机器人学

作业 4：微分运动学

Robotics (2023-2024-2)

Homework 4: Differential Kinematics

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2024 年 3 月 25 日

1 微分运动基础

已知坐标系 C 对基坐标系的变换为

$$\begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

而且对于基坐标系的微分平移分量分别为沿 x 轴移动 0.5, 沿 y 轴移动为 0, 沿 z 轴移动 1; 微分旋转分量分别为 $0.1, 0.2$ 和 0。

1.1 求相应的微分变换

[Solution]:

根据题意，微分平移和旋转矢量表示为：

$$\begin{aligned} \mathbf{d} &= 0.5\mathbf{i} + 0\mathbf{j} + 1\mathbf{k} \\ \delta &= 0.1\mathbf{i} + 0.2\mathbf{j} + 0\mathbf{k} \end{aligned} \tag{2}$$

因此对应的 Δ 矩阵为：

$$\Delta = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.2 & 0.5 \\ 0 & 0 & -0.1 & 0 \\ -0.2 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

因此，对应的微分变换为：

$$dC = \Delta C = \begin{bmatrix} 0 & 0 & 0.2 & 0.5 \\ 0 & 0 & -0.1 & 0 \\ -0.2 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.2 & 0 & 0 & 0.5 \\ -0.1 & 0 & 0 & 0 \\ 0 & -0.2 & 0.1 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

1.2 求对应于坐标系 C 的等效微分平移与旋转

[Solution]: 记基座标系下的微分运动为 $\mathbf{D} = \begin{bmatrix} \mathbf{d} \\ \delta \end{bmatrix}$，相对于坐标系 C 的等效微分平移与旋转为 ${}^{C}\mathbf{D} = \begin{bmatrix} {}^{C}\mathbf{d} \\ {}^{C}\delta \end{bmatrix}$，则有：

$$\mathbf{D}=\begin{bmatrix}d_x\\d_y\\d_z\\\delta_x\\\delta_y\\\delta_z\end{bmatrix}=\begin{bmatrix}0.5\\0\\1\\0.1\\0.2\\0\end{bmatrix};\quad {}^{\mathbf{C}}\mathbf{D}=\begin{bmatrix}{}^{C}d_x\\{}^{C}d_y\\{}^{C}d_z\\{}^{C}\delta_x\\{}^{C}\delta_y\\{}^{C}\delta_z\end{bmatrix} \tag{5}$$

对于坐标系 C, 有 $\mathbf{n}=[0,0,1]^T,\mathbf{o}=[1,0,0]^T,\mathbf{a}=[0,1,0]^T,\mathbf{p}=[4,3,0]^T$，因此有：

$$\begin{aligned}(\mathbf{p}\times\mathbf{n})&=\begin{vmatrix}i&j&k\\4&3&0\\0&0&1\end{vmatrix}=[3,-4,0]^T\\(\mathbf{p}\times\mathbf{o})&=\begin{vmatrix}i&j&k\\4&3&0\\1&0&0\end{vmatrix}=[0,0,-3]^T\\(\mathbf{p}\times\mathbf{a})&=\begin{vmatrix}i&j&k\\4&3&0\\0&1&0\end{vmatrix}=[0,0,4]^T\end{aligned} \tag{6}$$

因此，对应坐标系 C 的等效微分平移与旋转为：

$$\begin{aligned}{}^{\mathbf{C}}\mathbf{D}&=\begin{bmatrix}n_x&n_y&n_z&(\mathbf{p}\times\mathbf{n})_x&(\mathbf{p}\times\mathbf{n})_y&(\mathbf{p}\times\mathbf{n})_z\\o_x&o_y&o_z&(\mathbf{p}\times\mathbf{o})_x&(\mathbf{p}\times\mathbf{o})_y&(\mathbf{p}\times\mathbf{o})_z\\a_x&a_y&a_z&(\mathbf{p}\times\mathbf{a})_x&(\mathbf{p}\times\mathbf{a})_y&(\mathbf{p}\times\mathbf{a})_z\\0&0&0&n_x&n_y&n_z\\0&0&0&o_x&o_y&o_z\\0&0&0&a_x&a_y&a_z\end{bmatrix}\begin{bmatrix}d_x\\d_y\\d_z\\\delta_x\\\delta_y\\\delta_z\end{bmatrix}\\&=\begin{bmatrix}0&0&1&3&-4&0\\1&0&0&0&0&-3\\0&1&0&0&0&4\\0&0&0&0&0&1\\0&0&0&1&0&0\\0&0&0&0&1&0\end{bmatrix}\begin{bmatrix}0.5\\0\\1\\0.1\\0.2\\0\end{bmatrix}=\begin{bmatrix}0.5\\0.5\\0\\0\\0.1\\0.2\end{bmatrix}\end{aligned} \tag{7}$$

