AST3220 - project 2

Candidate number 17
(Dated: October 16, 2024)

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Big Bang nucleosynthesis (BBN) is the time period in the Universe right after the Big Bang, where all the lightest elements in the Universe were produced. The Boltzmann equation is used to describe the change in neutron and proton number densities (n_n and n_p):

$$\frac{\mathrm{d}n_n}{\mathrm{d}t} + 3Hn_n = n_p\Gamma_{p\to n} - n_n\Gamma_{n\to p} \tag{1}$$

$$\frac{\mathrm{d}n_p}{\mathrm{d}t} + 3Hn_p = n_n\Gamma_{n\to p} - n_p\Gamma_{p\to n} \tag{2}$$

where $\Gamma_{i\to j}$ is the decay rate, and H the Hubble constant. The relative number densities are defined as

$$Y_n = \frac{n_n}{n_b}, \quad Y_p = \frac{n_p}{n_b} \tag{3}$$

where

$$n_b = \frac{n_{b0}}{a^3} = \frac{\rho_{b0}}{a^3 m_p}, \quad \rho_{b0} = \Omega_{b0}\rho_{c0} \tag{4}$$

is the total baryon nucleon number density and ρ_{b0} is the total baryon mass density [1].

I. A)

We want to show that the equations for $dY_n/d(\ln T)$ and $dY_p/d(\ln T)$, starting from equation 1 and 2 are

$$\frac{\mathrm{d}Y_n}{\mathrm{d}(\ln T)} = -\frac{1}{H}\left[Y_p\Gamma_{p\to n} - Y_n\Gamma_{n\to p}\right] \tag{5}$$

and

$$\frac{\mathrm{d}Y_p}{\mathrm{d}(\ln T)} = -\frac{1}{H}\left[Y_n\Gamma_{n\to p} - Y_p\Gamma_{p\to n}\right]. \tag{6}$$

where $\Gamma_{p\to n}$ and $\Gamma_{n\to p}$ are the decay rates of protons and neutrons and we use the logarithmic temperature $T = T_0/a$ as the time variable. We start by using Leibniz's notation for the chain rule to rewrite $dY_n/d(\ln T)$.

$$\frac{\mathrm{d}Y_n}{\mathrm{d}(\ln T)} = \frac{\mathrm{d}Y_n}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}(\ln T)} \tag{7}$$

Then we calculate dY_n/dt using that $Y_n = n_n/n_b$:

$$\frac{\mathrm{d}Y_n}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{n_n}{n_b}\right) = n_b\frac{\mathrm{d}n_n}{\mathrm{d}t}\frac{1}{n_b^2} - n_n\frac{\mathrm{d}n_b}{\mathrm{d}t}\frac{1}{n_b^2}$$

We use the definition of n_b in equation 4

$$\begin{aligned}\frac{\mathrm{d}Y_n}{\mathrm{d}t} &= \frac{\mathrm{d}n_n}{\mathrm{d}t}\frac{1}{n_b} - n_n\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{n_{b0}}{a^3}\right)\frac{1}{n_b^2} \\ &= \frac{\mathrm{d}n_n}{\mathrm{d}t}\frac{1}{n_b} + 3\frac{n_n}{n_b^2}\left(\frac{n_{b0}}{a^4}\right)\frac{\mathrm{d}a}{\mathrm{d}t},\end{aligned}$$

find an expression for dn_n/dt by rearranging the terms in equation 1 and used that $da/dt = Ha$.

$$
\begin{aligned}
&= (n_p\Gamma_{p\to n} - n_n\Gamma_{n\to p} - 3Hn_n)\frac{1}{n_b} + 3\frac{n_n}{n_b^2}n_bH \\
&= (n_p\Gamma_{p\to n} - n_n\Gamma_{n\to p} - 3Hn_n + 3Hn_n)\frac{1}{n_b} \\
&= Y_p\Gamma_{p\to n} - Y_n\Gamma_{n\to p}
\end{aligned}
$$

We now calculate $dt/d(\ln T)$:

$$
\frac{\mathrm{d}t}{\mathrm{d}(\ln T)} = \frac{\mathrm{d}t}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}(\ln T)}
$$

where the first factor is

$$
\begin{aligned}
\frac{\mathrm{d}t}{\mathrm{d}T} &= \left(\frac{T}{t}\right)^{-1} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{T_0}{a}\right)\right)^{-1} \\
&= \left(-\frac{T_0}{a^2}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^{-1} = -\frac{a^2}{T_0}\frac{\mathrm{d}t}{\mathrm{d}a}
\end{aligned}
$$

and the second factor is

$$
\frac{\mathrm{d}T}{\mathrm{d}(\ln T)} = \left(\frac{\mathrm{d}(\ln T)}{\mathrm{d}T}\right)^{-1} = \left(\frac{1}{T}\right)^{-1} = T.
$$

We combine the two factors and get that

$$
\frac{\mathrm{d}t}{\mathrm{d}(\ln T)} = -\frac{Ta^2}{T_0}\frac{\mathrm{d}t}{\mathrm{d}a} = -\frac{1}{H} \tag{8}
$$

where we used that $a/T_0 = 1/T$ and that the Friedmann equation for a completely radiation dominated Universe is

$$
H = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = H_0\frac{\sqrt{\Omega_{r0}}}{a^2}. \tag{9}
$$

