1 Chapter 12 Inference in a Changing World

Problem 12.4 *We build on the HMM in section 12.1. Suppose the observer wants to make a prediction for the future state* s_{t+1}*, given the measurements* $x_1, \ldots, x_t$ *(i.e., the measurement at time* $t+1$ *has not been made yet). Derive the mean and variance of the posterior. You can assume the mean and variance of* $p(s_t \mid x_1, \ldots, x_{t-1})$ *to be known.*

Answer

We assume that at time t, the observer has a posterior over s_t:

$$p(s_t \mid x_1, \ldots, x_t) = \mathcal{N}(s_t; \mu_{\text{post},t}, \sigma^2_{\text{post},t}).$$

The generative dynamics of the HMM define the transition model as:

$$p(s_{t+1} \mid s_t) = \mathcal{N}(s_{t+1}; s_t + \Delta, \sigma_s^2),$$

where Δ is the changing term and σ_s^2 is the process noise variance.

The posterior prediction for s_{t+1} would be:
Marginalizing over the uncertainty in s_t to get the predictive distribution over s_{t+1}:

$$p(s_{t+1} \mid x_1, \ldots, x_t) = \int p(s_{t+1} \mid s_t)\, p(s_t \mid x_1, \ldots, x_t)\, ds_t.$$

Since both $p(s_t \mid x_{1:t})$ and $p(s_{t+1} \mid s_t)$ are Gaussians, the result of the convolution is also a Gaussian:

$$p(s_{t+1} \mid x_1, \ldots, x_t) = \mathcal{N}(\mu_{\text{pred},t+1}, \sigma^2_{\text{pred},t+1}),$$

where:

$$\mu_{\text{pred},t+1} = \mu_{\text{post},t} + \Delta,$$
$$\sigma^2_{\text{pred},t+1} = \sigma^2_{\text{post},t} + \sigma_s^2.$$

All in all, the predicted future state s_{t+1} has a Gaussian distribution with:

- Mean: $\mu_{\text{post},t} + \Delta$
- Variance: $\sigma^2_{\text{post},t} + \sigma_s^2$

This concludes the derivation of the predictive posterior for s_{t+1}. ■