Building a Dictionary of Coupled Oscillators

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Abstract

Fractal geometry, the study of self-similar forms, is a deceptively pervasive field which can be linked to both the natural world and useful mathematical and statistical methods. Iterated Function Systems (or IFS) provide a way to define fractal generation based on the union of a finite collection of contractions, and it is known that such shapes uniquely converge to specific "attractors" independent of the starting set upon which the contractions were performed. Researchers in the field have often studied random IFS, which apply transformations according to randomly ordered sequences and have limit sets displaying predictable attractors, but a more complicated case is that of driven IFS. Here, transformations are chosen from a set based on a collection of rules about the sequence of values, in order to visualize patterns and hierarchical properties of the data set. This project aims to use logistic maps to characterize the behavior and synchronization of these driven IFS, ultimately creating a "dictionary" of identifiable characteristics. This is a novel approach that would give new meaning to the fractal patterns produced by driven IFS oscillators; the proposed dictionary would enable analysis of time series data from a number of biological processes, including heartbeat spacing. Ideally, matching synchronization patterns in chaotically behaving logistic maps would allow researchers to more easily identify properties of coupled oscillators, even while lacking a detailed behavioral model.

Introduction

From coastlines and galaxies to heartbeats and neural networks, natural systems often display the characteristic roughness of fractal patterning and chaos. Using Iterated Function Systems, it is possible to recognize these dominant scaling features within natural systems, and examine the underlying systems and networks which drive them [5].

Despite their importance in diverse aspects of the world, much remains unknown about chaos and the networks of oscillators which often give rise to it. One manifestation of this gap in mathematical and physical understanding of oscillators is in the difficulty of predicting synchronizing chaotic behavior among a series of oscillating systems [15]. Therefore, the first goal of this research project is to explore different combinations of system parameters to find the patterns and shared characteristics between members of each class of coupled oscillators. The proposed research project aims to use data-driven Iterated Function Systems to create a "dictionary" of coupled oscillators—that is, a network of related classes describing the behavior of synchronized chaotic systems under varying parameters. Lattices of coupled logistic maps will be used to drive IFS algorithms, the characteristics of which will be analyzed to find the repeated patterns and visual signatures of different dynamics. The network of related families produced from this work will comprise the dictionary of oscillators, which may be used to analyze and compare chaotic signals from a wide variety of sources.

Using the proposed dictionary, the research project will also analyze time series data from a number of biological processes. From observations about the frequency of data points occupying various regions in the IFS, it is possible to create categories and compare real data to our models. This dictionary will then be used and tested by matching biological data samples to synchronization patterns in chaotically behaving logistic maps, thus allowing for easier identification of the properties of systems behaving as coupled oscillators. In general, matching synchronization patterns in chaotically behaving logistic maps will allow for a simpler characterization of properties of coupled oscillators, even while the complete workings of the systems are too complex to characterize. This is crucial in the study of many instances of biological synchronization, in which oscillators may act together even while behaving chaotically [8]. Finding families of driven IFS to correspond to different types and degrees of coupling can help to discover differences between instances of synchronized behavior—for example, healthy versus diseased cardiac rhythms—without requiring a full model of the systems behavior, which can often be prohibitively complicated [10].

Iterated Function Systems

Definitions and Deterministic IFS

Iterated Function Systems (or IFSs) provide a way to define fractal generation based on the union of a finite collection of contractions [7]. In general, a fractal can be understood as simply a "self similar" shape; one made up of smaller pieces of itself. Mathematically, the simplest formalization of a fractal, and the one most useful to the concept of the IFS, is the "self-affine set" A, defined as a compact finite set such that:

$$A = T_1(A) \cup \ldots \cup T_n(A).$$

[7] The transformations T_n can be thought of as combinations of scalings, rotations, and translations of A; these affine, or linear, transformations are defined formally as

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} r\cos\theta & -s\sin\phi \\ r\sin\theta & s\cos\phi \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

