HW3_Q3_Chen

October 30, 2024

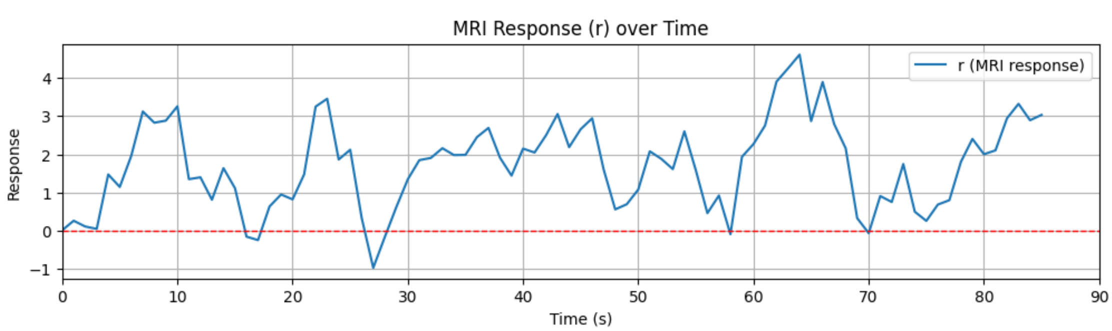
MathTools HW3 2024-10-21

Question 3

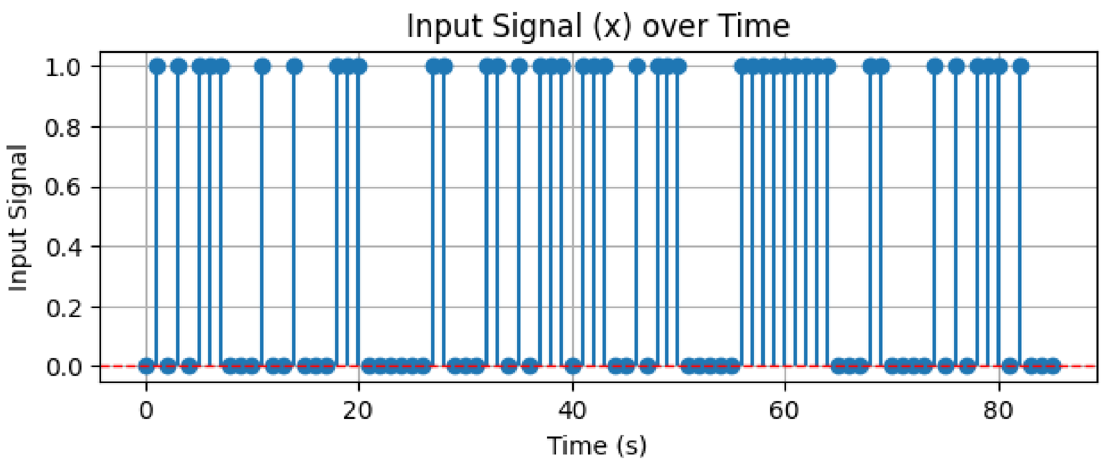
[1685]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy
import scipy.io
```

[1686]:
```
data = scipy.io.loadmat('hrfDeconv.mat')
r = data.get('r').flatten()
x = data.get('x').flatten()

time = range(len(x))
r_new = r[:len(x)]
```

[1687]:
```
plt.figure(figsize=(12, 6))
plt.subplot(2, 1, 1)
plt.plot(time, r_new, label='r (MRI response)')
plt.axhline(y=0, color='red', linestyle='--', linewidth=1)
plt.xlim(0, 90)
plt.title('MRI Response (r) over Time')
plt.xlabel('Time (s)')
plt.ylabel('Response')
plt.legend()
plt.grid(True)
```



[1688]:
```
plt.subplot(2, 1, 2)
plt.stem(time, x, basefmt=" ")
plt.axhline(y=0, color='red', linestyle='--', linewidth=1)
plt.title('Input Signal (x) over Time')
plt.xlabel('Time (s)')
plt.ylabel('Input Signal')
plt.grid(True)

plt.tight_layout()
plt.show()
```



(a)

[1689]:
```
# Define the function, create a convolution matrix X for an input vector x and␣
↪filter length M.
def createConvMat(x, M):
    N = len(x)

