Point Estimation

① Point Estimation

Estimator - statistical quantity $T = t(X_1, \dots X_n)$
Estimate - observed value $t = t(x_1, \dots t_n)$

① MME

j-th moment of X $\mu_j'(\vec{\theta}) = E[X^j]$

$M_j' = \frac{1}{n}\sum_{i=1}^{n}(X_i)^j \rightarrow$ functions of the sample r.v & ind. of model parameter

$\mu_j' = M_j'$

$\mu_1' = E[X] = \mu(\theta) \rightarrow M_1' = \overline{X_n}$

$M_j' = \mu_j'(\theta_1, \dots, \theta_k)$ $j = 1, \dots, k$ w.r.t $\theta_1 \dots \theta_k$ $\hat{\theta}_1 \dots \hat{\theta}_k$ are MMEs, $\theta_1 \dots \theta_k$ are function of r.v. of the sample

invariance property estimator of $g(\theta) = g$(estimator of θ)

② MLE

$L(\theta) = f(x_1; \theta) \times \dots \times f(x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$

$L(\hat{\theta}) = f(x_1, \dots, x_n; \hat{\theta}) = \max_{\theta \in \Omega} f(x_1, \dots x_n; \theta)$

invariance property: $\hat{\theta}$ is the MLE of θ & $\tau(\theta)$ is a function of θ, then $\tau(\hat{\theta})$ is the MLE of $\tau(\theta)$

③ Criteria for Estimators

① unbiased estimator: $E[T] = \tau(\theta)$ for all $\theta \in \Omega$

ie for $EXP(\theta)$, $E[\bar{x}] = \theta$, $E[\hat{\theta}] = \theta$

② Among unbiased estimators we prefer the one that has smallest variance.

③ UMVUE: $\begin{cases} \text{if } E[T^*] = \tau(\theta) \\ \text{for any other unbiased estimator } T \text{ of } \tau(\theta) \ Var[T^*] \le Var[T] \text{ for all } \theta \in \Omega. \end{cases}$

④ CRLB

$$Var[T] \ge \frac{(\tau'(\theta))^2}{E[\frac{\partial}{\partial\theta} \log f(X_1, \dots X_n; \theta)^2]} \qquad Var[T] \ge \frac{(\tau'(\theta))^2}{nE[(\frac{\partial}{\partial\theta}\log f(X;\theta))^2]}$$

Eg. $\frac{\partial}{\partial\theta}\log f(x;\theta) = -\frac{1}{\theta} + \frac{x}{\theta^2} = \frac{x-\theta}{\theta^2}$ $E[(\frac{x-\theta}{\theta^2})^2] = \frac{1}{\theta^4}E[(X - E(X))^2] = \frac{1}{\theta^4}Var[X] = \frac{1}{\theta^2}$

① $E[(\frac{\partial}{\partial\theta}\log f(X;\theta))^2] = \underline{-E[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)]}$ ② unbiased $T(\theta)$ attains CRLB. linear function of $\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log f(X_i; \theta)$

Thm: If there is an unbiased estimator of $\tau(\theta)$ with var = CRLB, then only for linear function of $\tau(\theta)$ unbiased estimators will exist with variance = CRLB

$E[(X-p)^2] = Var(X) + (E(X) - p)^2$

$\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log(f(x;\theta)) = \frac{1}{a}(T - \tau(\theta))$ as $E(X) \to p$

⑤ Efficiency

$re(T, T^*) = \frac{Var(T^*)}{Var(T)}$ $\qquad b(T) = E[T] - \tau(\theta)$ $\qquad MSE(T) = E(T - \tau(\theta))^2$
$= Var(T) + (b(T))^2$

⑥ Large-Sample properties

① Simple consistency: $\lim_{n\to\infty} P[|T_n - \tau(\theta)| \le \varepsilon] = 1$ $\quad T_n \xrightarrow{P} \tau(\theta)$

② MSE consistency: $\lim_{n\to\infty} E[T_n - \tau(\theta)]^2 = 0$

③ Asymptotically unbiased $\lim_{n\to\infty} E[T_n] = \tau(\theta)$

- 9.4.1 $\{T_n\}$ is MSE consistent if Asym. unbiased + $\lim_{n\to\infty} Var(T_n) = 0$
- 9.4.2 If $\{T_n\}$ is MSE consistent, it is also simply consistent
- 9.4.3 $\{T_n\}$ is simply consistent for $\tau(\theta)$ & $g(t)$ is continuous $\{g(T_n)\}$ is simply consistent for $g(\tau(\theta))$

④ Asymptotic Efficiency: $are(T_n, T_n^*) = \lim_{n\to\infty} \frac{Var(T_n^*)}{Var(T_n)}$

Let T be an estimator of $\tau(\theta)$

$E[T] = \tau(\theta)$

AND $\quad \Rightarrow$ T is an UMVUE of $\tau(\theta)$

$Var[T] = CRLB$