COUNTING 9-ARCS AND 10-ARCS IN THE PROJECTIVE PLANE

NATHAN KAPLAN, SUSIE KIMPORT, RACHEL LAWRENCE, LUKE PEILEN, AND MAX WEINREICH

The following is a preprint edition of a paper based on research done in the summer of 2015. The proof is complete except for the The final draft will be uploaded to the arXiv by December 15.

Abstract. An n-arc in a projective plane is a collection of n distinct points in the plane, no three of which lie on a line. Formulas counting n-arcs in a finite projective plane of order q were known up to $n = 8$ for general projective planes. In 1995, Iampolskaia, Skorobogatov, and Sorokin counted 9-arcs in $\mathbb{P}^2(\mathbb{F}_q)$ and showed that this count was a quasi-polynomial function of q [1]. We present a formula which counts 9-arcs even in non-Desarguesian planes, deriving their formula as a special case. Then we show that the number of 10-arcs in $\mathbb{P}^2(\mathbb{F}_q)$ cannot be expressed as a quasi-polynomial function of q, breaking the pattern demonstrated for values of n up to 9.

1. Introduction

An n-arc in a projective plane is a set of n distict points with no three on a line. This definition introduces the notion of "points in general position," fundamental to classical and algebraic geometry, to the finite setting. Over infinite fields, we expect almost all sets of points to be in general position, and further, we can find arbitrarily many points in general position. We have no such luck with finite projective planes, where the scarcity of points may make geometrical constructions more difficult.

Given the simplicity of the definition, it comes as no surprise that n-arcs arise across both classical and modern finite geometry. A 4-arc is just the same data as a choice of coordinates in a Desarguesian plane; a conic in $\mathbb{P}^2(\mathbb{F}_q)$ is surely also a $q+1$-arc. Arcs are so fundamental that one would hope to have a good supply of them in any projective space. This point is driven home by David Glynn's argument that in a putative projective plane of order 6, the number of 7-arcs would be at most -6, which is impossible [2].

When building an arc with many points, we risk running out of points to include. This question motivates Segre's problems on the existence of large arcs, which have remained central to finite geometry since they were first posed [3]. Furthermore, not all arcs have the same properties. An arc may be classical, meaning that it lies on a rational normal curve; it may also be complete, meaning that no larger arc contains it. A good understanding of n-arcs for small n can also prove useful in the study of larger arcs. For example, the key step in Glynn's discovery of the non-classical 10-arc in $\mathbb{P}^4(\mathbb{F}_9)$ depended on Hirschfeld's classification of 8-arcs in $\mathbb{P}^2(\mathbb{F}_9)$ [4].

Thus the question becomes: for fixed n, how many n-arcs exist in a given finite projective plane? In 1988, Glynn developed an inductive algorithm for counting the total number of n-arcs in a finite projective plane in terms of its order q. As a corollary, Glynn found that for $n \leq 13$, these counts C_n may be expressed as

$$C_n = f(q) + \sum_s g_s(q) A_s. \tag{*}$$

Here, $f(q)$ and the $g_s(q)$ are polynomials in q, the order of the projective plane; the s range over an finite set of highly determined configurations of points and lines, such as the Fano plane and the Mobius-Kantor configuration; and each A_s term counts the realizations of the configuration s in the projective plane under consideration. We call the polynomial $g_s(q)$ the *coefficient of influence* of s, since it measures the degree to which configuration s is relevant in the n-arcs formula.

Glynn then proved formulas for the number of n-arcs for $n \leq 8$ [2]:

(1) $C_1 = q^2 + q + 1$

(2) $C_2 = (q^2 + q + 1)(q^2 + q)$

(3) $C_3 = (q^2 + q + 1)(q^2 + q)q^2$

(4) $C_4 = (q^2 + q + 1)(q^2 + q)q^2(q - 1)^2$

(5) $C_5 = (q^2 + q + 1)(q^2 + q)q^2(q - 1)^2(q^2 - 5q + 6)$

(6) $C_6 = (q^2 + q + 1)(q^2 + q)q^2(q - 1)^2(q^2 - 5q + 6)(q^2 - 9q + 21)$

(7) $C_7 = (q^2 + q + 1)(q^2 + q)q^2(q - 1)^2(q - 3)(q - 5)(q^4 - 20q^3 + 148q^2 - 468q + 498) - A_7$

(8) $C_8 = (q^2 + q + 1)(q^2 + q)q^2(q - 1)^2(q - 5)$
$\times (q^7 - 43q^6 + 788q^5 - 7937q^4 + 47097q^3 - 162834q^2 + 299280q - 222960)$
$- (q^2 - 20q + 78)A_7 + A_8.$

Here, A_7 and A_8 count the number of Fano planes and Mobius-Kantor configurations contained in the projective plane in question. In the case of Desarguesian planes, the counts A_7 and A_8 are quasi-polynomials in q depending the residue class of q modulo 6. Therefore, Glynn's formulas demonstrate that the function $C_n(q)$ counting n-arcs in the Desarguesian plane of order q is a quasi-polynomial function of q, for $n \leq 8$. Glynn did not push his method further, noting "the complexity of the problem as the number of points approaches 10" [2].

In 1995, Iampolskaia, Skorobogatov, and Sorokin continued the work, counting 9-arcs as a means to counting linear MDS codes of length 9 and dimension 3 [1]. First, they observe using Glynn's theorem (*) that the 9-arcs count C_9 is polynomial in q and linear in twelve variables $A_7, A_8, A_{9_3}, A_{9_2}, ..., A_{9_{12}}$ which count instances of highly determined configurations of up to 9 points. Then, they restrict attention to $\mathbb{P}^2(\mathbb{F}_q)$. They show that the values A_s are quasipolynomial in q, proving that the 9-arc count is quasipolynomial in q as well. Finally, they interpolate the coefficients of the quasipolynomial, proving the following formula.

Theorem 1.1 (Iampolskaia, Skorobogatov, and Sorokin)**.** *The number of* 9*-arcs in the projective plane over the finite field* $\mathbb{F}_q$ *is given by*

$$\begin{aligned} C_9 = {} & (q^2+q+1)(q^2+q)(q^2)(q-1)^2 \\ & (q^{10} - 75q^9 + 2530q^8 - 50466q^7 + 657739q^6 - 5835825q^5 + 35563770q^4 \\ & - 146288034q^3 + 386490120q^2 - 588513120q + 389442480 \\ & - 1080(q^4 - 47q^3 + 807q^2 - 5921q + 15134)a(q) \\ & + 840(9q^2 - 243q + 1684)b(q) \\ & + 30240(-9c(q) + 9d(q) + 2e(q))) \end{aligned}$$

where the functions a, b, c, d, e *either check characteristic or count solutions to certain polynomials over* $\mathbb{F}_q$*:*

$$\begin{aligned} a(q) &= 1 \textit{ if } 2|q;\ 0 \textit{ otherwise} \\ b(q) &= \#(x \in \mathbb{F}_q : x^2 + x + 1 = 0) \\ c(q) &= 1 \textit{ if } 3|q;\ 0 \textit{ otherwise} \\ d(q) &= \#(x \in \mathbb{F}_q : x^2 + x - 1 = 0) \\ e(q) &= \#(x \in \mathbb{F}_q : x^2 + 1 = 0). \end{aligned}$$

In this paper, we continue the project of counting small arcs in projective planes, with the hope of illuminating the ways in which complexity explodes as one moves from 9-arcs to 10-arcs. To do so, we must return to Glynn's original algorithm. To our knowledge, our work is the first case of Glynn's algorithm actually being implemented, rather than used only to prove existence of the formula.

