COUNTING ARCS IN THE PROJECTIVE PLANE VIA GLYNN'S ALGORITHM

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Abstract. An n-arc in a projective plane is a collection of n distinct points in the plane, no three of which lie on a line. Formulas counting the number of n-arcs in a finite projective plane of order q are known for $n \leq 8$ for general projective planes. In 1995, Iampolskaia, Skorobogatov, and Sorokin counted 9-arcs in the projective plane over a finite field of order q and showed that this count is a quasipolynomial function of q. We present a formula that counts 9-arcs in any projective plane of order q, even those that are non-Desarguesian, deriving Iampolskaia, Skorobogatov, and Sorokin's formula as a special case. We obtain our formula from a new implementation of an algorithm due to Glynn; we give details of our implementation and discuss its consequences for larger arcs.

1. Introduction

We begin by recalling the basic definitions needed to describe the problem of counting n-arcs in finite projective planes.

Definition 1.1. A *projective plane* Π is a collection of points $\mathcal{P}$ and a collection of lines $\mathcal{L}$, where each $\ell \in \mathcal{L}$ is a subset of $\mathcal{P}$ such that:

(1) Every two points are incident with a unique line; that is, given distinct points $p_1, p_2, \in \mathcal{P}$ there exists a unique $\ell \in \mathcal{L}$ such that $\{p_1, p_2\} \subset \ell$.
(2) Every two lines are incident with a unique point; that is, given distinct lines $\ell_1, \ell_2 \in \mathcal{L}$ there exists a unique $p \in \mathcal{P}$ with $p \in \ell_1 \cap \ell_2$.
(3) There exist four points such that no three of them are contained in any line.

Let q be a positive integer. We say that Π has *order* q if each line contains exactly $q + 1$ points, and if each point is contained in exactly $q + 1$ lines.

The projective plane over a field k, denoted $\mathbb{P}^2(k)$, gives a well-studied algebraic class of examples. Projective planes that are isomorphic to $\mathbb{P}^2(k)$ for some field k are called *Desarguesian*, and all other projective planes are called *non-Desarguesian*. For an overview of the theory of non-Desarguesian projective planes, see [14].

We study functions that count special configurations of points called n-arcs.

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Definition 1.2. An *n-arc* in a projective plane Π is a collection of n distinct points, no three of which are collinear.

Arcs are collections of sets in *linear general position*, a fundamental concept in classical and algebraic geometry. In an infinite projective plane, most collections of points form arcs, but in finite projective planes, interesting enumerative problems arise.

For simplicity, throughout this paper we count *ordered n-arcs*, that is, n-tuples of points that form an arc, and we often omit the adjective ordered. The number of ordered n-arcs in a projective plane is equal to the number of n-arcs in the plane multiplied by a factor of $n!$.

Definition 1.3. Let Π be a projective plane of order q. Define $C_n(\Pi)$ as the number of ordered n-arcs of Π. In the case where Π is the projective plane $\mathbb{P}^2(\mathbb{F}_q)$, we write $C_n(q)$ in place of $C_n(\Pi)$.

For small values of n we can determine $C_n(q)$ using the algebraic structure of $\mathbb{P}^2(\mathbb{F}_q)$. For example, the automorphism group of $\mathbb{P}^2(\mathbb{F}_q)$ is $\mathrm{PGL}_3(\mathbb{F}_q)$, which acts sharply transitively on collections of four points, no three of which lie on a line. Therefore,

$$C_4(q) = |\mathrm{PGL}_3(\mathbb{F}_q)| = (q^2+q+1)(q^2+q)q^2(q-1)^2.$$

In fact, exactly the same formula holds in *any* projective plane Π of order q. Similar but more intricate ideas lead to similar polynomial formulas for all $n \le 6$.

Theorem 1.4. [3, Theorem 4.1] *For any finite projective plane Π of order q, we have*

(1) $C_1(\Pi) = C_1(q) = q^2+q+1$,

(2) $C_2(\Pi) = C_2(q) = (q^2+q+1)(q^2+q)$,

(3) $C_3(\Pi) = C_3(q) = (q^2+q+1)(q^2+q)q^2$,

(4) $C_4(\Pi) = C_4(q) = (q^2+q+1)(q^2+q)q^2(q-1)^2$,

(5) $C_5(\Pi) = C_5(q) = (q^2+q+1)(q^2+q)q^2(q-1)^2(q^2-5q+6)$,

(6) $C_6(\Pi) = C_6(q) = (q^2+q+1)(q^2+q)q^2(q-1)^2(q^2-5q+6)(q^2-9q+21)$.