    # Empty place holder
    X = np.zeros((N + M - 1, M))

    for i in range(M):
        X[i:i+N, i] = x

    return X
```

[1690]:
```
# Testing the convolution matrix
#x = np.random.rand(50)
M = 15
X = createConvMat(x , M)
```

[1691]:

```
# Testing with random h vector
h = np.random.rand(M)

# Manually calculated convolution
N_holder = len(X)
r_calculated = [0] * N_holder
for i in range(N_holder):
    for j in range(M):
        r_calculated[i] += X[i][j] * h[j]
# Python built in matrix multiplication operator: @; np.dot(); np.matmul()
#r_calculated = X @ h

r_python = np.convolve(x, h, mode='full')  #python numpy built in convolution
```

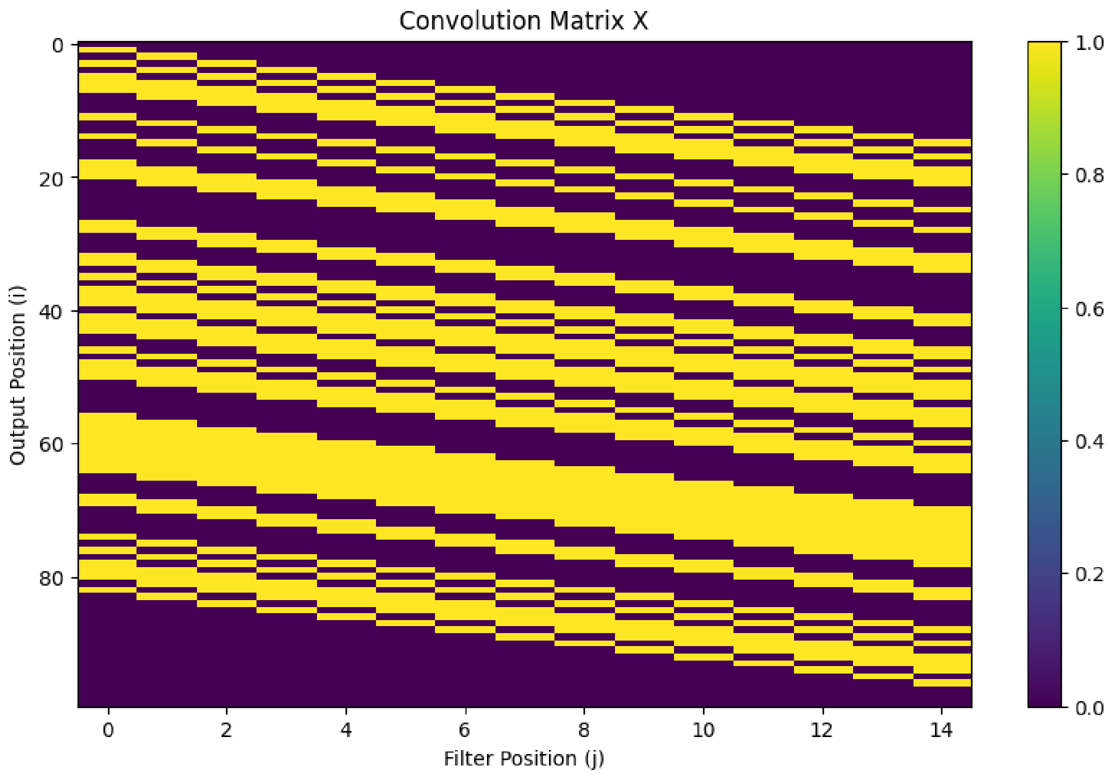
[1692]:

```
print("Check if the two responses are almost equal:", np.allclose(r_calculated,␣
↪r_python))
```