The problem of understanding n-arcs divides into two related problems: understanding the coefficients of influence $g_s(q)$, and reasoning about the realizations A_s of highly-determined configurations. We attack both sides of this problem in this paper, arriving at the following pair of theorems.

Theorem 1.2. *The number of* 9*-arcs in a general projective plane* $\mathbb{P}$ *is given by*

$$\begin{aligned} C_9 = {} & q^{18} - 75q^{17} + 2529q^{16} - 50392q^{15} + 655284q^{14} \\ & - 5787888q^{13} + 34956422q^{12} - 141107418q^{11} + 356715069q^{10} \\ & - 477084077q^9 + 143263449q^8 + 237536370q^7 + 52873326q^6 \\ & - 2811240q^5 - 588466080q^4 + 389304720q^3 \\ & + (-36q^4 + 1692q^3 - 29052q^2 + 212148q - 539784)A_7 \\ & + (9q^2 - 243q + 1647)A_8 \end{aligned}$$

$$- A_{9_3} - A_{9_4} - A_{9_5} + A_{9_7} - 3A_{9_{10}} + 3A_{9_{11}} - 9A_{9_{12}}.$$

Here A_7 counts the Fano plane, A_8 counts Mobius-Kantor, and A_{9_i} counts the i^{th} type of 9-point superfiguration as presented in [1].

Theorem 1.3. *Let $C_n(q)$ be the number of n-arcs in $\mathbb{P}^2(\mathbb{F}_q)$. Then $C_n(q)$ is a quasi-polynomial function of q for $1 \leq n \leq 9$, but not for $n = 10$.*

In Section 2, we describe the algorithm and discuss our implementation. In Section 3, we discuss the combinatorics of superfigurations, the geometric structures which prevent the arcs count from being polynomial in q. In Section 4, we discuss how classical projective geometry reduces the problem. In Section 5, we discuss the method of determinants, which describes how many realizations of superfigurations exist in projective planes. Section 6, which details the proof of our second theorem, is incomplete, and will be finished by December 15, 2016.

2. Algorithms

The algorithm used to arrive at Theorems 1.3 and 1.4 was first described in [2], and was further clarified in [5]. The latter form of the algorithm was a central component of Iampolskaia, Skorobogatov, and Sorokin's formula for counting 9-arcs in the projective plane over a finite field. We here present an exposition of the algorithm and prove that it works, following [5] and [1]; then we discuss how we altered the algorithm in order to achieve a manageable runtime.

Definition 2.1. A *boolean n-function* is a function taking subsets of $\{1, 2, 3, ..., n\}$ to {0,1}. Two boolean n-functions f and g are *isomorphic* if there is a permutation i of $\{1, ..., n\}$ so that $g = f \circ i$.

Particularly important is that f can be thought of as *labeled*: for instance, we must distinguish the boolean 2-function which just sends $\{1\}$ to 1 from the isomorphic boolean 2-function which just sends $\{2\}$ to 1. We also note that the boolean n-functions are in 1-1 correspondence with the power set of the power set of $\{1, ..., n\}$.

Definition 2.2. For boolean n-functions f and g, we say $f \geq g$ if $f(S) \geq g(S)$ for all $S \subseteq \{1, 2, ..., n\}$. Notice that $\geq$ is a partial order on the set of boolean n-functions.

Example 2.3. Suppose $f : 2^{\{1,...,7\}} \to \{0, 1\}$ sends just the sets {1,3,4,5}, {4,5,6} to 1 and every other set to 0. Then f is a boolean 7-function. If we let $f' : 2^{\{1,...,7\}} \to \{0, 1\}$ be the function that sends just the sets {1,3,4} and {4,5,6} to 1 and all other sets to 0, then we have $f \not\geq f'$.

We are working towards an abstraction of the axioms of geometry, in which the elements of $\{1, ..., n\}$ are points in some geometric setting, and the function f is an indicator function which evaluates to 1 for those subsets of {1,...,n} which are collinear. To make this precise:

Definition 2.4. Call a boolean n-function f a *linear space function* if it satisfies:

(1) If $f(I) = 1$, then $f(J) = 1$ for all $J \subseteq I$.

(2) If $\#(I) \leq 2$, then $f(I) = 1$.

(3) If $f(I) = f(J) = 1$ and $\#(I \cap J) \geq 2$, then $f(I \cup J) = 1$.

If f does not satisfy these laws, we call f *pathological.*

In words, these laws mean: subsets of lines are lines; any set of $0, 1$, or 2 points qualifies as a line; and if two lines intersect in at least two points, then the union of the sets is a line.

Thus, linear space functions capture the notion of collinearity which is common to affine and projective geometry. Indeed, finite affine and projective spaces of any dimension can be thought of as being just the data of some linear space function. Moreover, any subset of such a plane inherits collinearity relations from the plane, so it can also be encoded by some linear space function. Thus, a linear space function f defines a *linear space* with points $\{1, 2, ..., n\}$ and sets of collinear points defined by $f^{-1}(1)$. We can think as of a linear space as an isomorphism class of linear space functions.

For the rest of the section, fix some projective plane $\mathbb{P}$ of order q.

Definition 2.5. Suppose some n-tuple S of distinct points labeled $1, ..., n$ in $\mathbb{P}$ is such the sets which a given linear space function f sends to are exactly the collinear subsets of S. Such a tuple is called a *strong realization* of f.

Definition 2.6. Suppose some n-tuple S of distinct points labeled $1, ..., n$ in $\mathbb{P}$ is such the sets which a given linear space function f sends to are among the collinear subsets of S. Such a tuple is called a *weak realization* of f.

Consider the boolean n-function a for which $f^{-1}(1)$ is the set of subsets of $\{1, ..., n\}$ of size 0, 1, or 2. Then a is a boolean n-function, and is also a linear space function. Every tuple of n distinct points in $\mathbb{P}$ is a weak realization of a. A strong realization of a is an n-arc. Therefore, the goal of the algorithm is to calculate $m_q(a)$. We will do so indirectly, by examining weak realizations and working backwards.

Definition 2.7. For any boolean n-function f, define

$$n_q(f) = \sum_{g \geq f} m_q(g).$$

If f is a linear space function, then this number is the same as the number of weak realizations of f in $\mathbb{P}$. And if f is pathological, then $n_q(f)$ is still defined, although its interpretation in terms of weak realizations is less clear.

Now we reproduce the method described in [5] to calculate $n_q(g)$ in terms of the n_q numbers for linear space functions on fewer points.

Definition 2.8. Suppose that f is a linear space function. A *full line* of f is a subset $S \subseteq \{1, ..., n\}$ of size 3 or greater, so that $f(S) = 1$ and for all T which contain S, we have $f(T) = 1$ only if $T = S$. In other words, there is no larger set of collinear points which includes S. We say the *index* of a point p of f is the number of full lines which include p.