Formulas for $C_7(\Pi)$ and $C_8(\Pi)$ are known, but are no longer given by a single polynomial in q for any finite projective plane Π. These formulas involve the number of *strong realizations* of certain special configurations of points and lines.

Definition 1.5 ([1])**.** A *linear space* is a pair of sets $(\mathcal{P}, \mathcal{L})$, whose elements are referred to as *points* and *lines* respectively, such that each line is a subset of $\mathcal{P}$, and no two lines intersect in more than one point. A pair (p, l) consisting of a point p and a line l containing p is called an *incidence* of the linear space. Two linear spaces are *isomorphic* if there is a permutation of the points and a permutation of the lines of the second linear space such that its set of incidences exactly match those of the first.

Linear spaces capture the basic notions of incidence geometry without reference to a particular projective plane. Thus, a linear space may be thought of as a combinatorial blueprint whose incidence data may or may not be satisfied by any given set of points and lines in a projective plane. For instance, every n-arc encapsulates the data of the linear space $(\{1, 2, ..., n\}, \emptyset)$. To formalize this notion, we recall the notion of a *strong realization*.

Definition 1.6 ([4])**.** A *strong realization* of a linear space S in a projective plane Π is a collection of points and lines of Π such that the set of incidences between them is isomorphic to the set of incidences of S. We write $A_S(\Pi)$ for the number of strong realizations of S in Π, and in the particular case where $\Pi = \mathbb{P}^2(\mathbb{F}_q)$, we write $A_S(q)$ instead.

With this language, we can restate our problem as follows: Give a for $A_a(\Pi)$, where a is the linear space $(\{1, ..., n\}, \emptyset)$. In order to give such formulas, we recall the definition of a special class of linear spaces called *superfigurations*.

Definition 1.7 ([4])**.** A (combinatorial) n_k*-configuration* is a linear space on n points with n lines such that each line consists of exactly k points, and each point is contained in exactly k lines.

A linear space of n points and some number of lines, not necessarily n, in which each line contains *at least* k points and each point is contained in *at least* k lines is called an n_k-*superfiguration*. Since every configuration is a superfiguration, but not conversely, we state all of our results and definitions in terms of superfigurations.

Throughout this paper all n_k-configurations we encounter will have $k = 3$, so we refer to n_3-configurations simply as *configurations*. Similarly, we refer to n_3-superfigurations as *superfigurations*.

Remark 1.8. Glynn [3] refers to superfigurations as "variables", and Iampolskaia, Skorobogatov, and Sorokin [6] call them "overdetermined configurations". We have opted to call them superfigurations for the sake of consistency with Grünbaum's text [4] on classical configurations of points and lines, and to distinguish them from other uses of the terms "variable" and "configuration".

Glynn gives formulas for the number of 7-arcs and the number of 8-arcs in a projective plane Π of order q. Let $A_7(\Pi)$ denote the number of strong realizations of the *Fano plane*, the unique superfiguration with 7 points up to isomorphism. Similarly, let $A_8(\Pi)$ denote the number of strong realizations of the *Möbius-Kantor* configuration, the unique superfiguration with 8 points up to isomorphism.

Theorem 1.9. [3, Theorems 4.2 and 4.4] *Let Π be a projective plane of order q. Then*

(1) $C_7(\Pi) = (q^2+q+1)(q^2+q)q^2(q-1)^2(q-3)(q-5)(q^4-20q^3+148q^2-468q+498)-A_7(\Pi)$,

(2) $$\begin{aligned} C_8(\Pi) &= (q^2 + q + 1)(q^2 + q)q^2(q - 1)^2(q - 5) \\ &\cdot (q^7 - 43q^6 + 788q^5 - 7937q^4 + 47097q^3 - 162834q^2 + 299280q - 222960) \\ &- (q^2 - 20q + 78)A_7(\Pi) + A_8(\Pi). \end{aligned}$$

Glynn applies these formulas to the problem of classifying finite projective planes. Noting that $A_7(q) \geq 0$ shows that $C_7(\Pi)$ is at most the value of the degree 14 polynomial given in (1). For $q = 6$ this polynomial evaluates to -6, which implies that there is no projective plane of order 6 [3, Corollary 4.3]. It is not clear whether formulas for $C_n(\Pi)$ for larger n will have similar consequences for the classification of finite projective planes.

