8 Hypothesis Testing

8.1 hypothesis testing

① formulate null & alternative hypothesis

$H_0: \mu = 750 \quad H_1: \mu \neq 750$

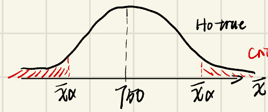
② determine test statistics and its distribution, given H_0

x_i test statistics: $\bar{x} = \frac{1}{25}(x_1 + \dots x_n)$ assume H_0 is true, $\bar{x} \sim N(750, \frac{400}{25} = 16)$, given H_0 true.

③ describe the assumptions

Sample is random; ppl is normally distributed; $\sigma^2 = 400$ known

④ determine the critical region for the test statistics.



$\bar{x}_{crit.} = P(Z < -Z_\alpha) = 750 - Z_\alpha 20 = 743.42$

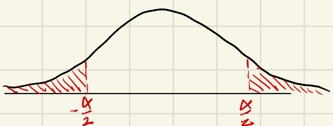
$(-\infty, 743.42)$

⑤ Compare sample outcome of test statistics with critical region

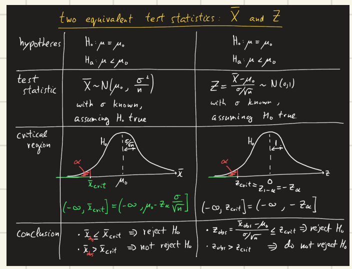
$\bar{x} = 741.2 \rightarrow$ reject H_0; $\bar{x} = 748.2 \rightarrow$ not reject H_0

⑥ reject or not reject the null hypothesis.

① One-sided vs Two-sided HT: 这是根据H_1 condition定的



critical region $= (-\infty, \bar{x}_L) \cup [\bar{x}_U, \infty)$



two equivalent test statistics: $\bar{X}$ and Z

hypotheses	$H_0: \mu = \mu_0$ $H_a: \mu < \mu_0$	$H_0: \mu = \mu_0$ $H_a: \mu < \mu_0$
test statistic	$\bar{X} \sim N(\mu_0, \frac{\sigma^2}{n})$ with σ known, assuming H_0 true	$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$ with σ known, assuming H_0 true.
critical region	H_0; $\frac{\sigma}{\sqrt{n}}$; α; $\bar{x}_{crit}$; μ_0; $\bar{X}$ $(-\infty, \bar{x}_{crit}] = (-\infty, \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}]$	H_0; 1; α; $z_{crit} = z_{1-\alpha} = -z_\alpha$; 0; z $(-\infty, z_{crit}] = (-\infty, -z_\alpha]$
conclusion	• $\bar{x}_{obs} \leq \bar{x}_{crit} \Rightarrow$ reject H_0 • $\bar{x}_{obs} > \bar{x}_{crit} \Rightarrow$ not reject H_0	• $z_{obs} = \frac{\bar{x}_{obs} - \mu_0}{\sigma/\sqrt{n}} \leq z_{crit} \Rightarrow$ reject H_0 • $z_{obs} > z_{crit} \Rightarrow$ do not reject H_0

② Composite

③ Discrete:

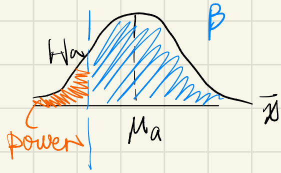
8.2 probability of type II error & power

① type I error: $P(\text{reject } H_0 \mid H_0 \text{ is true}) = \alpha$

② type II error: $P(\text{not reject } H_0 \mid H_0 \text{ false}) = \beta$

power of test: $P(\text{reject } H_0 \mid H_0 \text{ false}) = 1 - \beta$

O α之后算的带到值 $\leftarrow \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ $\quad$ μ给题目的值.



8.3 p-value

the probability of obtaining a result equal to or "more extreme" than what was actually observed assuming H_0 is true.

(theorem): p-value $\leq \alpha \iff H_0$ can be rejected with significance level α

8.4 hypothesis testing for a single pop.

test for:

① for μ (σ known) $\quad Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$

② for μ (σ unknown) $\quad T_{n-1} = \frac{\bar{x} - \mu_0}{S/\sqrt{n}} \sim t_{n-1}$

③ $\sigma = \sigma_0$ $\quad \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2(n-1)$

④ $\hat{p} = \frac{x}{n} \sim N(p_0, \frac{p_0(1-p_0)}{n}) \quad Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \sim N(0,1)$

section 8.4 **Hypotheses concerning a single population**

	"z-test"	"t-test"	–
hypotheses	$H_0: \mu=\mu_0$ vs. $H_a: \mu\neq\mu_0$ (σ known)	$H_0: \mu=\mu_0$ vs. $H_a: \mu\neq\mu_0$ (σ unknown)	$H_0: \sigma=\sigma_0$ vs. $H_a: \sigma\neq\sigma_0$
test statistic (assuming H_0 is true)	$\bar{X}\sim N(\mu_0, \frac{\sigma^2}{n})$ or $Z=\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\sim N(0,1)$	$T_{n-1}=\frac{\bar{X}-\mu_0}{S/\sqrt{n}} \sim t(n-1)$	$\frac{(n-1)S^2}{\sigma_0^2}\sim \chi^2(n-1)$
critical region	$(-\infty, \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}] \cup \ldots [\mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}})$ or $(-\infty, -z_{\alpha/2}] \cup [z_{\alpha/2}, \infty)$	$(-\infty, -t_{n-1;\alpha/2}] \cup \ldots [t_{n-1;\alpha/2}, \infty)$	$[0, \chi^2_{n-1;1-\alpha/2}] \cup \ldots [\chi^2_{n-1;\alpha/2}, \infty)$
conditions	population normal or $n\geq 30$ with CLT	population normal or $n \geq 30$ with CLT	population normal

section 8.4.4 **The hypothesis $p=p_0$ (binomial distribution)**

$p=$ probability of "success" in a Bernoulli

$H_0: p=p_0$ vs. $H_a: p>p_0$.