Notice that we can completely describe a linear space function just by giving its full lines.

Lemma 2.9. *Suppose the linear space function f on n points has a point of index 0, 1, or 2. Then we may construct $n_q(f)$ as a polynomial in q and the values $m_q(g)$, where g ranges over the linear space functions of $n-1$ points. Further, $n_q(f)$ is linear in the $m_q(g)$.*

Proof. Let f be as stated. Without loss of generality, say point n has index 0, 1, or 2. Define f' to be the linear space function which corresponds to the configuration of just the points $1, 2, ..., n-1$, inheriting collinearity data from f. Let G be the set of all linear space functions $g \geq f'$. We count weak realizations of f as follows: each one is a strong realization of some $g \geq f'$, together with point n. (In particular, notice that we do not need to range over choices of point n; just choices of g given a fixed n.) So we shall count, for each g, the number of ways to add point n such that the result is a weak realization of f. We let $\mu(g, f')$ denote the number of ways to add point n to g to get a weak realization of f. We thus get an equation

$$n_q(f) = \sum_{g \geq f'} m_q(g)\mu(g, f).$$

Then, it is enough to give an method which finds μ for any pair g, f as a polynomial in q. This method is described in [5]; we omit it. $\square$

Lemma 2.9 indicates the reason why Glynn's formula (*) is not just a polynomial in q. Since the method for inferring the value of n_q only applies to those linear space functions with a point of index 0, 1, or 2, the algorithm cannot inductively find n_q for linear space functions with all points of index at least 3. This suggests that these linear space functions belong to a special class, so we make a new definition.

Definition 2.10. Define a *superfiguration* to be a linear space function such that all points have index at least 3.

Glynn calls these structures "variables;" Iampolskaia, Skorobogatov, and Sorokin call them "overdetermined configurations." We have opted to call them superfigurations for the sake of consistency with Grunbaum's text on classical configurations of points and lines, and to distinguish them from other uses of the terms "variable" and "configuration" [6]. While Grunbaum takes a more expansive definition of "superfiguration," we restrict to the case of at least three lines per point and at least three points per line.

Now the following algorithm will inductively express each m_q and n_q in terms of just the values $m_q(f)$ for superfigurations f .

Algorithm 2.11.
(1) Find m_q and n_q for the linear space functions on 1 point.
(2) Assume that we have m_q and n_q for all linear space functions on k points.
(3) Use Lemma 2.9 to find n_q for every linear space function on $k+1$ points.
(4) Assume that f is a linear space function on $k+1$ points, and we assume that we know $m_q(g)$ for all linear space functions $g > f$. Then calculate $m_q(f)$, by writing

$$m_q(f) = n_q(f) - \sum_{g>f} m_q(g).$$

(5) Repeat the previous step until we have calculated $m_q(f)$ for all f on $k+1$ points.
(6) Continue by induction until $k = n$.

Thus, by running this algorithm, we can express the number of strong or weak realizations of any n-point linear space function L in a projective plane $\mathbb{P}$ as

$$f(q) + \sum_{s \in S} g_s(q) A_s(q)$$

where S is the set of superfigurations on up to n points, $A_s(q)$ is the number of realizations of superfiguration s in $\mathbb{P}$, and $f(q)$ and the $g_s(q)$ are polynomials in q.

Running this algorithm up to 9 points gives us the formula for 9-arcs claimed in Theorem 1.2.

3. Superfigurations

Definition 3.1. The *Levi graph* of a linear space S is the bipartite graph with: a black vertex for each point in S; a white vertex for each line in S; and an edge between a black vertex p and a white vertex l whenever p lies on l in S.

Note that if a point or line in S has index d, then its corresponding vertex in the Levi graph has degree d as well. We can see then that superfigurations are in 1-to-1 correspondence with the set of bipartite graphs having:

(1) a set of black vertices, all of degree at least 3;
(2) a set of white vertices, all of degree at least 3;
(3) girth at least 6, meaning that the graph contains no 4-cycles.

Besides giving us a new language with which to discuss superfigurations, this correspondence leads naturally to the notion of duality of superfigurations.

Definition 3.2. The **dual** of a linear space is the linear space obtained by interchanging black and white vertices in the Levi graph.

It is easy to verify that the dual of a superfiguration is a superfiguration.

To understand 9-arcs and 10-arcs, we need a list S of all n-point superfigurations, for $n \leq 10$.

Computation. The complete list of n-point superfigurations can be determined by computing the list of hypergraphs on n vertices under the constraints that the minimum vertex degree is 3, minimum set size is 3, and the intersection of any two sets is of size at most 1. For $n \leq 11$, McKay's *Nauty* software can quickly compute all such hypergraphs, and we use this software to verify our results [7].

Counts of Linear Spaces						
n	7	8	9	10	11	12
Linear spaces on n points	24	69	384	5250	232929	28872973
Superfigurations	1	1	10	151	16234	>>179894

We give the explicit definitions of the 163 superfigurations on up to 10 points on our website [8].

4. Realizability of Superfigurations I: Pappus and Desargues

We come now to one of the last steps in determining the 10-arcs formula: determining A_s, the number of strong realizations of each superfiguration $s \in S$ for a given $\mathbb{P}^2(\mathbb{F}_q)$. Given a finite field $\mathbb{F}_q$ and a superfiguration $\mathscr{C}$, it is not always the case that $\mathscr{C}$ can be constructed in the projective plane over $\mathbb{F}_q$. In the following section, we outline the method used in [1] for determining whether or not a superfiguration is realizable in $\mathbb{P}^2(\mathbb{F}_q)$ for a given prime power q, and refine it for the peculiar added difficulties that determining the number of strong realizations of a 10-point superfiguration presents. However, before we begin to implement and refine the method of [1] for determining the number of strong realizations of a certain superfiguration, we utilize theorems from classical synthetic projective geometry to prove that some superfigurations are unrealizable. We first recall the theorem of Pappus:

Theorem 4.1 (Pappus)**.** *Suppose that a_1, a_2, and a_3 is a set of collinear points, and that b_1, b_2, and b_3 is another set of collinear points. Let the intersection point of the lines $\overline{a_1b_2}$ and $\overline{a_2b_1}$ be c_1, the intersection point of the lines $\overline{a_1b_3}$ and $\overline{a_3b_1}$ be c_2, and the intersection point of the lines $\overline{a_2b_3}$ and $\overline{a_3b_2}$ be c_3. Then, the points c_1, c_2, and c_3 are collinear.*

It is known that this theorem is valid in the context of a projective plane over a field [9]. While this theorem may not appear to have immediate use, it allows us to directly show that sixteen of the superfigurations on 10 points are unrealizable in $\mathbb{P}^2(\mathbb{F}_q)$ for every prime power q. By simply attempting to construct superfigurations that do not obey Pappus's theorem, we can construct the Levi graphs of the sixteen superfigurations which disobey Pappus's theorem. However, for our purposes, we need only take a superfiguration and show that it is unrealizable, as we do in the following example.