We can also compare the values of $C_7(\Pi)$ for different finite projective planes of a fixed order q. In planes of odd order, we have $A_7(q) = 0$; however, if q is even, then any 4-arc together with its three diagonal points forms a Fano plane. Thus, $A_7(q) = C_4(q)$. Even in a non-Desarguesian plane, each Fano subplane contains a 4-arc, so $A_7(\Pi) \leq C_4(\Pi)$ in general. Theorem 1.9 shows that the sum of the number of 7-arcs and the number of Fano subplanes of Π depends only on the order of Π. Thus, for any $q = 2^r$, among all projective planes of order q, $\mathbb{P}^2(\mathbb{F}_q)$ is minimal with respect to number of 7-arcs.

A conjecture widely attributed to Neumann states that every non-Desarguesian finite projective plane contains a Fano subplane (see, for example, [13]). Theorem 1.9 leads to the following equivalent reformulations of Neumann's conjecture.

Conjecture 1.10. (1) Let Π be a finite non-Desarguesian plane. Then

$$C_7(\Pi) < (q^2+q+1)(q^2+q)q^2(q-1)^2(q-3)(q-5)(q^4-20q^3+148q^2-468q+498).$$

(2) Let q be a fixed odd prime power and let Π be a projective plane of order q with $\Pi \neq \mathbb{P}^2(\mathbb{F}_q)$. Then $C_7(\Pi) < C_7(q)$.

When $\Pi = \mathbb{P}^2(\mathbb{F}_q)$, we can compute $A_7(q)$ and $A_8(q)$ to get formulas for $C_7(q)$ and $C_8(q)$. We have seen that $A_7(q) = 0$ for all odd q, and is given by a polynomial in q when q is even.

Definition 1.11. A *quasipolynomial* of period m is a function $g(x)$ of the positive integers such that there is a collection of polynomials $f_0(x), \ldots, f_{m-1}(x)$ satisfying $g(x) = f_i(x)$ for all $x \equiv i \pmod{m}$.

Such functions are sometimes called *PORC*, or *polynomial on residue classes*. We see that $A_7(q)$ is a quasipolynomial of period 2. Similarly, $A_8(q)$ is a quasipolynomial of period 3. Glynn [3] computes these quasipolynomials and gives explicit quasipolynomial formulas for $C_7(q)$ and $C_8(q)$. He did not push this method further, noting "the complexity of the problem as the number of points approaches 10".

In order to study the problem of counting inequivalent linear MDS codes, Iampolskaia, Skorobogatov, and Sorokin give a formula for the number of ordered n-arcs in $\mathbb{P}^2(\mathbb{F}_q)$. For more on the connection between arcs and MDS codes, see [5]. There are 10 superfigurations on 9 points up to isomorphism, which we denote by $9_3, \ldots, 9_{12}$. Let $A_{9_i}(\Pi)$ denote the number of strong realizations of 9_i in the projective plane Π. When $\Pi = \mathbb{P}^2(\mathbb{F}_q)$ we write $A_{9_i}(q)$ instead of $A_{9_i}(\Pi)$.

Theorem 1.12. [6, Theorem 1] *The number of* 9*-arcs in the projective plane over the finite field* $\mathbb{F}_q$ *is given by*

$$\begin{aligned} C_9(q) \;=\; & (q^2+q+1)(q^2+q)(q^2)(q-1)^2 \\ & \Big[q^{10}-75q^9+2530q^8-50466q^7+657739q^6-5835825q^5+35563770q^4 \\ & -146288034q^3+386490120q^2-588513120q+389442480 \\ & -1080\big(q^4-47q^3+807q^2-5921q+15134\big)a(q) \end{aligned}$$

$$+840\big(9q^2-243q+1684\big)b(q)+30240\big(-9c(q)+9d(q)+2e(q)\big)\Big]$$

where

$$\begin{aligned}
a(q) &= \begin{cases} 1 & \textit{if } 2 \mid q, \\ 0 & \textit{otherwise}; \end{cases} \\
b(q) &= \#\{x \in \mathbb{F}_q : x^2+x+1=0\}, \\
c(q) &= \begin{cases} 1 & \textit{if } 3 \mid q, \\ 0 & \textit{otherwise}; \end{cases} \\
d(q) &= \#\{x \in \mathbb{F}_q : x^2+x-1=0\}, \\
e(q) &= \#\{x \in \mathbb{F}_q : x^2+1=0\}.
\end{aligned}$$

Iampolskaia, Skorobogatov, and Sorokin [6] give quasipolynomial formulas for each of the functions $a(q), b(q), c(q), d(q), e(q)$. For example, $e(q)$ depends only on q modulo 4. Substituting these formulas into Theorem 1.12 gives $C_9(q)$ as a quasipolynomial of period 60.

