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Stability and bifurcation for an SEIS epidemic model with the impact of media[☆]

Hai-Feng Huo [*], Peng Yang, Hong Xiang

Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu, 730050, People's Republic of China

HIGHLIGHTS

- A novel SEIS epidemic model with the impact of media is introduced.
- The occurrence of a forward, backward and Hopf bifurcation is derived.
- Numerical simulations and sensitivity analysis are performed.
- Our results show that media is a good indicator in controlling the disease.

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ABSTRACT

A novel SEIS epidemic model with the impact of media is introduced. By analyzing the characteristic equation of equilibrium, the basic reproduction number is obtained and the stability of the steady states is proved. The occurrence of a forward, backward and Hopf bifurcation is derived. Numerical simulations and sensitivity analysis are performed. Our results manifest that media can regard as a good indicator in controlling the emergence and spread of the epidemic disease.

1. Introduction

Nowadays, media coverage is changing the way that we communicate with each other in our daily life and work, and the media may be the most important source of healthy messages for the general public. Meanwhile, it also plays a significant role in the spread and control of epidemic disease by providing some healthy messages because people usually express their experiences of illness [1]. In the transmission process of epidemic disease, media coverage is a key factor. Media coverage about an epidemic disease gives a sense about the risk level and encourages the people to take precautionary measures such as wearing masks, avoiding public places and traveling when sick, frequent hand washing etc., to prevent the disease.

Massive news coverage and fast information flow can generate a profound psychological impact on public health. Moreover, a research showed that three tabloids and two broadsheets sent a total of 1153 messages about SARS in Britain [2], while the New Zealand Herald sent a total of 261 messages [3] from March to July 2003 during the spread of SARS. As the number of infected individuals increases, media coverage gives more reports about healthy messages of epidemic disease and cuts down the opportunity and probability of contact transmission among the alerted susceptible individuals, which is

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* Corresponding author.
E-mail address: hfhuo@lut.cn (H.-F. Huo).

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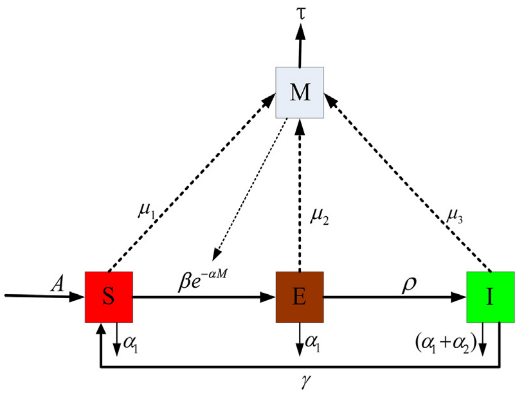

Fig. 1. Flowchart of the *SEIS* epidemic model with the impact of media.

beneficial to the control and prevention of disease for further spreading. Therefore, public-health organizations increasingly advocate for the use of all kinds of medias for their high-reach, low-cost messages dissemination potential of epidemic diseases.

Mathematical modeling has become an important tool in analyzing the spread and control of epidemic disease and is being used to indicate how the disease will spread over a period of time. Recently, many attempts have been made for investigating the transmission dynamics of epidemic disease [4–17]. Tchuenche et al. [9] studied the impact of media coverage on the spread and control of an influenza strain. Sun et al. [10] used non-linear contact rate to study media-induced social distancing in a two patch setting. Cui et al. [11] proposed a general contact rate $\beta(I) = c_1 - c_2 f(I)$ to reflect some intrinsic characters of media coverage. Liu et al. [12] proposed an *EIH* model to illustrate a possible mechanism for multiple outbreaks or even sustained periodic oscillations of emerging infectious diseases due to the psychological impact of the reported numbers of infectious and hospitalized individuals. K. Pawelek et al. [13] assumed that the disease transmission β was reduced by a factor $e^{-\alpha T}$ due to the behavior change of the public after reading tweets about influenza, where α determined how effective the disease information could influence the transmission rate. Their results showed that twitter might serve as a good indicator of seasonal influenza epidemics. Huo et al. [14] mainly studied the *SEI* model with the influence of positive and negative information of twitter. Their results showed that the impact posed by the negative information of twitter was not significant than the impact posed by the positive information of twitter on influenza.

Motivated by the documents [12–14], the goal of the present paper is to construct a more realistic SEIS epidemic model with the impact of media, in which we assume that infected people are temporary immunity and consider the effect of the natural and the disease-related death rates. The occurrence of a forward, backward and Hopf bifurcation is derived. Our results manifest that media can regard as a good indicator in controlling the emergence and spread of the epidemic disease.

The remain part of this paper is organized in the following: In Section 2, we introduce a new *SEIS* epidemic model with the impact of media. In Section 3, we calculate the basic reproductive number, and prove the stability of disease-free and endemic equilibria. A forward, backward and Hopf bifurcation are also investigated in this section. In Section 4, we carry out some numerical simulations. In Section 5, we perform sensitivity analysis on a few parameters. In the last section, we make some discussions.

2. Mathematical model

2.1. System description

The total population is divided into three compartments: $S(t)$, $E(t)$, $I(t)$. $S(t)$ represents the number of susceptible individuals; $E(t)$ represents the number of individuals exposed to the infected but unable to infect others; $I(t)$ represents the number of infected individuals who can infect other people. $M(t)$ represents the number of message that all of them provide about epidemic disease at time t, respectively. Since we consider the disease outbreak in the long time, we do not neglect the natural and the disease-related death rates. Further we assume that the susceptible individuals has a recruitment rate A. The total number of population at time t is given by

$$N(t) = S(t) + E(t) + I(t).$$

The model structure is shown in Fig. 1.

The transfer diagram leads to the following system of ordinary differential equations:

$$\begin{cases} \dot{S} = A + \gamma I - \beta S I e^{-\alpha M} - \alpha_1 S, \\ \dot{E} = \beta S I e^{-\alpha M} - \rho E - \alpha_1 E, \\ \dot{I} = \rho E - \gamma I - (\alpha_1 + \alpha_2) I, \\ \dot{M} = \mu_1 S + \mu_2 E + \mu_3 I - \tau M, \end{cases} \tag{2.1}$$

where α_1 is the natural death rate; α_2 is the disease-related death rate; We assume that all the people may send the message about epidemic disease at the rates μ_1, μ_2, and μ_3, respectively, during an epidemic season. β is the transmission rate between the susceptible individuals and the infected individuals, but owing to the behavior change of the public after reading messages about epidemic disease, the disease transmission rate β is reduced by a factor $e^{-\alpha M}$; α determines how effective the disease-related messages can influence the transmission rate; ρ is transmission coefficient from the exposed individuals to the infected individuals; γ is the transmission rate from the infected people to the susceptible people; τ is the rate that messages become outdated.

2.2. Basic properties

To show that the model (2.1) is epidemiologically meaningful, we will prove that all variables of system (2.1) are non-negative for all time $t > 0$. We thus have the following **Lemmas**.

2.2.1. Positivity of solutions

Lemma 2.2.1. *If $S(0) \geq 0$, $E(0) \geq 0$, $I(0) \geq 0$, and $M(0) \geq 0$, the solutions $S(t)$, $E(t)$, $I(t)$, and $M(t)$ of system (2.1) with initial conditions $S(0) \geq 0$, $E(0) \geq 0$, $I(0) \geq 0$, $M(0) \geq 0$ are positive for all $t > 0$.*

Proof. If $S(0) \geq 0$, according to the first equation of system (2.1), we have,

$$\frac{d\big(S(t)\big)}{dt} = A + \gamma I(t) - \beta S(t) I(t) e^{-\alpha M(t)} - \alpha_1 S(t).$$

It can be rewritten as:

$$\begin{aligned}
&\frac{d\big(S(t)\big)}{dt} exp\left\{\int_0^t \Big(\beta I(\tau) e^{-\alpha M(\tau)} + \alpha_1\Big) d\tau\right\} \\
&+ S(t)\Big(\beta I(\tau) e^{-\alpha M(\tau)} + \alpha_1\Big) exp\left\{\int_0^t \Big(\beta I(\tau) e^{-\alpha M(\tau)} + \alpha_1\Big) d\tau\right\} \\
&= \big(A + \gamma I(t)\big) exp\left\{\int_0^t \Big(\beta I(\tau) e^{-\alpha M(\tau)} + \alpha_1\Big) d\tau\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{d}{dt}\left(S(t) exp\left\{\int_0^t \Big(\beta I(\tau) e^{-\alpha M(\tau)} + \alpha_1\Big) d\tau\right\}\right) \\
&= \big(A + \gamma I(t)\big) exp\left\{\int_0^t \Big(\beta I(\tau) e^{-\alpha M(\tau)} + \alpha_1\Big) d\tau\right\}.
\end{aligned}$$

And then,

$$\begin{aligned}
&S(t) exp\left\{\int_0^t \Big(\beta I(\tau) e^{-\alpha M(\tau)} + \alpha_1\Big) d\tau\right\} - S(0) \\
&= \int_0^t \left(\big(A + \gamma I(\tau)\big) exp\left\{\int_0^\tau \Big(\beta I(\mu) e^{-\alpha M(\mu)} + \alpha_1\Big) d\mu\right\}\right) d\tau.
\end{aligned}$$

Hence,

$$\begin{aligned}
S(t) =\ & S(0) exp\left\{-\int_0^t \Big(\beta I(\tau) e^{-\alpha M(\tau)} + \alpha_1\Big) d\tau\right\} \\
& + exp\left\{-\int_0^t \Big(\beta I(\tau) e^{-\alpha M(\tau)} + \alpha_1\Big) d\tau\right\} \\
& \times \left(\int_0^t \left(\big(A + \gamma I(\tau)\big) exp\left\{\int_0^\tau \Big(\beta I(\mu) e^{-\alpha M(\mu)} + \alpha_1\Big) d\mu\right\}\right) d\tau\right) \\
& > 0.
\end{aligned}$$

Similarly, we can prove that $E(t) > 0$, $I(t) > 0$, $M(t) > 0$. So the solutions $S(t)$, $E(t)$, $I(t)$, $M(t)$ of system (2.1) with initial conditions $S(0) \geq 0$, $E(0) \geq 0$, $I(0) \geq 0$, $M(0) \geq 0$ are positive for all $t > 0$. This completes the proof of Lemma 2.2.1. □

2.2.2. Invariant region

Lemma 2.2.2. *The attracting region Ω defined by*

$$\Omega=\left\{(S,E,I,M)\in\mathbb{R}^4_+:0\le S,E,I\le N\le\frac{A}{\alpha_1},0\le M\le\frac{A(\mu_1+\mu_2+\mu_3)}{\tau\alpha_1}\right\},$$

with initial conditions $S(0)\ge 0$, $E(0)\ge 0$, $I(0)\ge 0$, $M(0)\ge 0$ and attracting all solutions initiating in the interior of the positive orthant is positive invariant for system (2.1).

Proof. Primarily, adding the former third equations of system (2.1), we have,

$$\frac{dN}{dt}=A-\alpha_1N-\alpha_2I\le A-\alpha_1N.$$

It follows that,

$$0\le N\le\frac{A}{\alpha_1}+N(0)e^{-\alpha_1t},$$

where $N(0)$ represents the initial value of the total population. Therefore, $\limsup_{t\to\infty}N\le\frac{A}{\alpha_1}$.