Furthermore, such a transformation is called a *contraction* if, intuitively, the image of the transformation is "smaller" than the initial object on which it was performed; that is, for Euclidean distance $d : \mathbb{R}^n \to \mathbb{R}$ and $0 \leq t < 1$,

$$d\left(T \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, T \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) \leq t\ d\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

[2] With this vocabulary, an Iterated Function System is a finite collection of contractions $(T_1, \ldots, T_n)$, which determines a collage map τ on compact subsets S of the plane given by

$$\tau(S) = \bigcup_{i=1}^{n} \{T_i(x, y) \mid (x, y) \in S\}$$

[7] Another important feature of Iterated Function Systems is that every IFS determines a unique nonempty compact set A, called the system's *attractor*, independent of the initial points upon which the contractions were performed [6]. This attractor is defined as the set such that

$$\tau(A) = A \text{ and } \tau^k(B) \to A \text{ as } k \to \infty$$

for any compact set B [6]. In other words, the collage map of the attractor is itself, and if the transformations defining the map are applied to any shape arbitrarily many times, the image set approaches the attractor. Thus, returning to the idea of a fractal generation, a fractal is simply the attractor generated by the deterministic algorithm prescribed by a given IFS [7].

Addresses

Using the deterministic IFS algorithm as described above, it follows that an IFS with transformations

$$T_1 = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{x} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}, T_2 = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{x} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, T_3 = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{x} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, T_4 = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{x} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

will have as its attractor the filled unit square [2]. Further, observe that from any point within this unit square, applying T_1 will result in an image in the bottom left quadrant of the unit square, T_2 the bottom right, and T_3 and T_4 the top left and right respectively. These regions are said to be specified by length 1 addresses. Continuing the process, each of these addresses can be further divided into length 2 addresses, so that a point in address 43 contains the images of all points in the unit square on which T_3 followed by T_4 were performed. The process of dividing the unit square may continue indefinitely, giving addresses of arbitrarily large length and precision for each point within the unit square [2]. Formally, each $x \in A$ for attractor A has an infinite address $i_1 i_2 \ldots$ representing the infinite composition of transformations $T_{i_1}(T_{i_2}(\ldots(p))\ldots)$ on some point p [2]. This concept of defining addresses based on transformations becomes useful in defining two other variations on the IFS: random and data-driven Iterated Function Systems.

Random IFS

Random Iterated Function Systems extend the deterministic IFS algorithm by incorporating the idea of using random sequences of transformations to "drive" the IFS. First developed by Michael Barnsley under the name the "Chaos Game", the random IFS algorithm was provided as a fast way for earlier computers to compute the attractors of deterministic IFSs; however, in subsequent years, numerous innovative implications of this algorithm and its variants have been realized [1].

To understand the random IFS algorithm, first consider an infinitely long random sequence composed of integers between 1 and n. Define an infinite random sequence as one which cannot be specified in any more compact way than a listing of every term. It then follows that such an infinite random sequence must contain all possible finite sequences; otherwise, some elements of the sequence could be predicted through elimination simply by knowing what numbers could not follow in the sequence [7].

Applying this fact to an IFS, consider an infinite random sequence of integers $I = \{i_1, i_2, \ldots\}$ for $1 \leq i_j \leq n$, and a list of transformations $T_1, \ldots, T_n$. Define the *forward orbit* of a point $O_I^+(x_0, y_0) = \{(x_0, y_0), T_{i_1}(x_0, y_0), T_{i_2}(T_{i_1}(x_0, y_0)), \ldots\}$ [7]. That is, a random combination of $T_1, \ldots, T_n$ has been performed on initial point (x_0, y_0).

