AST3220 - Project 1

Candidate number 17
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I. INTRODUCTION

One of the unsolved problems in cosmology today is the value of the cosmological constant. We will therefore in this project investigate if the quintessence field can serve as a better explanation for the acceleration of the universe. The quintessence field is a scalar field with energy density ρ_ϕ and pressure p_ϕ as described by equation 1 and equation 2

$$\rho_\phi = \frac{1}{2}\phi^2 + V(\phi) \tag{1}$$

$$p_\phi = \frac{1}{2}\phi^2 - V(\phi) \tag{2}$$

where $V(\phi)$ is the potential with units of energy density. The equation of state (EoS) parameter for the quintessence field is given as

$$w_\phi = \frac{p_\phi}{\rho_\phi}. \tag{3}$$

Since the quintessence field behaves as a fluid it follows the continuity equation

$$\dot{\rho_\phi} = -3H(\rho_\phi + p_\phi) = -3H(1 + w_\phi)\rho_\phi \tag{4}$$

where $\dot{\rho}_\phi = d\rho_\phi/dt$ [1].

II. PROBLEM 1

The continuity equation for the quintessence field is given in equation 4. To solve this equation we separate the variables and integrate on both sides:

$$\frac{d\rho_\phi}{dt} = -3H(1 + \omega_\phi)\rho_\phi$$

$$\int_{\rho_{\phi 0}}^{\rho_\phi} \frac{1}{\rho_\phi} d\rho_\phi = -3\int_{t_0}^{t} H(1 + \omega_\phi)dt$$

We then change the integration variable from t to z by using the relations

$$\frac{\dot{a}(t)}{a(t)} = H = \frac{da}{dt}\frac{1}{a(t)} \Rightarrow dt = \frac{da}{a(t)H}$$

$$z = \frac{a_0}{a(t)} - 1 \Rightarrow a(t) = \frac{a_0}{z+1} \Rightarrow da = \frac{-a_0}{(z+1)^2}dz$$

The integration limits changes to:

$$t_0 : z = \frac{a_0}{a(t_0)} - 1 = 1 - 1 = 0$$

$$t : z = \frac{a_0}{a(t)} - 1 = z$$

And the equation then becomes

$$\begin{aligned}\int_{\rho_{\phi 0}}^{\rho_\phi} \frac{1}{\rho_\phi} d\rho_\phi &= -3\int_{a_0}^{a} (1+\omega_\phi)\frac{da'}{a'} \\ &= -3\int_0^z (1+\omega_\phi)\frac{(z'+1)}{a_0}\frac{(-a_0)}{(z'+1)^2}dz' \\ &= 3\int_0^z \frac{(1+\omega_\phi)}{(z'+1)}dz'\end{aligned}$$

When performing the integration we get

$$\begin{aligned}\ln|\rho_\phi - \rho_{\phi 0}| &= \int_0^z \frac{3(1+\omega_\phi)}{(z'+1)}dz' \\ \rho_\phi &= \rho_{\phi 0}\ \exp\left[\int_0^z \frac{3(1+\omega_\phi)}{(z'+1)}dz'\right]\end{aligned}$$

which is exactly what we wanted to show.

III. PROBLEM 2

By using equation 1 and 2 for the energy density and pressure we want to show that the continuity equation for the scalar field can be written as

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{5}$$

We start by inserting equation 1 and 2 into equation 4 for the continuity equation, and get

$$\begin{aligned}\dot{\rho}_\phi &= -3H(\rho_\phi + p_\phi) \\ &= -3H\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}\dot{\phi}^2 - V(\phi)\right) \\ &= -3H\dot{\phi}^2.\end{aligned}$$

We then differentiate the expression for the energy density in equation 1:

$$\begin{aligned}\dot{\rho_\phi} &= \frac{d}{dt}\left[\frac{1}{2}\dot{\phi}(t)^2 - V(\phi(t))\right] \\ &= \frac{1}{2}2\frac{d}{dt}(\dot{\phi})\dot{\phi} + \frac{dV}{d\phi}\frac{d\phi}{dt} \\ &= \ddot{\phi}\dot{\phi} + \dot{\phi}V'(\phi)\end{aligned}$$

Setting the two expressions for $\dot{\rho}_\phi$ equal to each other and dividing by $1/\dot{\phi}$ (assuming $\dot{\phi} \neq 0$) we get the following:

$$\begin{aligned}-3H\dot{\phi}^2 = \dot{\phi}\ddot{\phi} + \dot{\phi}V'(\phi) \mid \cdot \frac{1}{\dot{\phi}} \\ \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0\end{aligned}$$

Which is the form of the continuity equation we wanted to show.