We can now combine what we have found to get the final expression of $dY_n/d(\ln T)$:

$$
\begin{aligned}
\frac{\mathrm{d}Y_n}{\mathrm{d}(\ln T)} &= \frac{\mathrm{d}Y_n}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}(\ln T)} && (10) \\
&= -\frac{1}{H}[Y_p\Gamma_{p\to n} - Y_n\Gamma_{n\to p}] && (11)
\end{aligned}
$$

which is what we wanted to show. The same method can be used to find $dY_p/d(\ln T)$ where the only difference is that when finding dY_n/dt we need to differentiate dn_p/dt which is given by rearranging the terms in equation 2 instead of equation 1 which we used previously. We then end up with

$$
\frac{\mathrm{d}Y_p}{\mathrm{d}t} = n_n\Gamma_{n\to p} - n_p\Gamma_{p\to n} \tag{12}
$$

and $dt/d(\ln T)$ stays the same. Therefore, we get that

$$
\frac{\mathrm{d}Y_p}{\mathrm{d}(\ln T)} = -\frac{1}{H}(n_n\Gamma_{n\to p} - n_p\Gamma_{p\to n}). \tag{13}
$$

II. BONUS QUESTION

At the time of BBN, where the temperature was $T \sim 10^9$ K, the baryon mass density can be estimated as follows:

The density is given as $\rho_i = \rho_{i0}\left(\frac{a_0}{a}\right)^{3(1+w)}$, where $w = 0$ for dust [2] which gives us

$$
\rho_b = \rho_{b0}\left(\frac{a_0}{a}\right)^3. \tag{14}
$$

We assume $a_0 = 1$ which gives

$$
\rho_b = \rho_{b0}\left(\frac{1}{a}\right)^3 = \rho_{b0}\left(\frac{T}{T_0}\right)^3 = \Omega_{b0}\rho_{c0}\left(\frac{T}{T_0}\right)^3 \tag{15}
$$

where we used that $T = T_0/a$ and $\Omega_{i0} = \rho_{i0}/\rho_{c0}$. We know that today's value of the baryon density parameter is $\Omega_{b0} = 0.05$, $\rho_{c0} \approx 9.2 \cdot 10^{-27}$ kg/m^3 and today's temperature of the Universe is $T_0 = 2.725$ K. Inserting these values in the equation above, equation 15 gives

$$
\begin{aligned}
\rho_b &\sim 0.05 \cdot 9.2 \cdot 10^{-27}\left(\frac{10^9\ \mathrm{K}}{2.725\ \mathrm{K}}\right)^3\ \mathrm{kg/m^3} \\
&\sim 10^{-2}\ \mathrm{kg/m^3}.
\end{aligned}
$$

When comparing this baryon density to the mean density of the Sun which is $\bar{\rho}_\odot \sim 10^3$ kg/m^3 we see that

$$
\frac{\rho_b}{\bar{\rho}_\odot} \sim \frac{10^{-2}}{10^3} = 10^{-5}. \tag{16}
$$

This means that the baryon density at the time of BBN is 5 orders of magnitude smaller than the mean density of the Sun. Lastly, we want to estimate the ratio ρ_b/ρ_r for $\Omega_{r0} \sim 10^{-4}$, at the time of BBN. The radiation density can be written as $\rho_r = \Omega_{r0}\rho_{c0}/a^4$, where the a^4 is because $w = 1/3$ for radiation. The ratio then becomes

$$
\begin{aligned}
\frac{\rho_b}{\rho_r} &= \frac{\Omega_{b0}\rho_{c0}a^4}{a^3\Omega_{r0}\rho_{c0}} = \frac{\Omega_{b0}}{\Omega_{r0}}a \\
&\sim \frac{0.05}{10^{-4}}\frac{2.725\ \mathrm{K}}{10^9\ \mathrm{K}} \sim 10^{-6}
\end{aligned}
$$

where we again used that $a = T_0/T$.

III. B)

We want to show why the relation $T_\nu = (4/11)^{1/3}T$ holds and consider the conservation of total entropy before and after electrons and positrons have become non-relativistic and annihilate. They annihilate through the reaction

$$
e^+ + e^- \to \gamma + \gamma \tag{17}
$$

which dominates over the reaction

$$
\gamma + \gamma \to e^+ + e^- \tag{18}
$$

as the temperature decreases, which it does when the Universe expands [2].

If we assume that entropy is conserved, then $S = g_{*s}(aT)^3 =$ constant, where g_{*s} is the effective number of relativistic degrees of freedom that contribute to the entropy. The total entropy before and after annihilation can then be written as

$$S_\nu^{\text{before}} + \left[g_{*s}(aT)^3\right]_{\text{before}} = S_\nu^{\text{after}} + \left[g_{*s}(aT)^3\right]_{\text{after}} \quad (19)$$

where S_ν is the entropy of the neutrinos which at this point are decoupled and no longer interacts with the rest of the Universe. The other term $[g_{*s}(aT)^3]$ is the entropy of the other particles (photons, positrons and electrons). The entropy of the neutrinos are conserved separately giving us two separate conservation laws:

$$S_\nu^{\text{before}} = S_\nu^{\text{after}} \quad (20)$$

and

$$\left[g_{*s}(aT)^3\right]_{\text{before}} = \left[g_{*s}(aT)^3\right]_{\text{after}} \quad (21)$$

