ASSIGNMENT-2.

1> Consider these vectors

$$U = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad V = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \quad W = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

i, Determine which two vectors are most similar to each other based on these norms:

a, l_2 norm: Euclidean distance

$$dist(x,y) = \|x-y\|_2 = \|y-x\|_2 = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$$

b, l_1 norm: Manhattan distance

$$dist(x,y) = \|x-y\|_1 = \|y-x\|_1 = \sum_{i=1}^{n}|x_i-y_i|$$

i, a, l_2 norm : Euclidean distance

i, $dist(\vec{U},\vec{V}) \Rightarrow \quad \vec{U} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{V} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}$

$$\vec{U}-\vec{V} = \begin{bmatrix} 1-0 \\ 0-2 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

$$\|\vec{U}-\vec{V}\| = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

$$dist(\vec{U},\vec{V}) = \sqrt{(1)^2+(-2)^2+(0)^2+(0)^2}$$

$$= \sqrt{1+4+0+0}$$

$$\boxed{dist(\vec{U},\vec{V}) = \sqrt{5}}$$

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ii, dist $(\vec{u},\vec{w}) \Rightarrow$

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\|\vec{u}-\vec{w}\| = \begin{bmatrix} 1-1 \\ 0-1 \\ 1-0 \\ 1-3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -2 \end{bmatrix}$$

$$dist(\vec{u},\vec{w}) = \sqrt{(0)^2+(-1)^2+(1)^2+(-2)^2}$$

$$= \sqrt{0+1+1+4}$$

$$\boxed{dist(\vec{u},\vec{w}) = \sqrt{6}}$$

iii, dist $(\vec{v},\vec{w}) \Rightarrow$

$$\vec{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\|\vec{v}-\vec{w}\| = \begin{bmatrix} 0-1 \\ 2-1 \\ 1-0 \\ 1-3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

$$dist(\vec{v},\vec{w}) = \sqrt{(-1)^2+(1)^2+(1)^2+(-2)^2}$$

$$= \sqrt{1+1+1+4}$$

$$\boxed{dist(\vec{v},\vec{w}) = \sqrt{7}}$$

the vectors $\vec{u}$ and $\vec{v}$ are small, so it is similar to each other

b) l_1 norm :- Manhattan Distance

i, ~~calcul dist~~

$$|\vec{U}-\vec{V}| = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} \quad |\vec{U}-\vec{W}| = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -2 \end{bmatrix}$$

$$|\vec{V}-\vec{W}| = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

i, $dist(\vec{U},\vec{V}) = |\vec{U}-\vec{V}| = |1| + |-2| + |0| + |0|$

$= 1 + 2 + 0 + 0$

$\boxed{dist(\vec{U},\vec{V}) = 3}$

ii, $dist(\vec{U},\vec{W}) = |\vec{U}-\vec{W}| = |0| + |-1| + |1| + |-2|$

$= 0 + 1 + 1 + 2$

$\boxed{dist(\vec{U},\vec{W}) = 4}$

iii, $dist(\vec{V},\vec{W}) = |\vec{V}-\vec{W}| = |-1| + |1| + |1| + |-2|$

$= 1 + 1 + 1 + 2$

$\boxed{dist(\vec{V},\vec{W}) = 5}$

∴ In l_2 norm U and V vectors are more similary vectors

$dist(\vec{U},\vec{V}) \leq dist(\vec{U},\vec{W}) \leq dist(\vec{V},\vec{W})$

$\boxed{\therefore \sqrt{5} \leq \sqrt{6} \leq \sqrt{7}}$

∴ In l_1 norm by Manhanttan distance U and V vectors are more vectors.

