Confidence Interval

Hypothesis Testing

Confidence Interval

① confidence interval

$l(X_1 \dots X_n)$, $u(X_1 \dots X_n)$ is called $100\gamma\%$ confidence interval for θ if $P[l(X_1 \dots X_n) \le \theta \le u(X_1 \dots X_n)] = \gamma$

- one-sided $100\gamma\%$ lower confidence limit for θ $\quad P[l(X_1 \dots X_n) < \theta] = \gamma$
- one-sided $100\gamma\%$ upper confidence limit for θ $\quad P[\theta < u(X_1 \dots X_n)] = \gamma$

Confidence interval for $\tau(\theta)$
- monotonic increasing $(\tau(\theta_L), \tau(\theta_U))$
- monotonic decreasing $(\tau(\theta_U), \tau(\theta_L))$

② Pivotal Quantity

$Q = q(X_1 \dots X_n; \theta)$ is called Pivotal Quantity if its pdf doesn't depend on θ or any other unknown parameters.

relation with CI. $\quad P[q_1 < Q < q_2] = \gamma$

① Definition

- $X_i \sim N(\mu, \sigma) \Rightarrow \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$
- $X_i \sim EXP(\theta) \Rightarrow \frac{2n\bar{X}_n}{\theta} \sim \chi^2(2n)$
- $X_i \sim UNIF(0,\theta) \Rightarrow Q = \frac{X_{n:n}}{\theta}$

② MLEs with location & Scale parameters

- location para. $\quad$ if $f(x;\theta) = f_0(x-\theta)$
- scale para. $\quad$ if $f(x;\theta) = \frac{1}{\theta} f_0\left(\frac{x}{\theta}\right)$
- location-scale para. $\quad$ if $f(x;\theta_1,\theta_2) = \frac{1}{\theta_2} f_0\left(\frac{x-\theta_1}{\theta_2}\right)$

Thm
- If θ is a location parameter $Q = \hat{\theta} - \theta$ is a pivotal quantity
- if θ is a scale parameter. $Q = \frac{\hat{\theta}}{\theta}$ is a pivotal quantity
- $\frac{\hat{\theta}_1 - \theta_1}{\hat{\theta}_2}$ & $\frac{\hat{\theta}_2}{\theta_2}$

} there is an MLE $\hat{\theta}$ for θ.

$F(X;\theta) \sim UNIF(0,1)$

$\longrightarrow -\log F(X;\theta) \sim EXP(1) \Rightarrow$ $-\Sigma \log F(X_i;\theta) \sim GAMMA$, $-2\Sigma \log F(X_i;\theta) \sim \chi^2$

③ probability integral transform

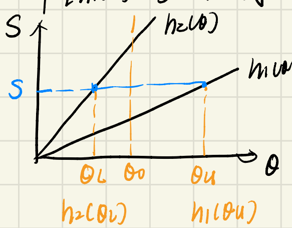
thm 6.3.3 if $X \sim f(x;\theta)$ & $F(x;\theta)$ is the CDF, then $F(X;\theta) \sim UNIF(0,1)$

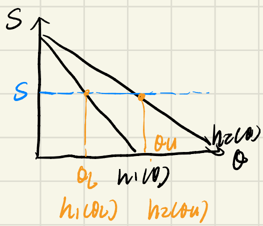
④ Approximate CI. use CLT

$\sqrt{n}\frac{\hat{p}-p}{\sqrt{p(1-p)}} \xrightarrow{d} Z \sim N(0,1)$ $\quad$ $\hat{p} \xrightarrow{P} p$ $\Rightarrow \underbrace{\sqrt{n}\frac{\hat{p}-p}{\sqrt{\hat{p}(1-\hat{p})}}}_{Q.} \xrightarrow{d} Z \sim N(0,1)$

③ General Method

$P[h_1(\theta) < S < h_2(\theta)] = \gamma = 1-\alpha$





① find $h_i(\theta)$

② find CL via $h_1(\theta_U) = s$, $h_2(\theta_L) = s$

$G(h_1(\theta);\theta) = \alpha_1 \quad G(h_2(\theta);\theta) = 1-\alpha_2 \quad (\uparrow)$

$G(s;\theta_L) = 1-\alpha_2 \quad G(s;\theta_U) = \alpha_1 \quad (\downarrow)$

$G(s;\theta_L) = \alpha_1 \quad G(s;\theta_U) = 1-\alpha_2 \quad (\uparrow)$

For $\uparrow$: $G(s;\theta_U) = \alpha$

$P[S \le s \mid \theta = \theta_U] = \alpha$

Conservative CI: $G(h_1(\theta);\theta) = \alpha_1 \quad P(S < s; \theta_L] = 1-\alpha_2, \quad P(S < s; \theta_U) = 1-\alpha_2$

④ Two Samples

- $\bar{Y} - \bar{X} - (\mu_2 - \mu_1) \sim N\left(0, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$
- $\frac{S_1^2 \sigma_2^2}{S_2^2 \sigma_1^2} \sim F(n_1 - 1, n_2 - 1)$
- $T = \frac{\bar{Y} - \bar{X} - (\mu_2 - \mu_1)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$

Hypothesis Testing:

① Test of hypothesis

① Simple hypothesis & composite hypothesis

② Null hypothesis & alternating hypothesis

critical region → α - significant level

two-sided alternative hypothesis test: $H_0: \mu=\mu_0$, $H_1: \mu\neq\mu_1$

① α test: $C=\{(x_1,\dots,x_n)\in\mathbb{R}^n \mid |\bar{x}-\mu_0|>c\}$

$\alpha = P[|\bar{x}-\mu_0|>c \mid \mu=\mu_0] = 2P[Z>\frac{c}{\sigma/\sqrt{n}}]$

② β-test: $\beta = P[|\bar{x}-\mu_0|<c \mid \mu_{true}=\mu] = P[\bar{x}<\mu_0+c \,/\, \bar{x}>\mu_0-c] = \Phi\left(\frac{\mu_0+c-\mu}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{\mu_0-c-\mu}{\sigma/\sqrt{n}}\right)$

② Composite hypotheses

③ Test for normal, binomial & Poisson

① test statistics

② critical region

③ $\alpha = P[\bar{x}>c \mid \theta_{true}=\theta_0]$; C

④ $\pi(\theta)$

④ Most Powerful Test

Neyman-Pearson: ① $\lambda(\vec{x};\theta_0,\theta_1)=\frac{f(\vec{x};\theta_0)}{f(\vec{x};\theta_1)}$ ② $C^*=\{\vec{x} \mid \lambda(\vec{x};\theta_0,\theta_1)\le k\}$

↳画图 k_1的取值范围。

⑤ UMP Tests

Simple + NP verify the final critical region is free of θ_1/H_1

UMP tests uniformly leads to UMA

⑥ GLR Test

$$\lambda(\vec{x}) = \frac{\max_{\vec{\theta}\in\Omega_0} f(\vec{x};\vec{\theta})}{\max_{\vec{\theta}\in\Omega} f(\vec{x};\vec{\theta})} = \frac{f(\vec{x};\hat{\vec{\theta}}_0)}{f(\vec{x};\hat{\vec{\theta}})}$$

restricted MLE

regular MLE

$\lambda(\vec{x})\le k$