Chapter **8**

Diffusively-Coupled Linear Systems

In this chapter we study diffusive interconnection among identical linear systems and linear control systems. As example system, we study the so-called second-order Laplacian flow.

8.1 Diffusively-coupled linear systems

In this chapter, we consider an agent to be a continuous-time linear single-input single-output dynamical systems with d-dimensional state and p-dimensional input, described by the matrices $\mathscr{A} \in \mathbb{R}^{d\times d}$, $\mathscr{B} \in \mathbb{R}^{d\times 1}$, and $\mathscr{C} \in \mathbb{R}^{1\times d}$. The dynamics of the ith agent, for $i \in \{1, \dots, n\}$, are

$$\begin{aligned} \dot{x}_i(t) &= \mathscr{A} x_i(t) + \mathscr{B} u_i(t), \\ y_i(t) &= \mathscr{C} x_i(t). \end{aligned} \tag{8.1}$$

Here, $x_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$, $u_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$, and $y_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$ are the state, input and output trajectories respectively.

The agents are interconnected through a weighted undirected graph G with edge weights $\{a_{ij}\}_{ij}$ and symmetric Laplacian matrix L. We assume that the input to each is based on information received from only its immediate neighbors in G. Specifically, we consider the *output-dependent diffusive coupling law*

$$u_i(t) = \sum_{j=1}^{n} a_{ij}\big(y_j(t) - y_i(t)\big). \tag{8.2}$$

Note: in control theory terms, this interconnection law amounts to a static output feedback controller. We illustrate this interconnection in Figure 8.1.

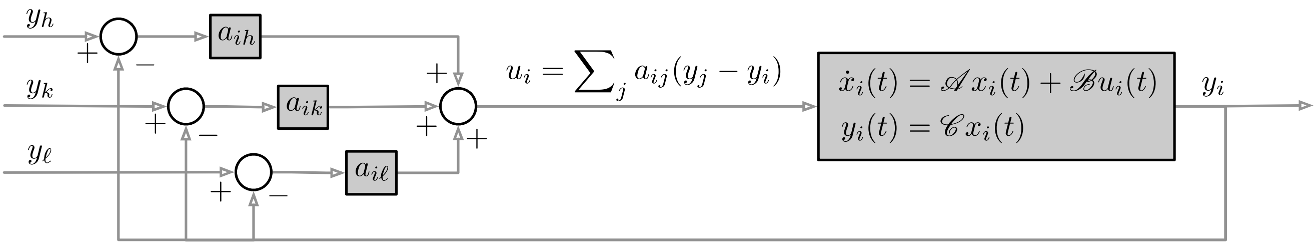


Figure 8.1: The output-dependent diffusive coupling law (8.2)

Definition 8.1. *A* network of diffusively-interconnected identical linear systems *is composed by n identical continuous-time linear single-input single-output systems described by the triplet $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ and a symmetric Laplacian matrix L.*

We aim to characterize when does the interconnected system achieve *asymptotic synchronization* in the sense that, for all agents i and j and all initial conditions,

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\|_2 = 0. \tag{8.3}$$

8.1.1 Second-order Laplacian flows

In this section we introduce an example of diffusively-coupled linear systems. We assume each node of the network obeys a so-called *system of double-integrators* (also referred to as *second-order dynamic*):

$$\ddot{q}_i = \bar{u}_i, \qquad \text{or, in first-order equivalent form,} \qquad \begin{cases} \dot{q}_i = v_i, \\ \dot{v}_i = \bar{u}_i, \end{cases} \tag{8.4}$$

where $\bar{u}_i$ is an appropriate control input signal to be designed.

