1 Chapter 4 The Response Distribution

Problem 4.15 ***A student claims, "To obtain the response distribution*** $p(\hat{s}_{PM}|s)$ ***in the model, I can average the posteriors*** $p(s|x_{obs})$ ***over all*** x_{obs} ***for a given*** s***." In other words,***

$$(wrong) \quad p(\hat{s}_{PM}|s) = \int p_{s|x}(\hat{s}_{PM}|x_{obs})p(x_{obs}|s). \tag{1}$$

(a) Although this would produce a distribution that is centered between the prior mean and the true stimulus, this claim is conceptually wrong. Why?

(b) Show mathematically that the variance of the resulting distribution is incorrect.

(c) As a specific example, determine what this student would predict in the infinite-noise limit ($\sigma \to \infty$), and what the correct calculation would predict. Explain both answers intuitively.

(a)

In the student's claim, they are basically saying that $\hat{s}_{PM}$ can act as a random variable (which is depend on x_{obs}).

However, $\hat{s}_{PM}$ is posterior mean estimate, which is a function of measurement x_{obs}, specifically:

$$\hat{s}_{PM} = E[s|x_{obs}]$$

So firstly, it's a function but not a variable.

This means that when x_{obs} is obtained, $\hat{s}_{PM}$ will not be a random/uncertain variable, but rather with certainty. So there is no conditioned probability distribution.

(b)

The student's claim treats the posterior mean estimate $\hat{s}_{PM}$ as if it were a random variable drawn from the posterior distribution $p(s \mid x_{obs})$, and thus averages over those posteriors. But this gives an incorrect variance.
Let us clarify what is actually meant by the variance of the response distribution $p(\hat{s}_{PM} \mid s)$. This is the variance across repeated trials, where in each trial the measurement x_{obs} is drawn from $p(x \mid s)$, and the observer then reports the posterior mean based on x_{obs}. So we can write:

$$Var(\hat{s}_{PM} \mid s) = Var_{x \sim p(x|s)} \left(E[s \mid x]\right).$$

This is **not equal** to the variance of the posterior distribution $Var(s \mid x) = \sigma^2_{post}$, which describes uncertainty about s in a single trial.
The student, by averaging over the entire posterior distribution $p(s \mid x)$ across x, computes something like:

$$\tilde{p}(\hat{s}_{PM} \mid s) = \int p(s \mid x)\, p(x \mid s)\, dx,$$

which is a mixture over posterior distributions. This overestimates the variance because it includes both the **variance within each posterior** and the **variance of the means across trials**.
The variance of the student's incorrect distribution becomes:

$$Var_{student}(\hat{s}_{PM} \mid s) = E_x \left[Var(s \mid x)\right] + Var_x \left(E[s \mid x]\right),$$

but in reality, the observer only reports $\hat{s}_{PM} = E[s \mid x]$, so the **true variance** is only:

$$Var_{correct}(\hat{s}_{PM} \mid s) = Var_x \left(E[s \mid x]\right),$$

i.e., the variability in posterior mean estimates due to the randomness in measurement x. The student's computation erroneously includes the posterior variance $Var(s \mid x)$ as if it were variability in the observer's response, which it is not.
Hence, the student's approach **overestimates** the variance in $\hat{s}_{PM}$, which leads to an incorrect response distribution.

(c) If measurement noise grows infinite, i.e., $\sigma \to \infty$, the student would predict that P($\hat{s}_{PM} \mid s) remainsamixofpriorandlikelihood, whichwillmakethevariancebesmallerandlessthanexpected.$

Since the student thinks that even when the measurement is very noise, it still provides some information. So they try to average over all possible posterior estimates based on different noisy measurements. This creates a mixture of posteriors centered between the prior and likelihood means. As a result, the student predicts a posterior mean estimate (PME) distribution that still reflects influence from both prior and likelihood.

But in the actual case, when $\sigma \to \infty$, the measurement noise is no longer containing useful information, then the Bayesian observer should fully revert to the prior, where P($\hat{s}_{PM} \mid s) shouldbeP(s)$.

The estimate is no longer "updated" by data, because the data has become useless.