$dist(\vec{U},\vec{V}) \leq dist(\vec{U},\vec{W}) \leq dist(\vec{V},\vec{W})$

$\boxed{\therefore 3 \leq 4 \leq 5}$

ii) Determine the two vectors which are similar to each other using Cosine Similarity.

$$\|\vec{u}\| = \sqrt{(1)^2 + (0)^2 + (1)^2 + (1)^2}$$

$$= \sqrt{1+0+1+1} = \sqrt{3}$$

$$\|\vec{v}\| = \sqrt{(0)^2 + (2)^2 + (1)^2 + (1)^2}$$

$$= \sqrt{0+4+1+1} = \sqrt{6}$$

$$\|\vec{w}\| = \sqrt{(1)^2 + (1)^2 + (0)^2 + (3)^2}$$

$$= \sqrt{1+1+0+9} = \sqrt{11}$$

$$\boxed{\|\vec{u}\| = \sqrt{3}} \quad \boxed{\|\vec{v}\| = \sqrt{6}}, \quad \boxed{\|\vec{w}\| = \sqrt{11}}$$

$$\cos\theta\,(u,v) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$= \frac{(1)(0) + (0)(2) + (1)(1) + (1)(1)}{\sqrt{3}\cdot\sqrt{6}}$$

$$= \frac{2}{\sqrt{18}}$$

$$= \frac{2}{3\sqrt{2}}$$

$$\boxed{\cos\theta = 0.4714}$$

$$\boxed{\theta \simeq 61.87^\circ}$$

$$\cos\theta\,(u,w) = \frac{\vec{u}\cdot\vec{w}}{\|\vec{u}\|\|\vec{w}\|}$$

$$= \frac{(1)(1) + (0)(1) + (1)(0) + (1)(3)}{\sqrt{3}\,\sqrt{11}}$$

$$= \frac{4}{\sqrt{33}}$$

$$\boxed{\cos\theta = 0.696}$$

$$\boxed{\theta \simeq 45.89^\circ}$$

$$\cos\theta(\vec{v},\vec{w}) = \frac{(0)(1)+(2)(1)+(1)(0)+(1)(3)}{\sqrt{6}\sqrt{11}}$$

$$= \frac{0+2+3}{\sqrt{66}}$$

$$= \frac{5}{\sqrt{66}}$$

$$\boxed{\cos\theta = 0.615}$$

$$\boxed{\theta \simeq 52.04^{\circ}}$$

ii, The most similar vectors based on cosine similarity is $\vec{v}$ and $\vec{w}$.

$\vec{v}$ and $\vec{w}$ are high in cosine similarity, but small in angle between the two vectors.

$$\cos(\theta u,v) \leq \cos(\theta v,w) \leq \cos(\theta u,w)$$

iii, Difference between the results (i) and (ii).

i, From determining the two similar vectors based on the l_2 norm and l_1 norm.

a, By observing l_2 norm, the distance is calculated by Euclidean distance. Among all the calculations of distance of 2 vectors. $\vec{u}$ and $\vec{v}$ is small compare to the other calculation of 2 vectors. So $\vec{u}$ and $\vec{v}$ is considered as most similar vectors for Euclidean distance

b, By observing l_1 norm, the distance is calculated by Manhattan distance. Among all the calculations of distance of 2 vectors. $\vec{u}$ and $\vec{v}$ is small compared to other calculation of 2 vectors. so $\vec{u}$ and $\vec{v}$ is considerd as most similar vectors for Manhattan distance.

c, But in Cosine Similarity between 2 vectors is considered when the cosine value got high, but the angle between two vectors is small. $\vec{v}$ and $\vec{w}$ is considered to be most similar vectors of cosine similarity.

ii, l_1 and l_2 norm considers the magnitude and direction.

iii, Cosine similarity, considers overall direction of the vectors, where it give angle between the 2 vectors, it points roughly in same direction.

iv, How to Resolve the difference:

Since the l_1 and l_2 norm is considered by magnitude and direction, we can use normalization to the vectors, divides by its magnitude, which align more closely with the directional nature of cosine similarity. The difference is resolved when vector is normalized.

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②. $A = \begin{bmatrix} -1 & 0 & a \\ b & 4 & c \\ d & 0 & 0 \end{bmatrix}$

i, find the rank of A based on the values a, b, c, d

By using gaussian elimination to transform into row echelon form.

i, In row 2, we multiply b with Row1 and adding in Row2.

