1 Chapter 7 Discrimination and Detection

Problem 7.3 ***In this problem, we explore the relationship between posterior probabilities and LPR numerically.***

(a) Create a vector of ninety-nine possible posterior probabilities of s_+, from 0.01 to 0.99 in steps of 0.01. For each value, calculate the LPR

$$d = \log \frac{p(s = s_+ \mid x)}{p(s = s_- \mid x)}.$$

Then plot this ratio as a function of the posterior probability of s_+. This should show that every posterior probability corresponds to exactly one LPR and the other way around (we are dealing with monotonic functions). Knowing one is as good as knowing the other.

(b) Why did we not include the posterior probabilities 0 and 1?

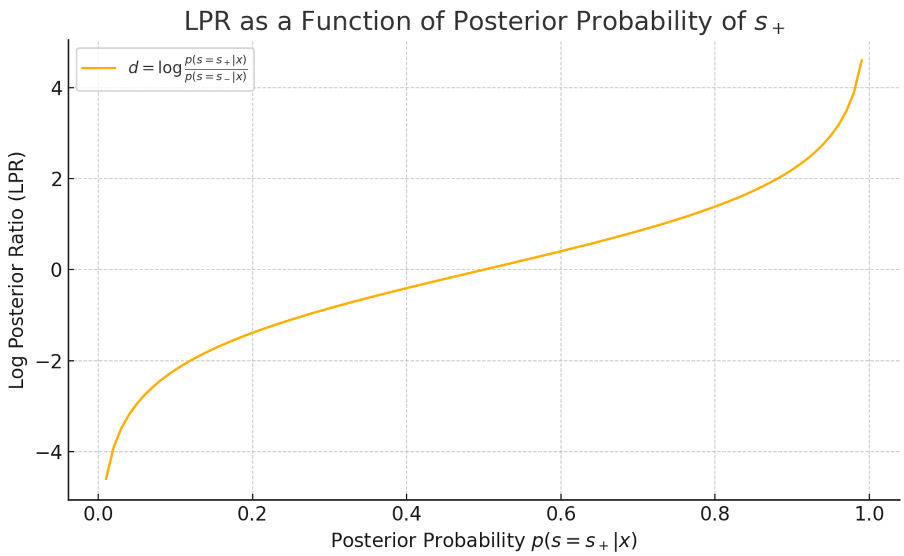
(c) Suppose you know the LPR d. Express the posterior probability of $s = s_+$, $p(s = s_+ \mid x)$, as a function of d only. Do the same for $p(s = s_- \mid x)$.

(d) If the LPR is 0.1, what are the posterior probabilities of s_+ and s_-? What if the LPR is 1?

(a)
To create the plotting, I created a vector of posterior probabilities for s_+ ranging from 0.01 to 0.99 in increments of 0.01. Then we can compute log posterior ratio (LPR) for each posterior probability $p(s = s_+ \mid x)$ using this formula:

$$d = \log \frac{p(s = s_+ \mid x)}{p(s = s_- \mid x)} = \log \frac{p(s = s_+ \mid x)}{1 - p(s = s_+ \mid x)}$$

From the plotting result we can see each posterior probability corresponds uniquely to a log posterior ratio.



(b)

LPR is computed as:

$$d = \log\left(\frac{p(s = s_+ \mid x)}{p(s = s_- \mid x)}\right)$$

If $p(s = s_+ \mid x) = 0$ or $p(s = s_- \mid x) = 0$, then the logarithm becomes:

$$\log(0) = -\infty \quad or \quad \log(\infty) = \infty$$

These values are not finite and thus cannot be represented.
Also, from a conceptual point, posterior probabilities of exactly 0 or 1 imply absolute certainty, which is not realistic.

(c)
Given that LPR is:

$$d = \log\left(\frac{p(s = s_+ \mid x)}{p(s = s_- \mid x)}\right)$$

We want to express posterior probability as a functyion of d.
Based on the two types of stimulus, we can define as following:

$$p(s = s_+ \mid x) = p_+ \quad and \quad p(s = s_- \mid x) = p_- = 1 - p_+$$

Then,

$$d = \log\left(\frac{p_+}{1 - p_+}\right)$$

And in order to solve this equation, here are the steps: $\mathrm{d} = \log\left(\frac{p_+}{1-p_+}\right)$
$e^d = \frac{p_+}{1-p_+}$

$e^d(1-p_+) = p_+$
$e^d - e^d p_+ = p_+$
$e^d = p_+ + e^d p_+$
$e^d = p_+(1+e^d)$
$p_+ = \frac{e^d}{1+e^d}$

Thus, the posterior probability for $s = s_-$ can be calculated as:

$$p(s = s_- \mid x) = 1 - p_+ = \frac{1}{1+e^d}$$

(d)
We use the formulas derived in part (c):

$$p(s = s_+ \mid x) = \frac{e^d}{1+e^d}, \quad p(s = s_- \mid x) = \frac{1}{1+e^d}$$

When: $d = 0.1$

$$p(s = s_+ \mid x) = \frac{e^{0.1}}{1+e^{0.1}} \approx \frac{1.1052}{1+1.1052} \approx \frac{1.1052}{2.1052} \approx 0.525$$

$$p(s = s_- \mid x) = 1 - 0.525 = 0.475$$

When: $d = 1$

$$p(s = s_+ \mid x) = \frac{e^1}{1+e^1} \approx \frac{2.718}{1+2.718} \approx \frac{2.718}{3.718} \approx 0.731$$

$$p(s = s_- \mid x) = 1 - 0.731 = 0.269$$

We can see that: larger values of d correspond to greater posterior belief in s_+, which is consistent with the interpretation of d as a log odds measure in favor of s_+.