_x(x)$.

substituting it in the $I(X:Y)$ mutual information

$$I(X:Y) = \sum_x \sum_y P_{x,y}(x,y)$$

$$= \sum_x \sum_y P_{x,y}(x,y) \ln(1)$$

$$I(X:Y) = 0 \qquad [\because \ln(1) = 0]$$

Mutual Information is zero. This shows the value of one variable doesn't provide any information about other variable

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Question 3) Probability Distribution:

Let x be a discrete random variable that takes values in $\{-2, -1, 0, 1, 2\}$ with equal probability. Also, Y is another discrete random variable defined as $Y = |x|$.

a) Construct the joint probability table. Are x and Y independent? Justify.

$$X = \{-2, -1, 0, 1, 2\}$$

$$Y = |x|$$

$$Y = \{0, 1, 2\}$$

Total probability is 5

the probabilities are

$$P(Y=0) = P(X=0) = \frac{1}{5}$$

$$P(Y=1) = P(X=-1) + P(X=1)$$
$$= \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

$$P(Y=2) = P(X=-2) + P(X=2)$$
$$= \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

Joint Probability:

i) $Y=0 \Rightarrow X=0$

$$P(X=0, Y=0) = P(X=0) = \frac{1}{5}$$

ii) $Y=1 \Rightarrow X=-1, X=1$

$$P(X=-1) \Rightarrow P(X=-1, Y=1) = \frac{1}{5}$$
$$P(X=1) \Rightarrow P(X=1, Y=1) = 1/5$$

iii) $Y=2 \Rightarrow X=-2, X=2$

$$P(X=-2) \Rightarrow P(X=-2, Y=2) = \frac{1}{5}$$
$$P(X=2) \Rightarrow P(X=2, Y=2) = \frac{1}{5}$$

Joint Probability Distribution Table:

$P(x,y)$	$y=0$	$y=1$	$y=2$	
$x=-2$	0	0	$1/5$	$1/5$
$x=-1$	0	$1/5$	0	$1/5$
$x=0$	$1/5$	0	0	$1/5$
$x=1$	0	$1/5$	0	$1/5$
$x=2$	0	0	$1/5$	$1/5$
	$1/5$	$2/5$	$2/5$	1

For checking independent :

By $P(X=x, Y=y) = P(X=x)\,P(Y=y)$

Product of Marginal probability against the joint probability

$$P(X=-2) \cdot P(Y=2) = \frac{1}{5} \cdot \frac{2}{5} = \frac{2}{25}$$

$$P(X=-2, Y=2) = \frac{1}{5}$$

$$\boxed{\frac{1}{5} \neq \frac{2}{25}}$$

Here we see the product of marginal probabilityes doesn't match the joint probability for all (x,y)

So X and Y are not independent

b, Find corr(x,y)

Formula for corr(x,y), to find correlation of x and y is by calculate covariance of x and y and standard deviation $\sigma_x \sigma_y$

$$corr(x,y) = \frac{cov(x,y)}{\sigma_x \sigma_y}$$

For calculating covariance of x and y, we need to get expected values of E(x), E(y) and E(x,y)

$$cov(x,y) = E(x,y) - E(x)\,E(y)$$

$$E(x) = \sum_x x \cdot P(X=x).$$

$$= (-2)(1/5) + (-1)(1/5) + (0)(1/5) + (1)\left(\tfrac{1}{5}\right) + 2\left(\tfrac{1}{5}\right)$$

$$= -\frac{2}{5} - \frac{1}{5} + \frac{1}{5} + \frac{2}{5}$$

$$E(x) = 0$$

$$E(y) = \sum_y y \cdot P(Y=y).$$

$$= (0)\left(\tfrac{1}{5}\right) + (1)(2/5) + (2)(2/5)$$

$$= \frac{2}{5} + \frac{4}{5}$$

$$E(y) = \frac{6}{5}$$

$$E(x,y) = (-2)(2)(1/5) + (-1)(1)\left(\tfrac{1}{5}\right) + 0 + (1)(1)\frac{1}{5} + (2)(2)\left(\tfrac{1}{5}\right)$$

$$= -\frac{4}{5} - \frac{1}{5} + \frac{1}{5} + \frac{4}{5}$$

$$E(x,y) = 0$$

$$cov(x,y) = E(x,y) - E(x)\,E(y)$$

$$= 0 - 0 \cdot (6/5).$$

$$\boxed{cov(x,y) = 0}$$

Calculating Standard Deviation

$$\sigma_x = \sqrt{E(x^2) - (E(x))^2}$$

$$\sigma_y = \sqrt{E(y^2) - (E(y))^2}$$

$$E(x^2) = \sum_x x^2 P(X=x) = (-2)^2\left(\tfrac{1}{5}\right) + (-1)^2\left(\tfrac{1}{5}\right) + 0 + (1)^2\left(\tfrac{1}{5}\right) + (2)^2\left(\tfrac{1}{5}\right)$$