Example 4.2. Consider superfiguration #71, given by the following definition. To represent the points of this superfiguration by the labels 1,2,...,10. The lines in this superfiguration are then given by triples of integers which represent lines in the superfiguration. These are all of the lines in our superfiguration; we are not allowed to draw any others. The lines are

$$(1,3,5),(1,2,6),(1,7,8),(1,9,10),(2,3,4),$$
$$(2,7,9),(2,5,8),(3,6,8),(3,7,10),(4,5,9),(4,6,10).$$

Suppose we view the line $(1,9,10)$ as $\overline{a_1a_2a_3}$ in the statement of Pappus's theorem, and the line $(4,3,2)$ as $\overline{b_1b_2b_3}$. Then, we have $c_1 = 5$, $c_2 = 6$, and $c_3 = 7$. Thus, by Pappus's theorem, any weak realization of this superfiguration must contain the line $(5,6,7)$. However, this is not a line in our superfiguration and thus there are no strong realizations of superfiguration #71 in $\mathbb{P}^2(\mathbb{F}_q)$ for all prime powers q.

The above example shows how Pappus's theorem can prove a superfiguration to be unrealizable. The classical theorem of Desargues is of similar use.

Theorem 4.3 (Desargues)**.** *Suppose we have two triangles, $\triangle a_1b_1c_1$ and $\triangle a_2b_2c_2$. Suppose further that $\overline{a_1b_1}$ meets $\overline{a_2b_2}$ at d_1, $\overline{a_1c_1}$ meets $\overline{a_2c_2}$ at d_2, and $\overline{b_1c_1}$ meets $\overline{b_2c_2}$ at d_3. Then, the points d_1, d_2, and d_3 are collinear if and only if the lines $\overline{a_1a_2}$, $\overline{b_1b_2}$, and $\overline{c_1c_2}$ are concurrent.*

This theorem also holds in any projective plane over a field, and thus we can use it in a way similar to Pappus's theorem to determine whether or not a superfiguration has any strong realizations in $\mathbb{P}^2(\mathbb{F}_q)$ [9]. As with Pappus, we can attempt to construct superfigurations which contradict Desargues's theorem in order to find more of the superfigurations which are unrealizable in $\mathbb{P}^2(\mathbb{F}_q)$ for all prime powers q. This rules out four more superfigurations.

Example 4.4. Superfiguration #61 has points labeled 1,2,...,10, and we describe the lines in the superfiguration using triples for the lines having three points and 4-tuples for the lines having four points, as follows:

$$(1,2,3,10),(1,4,5),(1,6,7),(8,9,10),(2,4,7,8)$$

$$(3,5,6,8),(2,5,9,),(3,7,9),(4,6,10),(5,7,10).$$

Recall that a strong realization of this superfiguration contains no lines besides the ones already named. Notice that $\triangle 542$ and $\triangle 763$ are both triangles in this superfiguration. Furthermore, they are centrally in perspective: the lines $\overline{57}$, $\overline{46}$, and $\overline{23}$ all intersect at point 10. Now, notice that $\overline{54}$ meets $\overline{76}$ at 1, $\overline{52}$ meets $\overline{73}$ at 9, and $\overline{42}$ meets $\overline{63}$ at 8. Thus, by Desargues' theorem, any weak realization of this superfiguration in $\mathbb{P}^2(\mathbb{F}_q)$ must contain $\overline{189}$. This line is not included in the superfiguration, and thus there are no strong realizations of this superfiguration in $\mathbb{P}^2(\mathbb{F}_q)$ for all prime powers q.

Via these classical geometric theorems, we are able to show that twenty of the superfigurations on 10 points are simply unrealizable. Having cleared these out of the way, we can proceed to outline the standard method with which we will approach the remaining superfigurations.

5. Realizability of Superfigurations II: Method of Determinants

The above techniques eliminate a fair number of superfigurations from our consideration in this particular problem, but there is still a much broader class of 131 superfigurations that we do need to consider. To tackle these remaining superfigurations, we apply a coordinate assignment technique utilized by Iampolskaia, Skorobogatov, and Sorokin. For the 9-arcs computation, this method was relatively straightforward and didn't need to be explicitly detailed; not too many difficulties with degenerate coordinate assignments arose and coordinate assignments yielded relatively simple conditions for the realizability of these superfigurations. However, the computations with the superfigurations on 10 points are a bit more nuanced, and thus we specifically outline the full algorithm used to compute the number of strong realizations of these in $\mathbb{P}^2(\mathbb{F}_q)$.

We begin by rephrasing Definitions 2.5 and 2.6 in more geometric language. A *collinearity* is a set of points all lying upon one line.

Definition 5.1. We say that an assignment of coordinates in a projective plane $\mathbb{P}$ of a superfiguration is a *weak realization* of that superfiguration if no two points are given the same coordinates, and sets of collinear points of the superfiguration are collinear in $\mathbb{P}$ after assigning coordinates.

This isn't quite enough, however. As we'll see later in this section, an assignment of coordinates to the points of a superfiguration can create additional collinearities and in doing so transform our superfiguration into a different linear space. To account for this, we need the following.

Definition 5.2. We say that a weak realization of a superfiguration in $\mathbb{P}^2(\mathbb{F}_q)$ is a *strong realization* of that superfiguration if no additional collinearities are created by the assignment of coordinates.

Lastly, we need to make sure that all of our points and lines in our assignment of coordinates are unique; as we'll also see later in this section, an assignment of coordinates which fails to preserve this uniqueness can lead to the failure of the realized structure in $\mathbb{P}^2(\mathbb{F}_q)$ to remain a superfiguration. For notational ease, we record the following definition.

Definition 5.3. A realization of a superfiguration in $\mathbb{P}^2(\mathbb{F}_q)$ is said to be *nondegenerate* if the assignment of coordinates preserves the uniqueness of all points and lines.

Having laid out this terminology, we are now in a position to clearly and easily outline the method for determining the number A_s in the 10-arcs formula for each superfiguration $s \in S$.

5.1. **Method for Determining A_s.** The idea for this method is to begin assigning coordinates, using variables for the coordinates when we cannot determine them explicitly, to determine how many different ways we can assign coordinates such that the resulting realization is a nondegenerate strong realization of our superfiguration. We begin by recalling a lemma from [1]:

Lemma 5.4. *Given a superfiguration with intersecting lines $\overline{125}$ and $\overline{345}$, any strong realization in $\mathbb{P}^2(\mathbb{F}_q)$ is projectively equivalent to one where the coordinates assigned to points $1,2,3,4,5$ are $C_1 = (1:0:0)$, $C_2 = (0:1:0)$, $C_3 = (0:0:1)$, $C_4 = (1:1:1)$, and $C_5 = (1:1:0)$, respectively.*

Essentially, we can begin to assign coordinates by selecting two intersecting cords of three points in our superfiguration. We label the intersection point $(1:1:0)$, and then label the points of one line $(1:0:0)$ and $(0:1:0)$ and the other line $(0:0:1)$ and $(1:1:1)$. Given this starting point, we now attempt to count the number of possible ways to assign coordinates to the remaining points of a superfiguration such that we end up with a nondegenerate strong realization in $\mathbb{P}^2(\mathbb{F}_q)$. Denote the number we arrive at for a given superfiguration $s \in S$ by $m'_s(q)$. Then, noticing that this process is essentially selecting a 4-arc in our superfiguration and beginning to assign coordinates from there, and furthermore recalling that all 4-arcs are projectively equivalent, we have that

$$A_s := \frac{m'_s(q)|PGL_3(\mathbb{F}_q)|}{|\mathrm{Aut}(s)|},$$

where Aut(s) denotes the automorphism group of the superfiguration s.

