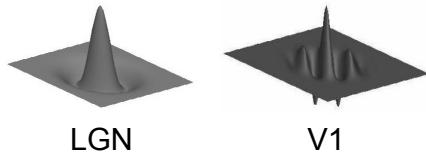


Spatial pattern vision and linear systems theory

- Receptive fields and neural images
- Shift-invariant linear systems and convolution
- Fourier transform and frequency response
- Applications of linear systems to spatial vision
 - Contrast sensitivity
 - Spatial frequency and orientation channels
 - Spatial frequency and orientation adaptation
 - Masking

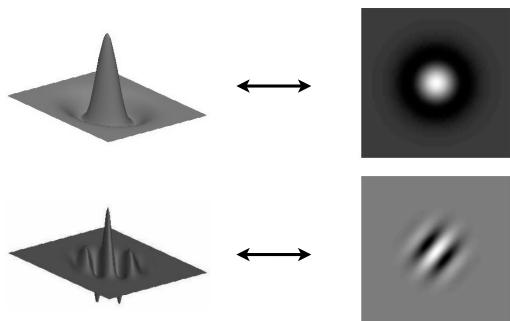
Receptive field

- In any modality: that region of the sensory apparatus that, when stimulated, can directly affect the firing rate of a given neuron
- Spatial vision: spatial receptive field can be mapped in visual space or on the retina
- Examples:

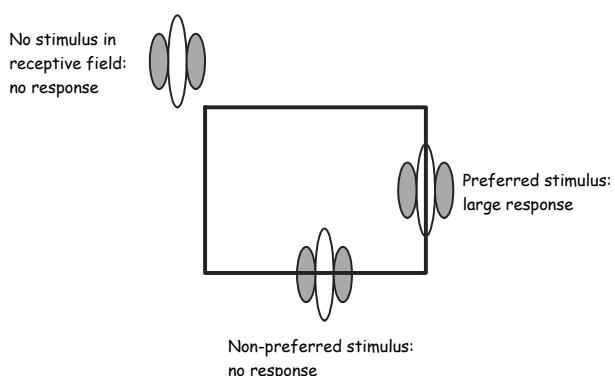


Receptive field

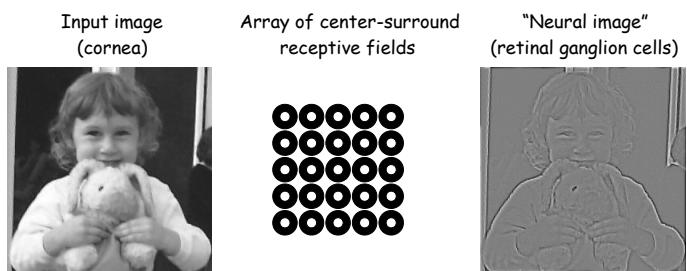
A spatial receptive field is an image:



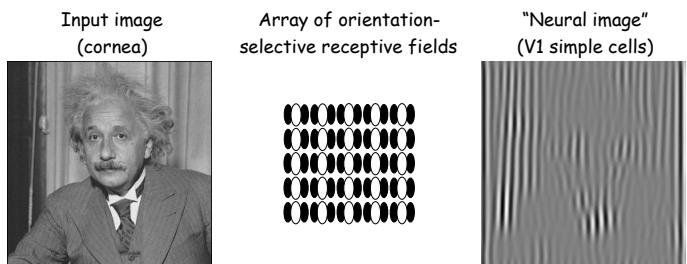
Orientation selective receptive field



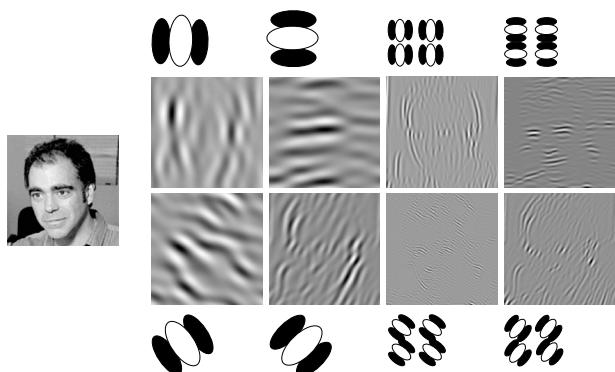
Neural image: retinal ganglion cell responses



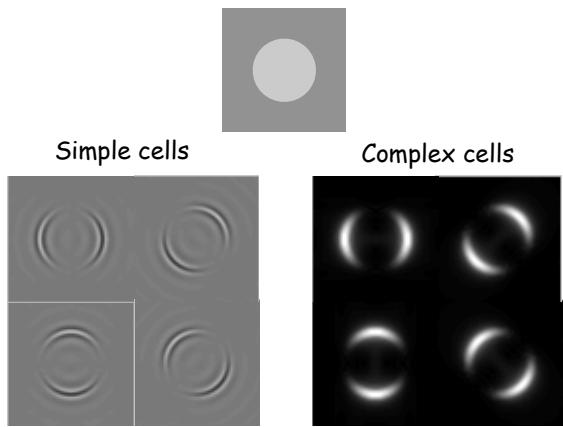
Neural image: simple cell responses



Lots of neural images: V1 simple cells

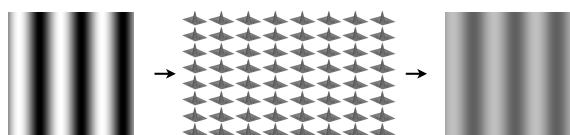


Lots of neural images



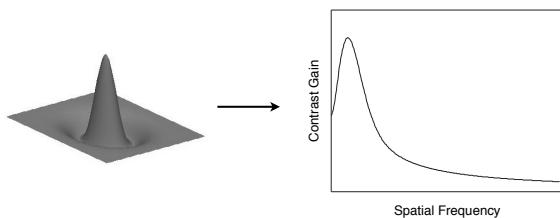
Neural image of a sine wave

For a linear, shift-invariant system such as a linear model of a receptive field, an input sine wave results in an identical output sine wave, except for a possible lateral shift and scaling.



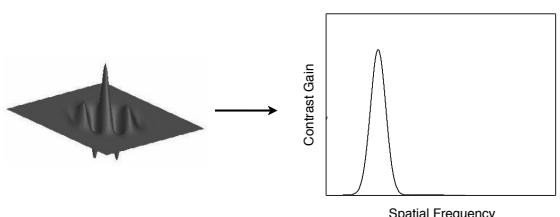
Frequency response

This scaling of contrast by a linear receptive field in the neural image is a function of spatial frequency determined by the shape of the receptive field.



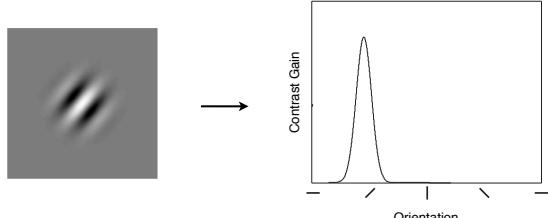
Frequency response

This scaling of contrast by a linear receptive field in the neural image is a function of spatial frequency determined by the shape of the receptive field.

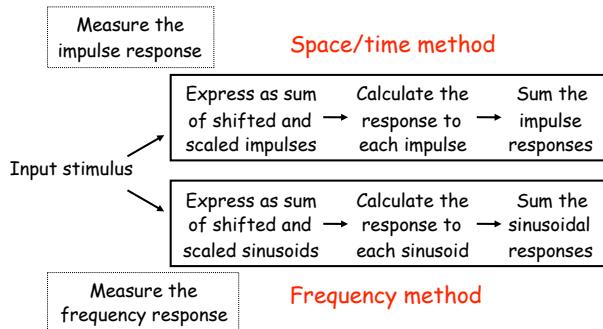


Orientation tuning

If a receptive field is not circularly symmetric, the scaling of contrast is also a function of orientation (for a given spatial frequency) determined by the shape of the receptive field.



Linear systems analysis



Spatial pattern vision and linear systems theory

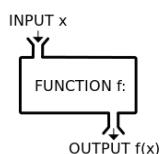
- Receptive fields and neural images
- **Shift-invariant linear systems and convolution**
- Fourier transform and frequency response
- Applications of linear systems to spatial vision
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 - Spatial frequency and orientation adaptation
 - Masking

Functions and Systems

A **function**: a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.