```
Check if the two responses are almost equal: True
```

[1693]:

```
plt.figure(figsize=(10, 6))
plt.imshow(X, aspect='auto', cmap='viridis')
plt.colorbar()
plt.title('Convolution Matrix X')
plt.xlabel('Filter Position (j)')
plt.ylabel('Output Position (i)')
plt.show()
```



In this image, each row contains shifted values of the input vector . This structure illustrates how the input signal (a series of impulses over time) is transformed to create the delayed MRI response .

(b)

[1694]:
```
# Manually calculating transpose
def transpose_matrix(matrix):
    rows = len(matrix)
    cols = len(matrix[0])
    transposed = [[0] * rows for _ in range(cols)] #since only cares about␣
 ↪number of rows to create the same column
    for i in range(rows):
        for j in range(cols):
            transposed[j][i] = matrix[i][j]
    return transposed
```

[1695]:
```
# Manually calculating matrices multiplication
def multiply_matrices(A, B):
    rows_A = len(A)
    cols_A = len(A[0])
    rows_B = len(B)
    cols_B = len(B[0])
```

```
    assert cols_A == rows_B, "Error"

    result = [[0] * cols_B for _ in range(rows_A)]
    for i in range(rows_A):
        for j in range(cols_B):
            for k in range(cols_A): #necessary, since k is accumulating the␣
↪total sum of all elements in accordance
                result[i][j] += A[i][k] * B[k][j]
    return result
```

[1696]:
```
# Manually calculating inverse matrix
def inverse_matrix(matrix):
    n = len(matrix)
    identity_matrix = [[float(i == j) for i in range(n)] for j in range(n)]␣
↪#float is for deciding whether it's a diagonal element based on the output,␣
↪1 for yes and 0 for no; and using inner layer and outer layer to avoid too␣
↪many for loops
    copy = [matrix[i] + identity_matrix[i] for i in range(n)]

    #Gauss-Jordan elimination
    for i in range(n):
        coef = copy[i][i]
        for j in range(2 * n):
            copy[i][j] /= coef

        for k in range(n):
            if k != i:
                coef = copy[k][i]
                for j in range(2 * n):
                    copy[k][j] -= coef * copy[i][j]

    inverse = [row[n:] for row in copy]
    return inverse
```

[1697]:
```
# Manually multiplying a matrix with a vector
def multiply_matrix_vector(matrix, vector):
    rows = len(matrix)
    cols = len(matrix[0])

    assert cols == len(vector), "Error"

    result = [0] * rows
    for i in range(rows):
        for j in range(cols):
            result[i] += matrix[i][j] * vector[j]
    return result
```

[1698]:
```
# Solving the least-squares regression problem MANUALLY
def leastSquares_calculated(X, r):

    #Step 1: X-transpose
    X_transpose = transpose_matrix(X)

    #Step 2: matrices mutiplications of X-transpose and X
    XT_X = multiply_matrices(X_transpose, X)

    #Step 3: MANUALLY CALCULATING INVERSE IN PYTHON (which is the cruelest␣
↪thing that I've ever been through) of step 2
    XT_X_inv = inverse_matrix(XT_X)

    # Step 4: X-transpose * r
    XT_r = multiply_matrix_vector(X_transpose, r)

    # Step 5: h_opt = (X^T * X)^-1 * (X^T * r)
    h_opt_calculated_test = multiply_matrix_vector(XT_X_inv, XT_r)

    return h_opt_calculated_test
```

[1699]:
```
# svd to calculate least square, manually
U, Sigma, VT = np.linalg.svd(X, full_matrices=False)
Sigma_pinv = np.diag(1 / Sigma)

#r = r[:len(X)].reshape(-1, 1)
h_opt_calculated = VT.T @ Sigma_pinv @ U.T @ r
```

[1700]:
```
h_opt_calculated_test = leastSquares_calculated(X, r_calculated)
print("Check if two ways of calculating least square are equal:", np.
↪allclose(h_opt_calculated, h_opt_calculated_test))
```