We continue with our determination of $m_s'(q)$.

Having assigned our first five coordinates, the next thing we must do is assign variables so as to determine the rest of the coordinates (in this computation, the remaining five coordinates). In order to ensure that our coordinate assignments preserve collinearity, we use the lines of our superfiguration to assign variables. For example, suppose $(2, 3, 9)$ is a line in our superfiguration, where 2 is given as $(0 : 1 : 0)$ and 3 is given by $(0 : 0 : 1)$. Note then that $\overline{23}$ is the line $x = 0$. If we normalize coordinate 9 to $y = 1$ and let $\alpha \in \mathbb{F}_q$, then $9 = (0 : 1 : \alpha)$ preserves the collinearity $(2, 3, 9)$. Assigning coordinates in this way allows us to assign coordinates to each of the 10 points while preserving collinearities.

Now that variables have been assigned to all of the coordinates, we must make sure that all remaining lines which we haven't yet considered are still preserved by our assignment of coordinates. To do this, we recall the characterization of $\mathbb{P}^2(\mathbb{F}_q)$ given in Cameron where points in $\mathbb{P}^2(\mathbb{F}_q)$ correspond to one-dimensional linear subspaces in affine space over $\mathbb{F}_q$. Then, three points in $\mathbb{P}^2(\mathbb{F}_q)$ are collinear if and only if the corresponding subspaces of affine space are coplanar. To check this, we merely see if the determinant of representative vectors is zero. In other words, three points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, and $(\alpha_3, \beta_3, \gamma_3)$ in $\mathbb{P}^2(\mathbb{F}_q)$ are collinear if and only if the determinant

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

is zero. These checks will give us polynomial constraints on the variables we used to assign coordinates in the previous step. For instance, suppose that after we've assigned coordinates we haven't ensured that $\overline{167}$ is in fact a line. Suppose that so far, we've assigned coordinates so that there are no constraints on $\alpha \in \mathbb{F}_q$, and 1,6, and 7 are given by $1 = (1 : 1 : 0)$, $6 = (1 : 0 : \alpha)$, and $7 = (\alpha : 1 : 1)$. Then, we must ensure that $\alpha \in \mathbb{F}_q$ is such that

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & \alpha \\ \alpha & 1 & 1 \end{vmatrix} = -\alpha - 1 + \alpha^2 = \alpha^2 - \alpha - 1 = 0.$$

This severely restricts the choices that we have for α. The number then of coordinate assignments that are still allowed after we've considered all collinearities is the number of weak realizations containing the initial coordinates.

We're not quite done yet, however. To count $m_s'(q)$, we seek the number of *nondegenerate strong realizations* given the first five coordinates. To check that our assignment of coordinates yields a strong realization, we must make sure that we haven't induced any extraneous collinearities. For instance, in the full example which we give below, we will have to make sure that our assignment of coordinates does not allow the points 5,6, and 8 to be collinear. This is not difficult to check; recalling that three coordinates are collinear if and only if the determinant of the 3×3 matrix they define is zero, we choose our coordinates such that their determinant is nonzero.

Lastly, we simply check that the coordinates are assigned in such a way that all points and lines are unique. It may seem trivial and redundant to state this, but important constraints can arise from these considerations which may not have been seen otherwise. To give full illustration of the entirety of this method, the potential complexity of the resulting constraints, and the eventual determination of the number of strong realizations of a superfiguration, consider the following full worked example.

Example 5.5. Consider Superfiguration #32, which has points $1, 2, 3, ..., 10$ and lines

$$(1,2,3),(1,4,5),(1,6,7),(1,8,9,10),(2,4,8),$$
$$(3,7,8),(2,5,9),(4,6,9),(3,6,10),(5,7,10).$$

Looking at the pairs of points which are not included in any of the lines given above, we note that it is possible to realize both these lines and $\overline{568}$ and remain a superfiguration. We note this, and will remember it later when considering the possible inputs for the coordinates we assign.

Following the method, we first choose two lines to assign our initial coordinates. We can choose any two lines and the method will work just fine, but to ease our calculations we choose the intersecting lines $\overline{167}$ and $\overline{5710}$. This choice makes our calculations easier as more lines of our superfiguration intersect these two lines than other pairs of lines. Consequently, we will be able to introduce fewer variables, which should make the final constraints more manageable. So, we begin by assigning the following coordinates:

$$1 := (1:0:0), 6 := (0:1:0), 7 := (1:1:0), 5 := (1:1:1), 10 := (0:0:1).$$

Given these, we assign coordinates to the remaining points, using variables when necessary. First, note that with this assignment the line $\overline{145}$ is given by the equation $y = z$. Note that $y, z \neq 0$ for point 5, otherwise the points 1 and 5 would be equivalent. Thus, we can normalize the coordinates for 5 to $y = z = 1$ and assign the x-coordinate $\alpha \in \mathbb{F}_q$. So, we have

$$5 := (\alpha : 1 : 1).$$

Now, we see also that the line $\overline{18910}$ is given by $y = 0$. x must also be nonzero for both points 8 and 9, otherwise they would be equivalent to point 10. Consequently, we can choose $\beta\gamma \in \mathbb{F}_q$, normalize $x = 1$, and arrive at

$$8 := (1:0:\beta),\ 9 := (1:0:\gamma).$$

Considering the last of the easy lines, we see that $\overline{3610}$ is given by $x = 0$. In point 3, y must also be nonzero, or else 3 would be equivalent to 10. Choosing $\delta \in \mathbb{F}_q$ and normalizing to $y = 1$ we see that

$$3 := (0:1:\delta).$$

It remains to give coordinates to point 2. Considering the coordinates we have already assigned, we see that $\overline{123}$ is now given by $z = \delta y$. y must be nonzero in point 2, otherwise z would also be zero and 2 would be equivalent to 1. Thus, normalizing to $y = 1$ and choosing $\epsilon \in \mathbb{F}_q$, we have

$$2 := (\epsilon : 1 : \delta).$$

To recap, we have the points

$$\begin{array}{lllll} 1 := (1:0:0) & 2 := (\epsilon:1:\delta) & 3 := (0:1:\delta) & 4 := (\alpha:1:1) & 5 := (1:1:1) \\ 6 := (0:1:0) & 7 := (1:1:0) & 8 := (1:0:\beta) & 9 := (1:0:\gamma) & 10 := (0:0:1), \end{array}$$

where α, β, γ, δ, and $\epsilon \in \mathbb{F}_q$.

At this point, α, β, γ, δ, and ϵ can be any elements of $\mathbb{F}_q$. However, we still need to make sure that they are chosen in such a way that preserves the collinearities (2,4,8), (3,7,8), (2,5,9), and (4,6,9). To do this, we consider each line separately.

