UCL Institute of Health Informatics

Generalized Linear Models

Session 1 – Introduction

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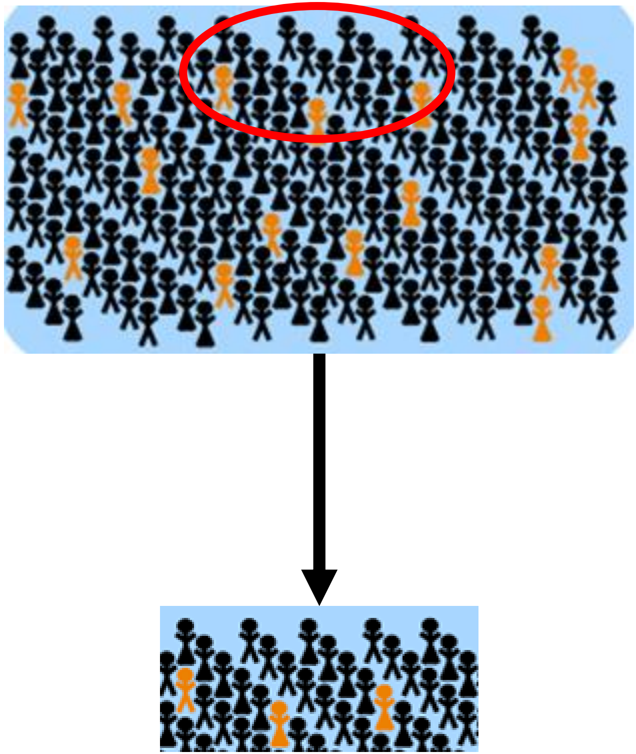
Today’s objectives

This session will introduce you to generalised linear models (GLMs). By the end of the morning you will be able to:

- Define the components of a GLM
- Identify when to use a GLM to model an outcome
- Fit GLMs in Stata for normal, poisson and binomial outcomes
- Interpret output from a GLM in Stata

MODEL

RECAP



Given a large, random sample of n of a population where a proportion (π) have some event – the proportion of the sample with the event, K, will follow an Bin(n, π) distribution.

MODEL

PARAMETERS

RECAP

observable

$$K \sim bin(n, \pi)$$

unknown

MODEL

PARAMETERS

RECAP

ESTIMATORS

Estimat<u>ator</u> for π: $\hat{\pi} = \frac{K}{n}$

Estim<u>ate</u> for π: $\frac{k}{n}$, where k is the observed K.

MODEL

PARAMETERS

RECAP

ESTIMATORS

simple hypothesis
(π=0.5 vs π=0.9)
composite hypothesis
(π <= 0.5 vs π > 0.5)

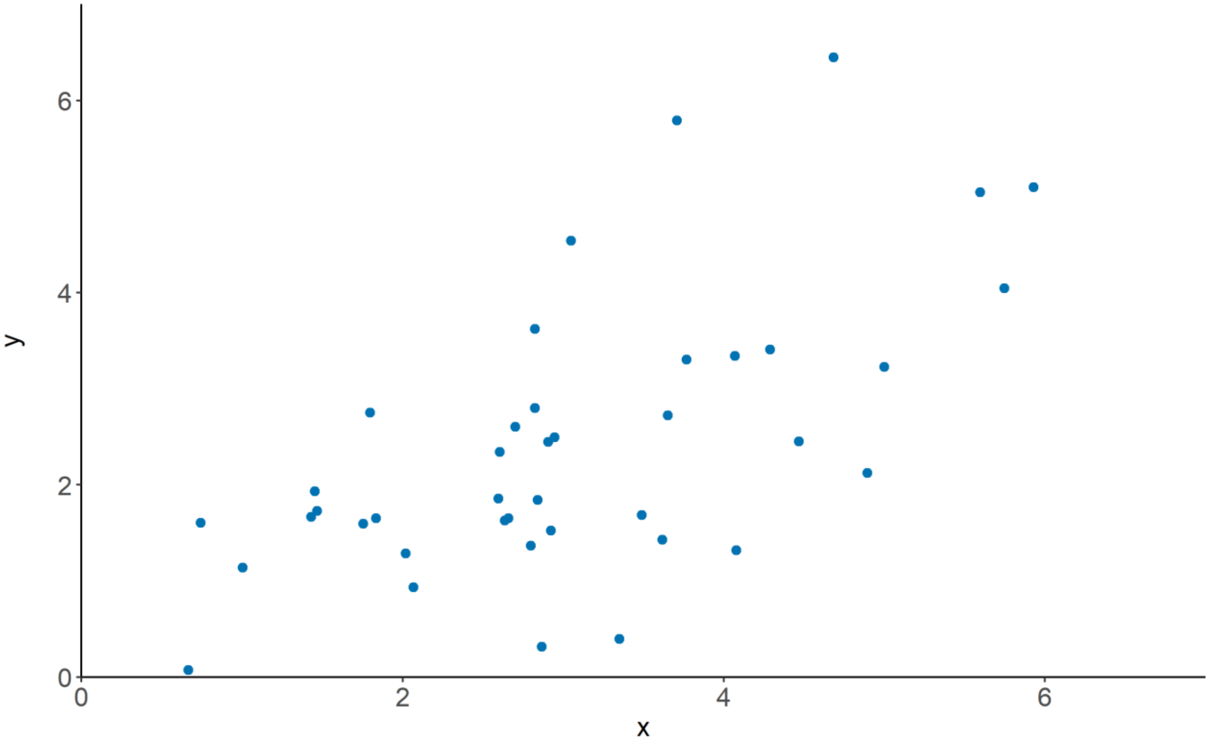
Standard Errors:

- measure of the precision with which the sample statistics approximates the true population

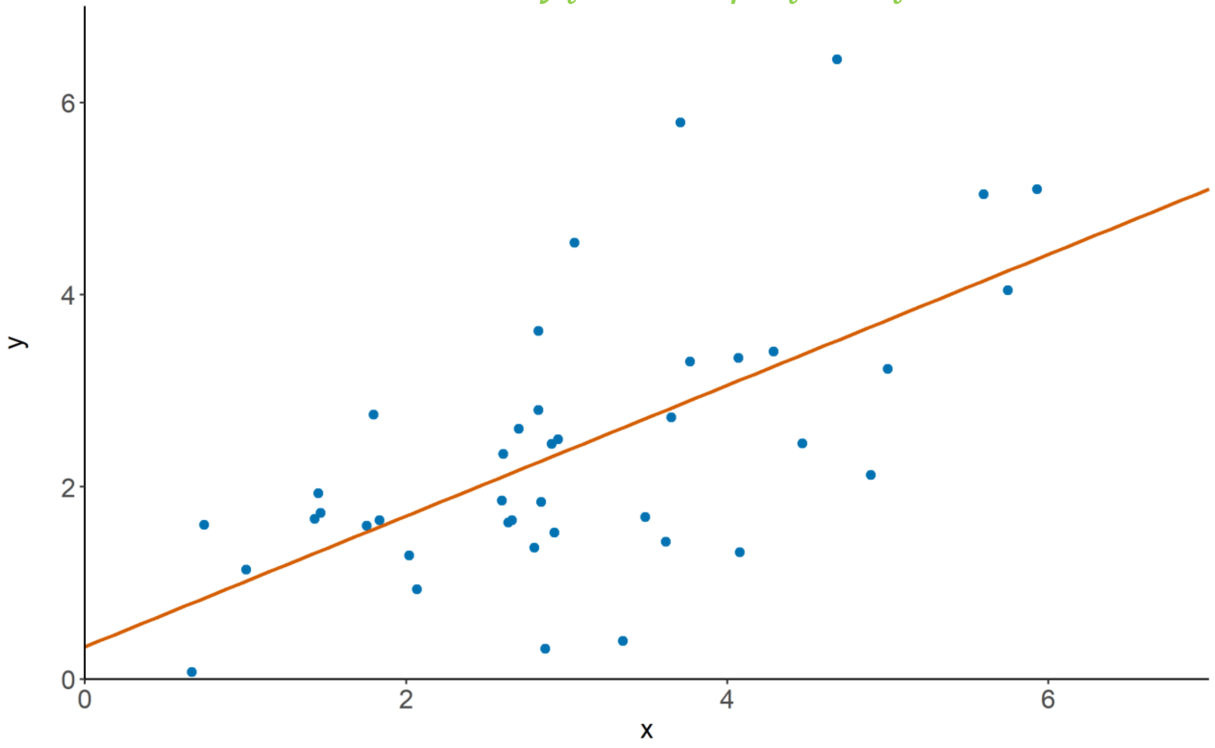
- There are different standard error formulae for different statistical measures (means, percentages, ORs)