We now find g_{*s}, leaving out the neutrinos as they are decoupled. Before the annihilation, $k_BT > m_ec^2$, meaning that we can assume that the photons are relativistic, and the only relativistic particles are photons giving $g_\gamma = 2$ and fermions (electron and positron), $g_{ferm} = \frac{7}{8}\cdot 2\cdot 2$ where the $2\cdot 2$ is because they have spin 1/2 and because they have anti-particles.

$$g_{*s} = 2 + \frac{7}{8}\cdot 2\cdot 2 = \frac{11}{2}. \quad (22)$$

After the annihilation we are left with only photons, which are the only relativistic particles contributing when excluding neutrinos. We then have that $g_{*s} = g_\gamma = 2$. We insert what we have found for g_{*s} in equation 21:

$$\frac{11}{2}(aT)^3_{\text{before}} = 2(aT)^3_{\text{after}} \quad (23)$$

$$\frac{11}{2}a^3_{\text{before}}T^3_{\text{before}} = 2a^3_{\text{after}}T^3_{\text{after}} \quad (24)$$

Since the process happens very fast, the Universe does not have time to expand very much and $a_{\text{before}} \simeq a_{\text{after}}$. In other words, the expansion rate will be much lower than the rate of the annihilation processes. Then we have that

$$\frac{11}{2}T^3_{\text{before}} = 2T^3_{\text{after}} \quad (25)$$

$$T_{\text{after}} = \left(\frac{11}{4}\right)^{1/3} T_{\text{before}}. \quad (26)$$

The temperature after annihilation is a bit higher than before. This seems reasonable because the energy from the annihilation of positrons and electrons is added to the photons and the photon mass is therefore heated a bit. The neutrinos does not participate and are not heated meaning that $T_\nu \neq T_\gamma$.

If we take this relation to be true throughout our treatment of BBN we assume that the positrons and electrons are annihilated before we start our simulations.

IV. C)

Assuming that photons and the number of neutrino species, N_{eff}, make up all of the radiation in our Universe, we want to show that

$$\Omega_{r0} = \frac{8\pi^3}{45}\frac{G}{H_0^2}\frac{(k_BT_0)^4}{\hbar^3c^5}\left[1 + N_{eff}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right]. \quad (27)$$

We start from the definition of the density parameter $\Omega_{r0} = \rho_{r0}/\rho_{c0}$ for radiation where

$$\rho_{c0} = \frac{3H_0^2}{8\pi G}. \quad (28)$$

Since we assume that photons and neutrinos make up all the radiation in the Universe we have that the total energy from radiation is $\rho_rc^2 = \rho_\gamma c^2 + \rho_\nu c^2$. We then express the total energy using equation (4.21) in Elgarøy (2024):

$$\rho_rc^2 = \frac{\pi^2}{30}\frac{(k_BT)^4}{(\hbar c)^3}g_* \quad (29)$$

where k_B is the Boltzmann constant, and g_* is again the effective number of relativistic degrees of freedom. For photons $g_\gamma = 2$ and for the neutrinos $g_\nu = 2N_{eff}$. We divide the energy by c^2 and find that the radiation density can then be expressed as

$$\begin{aligned}\rho_r &= \frac{\pi^2}{30}\frac{(k_BT)^4}{\hbar^3c^5}\left[g_\gamma\left(\frac{T_\gamma}{T}\right)^4 + \frac{7}{8}g_\nu\left(\frac{T_\nu}{T}\right)^4\right] \\ &= \frac{\pi^2}{30}\frac{(k_BT)^4}{\hbar^3c^5}\left[2 + \frac{7}{8}2N_{eff}\left(\frac{4}{11}\right)^{4/3}\right]\end{aligned}$$

where we used that $T = T_\gamma$ and $T_\nu = (4/11)^{1/3}T$. The density parameter then becomes

$$\begin{aligned}\Omega_{r0} = \frac{\rho_{r0}}{\rho_{c0}} &= \frac{\pi^2}{30}\frac{(k_BT_0)^4}{\hbar^3c^5}\frac{8\pi G}{3H_0^2}2\left[1 + \frac{7}{8}N_{eff}\left(\frac{4}{11}\right)^{4/3}\right] \\ &= \frac{8\pi^3}{45}\frac{(k_BT_0)^4}{\hbar^3c^5}\frac{G}{H_0^2}\left[1 + \frac{7}{8}N_{eff}\left(\frac{4}{11}\right)^{4/3}\right]\end{aligned}$$

which is what we wanted to show.

