THE NUMBER OF 10-ARCS IN THE PROJECTIVE PLANE OVER A FINITE FIELD IS NOT QUASIPOLYNOMIAL

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Abstract. An n-arc in a projective plane is a collection of n distinct points in the plane, no three of which lie on a line. Formulas counting n-arcs are known up to $n = 9$ for general projective planes; in the case of projective planes over finite fields, these counts are quasipolynomial functions of the order of the plane. We show that the number of 10-arcs in $\mathbb{P}^2(\mathbb{F}_q)$ cannot be expressed as a quasi-polynomial function of q, breaking the pattern demonstrated for values of n up to 9.

The following research paper is nearly complete, and is intended for submission within the next several months. In particular, Sections 1 through 4 are finished. The unfinished portion of Section 5 has been specified.

1. Introduction

An n-arc in a projective plane is a set of n distict points with no three on a line. In this paper, we study the problem of counting n-arcs in the projective plane over a finite field.

We will consider ordered n-arcs, that is, n-tuples of points which form arcs. Let $C_n(\Pi)$ count the number of ordered n-arcs in plane Π. In the case where $\Pi = \mathbb{P}^2(\mathbb{F}_q)$, we write $C_n(q)$ in place of $C_n(\Pi)$. Formulas for $C_n(q)$ were found for $n \leq 8$ by Glynn and for $n = 9$ by Iampolskaia, Skorobogatov, and Sorokin; we reproduce versions of their results below.

Theorem 1.1. [1, 2] *For $n \leq 9$, the number $C_n(q)$ of n-arcs arising in $\mathbb{P}^2(\mathbb{F}_q)$ is given by the following formulas.*

(1) $C_1(q) = q^2 + q + 1$,

(2) $C_2(q) = (q^2 + q + 1)(q^2 + q)$,

(3) $C_3(q) = (q^2 + q + 1)(q^2 + q)q^2$,

(4) $C_4(q) = (q^2 + q + 1)(q^2 + q)q^2(q - 1)^2$,

(5) $C_5(q) = (q^2 + q + 1)(q^2 + q)q^2(q - 1)^2(q^2 - 5q + 6)$,

(6) $C_6(q) = (q^2 + q + 1)(q^2 + q)q^2(q - 1)^2(q^2 - 5q + 6)(q^2 - 9q + 21)$.

(7) $C_7(q) = (q^2+q+1)(q^2+q)q^2(q-1)^2(q-3)(q-5)(q^4-20q^3+148q^2-468q+498)-30a(q)$,

(8) $C_8(q) = (q^2+q+1)(q^2+q)q^2(q-1)^2(q-5)$
$\cdot (q^7 - 43q^6 + 788q^5 - 7937q^4 + 47097q^3 - 162834q^2 + 299280q - 222960)$
$- 30(q^2 - 20q + 78)a(q) + 840b(q),$

(9) $C_9(q) = (q^2+q+1)(q^2+q)(q^2)(q-1)^2$
$(q^{10} - 75q^9 + 2530q^8 - 50466q^7 + 657739q^6 - 5835825q^5 + 35563770q^4$
$- 146288034q^3 + 386490120q^2 - 588513120q + 389442480$
$- 1080(q^4 - 47q^3 + 807q^2 - 5921q + 15134)a(q)$
$+ 840(9q^2 - 243q + 1684)b(q)$
$+ 30240(-9c(q) + 9d(q) + 2e(q))),$

where

$$\begin{aligned} a(q) &= \begin{cases} 1 & \textit{if } 2 \mid q, \\ 0 & \textit{otherwise}; \end{cases} \\ b(q) &= \#(x \in \mathbb{F}_q : x^2 + x + 1 = 0), \\ c(q) &= \begin{cases} 1 & \textit{if } 3 \mid q, \\ 0 & \textit{otherwise}; \end{cases} \\ d(q) &= \#\{x \in \mathbb{F}_q : x^2 + x - 1 = 0\} \\ e(q) &= \#\{x \in \mathbb{F}_q : x^2 + 1 = 0\}. \end{aligned}$$

While these formulas become more convoluted as n increases, they are all fundamentally of the same kind. The functions a, b, c, d, e are periodic, so these formulas are examples of *quasipolynomials* or *polynomials on residue classes* (PORC). This means that for each $n \leq 9$, there exists some integer m and a set of polynomials $f_0(q), ..., f_{m-1}(q)$ such that the value of $C_n(q)$ is equal to $f_i(q)$, where i is the residue of q modulo m.