$$= 4\left(\tfrac{1}{5}\right) + 1\left(\tfrac{1}{5}\right) + 0 + 1\left(\tfrac{1}{5}\right) + 4\left(\tfrac{1}{5}\right)$$

$$= \frac{4}{5} + \frac{1}{5} + \frac{1}{5} + \frac{4}{5}$$

$$= \frac{\cancel{10}\,2}{\cancel{5}}$$

$$E(x^2) = 2$$

$$\sigma_x = \sqrt{2 - 0^2} = \sqrt{2}$$

$$E(y^2) = \sum_y y^2 \cdot P(Y=y).$$

$$= (0)^2\left(\tfrac{1}{5}\right) + (1)^2\left(\tfrac{2}{5}\right) + (2)^2\left(\tfrac{2}{5}\right)$$

$$= 0 + \frac{2}{5} + \frac{8}{5}$$

$$= \frac{\cancel{10}\,2}{\cancel{5}}$$

$$E(y^2) = 2$$

$$\sigma_y = \sqrt{2 - \left(\tfrac{6}{5}\right)^2}$$

$$= \sqrt{2 - \frac{36}{25}}$$

$$= \sqrt{\frac{50-36}{25}}$$

$$= \sqrt{\frac{14}{25}}$$

$$= \frac{\sqrt{14}}{5}$$

$$\boxed{\sigma_x = \sqrt{2}} \quad \boxed{\sigma_y = \frac{\sqrt{14}}{5}}$$

$$\boxed{corr(x,y) = \frac{0}{\sqrt{2} \times \sqrt{\frac{14}{5}}} = 0}$$

The correlation between x & y is 0. So, there is no linear relationship between x & y.

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c) Based on your answer to part (a), can you explain the result in part (b)?

From the part (a) result, we have concluded that $P(X=x, Y=y) \neq P(X=x) \cdot P(Y=y)$, where the values of x and y are not independent.

From the part (b) result, the calculation we done using $corr(x,y)$ where $cov(x,y)$ is 0, which the product of standard deviation of x and y, so correlation of x and y is zero, there is no linear relationship between x and y.

The joint distribution, we calculate the product of the marginals for all the pair, here x and y are not independent, but they are not correlated with each other.

Since $Y=|x|$, where x and y should have relationship that is not captured by the linear correlation due to the non linear transformation.

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d, Although you have solved similar questions in question 2 and 3, can you explain the fundamental ~~[illegible]~~ difference between these two questions?

In Q_2 ~~y~~ is defined as linear equation $y = mx + c$ which explicitly defines the linear relationship between the variables. The correlation of two linearly related variables would be non-zero.

In Q_3 y is defined as absolute value of ~~x~~, $Y = |x|$ it is in a non linear transformation. This means the transformation is non linear because y is dependent on x, the linear correlation might be zero.

For the Q_2 case the independency is not applicable due to variable are related through the linear equation. $(y = mx + c)$.

For the Q_3 case it is independent because of Probability distribution, it will provide value of one variable gives no information about the other variable.

The Correlation helps with the relation between 2 variables, especially linear relationship. Here X and Y are not independent. So there will no correlation with x and y.

For the Probability distribution it depends on random variables of 2, it focuses on linear relation.

These are the fundamental difference between ~~[illegible]~~ two questions.

Question 4) Now, let's Solve some optimisation problems.

a, Linearise the function.

$$f(x,y) = \sqrt{2x+3y} - \frac{x}{y} \quad \text{at the point } (3,1).$$

For finding the linear approximation at that point or first order Taylor expansion of $f(x,y)$ and (a,b)

$$\boxed{L(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b)} \quad — ①$$

f_x and f_y is partial derivatives of x and y.

At the point $(3,1)$, compute with function

$$f(3,1) = \sqrt{2(3)+3(1)} - \frac{3}{1}$$

$$= \sqrt{6+3} - 3$$

$$= \sqrt{9} - 3$$

$$= 3 - 3$$

$$\boxed{f(3,1) = 0}$$

Now lets differentiate with f_x:

$$f_x(x,y) = \frac{d}{dx}\left(\sqrt{2x+3y} - \frac{x}{y}\right)$$

$$\frac{d}{dx}\left(\sqrt{2x+3y}\right) = \frac{1}{2}(2x+3y)^{-1/2} \cdot \frac{d}{dx}(2x+3y)$$

$$= \frac{1}{2\sqrt{2x+3y}} \cdot \frac{d}{dx}(2x+3y)$$

$$= \frac{2}{2\sqrt{2x+3y}}$$

$$= \frac{1}{\sqrt{2x+3y}}$$

$$\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{-1}{y}$$

$$\boxed{f_x(x,y) = \frac{1}{\sqrt{2x+3y}} - \frac{1}{y}}$$

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Let's differentiate w.r.t ~~y~~ of f_y:

$$\frac{d}{dy}\left(\sqrt{2x+3y}\right) = \frac{1}{2\sqrt{2x+3y}} \frac{d}{dy}(2x+3y)$$