We assume a weighted undirected graph describes the sensing and/or communication interactions among the agents with adjacency matrix A and Laplacian L. We also introduce constants $k_{\mathrm{p}}, k_{\mathrm{d}} \geq 0$ describing so-called *spring* and *damping* coefficients respectively, as well as constants $\gamma_{\mathrm{p}}, \gamma_{\mathrm{d}} \geq 0$ describing *position-averaging* and *velocity-averaging coefficients*. In summary, we consider the *proportional, derivative, position-averaging, and velocity-averaging control law*

$$\bar{u}_i = -k_{\mathrm{p}} q_i - k_{\mathrm{d}} \dot{q}_i + \sum_{j=1}^{n} a_{ij} \big(\gamma_{\mathrm{p}} (q_j - q_i) + \gamma_{\mathrm{d}} (\dot{q}_j - \dot{q}_i) \big). \tag{8.5}$$

A physical realization of this system as a spring/damper network is illustrated in Figure 8.2.

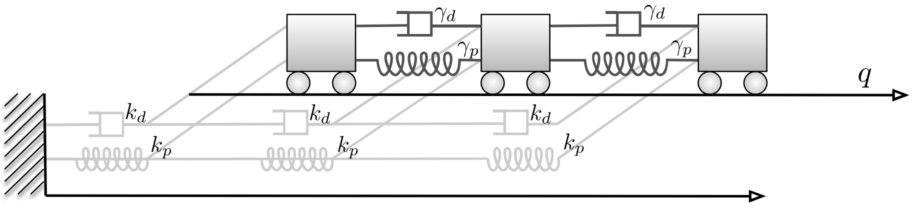


Figure 8.2: A network of unit-mass carts subject to spring and dampers gives rise to the second-order Laplacian flow (8.6). For illustration purposes, the springs and dampers connecting each cart to the left wall are drawn.

It is useful to rewrite the systems (8.4) interconnected via the law (8.5) in two useful manners. First, simply stacking each component into a vector, the corresponding closed-loop systems, called the *second-order Laplacian flow*, is

$$\ddot{q}(t) + (k_{\mathrm{d}} I_n + \gamma_{\mathrm{d}} L)\dot{q}(t) + (k_{\mathrm{p}} I_n + \gamma_{\mathrm{p}} L) q(t) = \mathbb{0}_n. \tag{8.6}$$

Second, we can rewrite $\bar{u}_i = -k_{\mathrm{p}} q_i - k_{\mathrm{d}} \dot{x}_i + u_i$, where $u_i = \sum_{j=1}^{n} a_{ij} \big(\gamma_{\mathrm{p}} (q_j - q_i) + \gamma_{\mathrm{d}} (\dot{q}_j - \dot{q}_i) \big)$ and define the matrices

$$\mathscr{A}_{\mathrm{msd}} = \begin{bmatrix} 0 & 1 \\ -k_{\mathrm{p}} & -k_{\mathrm{d}} \end{bmatrix}, \mathscr{B}_{\mathrm{msd}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } \mathscr{C}_{\mathrm{msd}} = \begin{bmatrix} \gamma_{\mathrm{p}} & \gamma_{\mathrm{d}} \end{bmatrix}. \tag{8.7}$$

With these definitions one can see that the systems (8.4) interconnected via the law (8.5) is equivalent to:

$$\frac{d}{dt}\begin{bmatrix} q_i \\ v_i \end{bmatrix} = \mathscr{A}_{\text{msd}} \begin{bmatrix} q_i \\ v_i \end{bmatrix} + \mathscr{B}_{\text{msd}} u_i,$$

where u_i is defined as in equation (8.2) and $y_i = \mathscr{C}_{\text{msd}} \begin{bmatrix} q_i \\ v_i \end{bmatrix}$. In other words, the matrices in equation (8.7) describe the second-order Laplacian system as a network of diffusively-interconnected identical linear systems.

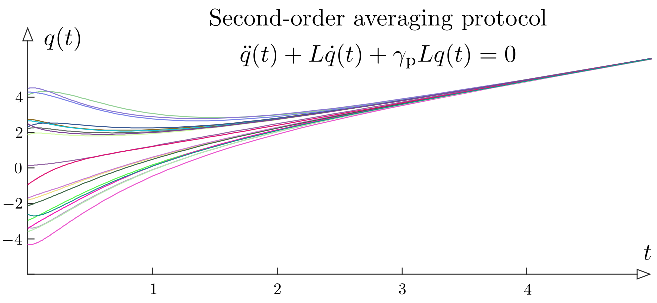
In Table 8.1 we catalog some interesting special cases and we illustrate in Figure 8.3 the behavior of the systems corresponding to the first three rows of Table 8.1. The next sections in this chapter focus on establishing rigorously the collective emerging behavior observed in these simulations.