Our main result is to show that this pattern breaks when $n = 10$.

Theorem 1.2. *Let $C_n(q)$ be the number of n-arcs in $\mathbb{P}^2(\mathbb{F}_q)$. Then $C_n(q)$ is a quasi-polynomial function of q for $1 \leq n \leq 9$, but not for $n = 10$.*

In Section 2, we reduce the problem of counting 10-arcs to the problem of counting certain special configurations of points and lines called *superfigurations*. In Section 3, we use classical incidence geometry to prove that several superfigurations do not arise in projective planes over finite fields. In Section 4, we show how to view superfigurations as points on quasi-affine varieties defined over $\mathbb{F}_q$. In Section 5, we classify the varieties which arise in the 10-arcs formula, and then count the points on these varieties to show that the number of 10-arcs is not quasipolynomial.

2. Superfigurations

Glynn developed an algorithm which shows how to count small arcs in terms of special configurations of points and lines called superfigurations. In this section, we review the necessary definitions, which are discussed at much greater length in our paper [3].

Definition 2.1. A *linear space* is a pair of sets $(\mathcal{P}, \mathcal{L})$, whose elements are referred to as *points* and *lines* respectively, with the following properties: each line is a subset of $\mathcal{P}$, and no two lines intersect in more than one point. A pair (p, l) consisting of a point p and a line l containing p is called an *incidence* of the linear space. Two linear spaces are *isomorphic* if there is a permutation of the points and a permutation of the lines of the second linear space such that its set of incidences exactly match those of the first.

Definition 2.2. A *strong realization* of a linear space S in a projective plane Π is a collection of points and lines of Π such that the set of incidences between them is isomorphic to the set of incidences of S. We write $A_S(\Pi)$ for the number of strong realizations of S in Π, and in the particular case where $\Pi = \mathbb{P}^2(\mathbb{F}_q)$, we write $A_S(q)$.

Definition 2.3. A (combinatorial) *n_k-configuration* is a linear space such that each line consists of exactly k points, and each point is contained in exactly k lines.

A linear space of n points and some number of lines, not necessarily n, in which each line contains *at least* k points and each point is contained in *at least* k lines is called an *n_k-superfiguration*. Throughout the paper, we will focus on $k = 3$, taking "superfiguration" to mean n_3-superfiguration.

In [1], Glynn gives an inductive algorithm which counts arcs in terms of superfigurations. The following statement is a version of Theorem 3.6 in [1].

Theorem 2.4. *Suppose $n \leq 13$. There exist polynomials $f(q), g_s(q)$ such that*

$$C_n(q) = f(q) + \sum_s g_s(q) A_s(q), \tag{*}$$

where the sum is taken over all superfigurations s with at most n points.

We call the polynomial $g_s(q)$ the *coefficient of influence* of the superfiguration s, since it measures the degree to which s is relevant in the n-arcs formula.

In [3], we show that there are 163 superfigurations on up to 10 points. We define each of these superfigurations on our website [4]. The remainder of our paper is concerned with understanding the functions $A_s(q)$ as a means to describing $C_n(q)$. In particular, our goal is to find a superfiguration s for which $A_s(q)$ is not quasipolynomial, and then we will show that none of the other terms cancel out the influence of $A_s(q)$ in the formula of Theorem 2.4.

