AST3220 - project 3

Kandidatnummer 17
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CONTENTS

I. MISCELLANEOUS PROBLEMS

A. Problem 1

In this problem, we assume that the universe is described by the Einstein-de Sitter model (EdS).

1. a)

The EdS model considers a flat, dust-filled universe. We have previously seen in Elgarøy 1, pg. 32-33 that in flat, dust- or radiation-filled models the scale factor can be found from the first Friedmann equation to be

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3(w+1)}.$$

For a dust-filled universe we have $w = 0$ and the scale factor as a function of time becomes

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3}. \tag{1}$$

We want to use this expression to find an expression for the age of the Universe at redshift z. First, we rewrite equation 1 as

$$\frac{a(t)}{a_0} = \left(\frac{t}{t_0}\right)^{2/3} = \frac{1}{1+z} \tag{2}$$

where we used the relation $a/a_0 = 1+z$. Then, we rewrite the equation above, equation 2, to find an expression for the time t:

$$\left(\frac{t(z)}{t_0}\right)^{2/3} = \frac{1}{1+z} \tag{3}$$

$$\frac{t(z)}{t_0} = \frac{1}{(1+z)^{3/2}} \tag{4}$$

$$t(z) = \frac{t_0}{(1+z)^{3/2}} \tag{5}$$

2. b)

We now assume that we, at the present epoch, observe two objects, one with redshift $z_1 = 3$ and another with redshift $z_2 = 8$. Using this, we want to find when, in units of t_0, the light we observe today was emitted by these two objects.

We simply insert the redshift values into equation 5 from a) and get

$$t_1(z_1 = 3) = \frac{t_0}{(1+3)^{3/2}} = \frac{t_0}{4^{3/2}} = \frac{t_0}{8} \tag{6}$$

$$t_2(z_2 = 8) = \frac{t_0}{(1+8)^{3/2}} = \frac{t_0}{9^{3/2}} = \frac{t_0}{27}. \tag{7}$$

3. c)

Next, we want to find the comoving radial coordinates of these two objects. The comoving coordinate is in Elgarøy 1, pg. 18 defined as

$$r = S_k\left[\int_t^{t_0} \frac{cdt'}{a(t')}\right], \quad S_k(r) = \begin{cases} \sin(r), & k=1 \\ r, & k=0 \\ \sinh(r), & k=-1. \end{cases} \tag{8}$$

Since we in this problem assume that the universe is described by the Einstein-de Sitter model, $k = 0$ and $S_k(r) = r$. The radial comoving coordinate then becomes

$$\begin{aligned} r &= \int_{t_e}^{t_0} \frac{cdt}{a(t)} = c\int_{t_e}^{t_0} \frac{dt}{a_0\left(\frac{t}{t_0}\right)^{2/3}} \\ &= \frac{ct_0^{2/3}}{a_0}\int_{t_e}^{t_0} t^{-2/3}dt = \frac{ct_0^{2/3}}{a_0}\left[3t^{1/3}\right]_{t_e}^{t_0} \\ &= \frac{3ct_0^{2/3}}{a_0}\left(t_0^{1/3} - t_e^{1/3}\right) = \frac{3ct_0}{a_0}\left[1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right] \end{aligned}$$

Inserting the expressions for t_1 and t_2 as t_e in the expression for r above, we get the radial comoving coordinates:

$$r_1 = \frac{3ct_0}{a_0}\left(1 - \left(\frac{1}{8}\right)^{1/3}\right) = \frac{3ct_0}{2a_0}$$

$$r_2 = \frac{3ct_0}{a_0}\left(1 - \left(\frac{1}{27}\right)^{1/3}\right) = \frac{2ct_0}{a_0}.$$

4. d)

The light we recieve now from the object with $z = z_2 = 8$ was emitted at a time t_e. We wish to determine the comoving radial coordinate of the light ray heading towards us at an arbitrary time later, t. We start by writing the integral expression for the radial comoving coordinate. This time we integrate from t_e to an arbitrary later time t. Using the exact same method as in the previous problem gives us

$$r = \int_{t_e}^{t} \frac{cdt'}{a(t')} = \frac{3ct_0^{2/3}}{a_0}\left(t^{1/3} - t_e^{1/3}\right)$$

We have that $t_e = t_2 = t_0/27$, and by inserting this into the expression above, we get

$$\begin{aligned} r &= \frac{3ct_0^{2/3}}{a_0}\left(t^{1/3} - \left(\frac{t_0}{27}\right)^{1/3}\right) \\ &= \frac{3ct_0^{2/3}}{a_0}\left(t^{1/3} - \frac{t_0^{1/3}}{3}\right) = \frac{ct_0}{a_0}\left(3\left(\frac{t}{t_0}\right)^{1/3} - 1\right). \end{aligned}$$

5. e)

We now imagine that there was an observer situated at the object with $z = z_1 = 3$. We wish to find what redshift she measures for the light coming from the object we observe today with redshift $z_2 = 8$. Using the redshift and scale factor relation we get

$$1 + z_{1,2} = \frac{a(t_{1,2})}{a(t_e)}$$

where $z_{1,2}$ is the redshift the observer at object 1 measures for the light coming from object 2, $t_{1,2} = t_1 = t_0/8$ and $t_e = t_2 = t_0/27$. Solving for $z_{1,2}$ gives us

$$\begin{aligned} z_{1,2} &= \frac{a_0\left(\frac{t_0}{8t_0}\right)^{2/3}}{a_0\left(\frac{t_0}{27t_0}\right)^{2/3}} - 1 = \frac{(1/8)^{2/3}}{(1/27)^{2/3}} - 1 \\ &= \frac{1/4}{1/9} - 1 = \frac{9}{4} - 1 = \frac{5}{4} = 1.25. \end{aligned}$$

B. Problem 2

1. a)

We wish to use the Friedmann equations to explain why there must be a time in the past when the scale factor vanished in models where the total density and pressure satisfy $\rho + 3p/c^2 > 0$, the density decrease faster with a than $1/a^2$, and $H_0 > 0$.

The criterion that $H_0 > 0$ tells us that $\dot{a}_0 > 0$ because $H_0 = \dot{a}_0/a_0$ and we know that $a_0 > 0$.

Next, we use the second Friedmann equation, given as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{9}$$

[1]. Since the expression in the parenthesis is positive, making the right hand side of the equation negative, we must have that $\ddot{a} < 0$ since $a \geq 0$. The reason that $a \geq 0$ is that we are only looking at the time interval from today, where $a_0 > 0$ until some time where the scale factor possibly vanished, making a positive in the whole time interval.