Next, according to the fourth equation of system (2.1), we get,

$$\frac{dM}{dt}\le\frac{(\mu_1+\mu_2+\mu_3)A}{\alpha_1}-\tau M,$$

and then,

$$0\le M\le\frac{(\mu_1+\mu_2+\mu_3)A}{\tau\alpha_1}+M(0)e^{-\tau t},$$

where $M(0)$ represents the initial value of the message of media. Thus, $\limsup_{t\to\infty}M\le\frac{(\mu_1+\mu_2+\mu_3)A}{\tau\alpha_1}$. Throughout this paper, we will consider dynamics of system (2.1) on the region Ω. This completes the proof of Lemma 2.2.2. □

3. Analysis of the model

3.1. Disease-free equilibrium and the basic reproductive number

Let the right-hand sides of Eq. (2.1) equal zero, we have,

$$\begin{cases}A+\gamma I-\beta SIe^{-\alpha M}-\alpha_1S=0,\\ \beta SIe^{-\alpha M}-\rho E-\alpha_1E=0,\\ \rho E-\gamma I-(\alpha_1+\alpha_2)I=0,\\ \mu_1S+\mu_2E+\mu_3I-\tau M=0.\end{cases}\tag{3.1}$$

Meanwhile, let $E=I=0$ in Eq. (3.1), it is straightforward to see that the model has a disease-free equilibrium given by the following:

$$P_0=(S_0,0,0,M_0)=\left(\frac{A}{\alpha_1},0,0,\frac{\mu_1A}{\alpha_1\tau}\right).\tag{3.2}$$

Next, we will obtain the basic reproductive number $\mathcal{R}_0$ of the system (2.1) by using the next-generation method [18,19]. Here, we have the following matrix of new infection $\mathcal{F}(x)$, and the matrix of transfer $\mathcal{V}(x)$. Let $x=(S,\ E,\ I,\ M)^T$, then system (2.1) can be rewritten as:

$$\frac{dx}{dt}=\mathcal{F}(x)-\mathcal{V}(x),$$

where

$$\mathcal{F}(x)=\begin{pmatrix}0\\ \beta SIe^{-\alpha M}\\ 0\\ 0\end{pmatrix},\qquad \mathcal{V}(x)=\begin{pmatrix}\beta SIe^{-\alpha M}+\alpha_1S-A-\gamma I\\ \rho E+\alpha_1E\\ \gamma I+(\alpha_1+\alpha_2)I-\rho E\\ \tau M-\mu_1S-\mu_2E-\mu_3I\end{pmatrix}.$$

The Jacobian matrices of $\mathcal{F}(x)$ and $\mathcal{V}(x)$ at the disease-free equilibrium P_0 respectively are,

$$D\mathcal{F}(P_0)=\begin{pmatrix}0&0&0&0\\0&0&\beta S_0e^{-\alpha M_0}&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad D\mathcal{V}(P_0)=\begin{pmatrix}\alpha_1&0&\beta S_0e^{-\alpha M_0}-\gamma&0\\0&\alpha_1+\rho&0&0\\0&-\rho&\gamma+\alpha_1+\alpha_2&0\\-\mu_1&-\mu_2&-\mu_3&\tau\end{pmatrix},$$

$$D\mathcal{V}(P_0)^{-1}$$
$$=\begin{pmatrix} \frac{1}{\alpha_1} & -\frac{\rho(\beta S_0 e^{-\alpha M_0}-\gamma)}{\alpha_1(\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2)} & -\frac{\beta S_0 e^{-\alpha M_0}-\gamma}{\alpha_1(\gamma+\alpha_1+\alpha_2)} & 0 \\ 0 & \frac{1}{\alpha_1+\rho} & 0 & 0 \\ 0 & \frac{\rho}{(\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2)} & \frac{1}{\gamma+\alpha_1+\alpha_2} & 0 \\ \frac{\mu_1}{\tau\alpha_1} & \frac{\alpha_1(\rho\mu_3+\mu_2(\gamma+\alpha_1+\alpha_2))-\mu_1\rho(\beta S_0 e^{-\alpha M_0}-\gamma)}{\tau\alpha_1(\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2)} & \frac{\alpha_1\mu_3(\alpha_1+\rho)-\mu_1(\alpha_1+\rho)(\beta S_0 e^{-\alpha M_0}-\gamma)}{\tau\alpha_1(\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2)} & \frac{1}{\tau} \end{pmatrix}.$$

Therefore, the basic reproductive number $\mathcal{R}_0$ is

$$\mathcal{R}_0=\rho\left(D\mathcal{F}(P_0)D\mathcal{V}(P_0)^{-1}\right)=\max\left(|\lambda|;\,\lambda\in\sigma\left(D\mathcal{F}(P_0)D\mathcal{V}(P_0)^{-1}\right)\right), \tag{3.3}$$

where $\rho(\cdot)$ and $\sigma(\cdot)$ denote the spectral radius and the set of eigenvalues of a matrix, respectively, because it can be verified that system (2.1) satisfies hypotheses (A_1)–(A_5) of **Theorem** 2 of [18], then, the basic reproductive number denoted by $\mathcal{R}_0$ is thus given by

$$\mathcal{R}_0=\frac{\beta\rho A e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}}{\alpha_1(\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2)}. \tag{3.4}$$

Define

$$\mathcal{R}_{01}=\frac{A\rho\Theta}{(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)-\rho\gamma}, \tag{3.5}$$

$$\mathcal{R}_p=\mathcal{R}_{01}e^{1-\mathcal{R}_{01}}, \tag{3.6}$$

where

$$\Theta=\frac{-\alpha}{\alpha_1\rho\tau}\left(\mu_1\rho\gamma-\mu_1(\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2)+\alpha_1\mu_2(\gamma+\alpha_1+\alpha_2)+\alpha_1\mu_3\rho\right). \tag{3.7}$$

Remark 3.1.1. It is clear to check that: $\mathcal{R}_{01}>0$ if and only if $\Theta>0$; $\mathcal{R}_{01}=0$ if and only if $\Theta=0$; $\mathcal{R}_{01}<0$ if and only if $\Theta<0$.

3.2. Stability of disease-free equilibrium

Theorem 3.2.1. *Disease-free equilibrium P_0 of the system (2.1) is globally asymptotically stable if $\mathcal{R}_0<1$, and is unstable if $\mathcal{R}_0>1$.*

Proof. The characteristic equation system (2.1) at the disease-free equilibrium P_0 is

$$\begin{vmatrix} \lambda+\alpha_1 & 0 & \frac{\beta A}{\alpha_1}e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}-\gamma & 0 \\ 0 & \lambda+(\alpha_1+\rho) & -\frac{\beta A}{\alpha_1}e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}} & 0 \\ 0 & -\rho & \lambda+(\gamma+\alpha_1+\alpha_2) & 0 \\ -\mu_1 & -\mu_2 & -\mu_3 & \lambda+\tau \end{vmatrix}=0, \tag{3.8}$$

then,

$$(\lambda+\alpha_1)(\lambda+\tau)\left(\left(\lambda+(\alpha_1+\rho)\right)\left(\lambda+(\gamma+\alpha_1+\alpha_2)\right)-\frac{\rho\beta A}{\alpha_1}e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\right)=0. \tag{3.9}$$

Thus, the two eigenvalues of Eq. (3.9) are $\lambda_1=-\mu$, $\lambda_2=-\tau$ and the other are determined by

$$\left(\lambda+(\alpha_1+\rho)\right)\left(\lambda+(\gamma+\alpha_1+\alpha_2)\right)-\frac{\rho\beta A}{\alpha_1}e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}=0. \tag{3.10}$$

Due to the expression of $\mathcal{R}_0$ (3.4), the above equation can be rewritten as:

$$\lambda^2+(\gamma+\rho+\alpha_2+2\alpha_1)\lambda+(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)(1-\mathcal{R}_0)=0. \tag{3.11}$$

Then, we have,

$$\lambda_3+\lambda_4=-(\gamma+\rho+\alpha_2+2\alpha_1)<0,$$
$$\lambda_3\lambda_4=(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)(1-\mathcal{R}_0).$$

Therefore, when $\mathcal{R}_0 < 1$, diseased-free equilibrium P_0 is locally asymptotically stable; when $\mathcal{R}_0 > 1$, diseased-free equilibrium P_0 is unstable.

Next, we define a Lyapunov function

$$V(S,E,I,M)=e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\rho E(t)+(\alpha_1+\rho)I(t).$$

It is obvious that $V(S,E,I,M)\geq 0$ and the equality holds if and only if $E(t)=I(t)=0$. Differentiating $V(S,E,I,M)$ and using the expression of $\mathcal{R}_0$ (3.4), we obtain,

$$\begin{aligned}
\frac{dV(S,E,I,M)}{dt} &= e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\rho\frac{dE(t)}{dt}+(\alpha_1+\rho)\frac{dI(t)}{dt}\\
&= e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\rho\Big(\beta SIe^{-\alpha M}-\rho E-\alpha_1 E\Big)+(\alpha_1+\rho)\left(\rho E-\gamma I-(\alpha_1+\alpha_2)I\right)\\
&= e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\rho\beta SIe^{-\alpha M}-e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\rho^2E-e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\rho\alpha_1E+\alpha_1\rho E-\alpha_1\gamma I-\alpha_1^2I\\
&\quad -\alpha_1\alpha_2I+\rho^2E-\rho\gamma I-\rho\alpha_1I-\rho\alpha_2I\\
&\leq \frac{A}{\alpha_1}e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\rho\beta Ie^{-\alpha M}-e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\rho^2E-e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\rho\alpha_1E+\alpha_1\rho E-\alpha_1\gamma I-\alpha_1^2I\\
&\quad -\alpha_1\alpha_2I+\rho^2E-\rho\gamma I-\rho\alpha_1I-\rho\alpha_2I\\
&\leq \frac{A}{\alpha_1}e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\rho\beta I-e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\rho^2E-e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\rho\alpha_1E+\alpha_1\rho E-\alpha_1\gamma I-\alpha_1^2I\\
&\quad -\alpha_1\alpha_2I+\rho^2E-\rho\gamma I-\rho\alpha_1I-\rho\alpha_2I\\
&\leq \frac{A}{\alpha_1}e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\rho\beta I-\alpha_1\gamma I-\alpha_1^2I-\alpha_1\alpha_2I-\rho\gamma I-\rho\alpha_1I-\rho\alpha_2I\\
&= \frac{A}{\alpha_1}\rho\beta Ie^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}-(\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2)I\\
&= (\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2)I\left(\frac{\rho\beta Ae^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}}{\alpha_1(\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2)}-1\right)\\
&= (\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2)I(\mathcal{R}_0-1).
\end{aligned}$$

Thus, $\mathcal{R}_0\leq 1$ guarantees that $\frac{dV(S,E,I,M)}{dt}\leq 0$ for all $t\geq 0$, and it follows that $V(S,E,I,M)$ is bounded and non-increasing. Therefore, $\lim_{t\to\infty}V(S,E,I,M)$ exists. By LaSalle's Invariance Principle [20], the disease-free equilibrium P_0 is globally asymptotic stability when $\mathcal{R}_0<1$. This completes the proof of Theorem 3.2.1. □

3.3. Existence of endemic equilibria

Theorem 3.3.1. *The system (2.1) has:*
(i) *A unique positive endemic equilibrium* ${P_1}^*$*, when* $\mathcal{R}_0>\max(1,\mathcal{R}_{01})$*;*
(ii) *A unique positive endemic equilibrium* ${P_2}^*$*, when* $\mathcal{R}_p=\mathcal{R}_0<\min(1,\mathcal{R}_{01})$*;*
(iii) *Two different positive endemic equilibria* ${P_3}^*$ *and* ${P_4}^*$*, when* $\mathcal{R}_p<\mathcal{R}_0<\min(1,\mathcal{R}_{01})$*.*

where $\mathcal{R}_0$, $\mathcal{R}_{01}$, $\mathcal{R}_p$, and Θ are given by (3.4)–(3.7), and ${P_i}^*=\left({S_i}^*,{E_i}^*,{I_i}^*,{M_i}^*\right)$ $(i=1,2,3,4)$ meet (3.13)–(3.15).