Name	Dynamics	Asymptotic behavior
Second-order averaging protocol	$k_{\text{p}} = k_{\text{d}} = 0$, $\gamma_{\text{d}} = 1$, $\gamma_{\text{p}} > 0$ $\implies$ $\ddot{q}(t) + L\dot{q}(t) + \gamma_{\text{p}} Lq(t) = \mathbb{0}_n$	consensus on a ramp Example: car platooning Ref: Theorem 8.4(i)
Harmonic oscillators with velocity averaging	$k_{\text{d}} = \gamma_{\text{p}} = 0$, $\gamma_{\text{d}} = 1$, $k_{\text{p}} > 0$ $\implies$ $\ddot{q}(t) + L\dot{q}(t) + k_{\text{p}} q(t) = \mathbb{0}_n$	consensus on harmonic oscillations Example E8.5: resonant inductor/-capacitor circuits Ref: Theorem 8.4(ii)
Position-averaging with absolute velocity damping	$k_{\text{p}} = \gamma_{\text{d}} = 0$, $\gamma_{\text{p}} = 1$, $k_{\text{d}} > 0$ $\implies$ $\ddot{q}(t) + k_{\text{d}}\dot{q}(t) + Lq(t) = \mathbb{0}_n$	consensus on positions Example: rendezvous in multi-robot systems Example: swing dynamics in power networks Ref: Theorem 8.4(iii)
Laplacian oscillators	$k_{\text{p}} = k_{\text{d}} = \gamma_{\text{d}} = 0$, $\gamma_{\text{p}} = 1$ $\implies$ $\ddot{q}(t) + Lq(t) = \mathbb{0}_n$	superposition of ramp and harmonics Example 7.1.4: discretized wave equation Ref: Exercise E8.6

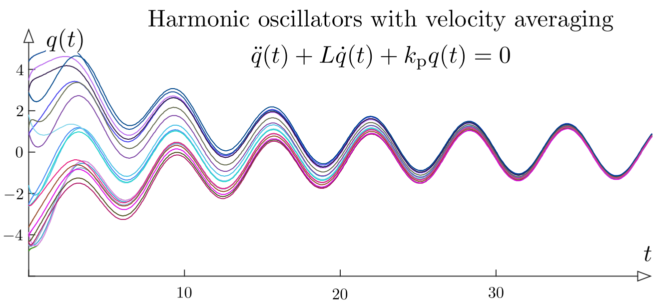
Table 8.1: Classification of second-order Laplacian flows arising from the general model in equation (8.6)

8.2 Modeling via Kronecker products

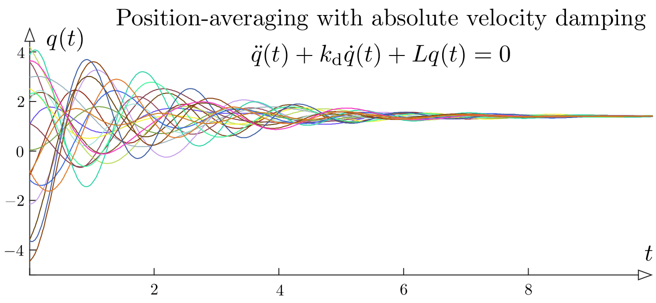
In this section we obtain a compact expression for the state matrix of a diffusively-coupled network of linear systems.



(a) asymptotic consensus on a ramp signal



(b) asymptotic synchronization on an harmonic signal



(c) asymptotic consensus on a weighted average

Figure 8.3: Representative trajectories of the second-order Laplacian flow (8.6) for a randomly-generated undirected graph with $n = 20$ nodes, random initial conditions, and the three choices of gains as cataloged in Table 8.1.

