Counting 10-arcs in the Projective Plane

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Projective Space over Finite Fields

Definition (Projective Plane over a Finite Field)

The **projective plane over** $\mathbb{F}_q$, denoted $\mathbb{P}^2(\mathbb{F}_q)$, is the set of all triples $(a : b : c)$ with $a, b, c \in \mathbb{F}_q$, except the triple $(0 : 0 : 0)$.

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For instance, in $\mathbb{F}_7$ we have $(1 : 2 : 3) = (3 : 6 : 2)$. This is an example of a **point** in $\mathbb{P}^2(\mathbb{F}_7)$.

Arcs in Projective Space

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Our Goal: Find a formula in q, the size of the finite field, for how many arcs exist in the projective plane over a given finite field.

Known formulae for the number of n-arcs

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- #(4-arcs), #(5-arcs) and #(6-arcs) of a similar form, but...

Known formulae for the number of n-arcs

Glynn [1988]:

- $\#$(7-arcs) $= \frac{1}{7!}(q^2+q+1)(q^2+q)q^3(q-1)^2(q-3)(q-5)(q^4-20q^3+148q^2-468q+498) - A_7$

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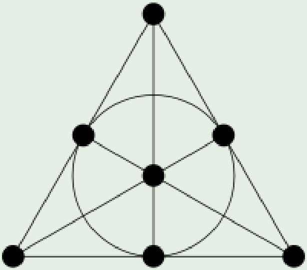
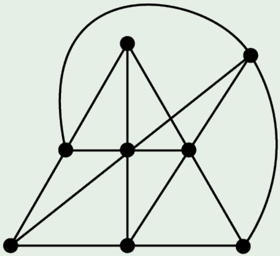
- $\#(\text{8-arcs}) = \frac{1}{8!}(q^2+q+1)(q+1)q^3(q-1)^2(q-5)(q^7-43q^6+788q^5-7937q^4+47097q^3-162834q^2+299280q-222960) - (q^2-20q+78)A_7 + A_8$

Example (A_7 and A_8)





9-arcs and Beyond

Iampolskaia, Skorobogatov, and Sorokin [1995]:

#(9-arcs)=

$$
\begin{aligned}
&(q-1)^8 \times (q^{10} - 75q^9 \\
&+2530q^8 - 50466q^7 + 657739q^6 - 5835825q^5 \\
&+35563770q^4 - 146288034q^3 + 386490120q^2 \\
&-588513120q + 389442480 \\
&-1080(q^4 - 47q^3 + 807q^2 - 5921q + 15134)a(q) \\
&+840(9q^2 - 243q + 1684)b(q) \\
&+30240(-9c(q) + 9d(q) + 2e(q))),
\end{aligned}
$$

where

$$
\begin{aligned}
a(q) &= 1, \text{ if } q \text{ is a power of } 2, \text{ and } a(q) = 0, \text{otherwise} \\
b(q) &= \#\{x \in \boldsymbol{F}_q \text{ such that } x^2 + x + 1 = 0\} \\
c(q) &= 1, \text{ if } q \text{ is a power of } 3, \text{ and } c(q) = 0, \text{otherwise} \\
d(q) &= \#\{x \in \boldsymbol{F}_q, \text{ such that } x^2 + x - 1 = 0\} \\
e(q) &= \#\{x \in \boldsymbol{F}_q, \text{ such that } x^2 + 1 = 0\}.
\end{aligned}
$$

Superfigurations

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- Two lines intersect each other at no more than one point.
- Two points are connected by no more than one line.
- There are at least 3 points on each line, and at least 3 lines through each point.

Game Plan

Theorem (Glynn)

Let S be the set of superfigurations on up to 10 points. In general, for #(10-arcs) we can expect an expression of the form

$$f(q) + \sum_{s \in S} g_s(q) A_s$$

where A_s is the number of copies of s in $\mathbb{P}^2(\mathbb{F}_q)$ and $g_s(q)$ and $f(q)$ are polynomials in q.