We will now consider spatially flat ($k = 0$) models of the Universe which contain radiation, non-relativistic matter and the quintessence field. The Hubble parameter for such a model can be written as

$$H^2 = H_0^2\left[\Omega_{m0}(1+z)^3 + \Omega(1+z)^4\right. \tag{6}$$

$$\left. + \Omega_{\phi 0}\text{exp}\left\{\int_0^z dz'\frac{3[1+w_\phi(z')]}{(1+z')}\right\}\right] \tag{7}$$

where $\Omega_{i0} = \rho_{i0}/\rho_{c0} = 8\pi G\rho_{i0}/(3H_0^2)$ for component i. The subscript 0 refers to values of the present-day [1].

IV. PROBLEM 3

We want to find the time derivative of the Hubble parameter by using the first and second Friedmann equations together with equation 1 and 2. We know that $H = \dot{a}/a$ and if we differentiate this with respect to the time t we get

$$\dot{H} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right) = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = \text{FII - FI}.$$

When we have several components contributing to the energy density with $k = 0$, the first and second Friedmann equations can be expressed as

$$FI: \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{\kappa^2}{3}\left(\rho_r + \rho_m + \rho_\phi\right)$$

$$\begin{aligned}FII: \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \\ &= -\frac{\kappa^2}{6}\left(\rho_r + \rho_m + \rho_\phi + \frac{3}{c^2}\left(p_r + p_m + p_\phi\right)\right)\end{aligned}$$

where we used that $\kappa^2 = 8\pi G$. When subtracting FI from FII we get the following:

$$\begin{aligned}\dot{H} &= FII - FI \\ &= -\frac{\kappa^2}{6}(\rho_r + \rho_m + \frac{1}{2}\dot{\phi}^2 + V(\phi)) - \frac{\kappa^2}{2c^2}\left(p_r + p_m + p_\phi\right) \\ &- \frac{\kappa^2}{3}(\rho_r + \rho_m + \frac{1}{2}\dot{\phi}^2 + V(\phi))\end{aligned}$$

We multiply the last term by $2/2$ so that we can add the first and last term:

$$\dot{H} = -\frac{\kappa^2}{2}(\rho_r + \rho_m + \frac{1}{2}\dot{\phi}^2 + V(\phi)) - \frac{\kappa^2}{2c^2}(p_r + p_m + p_\phi)$$

Then we use the equation of state relation $p_i = \omega_i\rho_i c^2$. This simplifies the second term and we can combine it with the first one.

$$\dot{H} = -\frac{\kappa^2}{2}(\rho_r + \rho_m + \frac{1}{2}\dot{\phi}^2 + V(\phi) - \omega_r\rho_r + \omega_m\rho_m + \omega_\phi\rho_\phi)$$

We know that $\omega_m = 0$ since our model contains non-relativistic matter and using equation 3 we get $\omega_\phi \rho_\phi = p_\phi$ which simplifies to the expression we wanted to show

$$\dot{H} = -\frac{\kappa^2}{2}\left[\rho_m + \rho_r(1+w_r) + \dot{\phi}^2\right]. \tag{8}$$

Before the next problems we define the density parameters $\Omega_i = \rho_i(t)/\rho_c(t)$ where

$$\rho_c(t) = \frac{3H^2}{8\pi G} = \frac{3H^2}{\kappa^2}$$

is the critical density at time t. We also define the dimensionless variables

$$x_1 = \frac{\kappa\dot{\phi}}{\sqrt{6}H},\ x_2 = \frac{\kappa\sqrt{V}}{\sqrt{3}H},\ x_3 = \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H} \quad [1] \tag{9}$$

V. PROBLEM 4

We want to show that the density parameters can be expressed as

$$\Omega_\phi = x_1^2 + x_2^2, \Omega_r = x_3^2, \Omega_m = 1 - x_1^2 - x_2^2 - x_3^2. \tag{10}$$

To do this, we start by rewriting the density parameters:

$$\Omega_\phi = \frac{\rho_\phi}{\rho_{c0}} = \rho_\phi\frac{\kappa^2}{3H^2}$$
$$\Omega_m = \frac{\rho_m}{\rho_{c0}} = \rho_m\frac{\kappa^2}{3H^2}$$
$$\Omega_r = \frac{\rho_r}{\rho_{c0}} = \rho_r\frac{\kappa^2}{3H^2}$$

Then we insert the expressions for the dimensionless variables x_i from equation 9 into equation 10:

$$\Omega_\phi = x_1^2 + x_2^2 = \frac{\kappa^2\dot{\phi}^2}{6H^2} + \frac{\kappa^2 V}{3H^2}$$
$$= \frac{\kappa^2}{3H^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right) = \rho_\phi\frac{\kappa^2}{3H^2}$$
$$\Omega_r = x_3^3 = \rho_r\frac{\kappa^2}{3H^2}$$

Since all the density parameters always adds up to 1, we have that

$$\Omega_m = 1 - \Omega_\phi - \Omega_r = 1 - x_1^2 - x_2^2 - x_3^2$$

and we have now shown that the density parameters can be expressed as in equation 10.

