Problem 10.3

Problem 10.3 Refer back to section 10.2.2 on same–different judgments of binary stimuli. Consider the case where the stimuli are the same in half of the cases: $p_{\text{same}} = 0.5$.

(a) Prove that the condition $d > 0$, with d in equation (10.18), reduces to equation (10.19),

$$\text{sign}(x_1) \;=\; \text{sign}(x_2).$$

(Hint: it might be helpful to use the definition and properties of the hyperbolic cosine function.)

Solution. From equation (10.18), with $p_{\text{same}} = 0.5$ (so $\log \frac{p_{\text{same}}}{1-p_{\text{same}}} = 0$), the log posterior ratio is

$$d = \log \frac{e^{\frac{\mu}{\sigma^2}(x_1+x_2)} + e^{-\frac{\mu}{\sigma^2}(x_1+x_2)}}{e^{\frac{\mu}{\sigma^2}(x_1-x_2)} + e^{-\frac{\mu}{\sigma^2}(x_1-x_2)}} = \log \frac{2\cosh\!\big(\frac{\mu}{\sigma^2}(x_1+x_2)\big)}{2\cosh\!\big(\frac{\mu}{\sigma^2}(x_1-x_2)\big)} = \log\cosh\!\big(A_+\big) - \log\cosh\!\big(A_-\big),$$

where

$$A_+ = \frac{\mu}{\sigma^2}(x_1+x_2), \quad A_- = \frac{\mu}{\sigma^2}(x_1-x_2).$$

Since $\cosh(u)$ is strictly increasing in $|u|$, the inequality $d > 0$ is equivalent to

$$\big|A_+\big| \;>\; \big|A_-\big| \quad\iff\quad \big|x_1+x_2\big| \;>\; \big|x_1-x_2\big|.$$

Squaring both sides,

$$(x_1+x_2)^2 \;>\; (x_1-x_2)^2 \implies 4\,x_1\,x_2 > 0 \implies x_1\,x_2 > 0 \implies \text{sign}(x_1) = \text{sign}(x_2),$$

which is exactly equation (10.19).

(b) In this case, derive an expression for $p(\hat{C} = 1 \mid s_1, s_2)$ in terms of cumulative standard normal distribution functions, Φ_{standard} (see Box 7.1).

Solution. Under binary stimuli $s_1, s_2 \in \{\pm\mu\}$ and Gaussian noise $x_i \sim \mathcal{N}(s_i, \sigma^2)$, part (a) shows $\hat{C} = 1 \iff x_1 x_2 > 0$. Hence

$$p(\hat{C} = 1 \mid s_1, s_2) = P(x_1 > 0, x_2 > 0 \mid s_1, s_2) \;+\; P(x_1 < 0, x_2 < 0 \mid s_1, s_2).$$

By independence,

$$= \big[P(x_1 > 0 \mid s_1)\big]\,\big[P(x_2 > 0 \mid s_2)\big] \;+\; \big[P(x_1 < 0 \mid s_1)\big]\,\big[P(x_2 < 0 \mid s_2)\big].$$

From Box 7.1,

$$P(x_i > 0 \mid s_i) = 1 - \Phi_{\text{standard}}\!\Big(\frac{0 - s_i}{\sigma}\Big) = \Phi_{\text{standard}}\!\Big(\frac{s_i}{\sigma}\Big),$$

$$P(x_i < 0 \mid s_i) = 1 - \Phi_{\text{standard}}\!\Big(\frac{s_i}{\sigma}\Big).$$

Therefore

$$p(\hat{C} = 1 \mid s_1, s_2) = \Phi\!\big(\tfrac{s_1}{\sigma}\big)\,\Phi\!\big(\tfrac{s_2}{\sigma}\big) \;+\; \big[1 - \Phi\!\big(\tfrac{s_1}{\sigma}\big)\big]\,\big[1 - \Phi\!\big(\tfrac{s_2}{\sigma}\big)\big].$$

(c) Based on your answer to (b), derive an expression for proportion correct. Simplify the expression until it has only a single Φ_{standard} in it.

Solution. Proportion correct under $p_{\text{same}} = 0.5$ is

$$P_{\text{correct}} = \tfrac{1}{2}\,P(\hat{C} = 1 \mid C = 1) \;+\; \tfrac{1}{2}\,P(\hat{C} = 2 \mid C = 2).$$

For $C = 1$, (s_1, s_2) is either (μ, μ) or $(-\mu, -\mu)$, each with probability $\frac{1}{2}$. But

$$p(\hat{C} = 1 \mid -\mu, -\mu) = p(\hat{C} = 1 \mid +\mu, +\mu) = \Phi\big(\tfrac{\mu}{\sigma}\big)^2 + \big[1 - \Phi\big(\tfrac{\mu}{\sigma}\big)\big]^2.$$

For $C = 2$, (s_1, s_2) is either $(\mu, -\mu)$ or $(-\mu, \mu)$, giving the same false-alarm rate

$$p(\hat{C} = 1 \mid \mu, -\mu) = 2\,\Phi\big(\tfrac{\mu}{\sigma}\big)\,\big[1 - \Phi\big(\tfrac{\mu}{\sigma}\big)\big].$$

Hence

$$P_{\text{correct}} = \tfrac{1}{2}\Big\{\,\Phi^2 + (1 - \Phi)^2\Big\} \;+\; \tfrac{1}{2}\Big\{1 - 2\Phi(1 - \Phi)\Big\}, \quad \Phi \equiv \Phi_{\text{standard}}\Big(\tfrac{\mu}{\sigma}\Big).$$

Combine terms:

$$P_{\text{correct}} = 1 \;-\; 2\,\Phi\,(1 - \Phi).$$

This form contains only a single Φ_{standard}, as required.

$$P_{\text{correct}} = 1 - 2\,\Phi_{\text{standard}}\Big(\tfrac{\mu}{\sigma}\Big)\,\Big[1 - \Phi_{\text{standard}}\Big(\tfrac{\mu}{\sigma}\Big)\Big].$$