Game Plan

Theorem (Glynn)

$$\#\text{(10-arcs)} = f(q) + \sum_{s \in S} g_s(q) A_s$$

In order to find all the terms of this polynomial, a number of subproblems arise:

1. Finding Superfigurations (Combinatorics problem)

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1. Finding Superfigurations (Combinatorics problem)
2. Realizability of Superfigurations (Algebra problem)
3. Finding Coefficients (Computational problem)

Finding Superfigurations

Theorem (LPW 2015)

There are 151 superfigurations on 10 points.

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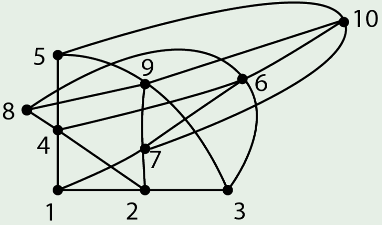
Example



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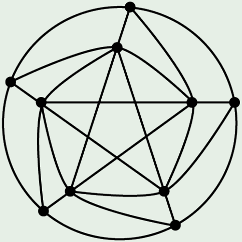

10-Arcs Formula

#(10-arcs) =

$$f(q) + \sum_{s \in S} g_s(q) A_s$$

We now know the contents of the set S of all superfigurations on up to 10 points. We still need to determine A_s for each $s \in S$, and the polynomials $g_s(q)$.

Realizability of Superfigurations

Definition (Realizability)

A superfiguration $\mathscr{C}$ is ***q*-realizable** if there exists some assignment φ of coordinates in $\mathbb{P}^2(\mathbb{F}_q)$ to points of the superfiguration such that φ preserves the collinearity (or lack of collinearity) of every subset of 3 points in $\mathscr{C}$.

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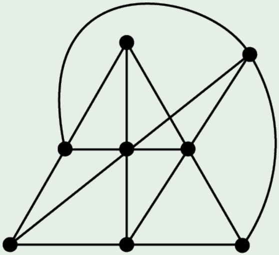
For each of the 151 superfigurations, we want to determine for which $\mathbb{F}_q$ the configuration can be realized, and if so, how many different ways there are to assign the coordinates to the points.

Realizability of Superfigurations

Example (Computing Realizability)



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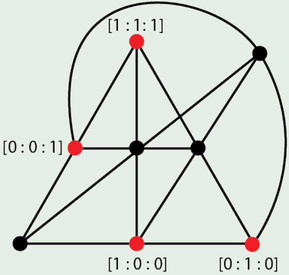

Add coordinates for 4 points in general position.

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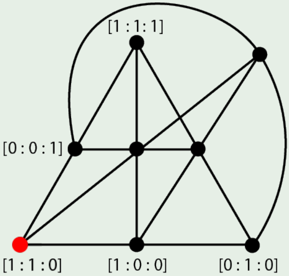

Deduce additional point.

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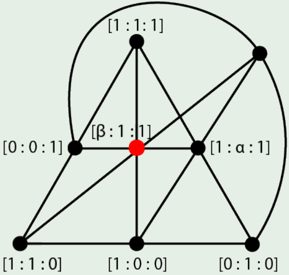

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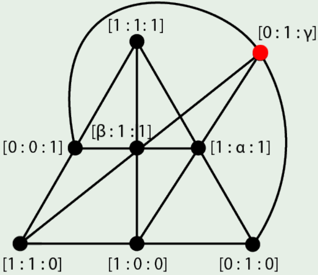

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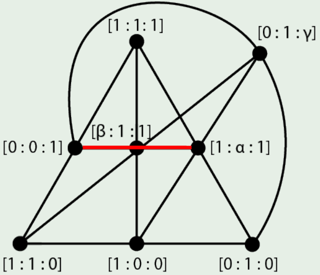

$$\begin{vmatrix} 0 & 0 & 1 \\ \beta & 1 & 1 \\ 1 & \alpha & 1 \end{vmatrix} = \alpha\beta - 1 = 0.$$