The first Friedmann equation is given as

$$\dot{a}^2 + kc^2 = \frac{8\pi G}{3}\rho a^2 \Rightarrow \frac{\dot{a}}{a} = \pm\sqrt{\frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}} \tag{10}$$

[1]. We choose the square root to be positive because $\dot{a}$ and a are always positive, making the left hand side of the equal sign positive. Why $\dot{a} > 0$ can be explained as follows: We know that $\dot{a}_0 > 0$ from the criterion $H_0 > 0$, and $\dot{a}$ therefore needs to stay positive until the scale factor vanish, otherwise we would get $\ddot{a} > 0$ when looking back in time, which it can not be as we saw from the second Friedmann equation. This is illustrated by the blue curve in figure 1, where we see that if $\dot{a} < 0$, we get $\ddot{a} > 0$ and a never becomes 0. We also

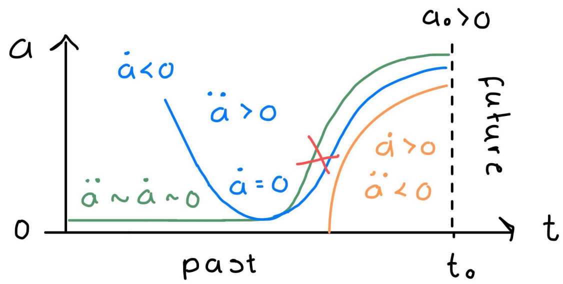

Figure 1. The orange curve shows the only possible shape the scale factor could have had in the past with the conditions given in problem 2a). The blue and green curve shows impossible curves. Note that all the curves should have started at the same value for a_0.

Let us look at what happens for different values of k in equation 10. For $k = 0, -1$ the expression under the square root becomes positive, causing no trouble, but we need to examine what happens for $k = 1$. For this, we use the criterion that the density decreases faster with a than with $1/a^2$ to determine if

$$\frac{8\pi G}{3}\rho > \frac{kc^2}{a^2} \tag{11}$$

for the whole time interval. We know from the first Friedmann equation that today

$$\frac{8\pi G}{3}\rho_0 > \frac{kc^2}{a_0^2} \tag{12}$$

because $\dot{a}_0 > 0$ and real. Now, we need to examine if the inequality sign could have changed direction some time in the past. Since the density ρ decreases faster with a than $1/a^2$, we see that the inequality sign must stay the way it is in inequality 11 until $a = 0$. The inequality sign can only change direction some time in the future for larger values of a. This again means that the expression under the square root needs to be positive also for $k = 1$. This also proves that $\dot{a} > 0$ for all values of k.

Since $\ddot{a} < 0$ and $\dot{a} > 0$ in the whole time interval from a_0 to $a = 0$, we know that the only possible shape the function $a(t)$ can have is the one shown in orange in figure 1. We also note that we can not have a curve for $a(t)$ where a has an asymptotic behaviour as shown by the green curve in figure 1 because then we would get $\ddot{a} \sim \dot{a} \sim 0$ which does not fulfill the given conditions.

2. b)

We seem to be living in a universe where the expansion is presently accelerating, $\Omega_{m0} = 0.3$, and $\Omega_{\Lambda 0} = 0.7$. We wish to figure out if there was a time in the past when $a = 0$ in this model too.

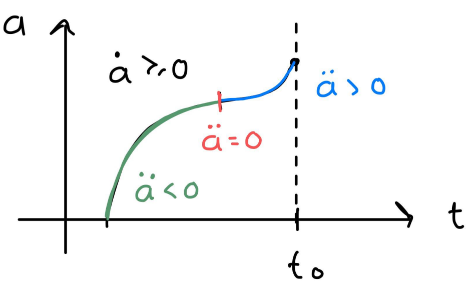

Figure 2. The curve shows the possible shape the scale factor can have if the expansion of the universe is accelerating today, but was decelerating in the past. At the turning point, both $\ddot{a}$ and $\dot{a}$ are 0.

As we saw in the previous problem, it is possible that a was 0 when the expansion of the universe was decelerating and $\dot{a} > 0$. We will now prove that this is also the case for a more realistic case where the expansion of the universe is accelerating today, as we know it is. Since we live in a universe where the expansion is accelerating, we have that $\ddot{a}_0 > 0$. We use the ΛCDM model to describe our current universe, as it is the best model we have compared to observations (although it also has some problems). This model has a cosmological constant and is matter dominated for a spatially flat universe. The first Friedmann equation 10 can then be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda).$$

Since ρ_m and ρ_Λ are both positive, the right hand side is positive, and since $a > 0$, then $\dot{a}$ must be positive as well.

We now use the second Friedmann equation 9, which in this model can be rewritten as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_{m0}\left(\frac{a_0}{a}\right)^3 + \rho_{\Lambda 0} + \frac{3p_{\Lambda 0}}{c^2}\right)$$

where we used the density expression $\rho_i = \rho_0(a_0/a)^{3(1+w)}$, where $w = 0$ for matter and $w = -1$ for the cosmological constant Λ making $\rho_\Lambda = \rho_{\Lambda 0}$. Since non-relativistic matter does not contribute to the pressure from the equation of state, we only get one pressure term from the cosmological constant. Using that $p_\Lambda = -\rho_\Lambda$ (since $w_\Lambda = -1$) and $c \approx 1$ we get

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho_{m0}\left(\frac{a_0}{a}\right)^3 - 2\rho_{\Lambda 0}\right) | \cdot \frac{1}{\rho_{c0}} \\ &= -\frac{H_0}{2}\left(\Omega_{m0}\left(\frac{a_0}{a}\right)^3 - 2\Omega_{\Lambda 0}\right)\end{aligned}$$

where we used that $\Omega_{i0} = \rho_{i0}/\rho_{c0}$ [1]. The only term that varies is a_0/a. Using this, we will now find if $\ddot{a}$ has ever been negative some time in the past. We find that when $\ddot{a} > 0$, we have

$$\begin{aligned}\Omega_{m0}\left(\frac{a_0}{a}\right)^3 - 2\Omega_{\Lambda 0} &< 0 \\ \Omega_{m0}\left(\frac{a_0}{a}\right)^3 &< 2\Omega_{\Lambda 0} \\ \frac{a_0}{a} &< \left(\frac{2\Omega_{\Lambda 0}}{\Omega_{m0}}\right)^{1/3} \\ (1+z) &< \left(\frac{2\Omega_{\Lambda 0}}{\Omega_{m0}}\right)^{1/3} \\ z &< \left(\frac{2\Omega_{\Lambda 0}}{\Omega_{m0}}\right)^{1/3} - 1 \simeq 0.67.\end{aligned}$$

This tells us that $\ddot{a}$ became positive at redshift $z = 0.67$, which was not very long a ago. Before this, $\ddot{a}$ was negative. Since $\ddot{a} < 0$ most of the past time, and $\dot{a} \geq 0$ we see that the arguments from the previous problem holds, and also in this model will a be zero at some point. In figure 2 we see the possible shape of the curve of $a(t)$ for this case.