In order to prove Theorem 1.12, Iampolskaia, Skorobogatov, and Sorokin use the fact that there is a natural way to assign coordinates to the points of $\mathbb{P}^2(\mathbb{F}_q)$. For example, by taking an appropriate change of coordinates, every superfiguration on at most 9 points is projectively equivalent to one where five points are chosen to be $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, $[1:1:1]$, and $[1:1:0]$. Therefore, we need only count the number of strong realizations of this configuration where these five points are fixed (see [6, 12]).

We cannot generally assign coordinates to the points of a non-Desarguesian projective plane. In particular, we note that Theorem 1.12 is insufficient to describe the count of 9-arcs in non-Desarguesian planes, that is, planes which are not coordinatized by a field.

The main result of this paper is to extend Theorem 1.12 to any projective plane of order q.

Theorem 1.13. *Let Π be a projective plane of order q. The number of 9-arcs in Π is*

$$\begin{aligned}
C_9(\Pi) = \; & q^{18}-75q^{17}+2529q^{16}-50392q^{15}+655284q^{14} \\
& -5787888q^{13}+34956422q^{12}-141107418q^{11}+356715069q^{10} \\
& -477084077q^9+143263449q^8+237536370q^7+52873326q^6 \\
& -2811240q^5-588466080q^4+389304720q^3 \\
& +(-36q^4+1692q^3-29052q^2+212148q-539784)A_7(\Pi) \\
& +(9q^2-243q+1647)A_8(\Pi) \\
& -A_{9_3}(\Pi)-A_{9_4}(\Pi)-A_{9_5}(\Pi)+A_{9_7}(\Pi)-3A_{9_{10}}(\Pi)+3A_{9_{11}}(\Pi)-9A_{9_{12}}(\Pi).
\end{aligned}$$

Computing $A_7(q), A_8(q)$, and $A_{9_i}(q)$ for each i recovers Theorem 1.12 as a corollary. These functions are computed in [6], but we note that there is a minor error in the calculation of $A_{9_3}(q)$. Despite this error, the final formula for $C_9(q)$ is correct. We include these counts for completeness:

$$
\begin{aligned}
A_7(q) &= 30a(q)C_4(q) & A_{9_7}(q) &= 60480(e(q)-a(q))C_4(q)\\
A_8(q) &= 840b(q)C_4(q) & A_{9_8}(q) &= 10080(q-2-b(q))C_4(q)\\
A_{9_3}(q) &= 3360((q-2-b(q))(q-5)+(q-3)b(q))C_4(q) & A_{9_9}(q) &= 0\\
A_{9_4}(q) &= 40320(q-2-b(q))C_4(q) & A_{9_{10}}(q) &= 1680b(q)C_4(q)\\
A_{9_5}(q) &= 30240(q-3)(1-a(q))C_4(q) & A_{9_{11}}(q) &= 90720d(q)C_4(q)\\
A_{9_6}(q) &= 30240(q-2)a(q)C_4(q) & A_{9_{12}}(q) &= 30240c(q)C_4(q)
\end{aligned}
$$

There are q^2+q+1 points in a projective plane of order q, so counting 9-arcs naively requires checking $O(q^{18})$ configurations. The formula from Theorem 1.13 reduces the problem to counting strong realizations of twelve superfigurations. Each of these superfigurations is so highly determined that at most q^2 sets of points must be considered following a selection of four initial points. Therefore, Theorem 1.13 reduces the total number of collections of points that we must consider to $O(q^{10})$.

Theorem 1.13 demonstrates non-obvious relationships between the function $C_9(\Pi)$ and the number of strong realizations of certain superfigurations, such as the Pappus configuration 9_3. Interestingly, the superfigurations $9_6, 9_8$, and 9_9 do not influence $C_9(\Pi)$. These relationships could lead to more conjectures along the lines of Conjecture 1.10.

In Section 2, we describe the algorithm used to prove Theorem 1.13. In Section 3 we discuss computational aspects of the implementation and the difficulty of extending our results to larger arcs. Finally, in Section 4 we discuss related work on counting 10-arcs in $\mathbb{P}^2(\mathbb{F}_q)$.

2. Algorithms for calculating arc formulas

In [3], Glynn gives an inductive algorithm for counting $C_n(\Pi)$ in terms of the number of strong realizations in Π of all superfigurations on at most n points. The following statement is a version of Theorem 3.6 in [3].

Theorem 2.1. *Suppose $n \leq 13$. There exist polynomials $p(q), p_s(q)$ such that for any finite projective plane Π of order q, we have*

$$C_n(\Pi) = p(q) + \sum_s p_s(q) A_s(\Pi),$$

where the sum is taken over all superfigurations s with at most n points.