The implication of this process, and the reason why the random IFS is of interest, lies in the closure of the set of points in the forward orbit. Taking a fixed point (x_f, y_f) such that $T_1(x_f, y_f) = (x_f, y_f)$, the closure of the forward orbit $O_I^+(x_f, y_f)$ is the attractor of the associated deterministic IFS under the same set of transformations [7]. In other words, driving an IFS is a series of random transformations starting at a fixed point will, after sufficient iterations, produce the same attractor as the deterministic IFS under these transformations; the limit set of the orbit is the deterministic fractal.

A final note about the random algorithm: starting at a fixed point is in fact not necessary to produce what appears as the same attractor. If the first point chosen is instead not a fixed point - and as a result, not necessarily contained within the attractor - some randomly distributed points will be generated until one falls in the limit set of the attractor [6]. After this point, all subsequently generated points will also fall in the limit set of the attractor, thus generating the same image, up to visual precision, as the random IFS generated with a fixed point. (Incidentally, this idea of plotting limit sets is the same reason why an image of a Cantor set or a filled-in square can be produced by the random inclusion of points from the set, despite the fact that a sequence of transformations is of countably infinite size, while the points in a square or the Cantor set are uncountably infinite [6].)

The Driven IFS: Visualizing Patterns in Data

A still more complicated and interesting variation of Iterated Function System generation is that of a driven IFS. Here, a set of transformations is again composed in a specific order, but rather than randomly, their order is chosen based on a collection of rules about the sequence of values. The technique was originally conceived by Ian Stewart, who applied the various "driving functions" to take the place of the random sequence used in the random IFS algorithm [14]. This achieved surprising results, in which the attractors produced had vastly different shapes as compared to the attractors of their randomly generated counterparts [14]. Furthermore, driving an IFS using a series of rules depending on a time series or other such data set is of significant utility in identifying the behaviors of the data points [7]. While Stewart failed to realize the full implications of his innovation at the time, the driven IFS can be applied to vastly complicated data and series, in order to visualize patterns and properties of the data set.

Applying Transformations

To apply transformations based on the driven IFS, data must first be divided into some small number of bins; the most sensible number of bins to use will depend upon the categorical nature of the data under investigation [7]. Several different binning schemes can be used as well, depending upon their effectiveness for encompassing different distributions of data; these include “equal size” bins which all have the same range, “equal weight” bins which all contain the same number of data points, and“’median centered” bins, with bin division based on the median and quartiles of the data [5].

Next, the transition of time series data from one bin to a specific bin numbered i can be assigned a transition $T_i(x, y)$ analogously to the random IFS algorithm. As in the random IFS algorithm, this scheme can be used to create a forward orbit of composed transformations across all bin transitions in the data [7].

Using the random IFS algorithm with, for example, the transformations

$$T_1 = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}, T_2 = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, T_3 = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, T_4 = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

will produce a unit square, allowing departures from randomness produced with the driven IFS to produce easily visible fractal patterns, which may manifest throughout the plot as forbidden addresses [5].

Forbidden Addresses

The idea of forbidden addresses is quite simple; it posits that, in a certain driven IFS, some sequence of transformations is by some mechanism prevented from occurring [3]. For example, a rule may be that the transformation T_4 may never immediately follow T_1.

Furthermore, if some combination $T_j, \ldots, T_k$ of transformations never occurs, then all subsections of the plot with addresses containing the sequence $i_j \ldots i_k$ must also be empty, as that sequence's presence would indicate that the forbidden composition had occurred [3]. This idea becomes very useful in analyzing the properties of a data set, providing information about the relative frequencies of various transitions in a way that might not otherwise be intuitively obvious; for example, a study analyzing a time series based on genetic code sequences found exactly such forbidden addresses representing transitions between different base pairs [11].