Examples

- $f(x) = x^2$
- $f(x) = \cos(x)$
- $f(x,y) = \sqrt{x^2 + y^2}$



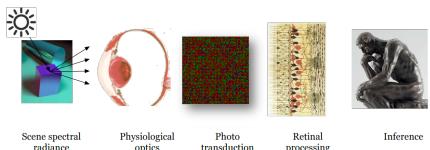
wikipedia

Functions and Systems

A **system**: a generalization of a function whereby inputs and outputs can be numbers or objects, discrete or continuous, any dimensionality

Examples

- Physiological optics: *Input*: scene spectral irradiance; *Output*: retinal image
- Eye: *Input*: scene spectral irradiance; *Output*: retinal ganglion cell firing rate
- Human: *Input*: scene spectral irradiance; *Output*: knob adjustment (eg color match)
- fMRI: *Input*: neuronal activity *Output*: T2*w image

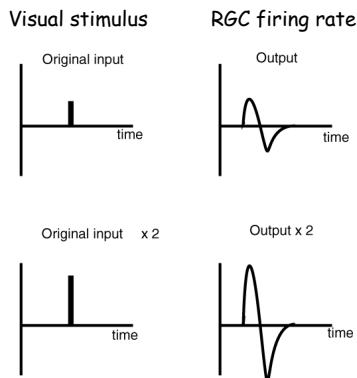


Linear Systems Analysis

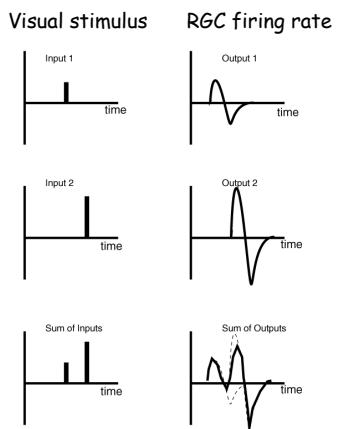
Systems with signals as input and output

- **1-d**: low- and high-pass filters in electronic equipment, fMRI data analysis, or in sound production (articulators) or audition (the ear as a filter)
 $y(t) = T\{x(t)\}$
- **2-d**: optical blur, spatial receptive field
 $g(x,y) = T\{f(x,y)\}$
- **3-d**: spatio-temporal receptive field
 $g(x,y,t) = T\{f(x,y,t)\}$

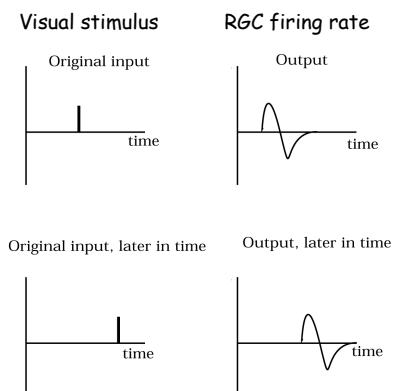
Homogeneity (scaling)



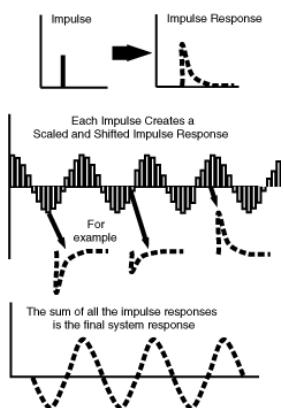
Additivity



Shift invariance



Shift-invariant linear systems and impulses



Convolution

Discrete-time signal: $x[n] = [x_1, x_2, x_3, \dots]$

A system or transform maps an input signal into an output signal:

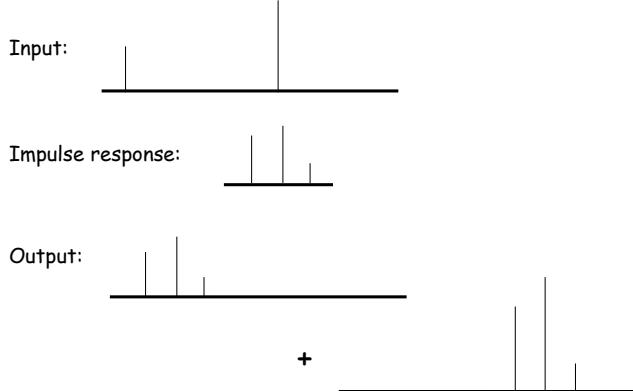
$$y[n] = T\{x[n]\}$$

A shift-invariant, linear system can always be expressed as a convolution:

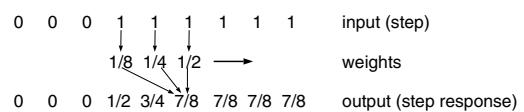
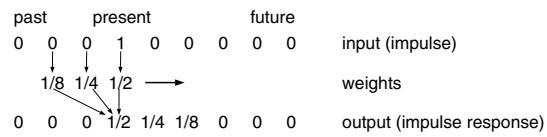
$$y[n] = \sum_m x[m] h[n-m]$$

where $h[n]$ is the impulse response.

Convolution as sum of impulse responses



Convolution as sequence of weighted sums



Convolution as matrix multiplication

Linear system \Leftrightarrow matrix multiplication

Shift-invariant linear system \Leftrightarrow Toeplitz matrix

$$\begin{bmatrix} \vdots \\ 5 \\ 2 \\ -3 \\ 4 \\ \vdots \end{bmatrix} = \begin{bmatrix} & & & \vdots \\ & 1 & 2 & 3 & 0 & 0 & 0 \\ \cdots & 0 & 1 & 2 & 3 & 0 & 0 \\ & 0 & 0 & 1 & 2 & 3 & 0 \\ & 0 & 0 & 0 & 1 & 2 & 3 \\ & & & \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ 1 \\ 2 \\ 0 \\ 0 \\ -1 \\ 2 \\ \vdots \end{bmatrix}$$

Columns contain shifted copies of the impulse response.
Rows contain time-reversed copies of impulse response.

Convolution as matrix multiplication

$$\begin{bmatrix} \vdots \\ 3 \\ 2 \\ 1 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} & & & \vdots \\ & 1 & 2 & 3 & 0 & 0 & 0 \\ \cdots & 0 & 1 & 2 & 3 & 0 & 0 \\ & 0 & 0 & 1 & 2 & 3 & 0 \\ & 0 & 0 & 0 & 1 & 2 & 3 \\ & & & \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Columns contain shifted copies of the impulse response.
Rows contain time-reversed copies of impulse response.

Derivation: shift-invariant linear system \Rightarrow convolution

Homogeneity:

$$T\{a x[n]\} = a T\{x[n]\}$$

Additivity:

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\}$$

Superposition:

$$T\{a x_1[n] + b x_2[n]\} = a T\{x_1[n]\} + b T\{x_2[n]\}$$

Shift-invariance:

$$y[n] = T\{x[n]\} \Rightarrow y[n-m] = T\{x[n-m]\}$$

Convolution derivation (cont)

Impulse sequence:

$d[n] = 1$ for $n = 0$, $d[n] = 0$ otherwise

Any sequence can be expressed as a sum of impulses:

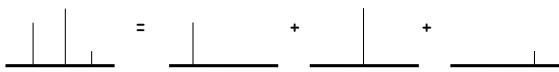
$$x[n] = \sum_m x[m] d[n-m]$$

where

$d[n-m]$ is impulse shifted to sample m

$x[m]$ is the height of that impulse

Example:



Convolution derivation (cont)

$x[n]$: input

$y[n] = T\{x[n]\}$: output

$h[n] = T\{d[n]\}$: impulse response

1) Represent input as sum of impulses:

$$y[n] = T\{x[n]\}$$

$$y[n] = T\left\{\sum_m x[m] d[n-m]\right\}$$

2) Use superposition:

$$y[n] = \sum_m x[m] T\{d[n-m]\}$$

3) Use shift-invariance:

$$y[n] = \sum_m x[m] h[n-m]$$

Matrix multiplication => scaling

$$\begin{matrix} \text{Nx1 vector} & \text{NxN Toeplitz} \\ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{bmatrix} & = \begin{bmatrix} & & & & x_1 \\ & & & & x_2 \\ & & & & x_3 \\ & & & \ddots & \vdots \\ & & & & x_N \end{bmatrix} \end{matrix} \quad \text{Impulse response}$$

$$\begin{bmatrix} \alpha y_1 \\ \alpha y_2 \\ \alpha y_3 \\ \vdots \\ \alpha y_N \end{bmatrix} = \begin{bmatrix} & & & & \alpha x_1 \\ & & & & \alpha x_2 \\ & & & \ddots & \vdots \\ & & & & \alpha x_N \end{bmatrix} \quad \text{Impulse response}$$