$R_2 \to R_2 + bR_1 \to \begin{bmatrix} -1 & 0 & a \\ b & 4 & c \\ d & 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -1 & 0 & a \\ 0 & 4 & c+ba \\ d & 0 & 0 \end{bmatrix}$

$R_2 \Rightarrow b \quad 4 \quad c$
$\cdot\, b \times R_1 \Rightarrow -b \quad 0 \quad ba$
$0 \quad 4 \quad c+ba$

ii, In Row3, we multiply d with R_1 and adding in Row 3.

$R_3 \to R_3 + dR_1 \to \begin{bmatrix} -1 & 0 & a \\ 0 & 4 & c+ba \\ d & 0 & 0 \end{bmatrix}$

$d \times R_1 \Rightarrow -d \quad 0 \quad ad$
$d \quad 0 \quad 0$
$0 \quad 0 \quad ad$

$\Rightarrow \begin{bmatrix} -1 & 0 & a \\ 0 & 4 & c+ba \\ 0 & 0 & ad \end{bmatrix}$

1^{st} row is alway non zero row, because it consists of non-zero values.

2^{nd} row will become zero if $c+ba=0$, if a is not equal to zero, it will consider to the rank.

3^{rd} row will become zero when $a=0$ or $d=0$. If $a\neq 0$ or $d\neq 0$, it will consider to the rank.

<u>cases:</u>

i, If $a\neq 0$ and $d\neq 0$ all rows will become non-zero rows. so the rank will be 3.

ii, If $a=0$ and $d=0$ there will be 2 non-zero rows, the rank will be 2.

ii, Solving the eigen values by the

$$\det(A-\lambda I)=0$$

$$A=\begin{bmatrix} -1 & 0 & a \\ b & 4 & c \\ d & 0 & 0 \end{bmatrix}$$

$$\det[A-\lambda I]=\begin{vmatrix} -1-\lambda & 0 & a \\ b & 4-\lambda & c \\ d & 0 & -\lambda \end{vmatrix}$$

Finding the co-factor of the determinant with Row 2.

$$=b\begin{vmatrix} 0 & a \\ 0 & -\lambda \end{vmatrix}-(4-\lambda)\begin{vmatrix} -1-\lambda & a \\ d & -\lambda \end{vmatrix}+c\begin{vmatrix} -1-\lambda & 0 \\ d & 0 \end{vmatrix}$$

$= b(0) - (4-\lambda)(-\lambda(-1-\lambda) - da) + c(0-0)$

$= -4+\lambda(\lambda+\lambda^2-da)+0$

$\det(A-\lambda I) = (4-\lambda)(\lambda+\lambda^2-da)$

case i) If $a \neq 0$ and $d \neq 0$

$$(4-\lambda)(\lambda+\lambda^2-ad)=0$$

$$4\lambda+4\lambda^2-4ad-\lambda^2-\lambda^3+\lambda ad=0$$

$$-\lambda^3+3\lambda^2+\lambda(4+ad)-4ad=0$$

$(-x)$

$$\lambda^3-3\lambda^2-\lambda(4+ad)+4ad=0$$

If $\boxed{\lambda=1}$

$$1-3-(4+ad)+4ad=0$$

$$1-3-4-ad+4ad=0$$

$$1-7+3ad=0$$

$$-6+3ad=0$$

$$+6=+3ad$$

$$\boxed{ad=2}$$

Factoring out $(\lambda-1)$ using synthetic division

if $ad=2$,

$$\lambda^3-3\lambda^2-(4+2)\lambda+4(2) = (\lambda-1)Q(\lambda)$$

$$\lambda^3-3\lambda^2-(4+2)\lambda+8 = (\lambda-1)Q(\lambda)$$

$$\lambda^3-3\lambda^2-4\lambda-2\lambda+8 = (\lambda-1)Q(\lambda)$$

$$\lambda^3-3\lambda^2-6\lambda+8 = (\lambda-1)Q(\lambda)$$

$Q(\lambda) = \lambda^2 - 2\lambda - 8$

Solving the equation by Quadratic equation

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1, \quad b = -2, \quad c = -8$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-8)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4 + 32}}{2}$$