First, consider (2,4,8). Recall that an assignment of coordinates causes these three points to be collinear if and only if the corresponding determinant is zero. Thus, our variables must satisfy

$$\begin{vmatrix} \epsilon & 1 & \delta \\ \alpha & 1 & 1 \\ 1 & 0 & \beta \end{vmatrix} = 1(1-\delta) + \beta(\epsilon - \alpha) = 0,$$

or,

$$\beta(\epsilon - \alpha) = \delta - 1. \tag{1}$$

Now, consider (3,7,8). As with (2,4,8), our variables must satisfy

$$\begin{vmatrix} 0 & 1 & \delta \\ 1 & 1 & 0 \\ 1 & 0 & \beta \end{vmatrix} = 1(-\delta) + \beta(-1) = 0,$$

or,

$$\delta = -\beta. \tag{2}$$

We move on to (2,5,9). Here, our variables need to satisfy

$$\begin{vmatrix} \epsilon & 1 & \delta \\ 1 & 1 & 1 \\ 1 & 0 & \gamma \end{vmatrix} = 1(1-\delta) + \gamma(\epsilon - 1) = 0,$$

or,

$$\gamma(\epsilon - 1) = \delta - 1. \tag{3}$$

Lastly, we have to consider (4,6,9). Once more, our variables need to satisfy

$$\begin{vmatrix} \alpha & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \gamma \end{vmatrix} = 1(\alpha\gamma - 1) = 0,$$

or,

$$\alpha\gamma = 1. \tag{4}$$

It is necessary for all our variables to satisfy equations (1)-(4); as we will see, these conditions will uniquely determine γ, δ, and ϵ in terms of α and β. Equation (2) immediately gives us $\delta = -\beta$. Examining equation (4), we immediately see that $\gamma = \alpha^{-1}$ and that we must have $\alpha \neq 0$. Substituting these into equation (3), we have that

$$\begin{aligned} \alpha^{-1}(\epsilon - 1) &= -\beta - 1, \\ \epsilon - 1 &= -\alpha(\beta + 1), \\ \epsilon &= 1 - \alpha(\beta + 1). \end{aligned}$$

Lastly, substituting these expressions for γ, δ, and ϵ into equation (1), we have

$$\begin{aligned}
\beta\{[1-\alpha(\beta+1)]-\alpha\} &= -\beta-1,\\
\beta\{\alpha-[1-\alpha(\beta+1)]\} &= \beta+1,\\
\beta(\alpha-1+\alpha\beta+\alpha) &= \beta+1,\\
\alpha\beta-\beta+\alpha\beta^2+\alpha\beta &= \beta+1,\\
\alpha\beta^2+2\alpha\beta-2\beta-1 &= 0.
\end{aligned}$$

So, we currently have that the number of weak realizations of this superfiguration in $\mathbb{P}^2(\mathbb{F}_q)$ is precisely the number of ordered pairs $(\alpha,\beta)\in\mathbb{F}_q^2$ s.t. $\alpha\beta^2+2\alpha\beta-2\beta-1=0$. To ensure that our coordinate assignment yields a strong realization of this superfiguration, however, recall that we need to make sure that $\overline{568}$ is not a line. Consequently, we must choose $(\alpha,\beta)\in\mathbb{F}_q^2$ s.t.

$$\begin{vmatrix} 1 & 1 & 1\\ 0 & 1 & 0\\ 1 & 0 & \beta \end{vmatrix} = \beta-1\neq 0,$$

which is true if and only if

$$\beta\neq 1.$$

We see now that precisely any choice of $(\alpha,\beta)\in\mathbb{F}_q^2$ s.t. $\alpha\beta^2+2\alpha\beta-2\beta-1=0$, $\alpha\neq 0$, and $\beta\neq 1$ will yield a unique strong realization of this superfiguration in $\mathbb{P}^2(\mathbb{F}_q)$. Lastly, we must check to make sure that our choice of (α,β) is such that all points and lines in the superfiguration are unique. Substituting in the above calculations for γ, δ, and ϵ and doing a bit of simple algebra gives the following expressions for the points of our superfiguration:

$$\begin{array}{lllll}
1=(1:0:0) & 2=(1-\alpha(\beta+1):1:-\beta) & 3=(0:1:-\beta) & 4=(\alpha:1:1) & 5=(1:1:1)\\
6=(0:1:0) & 7=(1:1:0) & 8=(1:0:\beta) & 9=(\alpha:0:1) & 10=(0:0:1),
\end{array}$$

and the following expressions for the lines of our superfiguration:

$$\begin{aligned}
\overline{123}&:\beta y+z=0 & \overline{145}&:y-z=0\\
\overline{167}&:z=0 & \overline{18910}&:y=0\\
\overline{248}&:\beta x+(1-\alpha\beta)y-z=0 & \overline{378}&:\beta x-\beta y-z=0\\
\overline{259}&:x+(\alpha-1)y-\alpha z=0 & \overline{469}&:x-\alpha z=0\\
\overline{3610}&:x=0 & \overline{5710}&:x-y=0.
\end{aligned}$$

Examining these points and lines, we see that in order for our assignment to be a nondegenerate strong realization we must choose $(\alpha,\beta)\in\mathbb{F}_q$ s.t. all of the following conditions are satisfied:

$$\begin{aligned}
\alpha\beta^2+2\alpha\beta-2\beta-1=0,\\
\alpha\notin\{0,1\},\\
\beta\notin\{-1,0,1\},\\
\beta\notin\left\{\alpha^{-1},\alpha^{-1}-1,(\alpha-1)^{-1}\right\}.
\end{aligned}\tag{5}$$

In what follows, we utilize elementary algebraic and number theoretic tools to count the number of $(\alpha, \beta) \in \mathbb{F}_q^2$ satisfying the above conditions. We begin first with the equation that (α, β) needs to satisfy; noting that it is linear in α, we rewrite it as

$$\alpha = \frac{2\beta + 1}{\beta(\beta + 2)}. \tag{6}$$

Consequently, any choice of $\beta \in \mathbb{F}_q$ s.t. $\beta \notin \{0, -2\}$ will yield a unique α. This is obvious in all finite fields other than those of characteristic three, where a choice of $\beta = -2$ leaves our original equation with 0=0 and thus allowing any choice of $\alpha \in \mathbb{F}_q$. However, in characteristic three, $-2 = 1$, and thus we are not allowed to choose such a β. Consequently, we get a unique α defined by equation (6) so long as $\beta \notin \{0, -2\}$. Taking this realization from equation (6), we can simplify the conditions in (5) so that we only need to count the number of $\beta \in \mathbb{F}_q$ s.t.

$$\begin{aligned} \beta &\notin \{-2, -1, 0, 1\}, \\ \alpha &\notin \{0, 1\}, \\ \beta &\notin \left\{\alpha^{-1}, \alpha^{-1} - 1, (\alpha - 1)^{-1}\right\}, \end{aligned} \tag{7}$$

where α is given by (6).