V. D)

We want to find an expression for $a(t)$ and $t(T)$ by integrating the Friedmann equation

$$H = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = H_0\sqrt{\Omega_{r0}}a^{-2} \quad (30)$$

where $\Omega_{r0} = \rho_{r0}/\rho_{c0}$ is the fraction of radiative energy in our Universe today that we found an expression for in the previous exercise. We separate and integrate equation 30:

$$\begin{aligned} a da &= H_0\sqrt{\Omega_{r0}}dt \\ \int_0^a a' da' &= H_0\sqrt{\Omega_{r0}}\int_0^t dt \\ \frac{1}{2}a^2 &= H_0\sqrt{\Omega_{r0}}t \\ \Rightarrow a(t) &= \sqrt{2H_0 t}(\Omega_{r0})^{1/4} \end{aligned}$$

To find $t(T)$ we use the relation $T = T_0/a$ and get

$$\begin{aligned} \frac{1}{2}\left(\frac{T_0}{T}\right)^2 &= H_0\sqrt{\Omega_{r0}}t \\ t(T) &= \left(\frac{T_0}{T}\right)^2 \frac{1}{2H_0\sqrt{\Omega_{r0}}} \end{aligned}$$

We use $t(T)$ to find how old the Universe was at $T = 10^{10}$ K, $T = 10^9$ K, and $T = 10^8$ K. The value of the Hubble constant today is $H_0 = 100 \cdot h$ km s^{-1} Mpc^{-1} and to find Ω_{r0} we insert in equation 27. The age of the Universe at the different temperatures is in the script `question_d.py` found to be

$$\begin{aligned} t(10^{10}\ \mathrm{K}) &= 1.778\ \mathrm{s} \\ t(10^9\ \mathrm{K}) &= 177.74\ \mathrm{s} \approx 2\ \mathrm{min}\ 57\ \mathrm{s} \\ t(10^8\ \mathrm{K}) &= 17773.69\ \mathrm{s} \approx 4\ \mathrm{h}\ 56\ \mathrm{min}\ 13\ \mathrm{s} \end{aligned}$$

VI. E)

Assuming that all of the baryonic mass ρ_b at the initial temperature T_i is in neutrons and protons and that they are all in thermal equilibrium at this temperature, we want to show that

$$Y_n(T_i) = \left[1 + e^{(m_n - m_p)c^2/k_B T_i}\right]^{-1} \tag{31}$$

and

$$Y_p(T_i) = 1 - Y_n(T_i). \tag{32}$$

The relative mass densities Y_n and Y_p are defined in equation 3 as $Y_n = n_n/n_b$ and $Y_p = n_p/n_b$. Initially the neutrons and protons are in equilibrium, with a preference for protons as they are a bit lighter, which can be expressed as

$$\frac{n_n^{(0)}}{n_p^{(0)}} = \left(\frac{m_p}{m_n}\right)^{3/2} e^{-(m_n - m_p)c^2/k_B T} \tag{33}$$

[1] where $n^{(0)}$ is the equilibrium densities and m_n, m_p is the mass of neutrons and protons respectively. If we divide the relative mass densities by each other we get

$$\frac{Y_n(T_i)}{Y_p(T_i)} = \frac{n_n^{(0)}}{n_p^{(0)}} = \left(\frac{m_p}{m_n}\right)^{3/2} e^{-(m_n - m_p)c^2/k_B T}.$$

Using the approximation $m_p \approx m_n$ outside of the exponent and that $Y_n + Y_p = 1$ since all the baryonic mass at T_i consists of neutrons and protons gives

$$\begin{aligned} \frac{Y_n(T_i)}{1 - Y_n(T_i)} &= e^{-(m_n - m_p)c^2/k_B T} \\ Y_n(T_i) &= (1 - Y_n(T_i))e^{-(m_n - m_p)c^2/k_B T} \\ Y_n(T_i)(1 + e^{-(m_n - m_p)c^2/k_B T}) &= e^{-(m_n - m_p)c^2/k_B T} \\ Y_n(T_i) &= \frac{1}{1 + e^{-(m_n - m_p)c^2/k_B T}} \end{aligned}$$

which is what we wanted to show. Since, as previously stated, $Y_n + Y_p = 1$, we see that

$$Y_p(T_i) = 1 - Y_n(T_i) \tag{34}$$

which we also wanted to show.

VII. F)

We write a code (script `question_f.py`) that solves equation 5 and 6 from $T_i = 100 \cdot 10^9$ K to $T_f = 0.1 \cdot 10^9$ K, with the initial conditions given in equation 31 and 32. To solve for Y_n and Y_p we implement the weak reactions 1)-3) listed in table 2a) in Wagoner *et al.* (1967) and integrate numerically the reaction rates which are given as

$$\begin{aligned} \Gamma_{n\to p}(T, q) &= \frac{1}{\tau}\int_1^\infty \frac{(x+q)^2(x^2-1)^{1/2}x}{[1+e^{xZ}][1+e^{-(x+q)Z_\nu}]}dx \qquad (35) \\ &+ \frac{1}{\tau}\int_1^\infty \frac{(x-q)^2(x^2-1)^{1/2}x}{[1+e^{-xZ}][1+e^{(x-q)Z_\nu}]}dx \qquad (36) \\ \Gamma_{p\to n}(T, q) &= \Gamma_{n\to p}(T, -q) \qquad (37) \end{aligned}$$

where $\tau = 1700$ s is the free neutron decay time, $Z = m_e c^2/k_B T = 5.93/T9$, $Z_\nu = m_e c^2/k_B T_\nu = 5.93/T9$ and $q = (m_n - m_p)/m_e = 2.53$. The solution of Y_n and Y_p can be seen plotted in figure 1. Note that the code is inspired by the code example written by Jakob Borg and has the same structure.