VI. PROBLEM 5

We want to derive the following expression:

$$\frac{\dot{H}}{H^2} = -\frac{1}{2}\left[3 + 3x_1^2 - 3x_2^2 + x_3^2\right]$$

By using the definition of the dimensionless variables we get the following:

$$\rho_m = \Omega_m\rho_c = (1 - x_1^2 - x_2^2 - x_3^2)\frac{3H^2}{\kappa^2}$$
$$\rho_r = \Omega_r\rho_c = x_3^2\frac{3H^2}{\kappa^2}$$

Inserting these expressions in equation 8 for the time derivative of the Hubble parameter and using that $\omega_r = 1/3$ we get

$$\dot{H} = -\frac{\kappa^2}{2}\left[(1 - x_1^2 - x_2^2 - x_3^2)\frac{3}{\kappa^2} + x_3^2\frac{3}{\kappa^2}(1 + \frac{1}{3}) + \dot{\phi}^2\right].$$

We then divide by H^2 and to eliminate $\dot{\phi}^2$ from the equation above we use the expression for x_1 in equation 9 to find an expression for $\dot{\phi}^2$

$$x_1^2 = \frac{\kappa^2\dot{\phi}^2}{6H^2} \Rightarrow\ \dot{\phi}^2 = x_1^2\frac{6H^2}{\kappa^2}$$

which we insert in the expression for $\dot{H}$ and we get

$$\frac{\dot{H}}{H^2} = -\frac{1}{2}\left[3 - 3x_1^2 - 3x_2^2 - 3x_3^2 + 3x_2^2 + x_3^2 + 6x_1^2\right]$$
$$= -\frac{1}{2}\left[3 + 3x_1^2 - 3x_2^2 + x_3^2\right]$$

which is what we wanted to find.

The next thing we do is introduce two quantities λ and Γ which will be used to expressed the equations of motion for the dimensionless variables.

$$\lambda = -\frac{V'}{\kappa V} \tag{11}$$
$$\Gamma = \frac{VV''}{(V')^2} \tag{12}$$

where $V'' = \frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}$. We also introduce a new time variable

$$N = \ln\frac{a}{a_0} \tag{13}$$

where a_0 is the present value of the scalar factor. We introduce N since we want to track the evolution of the densities over almost the entire history of the Universe and t will therefore vary with many orders of magnitude. The derivative of N is

$$\dot{N} = \frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\ln\frac{a}{a_0}\right] = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = H$$

which we can use to simplify derivatives with respect to N of any quantity f to a derivative over the time t instead

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}N}{\mathrm{d}t}\frac{\mathrm{d}f}{\mathrm{d}N} = H\frac{\mathrm{d}f}{\mathrm{d}N} \quad [1] \tag{14}$$

VII. PROBLEM 6

We want to show that the equations of motion for the dimensionless variables can be expressed as follows

$$\frac{dx_1}{dN} = -3x_1 + \frac{\sqrt{6}}{2}\lambda x_2^2 + \frac{1}{2}x_1(3 + 3x_1^2 - 3x_2^2 + x_3^2) \quad (15)$$

$$\frac{dx_2}{dN} = -\frac{\sqrt{6}}{2}\lambda x_1 x_2 + \frac{1}{2}x_2(3 + 3x_1^2 - 3x_2^2 + x_3^2) \quad (16)$$

$$\frac{dx_3}{dN} = -2x_3 + \frac{1}{2}x_3(3 + 3x_1^2 - 3x_2^2 + x_3^2) \quad (17)$$

where λ and Γ are given by equation 11, 12.

We start with the first equation and use the relation in equation 14 to differentiate with respect to the time t instead of N. For the first equation (15) we get

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}N} &= \frac{1}{H}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\kappa\dot{\phi}}{\sqrt{6}H}\right) = \frac{\kappa}{\sqrt{6}H}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\phi}}{H}\right) \\ &= \frac{\kappa}{\sqrt{6}H}\left(\frac{H\ddot{\phi} - \dot{\phi}\dot{H}}{H^2}\right) = \frac{\kappa}{\sqrt{6}H}\frac{\ddot{\phi}}{H} - \frac{\kappa\dot{\phi}}{\sqrt{6}H}\frac{\dot{H}}{H^2} \\ &= \frac{\kappa}{\sqrt{6}H}\frac{\ddot{\phi}}{H} - x_1\frac{1}{2}(3 + 3x_1^2 - 3x_2^2 + x_3^2)\end{aligned}$$

Where we recognized that $\kappa\dot{\phi}/\sqrt{6}H = x_1$ from equation 9 and that $\dot{H}/H^2$ is the expression we found in problem 5. We can express $\kappa\ddot{\phi}/\sqrt{6}H^2$ in a different way using the expression for the continuity equation in equation 5:

$$\begin{aligned}\frac{\kappa}{\sqrt{6}H}\frac{\ddot{\phi}}{H} &= -\frac{\kappa}{\sqrt{6}H}\frac{3H\dot{\phi} + V'(\phi)}{H} \\ &= -\frac{3\kappa}{\sqrt{6}H}\dot{\phi} - \frac{\kappa}{\sqrt{6}H^2}V'(\phi) = 3x_1 - \frac{\kappa}{\sqrt{6}H^2}V'(\phi)\end{aligned}$$

