Counting Hamiltonian Cycles in Dense and Regular Directed Graphs

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1 Background

The problem of counting Hamiltonian Cycles in general graphs, both directed and undirected, is notoriously difficult; finding even a single Hamiltonian Cycles in a general graph is famously NP-complete. However, much interest remains in proving sufficient conditions for the existence of Hamiltonian Cycles in particular classes of graphs, including Erdős-Rényi random graphs and their directed analogues, graphs with density or regularity requirements, as well as in considering related problems including cycle counting, covering, and packing.

One very simple such counting question is to place bounds on the quantity $\Phi(G)$ for a given graph G, where $\Phi(G)$ is total number of distinct Hamiltonian cycles that can be constructed in that graph. In particular, this report investigates a lower bound on $\Phi(G)$ in graphs which are likely to have many Hamiltonian Cycles: those which have a high density and are additionally regular.

Many strong results already exist in this field of inquiry. Starting from the classical result by Dirac [1] that every graph G on n vertices with minimum degree at least $n/2$ contains at least one Hamiltonian cycle, results regarding counting Hamiltonian cycles were incrementally improved over the course of many years. Sárközy, Selkow and Szemerédi [2], for example, improved this result to show that such "Dirac Graphs" in fact contain at least $c^n n!$ Hamilton cycles for some small positive constant c; Cukler and Kahn [3] improved this to show that for such graphs,

$$\Phi(G) \geq (\frac{\delta(G)}{e})^n (1 - o(1))^n$$

(where $\delta(G)$ is the minimum degree in the graph).

This idea extends to random graphs as well: for $G \sim \mathcal{G}(n,p), p > 1/2$ the previous estimate is sharp, and with high probability, $\delta(G) = pn + o(n)$ and the expected number of Hamiltonian cycles is

$$p^n(n-1)! < (pn/e)^n$$

Furthermore, Krivelevich and Glebov [4] proved for a random graph

$$G \sim \mathcal{G}(n,p), \text{ with } p \geq \frac{\log n + \log\log n + \omega(1)}{n}, \text{ that } \Phi(G) = (1 - o(1))^n n! p^n$$

and it was later shown that for larger p-values, $\Phi(G)$ has a log-normal distribution.

Additionally, Ferber, Krivelevich, and Sudakov [5] have recently shown a permanent-based approach for counting Hamiltonian cycles in non-random oriented graphs based on the number of r-factors in the graph (where an r-factor is an r-regular spanning subgraph), an approach which had previously only been successful on investigation of random graphs. It is a similar approach of counting r-factors upon which the results of this report are based.

This paper presents a simple method for counting and constructing Hamiltonian cycles in dense, regular directed graphs, and shows that the method may also be extended to produce counting results on a particular class of oriented graphs. As a proof of concept, we reproduce two recent results using this simpler method, including the strongest known result on regular tournaments, and examine ways in which this technique might be used to produce new bounds on related Hamiltonian Cycle counting problems.

2 Basic Terminology

First, as a reminder, we recall some fundamental language of graph theory that will play a central role in this paper:

Definition. *A* ***directed*** *graph G is a collection of vertices $V(G)$ and edges $E(G)$ such that each edge $(u, v) \in E(G)$ is an ordered pair with $u, v \in V(G)$.*

Definition. *The* ***in- (or out-) degree*** *of a vertex in a directed graph is the number of edges entering (or leaving) that vertex; the* ***minimum (or maximum) semi-degree*** *of a vertex is the lower (or higher) of the in- and out-degrees of that vertex.*

Definition. *An* ***oriented*** *graph G is a directed graph where for any pair of vertices $u, v \in V(G)$, at most one of (u, v) and (v, u) is in $E(G)$.*

Definition. *A* ***d-regular*** *directed or oriented graph G is a graph such that for all $v \in V(G)$, the in-degree $\delta^+(v)$ and out-degree $\delta^-(v)$ are both d.*

Definition. *A* ***tournament*** *is any orientation of a complete graph; that is, the directed graph obtained by assigning an orientation to each edge of an undirected complete graph.*

Definition. *A* ***Hamiltonian Cycle*** *in a directed graph is a cycle spanning all vertices within a graph, such that all edges of the cycle are oriented in the same direction when listed in a cyclic order. An* ***arbitrary oriented Hamiltonian Cycle****, by contrast, may have any prescribed sequence of orientations for the edges of the cycle [6].*