```
Check if two ways of calculating least square are equal: False
```

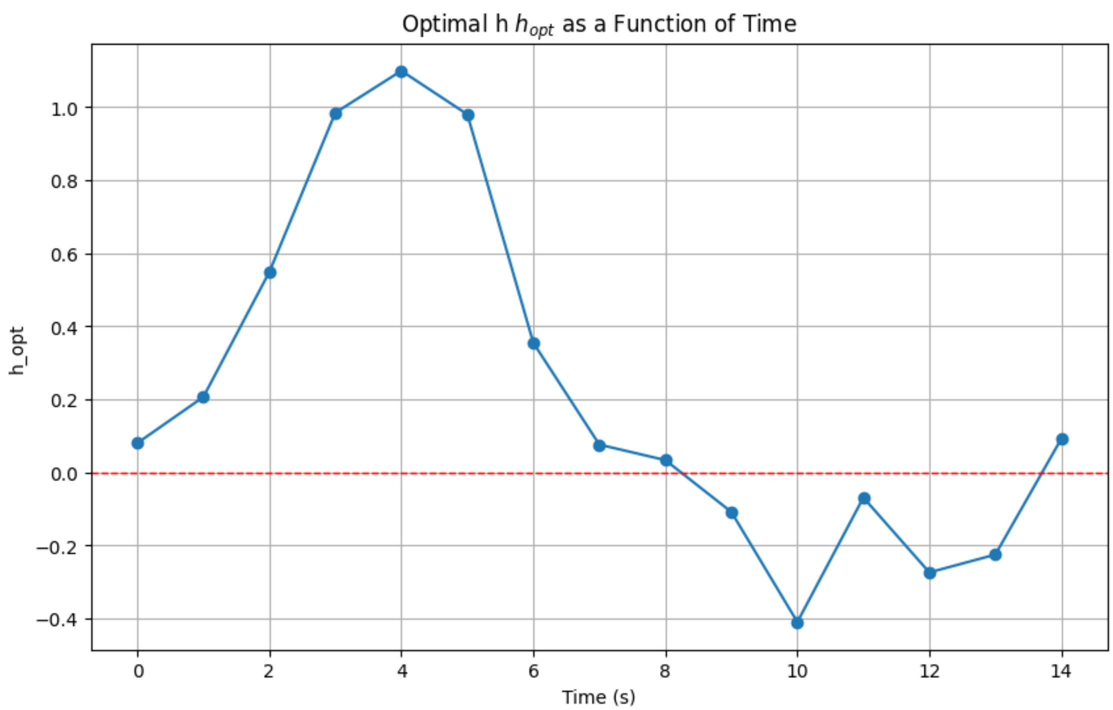
[1701]:
```
## Sanity check sections ##
# Solving the least-squares regression problem with all build in functions
def leastSquares_python(X, r):
    X_transpose_python = np.transpose(X)
    h_opt_python = np.linalg.inv(X_transpose_python @ X) @ (X_transpose_python␣
↪@ r)
    return h_opt_python
```

[1702]:
```
#h_opt_calculated = leastSquares_calculated(X, r_calculated)
h_opt_python = leastSquares_python(X, r)
h_opt_calculated = h_opt_calculated.flatten()
```

[1703]:
```
print("Check if the two responses are almost equal:", np.
↪allclose(h_opt_calculated, h_opt_python))
```