Comparing what we have found so far with equation 15 for dx_1/dN we see that the only term we are missing is $\sqrt{6}\lambda x_2^2/2$ and we therefore see if this term is equal the one we are left with.

$$\frac{\sqrt{6}}{2}\lambda x_2^2 = \frac{\sqrt{6}}{2}\left(-\frac{V'}{\kappa V}\right)\frac{\kappa^2 V}{3H^2} = -\frac{\kappa V'}{\sqrt{6}H^2}$$

We see that this was indeed the last term we were missing and we have therefore verified that equation 15 is one equation of motion.

Next we want to show that the second equation of motion can be written as in equation 16. We start by again using the relation in equation 14 to differentiate with respect to t instead of N. This gives us

$$\begin{aligned}\frac{\mathrm{d}x_2}{\mathrm{d}N} &= \frac{1}{H}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\kappa\sqrt{V}}{\sqrt{3}H}\right) = \frac{\kappa}{\sqrt{3}H}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sqrt{V}}{H}\right) \\ &= \frac{\kappa}{\sqrt{3}H}\left(\frac{H\dot{\sqrt{V}} - \sqrt{V}\dot{H}}{H^2}\right) \\ &= \frac{\kappa\dot{\sqrt{V}}}{\sqrt{3}H^2} + \frac{\kappa\sqrt{V}}{\sqrt{3}H}\frac{\dot{H}}{H^2} \\ &= \frac{\kappa\dot{\sqrt{V}}}{\sqrt{3}H^2} + x_2\frac{1}{2}(3 + 3x_1^2 - 3x_2^2 + x_3^2).\end{aligned}$$

We compute $\dot{\sqrt{V}}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{V} = \frac{1}{2\sqrt{V}}\frac{\mathrm{d}}{\mathrm{d}t}V = \frac{1}{2\sqrt{V}}\frac{\mathrm{d}V}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{1}{2\sqrt{V}}V'\dot{\phi}$$

The second equation of motion then becomes

$$\frac{\mathrm{d}x_2}{\mathrm{d}N} = \frac{\kappa}{2\sqrt{3}H^2}\frac{V'\dot{\phi}}{\sqrt{V}} - x_2\frac{1}{2}\left[3 + 3x_1^2 - 3x_2^2 + x_3^2\right].$$

We check if the first term is equal the one we are missing when comparing with equation 16

$$-\frac{\sqrt{6}}{2}\lambda x_1 x_2 = -\frac{\sqrt{6}}{2}\left(-\frac{V'}{\kappa V}\right)\frac{\kappa\dot{\phi}}{\sqrt{6}H}\frac{\kappa\sqrt{V}}{\sqrt{3}H} = \frac{\kappa V'\dot{\phi}}{2\sqrt{3}\sqrt{V}H^2}$$

We see that it is, and we have then shown that equation 16 is another equation of motion.

Lastly we want to show that the third equation of motion can be expressed as in equation 17. We again differentiate with respect to time.

$$\begin{aligned}\frac{\mathrm{d}x_3}{\mathrm{d}N} &= \frac{1}{H}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H}\right) = \frac{\kappa}{\sqrt{3}H}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sqrt{\rho_r}}{H}\right) \\ &= \frac{\kappa}{\sqrt{3}H}\left(\frac{H\dot{\sqrt{\rho_r}} - \sqrt{\rho_r}\dot{H}}{H^2}\right) \\ &= \frac{\kappa}{\sqrt{3}H^2}\dot{\sqrt{\rho_r}} - \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H}\frac{\dot{H}}{H^2} \\ &= \frac{\kappa}{\sqrt{3}H^2}\frac{\dot{\rho}_r}{2\sqrt{\rho_r}} + x_3\frac{1}{2}(3 + 3x_1^2 - 3x_2^2 + x_3^2)\end{aligned}$$

We use equation 4 to express $\dot{\rho}_r$ as $\dot{\rho}_r = -3H(1 + 1/3)\rho_r = -4H\rho_r$ where we used that $\omega_r = 1/3$. Inserting this into the expression above we get

$$\begin{aligned}\frac{\mathrm{d}x_3}{\mathrm{d}N} &= -\frac{2\kappa\sqrt{\rho_r}}{\sqrt{3}H} + x_3\frac{1}{2}\left[3 + 3x_1^2 - 3x_2^2 + x_3^2\right] \\ &= -2x_3 + \frac{1}{2}x_3(3 + 3x_1^2 - 3x_2^2 + x_3^2)\end{aligned}$$

Which is exactly what we wanted to show.

VIII. PROBLEM 7

We want to find out what V must look like if λ is constant and what the corresponding value of Γ must be.