Matrix multiplication => additivity

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} & & & & \\ & \text{Impulse} & & & \\ & \text{response} & & & \\ & & x_1 \\ & & x_2 \\ & & x_3 \\ & & \vdots \\ & & x_N \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_N \end{bmatrix} = \begin{bmatrix} & & & & \\ & \text{Impulse} & & & \\ & \text{response} & & & \\ & & w_1 \\ & & w_2 \\ & & w_3 \\ & & \vdots \\ & & w_N \end{bmatrix}$$

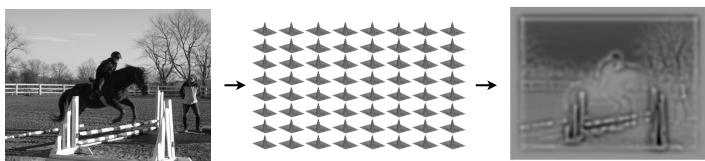
$$\begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \\ y_3 + z_3 \\ \vdots \\ y_N & z_N \end{bmatrix} = \begin{bmatrix} & & & & \\ & \text{Impulse} & & & \\ & \text{response} & & & \\ & & x_1 & w_1 \\ & & x_2 & w_2 \\ & & x_3 + w_3 \\ & & \vdots & \vdots \\ & & x_N & w_N \end{bmatrix}$$

Toeplitz matrix => shift invariance

$$\begin{bmatrix} \vdots \\ 3 \\ 2 \\ 1 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} & & & & & \vdots \\ & 1 & 2 & 3 & 0 & 0 & 0 \\ & 0 & 1 & 2 & 3 & 0 & 0 \\ & 0 & 0 & 1 & 2 & 3 & 0 \\ & 0 & 0 & 0 & 1 & 2 & 3 \\ & & & & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Neural image and convolution

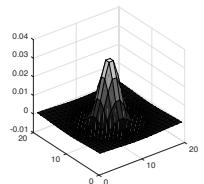
A spatial receptive field may also be treated as a linear system, by assuming a dense collection of neurons with the same receptive field translated to different locations in the visual field. In this view, it is a linear, shift-invariant system*[§].



* This is the basis of CNNs (convolutional neural networks).

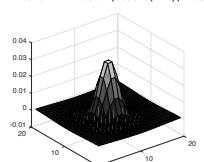
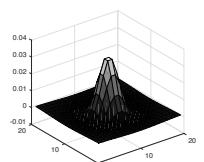
[§] The nervous system doesn't actually work like this. (It's not linear and it's not shift invariant!)

Neural image and convolution



```
% Make a Difference of Gaussian receptive field using fspecial  
DoG = fspecial('gaussian', 20,2) - fspecial('gaussian', 20,5);  
im = imread('cameraman.tif');  
  
% Make a neural image by convolution  
neuralim = conv2(double(im), DoG);  
  
% Show the image, the RF, and the neural image  
figure, subplot(1,3,1), imshow(im), subplot(1,3,2), surf(DoG)  
subplot(1,3,3), imagesc(neuralim), colormap gray, axis image off
```

Neural image and convolution



Continuous-time derivation of convolution

Linear systems requirements

A system (or transform) converts (or maps) an input signal into an output signal:

$$y(t) = T[x(t)]$$

A linear system satisfies the following properties.

1) Homogeneity (scalar rule):

$$T[a x(t)] = a y(t)$$

2) Additivity:

$$T[x_1(t) + x_2(t)] = y_1(t) + y_2(t)$$

Often, these two properties are written together and called superposition:

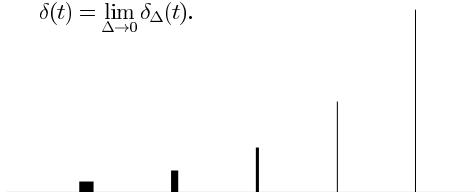
$$T[a x_1(t) + b x_2(t)] = a y_1(t) + b y_2(t)$$

Pulses and impulses

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & \text{if } 0 < t < \Delta \\ 0 & \text{otherwise} \end{cases}$$

$$\delta(t) = \lim_{\Delta \rightarrow 0} \delta_\Delta(t).$$



Staircase approximation to continuous time signal

$$\left| \text{Continuous Signal} \right| = \dots + \left| \text{Pulse 1} \right| + \left| \text{Pulse 2} \right| + \left| \text{Pulse 3} \right| + \dots$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta) \delta_\Delta(t - k\Delta) \Delta.$$

$$x(t) = \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{\infty} x(k\Delta) \delta_\Delta(t - k\Delta) \Delta.$$

$$x(t) = \int_{-\infty}^{\infty} x(s) \delta(t - s) ds.$$

Shift invariance

For a system to be shift-invariant (or time-invariant) means that a time-shifted version of the input yields a time-shifted version of the output:

$$y(t) = T[x(t)]$$

$$y(t-s) = T[x(t-s)]$$

The response $y(t-s)$ is identical to the response $y(t)$, except that it is shifted in time.

Convolution

Representing the input signal as a sum of pulses:

$$\begin{aligned} y(t) = T[x(t)] &= T \left[\int_{-\infty}^{\infty} x(s) \delta(t-s) ds \right] \\ &= T \left[\lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{\infty} x(k\Delta) \delta_{\Delta}(t-k\Delta) \Delta \right]. \end{aligned}$$

Using additivity,

$$y(t) = \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{\infty} T[x(k\Delta) \delta_{\Delta}(t-k\Delta) \Delta].$$

Taking the limit,

$$y(t) = \int_{-\infty}^{\infty} T[x(s) \delta(t-s)] ds.$$

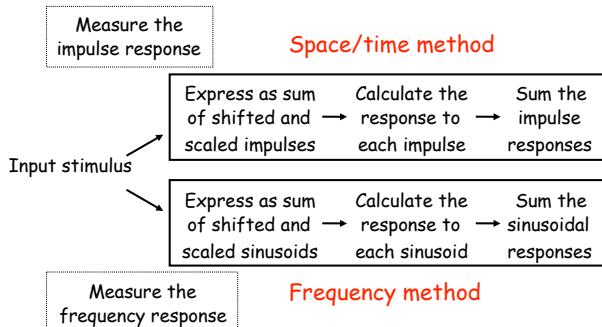
Using homogeneity (scalar rule),

$$y(t) = \int_{-\infty}^{\infty} T[x(s) \delta(t-s)] ds.$$

Defining $h(t)$ as the impulse response,

$$y(t) = \int_{-\infty}^{\infty} x(s) h(t-s) ds.$$

Linear systems analysis



Spatial pattern vision and linear systems theory

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Fourier transform and frequency response: summary

- Signals can be represented as sums of sine waves
- Linear, shift-invariant systems operate "independently" on each sine wave, and merely scale and shift them.
- A simplified model of neurons in the visual system, the linear receptive field, results in a neural image that is linear and shift-invariant.
- Psychophysical models of the visual system might be built of such mechanisms.
- It is therefore important to understand visual stimuli in terms of their spatial frequency content.
- The same tools can be applied to other modalities (e.g., audition) and other signals (EEG, MRI, MEG, etc.).

Temporal frequency and Fourier decomposition

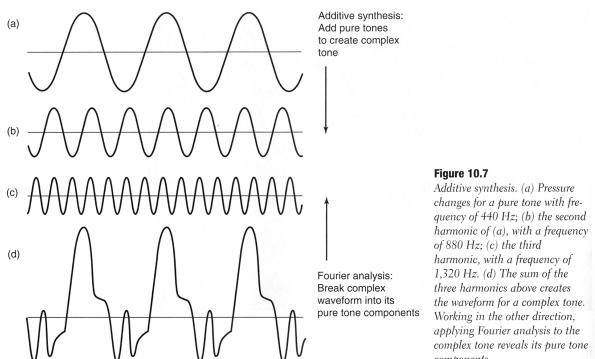
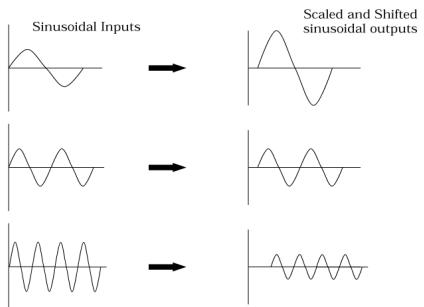


Figure 10.7
Additive synthesis: (a) Pressure changes for a pure tone with frequency of 440 Hz; (b) the second harmonic of (a), with a frequency of 880 Hz; (c) the third harmonic, with a frequency of 1,320 Hz; (d) The sum of the three harmonics above creates the waveform for a complex tone. Working in the other direction, applying Fourier analysis to the complex tone reveals its pure tone components.