We plot the mass fraction $A_i Y_i$ where A_i is the mass number of particle i, because if we had only plotted Y_i, the total number of particles would decrease as we get fewer but heavier particles when the temperature decreases. We therefore plot the fraction of total baryon mass in the different particle species instead. From figure 1 we see that most of the neutrons decay to protons as the temperature decreases. We also see that the total number of nucleons is conserved throughout the simulation as $\sum A_i Y_i = 1$, where A_i is the particle mass number.

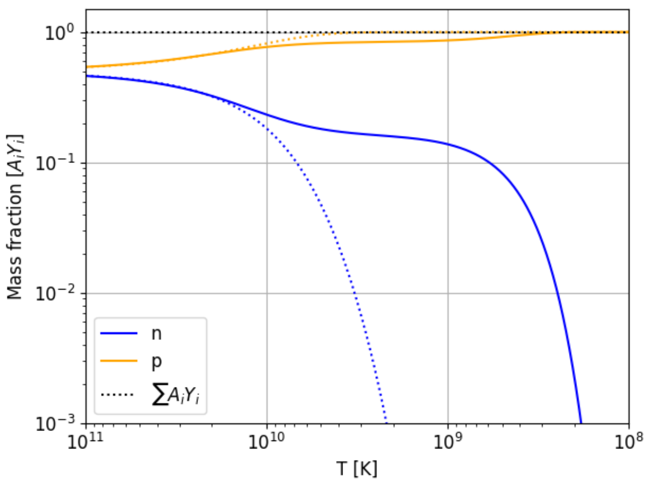

Figure 1. The solid lines show the solution of Y_n and Y_p plotted as a function of temperature. The dotted orange and blue lines represent the equilibrium values given in equation 31 and 32.

VIII. G)

The general Boltzmann equation is

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} + 3Hn_i = \sum_{j\neq i}\left[n_j\Gamma_{j\to i} - n_i\Gamma_{i\to j}\right] + \sum_{jkl}\left[n_k n_l \gamma_{kl\to ij} - n_i n_j \gamma_{ij\to kl}\right] \tag{38}$$

where i represents a particle which interacts with any number of other particles j, through two-body reactions and decays. We want to show that the equation for $dY_i/d(\ln T)$, starting from equation 38 can be expressed as

$$\frac{\mathrm{d}Y_i}{\mathrm{d}(\ln T)} = -\frac{1}{H}\left\{\sum_{i\neq j}\left[Y_j\Gamma_{j\to i} - Y_i\Gamma_{i\to j}\right] + \sum_{jkl}\left[Y_k Y_l \Gamma_{kl\to ij} - Y_i Y_j \Gamma_{ij\to kl}\right]\right\}. \tag{39}$$

We know from exercise A in section I that

$$\frac{\mathrm{d}Y_i}{\mathrm{d}(\ln T)} = \frac{\mathrm{d}Y_i}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}(\ln T)} = -\left(\frac{\mathrm{d}n_i}{\mathrm{d}t}\frac{1}{n_b} + \frac{3Hn_i}{n_b}\right)\frac{1}{H}.$$

Using the general Boltzmann equation 38, that $n_i/n_b = Y_i$, $n_k/n_b = Y_k$ and defining $Y_j\Gamma_{ij\to kl} = n_b\frac{n_j}{n_b}\gamma_{ij\to kl}$ and $Y_l\Gamma_{kl\to ij} = n_b\frac{n_l}{n_b}\gamma_{kl\to ij}$ we get

$$\begin{aligned}
\frac{\mathrm{d}Y_i}{\mathrm{d}(\ln T)} &= -\frac{1}{Hn_b}\left[\sum_{j\neq i}\left[n_j\Gamma_{j\to i} - n_i\Gamma_{i\to j}\right] + \sum_{jkl}\left[n_k n_l\gamma_{kl\to ij} - n_i n_j\gamma_{ij\to kl}\right]\right] \\
&= -\frac{1}{H}\left[\sum_{j\neq i}\left[\frac{n_j}{n_b}\Gamma_{j\to i} - \frac{n_i}{n_b}\Gamma_{i\to j}\right] + \sum_{jkl}\left[\frac{n_k}{n_b}n_l\gamma_{kl\to ij} - \frac{n_i}{n_b}n_j\gamma_{ij\to kl}\right]\right] \\
&= \frac{1}{H}\left\{\sum_{i\neq j}\left[Y_j\Gamma_{j\to i} - Y_i\Gamma_{i\to j}\right] + \sum_{jkl}\left[Y_k Y_l\Gamma_{kl\to ij} - Y_i Y_j\Gamma_{ij\to kl}\right]\right\}
\end{aligned}$$

which is what we wanted to show.

IX. H)

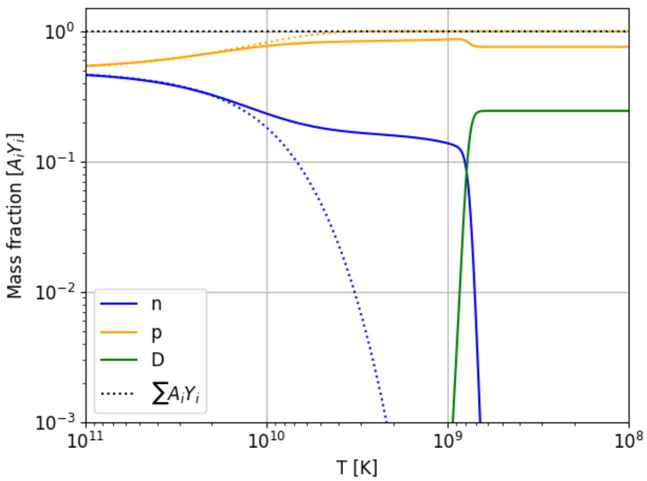

Figure 2. The solid lines show the solution of Y_n, Y_p and $2Y_D$ as a function of temperature. The dotted orange and blue lines show the equilibrium values from equation 31 and 32.