95% Confidence Intervals:

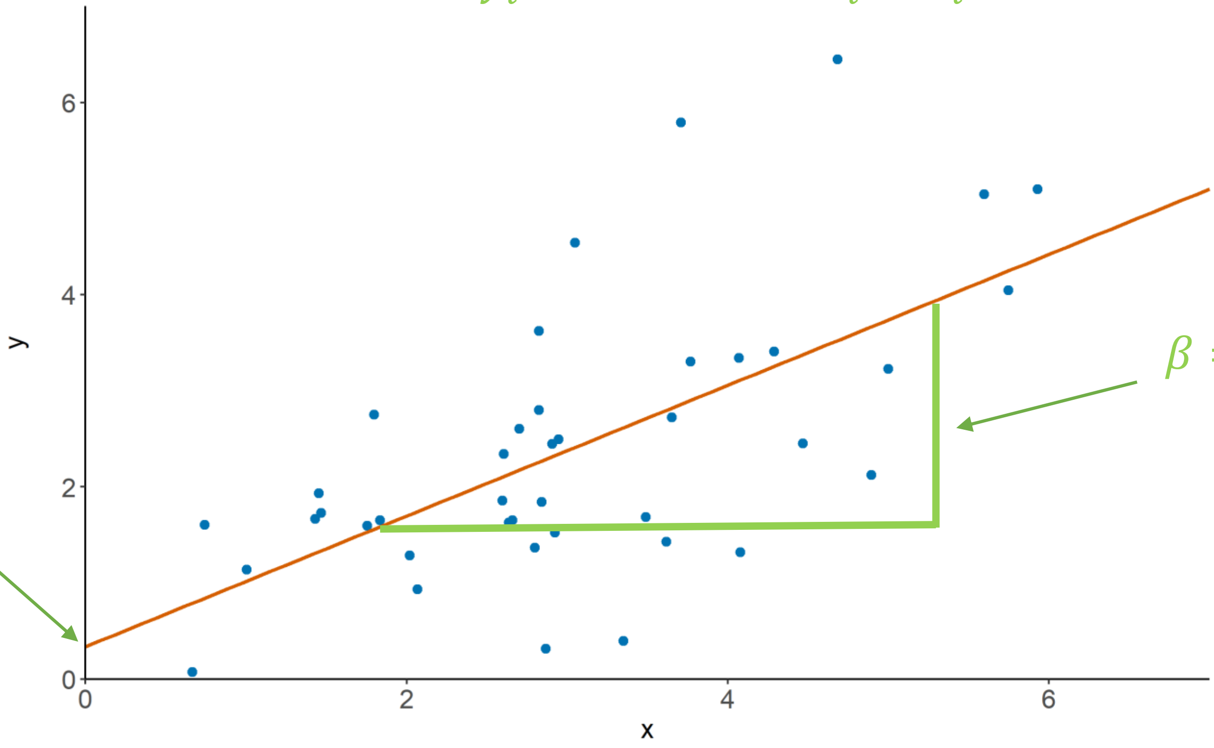
The interval (sample statistic +/- 1.96 SE) will contain the 'true value' for 95% of random samples


