1 Chapter 5 Cue Combination and Evidence Accumulation

Problem 5.6: Suboptimal Estimation in Cue Combination ***In this problem, we examine suboptimal estimation in the context of cue combination. Suppose an observer estimates a stimulus s from two conditionally independent, Gaussian-distributed measurements, $x_{obs,1}$ and $x_{obs,2}$. The prior is flat.***

(a) We start with a reminder of optimal estimation. Express the PME in terms of the measurements.

(b) What is the variance of the PME across trials?

(c) Now suppose the observer uses an estimator of the form $\hat{s} = wx_{obs,1} + (1-w)x_{obs,2}$, where w can be *any constant*. Show that this estimate is unbiased (just like the PME); this means that the mean of the estimate for given s is equal to s.

(d) What is the variance of this estimate as a function of w? Plot this function. At which value of w is it minimal, and does this value make sense? State your final conclusion in words.

(a) Since the prior is flat, which means it's uninformative, the PME for combining two Gaussian measurements is the weighted average of the measurements, with weights proportional to their reliabilities, which is the inverse variances. Let the variances of x_{obs_1} be σ_1^2 and x_{obs_2} be σ_2^2. Then the PME is:

$$\hat{s}_{PME} = \frac{\frac{1}{\sigma_1^2}x_{obs_1} + \frac{1}{\sigma_2^2}x_{obs_2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}$$

From the formula we can see that the more reliable measurement with smaller variance would contribute more to the estimate.

(b) Since the measurements are independent, we can get:

$$Var(\hat{s}_{PME}) = \left(\frac{1}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\right)$$

which can also be written as:

$$Var(\hat{s}_{PME}) = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

This is the variance of the PME and it is always smaller than either σ_1^2 or σ_2^2.

(c) Given the estimator:

$$\hat{s} = wx_{obs_1} + (1-w)x_{obs_2}$$

Now now we want to show that this estimator is unbiased for any constant $w \in R$.

Since both measurements are unbiased estimators of s, we can say that:

$$E[x_{obs_1}] = E[x_{obs_2}] = s,$$

we compute the expectation of the linear estimator:

$$E[\hat{s}] = E[wx_{obs,1} + (1-w)x_{obs,2}] = wE[x_{obs,1}] + (1-w)E[x_{obs,2}].$$

Substituting in the expectations:

$$E[\hat{s}] = ws + (1-w)s = s.$$

Therefore, the estimator $\hat{s}$ is unbiased for all $w \in R$.

(d) The estimator:

$$\hat{s} = wx_{obs,1} + (1-w)x_{obs,2},$$

Since the measurements $x_{obs_1} \sim \mathcal{N}(s, \sigma_1^2)$ and $x_{obs_2} \sim \mathcal{N}(s, \sigma_2^2)$ are independent, we can know that the variance of a linear combination is:

$$Var(\hat{s}) = w^2\sigma_1^2 + (1-w)^2\sigma_2^2.$$

To minimize this variance, we take the derivative with respect to w:

$$\frac{d}{dw}Var(\hat{s}) = 2w\sigma_1^2 - 2(1-w)\sigma_2^2.$$

Set the derivative equal to zero:

$$2w\sigma_1^2 - 2(1-w)\sigma_2^2 = 0.$$

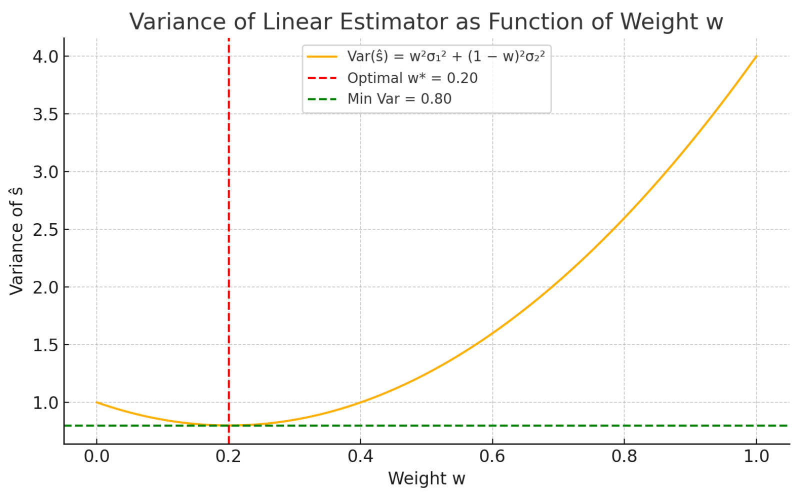
Solve for w:

$$w\sigma_1^2 = (1-w)\sigma_2^2 \Rightarrow w(\sigma_1^2 + \sigma_2^2) = \sigma_2^2 \Rightarrow w^* = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$$

This is the same optimal weight derived from PME. At this optimal weight, the variance becomes:

$$Var(\hat{s})_{min} = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{-1}.$$

Conclusion: The variance of the weighted estimator is minimized when the weights match the relative reliabilities of the measurements. This value of w recovers the PME.


Variance of Linear Estimator as Function of Weight w
Var(ŝ) = w²σ₁² + (1 − w)²σ₂²
Optimal w* = 0.20
Min Var = 0.80
Variance of ŝ
Weight w
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.0
0.2
0.4
0.6
0.8
1.0