8.2.1 The Kronecker product

We start by introducing a useful tool. The *Kronecker product* of $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{q\times r}$ is the $nq \times mr$ matrix $A \otimes B$ given by

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \ddots & a_{nm}B \end{bmatrix}. \tag{8.8}$$

For example, two simple cases are

$$(I_n \otimes B) = \begin{bmatrix} B & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ddots & B \end{bmatrix} \in \mathbb{R}^{nq\times nr} \quad \text{and } (A \otimes I_q) = \begin{bmatrix} a_{11}I_q & \dots & a_{1m}I_q \\ \vdots & \ddots & \vdots \\ a_{n1}I_q & \ddots & a_{nm}I_q \end{bmatrix} \in \mathbb{R}^{nq\times mq}. \tag{8.9}$$

Horn and Johnson (1994) review many useful properties of the Kronecker product, e.g., including

$$\text{the bilinearity property:} \quad (\alpha A + \beta B) \otimes (\gamma C + \delta D) = \alpha\gamma A \otimes C + \alpha\delta A \otimes D + \beta B \otimes \gamma C + \beta\delta B \otimes D, \tag{8.10a}$$

$$\text{the associativity property:} \quad (A \otimes B) \otimes C = A \otimes (B \otimes C), \tag{8.10b}$$

$$\text{the transpose property:} \quad (A \otimes B)^{\mathsf{T}} = A^{\mathsf{T}} \otimes B^{\mathsf{T}}, \tag{8.10c}$$

$$\text{the mixed product property:} \quad (A \otimes B)(C \otimes D) = (AC) \otimes (BD), \tag{8.10d}$$

where the A, B, C, D matrices have appropriate compatible dimensions.

The mixed product property (8.10d) leads to many useful consequences. As first example, if $Av = \lambda v$ and $Bw = \mu w$, then property (8.10d) implies $(A \otimes B)(v \otimes w) = (\lambda v) \otimes (\mu w)$. Therefore, we know

$$\text{the spectrum property:} \qquad \operatorname{spec}(A \otimes B) = \{\lambda\mu \mid \lambda \in \operatorname{spec}(A), \mu \in \operatorname{spec}(B)\}. \tag{8.11}$$

A second consequence of property (8.10d) is that, for square matrices A and B, $A \otimes B$ is invertible if and only if both A and B are invertible, in which case

$$\text{the inverse property:} \qquad (A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \qquad . \tag{8.12}$$

We ask the reader to prove these properties and establish other ones in Exercises E8.1 and E8.2.

8.2.2 The state matrix for a diffusively coupled system

We are now ready to provide a concise closed-form expression for the state matrix of the network.

Theorem 8.2 (The transcription theorem for diffusively-coupled linear systems). *Consider a network of diffusively-interconnected identical linear systems described by the system $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ and the symmetric Laplacian matrix L. Then hold:*

(i) the open-loop system obeys $\dot{\mathbf{x}} = (I_n \otimes \mathscr{A})\mathbf{x} + (I_n \otimes \mathscr{B})\mathbf{u}$,

(ii) the diffusive coupling law is $\mathbf{u}(t) = -L\mathbf{y}(t)$, and

(iii) the closed-loop system obeys

$$\dot{\mathbf{x}} = (I_n \otimes \mathscr{A} - L \otimes \mathscr{B}\mathscr{C})\mathbf{x}, \tag{8.13}$$

where we adopt the notation $\mathbf{x} = \begin{bmatrix} x_1^\top, \dots, x_n^\top \end{bmatrix}^\top \in \mathbb{R}^{nd}$, $\mathbf{u} = \begin{bmatrix} u_1, \dots, u_n \end{bmatrix}^\top \in \mathbb{R}^n$, and $\mathbf{y} = \begin{bmatrix} y_1, \dots, y_n \end{bmatrix}^\top \in \mathbb{R}^n$.

