WCV@R applied to Portfolio Optimization

Risk Management

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Outline

Narrative

Using this cumulative probability, we can define

$$VaR_{\beta}(\mathbf{x}) \triangleq min\,\{\alpha \in R : \psi(\mathbf{x}, \alpha) \geq \beta\}$$

The deficiencies of Value-at-Risk?

A natural coherent alternative to Value at Risk

$$CVaR_{\beta}(\mathbf{x}) \triangleq \mathbb{E}(f(\mathbf{x}, \mathbf{y}) | f(\mathbf{x}, \mathbf{y}) > VaR_{\beta}(\mathbf{x})), \tag{1}$$

$$\triangleq \frac{1}{1-\beta} \int_{f(\mathbf{x},\mathbf{y}) \geq VaR_{\beta}(\mathbf{x})} f(\mathbf{x}, \mathbf{y}) p(\mathbf{y})\, d\mathbf{y} \tag{2}$$

What's wrong with the above formulas? Somewhat tricky in practice!

Consider the following function:

$$F_\beta(\mathbf{x},\alpha) \triangleq \alpha + \frac{1}{1-\beta}\int_{\mathbf{y}\in R^m}[f(\mathbf{x},\mathbf{y})-\alpha]^+ p(\mathbf{y})\,d\mathbf{y}$$

we have:

- **Properties**

1. $F_\beta(x,\alpha)$ is a convex and continuously differentiable function in α.
2. $VaR_\beta(x)$ is a minimizer of $F_\beta(x,\alpha)$.
3. The minimum value of $F_\beta(x,\alpha)$ is $CVaR_\beta(x)$.

In particular, the optimal $CVaR_\beta(x)$ can be found by solving the following optimization problem:

$$\min_{(x,\alpha)} F_\beta(x,\alpha)$$

WCVaR Minimization

Definition 1. The worst-case CVaR (WCVaR) for a fixed $x \in X$ with respect to P is defined as

$$WCVaR_\beta(x) = \sup_{p(\cdot)\in P} CVaR_\beta(x)$$

Questions

1. Does the worst-case CVaR remain a coherent risk measure?
2. What can the ambiguity set P be made of?

Mixture Distribution

We assume that the distribution of **y** is only known to belong to a set of distributions that consists of all the mixtures of some predetermined likelihood distributions, i.e.,

$$p(\cdot) \in P_M \triangleq \left\{ \sum_{i=1}^{l} \lambda_i p^i(\cdot) : \sum_{i=1}^{l} \lambda_i = 1, \lambda_i \geq 0, i = 1, ..., l \right\}$$

- $p^i(\cdot)$: the ith likelihood distribution

Define

$$F_\beta^i(\mathbf{x}, \alpha) \triangleq \alpha + \frac{1}{1-\beta} \int_{\mathbf{y} \in R^m} [f(\mathbf{x}, \mathbf{y}) - \alpha]^+ p^i(\mathbf{y})\, d\mathbf{y}, \qquad i = 1, ..., l$$

THEOREM 1.

For each $\mathbf{x}$, $WCVaR_\beta(\mathbf{x})$ w.r.t. P_M is given by

$$WCVaR_\beta(\mathbf{x}) = \min_{\alpha \in R} \max_{i \in L} F^i_\beta(\mathbf{x}, \alpha), \qquad L \triangleq \{1, 2, ..., l\}$$

Denote

$$F^L_\beta(\mathbf{x}, \alpha) \triangleq \max_{i \in L} F^i_\beta(\mathbf{x}, \alpha)$$

COROLLARY 1.

Minimizing $WCVaR_\beta(\mathbf{x})$ over X can be achieved by minimizing $F^L_\beta(\mathbf{x}, \alpha)$ over $X \times R$, i.e.,

$$\min_{\mathbf{x} \in X} WCVaR_\beta(\mathbf{x}) = \min_{(\mathbf{x}, \alpha) \in X \times \mathbf{R}} F^L_\beta(\mathbf{x}, \alpha)$$

$$F^i_\beta(\mathbf{x},\alpha) \triangleq \alpha + \frac{1}{1-\beta}\int_{\mathbf{y}\in R^m}[f(\mathbf{x},\mathbf{y})-\alpha]^+ p^i(\mathbf{y})\,d\mathbf{y}, \qquad i=1,\dots,l$$

The WCVaR minimization is equivalent to

$$\min_{(\mathbf{x},\alpha,\theta)\in X\times R\times R} \quad \theta$$

$$\text{subject to} \quad \alpha + \frac{1}{1-\beta}\int_{\mathbf{y}\in R^m}[f(\mathbf{x},\mathbf{y})-\alpha]^+ p^i(\mathbf{y})\,d\mathbf{y} \le \theta, \qquad i=1,\dots,l$$

Monte Carlo simulation to approximate $F_\beta(\mathbf{x}, \alpha)$

$$\min_{(\mathbf{x},\alpha,\theta)\in X\times R\times R} \quad \theta$$

$$\text{subject to} \quad \alpha + \frac{1}{S^i(1-\beta)} \sum_{k=1}^{S^i} [f(\mathbf{x}, \mathbf{y}^i_{[k]}) - \alpha]^+ \leq \theta, \qquad i = 1, ..., I$$

- $\mathbf{y}^i_{[k]}$: kth sample w.r.t. the ith likelihood distribution $p^i(\cdot)$.
- S^i: number of corresponding samples.