C. Problem 3

The proper distance to the particle horizon at time t is given by

$$d_{P,PH}(t) = a(t)\int_0^t \frac{cdt'}{a(t')}. \qquad (13)$$

1. a)

We want to show that the proper distance to the particle horizon at redshift z is

$$d_{P,PH}(z) = \frac{c}{1+z}\int_z^\infty \frac{dz'}{H(z')} \qquad (14)$$

[2]. We start from equation 13 and using that $dt = da/\dot{a}$ we change the integration variable from t to a:

$$d_{P,PH}(z) = a(t)\int_0^t \frac{cdt'}{a(t')} = a(t)\int_{a=0}^a \frac{cda'}{a'\dot{a}'}$$

We then change the integration variable to z by using that redshift is given by $1 + z = a_0/a$ which gives $dz = -(a_0/a^2)da$. By inserting $da = -(a^2/a_0)dz$ in the integral above, rewriting a in front of the integral using redshift and using that $H = \dot{a}/a$, we get

$$\begin{aligned}d_{P,PH}(z) &= \frac{a_0}{1+z}\int_\infty^z \left(-\frac{c}{a\dot{a}}\frac{a^2}{a_0}\right)dz' \\ &= \frac{c}{1+z}\int_z^\infty \frac{a}{\dot{a}}dz' = \frac{c}{1+z}\int_z^\infty \frac{dz'}{H(z')}\end{aligned}$$

which is what we wanted to show.

2. b)

Next, we want to calculate $d_{P,PH}(z)$ for a matter-dominated universe with

$$H(z) = H_0\sqrt{\Omega_{m0}}(1+z)^{3/2}, \qquad (15)$$

and for a radiation-dominated universe with

$$H(z) = H_0\sqrt{\Omega_{r0}}(1+z)^2. \qquad (16)$$

We also want to show that

$$d_{P,PH} \sim \frac{c}{H(z)} \qquad (17)$$

in both cases. To do this we use the formula for $d_{P,PH}(z)$ in equation 14 that we derived in the previous problem and insert the expression for $H(z)$. Starting with the matter-dominated universe we get

$$\begin{aligned}d_{P,PH}(z) &= \frac{c}{1+z}\int_z^\infty \frac{dz'}{H_0\sqrt{\Omega_{m0}}(1+z')^{3/2}} \\ &= \frac{c}{(1+z)H_0\sqrt{\Omega_{m0}}}\left[-\frac{2}{(1+z)^{1/2}}\right]_z^\infty \\ &= \frac{2c}{(1+z)^{3/2}H_0\sqrt{\Omega_{m0}}} = \frac{2c}{H(z)} \sim \frac{c}{H(z)}.\end{aligned}$$

Using the same method for the radiation-dominated universe we get

$$\begin{aligned}d_{P,PH}(z) &= \frac{c}{1+z}\int_z^\infty \frac{dz'}{H_0\sqrt{\Omega_{r0}}(1+z')^2} \\ &= \frac{c}{(1+z)H_0\sqrt{\Omega_{r0}}}\left[-\frac{1}{(1+z)}\right]_z^\infty \\ &= \frac{c}{(1+z)^2H_0\sqrt{\Omega_{r0}}} = \frac{c}{H(z)}.\end{aligned}$$

3. c)

We wish to find at what redshift the radius of the observable universe was equal to the radius of a typical neutron star. A typical neutron star has a radius of about 10 km and mass of about 1.5 $M_\odot$. We assume that the radius of the observable universe at any given time is equal to the proper distance to the particle horizon at that time. We also assume that $\Omega_{m0} = 0.3$, $\Omega_{r0} = 10^{-4}$ and that $h = 0.7$ [2].

Starting with the matter-dominated universe we set the expression for the proper distance to the particle horizon from the previous problem equal to the neutron radius $r = 10$ km and solve for z. This gives us

$$\begin{aligned} \frac{c}{(1+z)^{3/2} H_0 \sqrt{\Omega_{m0}}} &= r \\ (1+z)^{3/2} &= \frac{c}{r H_0 \sqrt{\Omega_{m0}}} \\ z &= \left(\frac{c}{r H_0 \sqrt{\Omega_{m0}}} \right)^{2/3} - 1. \end{aligned}$$

To calculate a value for the redshift, we first find what H_0 is in SI units. We have that $H_0 = 100\, h$ km s^{-1}Mpc^{-1} = 70 km s^{-1}Mpc^{-1}. To convert the units to per seconds we use that km s^{-1}Mpc^{-1} = $3.09 \cdot 10^{19}$s^{-1}. The value of H_0 in SI units is then $H_0 = 70/(3.09 \cdot 10^{19})$ s^{-1}. Calculating the redshift using $\Omega_{m0} = 0.3$ we get that

$$z = \left(\frac{3 \cdot 10^8 \text{ m/s}}{10^4 \text{ m} \frac{70}{3.09 \cdot 10^{19} \text{ s}} \sqrt{0.3}} \right)^{2/3} - 1 \simeq 8.36 \cdot 10^{14} \quad (18)$$

For the radiation-dominated universe the expression for the redshift becomes

$$\begin{aligned} \frac{c}{(1+z)^2 H_0 \sqrt{\Omega_{r0}}} &= r \\ (1+z)^2 &= \frac{c}{r H_0 \sqrt{\Omega_{r0}}} \\ z &= \left(\frac{c}{r H_0 \sqrt{\Omega_{r0}}} \right)^{1/2} - 1. \end{aligned}$$

Inserting the value for H_0 and $\Omega_{r0} = 10^{-4}$ we get

$$z = \left(\frac{3 \cdot 10^8 \text{ m/s}}{10^4 \text{ m} \frac{70}{3.09 \cdot 10^{19} \text{ s}} \sqrt{10^{-4}}} \right)^{1/2} - 1 \simeq 1.15 \cdot 10^{12} \quad (19)$$

We note that since the radius of the observable universe is only 10 km and it has a low mass of only 1.5 $M_\odot$, we find ourselves in the very early universe which is radiation-dominated. We will therefore use the redshift found for a radiation-dominated universe in the next calculations.

