1 Chapter 3 Bayesian Inference under Measurement Noise

Problem 3.11 ***Some stimulus variables are periodic (or circular), taking values between (for instance)*** 0 ***and*** 2π***. Examples are orientation, motion direction, and time of day. In appendix section B.7.6, we discuss the Von Mises distribution, which is suitable for such variables. Suppose that motion direction follows a Von Mises distribution with circular mean*** μ ***and concentration parameter*** κ_s***:***

$$p(s) = \frac{1}{2\pi I_0(\kappa_s)} e^{\kappa_s \cos(s-\mu)}.$$

Further assume that the measurement distribution is also Von Mises, with circular mean s and concentration parameter κ:

$$p(x|s) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(x-s)}.$$

(a) A Bayesian observer infers s from a measurement x_{obs}. Show that the posterior over s is a Von Mises distribution with (circular) mean μ_{post} given by:

$$\cos \mu_{post} = \kappa_s \cos \mu + \kappa \cos x_{obs};$$

$$\sin \mu_{post} = \kappa_s \sin \mu + \kappa \sin x_{obs};$$

and concentration parameter

$$\sqrt{\kappa_s^2 + \kappa^2 + 2\kappa\kappa_s \cos(x_{obs} - \mu)}.$$

(b) Compare and contrast with the Gaussian case.

1. (a) Substituting Von Mises distribution into the Bayes' theorem, we can get:

$$p(s|x_{obs}) \propto \left[\frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(x_{obs}-s)}\right] \times \left[\frac{1}{2\pi I_0(\kappa_s)} e^{\kappa_s \cos(s-\mu)}\right]$$

After simplification (getting rid of the term that doesn't depend on s):

$$p(s|x_{obs}) \propto e^{\kappa \cos(x_{obs}-s)+\kappa_s \cos(s-\mu)}$$

Because of the cosine symmetry property, we can re-write it:

$$\cos(A - B) = \cos(B - A)$$

$$p(s|x_{obs}) \propto e^{\kappa \cos(s-x_{obs})+\kappa_s \cos(s-\mu)}$$

Following the sum-to-product rule, and the property of summing different weighted cosines, we know that:

$$A\cos(s-\alpha) + B\cos(s-\beta) = C\cos(s-\gamma)$$

where:

$$C = \sqrt{A^2 + B^2 + 2AB\cos(\alpha-\beta)}$$

$$\cos\gamma = \frac{A\cos\alpha + B\cos\beta}{C}$$

$$\sin\gamma = \frac{A\sin\alpha + B\sin\beta}{C}$$

Then, in our case, we can know that:

- $A = \kappa_s, \alpha = \mu$
- $B = \kappa, \beta = x_{obs}$

After these, back to our original question, we can substitute and get that:

$$\kappa_s\cos(s-\mu) + \kappa\cos(s-x_{obs}) = R\cos(s-\mu_{post})$$

where,

$$R = \sqrt{\kappa_s^2 + \kappa^2 + 2\kappa\kappa_s\cos(x_{obs}-\mu)}$$

$$\cos\mu_{post} = \frac{\kappa_s\cos\mu + \kappa\cos x_{obs}}{R}$$

$$\sin\mu_{post} = \frac{\kappa_s\sin\mu + \kappa\sin x_{obs}}{R}$$

2. (b)

For similarities:
1) In both case, posterior mean is the weighted combination of prior mean and observation (where Gaussian is linear weighting, von Mises is circular weighting using cosine and sine components).
2) In both case, posterior is derived from combination of prior and likelihood.

For differences:

Feature	Gaussian Case	Von Mises Case
Variable Type	Linear	Circular
Prior Distribution	$\mathcal{N}(\mu, \sigma^2)$	$VM(\mu, \kappa_s)$
Likelihood	$\mathcal{N}(x_{obs}, \sigma^2)$	$VM(x_{obs}, \kappa)$
Posterior Mean Update	$\mu_{post} = \frac{x_{obs}\sigma_s^2+\mu\sigma^2}{\sigma^2+\sigma_s^2}$	$\cos\mu_{post} = \frac{\kappa_s\cos\mu+\kappa\cos x_{obs}}{R}$
Posterior Precision	$\frac{1}{\sigma_{post}^2} = \frac{1}{\sigma^2} + \frac{1}{\sigma_s^2}$	$R = \sqrt{\kappa_s^2 + \kappa^2 + 2\kappa\kappa_s\cos(x_{obs} - \mu)}$

(Note: for the differences part, I had some answers and asked ChatGPT, the contrast table is what ChatGPT provided me and I thought it's more comprehensive and thorough.)