3. Realizability of Superfigurations I: Pappus and Desargues

We come now to one of the last steps in determining the 10-arcs formula: determining A_s, the number of strong realizations of each superfiguration $s \in S$ for a given $\mathbb{P}^2(\mathbb{F}_q)$. Given a finite field $\mathbb{F}_q$ and a superfiguration $\mathscr{C}$, it is not always the case that $\mathscr{C}$ can be constructed in the projective plane over $\mathbb{F}_q$. In the following section, we outline the method used in [2] for determining whether or not a superfiguration is realizable in $\mathbb{P}^2(\mathbb{F}_q)$ for a given prime power q, and refine it for the peculiar added difficulties that determining the number of strong realizations of a 10-point superfiguration presents. However, before we begin to implement and refine the method of [2] for determining the number of strong realizations of a certain superfiguration, we utilize theorems from classical synthetic projective geometry to prove that some superfigurations are unrealizable. We first recall the theorem of Pappus:

Theorem 3.1 (Pappus). *Suppose that a_1, a_2, and a_3 is a set of collinear points, and that b_1, b_2, and b_3 is another set of collinear points. Let the intersection point of the lines $\overline{a_1b_2}$ and $\overline{a_2b_1}$ be c_1, the intersection point of the lines $\overline{a_1b_3}$ and $\overline{a_3b_1}$ be c_2, and the intersection point of the lines $\overline{a_2b_3}$ and $\overline{a_3b_2}$ be c_3. Then, the points c_1, c_2, and c_3 are collinear.*

It is known that this theorem is valid in the context of a projective plane over a field [5]. While this theorem may not appear to have immediate use, it allows us to directly show that sixteen of the superfigurations on 10 points are unrealizable in $\mathbb{P}^2(\mathbb{F}_q)$ for every prime power q. By simply attempting to construct superfigurations that do not obey Pappus's theorem, we can construct the sixteen superfigurations which disobey Pappus's theorem. However, for our purposes, we need only take a superfiguration and show that it is unrealizable, as we do in the following example.

Example 3.2. Consider superfiguration #71, given by the following definition. We represent the points of this superfiguration by the labels 1, 2, ... , 10. The lines in this superfiguration are then given by triples of integers which represent lines in the superfiguration. These are all of the lines in our superfiguration; we are not allowed to draw any others. The lines are

$$(1,3,5),(1,2,6),(1,7,8),(1,9,10),(2,3,4),$$
$$(2,7,9),(2,5,8),(3,6,8),(3,7,10),(4,5,9),(4,6,10).$$

Suppose we view the line $(1,9,10)$ as $\overline{a_1a_2a_3}$ in the statement of Pappus's theorem, and the line $(4,3,2)$ as $\overline{b_1b_2b_3}$. Then, we have $c_1 = 5$, $c_2 = 6$, and $c_3 = 7$. Thus, by Pappus's theorem, any weak realization of this superfiguration must contain the line $(5,6,7)$. However, this is not a line in our superfiguration and thus there are no strong realizations of superfiguration #71 in $\mathbb{P}^2(\mathbb{F}_q)$ for all prime powers q.

The above example shows how Pappus's theorem can prove a superfiguration to be unrealizable. The classical theorem of Desargues is of similar use.

Theorem 3.3 (Desargues). *Suppose we have two triangles, $\triangle a_1b_1c_1$ and $\triangle a_2b_2c_2$. Suppose further that $\overline{a_1b_1}$ meets $\overline{a_2b_2}$ at d_1, $\overline{a_1c_1}$ meets $\overline{a_2c_2}$ at d_2, and $\overline{b_1c_1}$ meets $\overline{b_2c_2}$ at d_3. Then, the points d_1, d_2, and d_3 are collinear if and only if the lines $\overline{a_1a_2}$, $\overline{b_1b_2}$, and $\overline{c_1c_2}$ are concurrent.*

This theorem also holds in any projective plane over a field, and thus we can use it in a way similar to Pappus's theorem to determine whether or not a superfiguration has any strong realizations in $\mathbb{P}^2(\mathbb{F}_q)$ [5]. As with Pappus, we can attempt to construct superfigurations which contradict Desargues's theorem in order to find more of the superfigurations which are unrealizable in $\mathbb{P}^2(\mathbb{F}_q)$ for all prime powers q. This rules out four more superfigurations.