In addition to the redshift, we want to find the mass density and the radiation density at this redshift. We first find the mass density by using the expression $\rho_i = \rho_0 (a_0/a)^{3(1+w)}$ for density [1], where $w = 0$ for matter.

Using also the relations $\rho_0 = \rho_{c0}\Omega_{i0}$ and $a_0/a = 1 + z$ we get

$$\begin{aligned} \rho_m &= \rho_{c0}\Omega_{m0}(1+z)^3 \\ &= 1.879 \cdot 10^{-29} \cdot 0.7^2 \cdot 10^3 \text{ kg/m}^3 \cdot 0.3\,(1 + 1.15 \cdot 10^{12})^3 \\ &\simeq 4.20 \cdot 10^9 \text{ kg/m}^3 \end{aligned}$$

where we used that $\rho_{c0} = 1.879 \cdot 10^{-29}$ h^{2}g/cm. The radiation density is found the same way using $w = 1/3$:

$$\begin{aligned} \rho_r &= \rho_{c0}\Omega_{r0}(1+z)^4 \\ &= 1.879 \cdot 10^{-29} \cdot 0.7^2 \cdot 10^3 \text{ kg/m}^3 \cdot 10^{-4}(1 + 1.15 \cdot 10^{12})^4 \\ &\simeq 1.61 \cdot 10^{18} \text{ kg/m}^3. \end{aligned}$$

We mentioned that we find ourselves in the very early universe, where radiation dominates. This is clearly the case since the radiation density is much higher than the matter density. Using the expression for ρ_m and ρ_r we can find at which redshift the universe changed from being radiation dominated to matter dominated. We find this to be when

$$\begin{aligned} \rho_m &= \rho_r \\ \rho_{c0}\Omega_{m0}(1+z)^3 &= \rho_{c0}\Omega_{r0}(1+z)^4 \\ z &= \frac{\Omega_{m0}}{\Omega_{r0}} - 1 = \frac{0.3}{10^{-4}} - 1 = 2999. \end{aligned}$$

We see that this redshift has a much lower value than the one we found previously, confirming that the universe is radiation-dominated when the radius is only 10 km.

We compare our density-results with the average density of a typical neutron star which is

$$\begin{aligned} \rho_n &= \frac{1.5 M_\odot}{(10 \text{ km})^3} = \frac{1.5 \cdot 2 \cdot 10^{30} \text{ kg}}{(10^4 \text{ m})^3} \\ &= 3 \cdot 10^{18} \text{ kg/m}^3. \end{aligned}$$

We see that the typical neutron star has a density which is very close to the radiation density, as we would expect with the assumptions we made at the beginning of this problem.

4. d)

We now find the CMB temperature at this time in the universe from the photon temperature[1]:

$$\begin{aligned} T &= T_0(1+z) = (1 + 1.15 \cdot 10^{12}) \cdot 2.73 \text{ K} \\ &\simeq 3.14 \cdot 10^{12} \text{ K} \end{aligned}$$

This is much warmer than the average CMB temperature we have today, but to no surprise as the universe was much warmer in the early stages when it was much denser

[1] Formula found in personal notes from lecture about recombination and CMB

5. e)

We want to show that the age of the Universe at redshift z is given by

$$t(z) = \int_z^\infty \frac{dz'}{(1+z')H(z')} \tag{20}$$

and that it for the radiation-dominated universe is

$$t(z) = \frac{1}{H(z)}. \tag{21}$$

We start by using the definition of the Hubble constant to find an expression for the change in time dt:

$$H = \frac{\dot{a}}{a} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \Rightarrow dt = \frac{da}{aH(a)} \tag{22}$$

We then find the expression for the age by integrating both sides of equation 22:

$$\int_0^t dt = \int_0^a \frac{da'}{a'H(a)} \tag{23}$$

$$t(a) = \int_0^a \frac{da'}{a'H(a')} \tag{24}$$

To get to the expression we want to show, we use the relation $1+z = a_0/a$ to integrate over redshift instead. We then find

$$\begin{aligned} dz &= -\frac{a_0}{a^2}da \\ \Rightarrow da &= -\frac{a^2}{a_0}dz = -\frac{a}{(1+z)}dz = -\frac{a_0}{(1+z)^2}dz \end{aligned}$$

which we insert in equation 24, giving us

$$t(z) = \int_\infty^z \left(-\frac{\frac{a_0}{(1+z')^2}}{\frac{a_0}{(1+z')}H(z')} \right) dz' = \int_z^\infty \frac{dz'}{(1+z')H(z')}.$$

Having arrived at the expression we wanted to show, we use this expression to find the age of the radiation-dominated universe. Using $H(z)$ from equation 16 we get

$$\begin{aligned} t(z) &= \int_z^\infty \frac{dz'}{(1+z')H_0\sqrt{\Omega_{r0}}(1+z')^2} \\ &= \frac{1}{H_0\sqrt{\Omega_{r0}}} \int_z^\infty \frac{dz'}{(1+z')^3} \\ &= \frac{1}{H_0\sqrt{\Omega_{r0}}} \left[-\frac{1}{2(1+z)^2} \right]_z^\infty \\ &= \frac{1}{2H_0\sqrt{\Omega_{r0}}(1+z)^2} = \frac{1}{2H(z)} \end{aligned}$$

Inserting the value for Ω_{r0}, H_0 and the redshift we found in *c*) which was $z = 1.15 \cdot 10^{12}$, we find that the age of the radiation-dominated universe is

$$\begin{aligned} t(z) &= \frac{1}{2H_0\sqrt{\Omega_{r0}}(1+z)^2} \\ \Rightarrow t(1.15 \cdot 10^{12}) &= \frac{1}{2 \cdot \frac{70}{3.09 \cdot 10^{19}\ \mathrm{s}} \sqrt{10^{-4}}(1 + 1.15 \cdot 10^{12})^2} \\ &\simeq 19.19 \cdot 10^6\ \mathrm{s}. \end{aligned}$$

This corresponds to about 222 days, which is indeed early in the life of the Universe.

II. ON INFLATION

In the following problems we will use units where $\hbar = c = 1$.

A. Problem 4

Here I will give a brief description of the horizon problem and the flatness problem.