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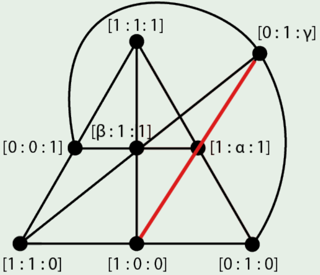

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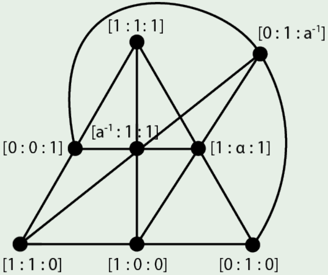

$$\begin{vmatrix} 1 & 1 & 0 \\ \alpha^{-1} & 1 & 1 \\ 0 & 1 & \alpha^{-1} \end{vmatrix}$$

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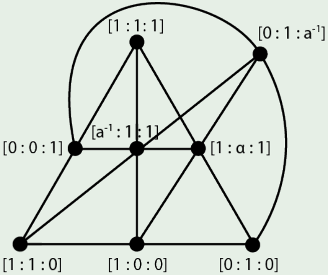

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$$\implies \alpha^2 - \alpha + 1 = 0$$

Weak and Strong Realizations

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Definition (Weak Realization)

A **weak realization** of a superfiguration $\mathscr{C}$ is an assignment φ of coordinates in $\mathbb{P}^2(\mathbb{F}_q)$ to points of $\mathscr{C}$ such that φ at least preserves all collinearities of $\mathscr{C}$.

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Definition (Strong Realization)

A **strong realization** of a superfiguration $\mathscr{C}$ is a weak realization of $\mathscr{C}$ that induces no additional collinearities.

- We say that a superfiguration is **nondegenerate** if all points and lines remain unique after assigning coordinates.

Realizability Results

Theorem (LPW 2015)

Each superfiguration S may be associated with a system of polynomial equations so that S appears in $\mathbb{P}^2(\mathbb{F}_q)$ if and only if that system of equations has solutions over $\mathbb{F}_q$.

Example

Mobius-Kantor appears in $\mathbb{P}^2(\mathbb{F}_q)$ when $x^2 - x + 1$ has solutions over $\mathbb{F}_q$.

Realizability Results

Of the 151 superfigurations, we have computed all of the previously unknown polynomial systems. We find that:

Theorem

There are 31 superfigurations which cannot be realized in $\mathbb{P}^2(\mathbb{F}_q)$ for any q.

Realizability Results

Example

The following polynomials show up in the realizability computations of superfigurations:

$$5 = 0$$

$$x^3 + x^2 - 1 = 0$$

$$x^2y^3 - (2x^3 - x^2)y^2 + (4x^3 - 7x^2 + 3x)y$$
$$-(2x^3 - 5x^2 + 4x - 1) = 0$$

The numbers A_S mentioned earlier may be characterized as the number of distinct solutions of each polynomial over $\mathbb{F}_q$.

10-arcs Formula

#(10-arcs) =

$$f(q) + \sum_{s \in S} g_s(q) A_s$$

We now have a way to find A_s for each $s \in S$, and just need the polynomials $g_s(q)$.

Computing $g_s(q)$ and $f(q)$

Definition (Boolean n-function)

Generalizing the notion of superfigurations, we define a boolean n-function

$$B : \mathcal{P}(\{1, 2, \cdots, n\}) \to \mathbb{1}_C$$

where $\mathbb{1}_C$ is the indicator function denoting whether the points in the set are collinear.

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We adapt an algorithm from Glynn, Rolland, and Skorobogatov to compute the $g_s(q)$ and $f(q)$ by examining a partial ordering on all boolean n-functions for $n \leq 10$.

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$$m_q(b_f) = \sum_{b_g \geq b_f} (-1)^{\#b_g^{-1}(1) - \#b_f^{-1}(1)} n_q(b_g)$$

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- Increment n for $n \leq 10$.

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- Existence of abstract projective planes
  - There is no projective plane of order six!

Thank you!