Proof. We write the n coupled systems in a single vector-valued equation on the state space $\mathbb{R}^{dn}$ using the Kronecker product. As in equation (8.9), we stack the n dynamical systems to write

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathscr{A} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ddots & \mathscr{A} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathscr{B} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ddots & \mathscr{B} \end{bmatrix} \mathbf{u} = (I_n \otimes \mathscr{A})\mathbf{x} + (I_n \otimes \mathscr{B})\mathbf{u}.$$

Next, recalling the definition of Laplacian, we write the output-dependent diffusive coupling law (8.2) as $\mathbf{u} = -L\mathbf{y}$. Moreover, we note $\mathbf{y} = (I_n \otimes \mathscr{C})\mathbf{x}$ and, plugging in, we obtain

$$\dot{\mathbf{x}} = (I_n \otimes \mathscr{A})\mathbf{x} - (I_n \otimes \mathscr{B})L(I_n \otimes \mathscr{C})\mathbf{x}.$$

From the remarkable mixed product property in equation (8.10d), we obtain

$$(I_n \otimes \mathscr{B})L(I_n \otimes \mathscr{C}) = (I_n \otimes \mathscr{B})(L \otimes 1)(I_n \otimes \mathscr{C}) = (L \otimes \mathscr{B}\mathscr{C})$$

and, in turn, the closed loop (8.13). Note that $L \in \mathbb{R}^{n \times n}$, $\mathscr{B} \in \mathbb{R}^{d \times 1}$, and $\mathscr{C} \in \mathbb{R}^{1 \times d}$ together imply that $\mathscr{B}\mathscr{C}$ has dimensions $d \times d$ and that $L \otimes \mathscr{B}\mathscr{C}$ has dimensions $nd \times nd$, the same as $I_n \otimes \mathscr{A}$. Hence, equation (8.13) is dimensionally correct. This concludes the proof. ■

8.3 The synchronization theorem

In this section we present the main result of this chapter. Define the *state average* $x_{\text{ave}} : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ by

$$\dot{x}_{\text{ave}}(t) = \mathscr{A} x_{\text{ave}}(t), \quad x_{\text{ave}}(0) = \frac{1}{n}\sum_{j=1}^{n} x_j(0),$$

and note that $x_{\text{ave}}(t) = \exp(\mathscr{A}t)x_{\text{ave}}(0)$.

Theorem 8.3 (Synchronization of output-dependent diffusively-coupled linear systems). *Consider a network of diffusively-interconnected identical linear systems described by the system $(\mathscr{A}, \mathscr{B}, \mathscr{C})$, the symmetric Laplacian L, and the closed-loop dynamics (8.13). Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ denote the eigenvalues of L. The following statements hold:*

(i) if each matrix $\mathscr{A} - \lambda_i \mathscr{B}\mathscr{C}$, $i \in \{2, \dots, n\}$, is Hurwitz, then

a) the system (8.1) achieves asymptotic synchronization as in equation (8.3), that is,

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\|_2 = 0, \quad \textit{for all } i, j \in \{1, \dots, n\};$$

b) each state trajectory asymptotically converges to the state average trajectory in the sense that

$$\lim_{t\to\infty} \|x_i(t) - x_{\text{ave}}(t)\|_2 = 0, \quad \textit{for all } i \in \{1, \dots, n\};$$

(ii) the system (8.13) is exponentially stable if and only if each matrix $\mathscr{A} - \lambda_i \mathscr{B}\mathscr{C}$, $i \in \{1, \dots, n\}$, is Hurwitz.

Note: recall that the graph associated to L is disconnected if and only if $\lambda_2 = 0$. For such graphs, the Hurwitz conditions on the matrices $\mathscr{A} - \lambda_i \mathscr{B}\mathscr{C}$ in statements (i) and (ii) are equivalent and include that requirement that $\mathscr{A}$ be Hurwitz. If instead the graph is connected, than it is possible for each matrix $\mathscr{A} - \lambda_i \mathscr{B}\mathscr{C}$, $i \in \{2, \dots, n\}$, is Hurwitz while $\mathscr{A}$ is not.