$$= \frac{2 \pm \sqrt{36}}{2}$$

$$\lambda = \frac{2 \pm 6}{2}$$

$\lambda = \frac{2+6}{2}$ $\quad\quad$ $\lambda = \frac{2-6}{2}$

$= \frac{8}{2}$ $\quad\quad$ $= \frac{-4}{2}$

$\boxed{\lambda = 4}$ $\quad\quad$ $\boxed{\lambda = -2}$

Distinct Eigen values are $\lambda = 1, \lambda = 4, \lambda = -2$

Case ii, If $a = 0$ and $d = 0$

$(4-\lambda)(\lambda + \lambda^2 - ad) = 0$

$a = 0, d = 0 \Rightarrow (4-\lambda)(\lambda + \lambda^2) = 0$

$4\lambda + 4\lambda^2 - \lambda^2 - \lambda^3 = 0$

$-\lambda^3 + 3\lambda^2 + 4\lambda = 0$

$-\lambda(\lambda^2 - 3\lambda - 4) = 0$

$-\lambda(\lambda^2 - 4\lambda + \lambda - 4) = 0$

$-\lambda(\lambda+1)(\lambda-4) = 0$

$\lambda = 0, \lambda = -1, \lambda = 4$

Distinct Eigen values are $\lambda = 0, \lambda = -1, \lambda = 4$

(3). $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

i, find the full solution set for $Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x \in R^3$

$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Ax = b$

$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$-1x_1 + 1x_2 + 0x_3 = 1$

$-x_1 + x_2 = 1$ — (1)

$0x_1 - 1x_2 + 1x_3 = 1$

$-x_2 + x_3 = 1$

$-x_2 = -x_3$

$x_2 = x_3$ — (2)

$x_3 = 1$, because x_3 is a free vector

then,

$$x_2 = 1 - 1 = 0$$

$$x_1 = x_2 - 1 = -1$$

let $x_3 = t$

$$x_2 = t - 1$$

$$x_1 = x_2 - 1$$

$$= (t-1) - 1$$

$$= t - 2$$

$$x = \begin{bmatrix} t-2 \\ t-1 \\ t \end{bmatrix}$$

ii, Find Rank(A)

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & +1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$R_1 \to -R_1 \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$R_2 \to R_2 + R_1 \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank(A) = 2

Because there are 2 non-zero rows.

. Basis for the column space

$C(A)$ is spanned by pivot columns

$$C(A) = \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} +1 \\ -1 \end{bmatrix} \right\}$$

Row space Basis $C(A^T)$.

$$C(A^T) = \text{Span}\{[1\ -1\ 0],\ [0\ 0\ 1]\}$$

Basis of Null space $N(A)$

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{ll} -x_1 + x_2 = 0 & -① \\ -x_2 + x_3 = 0 & -② \end{array}$$

so $x_1 = x_2$ from eq ①

$x_2 = x_3$ from eq ②

$\therefore x_1 = x_2 = x_3$

Thus Null space is

$$N(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Finding Left Null space $N(A^T)$.

$$A^T y = 0$$

$$A^T = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

$-y_1 = 0$

$y_1 - y_2 = 0$

$y_2 = 0$

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$y_1 = y_2 = 0.$

$N(A^T) = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

Since it is empty set it contains only zero values.

iii, $B = AA^T$

$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$

$B = AA^T$

$B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} (-1)(-1) + (1)(1) + (0)(0) & (-1)(0) + (1)(-1) + (0)(1) \\ (0)(-1) + (-1)(1) + (1)(0) & (0)(0) + (-1)(-1) + (1)(1) \end{bmatrix}$

$B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

For finding the eigenvalues

$\det(B - \lambda I) = 0$

$B - \lambda I = \begin{bmatrix} 2-\lambda & -\lambda \\ -1 & 2-\lambda \end{bmatrix}$

$$= (2-\lambda)^2 - (-1)^2$$

$$= 4 + \lambda^2 - 4\lambda - 1$$

$$= \lambda^2 - 4\lambda + 3$$

$$\Rightarrow (\lambda - 3)(\lambda - 1) = 0$$

Eigen values are

$\lambda_1 = 3$

$\lambda_2 = 1$

For $\lambda_1 = 3$

$$(B - 3I) = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$-x - y = 0 \Rightarrow x = -y$$