If λ is a constant then

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{V'}{\kappa V} = 0$$

We see from this expression that V must be an exponential function. To find the exact expression we separate variables and integrate while keeping λ constant:

$$\begin{aligned} \frac{V'}{V} &= -\kappa\lambda \\ \int_{V_0}^{V} \frac{dV}{V} &= -\kappa\lambda \int_0^{\phi} d\phi' \\ \ln|V - V_0| &= -\kappa\lambda\phi \\ V &= V_0 e^{-\kappa\lambda\phi} \end{aligned}$$

As predicted, V is an exponential function when λ is a constant. Now we find the corresponding value of Γ by differentiating the potential:

$$\begin{aligned} V' &= -\kappa\lambda V_0 e^{-\kappa\lambda\phi} = -\kappa\lambda V \\ V'' &= \kappa^2\lambda^2 V_0 e^{-\kappa\lambda\phi} = \kappa^2\lambda^2 V \\ \Rightarrow \Gamma &= \frac{VV''}{(V')^2} = \frac{V\kappa^2\lambda^2 V}{(-\kappa\lambda V)^2} = 1 \end{aligned}$$

The value of Γ is 1.

IX. PROBLEM 8

We want to show that

$$\frac{\mathrm{d}\lambda}{\mathrm{d}N} = -\sqrt{6}\lambda^2(\Gamma - 1)x_1 \tag{18}$$

for the case when Γ is not constant. We start by differentiating λ:

$$\begin{aligned} \frac{\mathrm{d}\lambda}{\mathrm{d}N} &= \frac{1}{H}\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{V'}{\kappa V}\right) = -\frac{1}{H\kappa}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{V'}{V}\right) \\ &= -\frac{1}{H\kappa}\left(\frac{V\dot{V}' - V'\dot{V}}{V^2}\right) \\ &= -\frac{1}{H\kappa}\left(\frac{VV''\dot{\phi}}{V^2} + \frac{(V')^2\dot{\phi}}{V^2}\right) \end{aligned}$$

We check if the expression we have found is the same as the one in equation 18 that we wanted to show:

$$\begin{aligned} \frac{\mathrm{d}\lambda}{\mathrm{d}N} &= -\sqrt{6}\lambda^2(\Gamma - 1)x_1 \\ &= -\sqrt{6}\frac{(V')^2}{\kappa^2 V^2}\left(\frac{VV''}{(V')^2} - 1\right)\frac{\kappa\dot{\phi}}{\sqrt{6}H} \\ &= -\sqrt{6}\frac{\kappa\dot{\phi}}{\sqrt{6}H}\left(\frac{(V')^2}{\kappa^2V^2}\frac{VV''}{(V')^2} - \frac{(V')^2}{\kappa^2V^2}\right) \\ &= -\frac{\dot{\phi}}{H\kappa}\left(\frac{VV''}{V^2} - \frac{(V')^2}{V^2}\right) \end{aligned}$$

We see that the two expressions match and $d\lambda/dN$ can be expressed as in equation 18.

X. PROBLEM 9

We now wish to study the numerical solutions to the equations of motion, equation 15, 16, 17. We choose to look at the solutions for two different potentials $V(\phi)$. One of them being an inverse power law

$$V(\phi) = M^{4+\alpha}\phi^{-\alpha}$$

where M is a mass scale and $\alpha = 1$, and the other potential being an exponential potential

$$V(\phi) = V_0 e^{-\kappa\zeta\phi}$$

where we let $\zeta = 3/2$ [1]. For the power-law potential we find Γ to be

$$\begin{aligned} \Gamma &= \frac{VV''}{(V')^2} = \frac{M^{4+\alpha}\phi^{-\alpha}\cdot M^{4+\alpha}\alpha(\alpha+1)\phi^{-\alpha-2}}{(M^{4+\alpha})^2(-\alpha)^2\phi^{-2\alpha-2}} \\ &= \frac{\alpha(\alpha+1)}{\alpha^2} = \frac{\alpha+1}{\alpha}. \end{aligned}$$

For the exponential potential the Γ function becomes

$$\Gamma = \frac{V_0 e^{-\kappa\zeta\phi}V_0(\kappa\zeta)^2 e^{-\kappa\zeta\phi}}{V_0^2(-\kappa\zeta)^2 e^{-2\kappa\zeta\phi}} = 1. \tag{19}$$

Numerically integrating the equations of motion for both potentials with the corresponding Γ functions gives the resulting plot shown in figure 5. We also plotted the sum of all the density parameters to confirm that it is always 1.

We have that $z = 0$ is today and higher values of z correspond earlier space time. From figure 5 we can see what the universe consisted of in different epochs. We see that the early universe, from $z = 2\cdot 10^7$, was radiation dominated, we then had a period of both matter and radiation (which is consistent with what we have learnt). At $z \approx 1000$ to $z \approx 10$ the universe was matter dominated and today the universe consists of both matter and the quintessence field.