Example 3.4. Superfiguration #61 has points labeled 1,2,...,10, and we describe the lines in the superfiguration using triples for the lines having three points and 4-tuples for the lines having four points, as follows:

$$(1,2,3,10),(1,4,5),(1,6,7),(8,9,10),(2,4,7,8)$$
$$(3,5,6,8),(2,5,9,),(3,7,9),(4,6,10),(5,7,10).$$

Recall that a strong realization of this superfiguration contains no lines besides the ones already named. Notice that $\triangle 542$ and $\triangle 763$ are both triangles in this superfiguration. Furthermore, they are centrally in perspective: the lines $\overline{57}$, $\overline{46}$, and $\overline{23}$ all intersect at point

10. Now, notice that $\overline{54}$ meets $\overline{76}$ at 1, $\overline{52}$ meets $\overline{73}$ at 9, and $\overline{42}$ meets $\overline{63}$ at 8. Thus, by Desargues' theorem, any weak realization of this superfiguration in $\mathbb{P}^2(\mathbb{F}_q)$ must contain $\overline{189}$. This line is not included in the superfiguration, and thus there are no strong realizations of this superfiguration in $\mathbb{P}^2(\mathbb{F}_q)$ for all prime powers q.

Via these classical geometric theorems, we are able to show that twenty of the superfigurations on 10 points are simply unrealizable. Having cleared these out of the way, we can proceed to outline the standard method with which we will approach the remaining superfigurations.

4. Realizability of Superfigurations II: Method of Determinants

The above techniques eliminate a fair number of superfigurations from our consideration in this particular problem, but there is still a much broader class of 131 superfigurations that we do need to consider. To tackle these remaining superfigurations, we apply a coordinate assignment technique utilized by Iampolskaia, Skorobogatov, and Sorokin. For the 9-arcs computation, this method was relatively straightforward and didn't need to be explicitly detailed; not too many difficulties with degenerate coordinate assignments arose and coordinate assignments yielded relatively simple conditions for the realizability of these superfigurations. However, the computations with the superfigurations on 10 points are a bit more nuanced, and thus we specifically outline the full algorithm used to compute the number of strong realizations of these in $\mathbb{P}^2(\mathbb{F}_q)$.

Definition 4.1. We say that an assignment of coordinates in a projective plane $\mathbb{P}$ of a superfiguration is a *weak realization* of that superfiguration if no two points are given the same coordinates, and sets of collinear points of the superfiguration are collinear in $\mathbb{P}$ after assigning coordinates.

This isn't quite enough, however. As we'll see later in this section, an assignment of coordinates to the points of a superfiguration can create additional collinearities and in doing so transform our superfiguration into another configuration of points and lines. To account for this, we need the following.

Definition 4.2. We say that a weak realization of a superfiguration in $\mathbb{P}^2(\mathbb{F}_q)$ is a *strong realization* of that superfiguration if no additional collinearities are created by the assignment of coordinates.

Lastly, we need to make sure that all of our points and lines in our assignment of coordinates are unique; as we'll also see later in this section, an assignment of coordinates which fails to preserve this uniqueness can lead to the failure of the realized structure in $\mathbb{P}^2(\mathbb{F}_q)$ to remain a superfiguration. For notational ease, we record the following definition.

Definition 4.3. A realization of a superfiguration in $\mathbb{P}^2(\mathbb{F}_q)$ is said to be *nondegenerate* if the assignment of coordinates preserves the uniqueness of all points and lines.

Having laid out this terminology, we are now in a position to clearly and easily outline the method for determining the number A_s in the 10-arcs formula for each superfiguration $s \in S$.