Proof. We assume that $P^*=(S^*,E^*,I^*,M^*)$ is a solution of Eq. (3.1), that is,

$$\begin{cases}A+\gamma I^*-\beta S^*I^*e^{-\alpha M^*}-\alpha_1S^*=0,\\ \beta S^*I^*e^{-\alpha M^*}-\rho E^*-\alpha_1E^*=0,\\ \rho E^*-\gamma I^*-(\alpha_1+\alpha_2)I^*=0,\\ \mu_1S^*+\mu_2E^*+\mu_3I^*-\tau M^*=0.\end{cases}\tag{3.12}$$

Next, we suppose that S^*, E^*, M^* are the linear function in regard to I^* respectively. We obtain,

$$S^*=\frac{A}{\alpha_1}+\left(\frac{\gamma}{\alpha_1}-\frac{(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)}{\alpha_1\rho}\right)I^*,\tag{3.13}$$

$$E^*=\frac{(\gamma+\alpha_1+\alpha_2)I^*}{\rho},\tag{3.14}$$

$$M^* = \frac{\mu_1 A}{\alpha_1 \tau} + \frac{I^*}{\alpha_1 \rho \tau}\big(\mu_1 \rho \gamma - \mu_1 (\alpha_1 + \rho)(\gamma + \alpha_1 + \alpha_2) + \alpha_1 \mu_2 (\gamma + \alpha_1 + \alpha_2) + \alpha_1 \mu_3 \rho\big). \tag{3.15}$$

Adding the (3.13)–(3.15) to the first equation of Eq. (3.12), we have,

$$\left(1 - \left(\frac{(\alpha_1 + \rho)(\alpha_2 + \gamma + \alpha_1)}{A\rho} - \frac{\gamma}{A}\right) I^*\right)\mathcal{R}_0 = e^{-\Theta I^*}. \tag{3.16}$$

By (3.5) and (3.16), we have,

$$\mathcal{R}_0 - \frac{\mathcal{R}_0}{\mathcal{R}_{01}}\Theta I^* - e^{-\Theta I^*} = 0. \tag{3.17}$$

We denote,

$$H(I) = \mathcal{R}_0 - \frac{\mathcal{R}_0}{\mathcal{R}_{01}}\Theta I - e^{-\Theta I}. \tag{3.18}$$

By (3.18), we can get,

$$H(0) = \mathcal{R}_0 - 1, H(\infty) = -\infty, H'(I) = -\frac{\mathcal{R}_0}{\mathcal{R}_{01}}\Theta + \Theta e^{-\Theta I},$$
$$H'(0) = -\frac{\mathcal{R}_0}{\mathcal{R}_{01}}\Theta + \Theta, H''(I) = -\Theta^2 e^{-\Theta I}.$$

(i) When $\mathcal{R}_0 > 1$, $H(0) = \mathcal{R}_0 - 1 > 0$. Meanwhile, $H(\infty) < 0$. If $\Theta \neq 0$

$$H''(I) = -\Theta^2 e^{-\Theta I} < 0,$$

we have $H'(I) < H'(0)$, namely, $\Theta e^{-\Theta I} < \Theta$, then

$$H'(I) = -\frac{\mathcal{R}_0}{\mathcal{R}_{01}}\Theta + \Theta e^{-\Theta I} < \Theta(1 - \frac{\mathcal{R}_0}{\mathcal{R}_{01}}).$$

When $\mathcal{R}_0 > \mathcal{R}_{01}$, then $H(I) = 0$ has a unique positive solution.

If $\Theta = 0$, according to Eq. (3.16), we have

$$I = \frac{A\rho}{(\alpha_1 + \rho)(\alpha_2 + \gamma + \alpha_1) - \gamma\rho}(1 - \frac{1}{\mathcal{R}_0}). \tag{3.19}$$

Similarly, when $\mathcal{R}_0 > 1$, $I > 0$. Thus, the endemic equilibrium $P_1{}^* = (S_1{}^*, E_1{}^*, I_1{}^*, M_1{}^*)$ can be obtained.

(ii) When $\mathcal{R}_0 < 1$, $H(0) = \mathcal{R}_0 - 1 < 0$, and $H(\infty) < 0$. We suppose that $H'(I) = 0$, then $I = \frac{1}{\Theta}\ln\frac{\mathcal{R}_{01}}{\mathcal{R}_0}$. When $\mathcal{R}_0 < \mathcal{R}_{01}$, I is positive. Further, I also is a positive solution $H(I) = 0$ if and only if $\mathcal{R}_0 = \mathcal{R}_p$. Thus, the endemic equilibrium $P_2{}^* = (S_2{}^*, E_2{}^*, I_2{}^*, M_2{}^*)$ can be obtained.

(iii) Based on (ii), when $\mathcal{R}_0 > \mathcal{R}_p$, $H(I) > 0$, then $H(I) = 0$ has two different positive solutions. Thus, the endemic equilibrium $P_i{}^* = (S_i{}^*, E_i{}^*, I_i{}^*, M_i{}^*)(i = 3, 4)$ can be obtained.

This completes the proof of Theorem 3.3.1. □

3.4. Stability of the endemic equilibria

Theorem 3.4.1. *The endemic equilibria $P_i{}^*(i = 1, 2, 3, 4)$ of the system (2.1) have:*

(i) *The endemic equilibrium $P_1{}^*$ is locally asymptotically stable, when $\mathcal{R}_0 > \max(1, \mathcal{R}_{01})$, and $a_1(I_1{}^*)a_2(I_1{}^*) - a_3(I_1{}^*) > 0$, $a_1(I_1{}^*)\big(a_2(I_1{}^*)a_3(I_1{}^*) - a_1(I_1{}^*)a_4(I_1{}^*)\big) - \big(a_1(I_3{}^*)\big)^2 > 0$, $a_4(I_1{}^*) > 0$;*

(ii) *The endemic equilibrium $P_2{}^*$ is a saddle node point, when $\mathcal{R}_p = \mathcal{R}_0 < \min(1, \mathcal{R}_{01})$;*

(iii) *The endemic equilibrium $P_3{}^*$ is an unstable saddle point, when $\mathcal{R}_p < \mathcal{R}_0 < \min(1, \mathcal{R}_{01})$;*

(iv) *The endemic equilibrium $P_4{}^*$ is an stable node point, when $\mathcal{R}_p < \mathcal{R}_0 < \min(1, \mathcal{R}_{01})$.*

Proof. The characteristic equation of system (2.1) at the endemic equilibria $P_i{}^*(i = 1, 2, 3, 4)$ is

$$\begin{vmatrix} \lambda + (\alpha_1 + \beta I_i{}^* e^{-\alpha M_i{}^*}) & 0 & \beta S_i{}^* e^{-\alpha M_i{}^*} - \gamma & -\alpha\beta I_i{}^* S_i{}^* e^{-\alpha M_i{}^*} \\ -\beta I_i{}^* e^{-\alpha M_i{}^*} & \lambda + (\alpha_1 + \rho) & -\beta S_i{}^* e^{-\alpha M_i{}^*} & \alpha\beta I_i{}^* S_i{}^* e^{-\alpha M_i{}^*} \\ 0 & -\rho & \lambda + (\gamma + \alpha_1 + \alpha_2) & 0 \\ -\mu_1 & -\mu_2 & -\mu_3 & \lambda + \tau \end{vmatrix} = 0. \tag{3.20}$$

We set $\Phi = \beta e^{-\alpha M_i{}^*}$, then the characteristic equation can be rewritten as:

$$G(\lambda) = \lambda^4 + a_1(I_i{}^*)\lambda^3 + a_2(I_i{}^*)\lambda^2 + a_3(I_i{}^*)\lambda + a_4(I_i{}^*) = 0, \tag{3.21}$$

where

$$a_1(I_i{}^*) = 3\alpha_1 + \rho + \gamma + \alpha_2 + \tau + \Phi I_i{}^*, \tag{3.22}$$

$$\begin{aligned} a_2(I_i{}^*) &= (3\alpha_1 + \rho + \gamma + \alpha_2 + \Phi I_i{}^*)\tau + (\Phi I_i{}^* + \alpha_1)(\alpha_2 + 2\alpha_1 + \rho + \gamma) \\ &\quad - \left(\frac{A\Theta}{\mathcal{R}_{01}} + \gamma\right)\alpha I_i{}^*(\mu_1 - \mu_2), \end{aligned} \tag{3.23}$$

$$\begin{aligned} a_3(I_i{}^*) &= (2\alpha_1 + \rho + \gamma + \alpha_2)(\alpha_1\tau + \Phi I_i{}^*) + \Phi I_i{}^*(\alpha_1 + \rho)(\alpha_2 + \gamma + \alpha_1) \\ &\quad - \left(\frac{A\Theta}{\mathcal{R}_{01}} + \gamma\right) I_i{}^*\alpha\big((\alpha_2 + \gamma + 2\alpha_1)(\mu_1 - \mu_2) + \rho(\mu_1 - \mu_3)\big), \end{aligned} \tag{3.24}$$

$$a_4(I_i{}^*) = \tau(\alpha_1 + \rho)(\alpha_2 + \gamma + \alpha_1)\big((\alpha_1 + \Phi I_i{}^*) - \alpha_1\left(1 + \Theta I_i{}^*\right)\big). \tag{3.25}$$

(i) According to (3.22)–(3.25), we have,

$$a_1(I_1{}^*) = 3\alpha_1 + \rho + \gamma + \alpha_2 + \tau + \Phi I_1{}^*, \tag{3.26}$$

$$\begin{aligned} a_2(I_1{}^*) &= (3\alpha_1 + \rho + \gamma + \alpha_2 + \Phi I_1{}^*)\tau + (\Phi I_1{}^* + \alpha_1)(\alpha_2 + 2\alpha_1 + \rho + \gamma) \\ &\quad - \left(\frac{A\Theta}{\mathcal{R}_{01}} + \gamma\right)\alpha I_1{}^*(\mu_1 - \mu_2), \end{aligned} \tag{3.27}$$

$$\begin{aligned} a_3(I_1{}^*) &= (2\alpha_1 + \rho + \gamma + \alpha_2)(\alpha_1\tau + \Phi I_1{}^*) + \Phi I_1{}^*(\alpha_1 + \rho)(\alpha_2 + \gamma + \alpha_1) \\ &\quad - \left(\frac{A\Theta}{\mathcal{R}_{01}} + \gamma\right) I_1{}^*\alpha\big((\alpha_2 + \gamma + 2\alpha_1)(\mu_1 - \mu_2) + \rho(\mu_1 - \mu_3)\big), \end{aligned} \tag{3.28}$$

$$a_4(I_1{}^*) = \tau(\alpha_1 + \rho)(\alpha_2 + \gamma + \alpha_1)\big((\alpha_1 + \Phi I_1{}^*) - \alpha_1\left(1 + \Theta I_1{}^*\right)\big). \tag{3.29}$$

It is clear that $a_1(I_1{}^*) > 0$, according to Routh–Hurwitz criteria [21], the proof (i) of Theorem 3.4.1 is obtained.