Proposed Research

Advantages of Binning Data

One major issue with analyzing experimental data relating to chaotic processes is that experimental measurements inherently incorporate random fluctuations, noise and experimental error under even the most controlled conditions, and standard tests are very sensitive to these fluctuations [13]. In order to describe the symbolic dynamics of the attractor of chaotic data, one solution is to analyze data with respect to several large bins, rather than using the error-prone exact measurements. A time series can then be represented by a matrix of directional transitions between states, which characterizes the system while remaining robust with respect to noise [13]. For this reason, it is very appropriate to coarse-grain time series data into a small number of bins, and carry out the driven IFS algorithm using transformations between only these few bins.

Logistic Maps and Synchronization

Chaotic processes are exceedingly difficult to analyze and predict; however, even chaotically oscillating systems may in certain cases display synchronized behaviors when coupled in an interacting system. The scientific discovery of physical synchronization is often attributed to a 17th century physicist named Christiaan Huygens, who realized that two pendulum clocks hanging next to each other on his wall tended to synchronize their motions over time, and surprising instances of synchronization have continuously been the subject of much mathematical thought ever since [14]. Questions remaining to this day include what equations, under what parameters and conditions, lead to this chaotic synchronization, and how often in coupled systems this behavior can be observed.

In this experiment, synchronization is to be explored using *logistic maps*. A logistic map is a recurrence relation, first proposed as a model of population sizes, definted by the equation $x_{n+1} = rx_n(1 - x_n)$ [12]. The logistic map is useful in that it represents one of the simplest situations in which chaotic behaviors arise. The parameter r in this equation is crucial in determining the degree of complex behavior of x_n, to the extent that the following ranges of behaviors dependent on r-values have been determined [7].

1. For $0 \leq r \leq 3$, x_n will converge to a point as $n \to \infty$
2. For $3 < r \leq$ approximately 3.56995, x_n will exhibit periodicity as $n \to \infty$. In this range, the value oscillates between different values, all multiples of two.
3. For approximately $3.56995 \leq r \leq 4$, chaotic behavior is often, but not always, observed. This is the range of interest for research and simulations.
4. For $r > 4$, x_n diverges.

Logistic maps may also be coupled with other maps. That is, a group of maps may depend on each other according to a coupling constant c, which provides the weight to be given to the recurrence relation itself versus all other coupled functions. For example, two coupled maps would have the recurrence relations

$$x_{n+1} = (1 - c)rx_n(1 - x_n) + cry_n(1 - y_n)$$

$$y_{n+1} = (1 - c)ry_n(1 - x_n) + crx_n(1 - y_n)$$

such that, for larger values of c, the functions would be more dependent upon each other and less dependent on themselves [7].

An important feature of coupled logistic maps is that even in ranges in which coupled logistic maps display chaotic features, it is possible for the maps to synchronize; in fact, this happens in a large number of cases. As an example, consider the two coupled logistic maps described above. To search for increasing synchronization between the functions over increasing values of n, one strategy is to test whether, after each iteration, x_{n+1} and y_{n+1} are closer together than x_n and y_n:

$$|x_{n+1} - y_{n+1}| < |x_n - y_n|$$

Through algebraic manipulation and the knowledge that $0 < x_n, y_n < 1$, the coupling equations give that:

$$|x_{n+1} - y_{n+1}| = |1 - 2c|r|x_n - y_n|$$

and thus, $|x_{n+1} - y_{n+1}| < |x_n - y_n|$ when $|1 - 2c|r < 1$, and so x and y synchronize under this condition.

However, while it is proven that synchronization holds for all c and r fulfilling the above condition, this is a conservative estimate in that it does not encompass all possible windows of synchronization. For example, processes may progress such that their difference fails to decrease monotonically, yet synchronization between the two does occur. Consequently, part of the necessary investigation of these dynamics involves expanding the cataloged windows of synchronization to a more comprehensive understanding of the parameters, using methods described below.