Proof. To prove statements (i) and (ii), we proceed as follows. Let $L = U\Lambda U^\top$ be the singular value decomposition of the L, where U is an orthonormal matrix and $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Consider the change of variable $\mathbf{z} = (U^\top \otimes I_d)\mathbf{x}$ and note $(U^\top \otimes I_d)(U \otimes I_d) = I_{nd}$. Compute:

$$\begin{aligned}
\dot{\mathbf{z}} &= (U^\top \otimes I_d)(I_n \otimes \mathscr{A} - L \otimes \mathscr{B}\mathscr{C})(U \otimes I_d)\mathbf{z} \\
&= \Big((U^\top I_n U) \otimes (I_d \mathscr{A} I_d) - (U^\top L U) \otimes (I_d \mathscr{B}\mathscr{C} I_d)\Big)\mathbf{z} \\
&= (I_n \otimes \mathscr{A} - \Lambda \otimes \mathscr{B}\mathscr{C})\mathbf{z}.
\end{aligned}$$

Now, note that the matrix $(I_n \otimes \mathscr{A} - \Lambda \otimes \mathscr{B}\mathscr{C})$ is block diagonal because

$$I_n \otimes \mathscr{A} = \begin{bmatrix} \mathscr{A} & & \\ & \ddots & \\ & & \mathscr{A} \end{bmatrix}, \quad \text{and} \quad \Lambda \otimes \mathscr{B}\mathscr{C} = \begin{bmatrix} \lambda_1 \mathscr{B}\mathscr{C} & & \\ & \ddots & \\ & & \lambda_n \mathscr{B}\mathscr{C} \end{bmatrix}.$$

This block diagonal form immediately implies statement (ii).

Next, assuming that only the $n-1$ matrices $\mathscr{A} - \lambda_i \mathscr{B}\mathscr{C}$, $i \in \{2, \dots, n\}$ are Hurwitz, we know that there exist a vector-valued function $\mathbf{r} : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ that is exponentially vanishing as $t \to \infty$ and such that

$$\begin{aligned}\mathbf{z}(t) &= \begin{bmatrix} \exp(\mathscr{A}t) z_1(0) \\ \mathbb{0}_d \\ \vdots \\ \mathbb{0}_d \end{bmatrix} + \mathbf{r}(t) = \begin{bmatrix} \exp(\mathscr{A}t) & & \\ & \mathbb{0}_{d\times d} & \\ & & \ddots & \\ & & & \mathbb{0}_{d\times d} \end{bmatrix} \mathbf{z}(0) + \mathbf{r}(t) \\ &= \Big((\mathbb{e}_1 \mathbb{e}_1^\top) \otimes \exp(\mathscr{A}t) \Big) \mathbf{z}(0) + \mathbf{r}(t).\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{t\to\infty} \mathbf{x}(t) &= (U^\top \otimes I_d) \lim_{t\to\infty} \mathbf{z}(t) \\ &= (U^\top \otimes I_d) \Big((\mathbb{e}_1 \mathbb{e}_1^\top) \otimes \exp(\mathscr{A}t) \Big) (U \otimes I_d) \mathbf{x}(0) \\ &= \Big((U^\top \mathbb{e}_1 \mathbb{e}_1^\top U) \otimes \exp(\mathscr{A}t) \Big) \mathbf{x}(0).\end{aligned}$$

Now, recall that $U^\top \mathbb{1}_n$ is the unit-length eigenvector of L corresponding to $\lambda_1 = 0$, that is, $U^\top \mathbb{1}_n = \mathbb{1}_n/\sqrt{n}$. In summary, with $\mathbf{h}(t) = (U^\top \otimes I_d)\mathbf{r}(t)$, we obtain