Generally,

$$\alpha + \frac{1}{1-\beta} \sum_{k=1}^{S^i} \pi^i_k [f(\mathbf{x}, \mathbf{y}^i_{[k]}) - \alpha]^+ \leq \theta, \qquad i = 1, ..., I$$

- π^i_k: probability according to the kth sample w.r.t. $p^i(\cdot)$.

By denoting $\pi^i = (\pi^i_1, \dots, \pi^i_{S^i})^T$ and introducing an auxiliary vector $\mathbf{u} = (\mathbf{u}^1; \mathbf{u}^2; \dots; \mathbf{u}^l) \in R^m$, where $m = \sum_{i=1}^{l} S^i$, we have:

$$
\begin{aligned}
\min \quad & \theta \\
\text{s.t.} \quad & \mathbf{x} \in \mathscr{X}, \\
& \alpha + \frac{1}{1-\beta} (\boldsymbol{\pi}^i)^T \mathbf{u}^i \leqslant \theta, \quad i = 1, \dots, l, \\
& u^i_k \geqslant f(\mathbf{x}, \mathbf{y}^i_{[k]}) - \alpha, \quad k = 1, \dots, S^i, \ i = 1, \dots, l, \\
& u^i_k \geqslant 0, \quad k = 1, \dots, S^i, \ i = 1, \dots, l.
\end{aligned}
$$

Discrete Distribution

Again: $F_\beta(\mathbf{x},\alpha) \triangleq \alpha + \frac{1}{1-\beta}\int_{\mathbf{y}\in R^m}[f(\mathbf{x},\mathbf{y})-\alpha]^+ p(\mathbf{y})\,d\mathbf{y}$

Let:

- $\left\{\mathbf{y}_{[1]},...,\mathbf{y}_{[S]}\right\}$ be the sample space of random vector $\mathbf{y}$
- $\pi = (\pi_1,...,\pi_S)^T$ the vector of the corresponding probabilities.

By denoting P_π as P, we can define:

$$\begin{aligned} WCVaR_\beta(\mathbf{x}) &\triangleq \ sup_{\pi\in P_\pi} CVaR_\beta(\mathbf{x},\pi), \\ &\triangleq sup_{\pi\in P_\pi} min_{\alpha\in R} G_\beta(\mathbf{x},\alpha,\pi) \end{aligned}$$

$$G_\beta(\mathbf{x},\alpha,\pi) \triangleq \alpha + \frac{1}{1-\beta}\sum_{k=1}^{s}\pi_k[f(\mathbf{x},\mathbf{y}_{[k]})-\alpha]^+$$

THEOREM 2. Suppose that P_π is a compact convex set. Then, for each $\mathbf{x}$, we have

$$WCVaR_\beta(\mathbf{x}) \triangleq \min_{\alpha\in R}\max_{\pi\in P_\pi} G_\beta(\mathbf{x},\alpha,\pi)$$

Thus, the minimization problem can be written as:

$$\min_{x} WCVaR_{\beta}(x) \triangleq \min_{(\alpha,x)} \max_{\pi} G_{\beta}(\mathbf{x},\alpha,\pi)$$

or, equivalently

$$\begin{aligned} \min \quad & \theta \\ \text{s.t.} \quad & \mathbf{x} \in \mathscr{X}, \\ & \max_{\boldsymbol{\pi}\in\mathscr{P}_{\pi}} \alpha + \frac{1}{1-\beta}\boldsymbol{\pi}^{T}\mathbf{u} \leqslant \theta, \\ & u_k \geqslant f(\mathbf{x}, \mathbf{y}_{[k]}) - \alpha, \quad k = 1,\dots,S, \\ & u_k \geqslant 0, \quad k = 1,\dots,S. \end{aligned}$$

Box Uncertainty

Under

$$\boldsymbol{\pi} \in \mathscr{P}_\pi^B \triangleq \{\boldsymbol{\pi} \colon \boldsymbol{\pi} = \boldsymbol{\pi}^0 + \boldsymbol{\eta}, \mathbf{e}^T\boldsymbol{\eta} = 0, \underline{\boldsymbol{\eta}} \leqslant \boldsymbol{\eta} \leqslant \bar{\boldsymbol{\eta}}\},$$