$$= \frac{1}{2\sqrt{2x+3y}} \times 3$$

$$= \frac{3}{2\sqrt{2x+3y}}$$

$$\frac{d}{dy}\left(-\frac{x}{y}\right) = -x \frac{d}{dy}\left(\frac{1}{y}\right)$$

$$= -x\left(-y^{-2}\right)$$

$$= \frac{x}{y^2}$$

$$\boxed{f_y(x,y) = \frac{3}{2\sqrt{2x+3y}} + \frac{x}{y^2}}$$

Apply $fx(x,y)$ with $(3,1)$:

$$fx(3,1) = \frac{1}{\sqrt{2(3)+3(1)}} - \frac{1}{1}$$

$$= \frac{1}{\sqrt{9}} - 1$$

$$= \frac{1}{3} - 1$$

$$\boxed{f_x(3,1) = \frac{-2}{3}}$$

Apply point $(3,1)$ at $f_y(x,y)$

$$= \frac{3}{2\sqrt{2(3)+3(1)}} + \frac{3}{1^2}$$

$$= \frac{3}{2\sqrt{6+3}} + \frac{3}{1}$$

$$= \frac{3}{2\sqrt{9}} + \frac{3}{1}$$

$$= \frac{3}{2\times 3} + 3$$

$$= \frac{\cancel{3}^{1}}{\cancel{6}_{2}} + 3$$

$$= \frac{1}{2} + 3$$

$$= \frac{7}{2}$$

$$\boxed{f_y(3,1) = 3.5}$$

substituting all the values in eq ①.

$$L(x,y) = f(3,1) + f_x(3,1)(x-3) + f_y(3,1)(y-1)$$

$$= 0 + \left(\frac{-2}{3}\right)(x-3) + 3.5(y-1)$$

$$= -\frac{2}{3}(x-3) + 3.5(y-1)$$

$$= -\frac{2}{3}x + \frac{\cancel{6}^{2}}{\cancel{3}} + 3.5y - 3.5$$

$$= \frac{-2}{3}x + 2 + 3.5y - 3.5$$

$$= \frac{-2}{3}x + 2 + \frac{7}{2}y - \frac{7}{2}$$

$$= -\frac{2}{3}x + \frac{7}{2}y + 2 - \frac{7}{2}$$

$$= \frac{-2}{3}x + \frac{7}{2}y + \left(\frac{4-7}{2}\right)$$

$$\Rightarrow \boxed{L(x,y) = \frac{-2}{3}x + \frac{7}{2}y - \frac{3}{2}}$$ is the linearization function at the point (3,1).

b) Find the second order Taylor polynomial for

$f(x,y) = e^{5x} \ln(1+y)$ at the point $(0,0)$.

For this problem we need to calculate first order partial derivative and second order partial derivative

The second order Taylor polynomial of a function $f(x,y)$ around the point (a,b).

$$\boxed{\begin{aligned} Q(x,y) = {} & f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \\ & \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \\ & \frac{1}{2} f_{yy}(a,b)(y-b)^2 \quad \text{— (1)} \end{aligned}}$$

Apply the point $(0,0)$ for function.

$f(0,0) = e^{5(0)} \ln(1+0)$ $\quad [e_; \ e^0 = 1]$

$= e^0 \ln(1)$ $\quad [\because \ln(1) = 0]$

$= 1 \cdot 0$

$= 0$

$\boxed{f(0,0) = 0}$

(1) First order partial derivative with function w.r.t

$f_x(x,y) = \frac{d}{dx}(e^{5x} \ln(1+y))$.

$= \frac{d\, e^{5x}}{dx}$

$= 5e^{5x}$ $\quad [\because$ Power rule of differenti... $\frac{de^{ax}}{dx} = ae^{ax}]$

Since we differentiate w.r.t x, $\ln(1+y)$ become constant.

$\boxed{f_x(x,y) = \ln(1+y)\, 5e^{5x}}$

First order derivative with respect y

$$f_y(x,y) = \frac{d}{dy}(e^{5x}\ln(1+y)).$$

$$\frac{de^{5x}}{dy} = e^{5x} \text{ becomes constant}$$

$$\frac{d}{dy}(\ln(1+y)) = 1(1+y)^{-1}$$

$$= \frac{1}{1+y}$$

$$\boxed{f_y(x,y) = \frac{1}{1+y} e^{5x}}$$

Apply the point $(0,0)$ for $f_x(x,y)$ and $f_y(x,y)$.