3 Useful Tools and Theorems

3.1 The Chernoff Bound

We will also frequently make use of the following version of Chernoff's bound on the lower and upper tails of the Binomial distribution:

Lemma 3.1 (Chernoffs inequality)**.** *Let $X \sim Bin(n,p)$ and let $\mathbb{E}(X) = \mu$. Then*

- $\mathbb{P}[X < (1-a)\mu] < e^{-a^2\mu/2}$ *for every* $a > 0$*;*
- $\mathbb{P}[X > (1+a)\mu] < e^{-a^2\mu/3}$ *for every* $0 < a < 3/2$*.*

Remark 3.2. *These bounds also hold when X is hypergeometrically distributed with mean μ.*

3.2 R-factors and Regular Bipartite Graphs

It will also be necessary in later proofs to count the number of regular spanning subgraphs of a given graph, which can be accomplished in the bipartite case using some useful theorems by Gayle, Ryser, and Ferber:

Definition. *An **r-factor** of a graph G is an r-regular, spanning subgraph of G.*

Theorem 3.3 (Gale and Ryser [7])**.** *Let $G = (A \cup B, E)$ be a bipartite graph with $|A| = |B| = n$, and let r be any integer. Then, G contains an r-factor if and only if*

$$e_G(X,Y) \geq r(|X| + |Y| - n) \text{ holds for all } X \subseteq A \text{ and } Y \subseteq B.$$

The following two theorems as conceived by Ferber are not yet published, so we provide a quick outline of the proof of each.

Theorem 3.4 (Ferber [8]). *Let $\alpha \geq 1/2$, and let n be a sufficiently large integer and let $\xi := \xi(n) \geq 0$. Let $G = (A \cup B, E)$ be a bipartite graph with $|A| = |B| = n$ with*

$$\alpha n + \xi \leq \delta(G) \leq \Delta(G) \leq \alpha n + \xi + \frac{2\xi^2}{n}$$

Then, G contains an αn-factor.

Proof sketch. Let $X \subseteq A$ and $Y \subseteq B$ of size x and y, respectively. Theorem 3.4 is then proved by showing that the result of Theorem 3.3 holds under the condition that $\alpha n + \xi \leq \delta(G) \leq \Delta(G) \leq \alpha n + \xi + \frac{2\xi^2}{n}$. This is accomplished using the estimates that

$$e_G(X, Y) \geq x(\delta(G) + y - n) \text{ and}$$

$$e_G(X, Y) = e_G(X, B) - e_G(X, B \setminus Y) \geq \delta(G)x - \Delta(G)(n - y)$$

and examining separately the cases where (1) $x + y \leq n$, (2) $x \leq y$ and $x \leq \delta(G)$, and (3) $x \leq y$ and $x > \delta(G)$. ■

Theorem 3.5 (Ferber [8]). *Let $G = (A \cup B, E)$ be a bipartite graph with parts of sizes $|A| = |B| =: m$, let $d \leq m/2$ and let $\xi := \xi(m)$. Suppose that $d - \xi - \xi^2/2m \leq \delta(G) \leq \Delta(G) \leq d - \xi$. Then, there exists a bipartite graph $H = (A \cup B, E')$ for which*

1. *H is d-regular, and*
2. *G is a subgraph of H.*

Proof sketch. Let G be a graph as in the assumptions of the theorem. Consider the graph $G^c = (A \cup B, E^*)$ for which $e \in E^*$ if and only if $e \notin E$. Clearly, $m - d + \xi \leq \delta(G^c) \leq \Delta(G^c) \leq m - d + \xi + \xi^2/2m$, and therefore, using Theorem 3.4 one can find an $(m-d)$-regular subgraph $S \subseteq G^c$. Letting $H := S^c$ we complete the proof. ■

3.3 (l, s)-Partitions

The proof of a lower bound on the number of Hamiltonian Cycles will also involve some useful properties of partitioning the vertices of a graph in a particular way known an $(l, s)-$ *partition*:

Definition. *An **(l,s)-partition** of a graph G on n vertices is a partition of the vertices $V(G)$ into l groups of size $\frac{n-s}{l}$ along with a single group of size s.*

Lemma 3.6. *For a given a d-regular graph G on n vertices, and a given s, $l < n$, with high probability, an (l, s)-partition of $V(G)$ into groups $V_1, \cdots, V_l, S$ satisfies*