1. The horizon problem

To understand the horizon problem, let us imagine that we are located at the center of the universe. We observe that the cosmic microwave background (CMB) approaches us from the surface of last scattering in all directions, and we observe that the temperature is even all over the sky. It can be shown that the particle horizon at last scattering is about 1 degree. This tells us that light can only travel 1 degree in angular distance from the beginning of the universe until the last scattering. This therefore limits how large areas physical mechanisms could have affected.

Imagine that we start out with an uneven temperature. The temperature can be evened out by physical processes like particle interactions. Photons on different sides of the particle horizon at last scattering will never have been in contact. Such that, if the temperature at two locations (separated by more than 1 degree) is not the same initially, we would expect them to still be different today. This is however not what we observe. We observe that the temperature of the CMB is the same with great precision over the whole sky. After the last scattering the photons do no longer interract and there is nothing that can even out the temperature. This is the horizon problem.

Summed up: Why is the temperature the same with very high precision over the whole sky, if physical processes could only have evened out the temperature in areas covering about 1 degree of the sky?

Note that if we had observed variations in the background radiation, it would have contradicted the cosmological principle, which we have assumed to be true. Inflation can help solve the problem. If we assume that there was a period of inflation and the number of e-foldings was greater than 60, then the horizon problem would be solved as the horizon at last scattering would be so large that it covers the whole observable universe.

2. *The flatness problem*

The flatness problem has to do with the geometry of the universe, which again depends on the total density of the universe. We therefore use the curvature density parameter Ω_k to explain this problem. Observations show that $|\Omega_{k0}| \lesssim 10^{-3}$ which tells us that the universe is very close to being spatially flat.

It can be shown that as we move backwards in time, Ω_k must become smaller and closer to 0. That means that the universe must have started out with a total density very close to the critical density. For us to have a density parameter today that is not very big compared to 1, then the density parameter must have been extremely small compared to 1 in the past. The problem is as follows: Does there exist a mechanism that works in such a way that no matter what the curvature started out as (positive, negative or 0), the universe would still end up with $k \simeq 0$ today? It turns out that inflation can solve this problem as well. During a period of inflation, where the universe expands exponentially, Ω_k will decrease to 0. After inflation the density parameter will start to increase again, but if it became close enough to 0 during inflation the increase will not make a significant difference of the value of Ω_k today. The flatness problem is also solved by having the number of e-foldings greater than 60 as was the case for the horizon problem as well.

B. Problem 5

We now assume that inflation is driven by a scalar field with the potential

$$V(\phi) = \lambda\phi^p, \tag{25}$$

where $p \geq 2$ and λ is a positive constant. We want to show that the total number of e-foldings during inflation is guaranteed to be large if the slow-roll conditions are fulfilled.

The slow-roll conditions are

$$\epsilon = \frac{E_p^2}{16\pi}\left(\frac{V'}{V}\right)^2 \ll 1, \quad |\eta| = \left|\frac{E_p^2}{8\pi}\left(\frac{V''}{V}\right)\right| \ll 1. \tag{26}$$

which when fulfilled guarantees that we have inflation [1]. We start by integrating the potential:

$$V' = p\lambda\phi^{p-1} = p\frac{V}{\phi}$$

$$V'' = p^2\lambda\phi^{p-2} = p^2\frac{V}{\phi^2}$$

Inserting this into the slow-roll parameters gives us

$$\epsilon = \frac{E_p^2}{16\pi}\left(\frac{V'}{V}\right)^2 = \frac{E_p^2}{16\pi}\left(\frac{pV}{\phi}\cdot\frac{1}{V}\right)^2 = \frac{E_p^2 p^2}{16\pi\phi^2} \tag{27}$$

$$\eta = \frac{E_p^2}{8\pi}\left(\frac{V''}{V}\right) = \frac{E_p^2}{8\pi}\left(\frac{p^2 V}{\phi^2}\frac{1}{V}\right) = \frac{E_p^2 p^2}{8\pi\phi^2} \tag{28}$$

The number of e-foldings can be calculated from

$$N_{tot} = \frac{8\pi}{E_p^2}\int_{\phi_{end}}^{\phi_i}\frac{V}{V'}d\phi \tag{29}$$

where we integrate backwards in time from ϕ_{end} where inflation ends and to the beginning of inflation at ϕ_i. The limit ϕ_{end} can be found from the criterion that $\epsilon(\phi_{end}) = 1$ which gives

$$\epsilon = \frac{p^2 E_p^2}{16\pi\phi_{end}^2} = 1 \Rightarrow \phi_{end} = \sqrt{\frac{p^2E_p^2}{16\pi}} = \frac{pE_p}{4\sqrt{\pi}}.$$

For inflation to start we need to satisfy the slow roll parameters, and we use $\epsilon \ll 1$ to find ϕ_i. From equation 27 and 28 we see that as long as $\epsilon \ll 1$ is fulfilled, so is $|\eta| \ll 1$. We therefore only need to use one of them and we use the criterion $\epsilon \ll 1$:

$$\frac{p^2E_p^2}{16\pi\phi_i^2} \ll 1 \Rightarrow \phi_i \gg \frac{pE_p}{4\sqrt{\pi}}$$

We then have that $\phi_i \gg \phi_{end}$. Calculating the total number of e-foldings, we find

$$\begin{aligned} N_{tot} &= \frac{8\pi}{E_p^2}\int_{\phi_{end}}^{\phi_i}\frac{V}{V'}d\phi = \frac{8\pi}{E_p^2}\int_{\phi_{end}}^{\phi_i}\frac{\lambda\phi^p}{p\lambda\phi^{p-1}}d\phi \\ &= \frac{8\pi}{E_p^2}\int_{\phi_{end}}^{\phi_i}\frac{\phi}{p}d\phi = \frac{8\pi}{E_p^2}\left[\frac{1}{p}(\phi_i^2 - \phi_{end}^2)\right]. \end{aligned}$$

Since we found that $\phi_i \gg \phi_{end}$ we see that the total number of e-foldings during inflation must be large if the slow-roll conditions are satisfied.