We first examine $\alpha \notin \{0, 1\}$. Via (6), $\alpha = 0$ if and only if $2\beta + 1 = 0$. Consequently, $\alpha = 0$ if and only if $\beta = (-2)^{-1}$. Consequently, the condition $\alpha \neq 0$ is equivalent to $\beta \neq (-2)^{-1}$. Note that we cannot have $\alpha = 0$ in characteristic two as such a β doesn't exist in those fields. Now, consider $\alpha \neq 1$. Note that

$$\alpha = 1 = \frac{2\beta + 1}{\beta(\beta + 2)}$$

if and only if

$$\beta^2 + 2\beta = 2\beta + 1$$

if and only if

$$\beta^2 = 1.$$

Thus, the constraint $\alpha \neq 1$ is equivalent to $\beta \notin \{-1, 1\}$, which we already have. With this, (7) becomes

$$\begin{aligned} \beta &\notin \left\{-2, -1, (-2)^{-1}, 0, 1\right\}, \\ \beta &\notin \left\{\alpha^{-1}, \alpha^{-1} - 1, (\alpha - 1)^{-1}\right\}, \end{aligned} \tag{8}$$

where α is given by (6).

Lastly, we consider $\beta \notin \{\alpha^{-1}, \alpha^{-1} - 1, (\alpha - 1)^{-1}\}$. We note that $\beta = \alpha^{-1}$ if and only if $\alpha\beta = 1$. Now,

$$\alpha\beta = \frac{2\beta + 1}{\beta + 2}$$

by equation (6), so $\alpha\beta = 1$ if and only if $2\beta + 1 = \beta + 2$, or $\beta = 1$. Thus, this constraint is equivalent to $\beta \neq 1$, which we already have. We move on to consider $\beta \neq \alpha^{-1} - 1$. This is clearly equivalent to $\alpha\beta \neq 1 - \alpha$. Recalling the characterization of α given in equation (6), we note that $\alpha\beta = 1 - \alpha$ if and only if

$$\begin{aligned}\frac{2\beta+1}{\beta+2} &= 1 - \frac{2\beta+1}{\beta(\beta+2)},\\ 2\beta+1 &= (\beta+2) - \frac{2\beta+1}{\beta},\\ 2\beta^2+\beta &= \beta^2+2\beta-2\beta-1,\\ \beta^2+\beta+1 &= 0.\end{aligned}$$

Consequently, this constraint is equivalent to $\beta^2+\beta+1 \neq 0$. We examine our final constraint, $\beta \neq (\alpha-1)^{-1}$. Obviously, $\beta = (\alpha-1)^{-1}$ if and only if $(\alpha-1)\beta = 1$, which is true if and only if $\alpha\beta = \beta+1$. Recalling equation (6), we have

$$\begin{aligned}\frac{2\beta+1}{\beta+2} &= \beta+1,\\ 2\beta+1 = (\beta+1)(\beta+2) &= \beta^2+3\beta+2,\\ \beta^2+\beta+1 &= 0.\end{aligned}$$

So, $\beta \neq (\alpha-1)^{-1}$ if and only if $\beta^2+\beta+1 \neq 0$; this constraint is equivalent to the previous one. Thus, we've reduced the problem to counting the number of $\beta \in \mathbb{F}_q$ s.t.

$$\begin{aligned}\beta &\notin \left\{-2,-1,(-2)^{-1},0,1\right\},\\ \beta^2+\beta+1 &\neq 0.\end{aligned} \tag{9}$$

We first check to see if there are any repeats in the set $\{-2,-1,(-2)^{-1},0,1\}$. In characteristic two, noting that $(-2)^{-1}$ doesn't exist and that $2 = 0$, this set becomes $\{0,1\}$. In characteristic three, we note that $-2 = 1 = (-2)^{-1}$, so this set becomes $\{-1,0,1\}$. It is a trivial verification that every element of the set $\{-2,-1,(-2)^{-1},0,1\}$ is unique in all other characteristics. Furthermmore, none of the elements of $\{-2,-1,(-2)^{-1},0,1\}$ satisfy $\beta^2+\beta+1 = 0$ in characteristic two or characteristic greater than three, and that $\beta^2+\beta+1 = 0$ has exactly one solution ($\beta = 1$) in characteristic three. Thus, letting $a(q)$ denote the number of solutions in $\mathbb{F}_q$ to $\beta^2+\beta+1 = 0$, we have

$$m'_s(q) = \begin{cases} q-(2+a(q)) & \text{if } 2|q;\\ q-3 & \text{if } 3|q;\\ q-(5+a(q)) & \text{otherwise.}\end{cases} \tag{10}$$

We use quadratic reciprocity to find a closed form for $a(q)$. First we examine the characteristic two case. Since $\beta^2+\beta+1$ is irreducible over $\mathbb{F}_2$, elementary field theory immediately tells us that for $2|q$ we have

$$a(q) = \begin{cases} 2 & \text{if } q = 2^{2k}, k \in \mathbb{N};\\ 0 & \text{otherwise.}\end{cases} \tag{11}$$

Now we consider the case where $\mathrm{char}(\mathbb{F}_q) \geq 5$. Applying the quadratic formula to $\beta^2+\beta+1 = 0$, we have

$$\beta = \frac{-1 \pm \sqrt{-3}}{2}.$$

As $\mathrm{char}(\mathbb{F}_q) \neq 2$, we have two distinct roots in the base field $\mathbb{F}_p$ if -3 is a square in $\mathbb{F}_p$, and 0 otherwise. To determine whether or not -3 is a square in $\mathbb{F}_p$, we use quadratic reciprocity. Note that

$$\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{3}{p}\right). \tag{12}$$

Furthermore, by quadratic reciprocity,

$$\left(\frac{3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{\frac{(p-1)(3-1)}{4}} = (-1)^{\frac{p-1}{2}}. \tag{13}$$

Substituting (13) into (12),

$$\left(\frac{-3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{\frac{p-1}{2}}(-1)^{\frac{p-1}{2}} = (-1)^{p-1} = 1, \tag{14}$$

as p is odd. Since $p \geq 5$, all Legendre symbols in (12) are ± 1, and thus we can conclude that

$$\left(\frac{-3}{p}\right) = \left(\frac{p}{3}\right)^{-1} = \left(\frac{p}{3}\right). \tag{15}$$

Equation (15) tells us that -3 is a square in $\mathbb{F}_p$ if and only if p is a square in $\mathbb{F}_3$. The squares in $\mathbb{F}_3$ are 0 and 1, so -3 is a square in $\mathbb{F}_p$ if and only if $p \equiv 0, 1(\mathrm{mod} 3)$. Using this and information from elementary field theory, we immediately see that for $\mathrm{char}(\mathbb{F}_q) \geq 5$,

$$a(q) = \begin{cases} 0 & \text{if } q = p^{2k+1}, p \equiv 2\ (\mathrm{mod} 3), k \in \mathbb{N}; \\ 2 & \text{if } q = p^{2k}, p \equiv 2\ (\mathrm{mod} 3), k \in \mathbb{N}; \\ 2 & \text{if } q = p^{k}, p \equiv 1\ (\mathrm{mod} 3), k \in \mathbb{N}. \end{cases} \tag{16}$$

In (16), note that if $p \equiv 2(\mathrm{mod} 3)$, then $p^{2k} \equiv 1(\mathrm{mod} 3)$ and $p^{2k+1} \equiv 2(\mathrm{mod} 3)$ for any $k \in \mathbb{N}$. Thus, we can rewrite (16) for the case where $\mathrm{char}(\mathbb{F}_q) \geq 5$:

$$a(q) = \begin{cases} 0 & \text{if } q \equiv 2(\mathrm{mod} 3); \\ 2 & \text{if } q \equiv 1(\mathrm{mod} 3). \end{cases} \tag{17}$$