$X=$ number of successes in a sample of size $n \Rightarrow X \overset{\text{if } H_0 \text{ true}}{\sim} Bin(n, p_0)$

test statistic	critical region/p-value	conditions
$\hat{p}=\frac{X}{n} \overset{\text{approx.}}{\sim} N\left(p_0, \frac{p_0(1-p_0)}{n}\right)$	$\left[p_0 + z_\alpha\sqrt{\frac{p_0(1-p_0)}{n}}, \infty\right)$	• $np_0 \geq 5$ • $n(1-p_0)\geq 5$
$Z=\frac{\hat{P}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \overset{\text{approx.}}{\sim} N(0,1)$	$[z_\alpha, \infty)$	" ——— "
$X \sim Bin(n, p_0)$	p-value $= P(X\geq x \mid p=p_0) \overset{\text{PTS1}}{=} \sum_{i=x}^{n}\binom{n}{i}p_0^i(1-p_0)^{n-i} = \ldots \Rightarrow$ compare with α.	n/a

8.5 hypothesis testing for differences between 2 populations

① for μ_1, μ_2

a) σ_1, σ_2 known Z

b) $\sigma_1 = \sigma_2$ unknown T

c) $\sigma_1 \neq \sigma_2$ unknown no test

② for σ^2

$H_0: \sigma_1^2 = a_0 \sigma_2^2$ $H_a: \sigma_1^2 \neq a_0 \cdot \sigma_2^2$

$\frac{S_1^2}{a_0 S_2^2} \overset{\text{th 6.6}}{\sim} F(n_1 - 1, n_2 - 1)$

section 8.5 **Hypotheses concerning differences between two populations**

$H_0: \mu_1 = \mu_2 + \Delta_0$ vs. $H_a: \mu_1 \ldots \mu_2 + \Delta$

	test statistic
σ_1, σ_2 known	$Z=\frac{\bar{X}_1-\bar{X}_2-\Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} \sim N(0,1)$, if H_0 true. if populations normal or CLT with $n\geq 30$
$\sigma_1 = \sigma_2$ unknown	$T=\frac{\bar{X}_1-\bar{X}_2-\Delta_0}{\sqrt{S_p^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \sim t(n_1+n_2-2)$ if H_0 true; S_p^2: pooled sample variance (sect. 7.4)
σ_1, σ_2 unknown	· no test statistic. ✗ · confidence interval ✓ · p-value ✓ } see section 7.4.4 for $\mu_1-\mu_2$ and check if Δ_0 is inside C.I.

Variances of two populations

$H_0: \sigma_1^2 = a_0 \cdot \sigma_2^2$ vs. $H_a: \sigma_1^2 \neq a_0 \cdot \sigma_2^2$.

test statistic: $\frac{S_1^2}{a_0 \cdot S_2^2} \overset{\text{th. 6.6}}{\sim} F(n_1-1, n_2-1)$ if H_0 true and populations must be normally distributed.

critical region: $[0, f_{n_1-1,n_2-1;1-\frac{\alpha}{2}}] \cup [f_{n_1-1,n_2-1;\alpha/2}, \infty)$

③ for $p_1 = p_2$

$$\frac{\hat{p}_1-\hat{p}_2-\Delta_0}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1}+\frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}} \overset{\text{approx}}{\sim} N(0,1)$$

The hypothesis $p_1 = p_2$

(from section 7.4.6)
$$\frac{\hat{P}_1-\hat{P}_2-(p_1-p_2)}{\sqrt{\frac{p_1(1-p_1)}{n_1}+\frac{p_2(1-p_2)}{n_2}}} \overset{\text{approx}}{\sim} N(0,1)$$

~~$H_0: p_1 = p_2 \ (=p)$~~

~~$\frac{\hat{P}_1-\hat{P}_2}{\sqrt{p(1-p)}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \overset{\text{approx.}}{\sim} N(0,1)$ if H_0 true.~~

not a good test statistic, because contains unknown parameter.

solution: replace p by $\hat{P}_p = \frac{n_1\hat{P}_1 + n_2\hat{P}_2}{n_1+n_2}$

$H_0: p_1 = p_2 + \Delta_0$

$$\frac{\hat{P}_1-\hat{P}_2-\Delta_0}{\sqrt{\frac{\hat{P}_1(1-\hat{P}_1)}{n_1}+\frac{\hat{P}_2(1-\hat{P}_2)}{n_2}}} \overset{\text{approx}}{\sim} N(0,1) \text{ if } H_0 \text{ true.}$$