C. Problem 6

1. *a)*

We wish to find out if inflation is possible if $V(\phi) = 0$ for all ϕ. To have inflation we need to have $\ddot{a} > 0$. We see what happens to the pressure and density of the scalar field for this potential. We find that

$$\rho_\phi c^2 = \frac{1}{2\hbar c^3}\dot{\phi}^2 + V(\phi) \Rightarrow \rho_\phi = \frac{1}{2}\dot{\phi}^2 \tag{30}$$

$$p_\phi = \frac{1}{2\hbar c^3}\dot{\phi}^2 - V(\phi) \Rightarrow p_\phi = \frac{1}{2}\dot{\phi}^2 \tag{31}$$

[1] where we used $\hbar = c = 1$. Inserting this into the second Friedmann equation 9 we get

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) = -\frac{4\pi G}{3}(\rho_\phi + 3p_\phi)\\ &= -\frac{4\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + \frac{3}{2}\dot{\phi}^2\right) = -\frac{8\pi G}{3}\dot{\phi}^2.\end{aligned}$$

Since $\ddot{a} > 0$ when we have inflation and $a > 0$, the right hand side is always positive. The left hand side, on the other hand, is always negative as $\dot{\phi}$ to the second power is always positive. We can therefore not have inflation if $V(\phi) = 0$ because we do not get accelerated expansion.

2. b)

We also want to find out if inflation is possible if the dynamics of the scalar field is such that we always have $\dot{\phi}^2 = 2V(\phi)$. We use the same method as in the previous problem. The density and pressure become

$$\begin{aligned}\rho_\phi &= \frac{1}{2}\cdot 2V(\phi) + V(\phi) = 2V(\phi)\\ p_\phi &= \frac{1}{2}\cdot 2V(\phi) - V(\phi) = 0.\end{aligned}$$

Inserting these expressions into the second Friedmann equation we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\dot{\phi}^2 + 0\right) = -\frac{4\pi G}{3}\dot{\phi}^2.$$

This is the same case as we had in the previous problem and inflation can not be possible with this potential either because we do not get accelerated expansion when the right hand side is always negative.

D. Problem 7

In this problem we will study inflation driven by a scalar field with the potential $V(\phi) = V_0 e^{-\lambda\phi}$, where λ and V_0 are both positive constants. We simplify the equations by introducing the reduced Planck mass $M_P = 1/\sqrt{8\pi G}$.

1. a)

We want to find the equations for ϕ and H in the slow-roll approximation with this potential. These are found by studying how the field varies in time. We start from the first Friedmann equation 10 which becomes

$$H^2 = \frac{8\pi G}{3}\rho_\phi = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{32}$$

when we assume $k = 0$ and that the scalar field dominates (is the dominating contribution to the energy density). Note that when we assume $k = 0$, we assume that the flatness problem does not exist, but we have seen in problem 4 that it does indeed exist. The other equation we will use to find ϕ and H is the continuity equation

$$\dot{\rho}_\phi = -3H(\rho_\phi + p_\phi). \tag{33}$$

We assume that the scalar field follows the continuity equation, and is therefore a perfect fluid [1]. Using these two equations, equation 32 and 33, it can be shown that the equation for the field must be

$$\ddot{\phi} + V'(\phi) + 3H\dot{\phi} = 0. \tag{34}$$

If the field varies so slowly that $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$ then $\rho_\phi = -p_\phi$ from equation 30 and 31 and field acts like a cosmological constant, giving accelerated expansion and exponential growth.

If the slow-roll conditions are satisfied, then $\epsilon \ll 1$ makes sure that $\dot{\phi}$ is small and $\eta \ll 1$ makes sure that $\ddot{\phi}$ is small. Equation 32 and 34 then simplifies to

$$3H\dot{\phi} = -V', \quad H^2 = \frac{8\pi G}{3}V \tag{35}$$

which is the equations for ϕ and H that we wished to find. These equations control the development of the field and the expansion of the universe in the slow-roll approximation. With the scalar field of this problem the equations become

$$3H\dot{\phi} = \lambda V, \quad H^2 = \frac{8\pi G}{3}V = \frac{1}{3M_P^2}V. \tag{36}$$

2. b)

To find an expression for $\phi(t)$ and $a(t)$, we use the equations found in problem *b*). We start by finding an expression for H by inserting the potential $V(\phi)$ of this problem into the second equation of 36. We then get

$$H^2 = \frac{1}{3M_P^2}V_0 e^{-\lambda\phi} \Rightarrow H = \frac{1}{M_P}\sqrt{\frac{V_0}{3}}e^{-\lambda\phi/2} \tag{37}$$

which we insert in the first equation of 36:

$$\begin{aligned}3\frac{1}{M_P}\sqrt{\frac{V_0}{3}}e^{-\lambda\phi/2}\dot{\phi} &= \lambda V_0 e^{-\lambda\phi}\\ \dot{\phi} &= \lambda V_0\sqrt{\frac{3}{V_0}}\frac{M_P}{3}e^{-\lambda\phi+\lambda\phi/2}\\ \dot{\phi} = \frac{\mathrm{d}\phi}{\mathrm{d}t} &= \lambda M_P\sqrt{\frac{V_0}{3}}e^{-\lambda\phi}\end{aligned}$$

We then separate the variables and integrate. We integrate from $\phi_i = -\infty$ to some ϕ in order for $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$ to be satisfied- The final expression for $\phi(t)$ then becomes

$$\begin{aligned}
\int_{-\infty}^{\phi} e^{\lambda\phi'/2} d\phi' &= \lambda M_P \sqrt{\frac{V_0}{3}} \int_0^t dt' \\
\frac{2}{\lambda}\left[e^{\lambda\phi/2} - e^{-\infty\lambda/2}\right] &= \lambda M_P \sqrt{\frac{V_0}{3}} t \\
e^{\lambda\phi/2} &= M_P \frac{\lambda^2}{2}\sqrt{\frac{V_0}{3}} t \\
\frac{\lambda}{2}\phi &= \ln\left(M_P \frac{\lambda^2}{2}\sqrt{\frac{V_0}{3}} t\right) \\
\phi(t) &= \frac{2}{\lambda}\ln\left(M_P \frac{\lambda^2}{2}\sqrt{\frac{V_0}{3}} t\right).
\end{aligned}$$

To find an expression for $a(t)$, we first insert the expression we found for $\phi(t)$ into equation 37 for H. We then get

$$\begin{aligned}
H &= \frac{1}{M_P}\sqrt{\frac{V_0}{3}} e^{-\lambda\phi/2} \\
&= \frac{1}{M_P}\sqrt{\frac{V_0}{3}} \exp\left(-\frac{\lambda}{2}\frac{2}{\lambda}\ln\left(M_P \frac{\lambda^2}{2}\sqrt{\frac{V_0}{3}} t\right)\right) \\
&= \frac{1}{M_P}\sqrt{\frac{V_0}{3}}\frac{2}{\lambda^2}\sqrt{\frac{3}{V_0}}\frac{1}{M_P t} = \frac{2}{M_P^2\lambda^2 t}
\end{aligned}$$