We call the polynomial $g_s(q)$ the *coefficient of influence* of the superfiguration s, since it measures the degree to which s is relevant in $C_n(\Pi)$. The 13 in this statement comes from the fact that there are no superfigurations on at most six points, but $C_{14}(\Pi)$ could involve a quadratic term in $A_7(\Pi)$, related to the number of strong realizations of two disjoint copies of the Fano plane in Π.

In order to prove Theorem 1.13 we return to Glynn's original algorithm from [3]. To our knowledge, this is the first time that the algorithm has actually been implemented to find new enumerative formulas, rather than just used as a theoretical tool to prove that formulas of a certain type exist. The implementation we describe has the potential to give analogues of Theorem 1.13, computing $C_n(\Pi)$ for larger values of n.

The algorithm used to arrive at Theorems 1.3 and 1.4 was first described in [3], and was further clarified by Rolland and Skorobogatov in [10]. The latter form of the algorithm was a central component of the proof of Theorem 1.12. We present an exposition of the algorithm and prove that it works, following [6] and [10]. We then discuss modifications to the algorithm that lead to a manageable runtime.

Definition 2.2 ([6])**.** A *boolean n-function* is a function taking subsets of $\{1, 2, 3, ..., n\}$ to {0,1}. Two boolean n-functions f and g are *isomorphic* if there is a permutation i of $\{1, ..., n\}$ so that $g = f \circ i$.

We highlight that f can be thought of as *labeled*: for instance, we must distinguish the boolean 2-function which just sends {1} to 1 from the isomorphic boolean 2-function which just sends {2} to 1. We also note that the boolean n-functions are in one to one correspondence with the power set of the power set of $\{1, ..., n\}$.

Definition 2.3 ([6])**.** For boolean n-functions f and g, we say $f \geq g$ if $f(S) \geq g(S)$ for all $S \subseteq \{1, 2, ..., n\}$. Notice that $\geq$ is a partial order on the set of boolean n-functions.

Example 2.4. Suppose $f : 2^{\{1,...,7\}} \to \{0, 1\}$ sends just the sets {1,3,4,5}, {4,5,6} to 1 and every other set to 0. Then f is a boolean 7-function. If we let $f' : 2^{\{1,...,7\}} \to \{0, 1\}$ be the function that sends just the sets {1,3,4} and {4,5,6} to 1 and all other sets to 0, then it is *not* the case that $f \geq f'$.

We are working towards an abstraction of the axioms of geometry, in which the elements of $\{1, ..., n\}$ are points and the function f is an indicator function that evaluates to 1 for the collinear subsets of $\{1, ..., n\}$.

Definition 2.5. Call a boolean n-function f a *linear space function* if it satisfies:

(1) If $f(I) = 1$, then $f(J) = 1$ for all $J \subseteq I$.

(2) If $\#(I) \leq 2$, then $f(I) = 1$.

(3) If $f(I) = f(J) = 1$ and $\#(I \cap J) \geq 2$, then $f(I \cup J) = 1$.

If f does not satisfy these requirements, we call f *pathological.*

In words, these requirements mean:

- Subsets of lines are lines;
- Any set of $0, 1$, or 2 points qualifies as a line; and
- If two lines intersect in at least two points, then the union of the sets is a line.

Thus, a linear space function f defines a linear space with points $\{1, 2, ..., n\}$ and sets of collinear points defined by $f^{-1}(1)$. If f defines a superfiguration, we sometimes say that f is a superfiguration.

For the rest of the section, fix some projective plane Π of order q. We now extend Definition 1.6 to boolean n-functions.

Definition 2.6. Suppose some n-tuple S of distinct points labeled $1, ..., n$ in Π is such that, for some boolean n-function f, the set $f^{-1}(1)$ is exactly the collection of collinear subsets of S. Such a tuple is called a *strong realization* of f. We denote the number of strong realizations of f in Π by $A_f(\Pi)$.

When f is a linear space function, this is equivalent to Definition 1.6; if f is pathological, then $A_f(\Pi) = 0$ by the projective plane axioms given in Definition 1.1.

Definition 2.7. Suppose some n-tuple S of distinct points labeled $1, ..., n$ in Π is such that $f^{-1}(1)$ is a subset of the collection of collinear subsets of S. Such a tuple is called a *weak realization* of f.