1. $\forall i, \forall v \in V_i, \delta^+(v, V_{i+1}) \in (1 \pm \epsilon)\frac{d}{l}\frac{n-s}{n}$
2. $\forall i, \forall v \in V_i, \delta^-(v, V_{i-1}) \in (1 \pm \epsilon)\frac{d}{l}\frac{n-s}{n}$
3. $\forall v \in V_l, \delta^+(v, S) \in (1 \pm \epsilon)\frac{ds}{n}$
4. $\forall v \in V_1, \delta^-(v, S) \in (1 \pm \epsilon)\frac{ds}{n}$

where, if μ is the expected value of the expression, $\epsilon = \frac{\log n}{\sqrt{\mu}}$. Specifically, these conditions hold with probability $\geq 1 - o(1)$.

Proof. The proof of this lemma is a straightforward application of the Chernoff bound. We prove statement 1 of the lemma, and the proofs for 2-4 follow identically.

First, fix the vertex $v \in V_i$. Then $\delta^+(v, V_{i+1})$ is distributed $\sim Hypergeom(n, \frac{n-s}{l}, d)$: out of a population of n vertices, exactly d of which have an edge from v, $\delta^+(v, V_{i+1})$ is the number of vertices in a random sample of size $|V_{i+1}| = \frac{n-s}{l}$ which have an edge from v. Then in expectation,

$$\mathbb{E}[\delta^+(v, V_{i+1}] = \mu = d \cdot \frac{n-s}{l} \cdot \frac{1}{n}$$

Using the Chernoff bound then gives

$$\mathbb{P}[\delta^+(v, V_{i+1}) \notin (1 \pm \epsilon)\mu] \leq e^{-c\epsilon^2\mu}$$

$$= e^{-c \log^2 n}$$

for a constant c. Noting that $e^{-c\log^2 n}$ is $o(\frac{1}{n^2}) = o(1)$, we have that statement 1 holds for the fixed vertex v. To extend this to hold for all vertices v, we can simply apply the union bound to see that

$$\mathbb{P}[\delta^+(v, V_{i+1}) \notin (1 \pm \epsilon)\mu \ \forall v \in V_i] \leq n \cdot e^{-c\log^2 n} = o(\frac{1}{n}) = o(1)$$

■

With these tools, we can demonstrate the method for counting Hamiltonian Cycles in regular graphs in the particular case of dense, directed graphs.

4 A lower bound on $\Phi(G)$ for sufficiently dense, d-regular directed graphs G

Theorem 4.1. *For a d-regular directed graph G on n vertices, with $d \geq \frac{n}{2} + \epsilon n$ ($\epsilon > 0$), the number of Hamiltonian cycles in G denoted by $\Phi(G)$ satisfies*

$$\Phi(G) \geq (1 - o(1))^n \cdot n!(\frac{d}{n})^n$$

The proof of this theorem relies on the construction of some additional structures based on the graph G, defined below.

4.1 Construction of S, P, and the Path System

First, select a set of vertices from $V(G)$ uniformly at random to create an induced subgraph S with $s = \frac{n}{\log^2 n}$ vertices. Additionally construct the induced subgraph P of G such that $V(P) = V(G) - S$, and define $m = |V(P)| = n - s$.

As a first step towards constructing a series of Hamiltonian Cycles, we would like to find sets of edge-disjoint paths in P, such that each vertex in P is on one such path. We will use the term **path system** to refer to any such collection of paths in G:

Definition. *A **path system** in a graph G is a set of edge-disjoint paths in G such that each vertex $v \in V(G)$ is on exactly one path.*

We can construct a series of path systems, with each path of length $log^4 n$, in in the following way:

1. Choose a random partition of the vertices of P into $\log^4 n$ groups of size $\frac{m}{log^4 n}$, labeling the groups $v_1, \cdots, v_{log^4 n}$
2. Create set E' such that for any edge $e = (u, w) \in E(G)$, $e \in E'$ if and only if $u \in v_i$ and $w \in v_{i+1}$ for some $1 \leq i \leq log^4 n - 1$.
3. Construct a perfect matching from v_i to v_{i+1} for each i using the edges in E'.
4. Use the edges of the $\log^4 n - 1$ perfect matchings to construct a path system using all vertices of P.