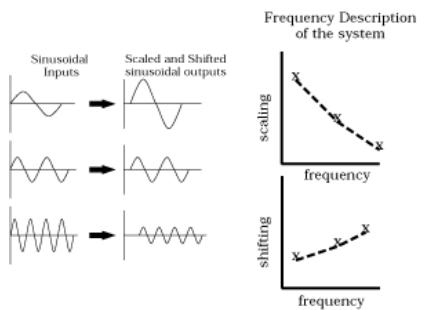
Shift-invariant linear systems & sinusoids

Measure the scaling and shifting for each sinusoid



Shift-invariant linear systems & sinusoids

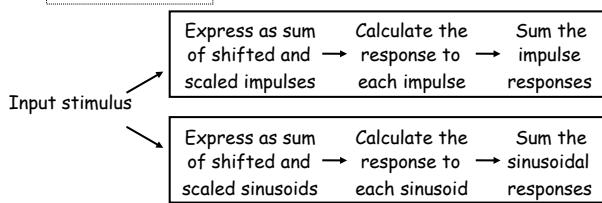
Frequency response



Linear systems analysis

Measure the impulse response

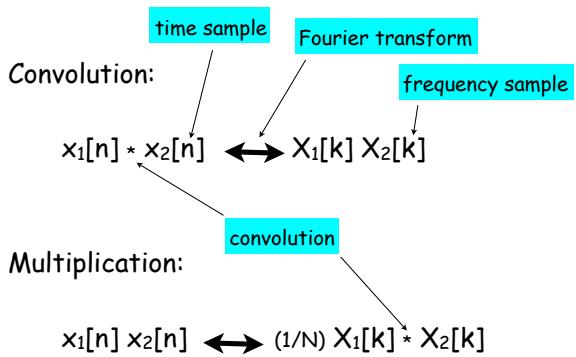
Space/time method



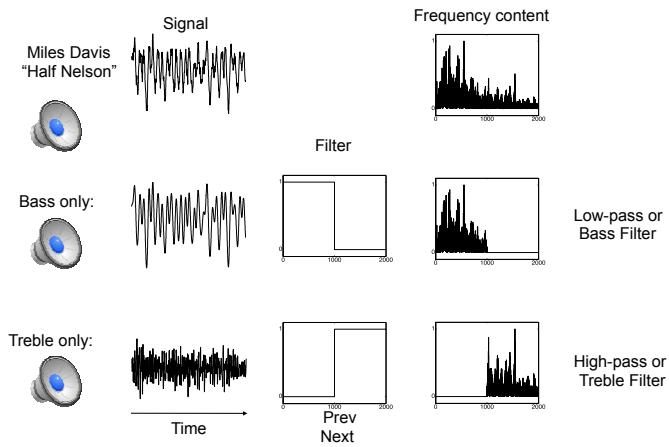
Measure the frequency response

Frequency method

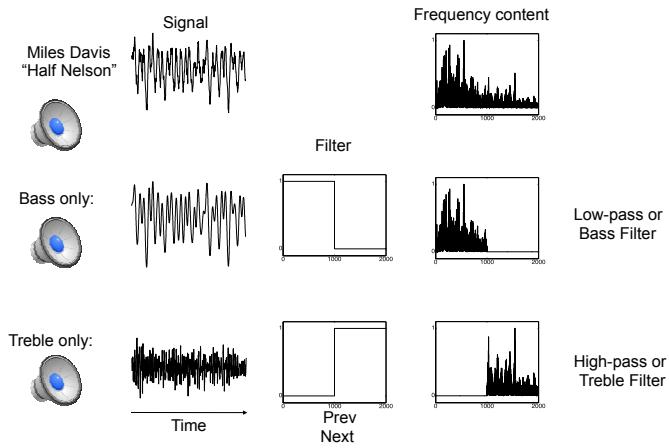
Convolution and multiplication



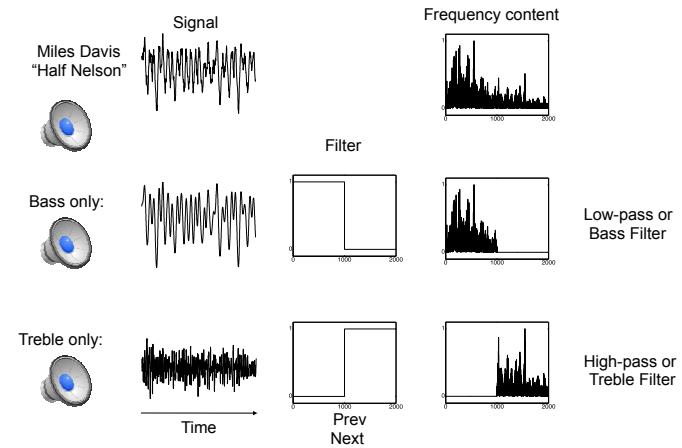
Linear filter example



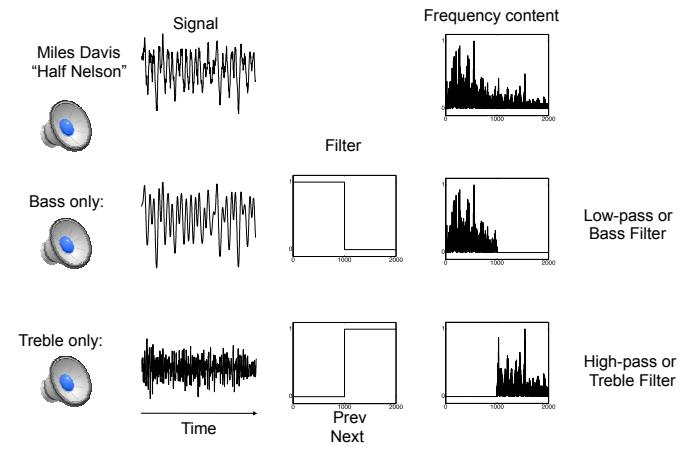
Linear filter example



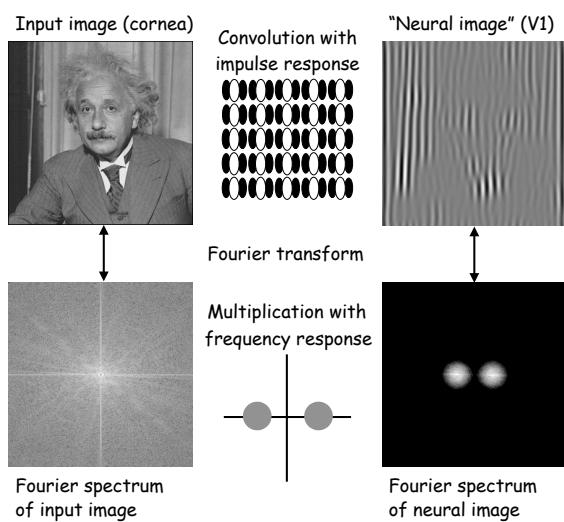
Linear filter example



Linear filter example



Linear filter

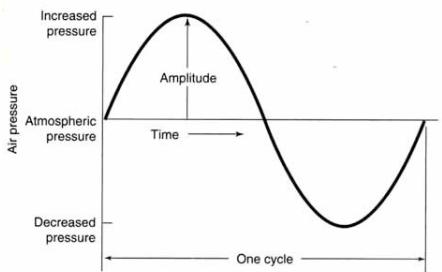


Fourier Analysis

Signals as sums of sine waves

- 1d: time series
 - fMRI signal from a voxel or ROI
 - mean firing rate of a neuron over time
 - auditory stimuli
- 2d: static visual image, neural image
- 3d: visual motion analysis

Auditory example: Pure tones



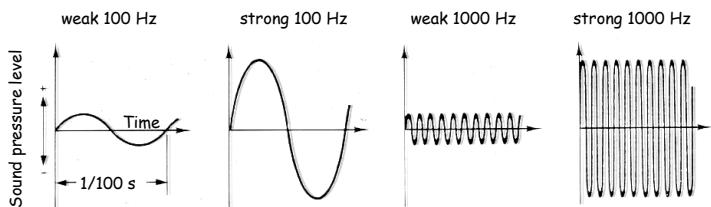
Pure tones can be described by 3 numbers:

Frequency = rate of air pressure modulation (related to pitch)

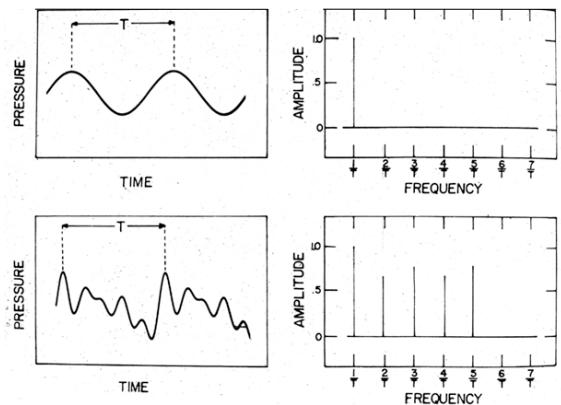
Amplitude = sound pressure level (related to loudness)

Phase = sin vs. cosine vs. another horizontal shift

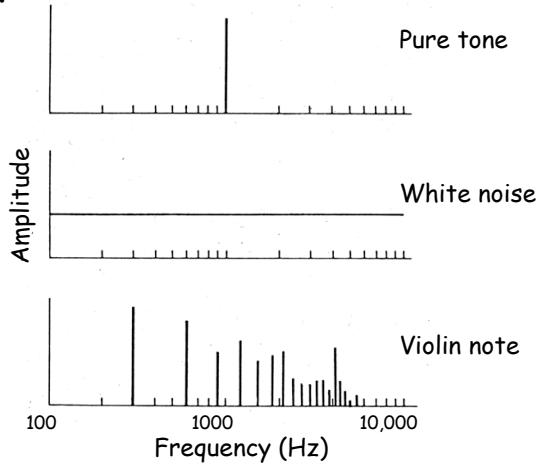
Frequency and amplitude



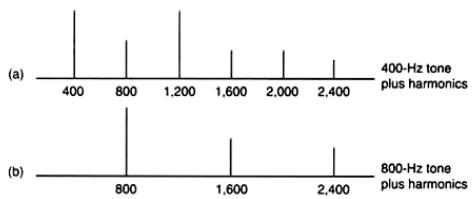
Fourier spectrum representation of sound



Fourier spectra of some sounds



Fundamental frequency and harmonics



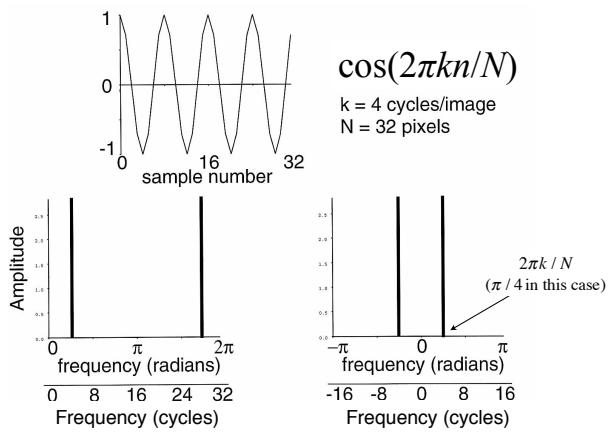
Lots of Fourier transforms

Name	Time domain	Freq domain
Fourier transform	continuous, infinite	continuous, infinite
Fourier series	continuous, periodic	discrete, infinite
DTFT	discrete, infinite	continuous, periodic
DFS	discrete, periodic	discrete, periodic
DFT	discrete, finite	discrete, finite

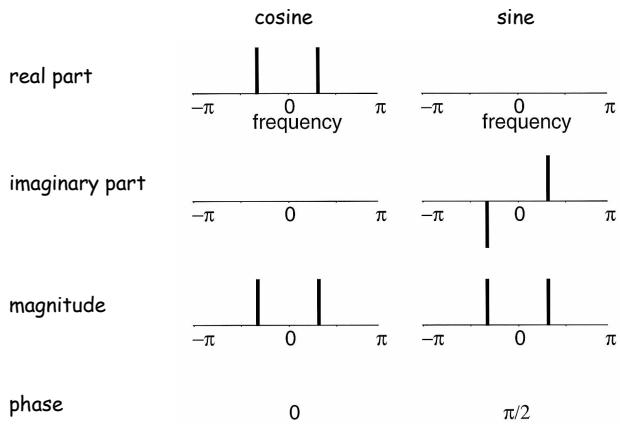
FFT algorithm

- Computes DFT of finite length input.
- Efficient for inputs of length $N = m^n$.
- Produces 2 outputs, each of size/length equal to that of the input: real part (cosine coeffs), imaginary part (sine coeffs).

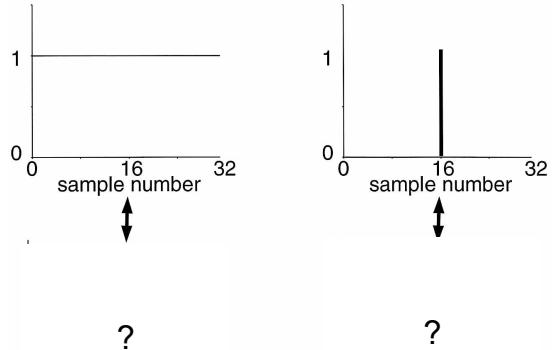
DFT of a cosine



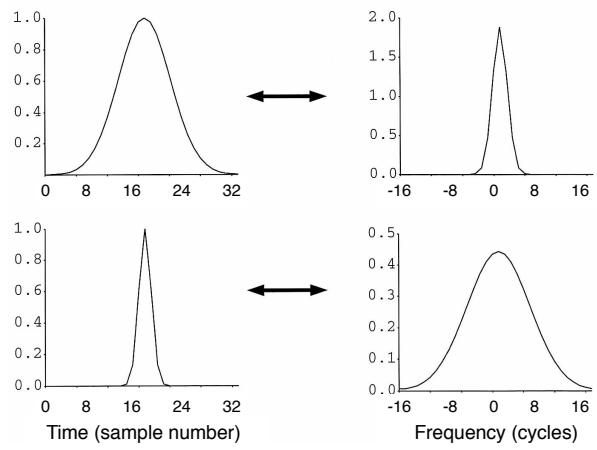
Real and imaginary parts



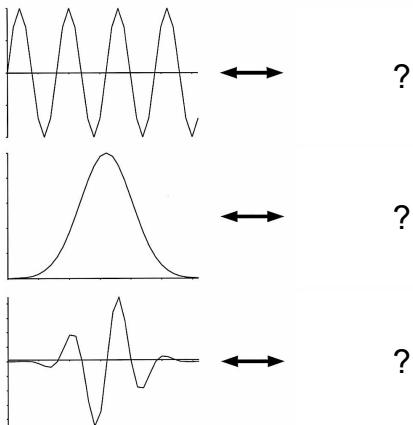
DFT of impulse signal and constant signal



Uncertainty principle



Multiplication and convolution



Discrete Fourier Transform (DFT)

Analysis:

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} & 0 \leq k \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

Synthesis:

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn} & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

$x[n]$: discrete, finite

$X[K]$: discrete, finite

Complex numbers and complex exponentials

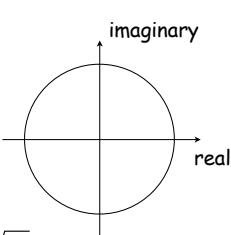
where:

$$\begin{aligned} z &= a + jb = Ae^{j\phi} = A[\cos(\phi) + j\sin(\phi)] \\ a &= A \cos(\phi) & A &= \sqrt{a^2 + b^2} \\ b &= A \sin(\phi) & \phi &= \tan^{-1}(b/a) & j &= \sqrt{-1} \end{aligned}$$

Why bother with complex exponentials?

$$(A_1 e^{j\phi_1})(A_2 e^{j\phi_2}) = A_1 A_2 e^{j(\phi_1 + \phi_2)}$$

amplitudes multiply phases add



Discrete Fourier transform matrix

$$\text{Analysis: } X[k] = \sum_n x[n] \exp(-j2\pi kn/N)$$

For real valued inputs:

$$X_d[k] = \sum_n x[n] \cos(\dots) \quad X_s[k] = \sum_n -x[n] \sin(\dots)$$

$$\begin{pmatrix} X_d[k] \\ X_s[k] \end{pmatrix} = \begin{pmatrix} \text{cosines} \\ \dots \\ \text{sines} \end{pmatrix} \begin{pmatrix} x[n] \end{pmatrix}$$

$$\text{Rows of } \mathbf{P} \text{ called projection functions: } \frac{1}{N} \mathbf{P}^T \mathbf{P} = \mathbf{I}$$