Next, we find an expression for the equation of state using the dimensionless variables in equation 9 together with equation 3 for the equation of state parameter. We get that

$$\omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi} - V}{\frac{1}{2}\dot{\phi} + V} = \frac{\dot{\phi} - 2V}{\dot{\phi} + 2V}$$
$$= \frac{\frac{x_1^2 6H^2}{\kappa^2} - 2\frac{x_2^2 3H^2}{\kappa^2}}{\frac{x_1^2 6H^2}{\kappa^2} + 2\frac{x_2^2 3H^2}{\kappa^2}} = \frac{\frac{6H^2}{\kappa^2}(x_1^2 - x_2^2)}{\frac{6H^2}{\kappa^2}(x_1^2 + x_2^2)}$$
$$= \frac{x_1^1 - x_2^2}{x_1^2 + x_2^2}$$

where we expressed $\dot{\phi}^2$ and V using the expressions for x_1 and x_2 in equation 9. The resulting plot of the equation of state parameter as a function of redshift z can be found in figure 1.

We observe that the density parameters seem to change the same way for the two different potential models. The equation of state model is on the other hand very different for the two models.

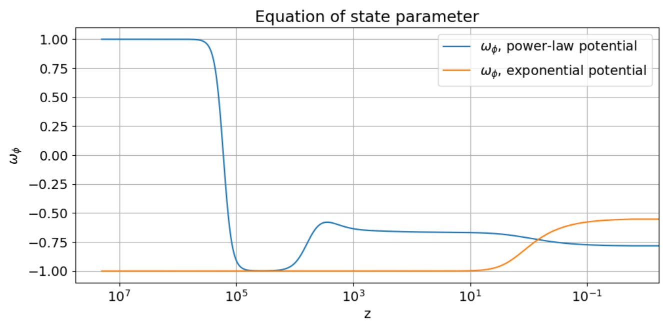

Figure 1. The equation of state parameter ω_ϕ as a function of redshift z, calculated using equation 3, for the power-law potential model and exponential potential model.

XI. PROBLEM 10

To calculate the Hubble parameter for the two models we use equation 7, divide by H_0^2 and square the expression. This gives us the dimensionless parameter $h = H/H_0$ instead of H. When calculating the integral in the expression we change variables from z to N. Since $z = e^{-N} - 1 \Rightarrow dz = -e^{-N}dN$ the integral becomes

$$I = \int_0^z dz' \frac{3(1+\omega_\phi)}{1+z'}$$
$$\Rightarrow I = \int_0^{-\ln(z+1)} \frac{3(1+\omega_\phi)}{1+e^{-N}-1}(-e^{-N'}dN')$$
$$= \int_{-\ln(z+1)}^0 3(1+\omega_\phi)dN$$

We also calculate the Hubble parameter for the ΛCDM model which is given as

$$H^2 = H_0^2\Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0}$$

where $\Omega_{\Lambda 0} = 1 - \Omega_{m0}$. We calculate and plot the Hubble parameter for both the power-law potential model and the exponential potential model together with the Hubble parameter for the ΛCDM model as a function of redshift. We let $z \in [2 \cdot 10^7, 0]$. The result can be seen in figure 2.

We plotted today's values for the density parameters because we use the critical density Ω_{c0} when defining the density parameters which is defined using today's value. We see from the plot that the Hubble parameters increase for higher redshifts which is what we expect from looking at the two formulas.

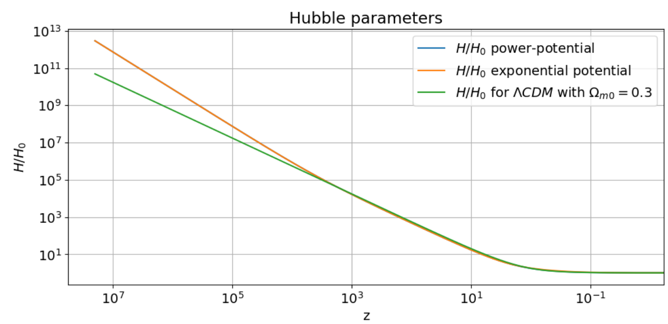

Figure 2. The Hubble parameter calculated for the power-law potential model, the exponential potential model and the ΛCDM model using the two Hubble parameters from problem 10.

XII. PROBLEM 11

To calculate the dimensionless age $H_0 t_0$ of the Universe we use that

$$H_0 t_0 = \int_0^a \frac{1}{a' h(a')} da$$

where $h(a) = H(a)/H_0$ [2]. We do a substitution first to z and then to N. We use that $1+z = a_0/a \Rightarrow a = a_0/(1+z)$ and that

$$da = -\frac{a_0 dz}{(1+z)^2}$$

The integral in terms of z becomes:

$$H_0 t_0 = -\int_0^\infty \frac{a_0 dz}{(1+z)^2}\frac{(z+1)}{a_0 h(z)} = -\int_0^\infty \frac{dz}{(1+z)h(z)}.$$

Using that $z = e^{-N} - 1$ and that $dz = -e^{-N}dN$ we get

$$H_0 t_0 = \int_0^\infty \frac{-e^{-N}dN}{(1+e^{-N}-1)h(N)} \approx \int_{-\ln(z+1)}^0 \frac{dN}{h(N)} \tag{20}$$

where we approximate the integral to go from $N = \ln 1/(z+1)$ to 0 instead of to infinity.