$$f_x(0,0) = \ln(1+0)\,5e^{5\times 0}$$

$$= \ln(1)\,5 \cdot e^{0}$$

$$= 0\times 5\times 1 \qquad [\because \ln(1)=0]$$

$$\boxed{f_x(0,0) = 0} \qquad [\because e^{0}=1].$$

$$f_y(0,0) = \frac{1}{1+0} e^{5(0)}$$

$$= \frac{1}{1} e^{0}$$

$$\boxed{f_y(0,0) = 1}$$

(2) Second Order partial derivative with respect to x

$$f_{xx}(x,y) = \frac{d}{dx}[5e^{5x}.\ln(1+y)]$$

$$= \frac{d}{dx} 5e^{5x} . \underbrace{\ln(1+y)}_{\text{becomes constant}}$$

$$\boxed{f_{xx}(x,y) = 25e^{5x}\ln(1+y)}$$

second derivative w.r.t to y.

$$f_{yy}(x,y) = \frac{d}{dy}\left[\frac{1}{1+y} e^{5x}\right]$$

$$= \frac{d}{dy}\left[\frac{1}{1+y}\right] \underbrace{e^{5x}}_{\text{becomes constant}}$$

$$= -\frac{1}{(1+y)^2} e^{5x}$$

$$\boxed{f_{yy}(x,y) = -\frac{e^{5x}}{(1+y)^2}}$$

$$f_{xy}(x,y) = \frac{d}{dy}\left[5e^{5x} \ln(1+y)\right]$$

$$= 5e^{5x}\left(\frac{1}{1+y}\right)$$

$$\boxed{f_{xy}(x,y) = \frac{5e^{5x}}{1+y}}$$

Apply the point $(0,0)$ for $f_{xx}(x,y)$, $f_{yy}(x,y)$ and $f_{xy}(x,y)$

$$f_{xx}(0,0) = 25e^{5(0)} \ln(1+0)$$

$$= 25e^{0} \cdot \ln(1)$$

$$= 25 \times 1 \times 0$$

$$\boxed{f_{xx}(0,0) = 0}$$

$$f_{yy}(0,0) = \frac{-e^{5(0)}}{(1+0)^2}$$

$$= \frac{-e^{0}}{1^2}$$

$$\boxed{f_{yy}(0,0) = -1}$$

$$f_{xy}(0,0) = \frac{5e^{5(0)}}{1+0}$$

$$= \frac{5e^{0}}{1}$$

$$= 5 \times 1$$

$$\boxed{f_{xy}(0,0) = 5}$$

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Substituting all the values at the equation ①

$$Q(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \frac{1}{2} f_{xx}(0,0)(x-0)^2 + f_{xy}(0,0)(x-0)(y-0) + \frac{1}{2} f_{yy}(0,0)(y-0)^2$$

$$= 0 + 0 \times (x-0) + 1 \times (y-0) + \frac{1}{2}(0)(x-0)^2 + 5 \times (x-0)(y-0) + \frac{1}{2}(-1)(y-0)^2$$

$$= 0 + 0 + y + 0 + 5xy - \frac{1}{2}(y)^2$$

$\Rightarrow$ $\boxed{Q(x,y) = y + 5xy - \frac{y^2}{2}}$ is the second order Taylor polynomial function at the point $(0,0)$.

---

c) For the multivariate function

$$f(x,y,z) = 4x^2 + 2y^2 + z^2 - 2yx + 2zy + y - z$$

i) find all the stationary points of this function.

the critical point are defined by solving the partial derivative equation

$$f_x \rightarrow \frac{d}{dx} f(x,y,z) = 0 \quad \text{— ①}$$

$$f_y \rightarrow \frac{d}{dy} f(x,y,z) = 0 \quad \text{— ②}$$

$$f_z \rightarrow \frac{d}{dz} f(x,y,z) = 0 \quad \text{— ③}$$

First let's solve $f_x(x,y,z) = 0$

$$\frac{d}{dx}\left[x^2y + y^2z + z^2 - 2xy + 2zy + y - z\right] = 0$$

$$2xy + 0 + 0 - 2y + 0 + 0 - 0 = 0$$

$$2xy - 2y = 0 \Rightarrow 2y(x-1) = 0$$

$$\boxed{y(x-1)=0} \quad \text{— (i)}$$

let solve $f_y(x,y,z)=0$

$$\frac{d}{dy}[x^2y+y^2z+z^2-2xy+2yz+y-z]=0$$

$$x^2+2yz+0-2x+2z+1-0=0$$

$$x^2+2yz-2x+2z+1=0$$

$$\boxed{x^2+2yz-2x+2z+1=0} \quad \text{— (ii)}$$

let solve $f_z(x,y,z)=0$

$$\frac{d}{dz}[x^2y+y^2z+z^2-2xy+2yz+y-z]=0$$

$$0+y^2+2z-0+2y+0-1=0$$

$$\boxed{y^2+2z+2y-1=0} \quad \text{— (iii)}$$

From (i) we can get

$$\boxed{x=1},\ \boxed{y=0.}$$

$\boxed{y=0} \Rightarrow$ sub $y=0$ in (iii)

$$0+2z+0-1=0$$

$$2z-1=0$$

$$2z=1$$

$$\boxed{z=\frac{1}{2}}$$

sub $z=\frac{1}{2}$ and $y=0$ in (ii)

$$x^2+2(0)\left(\frac{1}{2}\right)-2x+\cancel{2}\left(\frac{1}{\cancel{2}}\right)+1=0$$

$$x^2+0-2x+2=0$$

$$x^2-2x+2=0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{2^2 - 4(1)(2)}}{2}$$

$$= \frac{2 \pm \sqrt{4 - 8}}{2}$$

$$x = \frac{2 \pm \sqrt{-4}}{2}$$

$$x = \frac{2 + 2i}{2} \qquad x = \frac{2 - 2i}{2}$$

$$\boxed{x = 1 + i} \qquad \boxed{x = 1 - i}$$

For $y = 0$, we get complex values, there is no real root or critical point.