Using that $H = \dot{a}/a = da/(a\, dt)$ we find the final expression for $a(t)$:

$$\begin{aligned}
H &= \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{M_P^2\lambda^2 t} \\
\int_{a_0}^{a}\frac{1}{a'}da' &= \frac{2}{M_P^2\lambda^2}\int_{t_0}^{t}\frac{1}{t'}dt' \\
\ln\left(\frac{a}{a_0}\right) &= \frac{2}{M_P^2\lambda^2}\ln\left(\frac{t}{t_0}\right) \\
a(t) &= a_0\left(\frac{t}{t_0}\right)^{2/(M_P^2\lambda^2)}
\end{aligned}$$

3. c)

The full equations, without the slow-roll approximation, are

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{38}$$

$$H^2 = \frac{1}{3M_P^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \tag{39}$$

which have an exact analytical solution for the specific potential we are working with. The solution we found in b) suggests the ansatz

$$a(t) = Ct^{\alpha} \tag{40}$$

$$\phi(t) = \frac{2}{\lambda}\ln(Bt). \tag{41}$$

Using this, we find an expression for the constants α and B. We start by finding expressions for $V, V', \dot{\phi}, \ddot{\phi}$ and H which we will need in order to find α and B. We find

$$\begin{aligned}
V &= V_0 e^{-2\ln(Bt)} = V_0\left(e^{-\ln(Bt)}\right)^2 = \frac{V_0}{B^2t^2} \\
V' &= -\lambda V_0 e^{-\lambda\phi} = -\lambda V = -\frac{\lambda V_0}{B^2t^2} \\
\dot{\phi} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2}{\lambda}\ln(Bt)\right) = \frac{2}{\lambda}\frac{1}{Bt}\frac{\mathrm{d}(Bt)}{\mathrm{d}t} = \frac{2}{\lambda t} \\
\ddot{\phi} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2}{\lambda t}\right) = -\frac{2}{\lambda t^2} \\
H &= \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{1}{Ct^{\alpha}}\alpha CT^{\alpha-1} = \frac{\alpha}{t}
\end{aligned}$$

We insert the expressions found above into equation 38 to find an expression for B:

$$\begin{aligned}
-\frac{2}{\lambda t^2} + 3\frac{\alpha}{t}\frac{2}{\lambda t} - \frac{\lambda V_0}{B^2t^2} &= 0 \\
\frac{1}{t^2}\left(\frac{6\alpha-2}{\lambda} - \frac{\lambda V_0}{B^2}\right) &= 0
\end{aligned}$$

We see that $t = 0$ is a solution, but this is not the solution we are looking for. The other solution is

$$\frac{6\alpha-2}{\lambda} = \frac{\lambda V_0}{B^2} \Rightarrow B^2 = \frac{\lambda^2 V_0}{6\alpha-2} \tag{42}$$

To find α, we insert the expression for B^2 into equation 39:

$$\begin{aligned}
\frac{\alpha^2}{t^2} &= \frac{1}{3M_P^2}\left[\frac{1}{2}\frac{4}{\lambda^2t^2} + \frac{V_0}{B^2t^2}\right] \\
\alpha^2 &= \frac{1}{3M_P^2}\left[\frac{2}{\lambda^2} + \frac{V_0}{B^2}\right] \\
&= \frac{1}{3M_P^2}\left[\frac{2}{\lambda^2} + V_0\frac{6\alpha-2}{\lambda^2V_0}\right] \\
&= \frac{2}{3M_P^2\lambda^2} + \frac{6\alpha}{3M_P^2\lambda^2} - \frac{2}{3M_P^2\lambda^2} \\
\alpha &= \frac{2}{M_P^2\lambda^2}.
\end{aligned}$$

Then we insert the expression for α into equation 42 to find B:

$$\begin{aligned}
B^2 &= \frac{\lambda^2V_0}{6\alpha-2} = \lambda^2V_0\left(\frac{1}{6\frac{2}{M_P^2\lambda^2}-2}\right) = \frac{\lambda^2V_0}{\frac{12}{M_P^2\lambda^2}-2} \\
&= \frac{M_P^2\lambda^4V_0}{12-2M_P^2\lambda^2} = \frac{M_P^2\lambda^4}{4}\left(\frac{2V_0}{6-M_P^2\lambda^2}\right) \\
B &= \frac{\lambda^2M_P}{2}\sqrt{\frac{2V_0}{6-M_P^2\lambda^2}}
\end{aligned}$$

In the slow-roll approximation we have that $\epsilon \ll 1$ which we use to find the slow-roll solution.

$$\begin{aligned}\epsilon &= \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 = \frac{1}{16\pi G}\left(-\frac{\lambda V_0}{B^2t^2}\frac{B^2t^2}{V_0}\right)^2 \\ &= \frac{\lambda^2}{16\pi G} = \frac{M_P^2\lambda^2}{2} \ll 1\end{aligned}$$

From this we see that $M_P^2\lambda^2 \ll 2$. Since $M_P^2\lambda^2$ is smaller than 2 and therefore also smaller than 6, we can rewrite the expression we found for B where we let $M_P^2\lambda^2 \approx 0$. This gives

$$B \approx \frac{\lambda^2 M_P}{2}\sqrt{\frac{V_0}{3}} \tag{43}$$

which we see is the same that we got for B in the expression for $\phi(t)$ that we found in problem b) when assuming the solution given in equation 40. We also see that the expression for α that we found in this problem is the same as the α we found in problem b) in the expression for $a(t)$ when assuming the solution given in equation 41.

4. d)

The main problem with this potential as a model of inflation becomes clear if we study the slow-roll parameters for this potential. We have showed that $\epsilon = M_P^2\lambda^2/2 \ll 1$ and the other slow-roll parameter is

$$\eta = \frac{1}{8\pi G}\left(\frac{V''}{V}\right) = M_P^2\left(\frac{\lambda^2 V}{V}\right) = M_P^2\lambda^2 \ll 1. \tag{44}$$

Both slow-roll parameters are constant which means that the inflation period can never stop. Inflation ends when $\epsilon = 1$, such that we can either have $\epsilon \ll 1$ or $\epsilon = 1$ meaning a forever-lasting inflation period or no inflation period at all. This is the reason why the potential in this problem is problematic as a model of inflation.

E. Problem 8

The second-order differential equation for the scalar field can turn into a first-order equation with the slow-roll approximation. This is only true if the dynamics of the field makes the precise initial conditions on the field redundant [2]. We will assume that $\dot{\phi} > 0$ during inflation.