The problem can be formulated as follows:

$$\begin{aligned}
\min \quad & \theta \\
\text{s.t.} \quad & \mathbf{x} \in \mathscr{X}, \\
& \alpha + \frac{1}{1-\beta}(\boldsymbol{\pi}^0)^T\mathbf{u} + \frac{1}{1-\beta}(\bar{\boldsymbol{\eta}}^T\boldsymbol{\xi} + \underline{\boldsymbol{\eta}}^T\boldsymbol{\omega}) \leqslant \theta, \\
& \mathbf{e}z + \boldsymbol{\xi} + \boldsymbol{\omega} = \mathbf{u}, \\
& \boldsymbol{\xi} \geqslant 0, \boldsymbol{\omega} \leqslant 0, \\
& u_k \geqslant f(\mathbf{x}, \mathbf{y}_{[k]}) - \alpha, \quad k = 1, \ldots, S, \\
& u_k \geqslant 0, \quad k = 1, \ldots, S.
\end{aligned}$$

Ellipsoidal Uncertainty

Under

$$\boldsymbol{\pi} \in \mathscr{P}_{\pi}^{E} \triangleq \left\{\boldsymbol{\pi}:\ \boldsymbol{\pi} = \boldsymbol{\pi}^0 + A\boldsymbol{\eta},\ \mathbf{e}^T A\boldsymbol{\eta} = 0,\ \boldsymbol{\pi}^0 + A\boldsymbol{\eta} \geqslant 0,\ \|\boldsymbol{\eta}\| \leqslant 1\right\},$$

The problem can be formulated as follows:

$$\begin{aligned}
\min\quad & \theta \\
\text{s.t.}\quad & \mathbf{x} \in \mathscr{X}, \\
& \alpha + \frac{1}{1-\beta}(\boldsymbol{\pi}^0)^T \mathbf{u} + \frac{1}{1-\beta}[\zeta + (\boldsymbol{\pi}^0)^T \boldsymbol{\omega}] \leqslant \theta, \\
& -\boldsymbol{\xi} - A^T \boldsymbol{\omega} + A^T \mathbf{e} z = A^T \mathbf{u}, \\
& \|\boldsymbol{\xi}\| \leqslant \zeta,\ \boldsymbol{\omega} \geqslant 0, \\
& u_k \geqslant f(\mathbf{x}, \mathbf{y}_{[k]}) - \alpha, \quad k = 1, \ldots, S, \\
& u_k \geqslant 0, \quad k = 1, \ldots, S.
\end{aligned}$$

Robust Portfolio Management Using WCVaR

- $\mathbf{x} = (x_1, ..., x_n)^T \in \mathbb{R}^n$: amount of the investments in the n risky assets.
- $\mathbf{y} = (y_1, ..., y_n)^T \in \mathbb{R}^n$: uncertain returns of the n risky assets.

- Loss function: $f(\mathbf{x}, \mathbf{y}) = -\mathbf{x}^T\mathbf{y}$

The robust portfolio selection problem using WCVaR as a risk measure can be represented as

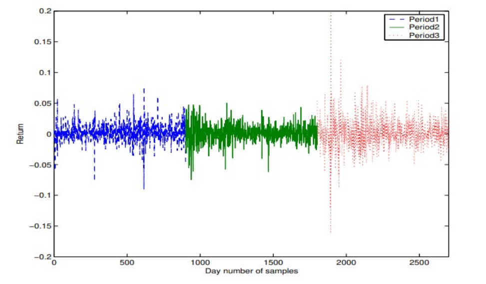
$$\min_{\mathbf{x} \in X} WCVaR(\mathbf{x})$$

We suppose

$$X \triangleq \left\{ \mathbf{x} : \mathbf{e}^T\mathbf{x} = w_0, \underline{\mathbf{x}} \leq \mathbf{x} \leq \bar{\mathbf{x}}, \min_{p(\cdot) \in P} E_p(\mathbf{x}^T\mathbf{y}) \geq \mu \right\}$$

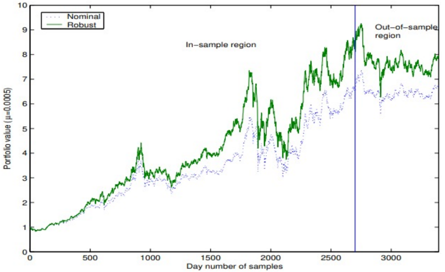
Results and Discussion

Market Data Simulation Analysis

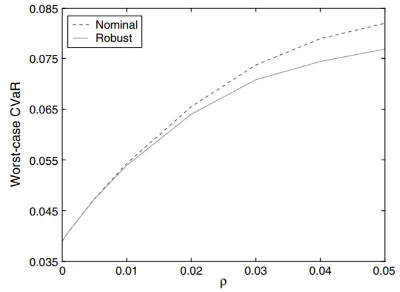


	Mean (10^{-3})				Variance (10^{-3})			
Period	HSNF	HSNU	HSNP	HSNC	HSNF	HSNU	HSNP	HSNC
Period1	1.8455	1.2522	1.5859	1.1383	0.2106	0.1947	0.2417	0.2062
Period2	0.7264	0.1648	0.2902	0.2625	0.1926	0.1823	0.2740	0.2110
Period3	0.5104	0.6743	−0.1271	0.1872	0.5346	0.4352	0.8138	0.8010