Additionally, investigation of coupled logistic maps need not only focus on two coupled maps. This project aims to generalize this result to higher numbers of coupled maps, including as a first example, the 3-map system. One challenge of this extension to higher dimensions is determining a valid measure of "distance", or similarity, between x_n, y_n, and third function z_n, analogous to the inequality determining distance in the 2-map system. Some proposed measures include:

1. The pairwise difference between each set of two logistic maps: $|x_{n+1} - y_{n+1}| < |x_n - y_n|$, $|x_{n+1} - z_{n+1}| < |x_n - z_n|$, and $|y_{n+1} - z_{n+1}| < |y_n - z_n|$

2. The sum of all these pairwise differences $|x_{n+1} - y_{n+1}| + |z_{n+1} - y_{n+1}| + |y_{n+1} - z_{n+1}| < |x_n - y_n| + |z_n - y_n| + |y_n - z_n|$

3. The sum of differences between the value of each logistic map and the average of the three maps:

$$\left|x_{n+1} - \frac{x_{n+1} + y_{n+1} + z_{n+1}}{3}\right| + \left|y_{n+1} - \frac{x_{n+1} + y_{n+1} + z_{n+1}}{3}\right| + \left|z_{n+1} - \frac{x_{n+1} + y_{n+1} + z_{n+1}}{3}\right|$$
$$\leq \left|x_n - \frac{x_n + y_n + z_n}{3}\right| + \left|y_n - \frac{x_n + y_n + z_n}{3}\right| + \left|z_n - \frac{x_n + y_n + z_n}{3}\right|$$

4. The individual differences between the value of each logistic map and the average of the three maps:

$$\left|x_{n+1} - \frac{x_{n+1} + y_{n+1} + z_{n+1}}{3}\right| \leq \left|x_n - \frac{x_n + y_n + z_n}{3}\right|$$
$$\left|y_{n+1} - \frac{x_{n+1} + y_{n+1} + z_{n+1}}{3}\right| \leq \left|y_n - \frac{x_n + y_n + z_n}{3}\right|$$
$$\left|z_{n+1} - \frac{x_{n+1} + y_{n+1} + z_{n+1}}{3}\right| \leq \left|z_n - \frac{x_n + y_n + z_n}{3}\right|$$

Dictionary of Coupled Oscillators

The main goal of this research is to create a dictionary of the different properties a network of oscillators can take on, when coupled in different numbers and modes of association. To achieve this, computer sampling of behavior for different values of c and r will be performed. Testing different values for r, however, can pose difficulties due to the difficulty of characterizing ranges of chaotic versus periodic behavior. As a result, the proposed method to accommodate the alternately chaotic and periodic results of different r values is a Monte Carlo simulation. The simulation will involve generating a list of r-values populated randomly in the chaotic region $3.6 \leq r \leq 4$; this allows the simulation to sample most or all of the behaviors of the system under varying r. We will then use these r-values, the measures of distance determined previously, and computer software to determine windows in which synchronization occurs for each parameter.

Finding repeated patterns for different values of the logistic map parameters will be achieved using a combination of existing and newly developed software. The testing of parameters under the Monte Carlo simulation will rely in part on existing Mathematica software, in addition to new C++ and Python code to be developed as a part of the project and serving to corroborate past results.

Specific goals for the dictionary include being able to determine between models which display isolated synchronized behavior and islands of synchronization, or larger windows of synchronized behavior grouped together. Furthermore, another goal lies in finding the minimal dictionary, containing the fewest number of entries to represent the varied coupling geometries of oscillator networks of a given size. Finally, in order to make use of the dictionary as a statistical tool, a measure of similarity between data and representative dictionary entries must be established; one way to achieve this could be by recording the relative fractional occurrences of different length three addresses, and determining which dictionary entry matches the distribution most closely.

Markov Chains

An important question to consider when analyzing empty addresses and emergent patterns in the graph of a driven IFS is whether or not a proposed series of rules will generate the same image as a plot graphed using data. When examining an IFS with memory, this concern corresponds to the idea of a Markov partition; that is a, system of dividing a map into bins and addresses such that the image may be determined solely by the forbidden pairs (or n-tuples) of digits in the addresses of points [7]. This implies that to fit the conditions for such a partition, every empty address in an image is one excluded as a result of a forbidden pair contained in its address.