4.1. **Method for Determining A_s.** The idea for this method is to begin assigning coordinates, using variables for the coordinates when we cannot determine them explicitly, to determine how many different ways we can assign coordinates such that the resulting realization is a nondegenerate strong realization of our superfiguration.

We begin by recalling a lemma from [2]:

Lemma 4.4. *Given a superfiguration with intersecting lines $\overline{125}$ and $\overline{345}$, any strong realization in $\mathbb{P}^2(\mathbb{F}_q)$ is projectively equivalent to one where the coordinates assigned to points $1, 2, 3, 4, 5$ are $C_1 = (1 : 0 : 0)$, $C_2 = (0 : 1 : 0)$, $C_3 = (0 : 0 : 1)$, $C_4 = (1 : 1 : 1)$, and $C_5 = (1 : 1 : 0)$, respectively.*

Essentially, we can begin to assign coordinates by selecting two intersecting cords of three points in our superfiguration. We label the intersection point $(1 : 1 : 0)$, and then label the points of one line $(1 : 0 : 0)$ and $(0 : 1 : 0)$ and the other line $(0 : 0 : 1)$ and $(1 : 1 : 1)$. Given this starting point, we now attempt to count the number of possible ways to assign coordinates to the remaining points of a superfiguration such that we end up with a nondegenerate strong realization in $\mathbb{P}^2(\mathbb{F}_q)$. Denote the number we arrive at for a given superfiguration $s \in S$ by $m'_s(q)$. Then, noticing that this process is essentially selecting a 4-arc in our superfiguration and beginning to assign coordinates from there, and furthermore recalling that all 4-arcs are projectively equivalent, we have that

$$A_s := \frac{m'_s(q)|PGL_3(\mathbb{F}_q)|}{|\mathrm{Aut}(s)|},$$

where Aut(s) denotes the automorphism group of the superfiguration s.

We continue with our determination of $m'_s(q)$.

Having assigned our first five coordinates, the next thing we must do is assign variables so as to determine the rest of the coordinates (in this computation, the remaining five coordinates). In order to ensure that our coordinate assignments preserve collinearity, we use the lines of our superfiguration to assign variables. For example, suppose $(2, 3, 9)$ is a line in our superfiguration, where 2 is given as $(0 : 1 : 0)$ and 3 is given by $(0 : 0 : 1)$. Note then that $\overline{23}$ is the line $x = 0$. If we normalize coordinate 9 to $y = 1$ and let $\alpha \in \mathbb{F}_q$, then $9 = (0 : 1 : \alpha)$ preserves the collinearity $(2, 3, 9)$. Assigning coordinates in this way allows us to assign coordinates to each of the 10 points while preserving collinearities.

Now that variables have been assigned to all of the coordinates, we must make sure that all remaining lines which we haven't yet considered are still preserved by our assignment of coordinates. To do this, we recall the characterization of $\mathbb{P}^2(\mathbb{F}_q)$ given in Cameron where points in $\mathbb{P}^2(\mathbb{F}_q)$ correspond to one-dimensional linear subspaces in affine space over $\mathbb{F}_q$. Then, three points in $\mathbb{P}^2(\mathbb{F}_q)$ are collinear if and only if the corresponding subspaces of affine space are coplanar. To check this, we merely see if the determinant of representative vectors is zero. In other words, three points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, and $(\alpha_3, \beta_3, \gamma_3)$ in $\mathbb{P}^2(\mathbb{F}_q)$ are collinear if and only if the determinant

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

is zero. These checks will give us polynomial constraints on the variables we used to assign coordinates in the previous step. For instance, suppose that after we've assigned coordinates we haven't ensured that $\overline{167}$ is in fact a line. Suppose that so far, we've assigned coordinates so that there are no constraints on $\alpha \in \mathbb{F}_q$, and 1,6, and 7 are given by $1 = (1 : 1 : 0)$, $6 = (1 : 0 : \alpha)$, and $7 = (\alpha : 1 : 1)$. Then, we must ensure that $\alpha \in \mathbb{F}_q$ is such that

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & \alpha \\ \alpha & 1 & 1 \end{vmatrix} = -\alpha - 1 + \alpha^2 = \alpha^2 - \alpha - 1 = 0.$$

This severely restricts the choices that we have for α. The number then of coordinate assignments that are still allowed after we've considered all collinearities is the number of weak realizations containing the initial coordinates of Lemma 2.4.