To find the age of the Universe for the two quintessence models we integrate numerically using the first Hubble parameter calculated in problem 10, and to find the age of the Universe for the ΛCDM model we integrate over the second Hubble parameter we found in problem 10. The results we get are shown in table I.

To find out if our results look reasonable, we compare

Model	$\mathbf{H_0 t_0}$
Power-law potential	0.993
Exponential potential	0.972
ΛCDM	0.964

Table I. The dimensionless age for lower-law potential mode, exponential potential model and ΛCDM model calculated using 20.

our value of the dimensionless age for the ΛCDM model with the value found by Avelino and Kirshner (2016). We see that they match and since we use the same method for all three models we can assume that the values are correct. From table I we see that the power-law model is the one which gives the oldest universe.

XIII. PROBLEM 12

We now want to calculate and plot the dimensionless luminosity distance $H_0 d_L(z)/c$ for the two models for $0 \leq z \leq 2$. The luminosity distance is given as

$$d_L = a_0(1+z)\frac{c}{a_0 H_0}\int_0^z \frac{dz'}{h(z')} \tag{21}$$

[2]. We make it dimensionless and substitute to N as done previously:

$$\begin{aligned}\frac{d_L H_0}{c} &= (1+z)\int_0^z \frac{dz'}{h(z')}\\ &= (1+e^{-N}-1)\int_0^N \frac{1}{h(N)}(-e^{-N}dN)\\ &= e^{-N}\int_N^0 \frac{e^{-N}}{h(N)}dN\end{aligned}$$

When integrating numerically and plotting the luminosity distance as a function of redshift z for the two quintessence models we get the result shown in figure 3. We see that the luminosity distance increases as we go backwards in spacetime. We also see that the luminosity distance increases more for z between 1 and 2 than for z between 0 and 1. This is due to the fact that as the redshift increases the light has to travel even further to reach us due to the expansion of the Universe.

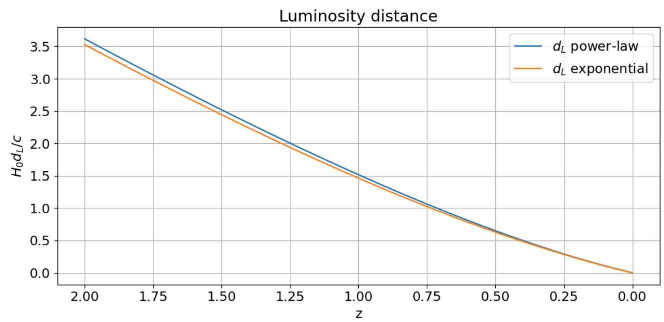

Figure 3. Graph showing the luminosity distance as a function of redshift z for the power-law potential model and exponential potential model.

XIV. PROBLEM 13

We now want to compare our models with a set of measured luminosity distances presented in the file `sndata.txt`. In the file, we are given the luminosity distances in Gpc with corresponding errors and redshifts. To calculate how similar our two quintessence models of the luminosity distance d_L are to the measured data we use the χ^2-test:

$$\chi^2 = \sum_i \frac{(\text{model}_i - \text{data}_i)^2}{\text{error}_i^2}. \tag{22}$$

Here, model_i is found by interpolating the luminosity distance data points we have for the power-law potential model and the exponential potential model when z is between 0 and 2. The values for data_i and error_i are given in the file with measured data. Before inserting in equation 22, we need to convert our d_L model data to Gpc. To do that we use that $H_0 = 100$ h km s^{-1}Mpc^{-1} [2]. Which means that we have to multiply our dimensionless values of $H_0 d_L/c$ by $c/(H_0 \cdot 10^6)$ to get d_L in units of Gpc.

Using that $h = 0.7$ and inserting our values in equation 22 we get that the χ^2 value for the inverse power-law potential is 35.14 and for the exponential potential we get a value of 98.26. The exponential potential model is therefore a better fit to the data.

XV. PROBLEM 14

To find which value of Ω_{m0} which provides the best fit for the spatially flat ΛCDM model to the measured data we perform the χ^2-test for a range of Ω_{m0} values between 0 and 1. We then find that the value of Ω_{m0} which gives the best fit for the ΛCDM model is $\Omega_{m0} = 0.30$ with a χ^2-value of 29.79. The χ^2-test results for all the different values of Ω_{m0} can be seen in figure 4.