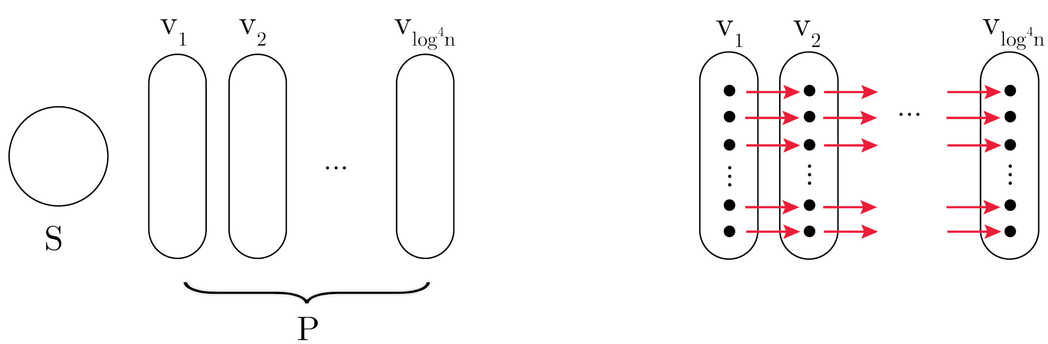

Figure 1: (left) Subgraphs S and P, with P partitioned into sets $v_1 \cdots v_{\log^4 n}$.
(right) A path system in P constructed as a set of perfect matchings between v_i's.

The exposition which follows will show that sufficiently many distinct path systems can be constructed in G using the above method, and that each one can be converted into a corresponding Hamiltonian Cycle.

4.2 Lower Bound for the Number of Perfect Matchings

Theorem 4.2. *Let ρ be the number of perfect matchings that can be created between v_i and v_{i+1} for any $1 \leq i \leq log^4 n - 1$. Then*

$$\rho \geq \left[\frac{1}{2} + \epsilon\right]^{\frac{m}{\log^4 n}} \left(\frac{m}{\log^4 n}\right)!$$

Proof. To prove this, we will use several of the results from Section 3:

Consider the graph G' consisting of $v_i \cup v_{i+1}$ and all edges (u, w) such that $u \in v_i$ and $w \in v_{i+1}$.

Lemma 4.3. *G' contains a $(\frac{1}{2} + \epsilon)\frac{m}{\log^4 n}$-factor.*

Proof. First, notice that G' is a bipartite graph on $2 \cdot \frac{m}{\log^4 n}$ vertices which is approximately regular: from Lemma 3.6, the degree of each vertex is

$$d'(v) \in (1 \pm \epsilon)\frac{d}{\log^4 n}\frac{n - \log^2 n}{n}$$

Noting that $d \geq \frac{n}{2} + \epsilon n$, the above expression is equivalently

$$d'(v) \in (1 \pm \epsilon_0)\frac{m(\frac{1}{2} + \epsilon)}{\log^4 n}$$

where $\epsilon_0 = \frac{\log n}{\sqrt{\mu}}$.

We can therefore apply Theorem 3.4 with $\alpha = \frac{1}{2} + \epsilon$ and $|A| = |B| = \frac{m}{\log^4 n}$ to conclude that G' contains a $(\frac{1}{2} + \epsilon)\frac{m}{\log^4 n}$-factor. ∎

Theorem 4.4 (Van der Waerden [9]). *For an r-regular bipartite graph G on $2n$ vertices, the number of perfect matchings in G is $\geq \left(\frac{r}{n}\right)^n \cdot n!$*

Using Theorem 4.4, we are able to complete the derivation of Theorem 4.2 to find a lower bound on the number of perfect matchings in G'. Let $r = (\frac{1}{2}+\epsilon)\frac{m}{\log^4 n}$. Then G' has an r-factor, which is by definition r-regular and, as a spanning subgraph of the bipartite graph G', is also bipartite on $2 \cdot \frac{m}{\log^4 n}$ vertices. Then the number of perfect matchings in the r-factor is

$$\geq \left(\frac{r}{n}\right)^n \cdot n! = \left[\frac{1}{2}+\epsilon\right]^{\frac{m}{\log^4 n}} \left(\frac{m}{\log^4 n}\right)!$$