y
6
4
2
0
0
2
4
6
x

$$y_i = \alpha + \beta x_i + \varepsilon_i$$



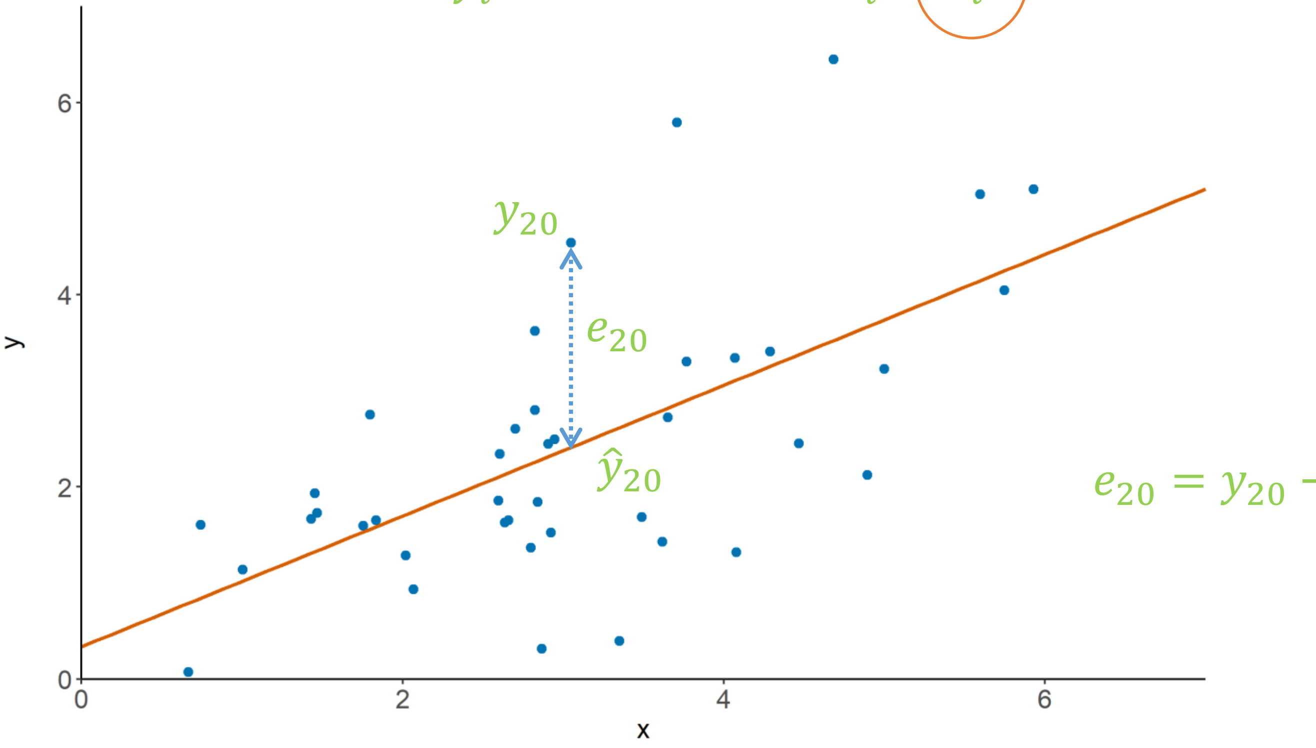
$$y_i = 0.29 + 0.65x_i + \varepsilon_i$$



$\alpha = 0.29$

$\beta = 0.65$

$$y_i = 0.29 + 0.65 * x_i + e_i$$



$$e_{20} = y_{20} - \hat{y}_{20}$$

$$\boldsymbol{y_i = \alpha + \beta_1 x_{1i} + \cdots + \beta_n x_{ni} + \varepsilon_i}$$

$$bmi_i = \alpha + \beta_1 age_i + \beta_2 sex_i + \varepsilon_i$$

```
. reg bmi age i.sex

      Source |       SS           df       MS      Number of obs   =     2,014
-------------+----------------------------------   F(2, 2011)      =    277.51
       Model |  7460.15291         2  3730.07645   Prob > F        =    0.0000
    Residual |  27029.9391     2,011  13.4410438   R-squared       =    0.2163
-------------+----------------------------------   Adj R-squared   =    0.2155
       Total |   34490.092     2,013  17.1336771   Root MSE        =    3.6662

------------------------------------------------------------------------------
         bmi |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   -.150729   .0064041   -23.54   0.000    -.1632882   -.1381697
       1.sex |  -.3156341   .1637774    -1.93   0.054    -.6368253    .0055571
       _cons |   28.87251   .3851415    74.97   0.000     28.11719    29.62783
------------------------------------------------------------------------------
```

$$\boldsymbol{bmi_i = \alpha + \beta_1 age_i + \beta_2 sex_i + \varepsilon_i}$$

```
. reg bmi age i.sex
```

Holding sex constant, a unit increase in age (i.e. a year) is associated with a 0.15 kg/m^2 decrease in BMI (95% CI: -0.16 to -0.14).

bmi	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
age	-.150729	.0064041	-23.54	0.000	-.1632882	-.1381697
1.sex	-.3156341	.1637774	-1.93	0.054	-.6368253	.0055571
_cons	28.87251	.3851415	74.97	0.000	28.11719	29.62783

$$bmi_i = \alpha + \beta_1 age_i + \beta_2 sex_i + \varepsilon_i$$

```
. reg bmi age i.sex
```