(ii) Due to the proof of (ii) of Theorem 3.3.1, we have $I_2{}^* = \frac{1}{\Theta}\ln\frac{\mathcal{R}_{01}}{\mathcal{R}_0}$ and $\Phi I_2{}^* = \alpha_1\Theta I_2{}^*$. Therefore, by (3.22), we have,

$$a_1(I_2{}^*) = 3\alpha_1 + \rho + \gamma + \alpha_2 + \tau + \Phi I_2{}^* > 0,$$

by (3.25) and $I_2{}^* = \frac{1}{\Theta}\ln\frac{\mathcal{R}_{01}}{\mathcal{R}_0}$, we know that,

$$a_4(I_2{}^*) = \tau\alpha_1(\alpha_1 + \rho)(\alpha_2 + \gamma + \alpha_1)\big((1 + \Theta I_2{}^*) - (1 + \Theta I_2{}^*)\big) = 0,$$

from (3.24), we obtain,

$$\begin{aligned} a_3(I_2{}^*) &= (2\alpha_1 + \rho + \gamma + \alpha_2)(\alpha_1\tau + \Phi I_2{}^*) + \Phi I_2{}^*(\alpha_1 + \rho)(\alpha_2 + \gamma + \alpha_1) \\ &\quad - \left(\frac{A\Theta}{R_{01}} + \gamma\right) I_2{}^*\alpha\big((\alpha_2 + \gamma + 2\alpha_1)(\mu_1 - \mu_2) + \rho(\mu_1 - \mu_3)\big). \end{aligned}$$

It is easy to know that $a_3(I_2{}^*) < 0$. Therefore we know that Eq. (3.30) has negative, positive and zero eigenvalues.

$$G(\lambda) = \lambda^4 + a_1(I_2{}^*)\lambda^3 + a_2(I_2{}^*)\lambda^2 + a_3(I_2{}^*)\lambda + a_4(I_2{}^*) = 0. \tag{3.30}$$

So the endemic equilibrium $P_2{}^*$ of the system (2.1) is a saddle node point.

(iii) By the proof of (iii) of Theorem 3.3.1, we have $I_3{}^* < I_2{}^* = \frac{1}{\Theta}\ln\frac{\mathcal{R}_{01}}{\mathcal{R}_0}$. Then $\Phi I_3{}^* < \alpha_1\Theta I_2{}^*$. Therefore, by (3.22), we can get,

$$a_1(I_3{}^*) = 3\alpha_1 + \rho + \gamma + \alpha_2 + \tau + \Phi I_3{}^* > 0,$$

substituting $I_3{}^* < \frac{1}{\Theta}\ln\frac{\mathcal{R}_{01}}{\mathcal{R}_0}$ to (3.25), we have,

$$a_4(I_3{}^*) < \tau\alpha_1(\alpha_1 + \rho)(\alpha_2 + \gamma + \alpha_1)\big((1 + \Theta I_3{}^*) - (1 + \Theta I_3{}^*)\big) = 0.$$

Let $\psi_j(I_3{}^*)(j = 1, 2, 3, 4)$ be the solutions of Eq. (3.31)

$$G(\lambda) = \lambda^4 + a_1(I_3{}^*)\lambda^3 + a_2(I_3{}^*)\lambda^2 + a_3(I_3{}^*)\lambda + a_4(I_3{}^*) = 0. \tag{3.31}$$

So we can obtain $\psi_j(I_3{}^*)(j = 1, 2, 3, 4)$ has properties:

$$\psi_1(I_3{}^*) + \psi_2(I_3{}^*) + \psi_3(I_3{}^*) + \psi_4(I_3{}^*) = -a_1(I_3{}^*) < 0,$$

and

$$\psi_1(I_3{}^*)\psi_2(I_3{}^*)\psi_3(I_3{}^*)\psi_4(I_3{}^*) = a_4(I_3{}^*) < 0.$$

Therefore, we know that Eq. (3.31) has negative and positive eigenvalues. Then the endemic equilibrium $P_3{}^*$ of the system (2.1) is an unstable saddle point.

(iv) By the proof (iii) of Theorem 3.3.1, we have $I_4{}^* > I_2{}^* = \frac{1}{\Theta}\ln\frac{\mathcal{R}_{01}}{\mathcal{R}_0}$, then $\Phi I_4{}^* > \mu\Theta I_4{}^*$. Further, by (3.22), we can yield,

$$a_1(I_4{}^*) = 3\alpha_1 + \rho + \gamma + \alpha_2 + \tau + \Phi I_4{}^* > 0,$$

next, from (3.25), we can get,

$$a_4(I_4{}^*) > \tau\alpha_1(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)\big((1+\Theta I_4{}^*) - (1+\Theta I_3{}^*)\big) = 0.$$

Let $\psi_j(I_4{}^*)(j = 1, 2, 3, 4)$ be the solutions of Eq. (3.32) with real parts satisfying $Re\big(\psi_1(I_4{}^*)\big) \le Re\big(\psi_2(I_4{}^*)\big) \le Re\big(\psi_3(I_4{}^*)\big) \le Re\big(\psi_4(I_4{}^*)\big)$, where Re denotes the real part of a complex number.

$$G(\lambda) = \lambda^4 + a_1(I_4{}^*)\lambda^3 + a_2(I_4{}^*)\lambda^2 + a_3(I_4{}^*)\lambda + a_4(I_4{}^*) = 0. \tag{3.32}$$

We know that Eq. (3.32) has properties:

$$\psi_1(I_4{}^*) + \psi_2(I_4{}^*) + \psi_3(I_4{}^*) + \psi_4(I_4{}^*) = -a_1(I_4{}^*) < 0, \tag{3.33}$$

and

$$\psi_1(I_4{}^*)\psi_2(I_4{}^*)\psi_3(I_4{}^*)\psi_4(I_4{}^*) = a_4(I_4{}^*) > 0. \tag{3.34}$$

Therefore, we can readily yield $\psi_j(I_4{}^*) < 0 (j = 1, 2, 3, 4)$. If we may take the suppose $Re\big(\psi_1(I_4{}^*)\big) \le Re\big(\psi_2(I_4{}^*)\big) < 0 < Re\big(\psi_3(I_4{}^*)\big) \le Re\big(\psi_4(I_4{}^*)\big)$, it leads to a contradiction with (3.33). Therefore the endemic equilibrium $P_4{}^*$ of the system (2.1) is a stable node point.

This completes the proof of Theorem 3.4.1. □

3.5. *Analysis of the bifurcation*

3.5.1. *A forward and backward bifurcation*

Theorem 3.5.1. (i) *If $\mathcal{R}_{01} < 1$, system (2.1) exhibits a forward bifurcation when $\mathcal{R}_0 = 1$;*
(ii) *If $\mathcal{R}_{01} > 1$, system (2.1) exhibits a backward bifurcation when $\mathcal{R}_0 = 1$.*

Proof. We make use of the center manifold approach as described in [22] and introduce $x_1 = S, x_2 = E, x_3 = I, x_4 = M$, system (2.1) becomes

$$\begin{cases} \dfrac{dx_1}{dt} = A + \gamma x_3 - \alpha_1 x_1 - \beta x_3 x_1 e^{-\alpha x_4} := f_1, \\ \dfrac{dx_2}{dt} = \beta x_3 x_1 e^{-\alpha x_4} - (\alpha_1+\rho)x_2 := f_2, \\ \dfrac{dx_3}{dt} = \rho x_2 - (\gamma+\alpha_1+\alpha_2)x_3 := f_3, \\ \dfrac{dx_4}{dt} = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 - \tau x_4 := f_4. \end{cases} \tag{3.35}$$

When $\mathcal{R}_0 = 1$, $\beta = \beta^* = \frac{\alpha_1(\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2)}{\rho A} e^{\frac{\alpha\mu_1 A}{\alpha_1\tau}}$, and the disease-free equilibrium $P_0 = x_0 = (x_{10}, x_{20}, x_{30}, x_{40}) = \left(\frac{A}{\alpha_1}, 0, 0, \frac{\mu_1 A}{\alpha_1\tau}\right)$. The linearization matrix of system (3.35) around the disease-free equilibrium x_0 when $\beta = \beta^*$ is

$$D_x f = \begin{pmatrix} -\alpha_1 & 0 & \gamma - \dfrac{\beta^* A}{\alpha_1} e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}} & 0 \\ 0 & -(\alpha_1+\rho) & \dfrac{\beta^* A}{\alpha_1} e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}} & 0 \\ 0 & \rho & -(\gamma+\alpha_1+\alpha_2) & 0 \\ \mu_1 & \mu_2 & \mu_3 & -\tau \end{pmatrix}. \tag{3.36}$$

Therefore, the characteristic equation of system (3.35) at the disease-free equilibrium x_0 when $\beta = \beta^*$ is

$$\begin{vmatrix} \lambda+\alpha_1 & 0 & \frac{\beta^* A}{\alpha_1}e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}-\gamma} & 0 \\ 0 & \lambda+(\alpha_1+\rho) & -\frac{\beta^* A}{\alpha_1}e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}} & 0 \\ 0 & -\rho & \lambda+(\gamma+\alpha_1+\alpha_2) & 0 \\ -\mu_1 & -\mu_2 & -\mu_3 & \lambda+\tau \end{vmatrix} = 0, \tag{3.37}$$

then,

$$(\lambda+\alpha_1)(\lambda+\tau)\left(\big(\lambda+(\alpha_1+\rho)\big)\big(\lambda+(\gamma+\alpha_1+\alpha_2)\big) - \frac{\rho\beta^* A}{\alpha_1}e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\right) = 0. \tag{3.38}$$

Substitute $\beta = \beta^* = \frac{\alpha_1(\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2)}{\rho A}e^{\frac{\alpha\mu_1 A}{\alpha_1\tau}}$ into Eq. (3.38), we obtain,

$$\lambda(\lambda+\alpha_1)(\lambda+\tau)\big(\lambda+(2\alpha_1+\rho+\alpha_2+\gamma)\big) = 0. \tag{3.39}$$

