1 Chapter 6 Learning as Inference

Problem 6.8 *In section 6.2 we assumed that the binary outcome was independent across days. Assume instead that the probability of rain on a day after a rain day is r_r and after a dry day is r_d.*

1. (a) Derive the posterior probability that it will rain on day $t+1$ after a series of observations $x_1, \ldots, x_t$.
2. (b) Give an intuition for the resulting expression.

Answer

Generative model

We let each day's weather $x_i \in \{0, 1\}$ (dry = 0, rain = 1) follow a first-order dependence:

$$\Pr\bigl(x_i = 1 \mid x_{i-1}\bigr) = \begin{cases} r_r, & x_{i-1} = 1, \\ r_d, & x_{i-1} = 0, \end{cases}$$

for $i = 2, \ldots, t+1$. No prior on x_1 is needed for the predictive step.

(a) Predictive posterior

We seek

$$\Pr\bigl(x_{t+1} = 1 \mid x_{1:t}\bigr) = \sum_{x_t \in \{0,1\}} \Pr\bigl(x_{t+1} = 1 \mid x_t\bigr) \Pr\bigl(x_t \mid x_{1:t}\bigr).$$

Because x_t is observed, $\Pr(x_t \mid x_{1:t}) = 1$ for the actual value of x_t and 0 otherwise. Hence the sum collapses:

$$\Pr\bigl(x_{t+1} = 1 \mid x_{1:t}\bigr) = \Pr\bigl(x_{t+1} = 1 \mid x_t\bigr) = \begin{cases} r_r, & x_t = 1, \\ r_d, & x_t = 0. \end{cases}$$

Equivalently, one can write in closed-form:

$$\Pr\bigl(x_{t+1} = 1 \mid x_{1:t}\bigr) \;=\; r_r^{x_t}\, r_d^{1-x_t}.$$

(b) Intuition

- **B**y assumption only the immediately preceding day x_t affects tomorrow.
- **Two regimes:**
  - After rain ($x_t = 1$), the persistence parameter r_r governs the chance of rain.
  - After dry ($x_t = 0$), the transition parameter r_d governs the chance of rain.
- Earlier days $x_{1:t-1}$ matter only insofar as they determined x_t.