综上，对应于坐标系 C 的等效微分平移为 ${}^{\mathbf{C}}\mathbf{d}=0.5\mathbf{i}+0.5\mathbf{j}$，等效微分旋转为 ${}^{\mathbf{C}}\delta=0.1\mathbf{j}+0.2\mathbf{k}$。

2 球形腕 Jacobi 矩阵的计算

已知如图所示球型腕及其 DH 参数表，计算其雅可比矩阵 ^3J 与 ^6J.

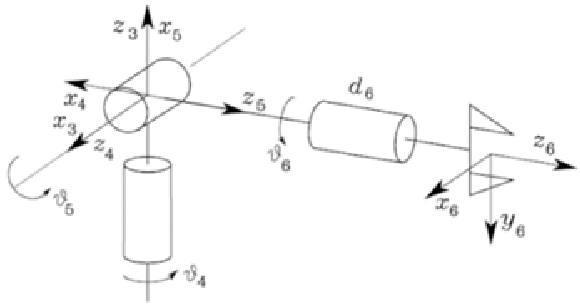

图 1: 球形腕示意图

Link	a_i	α_i	d_i	ϑ_i
4	0	$-\pi/2$	0	ϑ_4
5	0	$\pi/2$	0	ϑ_5
6	0	0	d_6	ϑ_6

2.1 ^6J 的计算

[Solution]: 由 Davis-Hartenberg 参数表，可得到各个关节的变换矩阵为：

$$^3T_4=\begin{bmatrix}cos\theta_4 & -sin\theta_4 & 0 & 0\\ 0 & 0 & 1 & 0\\ -sin\theta_4 & -cos\theta_4 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix};^4T_5=\begin{bmatrix}cos\theta_5 & -sin\theta_5 & 0 & 0\\ 0 & 0 & -1 & 0\\ sin\theta_5 & cos\theta_5 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix};^5T_6=\begin{bmatrix}cos\theta_6 & -sin\theta_6 & 0 & 0\\ sin\theta_6 & cos\theta_6 & 0 & 0\\ 0 & 0 & 1 & d_6\\ 0 & 0 & 0 & 1\end{bmatrix}$$

再进行关键变换矩阵的运算：

$$^3T_6={}^3T_4\cdot{}^4T_5\cdot{}^5T_6=\begin{bmatrix}c_4c_5c_6-c_4s_5s_6 & -c_4c_5s_6-c_4s_5c_6 & s_4 & d_6s_4\\ s_5c_6+c_5s_6 & -s_5s_6+c_5c_6 & 0 & 0\\ -s_4c_5c_6+s_4s_5s_6 & s_4c_5s_6+s_4s_5c_6 & c_4 & d_6c_4\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}c_4c_{56} & -c_4s_{56} & s_4 & d_6s_4\\ s_{56} & c_{56} & 0 & 0\\ -s_4c_{56} & s_4s_{56} & c_4 & d_6c_4\\ 0 & 0 & 0 & 1\end{bmatrix}$$

为了简化描述，上式中，$s_5=sin\theta_5, s_{56}=sin(\theta_5+\theta_6)$，以此类推。

同样地，

$$
{}^4T_6 = {}^4T_5 \cdot {}^5T_6 = \begin{bmatrix} c_{56} & -s_{56} & 0 & 0 \\ 0 & 0 & -1 & -d_6 \\ s_{56} & c_{56} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; {}^5T_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}
$$

由于全部都是旋转关节，因此有：

$$
{}^TJ(q) = \begin{bmatrix} {}^T\boldsymbol{J_{li}} \\ {}^T\boldsymbol{J_{ai}} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{p} \times \boldsymbol{n})_z \\ (\boldsymbol{p} \times \boldsymbol{o})_z \\ (\boldsymbol{p} \times \boldsymbol{a})_z \\ \boldsymbol{n}_z \\ \boldsymbol{o}_z \\ \boldsymbol{a}_z \end{bmatrix} \tag{9}
$$