We're not quite done yet, however. To count $m'_s(q)$, we seek the number of *nondegenerate strong realizations* given the first five coordinates. To check that our assignment of coordinates yields a strong realization, we must make sure that we haven't induced any extraneous collinearities. For instance, in the full example which we give below, we will have to make sure that our assignment of coordinates does not allow the points 5,6, and 8 to be collinear. This is not difficult to check; recalling that three coordinates are collinear if and only if the determinant of the 3×3 matrix they define is zero, we choose our coordinates such that their determinant is nonzero.

Lastly, we simply check that the coordinates are assigned in such a way that all points and lines are unique. It may seem trivial and redundant to state this, but important constraints can arise from these considerations which may not have been seen otherwise.

Thus, the claim that ten points are a nondegenerate strong realization of some superfiguration is equivalent to stating that their coordinates satisfy a certain set of determinantal equations and inequalities, all of which involve only polynomials. We call these *final polynomial systems*.

Each final polynomial system traces out a quasi-affine variety defined over $\mathbb{F}_q^m$, where m is the number of variables introduced during the process described above. A point on the quasi-affine variety corresponds to a strong realization of the original superfiguration. Thus, this method provides the dictionary which allows us to count realizations of superfigurations using algebraic geometry.

To give full illustration of the entirety of this method, the potential complexity of the resulting constraints, and the eventual determination of the number of strong realizations of a superfiguration, consider the following full worked example.

Example 4.5. Consider the 10-point superfiguration with points $1, 2, 3, ..., 10$ and lines

$$(1, 2, 3), (1, 4, 5), (1, 6, 7), (1, 8, 9, 10), (2, 4, 8),$$

$$(3, 7, 8), (2, 5, 9), (4, 6, 9), (3, 6, 10), (5, 7, 10).$$

(This is superfiguration #32 on our website.)

Looking at the pairs of points which are not included in any of the lines given above, we note that it is possible to realize both these lines and $\overline{568}$ and remain a superfiguration. We note

this, and will remember it later when considering the possible inputs for the coordinates we assign.

Following the method, we first choose two lines to assign our initial coordinates. We can choose any two lines and the method will work just fine, but to ease our calculations we choose the intersecting lines $\overline{167}$ and $\overline{5710}$. This choice makes our calculations easier as more lines of our superfiguration intersect these two lines than other pairs of lines. Consequently, we will be able to introduce fewer variables, which should make the final constraints more manageable. So, we begin by assigning the following coordinates:

$$1 := (1:0:0), 6 := (0:1:0), 7 := (1:1:0), 5 := (1:1:1), 10 := (0:0:1).$$

Given these, we assign coordinates to the remaining points, using variables when necessary. First, note that with this assignment the line $\overline{145}$ is given by the equation $y = z$. Note that $y, z \neq 0$ for point 5, otherwise the points 1 and 5 would be equivalent. Thus, we can normalize the coordinates for 5 to $y = z = 1$ and assign the x-coordinate $\alpha \in \mathbb{F}_q$. So, we have

$$5 := (\alpha : 1 : 1).$$

Now, we see also that the line $\overline{18910}$ is given by $y = 0$. x must also be nonzero for both points 8 and 9, otherwise they would be equivalent to point 10. Consequently, we can choose $\beta\gamma \in \mathbb{F}_q$, normalize $x = 1$, and arrive at

$$8 := (1:0:\beta),\ 9 := (1:0:\gamma).$$

Considering the last of the easy lines, we see that $\overline{3610}$ is given by $x = 0$. In point 3, y must also be nonzero, or else 3 would be equivalent to 10. Choosing $\delta \in \mathbb{F}_q$ and normalizing to $y = 1$ we see that