It is clear that 0 is a simple solution of Eq. (3.39), and then 0 is a simple eigenvalue of the linearization matrix $D_x f$ of system (3.35) around the disease-free equilibrium x_0 when $\beta = \beta^*$. A right eigenvector corresponding to the 0 eigenvalue is

$$R = (r_1, r_2, r_3, r_4)^T,$$

where

$$\begin{aligned} & r_1 = -\tau\big(\rho\gamma - (\alpha_1+\rho)(\gamma+\alpha_1+\alpha_1)\big), r_2 = \tau\alpha_1(\gamma+\alpha_1+\alpha_2), r_3 = \tau\alpha_1\rho, \\ & r_4 = \mu_1\rho\gamma - \mu_1(\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2) + \alpha_1\mu_2(\gamma+\alpha_1+\alpha_2) + \alpha_1\mu_3\rho. \end{aligned}$$

And the left eigenvector L corresponding to the 0 eigenvalue satisfying the equalities $LJ = 0$ and $LR = 1$ is

$$L = (l_1, l_2, l_3, l_4),$$

where

$$l_1 = 0, l_2 = \frac{1}{\alpha_1\tau(\gamma+\rho+\alpha_2+2\alpha_1)}, l_3 = \frac{\alpha_1+\rho}{\alpha_1\rho\tau(\gamma+\rho+\alpha_2+2\alpha_1)}, l_4 = 0.$$

Algebraic calculations show that

$$\begin{aligned} \mathcal{A} &= \sum_{k,i=1}^{4} l_k r_i \frac{\partial^2 f_k(P_0)}{\partial x_i \partial\beta} \\ &= l_2 r_3 \frac{\partial^2 f_2}{\partial x_3 \partial\beta} \\ &= \frac{A\rho}{\alpha_1(\gamma+\alpha_2+3\alpha_1)}e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}} > 0, \end{aligned}$$

and

$$\begin{aligned} \mathcal{B} &= \sum_{k,i,j=1}^{4} l_k r_i r_j \frac{\partial^2 f_k(P_0)}{\partial x_i \partial x_j} \\ &= l_2\left(r_1 r_3 \frac{\partial^2 f_2(P_0)}{\partial x_1\partial x_3} + r_3 r_1 \frac{\partial^2 f_2(P_0)}{\partial x_3\partial x_1} + r_3 r_4 \frac{\partial^2 f_2(P_0)}{\partial x_3\partial x_4} + r_4 r_3 \frac{\partial^2 f_2(P_0)}{\partial x_4\partial x_3}\right) \\ &= 2l_2\left(r_1 r_3 \beta e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}} + r_3 r_4 (-\frac{\alpha\beta A}{\alpha_1}e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}})\right) \\ &= \frac{2}{\alpha_1(\gamma+\rho+\alpha_2+2\alpha_1)}\beta e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}\Big(\tau\alpha_1\rho\big((\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2) - \rho\gamma\big) \\ &\quad - \alpha\rho A\big(\mu_1\rho\gamma - \mu_1(\alpha_1+\rho)(\gamma+\rho+\alpha_2) + \alpha_1\mu_2(\gamma+\alpha_1+\alpha_2) + \alpha_1\rho\mu_3\big)\Big) \end{aligned}$$

$$= \frac{2\alpha_1\tau(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)}{A(\alpha_2+\gamma+\rho+2\alpha_1)}(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)\left(\frac{A\rho\Theta}{(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)}-1\right)$$
$$= \frac{2\alpha_1\tau(\alpha_1+\rho)^2(\alpha_2+\gamma+\alpha_1)^2(\mathcal{R}_{01}-1)}{A(\alpha_2+\gamma+\rho+2\alpha_1)}.$$

According to Theorem 4.1 of [22], note that the coefficient $\mathcal{A}$ is always positive. If $\mathcal{R}_{01} < 1$, the coefficient $\mathcal{B}$ is negative. In this case, the direction of the bifurcation of the system (2.1) at $\mathcal{R}_0 = 1$ is forward (supercritical), as shown in the Fig. 4. If $\mathcal{R}_{01} > 1$, the coefficient $\mathcal{B}$ is positive. Under this circumstance, the direction of the bifurcation of the system (2.1) at $\mathcal{R}_0 = 1$ is backward (subcritical), as shown in the Fig. 5. This completes the proof of Theorem 3.5.1. □

3.5.2. A Hopf bifurcation

Theorem 3.5.2. *A Hopf bifurcation occurs around the endemic equilibrium $P_1{}^*$, when β increases and the β^{**} is crossed.*

Proof. We presume that characteristic equation (3.21) of system (2.1) around the endemic equilibrium $P_1{}^*$ has two real roots λ_1 , λ_2, and a couple of complex roots $a \pm bi$, where $\lambda_1 < 0$, $\lambda_2 < 0$ and $a, b \in \mathbb{R}$. Therefore, we have,

$$\begin{aligned} G(\lambda) &= (\lambda-\lambda_1)(\lambda-\lambda_2)\big(\lambda-(a+bi)\big)\big(\lambda-(a-bi)\big) \\ &= \lambda^4-(\lambda_1+\lambda_2+2a)\lambda^3+\Big(\lambda_1\lambda_2+2a(\lambda_1+\lambda_2)+\big(a^2+b^2\big)\Big)\lambda^2 \\ &\quad -\Big(\big(a^2+b^2\big)(\lambda_1+\lambda_2)+2a\lambda_1\lambda_2\Big)\lambda+\big(a^2+b^2\big)\lambda_1\lambda_2. \end{aligned}$$

Comparing with Eq. (3.21), we get,

$$a_1(I_1{}^*) = -(\lambda_1+\lambda_2+2a),\ a_2(I_1{}^*) = \Big(\lambda_1\lambda_2+2a(\lambda_1+\lambda_2)+\big(a^2+b^2\big)\Big),$$
$$a_3(I_1{}^*) = -\Big(\big(a^2+b^2\big)(\lambda_1+\lambda_2)+2a\lambda_1\lambda_2\Big),\ a_4(I_1{}^*) = \big(a^2+b^2\big)\lambda_1\lambda_2.$$

Therefore, we consider the case when the characteristic equation $G(\lambda) = 0$ has two real roots λ_1 , λ_2 and a pair of purely imaginary roots $\pm bi$; i.e., $a = 0$, where $\lambda_1 < 0$, $\lambda_2 < 0$ and $b \in \mathbb{R}$. Thus, we yield,

$$a_1(I_1{}^*) = -(\lambda_1+\lambda_2),\ a_2(I_1{}^*) = b^2+\lambda_1\lambda_2,\ a_3(I_1{}^*) = -b^2(\lambda_1+\lambda_2),\ a_4(I_1{}^*) = b^2\lambda_1\lambda_2.$$

Then,

$$a_3(I_1{}^*)(a_1(I_1{}^*)a_2(I_1{}^*)-a_3)(I_1{}^*)-\big(a_1(I_1{}^*)\big)^2a_4(I_1{}^*) = 0,$$

which leads to $\beta = \beta^{**}$, as shown in the proof of Theorem 3.4.1. As a result, the occurrence of a pair of purely imaginary roots corresponds to the threshold curve $\beta = \beta^{**}$.

In order to see how the real parts of the others eigenvalues $a \pm bi$ change their signs, we examine the transversality condition of the Hopf bifurcation. Substituting $a + bi$ into the characteristic equation (3.21), we obtain $G(a + bi) = 0$. Accordingly, $Re\big(G(a+bi)\big) = 0$, where Re means the real part of a complex number. Computing $Re\big(G(a+bi)\big) = 0$, we yield,

$$\begin{aligned} \Delta &= Re\Big((a+bi)^4+a_1(I_1{}^*)(a+bi)^3+a_2(I_1{}^*)(a+bi)^2+a_3(I_1{}^*)(a+bi)+a_4(I_1{}^*)\Big) \\ &= a^4-6a^2b^2+b^4+a_1(I_1{}^*)\big(a^3-3ab^2\big)+a_2(I_1{}^*)\big(a^2-b^2\big)+a_3(I_1{}^*)a+a_4(I_1{}^*) \\ &= 0. \end{aligned}$$

From (3.26)–(3.29), we know that $a_1(I_1{}^*)$, $a_2(I_1{}^*)$, $a_3(I_1{}^*)$, and $a_4(I_1{}^*)$ depend upon β owing to Φ contains β. Thus, Δ is a function with regard to a and β. Consequently, $\Delta(a, \beta) = 0$ defines an implicit function $a(\beta)$ with the independent variable β.

Differentiating Δ with respect to β, we obtain $\frac{\partial\Delta}{\partial\beta} = 0$, which leads to $\frac{\partial\Delta}{\partial a}\frac{\partial a}{\partial\beta}+\frac{\partial\Delta}{\partial\beta} = 0$. Hence, $\frac{\partial a}{\partial\beta} = -\frac{\partial\Delta}{\partial\beta}/\frac{\partial\Delta}{\partial a}$.

Next, we determine the sign of $\frac{\partial a}{\partial\beta}$ along the curve $\beta = \beta^{**}$. Since $a_1(I_1{}^*)$, $a_2(I_1{}^*)$, $a_3(I_1{}^*)$ and $a_4(I_1{}^*)$ count on β, and that $a = 0$ and $a_2(I_1{}^*) = b^2+\lambda_1\lambda_2$ on the curve $\beta = \beta^{**}$, we have,

$$\begin{aligned} \frac{\partial\Delta}{\partial\beta}\Big|_{\beta=\beta^{**}} &= \left(\big(a^3-3ab^2\big)\frac{\partial a_1(I_1{}^*)}{\partial\beta}+\big(a^2-b^2\big)\frac{\partial a_2(I_1{}^*)}{\partial\beta}+a\frac{\partial a_3(I_1{}^*)}{\partial\beta}+\frac{\partial a_4(I_1{}^*)}{\partial\beta}\right)\Big|_{\beta=\beta^{**}} \\ &= \left(-b^2\frac{\partial a_2(I_1{}^*)}{\partial\beta}+\frac{\partial a_4(I_1{}^*)}{\partial\beta}\right)\Big|_{\beta=\beta^{**}} \\ &= \big(-b^2(\alpha_2+\gamma+2\alpha_1+\rho+\tau)\big)I_1{}^*e^{-\alpha M_1{}^*}+\big(\tau(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)\big)I_1{}^*e^{-\alpha M_1{}^*} \\ &= \big(-b^2(\alpha_2+\gamma+2\alpha_1+\rho+\tau)+\tau(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)\big)I_1{}^*e^{-\alpha M_1{}^*}, \end{aligned}$$

and

$$\begin{aligned}\frac{\partial \Delta}{\partial a}|_{\beta=\beta^{**}} &= \left(4a^3 - 12ab^2 + 3a_1(I_1{}^*)(a^2 - b^2) + 2aa_2(I_1{}^*) + a_3(I_1{}^*)\right)|_{\beta=\beta^{**}} \\ &= -3b^2 a_1(I_1{}^*) + a_3(I_1{}^*) \\ &= 3b^2(\lambda_1+\lambda_2) - b^2(\lambda_1+\lambda_2) \\ &= 2b^2(\lambda_1+\lambda_2).\end{aligned}$$