In script `question_h.py` we have written a code which solves the Boltzmann equations for n, p and D up to the deuterium bottleneck. The deuterium bottleneck appears at around $T \approx 9 \cdot 10^8$ K where for higher temperatures than this there are many high-energy photons that instantaneously disintegrate the deuterium that is created. Because of this the BBN cannot proceed as deuterium is needed for further reactions to happen.

We implement the equations for the weak reactions 1)-3) from table 2 a) and the reaction which includes deuterium, equation 1) from table 2 b) in Wagoner *et al.* (1967). We integrate from $T_i = 100 \cdot 10^9$ K to $T_f = 0.1 \cdot 10^9$ K, with initial conditions for Y_n and Y_p given in equation 31 and 32 and the initial condition for $Y_D = 0$. The resulting solutions can be seen in figure 2.

Initially (for $T_i = 100 \cdot 10^9$) there is only protons and neutrons in equilibrium in the Universe as shown in figure 1. As the temperature decreases, we see that there is an increase in the mass fraction of protons and a decrease in the mass fraction of neutrons. The reason for this is that the protons are slightly lighter as explained in section VI.

When comparing the relative number densities to the equilibrium values we see that the reaction rate becomes lower than the expansion rate for n and p such that they fall out of equilibrium and freeze out. We see, however, that the neutron mass fraction starts to decrease even though they are frozen out when we reach the end of the deuterium bottleneck.

When the temperature reaches below $\sim 10^9$ K we see that the fraction of neutrons decreases rapidly, while the fraction of deuterium increases rapidly. This must be due to the fact that the temperature is now low enough for there to not be enough energetic photons to disintegrate all the deuterium. The reaction $p + n \rightarrow D + \gamma$ from table 2b) Wagoner *et al.* (1967) is the one who takes place at this time. We also notice a slight drop in the mass fraction of protons at this temperature. After these rapid changes both the deuterium and proton fractions remain constant as the mass fraction of neutrons becomes too low for the reactions which create protons and deuterium to happen.

X. I)

In script `question_i.py` we implement reactions 1)-3) from table 2a) and reactions 1)-11), 15)-18) and 20)-21) from table 2b) in Wagoner *et al.* (1967). Using these reactions we solve the Boltzmann equation for n, p, D, T, He^3, He^3, Li^7, Be^7. We integrate from $T_i = 100 \times 10^9$ K to $T_f = 0.01 \times 10^9$ K, with initial conditions for Y_n and Y_p given in equation 31 and 32 and the other initial conditions $Y_i = 0$.

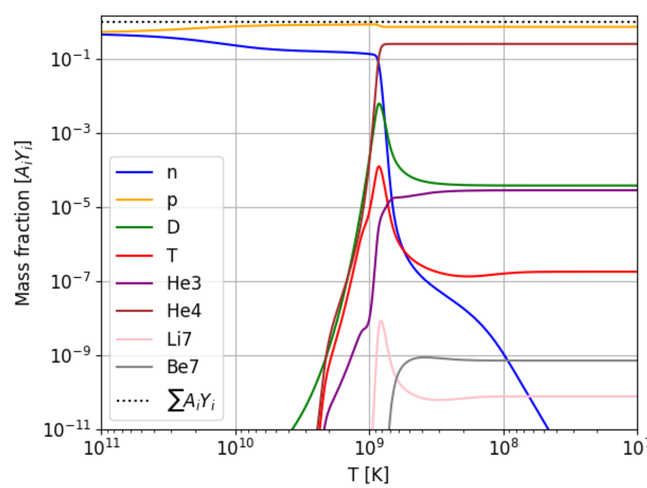

Figure 3. The mass fraction A_iY_i of all the 8 species of elements as a function of temperature using reactions from table 2 in Wagoner *et al.* (1967).

XI. J)

We compute the relic abundances of D, He^3, He^4 and Li^7 in the range $\Omega_{b0} = [0.01, 1]$ and interpolate them in log-space to find an accurate value for the best fit Ω_{b0}. We integrate from $T_i = 100 \cdot 10^9$ K to $T_f = 0.01 \cdot 10^9$ K. This is done in script `question_j.py`. The relic abundances are shown in the upper two plots in figure 4 in appendix A. We compare our theoretical predictions to observations of the abundance fractions of D and Li^7 relative to hydrogen:

$$
\begin{aligned}
Y_D/Y_p &= (2.57 \pm 0.03) \times 10^{-5} \\
4Y_{He^4} &= 0.254 \pm 0.003 \\
Y_{Li^7}/Y_p &= (1.6 \pm 0.3) \times 10^{-10}
\end{aligned}
$$

We then find that the most probable value for Ω_{b0} when comparing against the observations is $\Omega_{b0} = 0.05$ using Bayesian probability. This probability is shown in the bottom plot of figure 4 in appendix A. The corresponding χ^2-error was found to be 0.811.