$$3 := (0:1:\delta).$$

It remains to give coordinates to point 2. Considering the coordinates we have already assigned, we see that $\overline{123}$ is now given by $z = \delta y$. y must be nonzero in point 2, otherwise z would also be zero and 2 would be equivalent to 1. Thus, normalizing to $y = 1$ and choosing $\epsilon \in \mathbb{F}_q$, we have

$$2 := (\epsilon : 1 : \delta).$$

To recap, we have the points

$$\begin{array}{ccccc} 1 := (1:0:0) & 2 := (\epsilon:1:\delta) & 3 := (0:1:\delta) & 4 := (\alpha:1:1) & 5 := (1:1:1) \\ 6 := (0:1:0) & 7 := (1:1:0) & 8 := (1:0:\beta) & 9 := (1:0:\gamma) & 10 := (0:0:1), \end{array}$$

where α, β, γ, δ, and $\epsilon \in \mathbb{F}_q$.

At this point, α, β, γ, δ, and ϵ can be any elements of $\mathbb{F}_q$. However, we still need to make sure that they are chosen in such a way that preserves the collinearities (2,4,8), (3,7,8), (2,5,9), and (4,6,9). Utilizing the above outlined method of determinants, we obtain the equations

$$\beta(\epsilon - \alpha) = \delta - 1, \tag{1}$$

$$\delta = -\beta, \tag{2}$$

$$\gamma(\epsilon - 1) = \delta - 1, \tag{3}$$

and

$$\alpha\gamma = 1. \tag{4}$$

It is necessary for all our variables to satisfy equations (1)-(4). Algebraic manipulation yields γ, δ, and ϵ in terms of only α and β:

$$\begin{aligned}
\epsilon &= 1 - \alpha(\beta + 1), \\
\delta &= -\beta, \\
\gamma &= \alpha^{-1} && (\alpha \neq 0).
\end{aligned}$$

Substituting these expressions for γ, δ, and ϵ into equation (1), we obtain

$$\alpha\beta^2 + 2\alpha\beta - 2\beta - 1 = 0.$$

Thus, the number of weak realizations of this superfiguration in $\mathbb{P}^2(\mathbb{F}_q)$ is precisely the number of ordered pairs $(\alpha, \beta) \in \mathbb{F}_q^2$ s.t. $\alpha\beta^2 + 2\alpha\beta - 2\beta - 1 = 0$. To ensure that our coordinate assignment yields a strong realization of this superfiguration, however, recall that we need to make sure that $\overline{568}$ is not a line. Consequently, we must choose $(\alpha, \beta) \in \mathbb{F}_q^2$ s.t.

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \beta \end{vmatrix} = \beta - 1 \neq 0,$$

which is true if and only if

$$\beta \neq 1.$$

We see now that precisely any choice of $(\alpha, \beta) \in \mathbb{F}_q^2$ s.t. $\alpha\beta^2 + 2\alpha\beta - 2\beta - 1 = 0$, $\alpha \neq 0$, and $\beta \neq 1$ will yield a unique strong realization of this superfiguration in $\mathbb{P}^2(\mathbb{F}_q)$. Lastly, we must check to make sure that our choice of (α, β) is such that all points and lines in the superfiguration are unique. Substituting in the above calculations for γ, δ, and ϵ and doing a bit of simple algebra gives the following expressions for the points of our superfiguration:

$$\begin{array}{lllll}
1 = (1:0:0) & 2 = (1 - \alpha(\beta+1) : 1 : -\beta) & 3 = (0:1:-\beta) & 4 = (\alpha:1:1) & 5 = (1:1:1) \\
6 = (0:1:0) & 7 = (1:1:0) & 8 = (1:0:\beta) & 9 = (\alpha:0:1) & 10 = (0:0:1),
\end{array}$$

and the following expressions for the lines of our superfiguration:

$$\begin{aligned}
\overline{123} &: \beta y + z = 0 & \overline{145} &: y - z = 0 \\
\overline{167} &: z = 0 & \overline{18910} &: y = 0 \\
\overline{248} &: \beta x + (1 - \alpha\beta) y - z = 0 & \overline{378} &: \beta x - \beta y - z = 0 \\
\overline{259} &: x + (\alpha - 1) y - \alpha z = 0 & \overline{469} &: x - \alpha z = 0 \\
\overline{3610} &: x = 0 & \overline{5710} &: x - y = 0.
\end{aligned}$$

