6. Function of Multivariate R.V

6.1 Three methods for transformations:

① CDF method: $W=h(X,Y)$

① supporter of W:

② $F_W(w)=P(W\le w)=P(h(x,y)\le w)=\iint_{h(x,y)\le w} f_{X,Y}(x,y)\,dx\,dy \overset{ind.}{=} \iint f_X(x) f_Y(y)\,dx\,dy$ for domain of W.

③ $f_W(w)=F_W'(w)=\ldots=$ for domain of W

② Transformation method:

Discrete case:

(thm 6.1) If X is a k-dimensional discrete random variable with joint pdf $f_X(x)$ and $Y=u(X)$ defines 1-to-1 transformation, then the joint pdf of Y is: $f_Y(y_1,y_2,\ldots y_k)=f_X(x_1,x_2,\ldots x_k)$ where $x_1,x_2,\ldots x_k$ are solutions to $y=u(x)$

① inverse transformation: $y_1=x_1$ & $y_2=x_1+x_2$

② $f_{Y_1,Y_2}(y_1,y_2)=f_{X_1,X_2}(y_1,y_2-y_1)$

Continuous case:

(thm 6.2) if X is a k-dimensional continuous random variables with joint pdf $f_X(x)$ on the support set A. $Y=u(X)$ then the joint pdf of Y is given by:

$f_Y(y_1,y_2,y_3,\ldots y_k)=f_X(x_1,\ldots x_k)|J|$ where $x_1,x_2,\ldots x_k$ are solutions of $y=u(x)$

$J\rightarrow$ "Jacobian" = determinant of k×k-matrix $\begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \vdots \\ \vdots & & \vdots \\ \vdots & \cdots & \frac{\partial x_k}{\partial y_k} \end{pmatrix}$

① inverse function: $y_1=x_1$ & $y_2=x_1+x_2 \Rightarrow x_1=y_1$ & $x_2=y_2-y_1$

② $J=\det\begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix}=\det\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}=1$ $\quad f_Y(y)=f_X(y_1,y_2-y_1)|J|=e^{-y_1-(y_2-y_1)}|1|=e^{-y_2}$ for $0\le y_1\le y_2$

③ Support:

(Reduced Dimension): ① Choose "auxiliary" r.v $V=X_1$

② Apply trans method to $(X_1,X_2)\rightarrow(Y_1=X_1X_2,\ V=X_1)$

① inverse: $x_1=v,\ x_2=\frac{y}{v}$

② Support: $0\le v\le 1$ and $0\le y\le v$

③ $J=\det\begin{pmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{y}{v^2} \end{pmatrix}=-\frac{1}{v}$

④ $f_{Y,V}(y,v)=f_{X_1,X_2}(v,\frac{y}{v})|-\frac{1}{v}|=\ldots=\frac{1}{v}$ for $0\le y\le v\le 1$

③ $f_Y(y)=\int_{-\infty}^{\infty} f_{Y,V}(y,v)\,dv=\int_y^1 \frac{1}{v}\,dv=-\log y,\ 0\le y\le 1$

③ MGF-method: $W=X-Y,\ U=X+Y$

$M_{U,W}(t_1,t_2)=E[e^{t_1U+t_2W}]=E[e^{(t_1+t_2)X}e^{(t_1-t_2)Y}] \overset{ind.}{=} E[e^{(t_1+t_2)X}]\,E[e^{(t_1-t_2)Y}]$

$=M_X(t_1+t_2)\,M_Y(t_1-t_2)=e^{\frac{1}{2}(t_1+t_2)^2}e^{\frac{1}{2}(t_1-t_2)^2}=\exp[\frac{1}{2}(2t_1^2+2t_2^2)]$

$=\exp[0t_1+0t_2+\frac{1}{2}(2t_1^2+2t_2^2+0\sqrt{2}\sqrt{2}t_1t_2)]$

6.2 Distributions of sums of r.v.s

① Expected value & variance of sums of r.v.s:

$$E\left[\sum_{i=1}^{k} X_i\right] \begin{cases} \sum_{i=1}^{k} E[X_i] & \text{always} \\ kE[X] = k\mu & \text{i.i.d.} \end{cases}$$

$$Var\left[\sum_{i=1}^{k} X_i\right] \begin{cases} \sum_{i=1}^{k} Var[X_i] + 2\sum\sum_{i<j} Cov(X_i, X_j) & \text{always} \\ \sum_{i=1}^{k} Var[X_i] & \text{independence} \\ kVar[X] = k\sigma^2 & \text{i.i.d.} \end{cases}$$

② Distr. of sums of independent r.v.s

$$\begin{cases} \text{independent} & \Rightarrow M_Y(t) = \prod_{i=1}^{k} M_{X_i}(t) \\ \text{i.i.d} & \Rightarrow M_Y(t) = (M_X(t))^k \end{cases}$$

