Limiting Distribution

① limiting Distribution:

$Y_n \xrightarrow{d} y$, $\lim_{n\to\infty} G_n(y) = G(y)$ $\quad \left(\lim_{n\to\infty} (1+\frac{c}{n})^{bn} = e^{cb}\right)$

② Convergence in probability:

$Y_n \xrightarrow{P} y$, $\lim_{n\to\infty} P(|Y_n - y| < \varepsilon) = 1$

Stochastic convergence: $Y_n \xrightarrow{P} c$ $\quad P[Y=c]=1 \rightarrow \lim_{n\to\infty} P[|Y_n - c| < \varepsilon] = 1$

① The law of large number (for Bernoulli distribution) $\bar{x}_n \xrightarrow{P} \mu$ with $E(x_i)=\mu$

LLN $\quad \bar{x}_n \xrightarrow{P} \mu$ $\quad Var(x_i) = \sigma^2$

② Convergence in probability & asymptotic convergence.

If $\frac{Y_n - m}{c/\sqrt{n}} \xrightarrow{d} Z \sim N(0,1)$, then $Y_n \xrightarrow{P} m$

③ given $X_n \xrightarrow{P} c$, $Y_n \xrightarrow{P} d$, then:

a. $aX_n + bY_n \xrightarrow{P} ac + bd$

b. $X_n Y_n \xrightarrow{P} cd$

c. $\frac{X_n}{c} \xrightarrow{P} 1$ for $c \neq 0$

d. $\frac{1}{X_n} \xrightarrow{P} c^{-1}$

e. $\sqrt{X_n} \xrightarrow{P} \sqrt{c}$

④ Slutsky Theorem: Given $X_n \xrightarrow{P} c$, $Y_n \xrightarrow{d} Y$

a. $X_n + Y_n \xrightarrow{d} c + Y$

b. $X_n Y_n \xrightarrow{d} cY$

c. $\frac{Y_n}{X_n} \xrightarrow{d} \frac{Y}{c}$ $\quad c \neq 0$

③ Asymptotic Convergence:

$\frac{Y - m}{c/\sqrt{n}} \xrightarrow{d} Z \sim N(0,1) \Rightarrow Y \underset{\text{large } n}{\overset{\text{app.}}{\sim}} N(m, \frac{c^2}{n})$

Eg

① limiting distribution of central order statistics $X_{k:n}$: $X_{k:n} \overset{\text{large } n.}{\sim} N\left(x_p, \frac{p(1-p)}{nf(x)^2}\right)$

② Central Limit Theorem: $Z_n = \frac{\sum_{i}^{n} X_i - n\mu}{\sqrt{n}\sigma}$ or $\sqrt{n}\,\frac{\bar{X}_n - \mu}{\sigma} \xrightarrow{d} Z \sim N(0,1)$

$W_n = \sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} W \sim N(0, \sigma^2)$

④ Function of Sequence RV.

① $Y_n \xrightarrow{P} c$, $g(Y_n) \xrightarrow{P} g(c)$

② $Y_n \xrightarrow{d} y$, $g(Y_n) \xrightarrow{d} g(y)$

③ $Z \sim N(0,1)$ $\quad \frac{Y_n - m}{c/\sqrt{n}} \xrightarrow{d} Z$

$\frac{g(Y_n) - g(m)}{|c g'(m)|/\sqrt{n}} \xrightarrow{d} Z$

Example helping understand:

① limiting distribution of central order statistics:

$P[X \le x_p] = p$, x_p is percentile $f_X(x_p) \ne 0$

If $\frac{k}{n} \to p$ $X_{k:n}$ is asymptotic normally distributed with asymptotic mean x_p.

& variance $\frac{c^2}{n}$ where $c^2 = \frac{p(1-p)}{f_X(x_p)^2}$

Example $F(x) = 1 - e^{-x}$ $k = \frac{n+1}{2}$ (median)

$F(x_p) = p = 0.5$ $x_p = -\log(1-p)$

$1 - e^{-x_p} = p$ $x_{0.5} = -\log 0.5 = \log 2$

$c^2 = \frac{p(1-p)}{(f(x_p))^2} = \frac{0.5(1-0.5)}{(e^{-\log 2})^2} = 1.$

② Finding limiting distribution:

Let $Y = nX_{1:n}$

Then $G_n(y) = P[Y \le y] = P[nX_{1:n} \le y] = P[X_{1:n} \le \frac{y}{n}] = F_{X_{1:n}}(\frac{y}{n})$

③ Transformation method find pdf:

① Define $Y = g(X)$

② $x = g^{-1}(Y)$

③ $\left|\frac{dx}{dy}\right|$

④ $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$ $(x = g^{-1}(y))$

Ex. $X \sim U(0,1)$ $Y = X^2$

$X = \sqrt{Y}$

$\left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{y}}$

$f_Y(y) = \frac{1}{\sqrt{y}}$ For $0 < y < 1$

④ function of R.V.

(Theorem 7.7.2) If $Y_n \xrightarrow{P} c$, then for any function $g(y)$ that is continuous at $y = c$. $g(Y_n) \xrightarrow{P} g(c)$

proof: ① $g(y)$ is continuous at c, it follows that for every $\varepsilon > 0$, a $\delta > 0$ exists such that $|y - c| < \delta$ implies $|g(y) - g(c)| < \varepsilon$. Then $P[|g(Y_n) - g(c)| < \varepsilon] \ge P[|Y_n - c| < \delta]$

because $P[B] \ge P[A]$ when $A \subset B$

But because $Y_n \xrightarrow{P} c$ it follows for every $\delta > 0$. that

$\lim_{n \to \infty} P[|g(Y_n) - g(c)| < \varepsilon] = \lim_{n \to \infty} P[|Y_n - c| < \delta] = 1.$