sub $x = 1$ in (ii)

$\boxed{x = 1} \Rightarrow$

$$1^2 + 2yz - 2(1) + 2(z) + 1 = 0$$

$$1 + 2yz - 2 + 2z + 1 = 0.$$

$$2yz + 2z = 0.$$

$$2z(y+1) = 0$$

$$z(y+1) = 0$$

From this eq we get

$$\boxed{z = 0}, \boxed{y = -1}$$

sub $\boxed{z=0}$ in (iii)

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$$y^2+2z+2y-1=0$$

$$y^2+2(0)+2y-1=0$$

$$y^2+2y-1=0$$

solve this equation by

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-2\pm\sqrt{(2^2)-4(1)(-1)}}{2\times 1}$$

$$=\frac{-2\pm\sqrt{4+4}}{2}$$

$$=\frac{-2\pm\sqrt{8}}{2}$$

$$y=\frac{-2\pm 2\sqrt{2}}{2}$$

$\boxed{y=-1+\sqrt{2}}$ $\boxed{y=-1-\sqrt{2}}$

If $y=-1$ in (iii)

$$y^2+2z+2y-1=0$$

$$(-1)^2+2z+2(-1)-1=0$$

$$1+2z-2-1=0$$

$$2z-3+1=0$$

$$2z-2=0$$

$$2(z-1)=0$$

$$z-1=0$$

$$\boxed{z=1}$$

we got real values on partial derivatives for $x=1$, so we have critical point

critical points $\Rightarrow$ $\boxed{\begin{array}{l} a(1, -1-\sqrt{2}, 0) \\ b(1, -1+\sqrt{2}, 0) \\ c(1, -1, 1) \end{array}}$

---

ii) Find the Hessian Matrix

Hessian Matrix can be written as

$$H_f(x,y,z) = Q_f(x,y,z) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

$\Rightarrow$ $f(x,y) = yx^2 + zy^2 + z^2 - 2yx + 2zy + y - z$

lets calculate $\frac{\partial^2 f}{\partial x^2} \Rightarrow$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left[x^2 y + zy^2 + z^2 - 2yx + 2zy + y - z\right]\right]$$

$$= \frac{\partial}{\partial x}\left[2xy + 0 + 0 - 2y + 0 + 0 - 0\right]$$

$$= \frac{\partial}{\partial x}\left[2xy - 2y\right]$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = 2y}$$

lets calculate $\frac{\partial^2 f}{\partial x \partial y} \Rightarrow$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left[x^2 y + zy^2 + z^2 - 2yx + 2zy + y - z\right]\right]$$

$$= \frac{\partial}{\partial x}\left[x^2 + 2zy + 0 - 2x + 2z + 1 - 0\right]$$

$$= \frac{\partial}{\partial x}\left[x^2 + 2zy - 2x + 2z + 1\right]$$

$$= \frac{\partial}{\partial x}\left[x^2 + 2yz - 2x + 2z + 1\right]$$

$$= 2x + 0 - 2 + 0 + 0$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = 2x - 2}$$

let calculate $\frac{\partial^2 f}{\partial x \partial z} \Rightarrow$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial z}\left[x^2y + y^2z + z^2 - 2xy + 2yz + y - z\right]\right.$$

$$= \frac{\partial}{\partial x}\left[0 + y^2 + 2z - 0 + 2y + 0 - 1\right]$$

$$= \frac{\partial}{\partial x}\left[y^2 + 2z + 2y - 1\right]$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial z} = 0}$$

let calculate $\frac{\partial^2 f}{\partial y^2} \Rightarrow$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left[x^2y + y^2z + z^2 - 2xy + 2yz + y - z\right]\right]$$

$$= \frac{\partial}{\partial y}\left[x^2 + 2yz + 0 - 2x + 2z + 1 - 0\right]$$

$$= \frac{\partial}{\partial y}\left[x^2 + 2yz - 2x + 2z + 1\right]$$

$$= 0 + 2z - 0 + 0 + 0$$

$$\boxed{\frac{\partial^2 f}{\partial y^2} = 2z}$$

let's calculate $\frac{\partial^2 f}{\partial y \partial x}$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(x^2y + y^2z + z^2 - 2xy + 2yz + y - z\right)\right]$$

$$= \frac{\partial}{\partial y}\left[2xy + 0 + 0 - 2y + 0 + 0 - 0\right]$$

$$= \frac{\partial}{\partial y}\left[2xy - 2y\right]$$

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} = 2x - 2}$$

Let's calculate $\frac{\partial^2 f}{\partial y \partial z}$ =).

