1 Chapter 3: Bayesian Inference under Measurement Noise

Problem 3.4 We take a closer look at the central inference calculation of this chapter, which involves multiplying two Gaussians. The stimulus distribution $p_S(s)$ is Gaussian with mean μ and variance σ_s^2. The measurement distribution $p_{x|s}(x|s)$ is Gaussian with mean s and variance σ^2.

(a) Write down the equations for $p_{x|s}(x|s)$ and $p_S(s)$.

$$p_{x|s}(x|s) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-s)^2}{2\sigma^2}\right)$$

$$p_S(s) = \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left(-\frac{(s-\mu)^2}{2\sigma_s^2}\right)$$

(b) A Bayesian observer infers s from a measurement x_{obs}. Use Bayes' rule to write down the equation for the posterior, $p_{s|x}(s_{hyp}|x_{obs})$. Substitute the expressions for $p_{x|s}(x_{obs}|s_{hyp})$ and $p_S(s_{hyp})$, but do not simplify yet.

According to Bayes' rule:

$$p_{s|x}(s_{hyp}|x_{obs}) = \frac{p_{x|s}(x_{obs}|s_{hyp})p_S(s_{hyp})}{p_X(x_{obs})}$$

Substituting,

$$p_{s|x}(s_{hyp}|x_{obs}) = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_{obs}-s_{hyp})^2}{2\sigma^2}\right) \cdot \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left(-\frac{(s_{hyp}-\mu)^2}{2\sigma_s^2}\right)}{p_X(x_{obs})}$$

The numerator is a product of two Gaussians. As we discussed in section 3.3.3, the denominator, $p_X(x_{obs})$, is a normalization factor that ensures that the integral equals 1. For now, we will ignore it and focus on the numerator.

(c)Apply the rule $e^A e^B = e^{A+B}$ to simplify the numerator.

$$e^A e^B = e^{A+B}$$

Thus, the product of two Gaussian exponential would be:

$$e^{-\frac{(x_{obs}-s)^2}{2\sigma^2} - \frac{(s-\mu)^2}{2\sigma_s^2}}$$

(d) Expand the two quadratic terms in the exponent.

Expanding both quadratic terms:

$$-\frac{x_{obs}^2 - 2x_{obs}s_{hyp} + s_{hyp}^2}{2\sigma^2} - \frac{s_{hyp}^2 - 2\mu s_{hyp} + \mu^2}{2\sigma_s^2}$$

(e) Rewrite the exponent to the form as^2+bs+c, with $a, b,$ and c constants. Importantly, since c is just leading to a constant scaling, no need to calculate it.

To rewrite the exponent in the form as^2+bs+c, we take the exponent obtained in the previous step:

$$-\frac{s_{hyp}^2}{2\sigma^2} + \frac{2x_{obs}s_{hyp}}{2\sigma^2} - \frac{x_{obs}^2}{2\sigma^2} - \frac{s_{hyp}^2}{2\sigma_s^2} + \frac{2\mu s_{hyp}}{2\sigma_s^2} - \frac{\mu^2}{2\sigma_s^2}$$

And rewrite linearly,

$$-\left(\frac{1}{2\sigma^2} + \frac{1}{2\sigma_s^2}\right)s_{hyp}^2 + \left(\frac{x_{obs}}{\sigma^2} + \frac{\mu}{\sigma_s^2}\right)s_{hyp} + (constantterms)$$

(f) Rewrite the expression you obtained in (e) in a simpler form, $e^{c_1(s+c_2)^2+c_3}$, with $c_1, c_2,$ and c_3 constants. Hint: any quadratic function of the form as^2+bs+c can be written as $a\left(s+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}$; this rewriting is known as *completing the square.*

The expression of exponent is:

$$as_{hyp}^2 + bs_{hyp}$$

Thus, from part (e) we can derivate that:

$$a = -\left(\frac{1}{2\sigma^2} + \frac{1}{2\sigma_s^2}\right), \quad b = \frac{x_{obs}}{\sigma^2} + \frac{\mu}{\sigma_s^2}$$