As the r-factor is a subgraph of G', the perfect matchings in the r-factor are a subset of those in G'. Noting also that any perfect matching between v_i and v_{i+1} must use only those vertices and edges in G', we can conclude that the number of perfect matchings from v_i to v_{i+1} is also at least $\left[\frac{1}{2}+\epsilon\right]^{\frac{m}{\log^4 n}} \left(\frac{m}{\log^4 n}\right)!$. ■

4.3 Lower Bound for the Number of Path Systems in P

Theorem 4.5. *The number of distinct path systems which can be constructed in P is $\geq (1-o(1))^n \cdot n! \cdot (\frac{d}{n})^n$.*

Note. *There are $\frac{m!}{\left[\frac{m}{\log^4 n}!\right]^{\log^4 n}}$ distinct partitions of the vertices of P into groups of size $\frac{m}{\log^4 n}$.*

Proof. From Lemma 4.2 and the preceding note, we know that the process for generating path systems by partitioning vertices and finding perfect matchings between each successive group in the partition can result in $\frac{m!}{\left[\frac{m}{\log^4 n}!\right]^{\log^4 n}}$ distinct partitions, with $\left[\frac{1}{2}+\epsilon\right]^{\frac{m}{\log^4 n}} \left(\frac{m}{\log^4 n}\right)!$ possible perfect matchings between each of $\log^4 n - 1$ pairs of consecutive partition sets, and $\left(\frac{m}{\log^4 n}\right)!$ ways to order the partition sets (each resulting in perfect matchings between different pairs of sets). Therefore, in total there can be

$$\frac{m!}{\left[\frac{m}{\log^4 n}!\right]^{\log^4 n}} \cdot \left(\frac{m}{\log^4 n}\right)! \cdot \left[\left[\frac{1}{2}+\epsilon\right]^{\frac{m}{\log^4 n}} \left(\frac{m}{\log^4 n}\right)!\right]^{\log^4 n - 1}$$

$$= m!\left(\frac{d}{n}\right)^{m-\frac{m}{\log^4 n}}$$

$$\geq m!\left(\frac{d}{n}\right)^{m}$$

path systems constructed in this way. Recalling that $m = n - \frac{n}{\log^2 n}$, the above equation is

$$\geq m!\left(\frac{d}{n}\right)^{n}$$

Lemma 4.6. $\left(n - \frac{n}{\log^2 n}\right)! = (1 - o(1))^n \cdot n!$

Proof. First, note that $n - \frac{n}{\log^2 n} = n\left(1 - \frac{1}{\log^2 n}\right) = n(1 - o(1))$, as $\frac{1}{\log^2 n} = o(1)$. Then by Stirling's approximation:

$$\left(n - \frac{n}{\log^2 n}\right)! = (n(1 - o(1)))! = \sqrt{2\pi n(1 - o(1))}\left(\frac{n(1 - o(1))}{e}\right)^{n(1-o(1))}$$

$$= \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n(1-o(1))}\sqrt{1 - o(1)}$$

$$= n!(1 - o(1))^n\sqrt{1 - o(1)} = n!(1 - o(1))^n$$

And so we have that $\left(n - \frac{n}{\log^2 n}\right)! = (1 - o(1))^n \cdot n!$. ■

Using Lemma 4.6, the number of path systems which can be constructed in P using the provided construction is $\geq (1 - o(1))^n n! \left(\frac{d}{n}\right)^n$.

Noting that path systems generated using distinct vertex partitions or distinct perfect matchings cannot be identical to one another, it is the case that each of these path systems is distinct. Therefore we can construct at least $(1 - o(1))^n \cdot n! \left(\frac{d}{n}\right)^n$ distinct path systems in P, which completes the proof of Theorem 4.5. ■

Now that we have seen that there are at least as many path systems as the claimed lower bound for the number of Hamiltonian Cycles in G, it completes the proof of Theorem 4.1 to define a method for converting each path system into a distinct Hamiltonian Cycle.