We seek a way to combine equations (11) and (17) to obtain a quasi-polynomial expression for $m_s'(q)$. No prime power is divisible by six. Thus, if $3|q$, we necessarily have $q \equiv 3(\mathrm{mod}\, 6)$ (and vice versa). A simple inductive argument shows us that if $q = 2^{2k+1}, k \in \mathbb{N}$, we have $q \equiv 2(\mathrm{mod} 6)$. Similar reasoning tells us that if $q = 2^{2k}, k \in \mathbb{N}$, that $q \equiv 4(\mathrm{mod} 6)$. Lastly, if $q \equiv 1(\mathrm{mod} 6)$ then $q \equiv 1(\mathrm{mod} 3)$, and if $q \equiv 5(\mathrm{mod} 6)$ then $q \equiv 2(\mathrm{mod} 3)$. Combining this information with equations (10), (11), and (17), we obtain the following expression for $m_s'(q)$:

$$m_s'(q) = \begin{cases} q-7 & \text{if } q \equiv 1(\mathrm{mod} 6); \\ q-2 & \text{if } q \equiv 2(\mathrm{mod} 6); \\ q-3 & \text{if } q \equiv 3(\mathrm{mod} 6); \\ q-4 & \text{if } q \equiv 4(\mathrm{mod} 6); \\ q-5 & \text{if } q \equiv 5(\mathrm{mod} 6). \end{cases} \tag{18}$$

This result is verified by a computer program up to $q = 19$. A simple computer program can give us the size of the automorphism group of this superfiguration; substituting this result into the formula immediately after Lemma 5.4 gives A_s.

It can be seen in the above example that the number of strong realizations of a superfiguration may be quasi-polynomial. To derive Iampolskaia, Skorobogatov, and Sorokin's quasi-polynomial formula for 9-arcs in a Desarguesian plane, we calculated A_s for each superfiguration on up to 9 points and plugged these quasi-polynomials into our formula for general projective planes. We thus confirm their result.

6. Realizability of 10-point Superfigurations

Realizability conditions for 10-point superfigurations tend to be much more complicated than those up to 9 points. In particular, many 10-point superfigurations cannot be counted by quasipolynomials. Rather, they can be counted as the number of solutions to a final polynomial system, that is, a set of equalities and inequalities of polynomials in the variables which parameterize the superfiguration space. Thus, in algebro-geometric terms, the strong realizations of a superfiguration are in 1-1 correspondence with the points on some quasi-affine variety over $\mathbb{F}_q$.

The final polynomial systems for each superfiguration on up to 10 points may be found on our website [8].

Despite the wide variety of results in the above considerations and the varying difficulty of those results, they do fall into a small number of specific categories. We will discuss each of those and provide a full list of the polynomials appearing in those categories in this section.

In particular, according to Nathan Kaplan, the Desargues superfiguration has realizations in 1-1 correspondence with the points of a certain K3 surface. These surfaces generate nonquasipolynomial function. This will be described in more detail in the final version of the paper.

Therefore, in order to demonstrate that the 10-arc formula is not quasi-polynomial, it suffices to show that the Desargues superfiguration has a nonzero coefficient of influence. As it turns out, running Glynn's algorithm for 10-point linear spaces has a prohibitively high runtime because of the difficulty of calculating so many values of $n_q(s)$. We now present a new variant of Glynn's algorithm which circumvents this problem.

Recall that the number of weak realizations of a 10-arc is given by

$$n_q(a) = \sum_{g \geq a} m_q(g)$$

where g ranges over all linear space functions on 10 points. We may therefore express the strong realizations of the 10-arc linear space function as

$$\begin{aligned} m_q(a) &= n_q(a) + \sum_{g > a} (-1) m_q(g) \\ &= n_q(a) + \sum_{g > a} (-1) m_q(g) + \sum_{s > a} (-1) m_q(s), \end{aligned}$$

where the first sum ranges over linear space functions which are *not* superfigurations, and the second sum ranges over superfigurations only.

Choose a linear space function g which is minimal with respect to $>$ among the index set of the first sum. Apply the substitution

$$m_q(g) = n_q(g) - \sum_{h>g} m_q(h).$$

This eliminates the $m_q(g)$ term from our formula, leaving only terms $m_q(h)$ for $h > g$. Thus, by repeated applications of this substitution to the minimal non-superfiguration in the formula, we will arrive at a formula of the form

$$m_q(a) = \sum_g k(g)n_q(g) + \sum_s l(s)m_q(s),$$

where the $k(g)$ and $l(s)$ are integers.

From this point, we could calculate the Glynn formula for 10-arcs by replacing each $n_q(g)$ by a polynomial in q and $m_q(t)$ for supefigurations t on 9 or fewer points. But this substitution does not affect the coefficients of the $m_q(s)$ for superfigurations s on 10 points. Therefore, the already-calculated values $l(s)$ are the coefficients of influence for the 10-point superfigurations. So, this algorithm calculates 151 of the 163 coefficients of influence without finding any values of n_q. We also prove as a consequence the following lemma.

Lemma 6.1. *The coefficient of influence of each n-point superfiguration in the Glynn formula for n-arcs is a constant.*

Running this algorithm reveals that the X superfiguration has coefficient of influence Y. Since Y is not zero, we have proven our second theorem, namely that a the number of 10-arcs in $\mathbb{P}^2(\mathbb{F}_q)$ is not a quasi-polynomial function of the order.

References

[1] Anna V Iampolskaia, Alexei N Skorobogatov, and Evgenii A Sorokin. Formula for the number of [9, 3] mds codes. *IEEE Transactions on Information Theory*, 41(6):1667–1671, 1995.

[2] David G Glynn. Rings of geometries ii. *Journal of Combinatorial Theory, Series A*, 49(1):26–66, 1988.

[3] Beniamino Segre. Curve razionali normali e k-archi negli spazi finiti. *Annali di Matematica Pura ed Applicata*, 39(1):357–379, 1955.

[4] David G Glynn. The non-classical 10-arc of pg (4, 9). *Discrete mathematics*, 59(1):43–51, 1986.

[5] R. Rolland and A. N. Skorobogatov. Dénombrement des configurations dans le plan projectif. In Ruud Pellikaan, Marc Perret, and SG Vladut, editors, *Arithmetic, Geometry, and Coding Theory: Proceedings of the International Conference held at Centre International de Rencontres de Mathématiques (CIRM), Luminy, France, June 28-July 2, 1993*, pages 199–209. Walter de Gruyter, 1996.

[6] Branko Grünbaum. *Configurations of points and lines*, volume 103. American Mathematical Society Providence, RI, USA, 2009.

[7] Brendan D McKay and Adolfo Piperno. Practical graph isomorphism, ii. *Journal of Symbolic Computation*, 60:94–112, 2014.

[8] 10-arcs. `http://rachellawrence.github.io/TenArcs/`. Accessed: 2016-12-01.

[9] Emil Artin. *Geometric algebra*. Courier Dover Publications, 2016.