As the question stated that:

$$as^2 + bs + c = a\left(s + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Thus, we can re-write the equation to be:

$$a\left(s_{hyp} + \frac{b}{2a}\right)^2 + c_3$$

Thus,

$$c_3 = c - \frac{b^2}{4a}$$

Then, the equation transforms into:

$$e^{a(s_{hyp}+c_2)^2+c_3}$$

where,

$$c_2 = \frac{b}{2a}, \quad c_1 = a$$

(g) Now rewrite your expression into the form $e^Z e^{\frac{(s-\mu_{combined})^2}{2\sigma^2_{combined}}}$. Express $\mu_{combined}$ and $\sigma_{combined}$ in terms of $x, \mu,$ and σ_s.

From part (f), we obtained that,

$$c_1(s + c_2)^2 + c_3$$

And comparing with desired form, where

$$\frac{(s - \mu_{combined})^2}{2\sigma^2_{combined}}$$

So,

$$c_1 = \frac{1}{2\sigma^2_{combined}}, \quad c_2 = \frac{b}{2a} = -\mu_{combined}$$

Following, from previous parts of this question, we can equate and get that:

$$\frac{1}{2\sigma^2_{combined}} = -\left(\frac{1}{2\sigma^2} + \frac{1}{2\sigma_s^2}\right)$$

$$a = -\left(\frac{1}{2\sigma^2} + \frac{1}{2\sigma_s^2}\right), \quad b = \frac{x_{obs}}{\sigma^2} + \frac{\mu}{\sigma_s^2}$$

Thus,

$$\sigma^2_{combined} = \frac{\sigma^2\sigma_s^2}{\sigma^2 + \sigma_s^2}$$

$$\mu_{combined} = \frac{x_{obs}\sigma_s^2 + \mu\sigma^2}{\sigma^2 + \sigma_s^2}$$

(h) Recall that $p_{s|x}$ is a probability distribution and that its integral should therefore be equal to 1. However, the expression that you obtained in (g) is not properly normalized because we ignored $p_X(x_{obs})$. Modify the expression such that it is properly normalized, but without explicitly calculating $p_X(x_{obs})$ (Hint: does e^Z depend on s?)

Since $p_{s|x}(s|x_{obs})$ is a probability distribution,

$$\int p_{s|x}(s|x_{obs})ds = 1$$

$$p_{s|x}(s|x_{obs}) = e^Z e^{-\frac{(s-\mu_{combined})^2}{2\sigma^2_{combined}}}$$

So, the integration would be:

$$\int_{-\infty}^{\infty} e^Z e^{-\frac{(s-\mu_{combined})^2}{2\sigma^2_{combined}}} ds$$

Here, we know that e^Z is an known normalization constant, so we have to explicitly normalize the expression. And since e^Z is a constant independent of s, it can be forced out the integration:

$$e^Z \int_{-\infty}^{\infty} e^{-\frac{(s-\mu_{combined})^2}{2\sigma^2_{combined}}} ds$$

And since it's Gaussian, we can obtain that:

$$\int_{-\infty}^{\infty} e^{-\frac{(s-\mu_{combined})^2}{2\sigma^2_{combined}}} ds = \sqrt{2\pi\sigma^2_{combined}}$$

$$e^Z \cdot \sqrt{2\pi\sigma^2_{combined}}$$

Now, if we want $p_{s|x}$ to be 1, then we must set,

$$e^Z \cdot \sqrt{2\pi\sigma^2_{combined}} = 1$$

Then, we can obtain that the normalization factor is:

$$e^Z = \frac{1}{\sqrt{2\pi\sigma^2_{combined}}}$$

And after substituting, the properly normalized posterior is:

$$p_{s|x}(s|x_{obs}) = \frac{1}{\sqrt{2\pi\sigma^2_{combined}}} e^{-\frac{(s-\mu_{combined})^2}{2\sigma^2_{combined}}}$$

This ensures that,

$$\int_{-\infty}^{\infty} p_{s|x}(s|x_{obs}) ds = 1$$