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial z}\left\{x^2y + y^2z + z^2 - 2xy + 2yz + y - z\right]\right]$$

$$= \frac{\partial}{\partial y}\left[0 + y^2 + 2z - 0 + 2y + 0 - 1\right]$$

$$= \frac{\partial}{\partial y}\left[y^2 + 2z + 2y - 1\right]$$

$$= 2y + 0 + 2$$

$$\boxed{\frac{\partial^2 f}{\partial y \partial z} = 2y + 2}$$

let calculate $\frac{\partial^2 f}{\partial z \partial z}$ =)

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left\{\frac{\partial}{\partial x}\left(x^2y + y^2z + z^2 - 2xy + 2yz + y - z\right]\right]$$

$$= \frac{\partial}{\partial z}\left[2xy + 0 + 0 - 2y + 0 + 0 - 0\right]$$

$$= \frac{\partial}{\partial z}\left[2xy - 2y\right]$$

$$\boxed{\frac{\partial^2 f}{\partial z \partial x} = 0}$$

let calculate $\frac{\partial^2 f}{\partial z \partial y} \Rightarrow$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}\left[x^2y + y^2z + z^2 - 2xy + 2yz + y - z\right]\right]$$

$$= \frac{\partial}{\partial z}\left[x^2 + 2yz + 0 - 2x + 2z + 1 - 0\right]$$

$$= [0 + 2y - 0 + 2 + 0)$$

$$\boxed{\frac{\partial^2 f}{\partial z \partial y} = 2y + 2}$$

let calculate $\frac{\partial^2 f}{\partial z^2} \Rightarrow$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\left[x^2y + y^2z + z^2 - 2xy + 2yz + y - z\right]\right]$$

$$= \frac{\partial}{\partial z}\left[0 + y^2 + 2z - 0 + 2y + 0 - 1\right]$$

$$= [0 + 2 + 0 - 0]$$

$$\boxed{\frac{\partial^2 f}{\partial z^2} = 2}$$

substituting the second order partial derivative into the Hessian Matrix we get

$$\boxed{H_f(x,y,z) = \begin{bmatrix} 2y & 2x-2 & 0 \\ 2x-2 & 2z & 2y+2 \\ 0 & 2y+2 & 2 \end{bmatrix}}$$

iii, Classify the stationary points.

For classifying the stationary points we substitute the critical points with Hessian. Matrix to find the Eigen values

① $H_f(1, -1-\sqrt{2}, 0) = \begin{bmatrix} 2y & 2x-2 & 0 \\ 2x-2 & 2z & 2y+2 \\ 0 & 2y+2 & 2 \end{bmatrix}$

consider this as $A = \begin{bmatrix} 2(-1-\sqrt{2}) & 2(1)-2 & 0 \\ 2(1)-2 & 2(0) & 2(-1-\sqrt{2})+2 \\ 0 & 2(-1-\sqrt{2})+2 & 2 \end{bmatrix}$

$= \begin{bmatrix} -2-2\sqrt{2} & 0 & 0 \\ 0 & 0 & -2-2\sqrt{2}+2 \\ 0 & -2-2\sqrt{2}+2 & 2 \end{bmatrix}$

$A = \begin{bmatrix} -2-2\sqrt{2} & 0 & 0 \\ 0 & 0 & -2\sqrt{2} \\ 0 & -2\sqrt{2} & 2 \end{bmatrix}$

Eigen values A. can be obtained by

$\det [A - \lambda I] = 0$

$\left| \begin{bmatrix} -2-2\sqrt{2} & 0 & 0 \\ 0 & 0 & -2\sqrt{2} \\ 0 & -2\sqrt{2} & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0.$

$\begin{vmatrix} -2-2\sqrt{2}-\lambda & 0 & 0 \\ 0 & -\lambda & -2\sqrt{2} \\ 0 & -2\sqrt{2} & 2-\lambda \end{vmatrix} = 0$

$$(-2-2\sqrt{2}-\lambda)\left[(-\lambda)(2-\lambda)-(2\sqrt{2})(2\sqrt{2})\right]=0$$ (3)

$$(\lambda+2+2\sqrt{2})\left[(\lambda^2-2\lambda-8)\right]=0$$

$$(\lambda+2+2\sqrt{2})\left[\lambda^2-4\lambda+2\lambda-8\right]=0$$

$$(\lambda+2+2\sqrt{2})\ (\lambda-4)\ (\lambda+2)=0$$

$$\boxed{\lambda_A=(-2-2\sqrt{2}),4,-2}$$

the values ~~ofe~~ Eigen values of A are non-zero, we consider this as a saddle point.