$$\begin{aligned}\mathbf{x}(t) &= \frac{1}{n} \Big((\mathbb{1}_n \mathbb{1}_n^\top) \otimes \exp(\mathscr{A}t) \Big) \mathbf{x}(0) + \mathbf{h}(t) \\ &= \frac{1}{n} \Big(I_n \otimes \exp(\mathscr{A}t) \Big) \Big((\mathbb{1}_n \mathbb{1}_n^\top) \otimes I_d \Big) \mathbf{x}(0) + \mathbf{h}(t) \\ &= \frac{1}{n} \begin{bmatrix} \exp(\mathscr{A}t) & & \\ & \ddots & \\ & & \exp(\mathscr{A}t) \end{bmatrix} \begin{bmatrix} I_d & \dots & I_d \\ \vdots & \ddots & \vdots \\ I_d & \ddots & I_d \end{bmatrix} \begin{bmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{bmatrix} + \mathbf{h}(t),\end{aligned}$$

so that each system i satisfies

$$x_i(t) = \exp(\mathscr{A}t) \Big(\frac{1}{n} \sum_{j=1}^{n} x_j(0) \Big) + h_i(t),$$

where the transient evolution satisfies $\lim_{t\to\infty} h_i(t) = \mathbb{0}_d$. This concludes the proof of statement (i). ■

8.3.1 Synchronization in second-order Laplacian systems

We now apply to second-order Laplacian systems the theoretical results obtained in the synchronization Theorem 8.3. Unlike for the general case, it is possible to obtain quite explicit results.

First, recall that second-order Laplacian systems are diffusively-coupled linear systems with matrices $(\mathscr{A}_{\text{msd}}, \mathscr{B}_{\text{msd}}, \mathscr{C}_{\text{msd}})$ and with Laplacian interconnection matrix L. As before, let $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ denote the eigenvalues of L. We compute:

$$\begin{aligned}\mathscr{A}_{\text{msd}} - \lambda_i \mathscr{B}_{\text{msd}} \mathscr{C}_{\text{msd}} &= \begin{bmatrix} 0 & 1 \\ -k_{\text{p}} & -k_{\text{d}} \end{bmatrix} - \lambda_i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \gamma_{\text{p}} & \gamma_{\text{d}} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -(k_{\text{p}} + \lambda_i \gamma_{\text{p}}) & -(k_{\text{d}} + \lambda_i \gamma_{\text{d}}) \end{bmatrix}.\end{aligned}$$

In other words, the i-th subsystem $\mathscr{A}_{\mathrm{msd}} - \lambda_i \mathscr{B}_{\mathrm{msd}} \mathscr{C}_{\mathrm{msd}}$ is a spring/damper system with effective spring coefficient $k_{\mathrm{p}} + \lambda_i \gamma_{\mathrm{p}}$ and effective damper coefficient $k_{\mathrm{d}} + \lambda_i \gamma_{\mathrm{d}}$. Based on a well known result, it is easy to see that

$$\mathscr{A}_{\mathrm{msd}} - \lambda_i \mathscr{B}_{\mathrm{msd}} \mathscr{C}_{\mathrm{msd}} \quad \text{is Hurwitz} \qquad \iff \qquad k_{\mathrm{p}} + \lambda_i \gamma_{\mathrm{p}} > 0 \text{ and } k_{\mathrm{d}} + \lambda_i \gamma_{\mathrm{d}} > 0.$$

Next, as state average system, we define the average mass/spring/damper system by

$$\frac{d}{dt} \begin{bmatrix} q_{\mathrm{ave}}(t) \\ \dot{q}_{\mathrm{ave}}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_{\mathrm{p}} & -k_{\mathrm{d}} \end{bmatrix} \begin{bmatrix} q_{\mathrm{ave}}(t) \\ \dot{q}_{\mathrm{ave}}(t) \end{bmatrix}. \tag{8.14}$$

where we set $q_{\mathrm{ave}}(0) = \sum_{j=1}^n q_j(0)$ and $\dot{q}_{\mathrm{ave}}(0) = \sum_{j=1}^n \dot{q}_j(0)$.

These observations lead to the main synchronization result for second-order Laplacian flows.