4.4 Construction of Hamiltonian Cycles

Theorem 4.7. *Each distinct path system in P consisting of $\geq \frac{m}{\log^4 n}$ paths corresponds to a unique Hamiltonian Cycle in G.*

Proof. Given a path system $\{P_1, \cdots P_l\}$ where $l = \frac{m}{\log^4 n}$, we can create a Hamiltonian Cycle by linking the paths together using vertices from S. This construction works as follows:

1. Select a random partition of S into $\frac{m}{\log^4 n}$ sets $S_1, \cdots, S_l$ of equivalent size (up to rounding).
2. Let p_i^{start} and p_i^{end} be the first and last vertices in path P_i, respectively, and let j be $i-1$ for $i > 1$, and l otherwise. Designate two vertices s_i^{in} and s_i^{out} in each S_i, such that there exist edges $e_i^1 = (p_j^{end}, s_i^{in})$ and $e_i^2 = (s_i^{out}, p_i^{start})$. (See Fig. 3)
3. Construct a Hamiltonian path Q_i in S_i from s_i^{in} to s_i^{out} for each S_i.
4. $P_l,\ e_1^1,\ Q_1,\ e_1^2,\ P_1,\ e_2^1,\ Q_2,\ e_2^2,\ P_2,\ \cdots,\ e_l^1,\ Q_l,\ e_l^2$ is a Hamiltonian Cycle in G. (See Fig. 4 and 5)

First note that if such a construction is possible, the constructed Hamiltonian Cycle is distinct from the cycles created using any other path system of P. This is evident simply because each path in the path system occurs once in the Hamiltonian Cycle, and no other edges between two vertices of P exist in the Cycle. Therefore, a second Hamiltonian Cycle created from a different path system of P will contain some path between vertices of P that does not occur in the original Hamiltonian Cycle, and so the two Cycles must be distinct.

Now it remains to show that such a construction is guaranteed to be possible. For this, we rely on a theorem of Keevash, Kühn, and Osthus [10]:

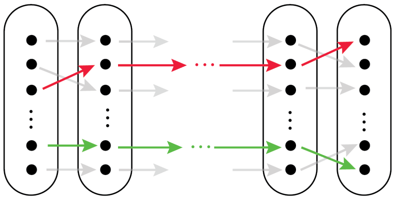

Figure 2: A path system with two paths P_i and P_j highlighted.

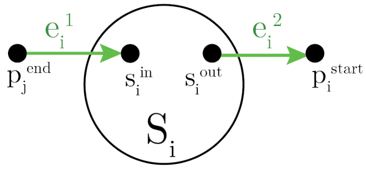

Figure 3: One subset $S_i \subset S$, along with vertices s_i^{in}, $s_i^{out} \in S_i$, $p_j^{end} \in P_j$, and $p_i^{start} \in P_i$.

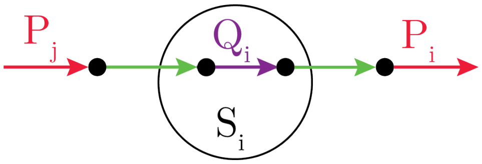

Figure 4: One link of the Hamiltonian Cycle: Path P_j from the path system, followed by edge e_i^1, connects to the Hamiltonian path through S_i, Q_i. Q_i is followed by edge e_i^2, which connects to P_i, the next path in the path system.

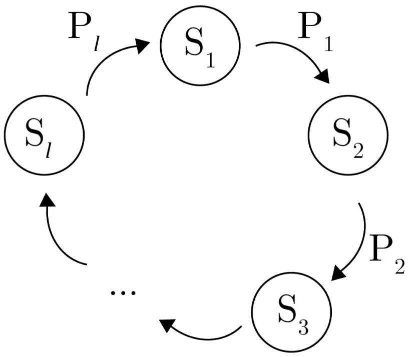

Figure 5: The fully constructed Hamiltonian Cycle (with internal edges Q_i and connecting edges e_i omitted for clarity).

Theorem 4.8. *There exists an n_0 such that any oriented graph G on $n \geq n_0$ vertices with minimum semi-degree $\delta^0(G) \geq \lceil \frac{3n-4}{8} \rceil$ contains a Hamiltonian Cycle.*

Remark 4.9. *It also follows from the proof of Theorem 4.8 as published by Keevash, Kühn, and Osthus that for a directed graph G on m vertices, with minimum semi-degree $\delta^0(G) \geq \frac{m}{2} + \epsilon' m$, then $\forall x, y \in V(G)$, $x \neq y$, there exists a Hamiltonian path from x to y.*

Step 3 is Possible

We consider the degree of each vertex in S_i. Using the Chernoff bound, as in Lemma 3.6, $\forall s \in S_i$,

$$\delta^-(s, S_i) \in (1 \pm \epsilon_0)\frac{d \cdot |S_i|}{n} \geq (1 \pm \epsilon_0)|S_i|(\frac{1}{2} + \epsilon)$$

with probability $\geq 1 - o(1)$, where μ is the expected value $\frac{d \cdot |S_i|}{n}$ and $\epsilon_0 = \frac{\log n}{\sqrt{\mu}}$.

