1 Chapter 7 Discrimination and Detection

Problem 7.5 ***Prove equation (7.10) for the log likelihood in the Gaussian measurement model.***

The Gaussian measurement model is:

$$p(x \mid s) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-s)^2}{2\sigma^2}\right)$$

The posterior log odds is:

$$d = \log \frac{p(s = s_+ \mid x)}{p(s = s_- \mid x)}$$

Using Bayes' rule:

$$d = \log \frac{p(x \mid s_+)p(s = s_+)}{p(x \mid s_-)p(s = s_-)} = \log \frac{p(s = s_+)}{p(s = s_-)} + \log \frac{p(x \mid s_+)}{p(x \mid s_-)}$$

Substitute the Gaussian likelihoods:

$$\log \frac{p(x \mid s_+)}{p(x \mid s_-)} = \log \left(\exp\left[-\frac{(x-s_+)^2 - (x-s_-)^2}{2\sigma^2}\right]\right) = -\frac{1}{2\sigma^2}\left[(x-s_+)^2 - (x-s_-)^2\right]$$

Simplify it:

$$(x-s_+)^2 - (x-s_-)^2 = -2x(s_+ - s_-) + s_+^2 - s_-^2$$

So:

$$\log \frac{p(x \mid s_+)}{p(x \mid s_-)} = \frac{(s_+ - s_-)}{\sigma^2}x - \frac{s_+^2 - s_-^2}{2\sigma^2}$$

Let:

$$\Delta s = s_+ - s_-, \quad \bar{s} = \frac{s_+ + s_-}{2} \Rightarrow s_+^2 - s_-^2 = \Delta s(s_+ + s_-) = 2\Delta s \bar{s}$$

Then we can get:

$$d = \log \frac{p(s = s_+)}{p(s = s_-)} + \frac{\Delta s}{\sigma^2}(x - \bar{s})$$