We see that $\Omega_{m0} = 0.3$ is a better fit than the previous two quintessence models we considered. However, this is not a fair comparison as we did not determine the values of the density parameters or equation of state parameter which gives the best fit for the two quintessence models. Had we done that then the ΛCDM model might not have been the one with the best fit, seeing that the inverse power-law model got a χ^2 value which was not much higher than what we got for the ΛCDM model.

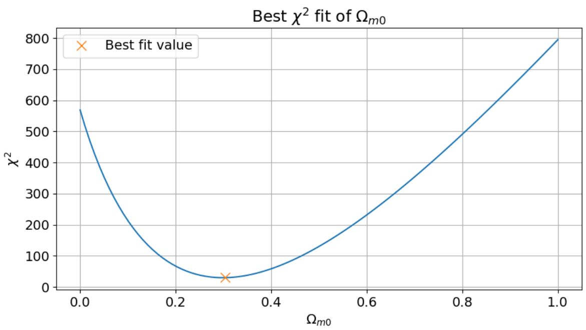

Figure 4. Graph showing which Ω_{m0} value which gives the best χ^2 fit. The orange "X" shows the lowest χ^2 value.

[1] Elgarøy, *AST3220, spring 2024: Project 1* (Institute of Theoretical Astrophysics, UiO, 2024).
[2] Elgarøy, *AST3220 – Cosmology I* (Institute of Theoretical Astrophysics, UiO, 2024).
[3] A. Avelino and R. P. Kirshner, The Astrophysical Journal **828**, 35 (2016).

Appendix A: Figures

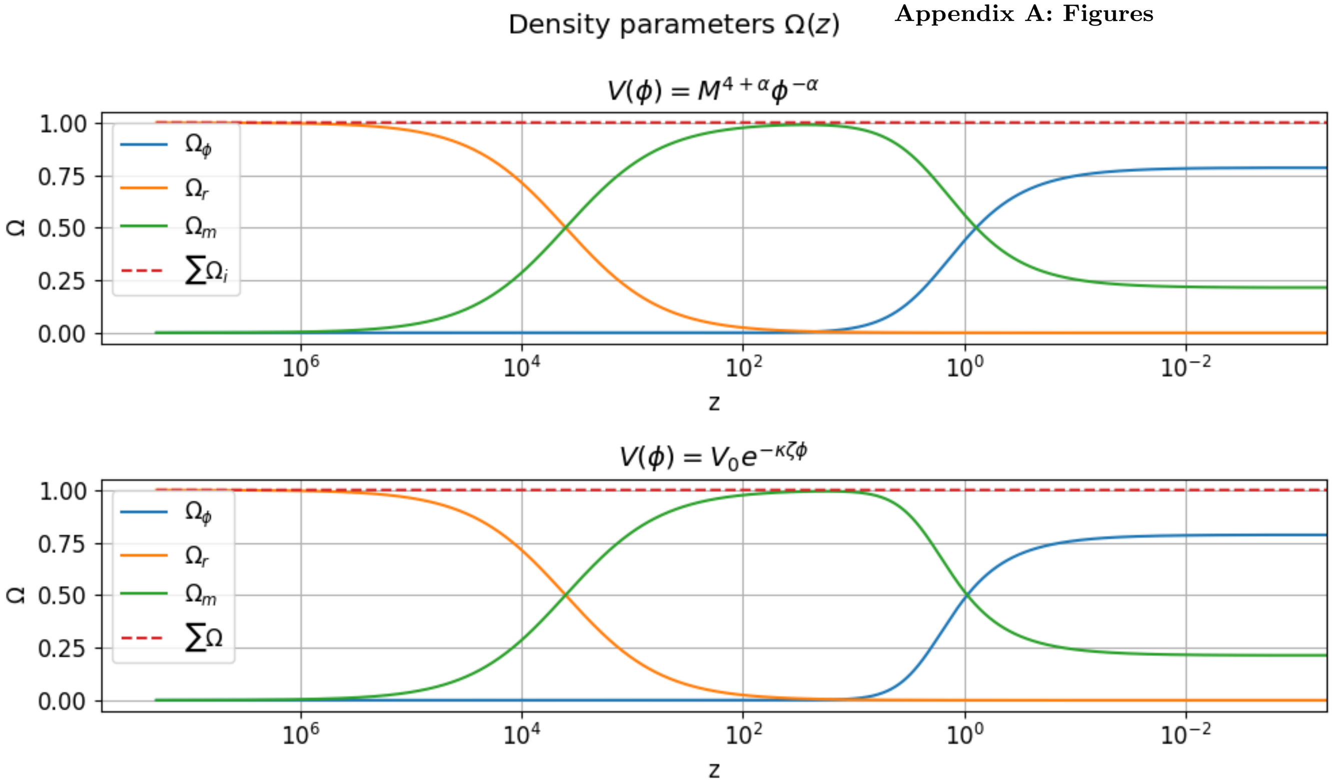

Figure 5. The density parameters Ω plotted for the the power-law potential model (top figure) and the exponential potential model (bottom figure). The red, dashed line shows the sum of all the density parameters.