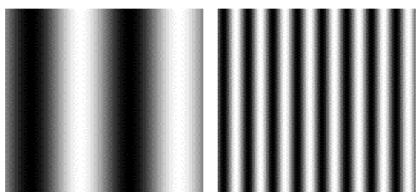
Discrete Fourier transform matrix

$$\text{Synthesis: } x[n] = \sum_k X[k] \exp(j2\pi kn/N)$$

$$\begin{pmatrix} x[n] \end{pmatrix} = \frac{1}{N} \begin{pmatrix} \text{cosines} & \dots & \text{sines} \end{pmatrix} \begin{pmatrix} X_d[k] \\ X_s[k] \end{pmatrix}$$

$$\text{Cols of } \mathbf{B} \text{ called basis functions. } \mathbf{B} = \mathbf{P}^T, \quad \frac{1}{N} \mathbf{B} \mathbf{B}^T = \mathbf{I}$$

Sine wave gratings and spatial frequency

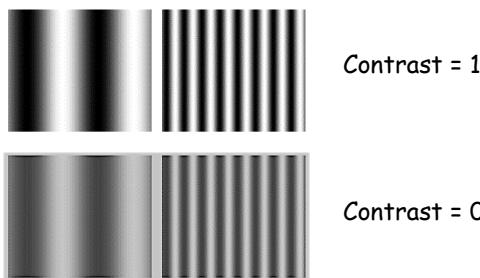


Low SF
1 cpd

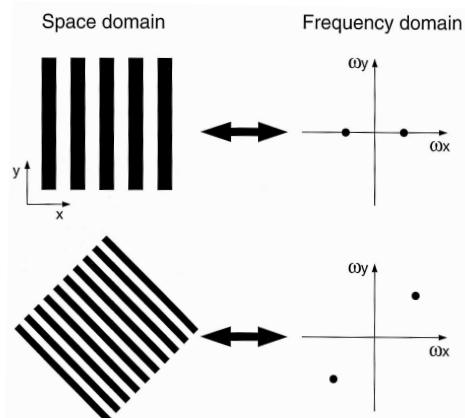
High SF
4 cpd

Measured in cycles per degree (cpd or c/deg or c/°) of visual angle.

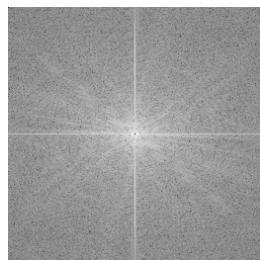
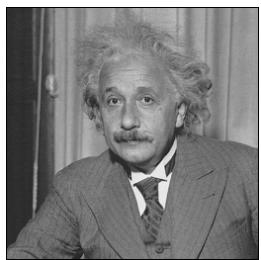
Contrast



Two-dimensional Fourier spectra

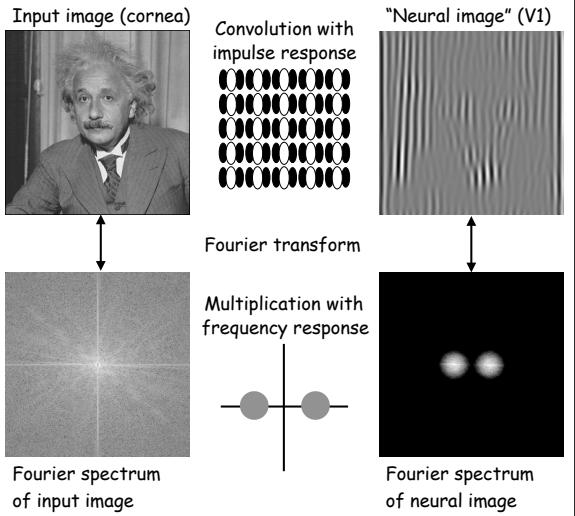


Two-dimensional Fourier spectrum



Intensity in the Fourier spectrum at each location indicates amount of contrast (in the original image) for each spatial frequency and orientation.

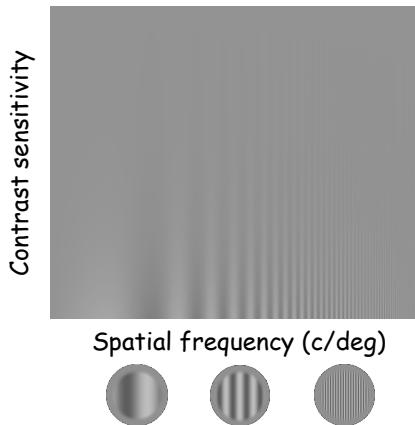
Linear systems analysis



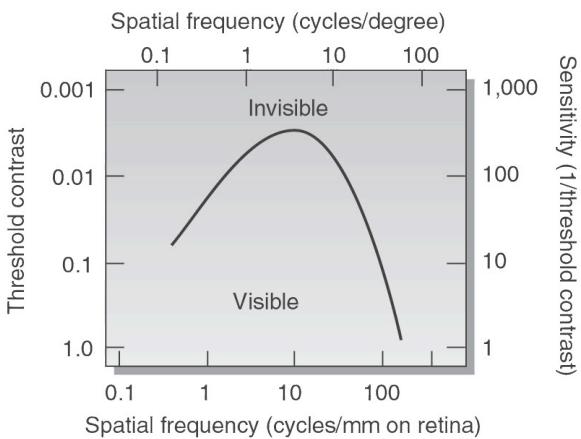
Spatial pattern vision and linear systems theory

- Receptive fields and neural images
- Shift-invariant linear systems and convolution
- Fourier transform and frequency response
- **Applications of linear systems to spatial vision**
 - Contrast sensitivity
 - Spatial frequency and orientation channels
 - Spatial frequency and orientation adaptation
 - Masking

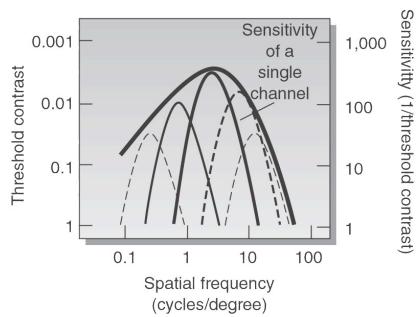
Spatial contrast sensitivity



Spatial contrast sensitivity

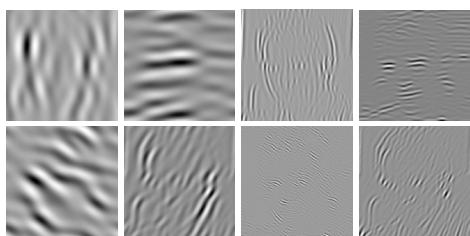


Spatial frequency channels



Each channel is sensitive to a narrow range of frequencies. Overall contrast sensitivity depends on all of them together.

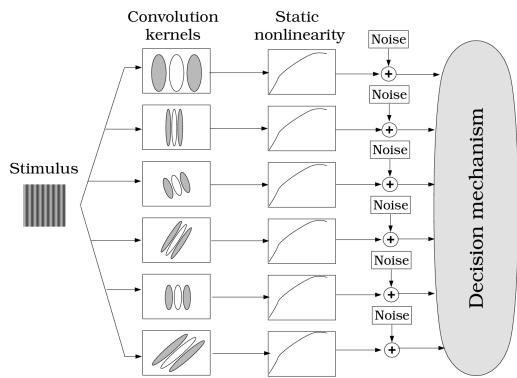
Theory of spatial pattern analysis by the visual system



Low sf filters encode coarse-scale information (large objects, overall shape)

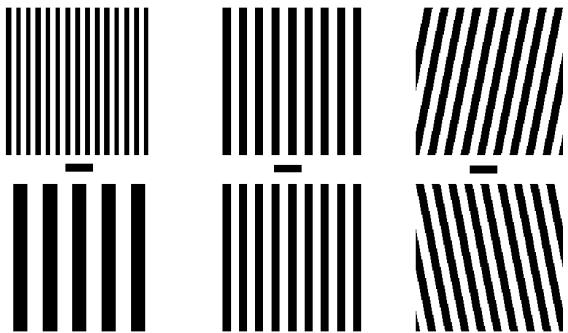
High sf filters encode fine-scale information (small objects, detail)