对于第三列（对应矩阵 ${}^5T_6$），有：

$$
\begin{gathered}
\boldsymbol{n} = \begin{bmatrix} c_6 & s_6 & 0 \end{bmatrix}^T, \boldsymbol{o} = \begin{bmatrix} -s_6 & c_6 & 0 \end{bmatrix}^T, \boldsymbol{a} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, \boldsymbol{p} = \begin{bmatrix} 0 & 0 & d_6 \end{bmatrix}^T \\
\boldsymbol{p} \times \boldsymbol{n} = \begin{vmatrix} i & j & k \\ 0 & 0 & d_6 \\ c_6 & s_6 & 0 \end{vmatrix}_z = 0, \boldsymbol{p} \times \boldsymbol{o} = \begin{vmatrix} i & j & k \\ 0 & 0 & d_6 \\ -s_6 & c_6 & 0 \end{vmatrix}_z = 0, \boldsymbol{p} \times \boldsymbol{a} = \begin{vmatrix} i & j & k \\ 0 & 0 & d_6 \\ 0 & 0 & 1 \end{vmatrix}_z = 0
\end{gathered} \tag{10}
$$

thus,

$$
{}^6J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T
$$

同样的道理，对于第二列（对应矩阵 ${}^4T_6$），有：

$$
\begin{gathered}
\boldsymbol{n} = \begin{bmatrix} c_{56} & 0 & s_{56} \end{bmatrix}^T, \boldsymbol{o} = \begin{bmatrix} -s_{56} & 0 & c_{56} \end{bmatrix}^T, \boldsymbol{a} = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T, \boldsymbol{p} = \begin{bmatrix} 0 & -d_6 & 0 \end{bmatrix}^T \\
\boldsymbol{p} \times \boldsymbol{n} = \begin{vmatrix} i & j & k \\ 0 & -d_6 & 0 \\ c_{56} & 0 & s_{56} \end{vmatrix}_z = d_6 s_{56}, \\
\boldsymbol{p} \times \boldsymbol{o} = \begin{vmatrix} i & j & k \\ 0 & -d_6 & 0 \\ -s_{56} & 0 & c_{56} \end{vmatrix}_z = -d_6 s_{56}, \\
\boldsymbol{p} \times \boldsymbol{a} = \begin{vmatrix} i & j & k \\ 0 & -d_6 & 0 \\ 0 & -1 & 0 \end{vmatrix}_z = 0
\end{gathered} \tag{11}
$$

thus,

$$ {}^6J_2 = \begin{bmatrix} d_6 s_{56} & -d_6 s_{56} & 0 & s_{56} & c_{56} & 0 \end{bmatrix}^T $$

最后，对于第一列（对应矩阵 ${}^3T_6$），有：

$$ \boldsymbol{n} = \begin{bmatrix} c_4 c_{56} & s_{56} & -s_4 c_{56} \end{bmatrix}^T, \boldsymbol{o} = \begin{bmatrix} -c_4 s_{56} & c_{56} & s_4 s_{56} \end{bmatrix}^T, \boldsymbol{a} = \begin{bmatrix} s_4 & 0 & c_4 \end{bmatrix}^T, \boldsymbol{p} = \begin{bmatrix} d_6 s_4 & 0 & d_6 c_4 \end{bmatrix}^T $$

$$ \begin{aligned} \boldsymbol{p} \times \boldsymbol{n} &= \begin{vmatrix} i & j & k \\ d_6 s_4 & 0 & d_6 c_4 \\ c_4 c_{56} & s_{56} & -s_4 c_{56} \end{vmatrix}_z = d_6 c_4 s_{56}, \\ \boldsymbol{p} \times \boldsymbol{o} &= \begin{vmatrix} i & j & k \\ d_6 s_4 & 0 & d_6 c_4 \\ -c_4 s_{56} & c_{56} & s_4 s_{56} \end{vmatrix}_z = d_6 s_4 c_{56}, \\ \boldsymbol{p} \times \boldsymbol{a} &= \begin{vmatrix} i & j & k \\ d_6 s_4 & 0 & d_6 c_4 \\ s_4 & 0 & c_4 \end{vmatrix}_z = 0 \end{aligned} \tag{12} $$

thus,

$$ {}^6J_1 = \begin{bmatrix} d_6 c_4 s_{56} & d_6 s_4 c_{56} & 0 & -s_4 c_{56} & s_4 s_{56} & c_4 \end{bmatrix}^T $$

综上所述，${}^6J$ 为：

$$ {}^6J(q) = \begin{bmatrix} {}^6J_1 & {}^6J_2 & {}^6J_3 \end{bmatrix} = \begin{bmatrix} d_6 c_4 s_{56} & d_6 s_{56} & 0 \\ d_6 s_4 c_{56} & -d_6 s_{56} & 0 \\ 0 & 0 & 0 \\ -s_4 c_{56} & s_{56} & 0 \\ s_4 s_{56} & c_{56} & 0 \\ c_4 & 0 & 1 \end{bmatrix} \tag{13} $$