① BIN, $Y \sim BIN(\Sigma m_i, p)$ / $Y \sim BIN(km, p)$

② POI, $Y \sim POI(\Sigma \mu)$ / $Y \sim POI(k\mu)$

③ NEGBIN, $Y \sim NEGBIN(\Sigma r_i, p)$ / $Y \sim NEGBIN(kr, p)$

④ EXP, $Y \sim GAM(\frac{1}{\lambda}, k)$

⑤ GAM, $Y \sim GAM(\theta, \Sigma r_i)$ / $Y \sim GAM(\theta, kr)$

⑥ N, $Y \sim N(\Sigma\mu_i, \Sigma\sigma_i^2)$ / $Y \sim N(k\mu, k\sigma^2)$

⑦ χ^2, $Y \sim \chi^2(\sum_{i=1}^{k} v_i)$ / $Y \sim \chi^2(kv)$

③ Convolution formula for $S = X+Y$.

(thm 6.3) If X, Y indep cont r.v.s and $S = X+Y$, then $f_S(s) = \int_{-\infty}^{\infty} f_X(x) f_Y(s-x)\,dx$

6.3 t-distribution F-distribution

① t-distribution:

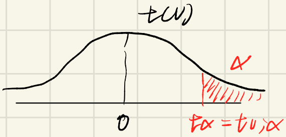
① (Def 6.1) $T \sim t_v$ / $T \sim t(v)$ $\quad f(x) = \frac{\Gamma(\frac{1}{2}v + \frac{1}{2})}{\Gamma(\frac{1}{2}v)\sqrt{\pi v}} \left(1 + \frac{x^2}{v}\right)^{-\frac{1}{2}(v+1)}$ $(-\infty < x < \infty)$ where $v = 1, 2, 3 \dots$ $\quad \Leftarrow \lim_{v \to \infty} t(v) = N(0,1)$

② Relation btw t-, normal & chi-square distributions:

(thm 6.4) $Z \sim N(0,1)$ $U \sim \chi^2(v)$ $T = \frac{Z}{\sqrt{\frac{U}{v}}} \sim t(v)$

③ mean & variance of t-distribution:

(thm 6.5) Let $T \sim t(v)$, with integer parameter $v > 2$ $\; E[T] = 0$, $Var[T] = \frac{v}{v-2}$



② F-distribution:

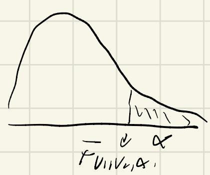
① $F \sim F_{v_1, v_2}$ / $F \sim F(v_1, v_2)$ $\quad f(x) = \frac{\Gamma(\frac{1}{2}v_1 + \frac{1}{2}v_2)}{\Gamma(\frac{1}{2}v_1)\Gamma(\frac{1}{2}v_2)} \sqrt{\frac{v_1^{v_1} v_2^{v_2} x^{v_1 - 2}}{(v_2 + v_1 x)^{v_1 + v_2}}}$ for $x > 0$ where $v_1, v_2 = 1, 2, 3$

v_1 is numerator v_2 is denominator

② properties: (thm 6.6) Let $U_1 \sim \chi^2(v_1)$ and $U_2 \sim \chi^2(v_2)$, U_1 & U_2 are independent.

(relation with χ^2) $\quad F_{v_1, v_2} = \frac{U_1 / V_1}{U_2 / V_2} \sim F(V_1, V_2)$

③ mean & variance: $E[F] = \frac{v_2}{v_2 - 2}$ $\quad Var[F] = \frac{2v_2^2(v_1 + v_2 - 2)}{v_1(v_2-2)^2(v_2-4)}$ $\quad F_{v_1, v_2; 1-\beta} = \frac{1}{F_{v_2, v_1; \beta}}$



6.4 Samples, Samples from normal distribution

① Def:

(def 6.3) $X_1 \dots X_n$ form a sample from distribution with sample size n, if the joint pdf can be written as

$f_{X_1 \dots X_n}(x_1 \dots x_n) \overset{i.i.d}{=} f(x_1) f(x_2) \dots f(x_n)$ $\longrightarrow$ $X_1 X_2 \dots X_n$ random drawn from ppl

(def 6.4) $X_1 \dots X_n$, so $T = t(X_1, \dots X_n)$, which donot involve any unknown parameters, is Sample statistics

Sample mean: $\bar{x}$; Sample variance: S^2 ; sample proportion: $\hat{p}$

② 1) Sample mean $\bar{x}$ (Def 6.5) $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$

(thm 6.8) If $x_1..x_n$ is a sample from a distribution $E(X)=\mu$, $Var[X]=\sigma^2$ then:

$E[\bar{x}]=\mu$ and $Var[\bar{x}]=\frac{\sigma^2}{n}$

2) Sample variance S^2 (Def 6.6) $S^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$

(thm 6.9) Alternative expressions: $S^2=\frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2-n\bar{x}^2\right)=\frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2-\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2\right)$