Consider the boolean n-function a for which $f^{-1}(1)$ is the set of subsets of $\{1, ..., n\}$ of size 0, 1, or 2. Then a is a boolean n-function, and is also a linear space function. Every tuple of n distinct points in Π is a weak realization of a. A strong realization of a is an n-arc. Therefore, the goal of the algorithm is to calculate $A_a(\Pi)$. We do this indirectly, by examining weak realizations and working backwards.

Definition 2.8. For any boolean n-function f, define

$$B_f(\Pi) = \sum_{g \geq f} A_g(\Pi).$$

If f is a linear space function, then $B_f(\Pi)$ is the number of weak realizations of f in Π. If f is pathological, then $B_f(\Pi)$ is still defined, although its interpretation in terms of weak realizations is less clear.

Now we reproduce the method described by Rolland and Skorobogatov [10] to calculate $B_g(\Pi)$ in terms of realizations of linear space functions on fewer points.

Definition 2.9. Suppose that f is a linear space function. A *full line* of f is a subset $S \subseteq \{1, ..., n\}$ with $\#S \geq 3$, so that $f(S) = 1$ and for all T that properly contain S, we have $f(T) = 0$. In other words, there is no larger set of collinear points that includes S. We say the *index* of a point p of f is the number of full lines that include p.

Notice that we can completely describe a linear space function by giving its full lines.

Lemma 2.10. *Suppose the linear space function f on n points has a point of index 0, 1, or 2. Then we may construct $B_f(\Pi)$ as a polynomial in q and the values $A_g(\Pi)$, where g ranges over the linear space functions of $n-1$ points. Further, $B_f(\Pi)$ is linear in the $A_g(\Pi)$.*

Proof. Let f be as stated. Without loss of generality, suppose point n has index 0, 1, or 2. Define f' to be the linear space function that corresponds to the configuration of just the points $1, 2, ..., n-1$, inheriting collinearity data from f. Each weak realization of f is a strong realization of some $g \geq f'$, together with point n. In particular, we do not need to range over choices of the point n; just choices of g given a fixed n. So we shall count, for each g, the number of ways to add point n to a strong realization of g such that the result is a weak realization of f. We let $\mu(g, f)$ denote the number of ways to add a new point to a strong realization of g to obtain a weak realization of f. We thus get an equation

$$B_f(\Pi) = \sum_{g \geq f'} \mu(g, f) A_g(\Pi).$$

Then, it is enough to give a method that finds μ for any pair g, f as a polynomial in q. The method described by Rolland and Skorobogatov [10] is subtle in the sense that two isomorphic boolean n-functions g, g' do not necessarily satisfy $\mu(g, f) = \mu(g', f)$.

If the index of n is 0, then n may be placed anywhere in the plane not already occupied by a point of g; thus $\mu(g, f) = q^2 + q + 1 - (n - 1)$.

If the index of n is 1, then let L be the line of f on which n must lie. Then any point on L not already occupied is a valid choice for n, so $\mu(g, f) = q + 1 - |L|$.

If the index of n is 2, then distinct lines L_1 and L_2 of f contain n, and there are multiple cases to consider. Lines defined by a linear space function that intersect in two or more points must be the same. So if in g, the lines L_1 and L_2 intersect in two or more points then $\mu(g, f) = q + 1 - |L_1|$. If in g, the lines L_1 and L_2 intersect in exactly one point of g, then the only possible spot to place n is already filled, so $\mu(g, f) = 0$. Finally, if L_1 and L_2 do not intersect, then there is a unique place to put n, so $\mu(g, f) = 1$. $\square$

Lemma 2.10 indicates the reason why superfigurations arise in Glynn's algorithm and in Theorem 2.1. Since the method for computing $B_f(\Pi)$ only applies to those linear space functions f with a point of index 0, 1, or 2, the algorithm cannot inductively find $B_f(\Pi)$ for linear space functions with all points of index at least 3. In other words, the algorithm cannot determine $B_f(\Pi)$ for superfigurations.

The following algorithm inductively expresses each $A_s(\Pi)$ and $B_s(\Pi)$ in terms of the values $A_f(\Pi)$ for superfigurations f.

Algorithm 2.11.

(1) Find $A_s(\Pi)$ and $B_s(\Pi)$ for the unique linear space function s on 1 point.
(2) Assume that we have $A_t(\Pi)$ and $B_t(\Pi)$ for all linear space functions t on k points.
(3) Use Lemma 2.10 to find $B_t(\Pi)$ for every non-superfiguration linear space function t on $k + 1$ points.
(4) Assume that f is a linear space function on $k + 1$ points, and assume that we know $A_g(\Pi)$ for all linear space functions $g > f$. If f is not a superfiguration, calculate $A_f(\Pi)$, by writing
$$A_f(\Pi) = B_f(\Pi) - \sum_{g>f} A_g(\Pi),$$
and then use the expression found in (3) to express $B_f(\Pi)$ in terms of $A_s(\Pi)$ for k-point superfigurations s.
(5) Repeat the previous step until we have calculated $A_f(\Pi)$ for all linear spaces f on $k + 1$ points.
(6) Continue by induction until $k = n$.