2.2 ${}^3J$ 的计算

[Solution]: 对于 ${}^3J$ 的计算，由关系式

$$ {}^3J(q) = \begin{bmatrix} {}^3R_6 & 0 \\ 0 & {}^3R_6 \end{bmatrix} \cdot {}^6 J(q) \tag{14} $$

因为旋转矩阵为**正交矩阵**, 有 ${}^3R_6 = ({}^6R_3)^{-1} = {}^3 R_6^T$，因此有：

$$ {}^3R_6 = \begin{bmatrix} c_4 c_{56} & s_{56} & -s_4 c_{56} \\ -c_4 s_{56} & c_{56} & s_4 s_{56} \\ s_4 & 0 & c_4 \end{bmatrix} $$

因此，^3J 为：

$$
\begin{aligned}
{}^3J(q) &= \begin{bmatrix} {}^3R_6 & 0 \\ 0 & {}^3R_6 \end{bmatrix} \cdot {}^6J(q) \\
&= \begin{bmatrix} c_4c_{56} & s_{56} & -s_4c_{56} & 0 & 0 & 0 \\ -c_4s_{56} & c_{56} & s_4s_{56} & 0 & 0 & 0 \\ s_4 & 0 & c_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4c_{56} & s_{56} & -s_4c_{56} \\ 0 & 0 & 0 & -c_4s_{56} & c_{56} & s_4s_{56} \\ 0 & 0 & 0 & s_4 & 0 & c_4 \end{bmatrix} \begin{bmatrix} d_6c_4s_{56} & d_6s_{56} & 0 \\ d_6s_4c_{56} & -d_6s_{56} & 0 \\ 0 & 0 & 0 \\ -s_4c_{56} & s_{56} & 0 \\ s_4s_{56} & c_{56} & 0 \\ c_4 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} {}^3J_{11} & {}^3J_{12} & {}^3J_{13} \\ {}^3J_{21} & {}^3J_{22} & {}^3J_{23} \\ {}^3J_{31} & {}^3J_{32} & {}^3J_{33} \\ {}^3J_{41} & {}^3J_{42} & {}^3J_{43} \\ {}^3J_{51} & {}^3J_{52} & {}^3J_{53} \\ {}^3J_{61} & {}^3J_{62} & {}^3J_{63} \end{bmatrix}
\end{aligned}
\tag{15}
$$

其中，各个元素的值为：

$$
\begin{aligned}
{}^3J_{11} &= c_4c_{56}d_6c_4s_{56} + s_{56}d_6s_4c_{56} = d_6s_{56}(c_4^2 + c_{56}s_4) \\
{}^3J_{12} &= c_4c_{56}d_6s_{56} - s_{56}d_6s_4c_{56} = d_6s_{56}(c_4c_{56} - s_{56}) \\
{}^3J_{21} &= -c_4s_{56}d_6c_4s_{56} - c_{56}d_6s_4s_{56} = -d_6s_{56}(c_4^2s_{56} + c_{56}) \\
{}^3J_{22} &= -c_4s_{56}d_6s_{56} - c_{56}d_6s_4s_{56} = -d_6s_{56}(c_4s_{56} + c_{56}) \\
{}^3J_{31} &= s_4d_6c_4s_{56} \\
{}^3J_{32} &= s_4d_6s_{56} \\
{}^3J_{41} &= -c_4c_{56}s_4c_{56} + s_{56}s_4s_{56} = -s_4(c_56c_4^2 + s_{56}^2) \\
{}^3J_{42} &= -c_4c_{56}s_{56} + s_{56}c_{56} \\
{}^3J_{51} &= c_4s_{56}s_4c_{56} + c_{56}s_4s_{56} + s_4s_{56}c_4 \\
{}^3J_{52} &= -c_4s_{56}s_{56} + c_{56}^2 \\
{}^3J_{61} &= -s_4^2c_{56} + c_4^2 \\
{}^3J_{62} &= s_4s_{56} \\
{}^3J_{13} &= 0,{}^3J_{23} = 0,{}^3J_{33} = 0 \\
{}^3J_{43} &= -s_4c_{56},{}^3J_{53} = s_4s_{56},{}^3J_{63} = c_4
\end{aligned}
\tag{16}
$$

以上，求解完毕。