Since $\lambda_1 < 0$ and $\lambda_2 < 0$, then $\lambda_1+\lambda_2 < 0$, combining $b^2 > 0$, we have $\frac{\partial \Delta}{\partial a}|_{\beta=\beta^{**}} < 0$. In order to ensure the sign of $\frac{\partial \Delta}{\partial \beta}|_{\beta=\beta^{**}}$ according to (3.11), we yield,

$$\lambda_{3,4} = \frac{-(\gamma+\rho+\alpha_2+2\alpha_1) \pm \sqrt{(\gamma+\rho+\alpha_2+2\alpha_1)^2 - 4(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)(1-\mathcal{R}_0)}}{2}.$$

We change the form of above equality so as to obtain b, namely,

$$\lambda_{3,4} = \frac{-(\gamma+\rho+\alpha_2+2\alpha_1) \pm \sqrt{4(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)(1-\mathcal{R}_0) - (\gamma+\rho+\alpha_2+2\alpha_1)^2}\,i}{2}.$$

Consequently, we obtain,

$$b = \sqrt{(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)(1-\mathcal{R}_0) - \frac{(\alpha_2+\gamma+\rho+2\alpha_1)^2}{4}}.$$

Substituting the expression of b into the expression of $\frac{\partial \Delta}{\partial \beta}|_{\beta=\beta^{**}}$, we get,

$$\begin{aligned}\frac{\partial \Delta}{\partial \beta}|_{\beta=\beta^{**}} &= \left(-b^2(\alpha_2+\gamma+2\alpha_1+\rho+\tau) + \tau(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)\right) I_1{}^* e^{-\alpha M_1{}^*} \\ &= \Bigg(\tau(\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1) - (\alpha_1+\rho)(\alpha_2+\gamma+\alpha_1)(1-\mathcal{R}_0)\times(\alpha_2+\gamma+2\alpha_1 \\ &\quad +\rho+\tau) + \frac{(\alpha_2+\gamma+\rho+2\alpha_1)^2(\alpha_2+\gamma+2\alpha_1+\rho+\tau)}{4}\Bigg) I_1{}^* e^{-\alpha M_1{}^*}.\end{aligned}$$

Since $\mathcal{R}_0 > 1$, thus, $\frac{\partial \Delta}{\partial \beta}|_{\beta=\beta^{**}} > 0$. In a word, we have,

$$\frac{\partial a}{\partial \beta}|_{\beta=\beta^{**}} = -\frac{\partial \Delta}{\partial \beta} \Big/ \frac{\partial \Delta}{\partial a}|_{\beta=\beta^{**}} > 0.$$

This demonstrates that when β crosses the curve $\beta = \beta^{**}$, a Hopf bifurcation occurs. The proof of Theorem 3.5.2 is completed. □

4. Numerical simulation

In this section, we present some numerical results of system (2.1) that support and extend our theoretical results. We choose some parameters based on the Table 1.

We choose a set of the following parameters:

$$\begin{aligned}&A = 0.8\text{ day}^{-1}, \alpha_1 = 0.6\text{ day}^{-1}, \beta = 0.8\text{ person}^{-1}\text{ day}^{-1}, \alpha = 0.08\text{ message}^{-1}, \mu_1 = 0.99\text{ day}^{-1}, \\ &\mu_2 = 0.4\text{ day}^{-1}, \mu_3 = 0.8\text{ day}^{-1}, \tau = 0.6\text{ day}^{-1}, \alpha_2 = 0.02\text{ day}^{-1}, \gamma = 0.7\text{ day}^{-1}, \rho = 0.09\text{ day}^{-1}.\end{aligned}$$

Thus, we get the following system:

$$\begin{cases}\dot{S} = 0.8 + 0.7I - 0.8SIe^{-0.08M} - 0.6S, \\ \dot{E} = 0.8SIe^{-0.08M} - 0.09E - 0.6E, \\ \dot{I} = 0.09E - 0.7I - (0.6+0.02)I, \\ \dot{M} = 0.99S + 0.4E + 0.8I - 0.6M.\end{cases} \tag{4.1}$$

It is easy to verify that $\mathcal{R}_0 = 0.0884 < 1$. Further, we have the unique disease-free equilibrium $P_0 = (1.3333, 0, 0, 2.2)$ of system (2.1). Then, from Theorem 3.2.1, the disease-free equilibrium $P_0 = (1.3333, 0, 0, 2.2)$ of system (2.1) is globally asymptotically stable when $\mathcal{R}_0 = 0.0884 < 1$ (Fig. 2).

Next, we select a set of the following parameters:

$$\begin{aligned}&A = 0.8day^{-1}, \alpha_1 = 0.2day^{-1}, \beta = 0.8person^{-1}day^{-1}, \alpha = 0.091message^{-1}, \mu_1 = 0.2day^{-1}, \\ &\mu_2 = 0.8day^{-1}, \mu_3 = 0.8day^{-1}, \tau = 0.6day^{-1}, \alpha_2 = 0.02day^{-1}, \gamma = 0.006day^{-1}, \rho = 0.4day^{-1}.\end{aligned}$$

Table 1
The parameters description of the epidemic model.

Parameter	Description	Estimated value	Source
A	The constant recruitment rate of the population	$0.8day^{-1}$	[16]
β	The disease transmission coefficient	0.0099–0.8 person^{-1} day^{-1}	[14]
α	The coefficient that determines how effective the disease information can influence the transmission rate	0.00091–0.8 day^{-1}	[14]
ρ	Transmission coefficient from the exposed individuals compartment to the infected individuals compartment	0.04–0.99 day^{-1}	[14]
μ_1	The rate that susceptible individuals may send message about influenza during an epidemic season	0–0.99 day^{-1}	[13]
μ_2	The rate that exposed individuals may send message about influenza during an epidemic season	0.008–0.8 day^{-1}	[13]
μ_3	The rate that infectious individuals may send message about influenza during an epidemic season	$0.6–0.8day^{-1}$	[13]
α_1	The natural death rate of the population	0.009–0.6 year^{-1}	Estimate
α_2	The death rate due to the disease	0.02–0.5 day^{-1}	Estimate
γ	The state transmission rate from the infected to the susceptible one	0.006–0.99 day^{-1}	Estimate
τ	The rate that message become outdated.	0.03–0.6 year^{-1}	[13]

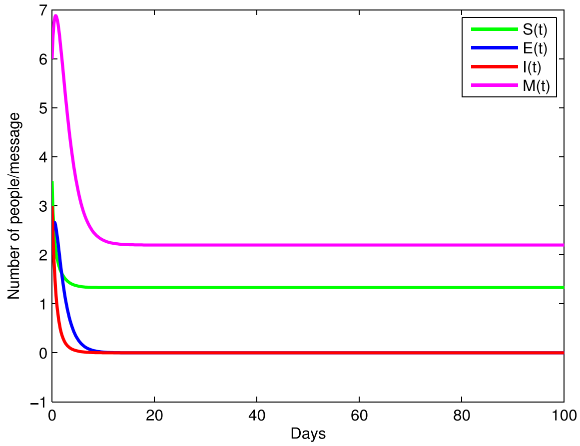

Fig. 2. Disease-free equilibrium $P_0 = (1.3333, 0, 0, 2.2)$ of system (2.1) is globally asymptotically stable when $\mathcal{R}_0 = 0.0884 < 1$.

Therefore, we get the following system:

$$\begin{cases} \dot{S} &= 0.8 + 0.006I - 0.8SIe^{-0.091M} - 0.2S, \\ \dot{E} &= 0.8SIe^{-0.091M} - 0.4E - 0.2E, \\ \dot{I} &= 0.04E - 0.006I - (0.2 + 0.02)I, \\ \dot{M} &= 0.2S + 0.8E + 0.8I - 0.6M. \end{cases} \tag{4.2}$$

It is ready to verify that $\mathcal{R}_0 = 2.4882 > 1$. Then, from Theorem 3.4.1, the endemic equilibrium ${P_1}^*$ of system (2.1) is locally asymptotically stable when $\mathcal{R}_0 > \max(1, \mathcal{R}_{01})$, where $\mathcal{R}_{01} = 1.6833$ (Fig. 3).

Since

$$\mathcal{R}_0 = \frac{\beta\rho A e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}}}{\alpha_1(\alpha_1 + \rho)(\gamma + \alpha_1 + \alpha_2)}.$$

So we can see $\mathcal{R}_0$ is an aggregate of parameters in the model. According to [23], a bifurcation parameter β in $\mathcal{R}_0$ is allowed to vary, and hence $\mathcal{R}_0$ itself varies. Thus we choose a set of parameters in $\mathcal{R}_0$:

$$A = 0.8day^{-1}, \alpha_1 = 0.6day^{-1}, \alpha = 0.08message^{-1}, \mu_1 = 0.99day^{-1}, \tau = 0.6day^{-1},$$
$$\alpha_2 = 0.02day^{-1}, \gamma = 0.7day^{-1}, \rho = 0.09day^{-1}, \beta \in \{0.035, 0.275\}person^{-1}day^{-1},$$

Therefore, $\mathcal{R}_0 \in \{0.90, 1.25\}$.

Fig. 4 suggests a forward bifurcation happens when $\mathcal{R}_0$ crosses unity from below for system (2.1). A small positive asymptotically stable equilibrium appears and the disease-free equilibrium losses its stability. The $x - axis$ shows $\mathcal{R}_0$, the average number of new infectious produced by an infectious individual near the disease-free equilibrium. In system (2.1),

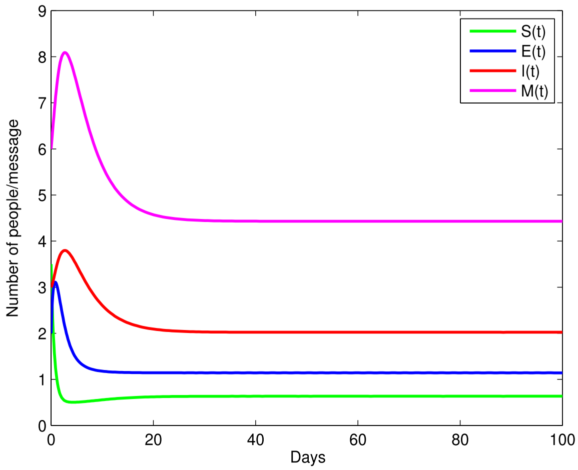

Fig. 3. The endemic equilibrium $P_1{}^*$ of system (2.1) is locally asymptotically stable when $\mathcal{R}_0 > \max(1, \mathcal{R}_{01})$, where $\mathcal{R}_{01} = 1.6833$.