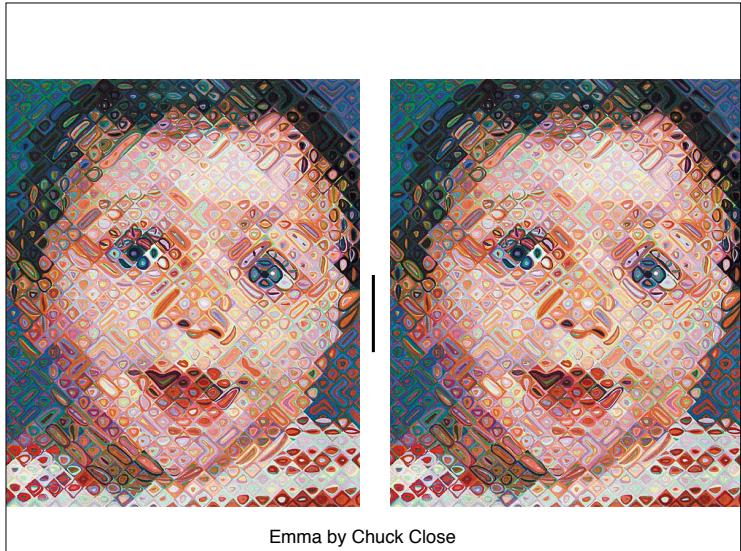
Multi-resolution model



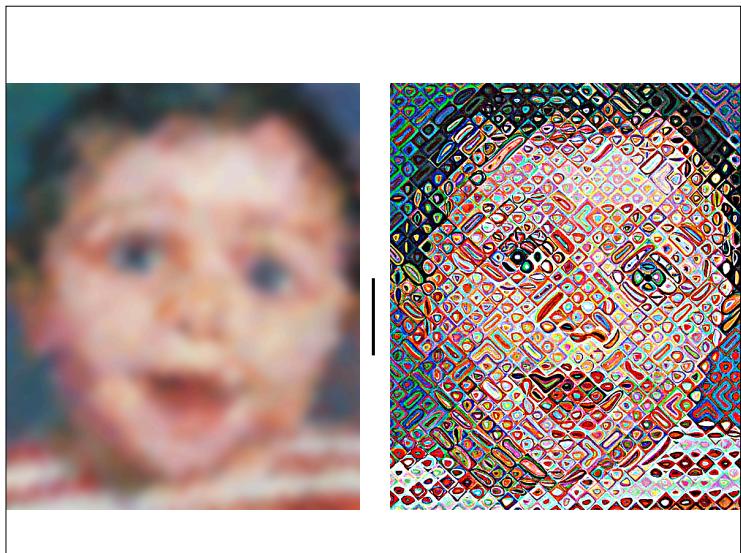
Psychophysical/perceptual evidence for spatial-frequency and orientation selective channels

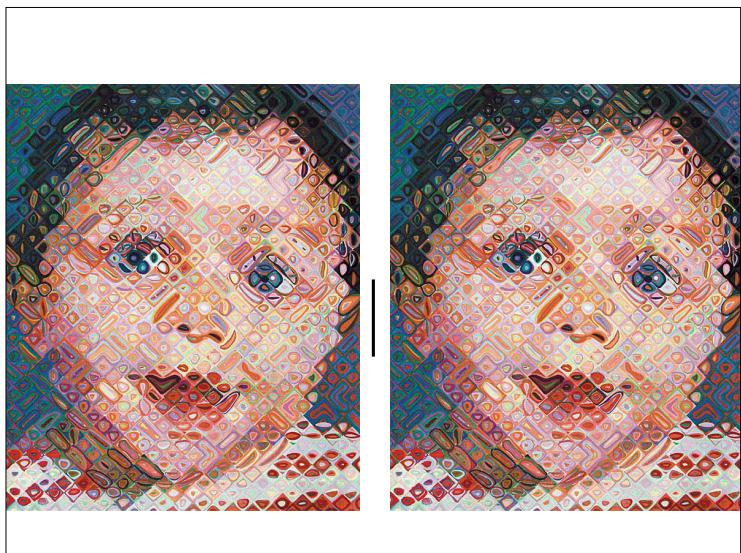
Orientation and spatial frequency selective adaptation: appearance



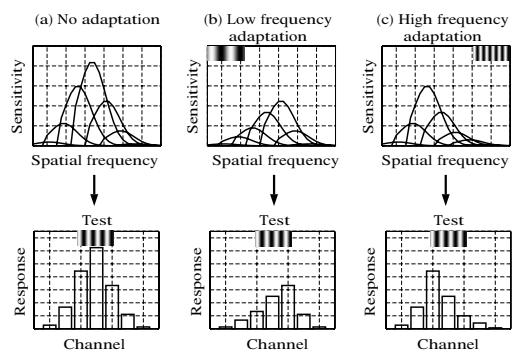


Emma by Chuck Close

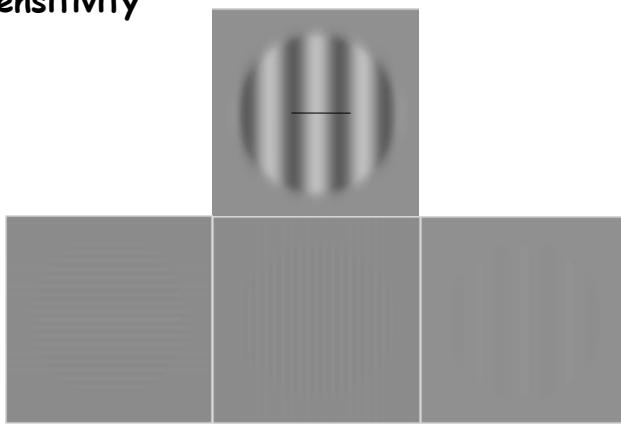




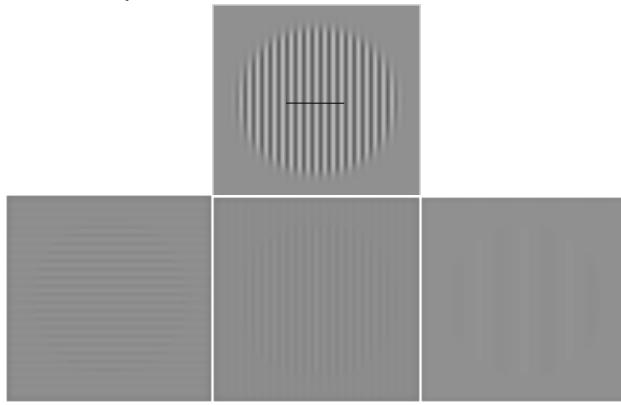
Spatial frequency selective adaptation: appearance



Spatial frequency selective adaptation: sensitivity



Spatial frequency selective adaptation: sensitivity



Contrast sensitivity before & after adaptation

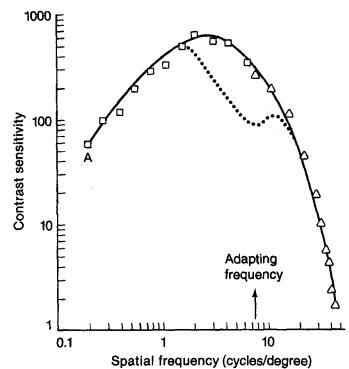
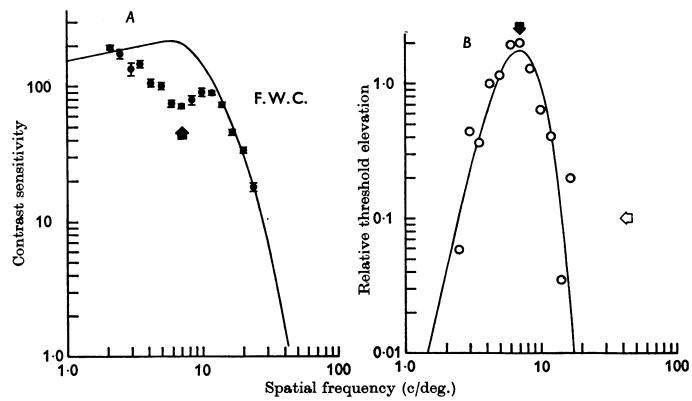
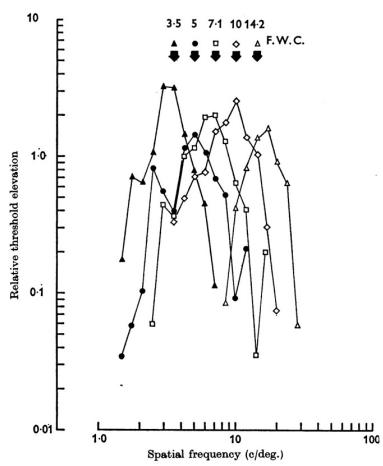


Figure 3.22
Squares and solid curve: Contrast sensitivity function for a sine-wave grating. (From Campbell & Robson, 1968.) Dotted curve: Contrast sensitivity measured after adaptation to a 7.5 cycles/degree grating.