Holding age constant, there is weak evidence that being female is associated with a 0.32 kg/m^2 lower BMI (95% CI: -0.64 to 0.01).

bmi	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
age	-.150729	.0064041	-23.54	0.000	-.1632882	-.1381697
1.sex	-.3156341	.1637774	-1.93	0.054	-.6368253	.0055571
_cons	28.87251	.3851415	74.97	0.000	28.11719	29.62783

$$bmi_i = \alpha + \beta_1 age_i + \beta_2 sex_i + \varepsilon_i$$

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    Residual |  27029.9391     2,011  13.4410438   R-squared       =    0.2163
-------------+----------------------------------   Adj R-squared   =    0.2155
       Total |   34490.092     2,013  17.1336771   Root MSE        =    3.6662
```

Age and Sex explain an estimated 22% of the variability in BMI

Assumptions of linear regression:

- $Y|x \sim N(\alpha + \beta x, \sigma^2)$
- The relationship between Y and the X's is linear
- The residuals (error terms) are normally distributed i.e. $e_i \sim N(0, \sigma^2)$
- The residuals are independent one of another
- The variance of the residuals is constant, independent of x's

- For many health-related outcomes, these assumptions will not be appropriate
- For example, number of days spent in hospital or whether a patient has diabetes or not

Generalized linear models *generalize* linear modelling and allow us to model a wider variety of outcome types.

The **Exponential Family** of distributions

$$ln\{f(y)\} = \frac{y\theta - b(\theta)}{\varphi} - c(y, \varphi)$$

$$ln\{f(y)\} = \frac{y\theta - b(\theta)}{\varphi} - c(y, \varphi)$$

$$f(y) = \sqrt{\frac{1}{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\}$$

$$ln\big(f(y)\big) = ln\left(\sqrt{\frac{1}{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\}\right)$$

$$ln\{f(y)\} = \frac{y\theta - b(\theta)}{\varphi} - c(y, \varphi)$$

$$ln(f(y)) = ln\left(\sqrt{\frac{1}{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2}(y - \mu)^2$$

$$ln(f(y)) = -\frac{1}{2}ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y^2 - 2y\mu + \mu^2)$$

$$ln(f(y)) = \frac{y\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \left(\frac{y^2}{2\sigma^2} + \frac{1}{2}ln(2\pi\sigma^2)\right)$$

$$ln\{f(y)\} = \frac{y\theta - b(\theta)}{\varphi} - c(y,\varphi)$$

Distributions that can be written in this form belong to the exponential distribution of families

$$\theta = \mu, \qquad b(\theta) = \frac{\mu^2}{2}, \qquad \varphi = \sigma^2,$$

$$c(y,\varphi) = \left(\frac{y^2}{2\sigma^2} + \frac{1}{2}ln(2\pi\sigma^2)\right)$$

$$ln\{f(y)\} = \frac{y\theta - b(\theta)}{\varphi} - c(y, \varphi)$$

The Poisson distribution:

$$f(y) = \Pr(Y = y) = \frac{\mu^y e^{-\mu}}{y!}, y = 0,1,2,3 \ldots$$

$$\ln\big(f(y)\big) = \mathrm{y}\ln(\mu) - \mu - \ln(y!)$$

$$\theta = \ln(\mu), \qquad b(\theta) = \mu, \qquad \varphi = 1, \qquad c(y, \varphi) = \ln(y!)$$

$$ln\{f(y)\} = \frac{y\theta - b(\theta)}{\varphi} - c(y, \varphi)$$

The Binomial distribution:

$$f(y) = \Pr(Y = y) = \binom{n}{y} \pi^y (1-\pi)^{n-y}, y = 0,1, \dots, n$$

$$\ln\big(f(y)\big) = \text{yln}\left(\frac{\pi}{1-\pi}\right) + \text{nln}(1-\pi) + ln\left\{\binom{n}{y}\right\}$$

$$\theta = \ln\left(\frac{\pi}{1-\pi}\right), \qquad b(\theta) = -\text{nln}(1-\pi), \qquad \varphi = 1,$$

$$c(y, \varphi) = -ln\left\{\binom{n}{y}\right\}$$

Three GLM components:

- Response distribution

  The $Y_i's$ $i = 1, \ldots, n$ are assumed to be independent and arising from an exponential family. $E(Y_i) = \mu_i$

- Linear predictor

$$n_i = \alpha + \beta_1 x_{1i} + \cdots + \beta_n x_{ni}$$

- Link function

$$g(\mu_i) = n_i$$

Exponential family	Canonical (natural) link	Response variable(s)	examples
Normal (gaussian)	y	Continuous numbers $(-\infty, \infty)$	Heights, weights, blood pressure, other lab values
Poisson	log(y) = log(μ)	Counts (0,1,2,3,4….) Rates [0,∞)	Counts, rates (e.g. average inpatient length of stay, number of emergency admissions…)
Binomial	logit(y) = $\log(\frac{\pi}{1-\pi})$	Binary [0,1] – logistic regression	i.e. “case” vs “non-case” (e.g. death, disease, receiving an intervention…)