(2) $Hf(1,-1+\sqrt{2},0)=\begin{bmatrix} 2y & 2x-2 & 0 \\ 2x-2 & 2z & 2y+2 \\ 0 & 2y+2 & 2 \end{bmatrix}$

Consider this as $B=\begin{bmatrix} 2(\sqrt{2}-1) & 2(1)-2 & 0 \\ 2(1)-2 & 2(0) & 2(\sqrt{2}-1)+2 \\ 0 & 2(\sqrt{2}-1)+2 & 2 \end{bmatrix}$

$$=\begin{bmatrix} 2(\sqrt{2}-1) & 0 & 0 \\ 0 & 0 & 2\sqrt{2} \\ 0 & 2\sqrt{2} & 2 \end{bmatrix}$$

Eigen values B can be obtained by

$$\det[B-\lambda I]=0$$

$$\left|\begin{bmatrix} 2(\sqrt{2}-1) & 0 & 0 \\ 0 & 0 & 2\sqrt{2} \\ 0 & 2\sqrt{2} & 2 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right|=0$$

$$\left|\begin{bmatrix} 2(\sqrt{2}-1) & 0 & 0 \\ 0 & 0 & 2\sqrt{2} \\ 0 & 2\sqrt{2} & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} 2(\sqrt{2}-1)-\lambda & 0 & 0 \\ 0 & -\lambda & 2\sqrt{2} \\ 0 & 2\sqrt{2} & 2-\lambda \end{vmatrix} = 0$$

$$2(\sqrt{2}-1)-\lambda)\left[(2-\lambda)(-\lambda) - (2\sqrt{2})(2\sqrt{2})\right] = 0$$

$$(2\sqrt{2}-2-\lambda)\left[-2\lambda+\lambda^2-8\right] = 0$$

$$(2\sqrt{2}-2-\lambda)\left[\lambda^2-2\lambda-8\right] = 0$$

$$(2\sqrt{2}-2-\lambda)\left[\lambda^2-4\lambda+2\lambda-8\right] = 0$$

$$(2\sqrt{2}-2-\lambda)(\lambda-4)(\lambda+2) = 0$$

8, -2, -4, +2

$$2\sqrt{2}-2-\lambda = 0$$

$$\boxed{\lambda_B = -2+2\sqrt{2},\ 4,\ -2}$$

the values of Eigen values of B are non-zero,

we can consider this as a saddle point.

③ $H_f(1,-1,1) = \begin{bmatrix} 2y & 2x-2 & 0 \\ 2x-2 & 2z & 2y+2 \\ 0 & 2y+2 & 2 \end{bmatrix}$

consider as $C = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Eigen values for c is given by det $[C - \lambda I] = 0$ ③

$$\left| \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)\left[(2-\lambda)(2-\lambda) - 0\right] = 0$$

$$(-2-\lambda)\left[4 - 2\lambda - 2\lambda + \lambda^2\right] = 0$$

$$(-2-\lambda)\left[4 - 4\lambda + \lambda^2\right] = 0$$

$$(-2-\lambda)\left[\lambda^2 - 4\lambda + 4\right] = 0$$

4, -4, -2 -2

$$(-2-\lambda)(\lambda^2 - 2\lambda - 2\lambda + 4) = 0$$

$$\boxed{\lambda_c = 2, 2, -2}$$

the values of Eigen values of c are non zero values, so this is also consider as saddle point.

$$\left\{ \begin{matrix} (1, -1-\sqrt{2}, 0) \\ (1, -1+\sqrt{2}, 0) \\ (1, -1, 1) \end{matrix} \right\}$$ are saddle points

d) Find all values for K so that $f(x,y) = x^4 + Kxy + y^4$ has a local minimum at (1,1). Give your answer in the form of an interval.

First lets calculate 1^{st} & 2^{nd} order condition at point (1,1).

$$f_x = \frac{df}{dx} = 4x^3 + Ky$$

$$f_y = \frac{df}{dy} = Kx + 4y^3$$

If f has local maximum or minimum at (a,b) and first order derivative, then

$$f_x(a,b) = 0 \text{ and } f_y(a,b) = 0$$

At point

$$f_x(1,1) = 0, \quad f_y(1,1) = 0$$

$$f_x(1,1) = 4(1)^3 + K(1) = 4 + K$$

$$f_y(1,1) = K(1) + 4(1) = K + 4$$

we get $\boxed{K = -4}$

the 1^{st} order condition is satisfied if $K = -4$

We use 2^{nd} condition using Hessian Matrix,

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$f_{xx} = \frac{d^2f}{dxx} = \frac{d}{dx}(4x^3 + Ky) = 12x^2$$

$$f_{xy} = \frac{d^2f}{dxy} = \frac{d}{dx}(4y^3 + Kx) = K$$

$$f_{yy} = \frac{d^2f}{dy^2} = \frac{d}{dy}(4y^3 + Kx) = 12y^2$$

At $(1,1) \to f_{xx}(1,1) = 12$

$f_{xy}(1,1) = K$

$f_{yy}(1,1) = 12$

$$\boxed{H = \begin{bmatrix} 12 & K \\ K & 12 \end{bmatrix}}$$

For (1,1) to be local minimum, the Hessian matrix positive definition at that point.