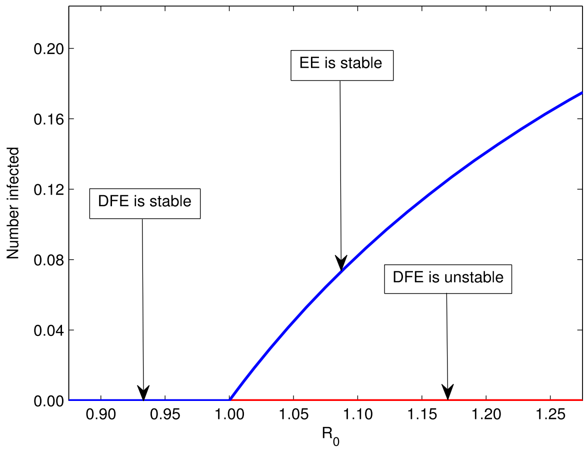

Fig. 4. A forward bifurcation. The curve shows the non-trivial bifurcating equilibrium. The arrows show the direction of flow for the model disease system, after the system reaches the manifold on which slow dynamics occur; When $\mathcal{R}_0 < 1$, the disease-free equilibrium is globally asymptotically stable; When $\mathcal{R}_0 > 1$, the disease-free equilibrium is unstable and there is a stable endemic equilibrium.

if everyone is susceptible at the disease-free equilibrium, $\mathcal{R}_0$ would be $\frac{\beta e^{-\alpha M}}{\rho}$. For a given set of parameters, $\mathcal{R}_0$ remains fixed, and the change in the number of infectives is shown by the diagram. When $\mathcal{R}_0 < 1$, the number of infectives reduces to zero, while when $\mathcal{R}_0 > 1$, the number of infectives will increase or decrease to the curved line that marks the endemic equilibrium. At the same time, the absence of positive equilibrium near the disease-free when $\mathcal{R}_0 < 1$, in other words, the disease-free equilibrium is often the only equilibrium for $\mathcal{R}_0 < 1$, and a low level of endemicity when $\mathcal{R}_0$ is slightly above 1. Fig. 5 implies a backward bifurcation happens when $\mathcal{R}_0$ is less than unity. A small positive unstable equilibrium appears while the disease-free equilibrium and a larger positive equilibrium are locally asymptotically stable. Epidemiologically, a backward bifurcation shows that it is not enough to only reduce the basic reproductive number to less than one to eliminate a disease and that when $\mathcal{R}_0$ crosses unity, hysteresis takes place.

Then, we elect a set of the following parameters:

$$A = 0.8day^{-1}, \alpha_1 = 0.009day^{-1}, \beta = 0.0099person^{-1}day^{-1}, \alpha = 0.00091message^{-1}, \mu_2 = 0.8day^{-1},$$
$$\mu_1 = 0.008day^{-1}, \mu_3 = 0.8day^{-1}, \tau = 0.03day^{-1}, \alpha_2 = 0.5day^{-1}, \gamma = 0.08day^{-1}, \rho = 0.99day^{-1}.$$

Therefore, we get the following system:

$$\begin{cases} \dot{S} &= 0.8 + 0.08I - 0.0099SIe^{-0.00091M} - 0.009S, \\ \dot{E} &= 0.0099SIe^{-0.00091M} - 0.99E - 0.009E, \\ \dot{I} &= 0.99E - 0.08I - (0.009 + 0.5)I, \\ \dot{M} &= 0.008S + 0.8E + 0.8I - 0.03M. \end{cases} \tag{4.3}$$

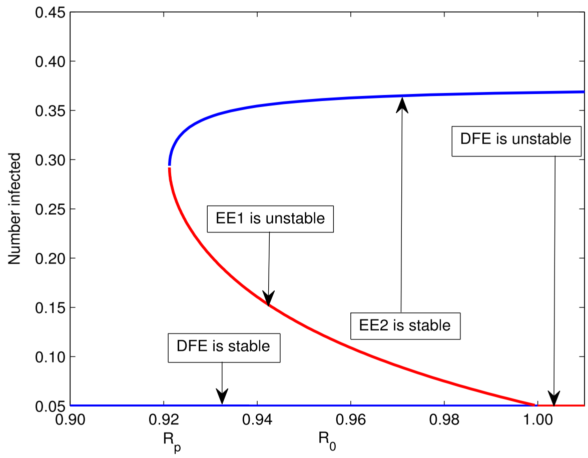

Fig. 5. A backward bifurcation. When $\mathcal{R}_0 < \mathcal{R}_p$, the disease-free equilibrium is globally asymptotically stable. However, when $\mathcal{R}_p < \mathcal{R}_0 < 1$, there are two endemic equilibrium. The upper one is stable and the lower one is unstable and when $\mathcal{R}_0 > 1$, the disease-free equilibrium is unstable.

The endemic equilibrium ${P_1}^*$ of the system (2.1) is locally asymptotically stable when $\mathcal{R}_0 > \max(1, \mathcal{R}_{01})$ and $\beta < \beta^*$ (Fig. 6).

Finally, we choose a set of the following parameters:

$$A = 0.8day^{-1}, \alpha_1 = 0.009day^{-1}, \beta = 0.0099person^{-1}day^{-1}, \alpha = 0.007massage^{-1}, \mu_2 = 0.8day^{-1},$$
$$\mu_1 = 0.008day^{-1}, \mu_3 = 0.8day^{-1}, \tau = 0.03day^{-1}, \alpha_2 = 0.5day^{-1}, \gamma = 0.99day^{-1}, \rho = 0.99day^{-1}.$$

Therefore, we get the following system (2.1):

$$\begin{cases} \dot{S} = 0.8 + 0.99I - 0.0099SIe^{-0.007M} - 0.009S, \\ \dot{E} = 0.0099SIe^{-0.007M} - 0.99E - 0.009E, \\ \dot{I} = 0.99E - 0.99I - (0.009 + 0.5)I, \\ \dot{M} = 0.008S + 0.8E + 0.8I - 0.03M. \end{cases} \tag{4.4}$$

When β passes through the critical value β^{**}, the positive endemic equilibrium ${P_1}^*$ loses its stability and a Hopf bifurcation occurs. This property can be seen from Fig. 7.

5. Sensitivity analysis

In this section, we perform sensitivity analyses of the basic reproductive number $\mathcal{R}_0$ and the infectious individuals I.

First, we perform the sensitivity analysis of the basic reproductive number $\mathcal{R}_0$. We study the influence of α, μ_1 and β to $\mathcal{R}_0$. It is straightforward from (3.4) that $\mathcal{R}_0$ increases as β increases. This agrees with the intuition that higher transmission coefficient increases the basic reproduction number. In order to see the relationship of these parameters and $\mathcal{R}_0$, we regard $\mathcal{R}_0$ as a function about those parameters. Note that

$$\frac{\partial \mathcal{R}_0}{\partial \alpha} = -\frac{\beta\rho A\mu_1}{\mu^2\tau(\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2)}e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}} < 0,$$
$$\frac{\partial \mathcal{R}_0}{\partial \mu_1} = -\frac{\beta\rho A\alpha}{{\alpha_1}^2\tau(\alpha_1+\rho)(\gamma+\alpha_1+\alpha_2)}e^{-\frac{\alpha\mu_1 A}{\alpha_1\tau}} < 0.$$

Therefore, we find that $\mathcal{R}_0$ decreases as α and μ_1 increase. The parameter values are

$$A = 0.8day^{-1}, \alpha = 0.8day^{-1}, \tau = 0.6day^{-1}, \beta = 0.8person^{-1}day^{-1}, \rho = 0.4day^{-1},$$
$$\mu_1 = 0.2day^{-1}, \mu_2 = 0.8day^{-1}, \mu_3 = 0.8day^{-1}, \alpha_1 = 0.2day^{-1}, \alpha_2 = 0.02day^{-1}, \gamma = 0.6day^{-1}.$$

Fig. 8(a) shows that the basic reproduction $\mathcal{R}_0$ is reducing when α is increasing. Fig. 8(b) suggests that the basic reproduction $\mathcal{R}_0$ is reducing when μ_1 is increasing.

Second, we main consider the effect of α, μ_1, τ, γ on the dynamics of infected individuals. The parameters are

$$A = 0.8day^{-1}, \alpha = 0.8day^{-1}, \tau = 0.6day^{-1}, \beta = 0.8person^{-1}day^{-1}, \rho = 0.4day^{-1},$$
$$\mu_1 = 0.2day^{-1}, \mu_2 = 0.8day^{-1}, \mu_3 = 0.8day^{-1}, \alpha_1 = 0.2day^{-1}, \alpha_2 = 0.02day^{-1}, \gamma = 0.6day^{-1}.$$

From Fig. 9, we know that infected number will decrease when α, μ_1 and γ increase, and increase when τ increases.

From Figs. 8 and 9, we find that media coverage has a great impact on the transmission of epidemic diseases.

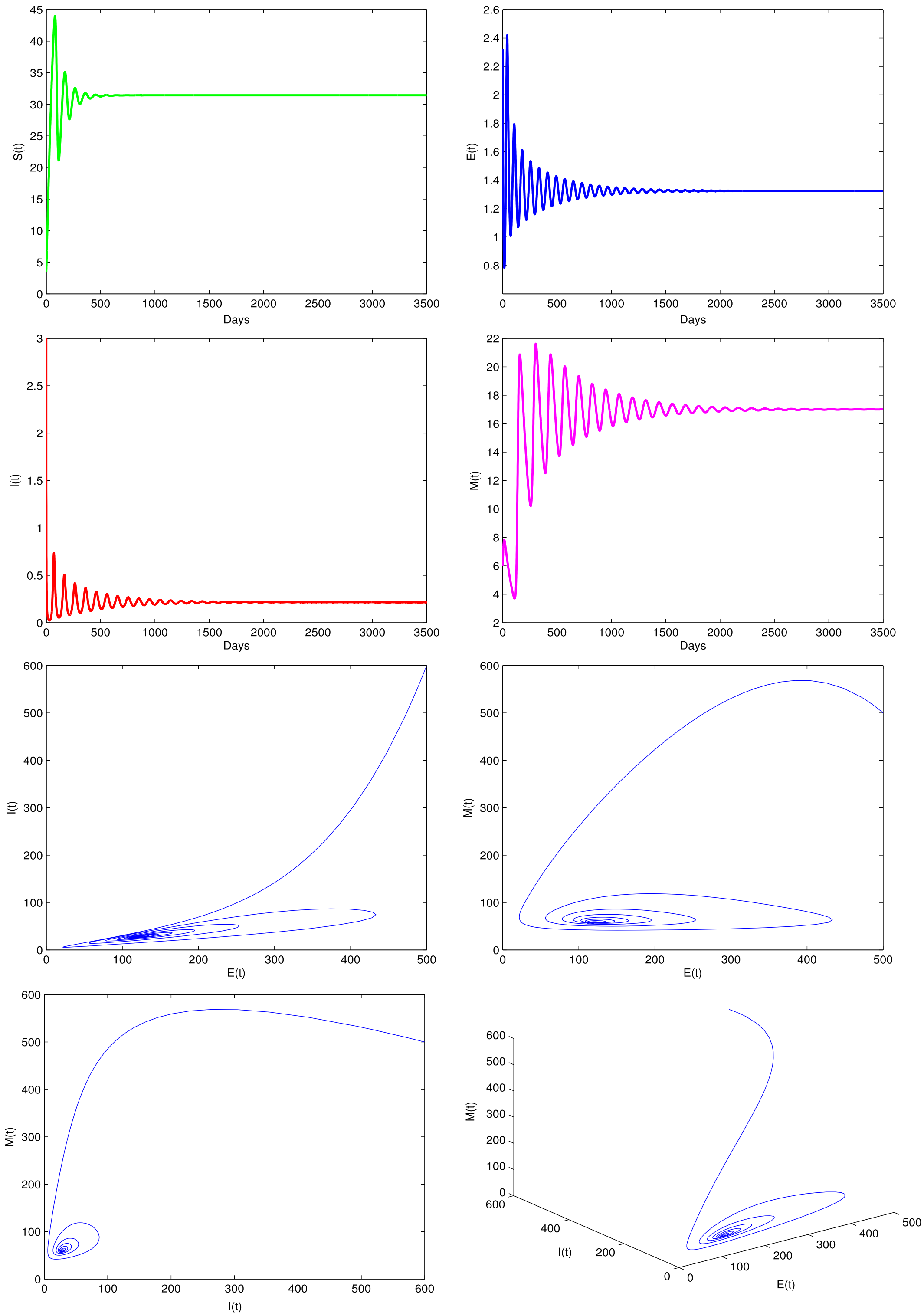

Fig. 6. The endemic equilibrium $P_1{}^*$ of the system (2.1) is locally asymptotically stable when $\mathcal{R}_0 > \max(1, \mathcal{R}_{01})$ and $\beta < \beta^*$.