Thus, applying the union bound for all vertices, there is a nonzero probability that this condition holds for all $s \in S_i$, and furthermore, that it holds for all i. Therefore by the probabilistic method, for a randomly chosen partition of S, there is a nonzero probability that each S_i possesses the degree condition required to guarantee the existence of a Hamiltonian path between any two vertices in S_i. Since this occurrence has a positive probability, there is some partition of S for which it does happen, and we can take this partition to be the one used in the construction [11].

In this case, Theorem 4.8 applies to S_i to show that for any $s_i^{in}, s_i^{out} \in S_i$, there exists a Hamiltonian Path from s_i^{in} to s_i^{out}.

Step 2 is Possible

Finally, we see that it is possible to choose s_i^{in} and s_i^{out} such that edges e_i^1 and e_i^2 exist in G. By an analogous application of the Chernoff Bound to that of Lemma 3.6, it is the case that $\forall v \in V_i$,

$$\delta^-(v, S_i) \in (1 \pm \epsilon_0)\frac{d \cdot |S_i|}{n}$$

with probability $\geq 1 - o(1)$, where μ is the expected value $\frac{d \cdot |S_i|}{n}$ and $\epsilon_0 = \frac{\log n}{\sqrt{\mu}}$.

Once again, applying the union bound, given a random partition of S, such an s_i^{in} and s_i^{out} can be chosen for all i with a positive probability. Thus, using the probabilistic method, since there is a nonzero probability of choosing all s_i^{in} and s_i^{out} such that edges e_i^1 and e_i^2 exist in G for some partition of S, then there is some partition of S for which this condition will indeed hold.

Therefore, it is guaranteed that each step of the construction is possible, and so the construction is necessarily possible on every path system. Recalling that

$$\bigcup_i V(P_i) \cup \bigcup_i V(Q_i) = V(P) \cup \bigcup_i V(S_i)$$

$$= V(P) \cup V(S) = V(G)$$

the cycle constructed in Step 4 contains each vertex of G exactly once; thus, we have demonstrated a construction which maps each distinct path system in P to a a unique Hamiltonian Cycle in G. ■

From this construction, the lower bound for $\Phi(G)$ from Theorem 4.1 follows quickly:

4.5 Lower Bound for $\Phi(G)$

Proof. (Theorem 4.1) From Theorem 4.5, the number of distinct path systems in P is $\geq (1-o(1))^n \cdot n! \cdot (\frac{d}{n})^n$. From Theorem 4.7, each of these path systems can be converted into a distinct Hamiltonian Cycle in G using the remaining vertices from S. Thus we are able to construct at least

$$(1-o(1))^n \cdot n! \left(\frac{d}{n}\right)^n$$

Hamiltonian Cycles in G, completing the proof that

$$\Phi(G) \geq (1-o(1))^n \cdot n!(\frac{d}{n})^n$$

∎

5 Extension to Tournament Graphs

Counting Hamiltonian Cycles is also of interest in regular tournament graphs, which have similar characteristics to the dense, d-regular graphs from the previous section. However, the minimum density requirement in Theorem 4.1 is not fulfilled for regular tournaments: in a regular tournament on n vertices (for odd n) each vertex has in- and out- degree of exactly $\frac{n-1}{2}$, whereas degree $\geq \frac{n}{2} + \epsilon n$ ($\epsilon > 0$) is required in Theorem 4.1.

With minor modification, however, a similar proof to that of the previous section can be shown to work for the case of Hamiltonian Cycles in regular tournaments. The key methodology will be deleting a small number of edges from the graph and then using Theorem 3.5 on the graph's complement to ultimately find the necessary r-factor to proceed with the proof.