Determinant of Hessian Matrix

$\det(H) = 144 - K^2$

H_{11} or $H_{22} = 12 > 0$

For Hessian matrix to be true we need

$144 - K^2 > 0$

$144 > K^2$

$K \leq \sqrt{144}$

$$\boxed{|K| < 12}$$

The values of K has local minimum at (1,1).
At the interval $(-12, 12)$.

e) Find the maximum and minimum values of $f(x,y,z) = -x-y+z$ subject to $x^2+y^2=z$ and $x+y+z=1$.

Objective Function:

$$f(x,y,z) = -x-y+z$$

The O.F Constraints:

$$g(x,y,z) \Rightarrow x^2+y^2-z=0$$

$$h(x,y,z) \Rightarrow x+y+z=0$$

For the optimization problem, our constraints need to be perpendicular to find that, we will be calculating on graph. For finding in graph, we will calculate ∇_f, ∇_g and ∇_h which should be parallel.

we calculate as,

$$\nabla_f = \lambda \nabla_g + \mu \nabla_h$$

$\nabla_f(x,y,z) = 0$ or it does not exist.

we solve,

$$f_x = \lambda g_x + \mu h_x \quad \text{—} (1)$$

$$f_y = \lambda g_y + \mu h_y \quad \text{—} (2)$$

$$f_z = \lambda g_z + \mu h_z \quad \text{—} (3)$$

(1) $f_x = \lambda g_x + \mu h_x$.

diff $\Rightarrow$

$$\frac{d}{dx} f(x,y,z) = \lambda\left[\frac{d}{dx} g(x,y,z)\right] + \mu\left[\frac{d}{dx} h(x,y,z)\right]$$

$$-1-0+0 = \lambda\left[2x+0-0\right] + \mu\left[1+0+0-1\right]$$

$$\boxed{-1 = \lambda(2x) + \mu} \quad \text{—} (4)$$

2. $f_y = \lambda g_y + \mu h_y$

$$\frac{d}{dy} f_y = \lambda \frac{dg}{dy} + \mu \frac{dh}{dy}$$

$$-1 = \lambda[2y + 0 - 0] + \mu[0 + 1 + 0 - 0]$$

$$-1 = \lambda(2y) + \mu$$

$$\boxed{-1 = 2y\lambda + \mu} \quad — (5)$$

3. $f_z = \lambda g_z + \mu h_z$

$$\frac{d}{dz} f_z = \lambda \frac{dg}{dz} + \mu \frac{dh}{dz}$$

$$1 = \lambda[0 + 0 + 0] + \mu[0 + 0 + 1 - 0]$$

$$1 = 0 + \mu$$

$$\boxed{\mu = 1} \quad — (6)$$

Sub $\boxed{\mu = 1}$ in (4) & (5)

$$\lambda(2x) + 1 = -1$$

$$\cancel{2}x\lambda = -\cancel{2}$$

$$x\lambda = 1$$

$$\lambda(2y) + 1 = -1$$

$$\cancel{2}y\lambda = -\cancel{2}$$

$$y\lambda = 1$$

$$\cancel{\lambda}x = \cancel{\lambda}y$$

$$\boxed{x = y} \quad — (7)$$

Sub $x = y$ in $g(x, y, z)$.

$$x^2 + y^2 - 2 = 0$$

$$x^2 + x^2 = 2$$

$$2x^2 = 2$$

$x^2 = \frac{2}{2}$

$x^2 = 1$

$x = \pm 1$

$(x+1)(x-1) = 0$

$\boxed{x=-1}, \boxed{x=1}$

since $\boxed{x=y}$ in (7)

$\boxed{y=-1}, \boxed{y=1}$

sub in $h(x,y,z)$

$x+y+z=1$

$2x+z=1$

$\boxed{z=1-2x}$ — (8)

For $x=-1 \Rightarrow z = 1-2(-1) = 1+2 = 3$

For $x=1 \Rightarrow z = 1-2(1) = 1-2 = -1$

$\boxed{z=3}, \boxed{z=-1}$

From (7) $x=y$ so we get only 2 points

$(-1,-1,3)$

$(1,1,-1)$.

Let sub those points in $f(x,y,z)$.

$f_1(-1,-1,3) = -x_1 - y_1 + z_1$

$= -(-1)-(-1)+3$

$= 1+1+3$

$= 5$

$f_2(+1,1,-1) = -x_2 - y_2 + z_2$

$= -(1)-(1)+(-1)$

$= -1-1-1$

$= -3$

From the results we conclude $\boxed{f_1(-1,-1,3) > f_2(1,1,-1)}$ where $f_1(-1,-1,3)$ is the global maxima and $f_2(1,1,-1)$ is the global minima.