1 Chapter 9 Top-Level Nuisance Variables and Ambiguity

Problem 9.8 *In section 9.2, we assumed a flat prior over* $\log D$*. We will now consider a more realistic extension in which this prior is a normal distribution,*

$$p(\log D) = \mathcal{N}(\log D; \mu_D, \sigma_D^2). \tag{9.24}$$

(a) Show that the posterior over $\log D$, from equation (9.11), now becomes

$$p(\log D \mid \log x) = \mathcal{N}(\log D; \mu_{\text{post}}, \sigma_{\text{post}}^2), \tag{9.25}$$

where

$$\mu_{\text{post}} = \frac{J_D \mu_D + J_w(\mu_w + \log l - \log x)}{J_D + J_w}, \tag{9.26}$$

$$\sigma_{\text{post}}^2 = \frac{1}{J_D + J_w}, \tag{9.27}$$

with $J_D \equiv \frac{1}{\sigma_D^2}$ and $J_w \equiv \frac{1}{\sigma_w^2}$.

(b) Show that the estimate of D, from equation (9.15), now becomes

$$\hat{D} = \text{Median}[D]^{\frac{J_D}{J_D + J_w}} \left(D \cdot \frac{\text{Median}[w]}{w} \right)^{\frac{J_w}{J_D + J_w}}. \tag{9.28}$$

(c) Interpret this expression.

(a)
We can rewrite the likelihood as a function of $\log D$:

$$\log x = \mu_w + \log l - \log D \quad \Rightarrow \quad \log D = \mu_w + \log l - \log x$$

So:

$$p(\log x \mid \log D) = \mathcal{N}(\log D; \mu_w + \log l - \log x, \sigma_w^2)$$

Then we can combine:

$$p(\log D \mid \log x) \propto p(\log x \mid \log D) \cdot p(\log D)$$

And all terms are Gaussian:

$$p_1(\log D) \propto \exp\left(-\frac{1}{2\sigma_D^2} (\log D - \mu_D)^2 \right)$$

$$p_2(\log D) \propto \exp\left(-\frac{1}{2\sigma_w^2} (\log D - [\mu_w + \log l - \log x])^2 \right)$$

$$p(\log D \mid \log x) \propto \exp\left(-\frac{1}{2} \left[\frac{(\log D - \mu_D)^2}{\sigma_D^2} + \frac{(\log D - \mu_L)^2}{\sigma_w^2} \right] \right)$$

To simplify, we can define:

$$\mu_L = \mu_w + \log l - \log x$$

Let $z = \log D$. Then the exponent becomes:

$$\frac{1}{\sigma_D^2}(z^2 - 2z\mu_D + \mu_D^2) + \frac{1}{\sigma_w^2}(z^2 - 2z\mu_L + \mu_L^2)$$

Combine:

$$\left(\frac{1}{\sigma_D^2} + \frac{1}{\sigma_w^2}\right) z^2 - 2\left(\frac{\mu_D}{\sigma_D^2} + \frac{\mu_L}{\sigma_w^2}\right) z + (\text{constants})$$

This is a quadratic in $z = \log D$. The posterior is therefore Gaussian in z.

Let:

$$J_D = \frac{1}{\sigma_D^2}, \quad J_w = \frac{1}{\sigma_w^2}$$

Then:

$$\sigma_{\text{post}}^2 = \frac{1}{J_D + J_w}$$

$$\mu_{\text{post}} = \frac{J_D\mu_D + J_w\mu_L}{J_D + J_w} = \frac{J_D\mu_D + J_w(\mu_w + \log l - \log x)}{J_D + J_w}$$

Then finally we can get the expression:

$$p(\log D \mid \log x) = \mathcal{N}(\log D; \mu_{\text{post}}, \sigma_{\text{post}}^2)$$

with

$$\mu_{\text{post}} = \frac{J_D\mu_D + J_w(\mu_w + \log l - \log x)}{J_D + J_w}, \quad \sigma_{\text{post}}^2 = \frac{1}{J_D + J_w}$$

(b)
From part (a), we know:

$$p(\log D \mid \log x) = \mathcal{N}(\mu_{\text{post}}, \sigma_{\text{post}}^2)$$

Since the posterior over $\log D$ is normal, then D follows a **log-normal distribution**. The **median** of a log-normal is:

$$\text{Median}[D] = e^{\mu_{\text{post}}}$$

So:

$$\hat{D} = \exp\left(\frac{J_D\mu_D + J_w(\mu_w + \log l - \log x)}{J_D + J_w}\right)$$

Split the exponent:

$$\hat{D} = \exp\left(\frac{J_D\mu_D}{J_D + J_w}\right) \cdot \exp\left(\frac{J_w(\mu_w + \log l - \log x)}{J_D + J_w}\right)$$

We can define that:

$$\text{Median}[D] = e^{\mu_D}, \quad \text{Median}[w] = e^{\mu_w}$$

And from equation 9.14, we know that:

$$\frac{l}{x} = \frac{D}{w} \Rightarrow \log l - \log x = \log\left(\frac{D}{w}\right)$$

Then:

$$\hat{D} = (\text{Median}[D])^{\frac{J_D}{J_D+J_w}} \cdot \left(\text{Median}[w] \cdot \frac{D}{w}\right)^{\frac{J_w}{J_D+J_w}}$$

Now, this obtained equation matches the equation from problem.

(c)
So from the equation we know that the final estimate of the distance is formed by two references of information:
1. The prior belief about distance: $\text{Median}[D] = e^{\mu_D}$ 2. The inferred distance based on retinal image size and assumed object size: $D \cdot \frac{\text{Median}[w]}{w}$

And from the equation we can see that each component is raised to a power corresponding to its inverse variance:

$$J_D = \frac{1}{\sigma_D^2}, \quad J_w = \frac{1}{\sigma_w^2}$$

Therefore, we can conclude that we make perceptions based on its prior beliefs and sensory evidence according to their relative reliability.