This algorithm gives the number of strong or weak realizations of any n-point linear space function L in a projective plane Π as
$$p(q) + \sum_{s \in S} p_s(q) A_s(q),$$

where S is the set of superfigurations on at most n points, $A_s(q)$ is the number of strong realizations of superfiguration s in Π, and $p(q)$ and the $p_s(q)$ are polynomials in q.

We implemented this algorithm in Sage. Running the algorithm up to 9 points, which takes several minutes of computation time, gives the formula for 9-arcs from Theorem 1.13.

3. Counting larger arcs

Algorithm 2.11 also computes the formula for the count of n arcs for $9 < n \leq 13$. In particular, the formula for 10-arcs in general projective planes is now within reach. However, we run into problems due to the complexity of the algorithm, whose runtime is roughly proportional to the square of the number of linear space functions on n points.

The complete list of n-point linear spaces can be determined by computing the list of hypergraphs on n vertices under the constraints that the minimum set size is 3 and the intersection of any two sets is of size at most 1. To restrict attention to superfigurations, we impose the additional condition that the minimum vertex degree is 3. For $n \leq 11$, McKay's *Nauty* software can quickly compute all such hypergraphs up to isomorphism (see [8]).

Counts of Linear Spaces

n	7	8	9	10	11	12
Linear spaces on n points	24	69	384	5250	232929	28872973
Superfigurations	1	1	10	151	16234	>179000
Configurations	1	1	3	10	31	229

The fast growth of these figures indicates the increasing difficulty of applying Algorithm 2.11. In particular, we found that the prohibitively high runtime comes from of the difficulty of calculating so many values of $B_s(\Pi)$ using Lemma 2.10. We introduce a variant this algorithm that partially circumvents this problem.

Recall that the number of weak realizations of a n-arc is given by

$$B_a(\Pi) = \sum_{g \geq a} A_g(\Pi)$$

where g ranges over all linear space functions on n points. We may therefore express the strong realizations of the n-arc linear space function as

$$\begin{aligned} A_a(\Pi) &= B_a(\Pi) - \sum_{g > a} A_g(\Pi) \\ &= B_a(\Pi) - \sum_{g > a} A_g(\Pi) - \sum_{s > a} A_s(\Pi), \end{aligned}$$

where the first sum ranges over linear space functions that are *not* superfigurations, and the second sum ranges over superfigurations only.

Choose a linear space function g that is minimal with respect to $>$ among the index set of the first sum, that is, there does not exist any linear space function g' occurring in the first sum with $g > g'$. We say that g is a minimal non-superfiguration of this formula. Apply the substitution

$$A_g(\Pi) = B_g(\Pi) - \sum_{h>g} A_h(\Pi).$$

This eliminates the $A_g(\Pi)$ term from our formula, leaving only terms $A_h(\Pi)$ for $h > g$. By repeated applications of this substitution to a minimal non-superfiguration in the formula, we arrive at an expression of the form

$$A_a(\Pi) = \sum_{g} k(g)B_g(\Pi) + \sum_{s} l(s)A_s(\Pi),$$

where the $k(g)$ and $l(s)$ are integers.

We can now give a formula for n-arcs where we replace each instance of $B_g(\Pi)$ by a polynomial in q and $A_t(\Pi)$ for superfigurations t on $n-1$ or fewer points. This substitution does not affect the coefficients of the $A_s(\Pi)$ for superfigurations s on n points. Therefore, the values $l(s)$, which we have already calculated, are the coefficients of influence for the n-point superfigurations. We state this as a lemma.

Lemma 3.1. *In the formula for n-arcs given in Theorem 2.1, the coefficient of influence of each n-point superfiguration is a constant.*

Let us consider the implications for $n = 10$. Of the 163 superfigurations on up to 10 points, 151 are on exactly 10 points. Therefore, the algorithm just described calculates 151 of the 163 coefficients of influence without finding any values of $B_f(\Pi)$. Computing the coefficients of influence for remaining the 12 superfigurations on at most 9 points would be quite computationally intensive. The following table gives the coefficients of influence for each of the superfigurations $10_{13}, 10_{14}, ..., 10_{163}$. These superfigurations are defined at the website [7], which we have created to organize information related to counting 10-arcs.