```
. reg bmi age i.sex

      Source |       SS           df       MS
-------------+----------------------------------
       Model |  7460.15291         2  3730.07645
    Residual |  27029.9391     2,011  13.4410438
-------------+----------------------------------
       Total |   34490.092     2,013  17.1336771

------------------------------------------------------
         bmi |      Coef.   Std. Err.      t    P
-------------+----------------------------------------
         age |   -.150729   .0064041   -23.54   0.
       1.sex |  -.3156341   .1637774    -1.93   0.
       _cons |   28.87251   .3851415    74.97   0
------------------------------------------------------
```

```
. glm bmi age i.sex, family(gaussian) link(identity)

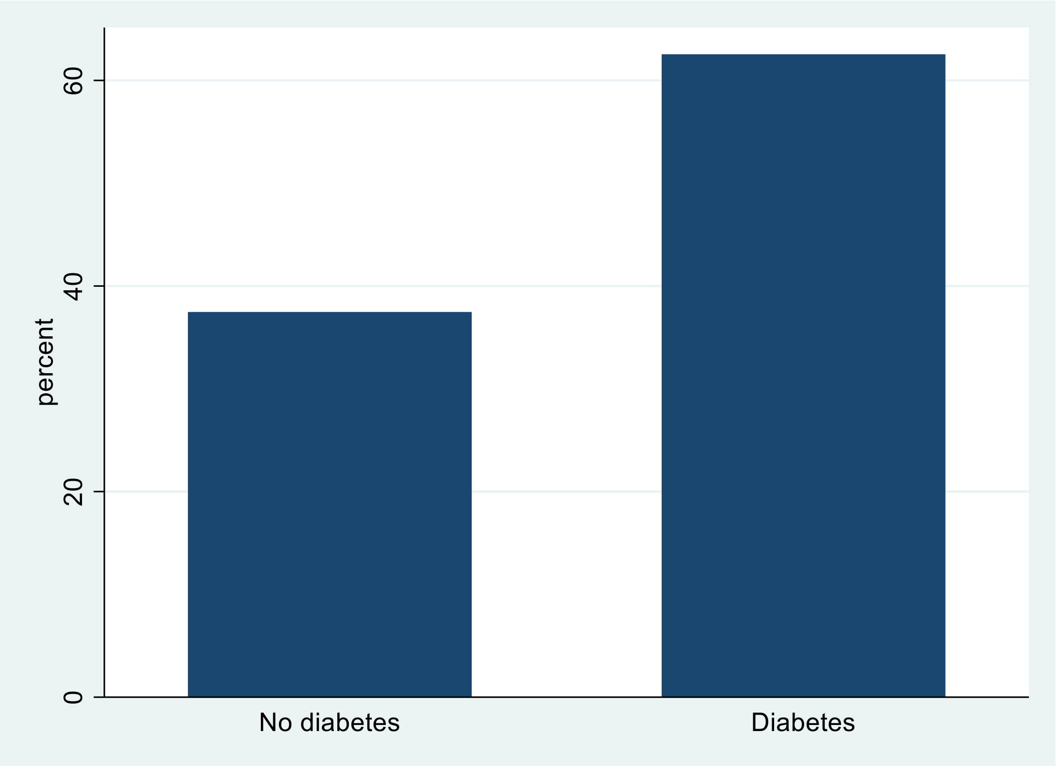
Iteration 0:   log likelihood = -5472.7423

Generalized linear models                         No. of obs      =      2,014
Optimization     : ML                             Residual df     =      2,011
                                                  Scale parameter =   13.44104
Deviance         =  27029.93914                   (1/df) Deviance =   13.44104
Pearson          =  27029.93914                   (1/df) Pearson  =   13.44104