Eigen vector for $\lambda_1 = 3$ is

$$u_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 1$

$$(B - 1\lambda) = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$x - y = 0 \Rightarrow x = y$

The eigen vectors for $\lambda_2 = 1$ is

$$u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

construct D

$$D = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}$$

$$\sigma_1 = \sqrt{\lambda_1}$$

$\sigma_1 = \sqrt{3}$, $\sigma_2 = \sqrt{1} = 1$

$$D = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

iv, If eigen vectors are orthonormal

Normalize them.

$$u_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Normalize u_1 & u_2:

$$\|u_1\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\|u_2\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$u_1 = \frac{u_1}{\|u_1\|}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} \\ +\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$u_2 = \frac{u_2}{\|u_2\|}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$u = [u_1, u_2]$$

$$U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

checking the orthogonality of $U \Rightarrow U^T U = I$.

$$U^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$U^T U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

$$U^T U = I$$

$\therefore$ U is orthogonal

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v) Find eigen values and eigen vectors of matrix $G = A^T A$ and orthonormalize them.

$$G = A^T A$$

$$A^T = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & -1+0 & 0+0 \\ -1+0 & 1+1 & 0-1 \\ 0+0 & 0-1 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

To find eigen values of G

$$\det \cdot (G - \lambda I) = 0$$

$$(G - \lambda I) = \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}$$

$$\det (G - \lambda I) = (1-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)((2-\lambda)(1-\lambda) - (-1)(-1)) - (-1)(-1)(1-\lambda)$$

$= (1-\lambda)(2-\lambda-2\lambda+\lambda^2-1)-(1-\lambda)$

$= (1-\lambda)(\lambda^2-3\lambda+1)-(1-\lambda)$

$= (\lambda^2-3\lambda+1-\lambda^3+3\lambda^2-\lambda)-(1-\lambda)$

$= -\lambda^3+4\lambda^2-4\lambda+1-1+\lambda$

$= -\lambda^3+4\lambda^2-3\lambda$

$= -(\lambda^3-4\lambda^2+3\lambda)$

$= \lambda(\lambda-3)(\lambda-1)$

$\lambda_1 = 3$

$\lambda_2 = 1$

$\lambda_3 = 0$

Eigen vectors of G for $\lambda_1 = 3$

$$(G-3I) = \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-2x-y=0$

$-x-y-z=0$

$-y-2z=0$

$x = \frac{y}{2}$

$z = \frac{y}{2}$ so $x = z$

Eigen vector for $\lambda = 3$ is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

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Eigen vectors of G for $\lambda_2 = 1$

$$[G - 1\lambda] = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-y = 0$

$-x + y - z = 0$

$-y = 0$

$y = 0$

$-x - z = 0$

$x = -z$

The Eigen vectors for $\lambda = 1$ is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Eigen vectors for $\lambda = 0$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$x - y = 0$

$-x + 2y - z = 0$

$-y + z = 0$

$x = y$

$y = -z$

$x = -z$

Eigen vectors of $\lambda = 0$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$V_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Normalizing V_1, V_2, V_3

$$V_1 \Rightarrow \frac{V_1}{\|V_1\|}$$

$$\|V_1\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$$

$$V_1 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$V_2 \Rightarrow \frac{V_2}{\|V_2\|}$$

$$\|V_2\| = \sqrt{1^2 + 0 + (-1)^2} = \sqrt{2}$$

$$V_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$V_3 \Rightarrow \frac{V_3}{\|V_3\|}$$

$$\|V_3\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$V_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

(vi) Orthogonalize (v_1, v_2, v_3) and construct V.