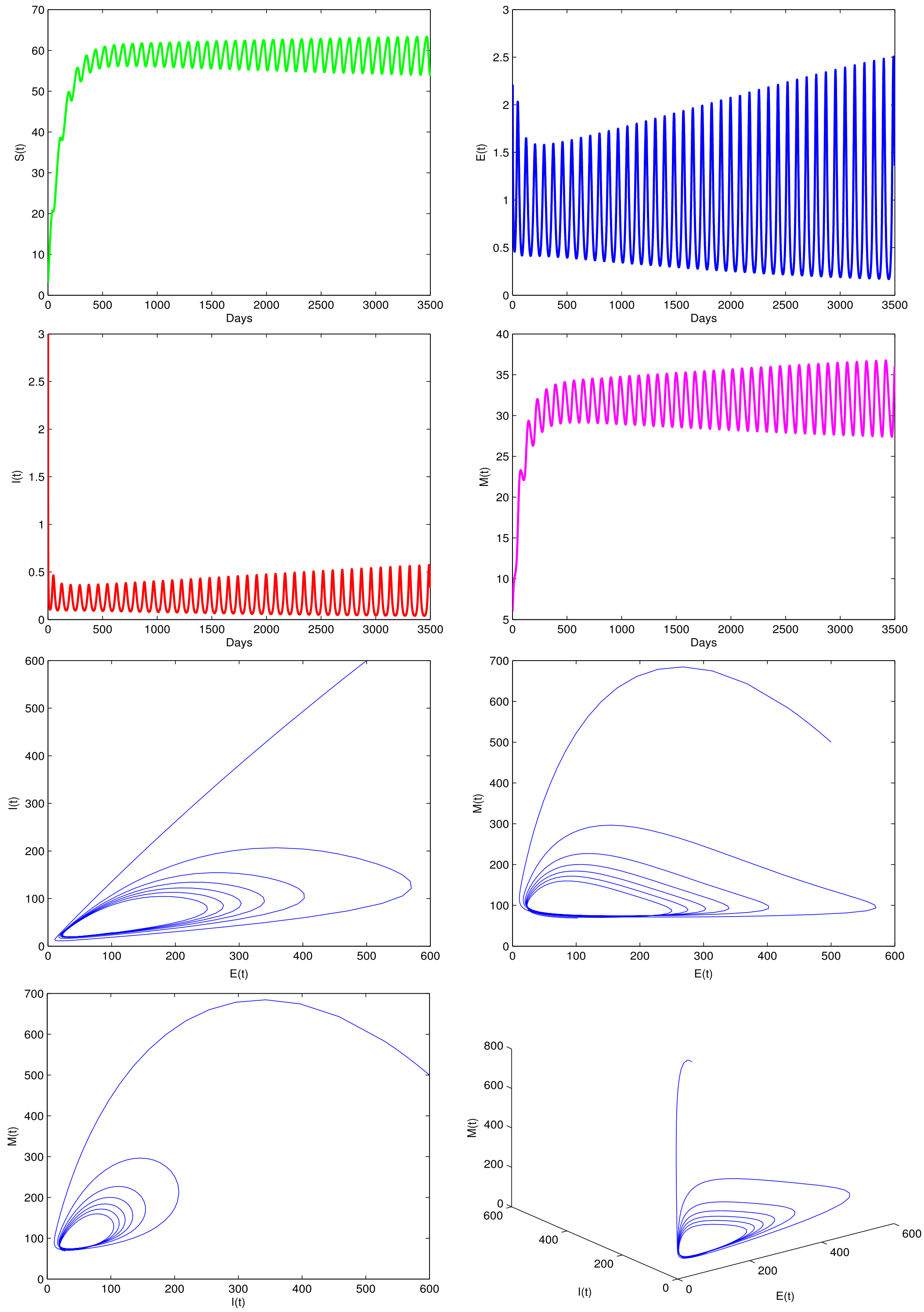

Fig. 7. The endemic equilibrium $P_1{}^*$ of the system (2.1) occurs a Hopf bifurcation when $\mathcal{R}_0 > \max(1, \mathcal{R}_{01})$ and $\beta > \beta^*$. In other word, the system (2.1) converges to a sustained periodic solution.

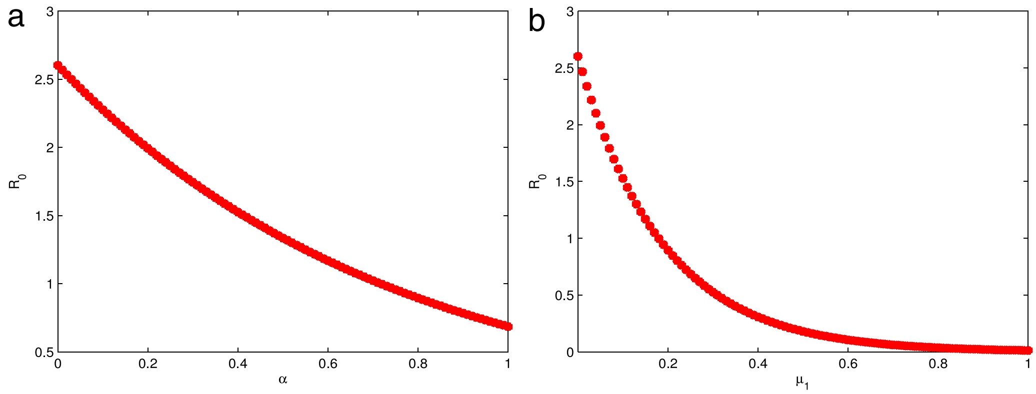

Fig. 8. The effect of media-related parameters on the dynamics of infectious individuals. The parameter that varies is indicated in each figure.

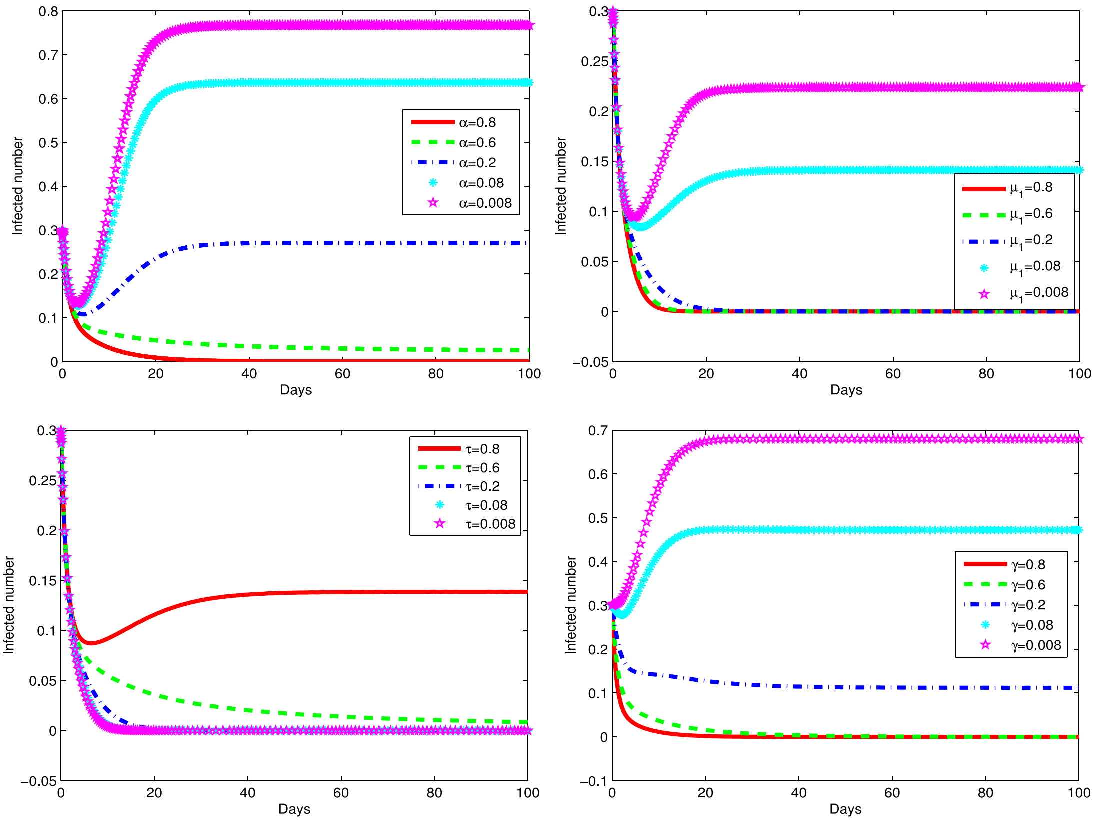

Fig. 9. The effect of message-related parameters on the dynamics of infectious individuals. The parameter that varies is indicated in each figure.

6. Discussion

A new *SEIS* epidemic disease with the impact of media is formulated and stability of the steady states is proved. When $\mathcal{R}_0 < 1$, the disease-free equilibrium is globally asymptotically stable; When $\mathcal{R}_0 > 1$, the disease-free equilibrium is unstable. Meanwhile, When $\mathcal{R}_0 = 1$, a forward and backward bifurcation occur, which show a more and more complicated dynamics behavior of disease transmission. A Hopf bifurcation occurs when a threshold curve is crossed, which implies the possibility of multiple outbreaks of epidemic disease. Our results show that the media coverage are helpful in reducing the spread of epidemic disease.

If we consider a multi-group model. The system (2.1) can be rewritten:

$$\begin{cases} \dot{S}_i &= A_i + \gamma_i I_i - S_i \sum_{j=1}^{n} \beta_{ij} e^{-\alpha_{ij} M_{ij}} I_j - \mu_i S_i, \\ \dot{E}_i &= S_i \sum_{j=1}^{n} \beta_{ij} e^{-\alpha_{ij} M_{ij}} I_j - \rho_i E_i - \mu_i E_i, \\ \dot{I}_i &= \rho_i E_i - \gamma_i I_i - (\mu_i + d_i) I_i, \\ \dot{T}_i &= \mu_{1i} S + \mu_{2i} E + \mu_{3i} I_i - \tau_i M_i. \end{cases} \tag{6.1}$$

Here S_i, E_i, I_i and M_i are the number of susceptible individuals, exposed individuals, infective individuals, messages, respectively, in group i. The other parameters have the same as meaning of the system (2.1). We leave these works for the future.

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