Projective Geometry

Ruichen Lu, University of Toronto

1 Abstract

Projective geometry has a rapid development during 19th century, and many neoteric mathematics is derived from prjective geometry. This paper mainly introduces two fundamental theorems, one is Pappus's hexagon theorem and the other one is Desargues's theorem which are the basis of projective geometry.

2 Introduction

Projective geometry is one of the oldest branches of mathematics and has a wide range of applications like cameras, theater movies. Its name implies that projective geometry deals with geometric objects with invariant properties under projection, mapping, and transformation. It differs from Euclidean geometry that two parallel lines will meet at the point infinity in the projective plane, thus angles are not preserved [1]. To study this interesting fact, I will introduce Menelaus's theorem as a base to Pappus's hexagon theorem, and Desargues's theorem.

3 Menelaus's Theorem and Pappus's Hexagon Theorem

3.1 Theorem. Menelaus's Theorem : Let $\triangle ABC$ be a triangle, and three points E, D, F lie on the lines AC, CB and AB respectively. Define the ratio $\frac{AF}{FB}$ is positive when F is between A and B or negative otherwise, same principle applies to the ratio $\frac{BD}{DC}$ and $\frac{CE}{EA}$. Then the three points E, D, F are collinear if and only if:

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = -1 \quad [2]$$

We will use the results of this theorem to prove the following theorems.

3.2 Theorem. Pappus's Hexagon Theorem: Given three points A, B, C lie on the same straight l_1, and other three points D, E, F lie on another straight l_2. Then the three points (X, Y, Z) formed by intersecting the six line pairs AE, BD, AF, CD, BF and CE are also collinear. See Figure 1 and the graph is adapted from [3].

Proof. Extend line LB and MC so that they meet at the point N, then we have the triangle $\triangle LMN$ with the following set of collinear points:

$$\{B, Z, F\}, \{C, Y, D\}, \{A, X, E\}, \{A, B, C\}, \{D, E, F\}$$

After applying Menelaus's Theorem, we have:

$$\frac{|ZN|}{|MZ|} \times \frac{|MF|}{|LF|} \times \frac{|LB|}{|NB|} = \frac{|YM|}{|LY|} \times \frac{|LD|}{|ND|} \times \frac{|CN|}{|MC|} = \frac{|LX|}{|NX|} \times \frac{|EN|}{|ME|} \times \frac{|AM|}{|AL|}$$

$$= \frac{|AL|}{|AM|} \times \frac{|CM|}{|CN|} \times \frac{|BN|}{|BL|} = \frac{|EM|}{|EN|} \times \frac{|DN|}{|DL|} \times \frac{|LF|}{|MF|} = 1$$

Noted that some points are outside of the line segment which results in a negative ratio, then it cancels out with the negative sign of -1, then we can consider the magnitude only. Then multiply all equations together and cancel out all same terms, we get:

$$\frac{|ZN|}{|MZ|} \times \frac{|YM|}{|LY|} \times \frac{|LX|}{|NX|} = 1 \Rightarrow \frac{|ZN|}{|MZ|} \times \frac{|YM|}{|LY|} \times (-\frac{|LX|}{|NX|}) = -1$$

Therefore, X, Y, Z are collinear, it completes the proof. $\square$

3.1 Remark. Collinearity is a projective invariant that would not change its property under projection.

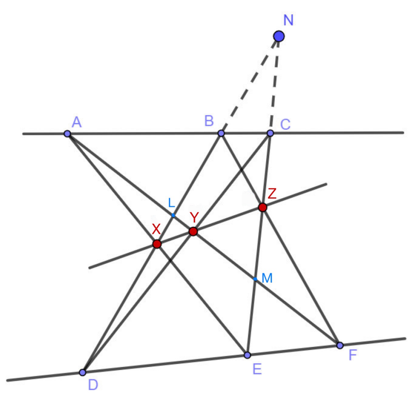

Figure 1: Pappus's Hexagon Theorem

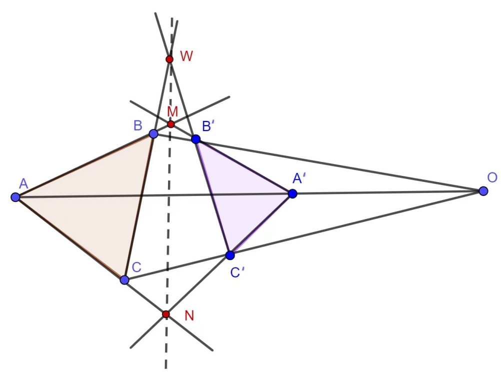

Figure 2: Desargues's Theorem

4 Desargues's Theorem

Unlike collinearity, angle is a projective variant which is also the key to why two parallel line would intersect at the infinity.

4.1 Theorem. Desargues's Theorem: Given two triangles $\triangle ABC$ and $\triangle A'B'C'$, if lines AA', BB', CC' formed by the two triangles vertices are concurrent (they meet at a common point O called center of perspectivity), then the intersections W, M, N of the corresponding sides are collinear (they lie on the axis of perspectivity). See Figure 2 and the graph is adapted from [4].

Proof. Let W=$BC \cup B^{'}C^{'}$, N=$AC \cup A^{'}C^{'}$, M=$AB \cup A^{'}B^{'}$. Then we have $\triangle BCO$, $\triangle ACO$ and $\triangle AOB$. Then we have sets of collinear points: $\{B^{'}, C^{'}, W\}$, $\{A^{'}, C^{'}, N\}$ and $\{A^{'}, B^{'}, M\}$. Then we can apply Menelaus's theorem [4] [5]:

$$\frac{|BW|}{|WC|} \times \frac{|CC^{'}|}{|OC'|} \times \frac{|OB^{'}|}{|BB'|} = \frac{|CN|}{|NA|} \times \frac{|AA^{'}|}{|A'O|} \times \frac{|OC^{'}|}{|CC^{,}|} = \frac{|AM|}{|MB|} \times \frac{|BB^{'}|}{|OB'|} \times \frac{|OA^{'}|}{|AA'|} = 1$$

We still cancel out the negative sign of -1 and consider magnitude only, then multiply all equations and simplify it:

$$\frac{|BW|}{|WC|} \times \frac{|CN|}{|NA|} \times \frac{|AM|}{|MB|} = 1 \Rightarrow (-\frac{|BW|}{|WC|}) \times (-\frac{|CN|}{|NA|}) \times (-\frac{|AM|}{|MB|}) = -1$$

Thus, W, M, N are collinear, it completes the proof. □

The inverse of the theorem is also true, it gives the insight none of the corresponding sides are parallel in the projective space, they will eventually meet at the point infinity. Desargues's discovery is a key foundation to projective geometry.

5 Application

Pinhole camera model is a typical type of projective geometry. There is a barrier in front of the camera sensor with an aperture to capture image. The goal of the model is to create a one-to-one mapping to project points in the 3D world frame to the 2D camera image plane. Let $P = [x, y, z]^T$ to be a point in the real world, and the corresponding projected point on the image plane is $P^{'} = [x^{'}, y^{'}]^T$ [6]. Then by doing mathematical calculation of the camera coordinate system and setting up the matrix, we are able to perform transformation and the projection. Pinhole camera model is widely used in movies or autonomous driving industry.

6 Reference

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