However, considering an IFS plot based on data rather than deterministically defined, it can be more difficult to determine whether an empty space is truly "empty", or whether it might be filled if more data were collected — in other words, whether or not the map is a Markov partition [7]. To compare time series data with dictionary entries, it is important to have statistical tools to assess the significance of empty addresses; that is, whether any unfilled bins of a given size indicate real exclusions in the data, or simply gaps caused by random variations. To investigate this, one can compare the probability of finding an empty address of this length in data for which the address is not forbidden to an alternative hypothesis asserting that the address is indeed forbidden.

For any given finite length of memory, the transitions between bins of time series data can be considered as a Markov chain [7]. In general, an empty length-three address in a time series divided into four bins can be modeled with a five state Markov process, with the specific graph varying depending upon whether or not any bins are repeated in the address. To see this, consider a data set with a potentially forbidden length-three address of "123" (recalling that the naming convention bin addresses introduces the newest transitions to the leftmost side of the address). Then the five states, relating to present or absent substrings in the sequence, are:

A. *No* 123 *substring, and the leftmost number is* 1 *or* 4

B. *No* 123 *substring, and the leftmost number is* 3

C. *No* 123 *substring, and the leftmost number is* 2*, and the previous number is not* 3

D. *No* 123 *substring, and the leftmost number is* 2*, and the previous number is* 3

E. 123 *substring is present*

To calculate the probability of a transition from state 1 to state 2 (which may or may not be the same as state 1) requires simply the summation of the frequency of transitions between 1 and 2 along single-step paths through the graph, divided by the total frequency of data points in each of bins 1 and 2. Putting these probabilities into a matrix M, then calculate

$M^n \vec{v}_i = \vec{R}$, where $\vec{v}_i$is a standard basis vector with 1 in the i-th place. The resultant vector represents the probabilities of states 1, 2, 3, 4, and 5 respectively in a data set of that size, if the length-two addresses in question were not forbidden; a value of 0.999975 in the fifth position, for example, represents that according to the null hypothesis, 99.9975% of all randomly sampled data sets generated from these rules will have trait 5 and thus contain the address in question.

Although this calculation deals solely with the anticipated question of empty length-three addresses in four-bin systems, expanding the process to arbitrarily long addresses in systems with arbitrarily many bins has no effect on the calculations, and simply necessitates a more complicated and list of independent states.

With such statistical tools, it is possible to determine the likelihood that any given empty length n address represents a forbidden n-tuple, and thus, gives a numerical measure of the validity of modeling a driven IFS with a memory of at least n transitions.

A further statistical technique to be explored is creating a histogram from our simulations of the number of iterations required to reach a state containing forbidden-tuples (state E in the previous example). If the results of these iterations can be said to be independent and identically distributed, the Central Limit Theorem states that this histogram will approximate the normal distribution, in which case simple hypothesis testing based on standard deviations will also suffice to determine whether or not an empty address is truly "forbidden".

However, it is likely that this will not be the case, because the transition graph from one state to another involves some "memory" of previous states. It is easy to see that an address at state D, for example, will be more likely to reach state E in fewer steps than an address at state B, because the address in state B must first transition to state D before ever reaching E. However, it is also possible that this dependence on previous states will only have a small effect on the independence of each sample run from the next, in which case the histogram distribution may still appear nearly normal. In order to test the normality of the experimentally determined distribution, the kurtosis of the distribution can be calculated; if it is not nearly-normal, a different distribution can be created to model the iterations and reach the same goal of simple hypothesis testing.