The total matter content of the Universe is about $\Omega_{m0} = 0.3$, where dark matter is a significant fraction of this. The value for Ω_{b0} that we find from BBN is much smaller than the total matter content of the Universe. This tells us that the dark matter cannot be baryonic matter. Instead it might be some other particle that must be stable and massless or approximately massless. The dark matter particles also cannot interact much with the baryonic matter and other standard model particles. One particle candidate for dark matter is neutrinos, as they exist in great abundance in the Universe, only have weak interactions, but neutrino-oscillations show that not all neutrinos are massless. It is also proven that the neutrinos alone do not contribute enough to explain all the dark matter, and can only be a small part of the total dark matter.

XII. K)

We now compute the same relic abundances as in the previous exercise and in the same way, but in the range $N_{eff} = [1, 5]$. We integrate from $T_i = 100 \cdot 10^9$ K to $T_f = 0.01 \cdot 10^9$ K. The relic abundances is shown in the three upper plots of figure 5 in appendix A. We compare against the same observations listed in section XI and find that the most probable value for N_{eff}, meaning the best fit to the data is $N_{eff} \approx 3$. The Bayesian probability curve is shown in the bottom plot of figure 4 in appendix A. The corresponding χ^2-error was found to be 0.874.

The best-fit value for the effective number of neutrino species $N_{eff} = 3$ is consistent with what we have learnt about neutrinos. In the standard model in particle physics it is stated that there are 3 types of neutrinos (electron-neutrino, muon-neutrino and tau-neutrino) [2], which is exactly what we find from the observations and simulation.

[1] *AST3220, spring 2024: Project 2* (Institute of Theoretical Astrophysics, UiO, 2022).

[2] Elgarøy, *AST3220 – Cosmology I* (Institute of Theoretical Astrophysics, UiO, 2024).

[3] R. V. Wagoner, W. A. Fowler, and F. Hoyle, Astrophys. J. **148**, 3 (1967).