Theorem 5.1. *For a d-regular tournament graph G on n vertices, the number of Hamiltonian cycles in G satisfies*

$$\Phi(G) \geq (1-o(1))^n \cdot n!(\frac{d}{n})^n$$

Proof. The proof proceeds exactly as in the previous section until the application of Theorem 3.4 to find an r-factor in the graph. It is at this point that the degree requirement is first used: for the graph G_i' consisting of $v_i \cup v_{i+1}$ and all edges (u,w) such that $u \in v_i$ and $w \in v_{i+1}$, the average degree of each vertex in G_i' is

$$\frac{d}{n} \cdot \frac{m}{\log^4 n} = \frac{m(n-1)}{2n\log^4 n}$$

rather than

$$\frac{m(\frac{1}{2}+\epsilon)}{\log^4 n}$$

as in the non-tournament graph. Therefore, the Chernoff bound can only show with high probability that, letting $m_0 = \frac{m}{\log^4 n}$, for any vertex $g \in G'$, $\delta(g) \in (\frac{m_0}{2} \pm \epsilon)$. This is not sufficient to use Theorem 3.4.

To circumvent this difficulty, we set p such that $p \cdot (\frac{m_0}{2} + \epsilon) = \frac{m_0}{2} - \xi - \frac{\xi^2}{4m_0}$ for some $\xi > 0$, and create a new graph B which includes each edge of graph G' with probability p.

Now, applying the Chernoff bound shows that each vertex $b \in B$ satisfies $\delta(b) \in (\frac{m_0}{2} - \xi - \frac{\xi^2}{4m_0} \pm \frac{\xi^2}{4m_0}$ with probability $\geq 1 - e^{-c\mu\frac{\xi^2}{4m_0}}$. We now see that the degrees of vertices in B are sufficiently low that the complement B^C of B satisfy the conditions of Theorem 3.5: for $b \in B^C$,

$$\delta(b) \in (m_0 - (\frac{m_0}{2} - \xi - \frac{\xi^2}{4m_0}), m_0 - (\frac{m_0}{2} - \xi - \frac{2\xi^2}{4m_0}))$$

$$\implies \delta(b) \in (\frac{m_0}{2} + \xi, \frac{m_0}{2} + \xi + \frac{\xi}{2m_0})$$

Now, the condition for Theorem 3.5 holds, and we have that there exists some H such that H is $\frac{m}{2}$-regular and B^C is a subgraph of H. Recalling the proof of Theorem 3.5, the complement H^C of this graph H is a $(m_0 - \frac{m_0}{2}) = (\frac{m_0}{2})$-regular graph which spans the same vertices as H, and additionally contains no edges that are not in G'. We use this H^C as the required $(\frac{d}{n} \cdot m_0)$-factor spanning graph G', and continue the proof as in the previous section, applying Theorem 4.4.

The remainder of the proof of the bound for directed graphs does not require the use of the extra ϵ in the degree of G, and so the remainder of the proof proceeds identically, showing the lower bound on the number of path systems and converting each path system into a Hamiltonian Cycle. ■

6 Discussion and Future Work

Considering this work in the context of what is known about counting Hamiltonian Cycles, it is important to note that Theorem 4.1 is not the strongest known for directed graphs. The central idea of this proof is dependent primarily on the use of concentration bounds: showing that in dense, regular graphs, with high probability subsets of the graph will act very similarly to the properties found in the entire graph. Furthermore, the technique of finding regular, spanning subgraphs of portions of the graph is a robust tool that can apply to graphs even with slightly weaker density conditions. The proof is therefore more significant as a proof of concept, as it is a far simpler technique than those used to prove similar bounds, and leaves much room to extend the methods to other types of graphs, on which such strong bounds are not yet known; for example, to tournament graphs or other graphs with weaker degree conditions.

In particular, in the coming months, we expect to continue working to extend this result to apply to the case of counting arbitrary oriented Hamiltonian cycles. The motivation behind this project was, in fact, based on a Hamiltonian Cycle-finding game on oriented graphs; results similar to those presented in this paper for arbitrary oriented Hamiltonian cycles with a required orientation perhaps chosen adversarially are generally unknown. While the exact method presented here is most directly applicable to ordinary Hamiltonian cycles in sufficiently dense directed graphs, and in this case it is only strong enough to re-prove a known bound, after confirming that the central mechanism used in this paper on directed graphs could also be used on tournament graphs, we believe that the same method could also be extended to apply to general oriented graphs and arbitrary oriented Hamiltonian cycles. Ultimately, we hope to continue work in this vein to prove a novel bound for arbitrary oriented Hamiltonian Cycles.

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