$$V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

Orthogonality of $V \Rightarrow V^T V = I$

$$V^T = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$V^T V = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$R_1 \Rightarrow \begin{cases} \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{-2}{\sqrt{6}}\right)\left(\frac{-2}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right) = 1 \\ \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{2}}\right) + 0 + \left(\frac{1}{\sqrt{6}}\right)\left(\frac{-1}{\sqrt{2}}\right) = 0 \\ \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{-2}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{3}}\right) = 0 \end{cases}$$

$$R_2 \Rightarrow \begin{cases} \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{6}}\right) + 0 + \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{6}}\right) = 0 \\ \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 0 + \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) = 1 \\ \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}\right) + 0 + \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}\right) = 0 \end{cases}$$

$$R_3 = \begin{cases} \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{-2}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}\right) = 0 \\ \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{2}}\right) + 0 + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{-1}{\sqrt{2}}\right) = 0 \\ \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) = 1 \end{cases}$$

$$V^T V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$V^T V = I$$

V is an orthogonal matrix.

vii, Find vectors $\{w_1, w_2, w_3\}$

$$w_i = \frac{1}{\sigma_i} A^T u_i, \quad i = 1, 2$$

$w_3 \perp w_1$, $w_3 \perp w_2$ and $\|w_3\| = 1$

obtained σ_1 & σ_2 from (iv)

$$u_1 = \begin{bmatrix} -1/\sqrt{2} \\ +1/\sqrt{2} \end{bmatrix} \quad u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\sigma_1 = \sqrt{3}$$

$$A^T = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A^T u_1 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} \\ -\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

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$\sigma_1 = 3$

$$\omega_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1/\sqrt{2} \\ -\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{6} \end{bmatrix}$$

$$A^T u_2 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$\sigma_2 = 1$

$$\omega_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

to find ω_3,

$\omega_3 \perp \omega_1$, $\omega_3 \perp \omega_2$ and $\|\omega_3\| = 1$

we can start with any vector v not in the span of ω_1 and ω_2 and orthogonalize against ω_1 & ω_2

$$\omega_3 = v - \frac{v \cdot \omega_1}{\omega_1 \cdot \omega_1} \omega_1 - \frac{v \cdot W_2}{\omega_2 \cdot \omega_2} \cdot \omega_2$$

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v \cdot \omega_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{6} \end{bmatrix}$$

$$= \left(-1/\sqrt{6} + \frac{\sqrt{2}}{\sqrt{3}} - 1/\sqrt{6}\right)$$

$= 0$

$$v \cdot w_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$= -1/\sqrt{2} + 0 + 1/\sqrt{2}$$

$$= 0.$$

v is already orthogonal to w_1 & w_2.

$$v \cdot w_1 = v \cdot w_2 = 0$$

w_3 can be normalised and given as,

$$w_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$w_1 = \begin{bmatrix} 1/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{6} \end{bmatrix}, \quad w_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad w_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$w_1 = v_1, \quad w_2 = v_2, \quad w_3 = v_3.$$

the sets of $\{w_1, w_2, w_3\}$ & $\{v_1, v_2, v_3\}$ are similar to each other.

viii). Compute the product UDV^T.

$$U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad D = \begin{bmatrix} +\sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$UD = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/\sqrt{2} & 1/\sqrt{2} & 0 \\ \sqrt{3}/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$UDV^T = \begin{bmatrix} -\sqrt{3}/\sqrt{2} & 1/\sqrt{2} & 0 \\ \sqrt{3}/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{6} & \frac{-2}{\sqrt{6}} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$\therefore UDV^T = A,$

ix) Compare the colums of U and V with the bases you found for $C(A)$, $C(A^T)$, $N(A)$, $N(A^T)$.

i) Column U: The u_1 and u_2 form an basis of $C(A^T)$, which is the row space.

ii) column V: The v_1, v_2 and v_3 form an bases for $C(A)$ which is the column space of A.

* In Column U. the orthonormal eigenvectors of AA^T that spans the rowspace of A.

* In column of V the orthonormal eigenvectors of A^TA, which spans the columnspace of A.

iii) In Column V - ① v_1 and v_2, form an orthonormal basis for $N(A^T)$, the left null space of A.

② v_2 and v_3 form an orthonormal basis for $N(A)$, the null space of A.

The columns of U & V are closely related to each other in terms of the basis for four subspaces of matrix A.