1 Chapter 4 The Response Distribution

Problem 4.6 ***An observer infers a stimulus s from a measurement x_{obs}. As in the chapter, the measurement distribution $p(x|s)$ is Gaussian with mean s and variance σ^2. Unlike in the chapter, we use the prior.***

$$p(s) = e^{-\lambda s}, \tag{1}$$

where λ is a positive constant. This is an *improper prior* (see section 3.5.2) but that does not stop us.

(a) Derive an equation for the PME.

(b) Derive an equation for the distribution of the PME for given s.

(a)
Bayes' rule:

$$p(s|x_{obs}) \propto p(x_{obs}|s)p(s)$$

Since the measurement distribution is a Gaussian, we can obtain that:

$$p(x_{obs}|s) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_{obs}-s)^2}{2\sigma^2}\right)$$

(Since we will be doing multiplication on exponential base, we can ignore the terms that are constants, following steps will be the same.)

Then, we can substitute what we've known from the question:

$$p(x_{obs}|s) \propto \exp\left(-\frac{(x_{obs}-s)^2}{2\sigma^2}\right)$$

For the prior, we know that:

$$p(s) \propto \exp(-\lambda s)$$

Then, multiplying them together, we can get that:

$$p(s|x_{obs}) \propto \exp\left(-\frac{s^2 - 2x_{obs}s}{2\sigma^2}\right) \cdot \exp(-\lambda s)$$

Identity of exponential, $e^A \cdot e^B = e^{A+B}$, we can get that:

$$p(s|x_{obs}) \propto \exp\left(-\frac{s^2 - 2x_{obs}s}{2\sigma^2} - \lambda s\right)$$

Where,

$$-\frac{s^2 - 2x_{obs}s}{2\sigma^2} - \lambda s = -\frac{s^2 - 2x_{obs}s + 2\lambda\sigma^2 s}{2\sigma^2} = -\frac{s^2 - 2(x_{obs} - \lambda\sigma^2)s}{2\sigma^2}$$

Then, we can recognize that:

$$-\frac{s^2 - 2x_{obs}s}{2\sigma^2} - \lambda s = -\frac{(s - \mu_{post})^2}{2\sigma_{post}^2} + (constant)$$

where:

$$\mu_{post} = x_{obs} - \lambda\sigma^2$$

$$\sigma_{post}^2 = \sigma^2$$

Since PME is the mean of the posterior distribution:

$$s_{PME} = x_{obs} - \lambda\sigma^2$$

(b)
Since the measurement follows a Gaussian likelihhod centerted at s, we can know that $x_{obs} \sim \mathcal{N}(s, \sigma^2)$, and we just derived PME, we can re-write that:

$$s_{PME} \sim \mathcal{N}(s - \lambda\sigma^2, \sigma^2)$$

The distribution would be:

$$p(s_{PME}|s) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(s_{PME} - (s - \lambda\sigma^2))^2}{2\sigma^2}\right)$$