(thm 6.10) If $x_1..x_n$ is a sample from a distribution with $E(X)=\mu$ and $Var[X]=\sigma^2$, then:

$E[S^2]=\sigma^2$

3) Sample proportion $\hat{p}$ (Def 6.7) $x_i\sim BIN(1,p)$, $Y=\sum_{i=1}^{n}x_i\sim BIN(n,p)$, $\hat{p}=\frac{Y}{n}$

(thm 6.11) If $x_1..x_n$ drawn from Bernoulli distribution with parameter p. i.e. then

$E[\hat{p}]=p$; $Var[\hat{p}]=\frac{p(1-p)}{n}=\frac{pq}{n}$

③ Samples from normal distribution:

1) (thm 6.12) Let $x_1,...x_n$ be a sample from a normal distribution with mean μ and variance σ^2, i.e. $X_i\sim N(\mu,\sigma^2)$. Then:

$$\bar{x}\sim N\left(\mu,\frac{\sigma^2}{n}\right) \quad / \quad \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}\sim N(0,1)$$

Compare with CLT: $x_1..x_n$ be a sample from a distribution with μ and variance σ^2. Then $\bar{x}\overset{\text{approx.}}{\underset{\text{large } n}{\sim}}N\left(\mu,\frac{\sigma^2}{n}\right)$ $\quad \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}\tilde{\sim}N(0,1)$

confidence interval: $100(1-\alpha)\%$ prob. interval for $\bar{x}$: $(\mu-Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}},\ \mu+Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}})$

2) (thm 6.13) Let $x_1,...x_n$ be a sample from a normal distribution with mean μ & variance σ^2 i.e. $X_i\sim N(\mu,\sigma^2)$. Then:

$$\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}\sim\chi^2(n) \text{ and } n\left(\frac{\bar{x}-\mu}{\sigma}\right)^2\sim\chi^2(1)$$

3) (thm 6.14) Let $x_1,...x_n$ be a sample from a normal distribution with mean μ & variance σ^2 i.e. $X_i\sim N(\mu,\sigma^2)$. Then:

① $\bar{x}$ and S^2 are independent ② $\frac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$ → ($100(1-\alpha)\%$ prediction interval)

$\left(\frac{\sigma^2}{n-1}\chi^2_{n-1,1-\frac{\alpha}{2}},\ \frac{\sigma^2}{n-1}\chi^2_{n-1,\frac{\alpha}{2}}\right)$

6.5 Order Statistics:

① First & last order statistics:

(thm 6.15) ① 1st order statistics: $W=\min(x_1...x_n)=x_{1:n}$ ② last order statistics: $V=\max(x_1,...,x_n)=x_{n:n}$

$F_W(w)=1-(1-F(w))^n$ $\qquad F_V(v)=(F(v))^n$

$f_W(w)=n(1-F(w))^{n-1}f(w)$ $\qquad f_V(v)=nF(v)^{n-1}f(v)$

② All order statistics:

① $y_1=x_{1:n}=\min(x_1...x_n)$ ② $y_2=x_{2:n}$ ③ $y_3=x_{3:n}$ .... ⓝ $y_n=x_{n:n}=\max(x_1,...x_n)$

(thm 6.16) Let $x_1..x_n$ be a sample from pdf $f(x)$. Then $f_{\underline{y}}(\underline{y})=n!\prod_{i=1}^{n}f(y_i)$ for $y_1<y_2<...y_n$

Ex. $Y_2=X_{2:3}$ $\quad f_{\underline{y}}(\underline{y})=3!f(y_1)f(y_2)f(y_3)=48y_1y_2y_3$ for $0<y_1<y_2<y_3<1$

$f_{y_2}(y_2)=\int_{y_2}^{1}\int_{0}^{y_2}48y_1y_2y_3\,dy_1dy_3=...=12(y_2^3-y_2^5)$ for $0<y_2<1$

CDF of kth order statistics: (thm 6.17) For any sample from a distribution with CDF $F(x)$, the CDF of the k-th order statistics is given by:

$F_{Y_k}(y)=\sum_{j=k}^{n}\binom{n}{j}[F(y)]^j[1-F(y)]^{n-j}$

pdf of kth order statistics: (thm 6.18) $f_{Y_k}(y)=\frac{n!}{(k-1)!(n-k)!}[F(y)]^{k-1}[1-F(y)]^{n-k}f(y)$

joint pdf: (thm 6.19) $f_{Y_i,Y_j}=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}f(y_i)f(y_j)[F(y_i)]^{i-1}[F(y_j)-F(y_i)]^{j-i-1}[1-F(y_j)]^{n-j}$

Sample range: $R = X_{n:n} - X_{1:n}$

Sample median: $\begin{cases} X_{\frac{n+1}{2}:n} & \text{if } n \text{ is odd} \\ \frac{1}{2}\left(X_{\frac{n}{2}:n} + X_{\frac{n}{2}+1:n}\right) & \text{if } n \text{ even} \end{cases}$