Appendix A: Figures of relic abundances

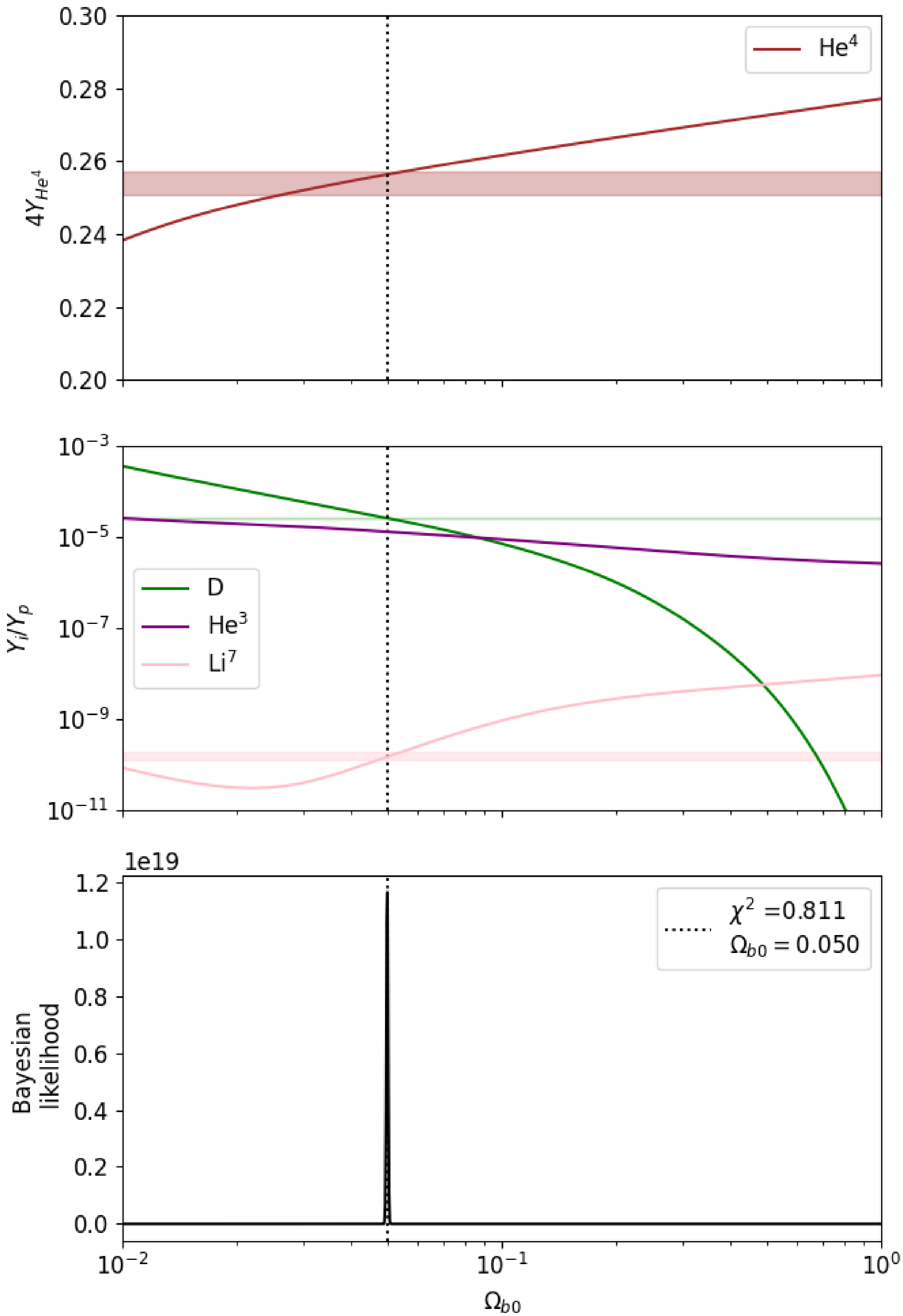

Figure 4. In the upper two plots we see the relic abundance of elements as a function of the baryon density Ω_{b0}. The shaded horizontal lines show the measurements listed in section XI. The bottom plot show the Bayesian probability and the best fit value of Ω_{b0} is shown by the dotted line with minimum value of χ^2.

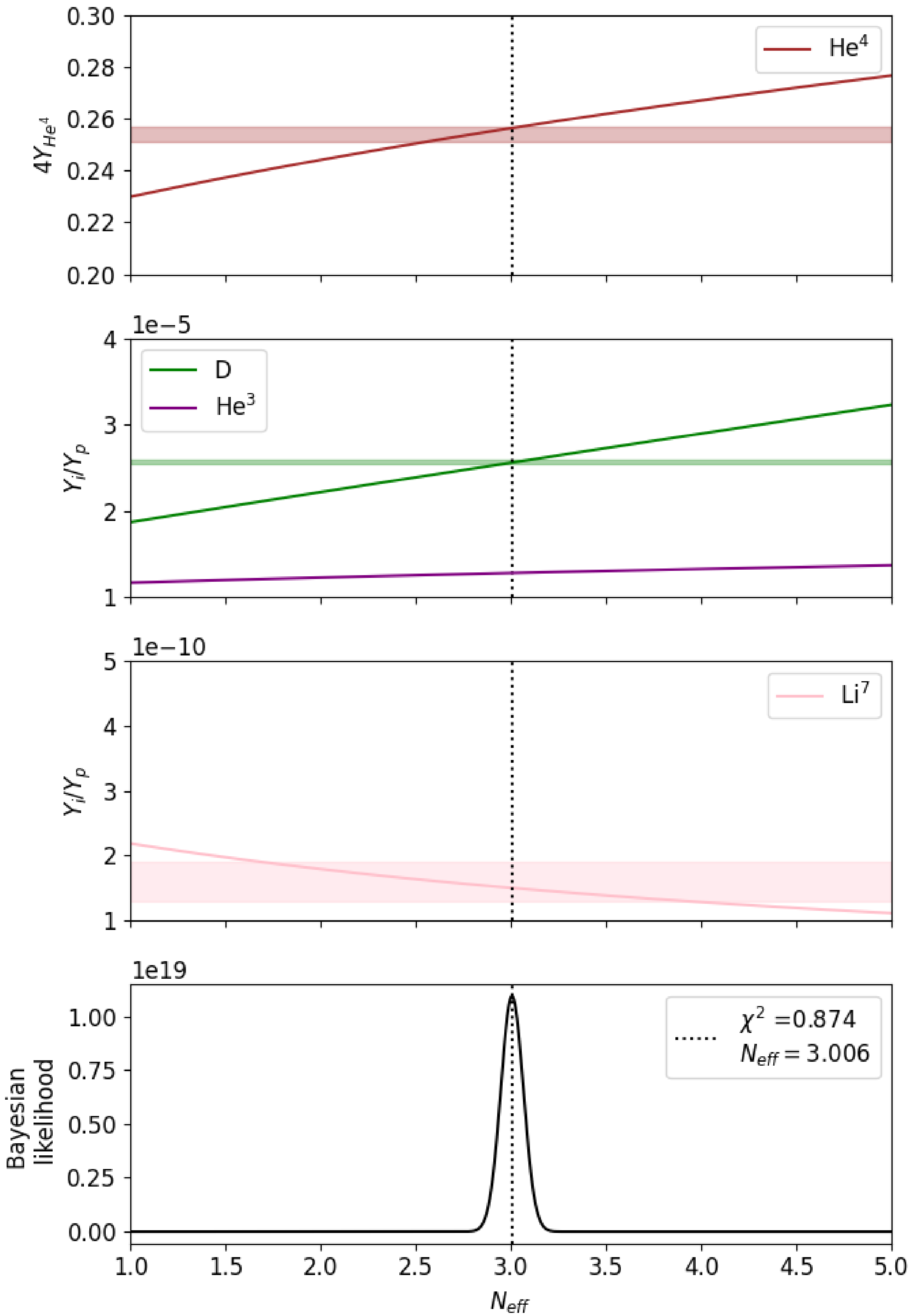

Figure 5. In the upper two plots we see the relic abundance of elements as a function of the effective number of neutrinos N_{eff}. The shaded horizontal lines show the measurements listed in section XI. The bottom plot show the Bayesian probability and the best fit value of N_{eff} is shown by the dotted line with minimum value of χ^2.