Examining these points and lines, we see that in order for our assignment to be a nondegenerate strong realization we must choose $(\alpha, \beta) \in \mathbb{F}_q$ s.t. all of the following conditions are satisfied:

$$\begin{aligned}
\alpha\beta^2 + 2\alpha\beta - 2\beta - 1 &= 0, \\
\alpha &\notin \{0, 1\}, \\
\beta &\notin \{-1, 0, 1\}, \\
\beta &\notin \left\{\alpha^{-1}, \alpha^{-1} - 1, (\alpha - 1)^{-1}\right\}.
\end{aligned} \tag{5}$$

Applying elementary techniques in number theory, we see that the number of nondegenerate strong realizations of this superfiguration in $\mathbb{P}^2(\mathbb{F}_q)$ is given by

$$m'_s(q) = \begin{cases} q-7 & \text{if } q \equiv 1(\text{mod}6); \\ q-2 & \text{if } q \equiv 2(\text{mod}6); \\ q-3 & \text{if } q \equiv 3(\text{mod}6); \\ q-4 & \text{if } q \equiv 4(\text{mod}6); \\ q-5 & \text{if } q \equiv 5(\text{mod}6). \end{cases} \tag{6}$$

This result is verified by a computer program up to $q = 19$. A simple computer program can give us the size of the automorphism group of this superfiguration; substituting this result into the formula immediately after Lemma 5.4 gives A_s.

On our website, we give final polynomial systems for each of the 163 superfigurations on up to 10 points [4].

5. Classification of 10-point Superfigurations

Realizability conditions for 10-point superfigurations tend to be much more complicated than those up to 9 points. In particular, many 10-point superfigurations cannot be counted by quasipolynomials. Rather, they can be counted as the number of solutions to a final polynomial system, that is, a set of equalities and inequalities of polynomials in the variables which parameterize the superfiguration space. Thus, in algebro-geometric terms, the strong realizations of a superfiguration are in 1-1 correspondence with the points on some quasi-affine variety over $\mathbb{F}_q$. In this section, we classify the quasi-affine varieties which arise from the superfigurations on up to 10 points.

(1) Empty: 9, 13, 19, 31, 43, 52, 53, 61, 63, 68, 81, 84, 90, 102, 105, 107, 108, 111, 115, 119, 126, 128, 131, 133, 136, 142, 143, 148, 153, 154, 157, 158, 161.

(2) Dimension 0: 1, 2, 7, 10, 11, 12, 14, 46, 47, 48, 49, 54, 56, 57, 62, 99, 103, 106, 112, 113, 114, 117, 120, 121, 123, 124, 127, 132, 134, 137, 139, 140, 141, 146, 147, 149, 150, 151, 163.

(3) Dimension 1: 4, 5, 6, 8, 15, 28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 44, 45, 50, 51, 55, 58, 59, 60, 65, 66, 67, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 82, 83, 85, 86, 87, 88, 89, 91, 92, 93, 94, 95, 96, 97, 98, 101, 145.

(4) Dimension 2: 3, 26, 27.

(5) K3 surfaces: 13.

(6) To be classified: 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25.

The above list has some errors, in particular because the notion of dimension is not well-defined for a variety due to the fact that we are considering multiple ground fields. However, the fact that a K3 surface arises is enough to show that the count is not quasipolynomial.

In [3], we demonstrate that the coefficient of influence of superfiguration #13 is nonzero, completing the proof.

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