Applications of the Method and Future Developments

While the focus of the research project is primarily to create the tools necessary to categorize driven IFSs and analyze time series data based on these categories, avenues for further research lie in extending this inquiry to different examples of coupled oscillators, which are abundant (and often little understood) in the natural world. Several data sources are suggested as ideal candidates for analysis, including biological systems which rely heavily on the synchronization of chaotic processes.

Heartbeat Spacing

Nonlinear dynamics have long been noted as a characteristic feature of many periodic biological functions, and characterizing these dynamic systems and networks may have medical value in differentiating working from malfunctioning dynamics, especially in body systems such as the heart or brain [10].

It is known that, as with many biological systems, healthy cardiac tissue can act as a network of coupled electrical oscillators as pacemaker cells react to the pulse frequency of those around them [15]. This behavior can cause synchronization or desynchronization of individual cells, affecting the behavior of the heart as a whole. Furthermore, experimentation and analysis of heart rate fluctuations has revealed that heart rates may be more accurately modeled as a chaotic system, with complex variability and without a constant steady state, than as a homeostatic system, in which all fluctuations departing from a steady state oscillation result from external influences [8]. Manifestations of this chaotic state have been experimentally observed in the fact that disruption and destabilization of heart rates tends to result in an overall loss of variability in heart rate dynamics, which fits with the chaotic model of high complexity rather than one in which a steady-state oscillation is considered healthy [10]. These changes in the degree of chaotic behavior in heart rates have been observed in cases of erratic arrhythmias, possibly indicating changes in coupled oscillator dynamics and synchronization between healthy and arrhythmic hearts [9].

As a result, this research further proposes to examine the spacing of heartbeats in various individuals as a process involving coupled oscillators, and to compare dictionary entries with cardiac data from PhysioBank archives. By observing whether data from healthy individuals identifies with a distinct grouping from the dictionary entry relating to a certain irregularity or condition, it can then be inferred whether the irregularity is related to some difference in the type of oscillating system created by the pulsing cells.

Seismic Data

While biological time series analysis is one of the most readily available applications for the proposed dictionary, the tools used in this analysis can be applied to a much wider variety of networks; time series data on the geographic locations of earthquakes, for example, can likely be subjected to this same analysis [16]. Recent research has modeled earthquake occurrences within defined geographical cells by recording the transitions, or "directed edges", between cells in which occurrences of earthquakes were found over time [4]. Furthermore, these transitions were found to behave in a way resembling small-world network coupling, further confirming that this data would be another suitable candidate for analysis using the dictionary of oscillator dynamics [4].

Other Directions

In addition to the most immediately available applications of the proposed dictionary, an abundance of other phenomena potentially displaying chaotic characteristics are also candidates for analysis. These include an extremely diverse range of fields; from the layout of the neural net to self-similar scaling properties of epidemic spreads, fractal patterns are surprisingly common [17]. Fluctuations in stock market rates and seismic events are often theorized to be based on chaotic processes and random concatenations of small associations, while sunspot patterns and even patterns in reversal of the earth's magnetic field have been found to display fractal characteristics [16]. Applications of the new tools of data analysis this research hopes to provide would potentially have the opportunity to break new ground and work towards understanding any of these complex networks.

Conclusions

The applications of tools to analyze and understand chaos are ubiquitous, aiding in understanding any natural and technological phenomena displaying the characteristic "roughness" seen in chaotic dynamics. From the patterns in heartbeats to genetic code to earthquakes and population dynamics, fractal patterns and chaos are inescapable features of much of the world. Through the analysis of coupled logistic maps and the proposed dictionary of coupled oscillator characteristics, this research project aims to develop a new tool for categorizing the synchronization of chaotic networks of oscillators, envisioning far-reaching applications to the analysis of time series data from a wide variety of sources. The prospect of identifying meaningful patterns—whether they may relate to warning characteristics of heartbeat arrhythmias, geological predictions, or other patterns of equally widespread significance—is a strong motivator for continuing this project and working to realize its tangible benefits on data analysis in complicated, chaotically behaving networks.

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