Variance function: V(u) = 1                       [Gaussian]
Link function    : g(u) = u                       [Identity]

                                                  AIC             =   5.437679
Log likelihood   = -5472.742277                   BIC             =    11730.5

------------------------------------------------------------------------------
             |                 OIM
         bmi |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   -.150729   .0064041   -23.54   0.000    -.1632807   -.1381772
       1.sex |  -.3156341   .1637774    -1.93   0.054     -.636632    .0053637
       _cons |   28.87251   .3851415    74.97   0.000     28.11765    29.62738
------------------------------------------------------------------------------
```


percent
0
20
40
60
No diabetes
Diabetes

```
. glm diabetes age sex, family(binomial) link(logit)

Iteration 0:   log likelihood = -3302
Iteration 1:   log likelihood = -3297
Iteration 2:   log likelihood = -3297
Iteration 3:   log likelihood = -3297

Generalized linear models
Optimization     : ML

Deviance         =  6594.233323
Pearson          =  4999.872144

Variance function: V(u) = u*(1-u)
Link function    : g(u) = ln(u/(1-u))


Log likelihood   = -3297.116661
```

Odds Ratio = exp(0.0411039) = 1.042

95% C.I. 1.023 to 1.061

Holding sex constant, there is evidence that, for a 1 year increase in age, the odds of developing diabetes increases by 4% (95% C.I. of 2 to 6%).

Log Odds Ratio

diabetes	Coef.	OIM Std. Err.	z	P>|z|	[95% Conf.	Interval]
age	.0411039	.0093775	4.38	0.000	.0227245	.0594834
sex	-.0154755	.0592913	-0.26	0.794	-.1316844	.1007334
_cons	-2.356755	.6561699	-3.59	0.000	-3.642824	-1.070686

```
. glm diabetes age sex, family(binomial) link(logit)

Iteration 0:   log likelihood = -3302
Iteration 1:   log likelihood = -3297
Iteration 2:   log likelihood = -3297
Iteration 3:   log likelihood = -3297

Generalized linear models
Optimization     : ML

Deviance         =  6594.233323
Pearson          =  4999.872144

Variance function: V(u) = u*(1-u)
Link function    : g(u) = ln(u/(1-u))


Log likelihood   = -3297.116661
```

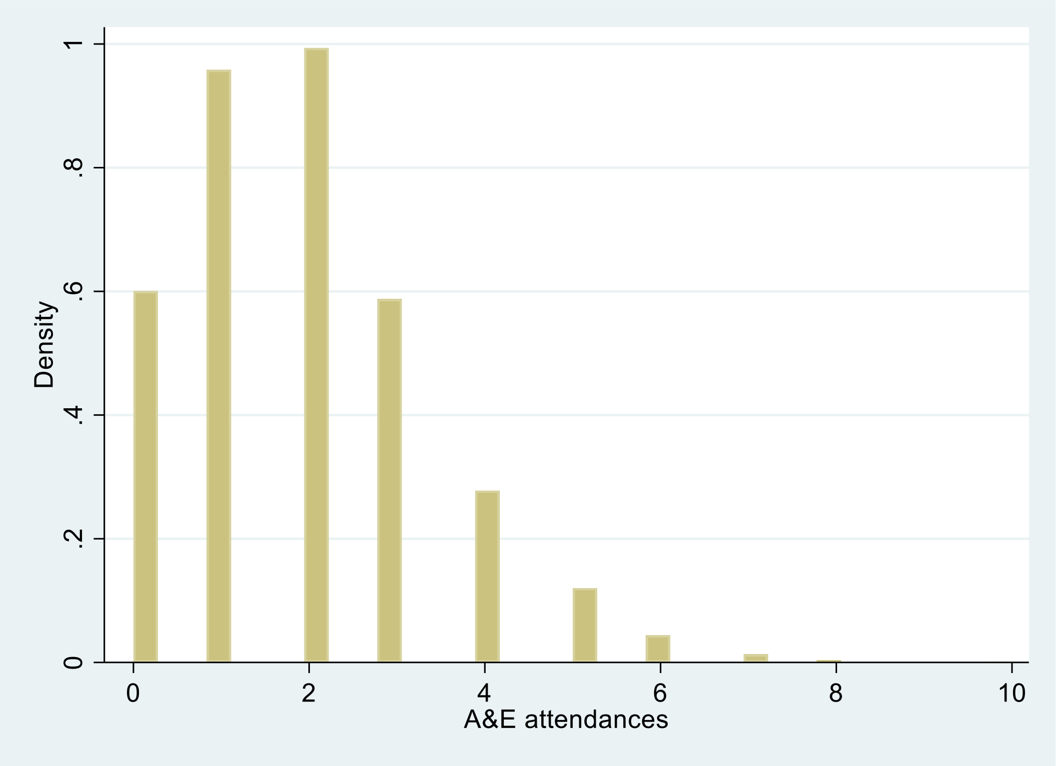
Odds Ratio = exp(-0.0154755) = 0.985

95% C.I. 0.877 to 1.106

Holding age constant, being female is associated with a 1% decrease in the odds of developing diabetes (95% C.I. of -12% to +11%).

Log Odds Ratio

diabetes	Coef.	OIM Std. Err.	z	P>|z|	[95% Conf.	Interval]
age	.0411039	.0093775	4.38	0.000	.0227245	.0594834
sex	-.0154755	.0592913	-0.26	0.794	-.1316844	.1007334
_cons	-2.356755	.6561699	-3.59	0.000	-3.642824	-1.070686


Density
0
.2
.4
.6
.8
1
0
2
4
6
8
10
A&E attendances