1. a)

By using equation 38 and 39 from problem 7, we will show that

$$\dot{\phi} = -2M_P^2 H'(\phi) \tag{45}$$

where $H'(\phi) = dH/d\phi$. We see that the expression we want to find has the derivative of $H(\phi)$, so we start by differentiating equation 39 with respect to ϕ:

$$2HH' = \frac{1}{3M_P^2}\left(\dot{\phi}\frac{\mathrm{d}\dot{\phi}}{\mathrm{d}\phi} + V'(\phi)\right) \tag{46}$$

$$6HH'M_P^2 = \frac{\mathrm{d}}{\mathrm{d}t}\dot{\phi} + V'(\phi) = \ddot{\phi} + V'(\phi) \tag{47}$$

We then rewrite equation 38 as

$$\ddot{\phi} + V'(\phi) = -3H\dot{\phi}$$

and insert this as the right hand side of equation 47. This gives

$$6HHM_P^2 = -3H\dot{\phi} \tag{48}$$

$$-2H'M_P^2 = \dot{\phi} \tag{49}$$

which is what we wanted to show.

2. b)

Using the result from a) we will show that the first Friedmann equation 10 can be written as

$$[H'(\phi)]^2 - \frac{3}{2M_P^2}H^2(\phi) = -\frac{1}{2M_P^4}V(\phi). \tag{50}$$

We insert equation 49 into the first Friedmann equation 10 and get

$$\begin{aligned}H^2 &= \frac{1}{3M_P^2}\left(\frac{1}{2}(-2H'M_P^2)^2 + V(\phi)\right) \\ 3M_P^2H^2 &= 2(H')^2M_P^4 + V(\phi) \\ M_P^4\left(\frac{3H^2}{M_P^2} - 2(H')^2\right) &= V(\phi) \\ \frac{3H^2}{M_P^2} - 2(H')^2 &= \frac{V(\phi)}{M_P^4} \\ (H')^2 - \frac{3}{2M_P^2}H^2 &= -\frac{1}{2M_P^4}V(\phi)\end{aligned}$$

which is want we wanted to show.

3. c)

We consider a linear perturbation around a solution of equation 50 from b):

$$H(\phi) = H_0(\phi) + \delta H(\phi) \tag{51}$$

where H_0 is a solution of equation 50 [2]. We assume that H is also a solution of equation 50 and we want to show that for a first order perturbation we have

$$H_0'\delta H' = \frac{3}{2M_P^2}H_0\delta H. \tag{52}$$

We start by inserting equation 51 into equation 50 and get the following:

$$\left(H_0' + (\delta H)'\right)^2 - \frac{3}{2M_P^2}\left(H_0 + \delta H\right)^2 = -\frac{V}{2M_P^4}$$

We multiply out the parentheses and use that since we have a linear first order perturbation, we can assume higher order terms of the perturbation to be 0. The expression then reduces to

$$H_0'^2 + 2H_0'(\delta H)' - \frac{3}{2M_P^2}(H_0^2 + 2H_0\delta H) = -\frac{V}{2M_P^4}.$$

We recognize the first, third and last term as equation 50 only with $H_0 = H$ and since H_0 solves equation 50, these terms cancel giving us

$$2H_0'(\delta H)' = \frac{3}{2M_P^2}(2H_0\delta H)$$
$$H_0'\delta H' = \frac{3}{2M_P^2}H_0\delta H$$

which is exactly what we wanted to show. This equation is an equation of motion for how the perturbation changes on top of the homogeneous solution H_0 through the inflation period.

4. d)

We want to show that equation 52 from *c*) has the general solution

$$\delta H(\phi) = \delta H(\phi_i)\exp\left[\frac{3}{2M_P^2}\int_{\phi_i}^{\phi}\frac{H_0(\varphi)}{H_0(\varphi)}d\varphi\right] \tag{53}$$

where φ is some integration variable and ϕ_i is the initial value of the scalar field. We start by separating the variables in equation 52 which gives us

$$\frac{\mathrm{d}\delta H}{\mathrm{d}\phi}\frac{1}{\delta H} = \frac{3}{2M_P^2}\frac{H_0}{H_0'}$$

and then integrate:

$$\int_{\delta H(\phi_i)}^{\delta H}\frac{1}{\delta H'}d\delta H' = \frac{3}{2M_P^2}\int_{\phi_i}^{\phi}\frac{H_0(\varphi)}{H_0'(\varphi)}d\varphi$$
$$\ln\left(\frac{\delta H(\phi)}{\delta H(\phi_i)}\right) = \frac{3}{2M_P^2}\int_{\phi_i}^{\phi}\frac{H_0(\varphi)}{H_0'(\varphi)}d\varphi$$
$$\delta H(\phi) = \delta H(\phi_i)\,\exp\left(\frac{3}{2M_P^2}\int_{\phi_i}^{\phi}\frac{H_0(\varphi)}{H_0'(\varphi)}d\varphi\right)$$

This is the general solution of equation 52 that we wanted to show. Now we want to use this result to explain why the perturbation δH quickly dies out. We start by rewriting the exponential function using equation 45 from 8 *a*):

$$-2H_0'M_P^2 = \dot{\varphi} \Rightarrow H_0' = -\frac{\dot{\varphi}}{2M_P^2}$$

Inserting this into equation 53 we get

$$\delta H = \delta H(\phi_i)\,\exp\left(-3\int_{\phi_i}^{\phi}\frac{H_0(\varphi)}{\dot{\varphi}}d\varphi\right).$$

Using that $d\varphi/\dot{\varphi} = d\varphi/(d\varphi/dt) = dt$ and that $H_0 dt = (\dot{a}_0/a_0)dt = da_0/a_0$ the perturbation becomes

$$\delta H = \delta H(\phi_i)\,\exp\left(-3\int_{t_i}^{t}H_0(t')dt'\right)$$
$$= \delta H(\phi_i)\,\exp\left(-3\int_{a_0}^{a}\frac{da_0}{a_0}\right)$$
$$= \delta H(\phi_i)\,\exp\left(-3\ln\left(\frac{a(t)}{a_0}\right)\right).$$

We see that as $a(t)$ increases the exponent gets a larger negative value making the whole expression for δH decrease. We know that the scale factor increases rapidly during inflation and the perturbation δH does then quickly die out.

[1] Elgarøy, *AST3220 – Cosmology I* (Institute of Theoretical Astrophysics, UiO, 2024).
[2] *Part 2 of project 3 in AST3220, 2024* (Institute of Theoretical Astrophysics, UiO, 2024).