```
Check if the two responses are almost equal: True
```

[1704]:
```
#Plotting
time_h = np.arange(len(h_opt_calculated))

plt.figure(figsize=(10, 6))
plt.plot(time_h, h_opt_calculated, marker='o')
plt.axhline(y=0, color='red', linestyle='--', linewidth=1)
plt.title("Optimal h $h_{opt}$ as a Function of Time")
plt.xlabel("Time (s)")
plt.ylabel("h_opt")
plt.grid(True)
plt.show()
```



The plot of h_{opt} as a function of time represents a temporal response of an impulse response. From the graph we can see that it's fluctuating. And it's lating for 14s.

(c)

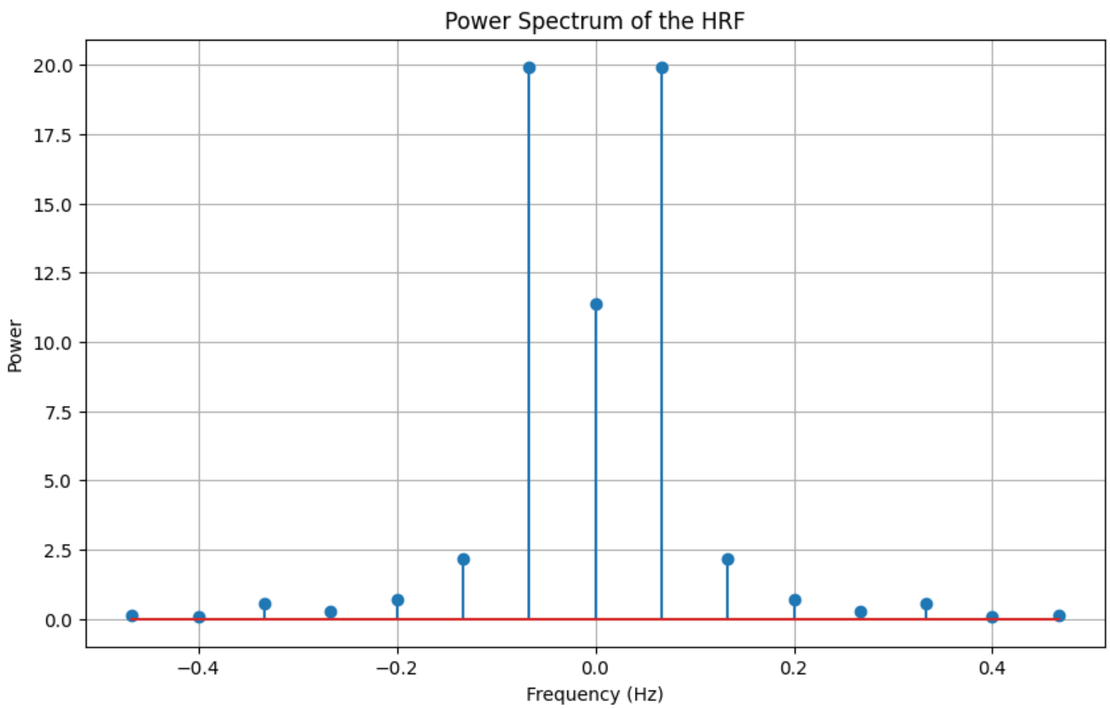
[1705]:
```
# FFT to HRF
h_opt_calculated = np.array(h_opt_calculated)
H = np.fft.fft(h_opt_calculated)
```

[1706]:
```
# Get the power spectrum
power_spectrum = np.abs(H) ** 2
```