```
. glm ae_atd age i.sex, family(poisson) link(log)

Iteration 0:   log likelihood = -8522.4235
Iteration 1:   log likelihood
Iteration 2:   log likelihood
Iteration 3:   log likelihood

Generalized linear models
Optimization     : ML

Deviance         =  6127.4037
Pearson          =  5375.7933

Variance function: V(u) = u
Link function    : g(u) = ln(

Log likelihood   = -8499.5734
```

Risk Ratio = exp(0.0101182) = 1.01

95% C.I. 1.00 to 1.02

Holding sex constant, there is evidence that, for a 1 year increase in age, the risk of A&E attendances increases by 1% (95% C.I. of 0 to 2%).

Log Risk Ratio

ae_atd	Coef.	OIM Std. Err.	z	P>|z|	[95% Conf.	Interval]
age	.0101182	.0032787	3.09	0.002	.003692	.0165444
1.sex	.0095216	.0207984	0.46	0.647	-.0312425	.0502857
_cons	-.0742624	.2300892	-0.32	0.747	-.525229	.3767041

```
. glm ae_atd age i.sex, family(poisson) link(log)

Iteration 0:   log likelihood = -8522.4235
Iteration 1:   log likelihood = -8499.5987
Iteration 2:   log likelihood
Iteration 3:   log likelihood

Generalized linear models
Optimization     : ML

Deviance         =  6127.4037
Pearson          =  5375.7933

Variance function: V(u) = u
Link function    : g(u) = ln(


Log likelihood   = -8499.5734
```

Risk Ratio = exp(0.0095216) = 1.01

95% C.I. 0.97 to 1.05

Holding age constant, being female is associated with a 1% increase in risk of A&E attendances (95% C.I. of -3% to +5%).

Log Risk Ratio

ae_atd	Coef.	OIM Std. Err.	z	P>|z|	[95% Conf.	Interval]
age	.0101182	.0032787	3.09	0.002	.003692	.0165444
1.sex	.0095216	.0207984	0.46	0.647	-.0312425	.0502857
_cons	-.0742624	.2300892	-0.32	0.747	-.525229	.3767041

Summary

- We have recapped some of the key definitions underpinning statistical inference
- We have recapped linear regression modelling
- We have learnt about the exponential family of distributions and its properties
- We have learnt about GLMS and its three key components
- We have learnt how to use GLMs to model binary and count outcomes
- We have learnt how to interpret parameter estimates in GLMs

References

- Nelder, J., & Wedderburn, R. (1972). Generalized Linear Models. *Journal of the Royal Statistical Society. Series A (General),135*(3), 370-384. doi:10.2307/2344614

- Dobson AJ. *An Introduction to Generalized Linear Models*. Chapman & Hall/CRC: Boca Raton, FL, 1990.

Fitting GLMs in Stata:

glm <response variable> <explanatory variables to form linear predictor>, family (<name of distribution>) link(<link function>).