Coefficients of Influence in the Formula for $C_{10}(\Pi)$											
s	$p_s(q)$	s	$p_s(q)$	s	$p_s(q)$	s	$p_s(q)$	s	$p_s(q)$	s	$p_s(q)$
10_{13}	27	10_{39}	-3	10_{65}	0	10_{91}	-1	10_{117}	2	10_{143}	-2
10_{14}	27	10_{40}	-3	10_{66}	0	10_{92}	-1	10_{118}	-1	10_{144}	-2
10_{15}	27	10_{41}	-3	10_{67}	0	10_{93}	-1	10_{119}	1	10_{145}	-1
10_{16}	1	10_{42}	-3	10_{68}	0	10_{94}	-1	10_{120}	-1	10_{146}	-2
10_{17}	1	10_{43}	-3	10_{69}	0	10_{95}	-1	10_{121}	-1	10_{147}	-2
10_{18}	1	10_{44}	-3	10_{70}	0	10_{96}	-1	10_{122}	-1	10_{148}	-2
10_{19}	1	10_{45}	-3	10_{71}	0	10_{97}	-1	10_{123}	-1	10_{149}	-2
10_{20}	1	10_{46}	-3	10_{72}	-1	10_{98}	-1	10_{124}	2	10_{150}	1
10_{21}	1	10_{47}	9	10_{73}	-1	10_{99}	3	10_{125}	-1	10_{151}	9
10_{22}	1	10_{48}	9	10_{74}	-1	10_{100}	3	10_{126}	-1	10_{152}	9
10_{23}	1	10_{49}	9	10_{75}	-1	10_{101}	3	10_{127}	2	10_{153}	2
10_{24}	1	10_{50}	9	10_{76}	-1	10_{102}	3	10_{128}	3	10_{154}	1
10_{25}	1	10_{51}	9	10_{77}	-1	10_{103}	3	10_{129}	5	10_{155}	2
10_{26}	-2	10_{52}	4	10_{78}	-1	10_{104}	-1	10_{130}	4	10_{156}	4
10_{27}	-2	10_{53}	4	10_{79}	-1	10_{105}	0	10_{131}	12	10_{157}	5
10_{28}	-2	10_{54}	4	10_{80}	-1	10_{106}	-1	10_{132}	0	10_{158}	6
10_{29}	-2	10_{55}	6	10_{81}	-1	10_{107}	-1	10_{133}	0	10_{159}	18
10_{30}	-2	10_{56}	6	10_{82}	-1	10_{108}	0	10_{134}	0	10_{160}	1
10_{31}	16	10_{57}	6	10_{83}	-1	10_{109}	-1	10_{135}	0	10_{161}	1
10_{32}	-2	10_{58}	6	10_{84}	-1	10_{110}	1	10_{136}	0	10_{162}	10
10_{33}	-2	10_{59}	19	10_{85}	-1	10_{111}	2	10_{137}	-1	10_{163}	-6
10_{34}	-2	10_{60}	-8	10_{86}	-1	10_{112}	1	10_{138}	-1		
10_{35}	-3	10_{61}	19	10_{87}	-1	10_{113}	2	10_{139}	0		
10_{36}	-3	10_{62}	19	10_{88}	-1	10_{114}	2	10_{140}	-1		
10_{37}	-3	10_{63}	-12	10_{89}	-1	10_{115}	1	10_{141}	-2		
10_{38}	-3	10_{64}	-8	10_{90}	-1	10_{116}	1	10_{142}	-1		

These values were obtained by running an implementation of our algorithm in Sage on the "Grace" High Performance Computing cluster at Yale University. Thirty-two IBM NeXtScale nx360 M4 nodes each running twenty Intel Xeon E5-2660 V2 processor cores completed the parallelized algorithm in several hours.

4. Application to Counting 10-arcs

We noted in the introduction that $C_n(q)$ is given by a polynomial in q for all $n \leq 6$, and that $C_7(q), C_8(q)$, and $C_9(q)$ have quasipolynomial formulas. It is natural to ask how $C_n(q)$ varies with q for larger fixed values of n. In forthcoming work, Elkies [2] analyzes $A_s(q)$ as a function of q for each of the 10_3-configurations and shows that several of these functions are not quasipolynomial. In a follow-up paper [9], we analyze $A_s(q)$ for each of the 151 superfigurations on 10 points, finding several more non-quasipolynomial examples. As a consequence, we conclude that $C_{10}(q)$ is not quasipolynomial (see [2, 9]).

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