μ (10^{-3})	Robust (I) Nominal (II)	Mean (10^{-3}) Period1	Period2	Period3	$\mathrm{CVaR}_{0.95}$ Period1	Period2	Period3
0	I	1.3546	0.2618	0.6460	0.0299	0.0299	0.0425
	II	1.4455	0.3478	0.6209	0.0293	0.0295	0.0427
0.5	I	1.6063	0.5000	0.5765	0.0292	0.0299	0.0441
	II	1.4455	0.3478	0.6209	0.0293	0.0295	0.0427
0.55	I	1.6591	0.5500	0.5619	0.0294	0.0304	0.0448
	II	1.4455	0.3478	0.6209	0.0293	0.0295	0.0427
0.95	I	—	—	—	—	—	—
	II	1.7064	0.5948	0.5488	0.0297	0.0308	0.0455

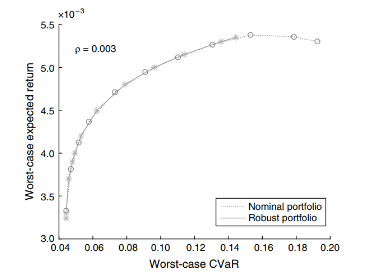


Monte Carlo Simulation Analysis



	ρ					
	0.001		0.003		0.005	
μ	Robust	Nominal	Robust	Nominal	Robust	Nominal
0	0.040870	0.040874	0.044218	0.044257	0.047295	0.047370
0.002	0.040870	0.040874	0.044218	0.044257	0.047445	—
0.004	0.040870	0.040874	0.049347	—	—	—
0.005	0.041453	—	0.096214	—	—	—
0.007	0.069936	—	—	—	—	—

Figure3 — Comparison of nominal optimal and robust optimal portfolios in the worst-case return-risk plane



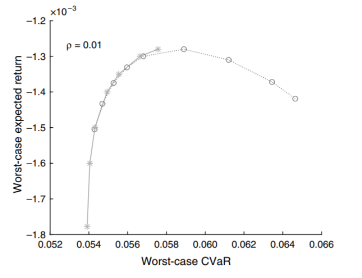
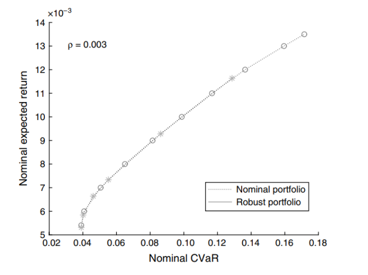
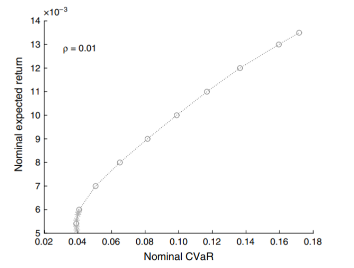


Figure4 — Comparison of nominal optimal and robust optimal portfolios in the nominal return-risk plane





Conclusion

Following the aforementioned comparison between the Robust and the Nominal portfolio optimization policies, our numerical experience indicates that using the WCCV@R as a measure of risk provides a robust performance and more flexibility in portfolio decision analysis.

NB — We could've handled the problem using this alternative approach:

$$
\begin{aligned}
\max \quad & \mu \\
s.t. \quad & \boldsymbol{x} \in X \\
& \alpha + \frac{1}{1-\beta} (\boldsymbol{\pi}^i)^T \boldsymbol{u}^i \leq \theta, \qquad i = 1, \dots, l, \\
& u_k^i \geq f\left(\boldsymbol{x}, \boldsymbol{y}_{[k]}^i\right) - \alpha, \quad k = 1, \dots, S^i, i = 1, \dots, l, \\
& u_k^i \geq 0, \qquad k = 1, \dots, S^i, i = 1, \dots, l, \\
& \boldsymbol{e}^T \boldsymbol{x} = w_0 \\
& \underline{\boldsymbol{x}} \leq \boldsymbol{x} \leq \overline{\boldsymbol{x}} \\
& \boldsymbol{x}^T \overline{\boldsymbol{y}}^i \geq \mu, \qquad i = 1, \dots, l
\end{aligned}
$$

Thanks for your attention!

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