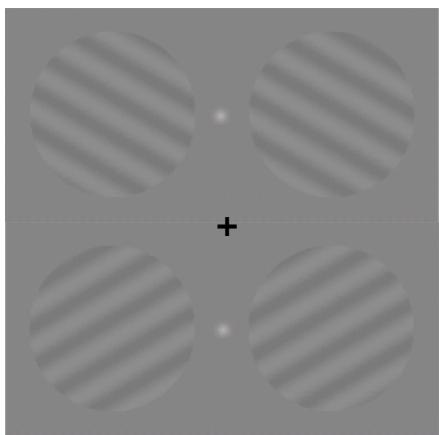
Contrast sensitivity before & after adaptation



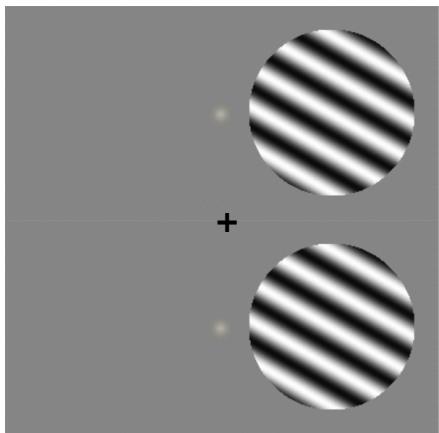
Contrast sensitivity before & after adaptation



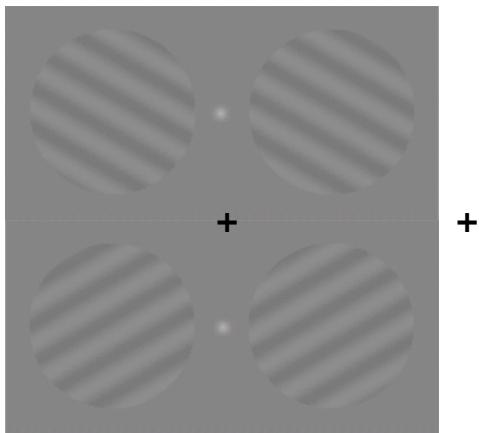
Orientation selective adaptation



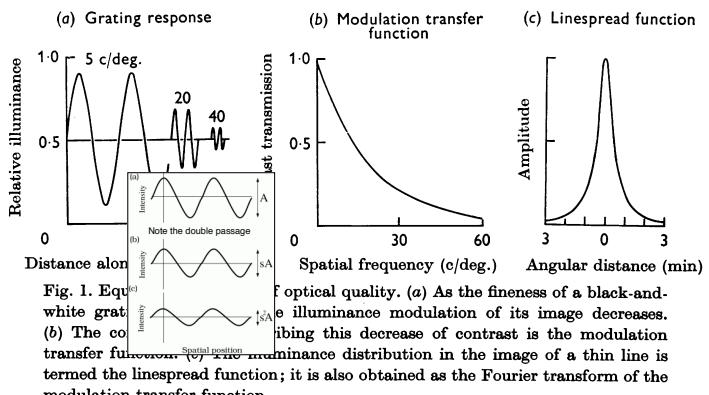
Orientation selective adaptation



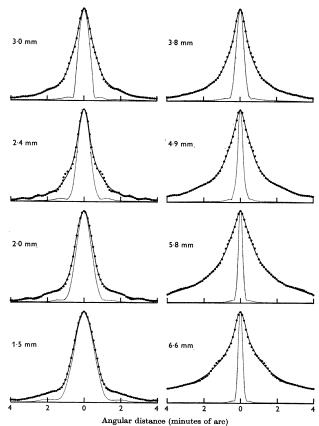
Orientation selective adaptation



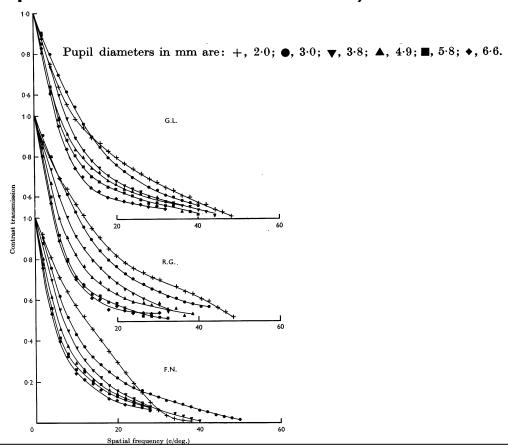
Applications: Optical Line Spread Function (Campbell & Gubisch, 1966)



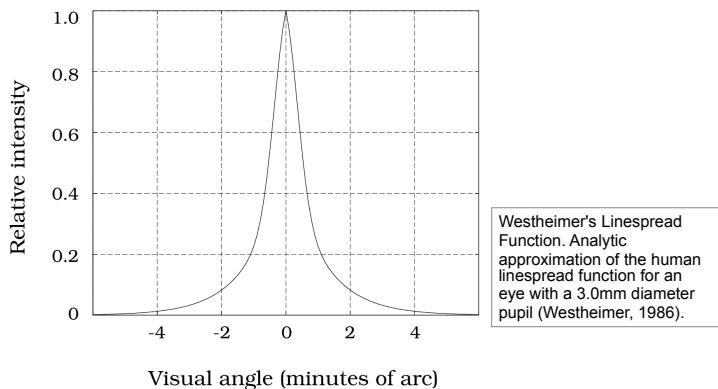
Applications: Optical Line Spread Function (Campbell & Gubisch, 1966)



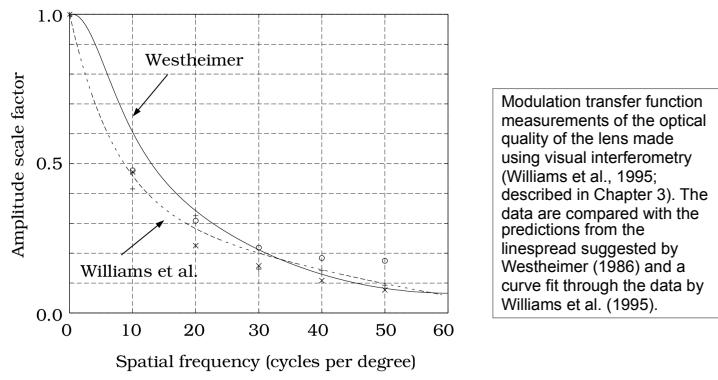
Applications: Optical Line Spread Function (Campbell & Gubisch, 1966)



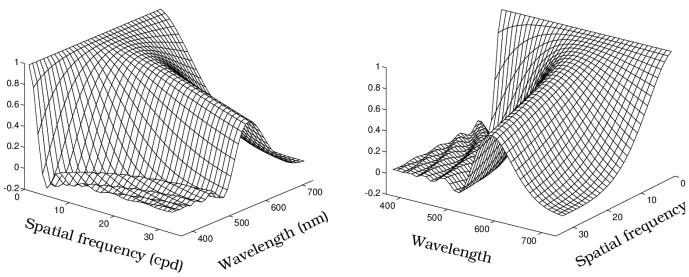
Applications: Optical Line Spread Function (Westheimer, 1986)



Applications: Optical Line Spread Function (Westheimer, 1986)



Applications: Optical Line Spread Function, Wavelength Dependency



OTF of Chromatic Aberration: Two views of the modulation transfer function of a model eye at various wavelengths. The model eye has the same chromatic aberration as the human eye (see Figure 2.23) and a 3.0mm pupil diameter. The eye is in focus at 580nm; the curve at 580nm is diffraction limited. The retinal image has no contrast beyond four cycles per degree at short wavelengths. (From Marimont and Wandell, 1993).

Applications: Optical Line Spread Function, Wavelength Dependency