```
# Shiifting the zero-value to center
power_spectrum_shifted = np.fft.fftshift(power_spectrum)
```

[1707]:
```
# Get the frequency
nn = len(h_opt_calculated)
freuency = np.fft.fftfreq(nn) #sampling rate is at 1hz
freq_shifted = np.fft.fftshift(freuency)
```

[1708]:
```
plt.figure(figsize=(10, 6))
plt.stem(freq_shifted, power_spectrum_shifted)
plt.title('Power Spectrum of the HRF')
plt.xlabel('Frequency (Hz)')
plt.ylabel('Power')
plt.grid(True)
plt.show()
```



Based on th plot, we can tell that the HRF is a low-pass filter, allowing low frequencies (including the DC component) to pass through. I think it make sense, since HRD is corresponding to slow and sustained changes, such as blood flow changes in response to neural activities.

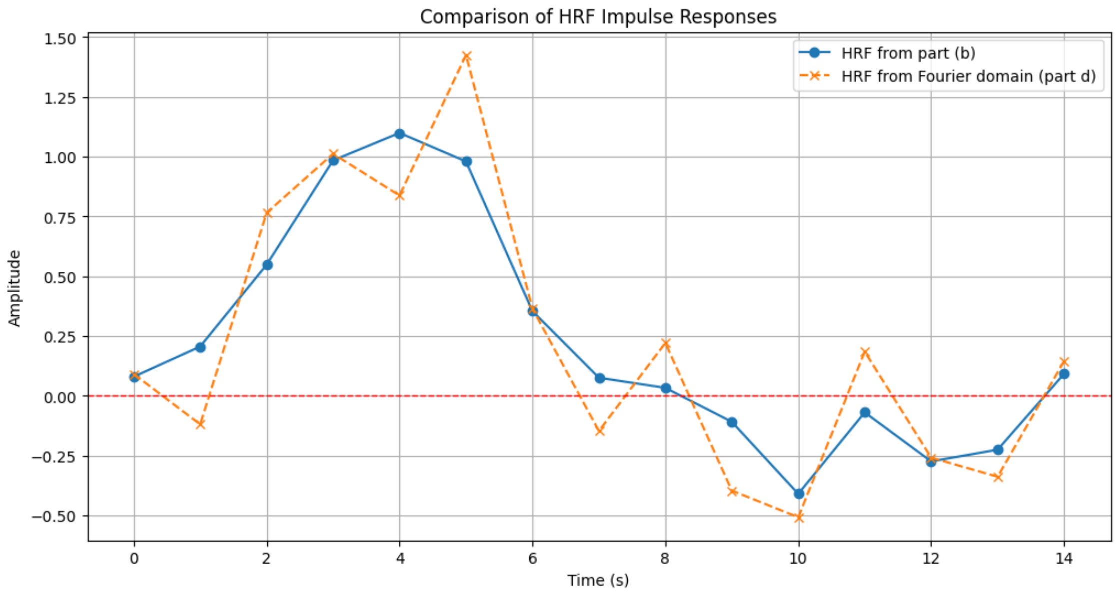
(d)

[1709]:
```
# Convolution theorem in fourier domain
```

```python
# The Convolution Theorem states that the convolution of two signals in the
↪time domain is equivalent to multiplying their Fourier transforms in the
↪frequency domain.
r_questiond = np.array(r_calculated)
maxlength = max(len(x), len(r_questiond))
x_pad = np.pad(x, (0, maxlength - len(x)), 'constant')
r_pad = np.pad(r, (0, maxlength - len(X)), 'constant')

x_fft = np.fft.fft(x_pad)
r_questiond_fft = np.fft.fft(r_pad)
```

[1710]:
```python
# Calculate the fourier transform of $h_{opt}$ in fourier domain
h_questiond_fft = r_questiond_fft / x_fft

# In time domain
h_questiond_fft_recovered = np.fft.ifft(h_questiond_fft)[:M].real
```

[1712]:
```python
#Plotting (b) and (d)
time_questiond = np.arange(len(h_questiond_fft_recovered))

plt.figure(figsize=(12, 6))
plt.plot(time_questiond, h_opt_calculated, '-o', label='HRF from part (b)')
plt.plot(time_questiond, h_questiond_fft_recovered, '--x', label='HRF from
↪Fourier domain (part d)')
plt.axhline(y=0, color='red', linestyle='--', linewidth=1)
plt.title('Comparison of HRF Impulse Responses')
plt.xlabel('Time (s)')
plt.ylabel('Amplitude')
plt.legend()
plt.grid(True)
plt.show()
```



From the graph, after zero-padding, I can tell that the Convolution theorem in fourier domain is providing a only-okay approximation of the HRF, providing a similar tendency.

However, this method would fail if the input signal x has zero values at any frequency. Or, if the input and response signal have too high noise level, which will get amplified during the division step.