# 1 Chapter 12 Inference in a Changing World

**Problem 12.5** *A time series $y_1, \dots, y_t$ is said to asymptote if it monotonically increases or decreases, but as $t$ grows very large, approaches a finite value.*

(a) Explain intuitively why the standard deviation asymptotes.

(b) Prove that in our treatment of the HMM in section 12.1, the variance of the posterior asymptotes at

$$\sigma^2_{\text{posterior}} = \frac{\sigma_s^2}{2}\left(\sqrt{1+\frac{4\sigma^2}{\sigma_s^2}} - 1\right).$$

## Answer

(a)
As we accumulate more observations in a time series, the influence of each individual measurement becomes less significant.And in the context of the Hidden Markov Model (HMM), the uncertainty about the hidden state is reduced by integrating evidence across time.
However, the observations are noisy and conditionally independent given the hidden state, so the information gain saturates over time. Eventually, the posterior variance approaches a stable limit, which represents the best achievable precision given the balance of process noise and observation noise.

(b)
Since the question is asking when posterior is asymptoted, that's where we entered a steady state of the recursive variance update.
We can name the steady-state posterior variance as $x = \sigma^2_{\text{posterior}}$ Let $x = \sigma^2_{\text{posterior}}$. The fixed-point equation (steady-state) is:

$$x = \left(\frac{1}{\sigma_s^2 + x} + \frac{1}{\sigma^2}\right)^{-1}$$

$$\frac{1}{x} = \frac{1}{\sigma_s^2 + x} + \frac{1}{\sigma^2}$$

Multiply through by $x(\sigma_s^2 + x)\sigma^2$:

$$\sigma_s^2\sigma^2 = x\sigma_s^2 + x^2$$

$$x^2 + \sigma_s^2 x - \sigma_s^2\sigma^2 = 0$$

$$x = \frac{-\sigma_s^2 + \sqrt{\sigma_s^4 + 4\sigma_s^2\sigma^2}}{2} = \frac{\sigma_s^2}{2}\left(-1 + \sqrt{1+\frac{4\sigma^2}{\sigma_s^2}}\right)$$

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