Theorem 8.4 (Synchronization of second-order Laplacian flows). *Consider the second-order Laplacian flow (8.6). Assume that the undirected graph associated to L is connected. If $k_{\mathrm{p}} + \gamma_{\mathrm{p}} > 0$ and $k_{\mathrm{d}} + \gamma_{\mathrm{d}} > 0$, then*

(i) the system (8.6) achieves asymptotic synchronization in the sense that

$$\lim_{t\to\infty} \|q_i(t) - q_j(t)\|_2 = \lim_{t\to\infty} \|\dot{q}_i(t) - \dot{q}_j(t)\|_2 = 0, \qquad \textit{for all } i, j \in \{1, \dots, n\};$$

(ii) each trajectory asymptotically converges to the state average trajectory in the sense that

$$\lim_{t\to\infty} \|q_i(t) - q_{\mathrm{ave}}(t)\|_2 = 0, \qquad \textit{for all } i \in \{1, \dots, n\}.$$

Specifically:

(i) for the second-order averaging protocol ($k_{\mathrm{p}} = k_{\mathrm{d}} = 0$, $\gamma_{\mathrm{d}} = 1$, $\gamma_{\mathrm{p}} > 0$, first row Table 8.1), asymptotic consensus on a ramp signal is achieved, that is, as $t \to \infty$,

$$q(t) \to \Big(q_{\mathrm{ave}}(0) + \dot{q}_{\mathrm{ave}}(0)t\Big)\mathbb{1}_n;$$

(ii) for the harmonic oscillators with velocity averaging ($k_{\mathrm{d}} = \gamma_{\mathrm{p}} = 0$, $\gamma_{\mathrm{d}} = 1$, $k_{\mathrm{p}} > 0$, second row Table 8.1), asymptotic consensus on an harmonic signal with frequency $\sqrt{k_{\mathrm{p}}}$ is achieved, that is, as $t \to \infty$,

$$q(t) \to \Big(q_{\mathrm{ave}}(0)\cos(\sqrt{k_{\mathrm{p}}}t) + \frac{1}{\sqrt{k_{\mathrm{p}}}}\dot{q}_{\mathrm{ave}}(0)\sin(\sqrt{k_{\mathrm{p}}}t)\Big)\mathbb{1}_n;$$

(iii) for the position-averaging flow with absolute velocity damping ($k_{\mathrm{p}} = \gamma_{\mathrm{d}} = 0$, $\gamma_{\mathrm{p}} = 1$, $k_{\mathrm{d}} > 0$, third row Table 8.1), asymptotic consensus on a weighted average value is achieved, that is, as $t \to \infty$

$$q(t) \to \Big(q_{\mathrm{ave}}(0) + \dot{q}_{\mathrm{ave}}(0)/k_{\mathrm{d}}\Big)\mathbb{1}_n.$$

The asymptotic behavior of the dynamical systems as classified in the three scenarios of this theorem and defined in the first three rows of Table 8.1 is consistent with the empirical observations in Figure 8.3.

8.4 Control design for synchronization

We now generalize the study of diffusively-coupled systems in three ways: (1) we assume the interconnection graph is directed, (2) we consider a multi-input multi-output interconnection, and, most importantly, (3) we consider a control design problem, instead of a stability analysis problem.

For simplicity, we consider the setting of state feedback. While the transcription and stability analysis method is very similar to that in the previous sections, the method of proof for digraph interconnections relies upon a transcription into Jordan normal form instead of a diagonalization procedure. We also review various stabilizability notions from linear control theory.

8.4.1 Problem statement

In this section, we consider an agent to be a continuous-time linear control systems with d-dimensional state and p-dimensional input, described by the matrices $\mathscr{A} \in \mathbb{R}^{d\times d}$ and $\mathscr{B} \in \mathbb{R}^{d\times p}$. The dynamics of the ith agent, for $i \in \{1,\dots,n\}$, are

$$\dot{x}_i = \mathscr{A} x_i + \mathscr{B} u_i, \tag{8.15}$$

where $x_i \in \mathbb{R}^d$ is the state and $u_i \in \mathbb{R}^p$ is the control input.

The agents communicate along the edges of a weighted directed graph G with edge weights $\{a_{ij}\}_{ij}$ and Laplacian matrix L. We assume each agent regulates its own control signal based on information received from only its immediate in-neighbors in G.

The problem statement is as follows: design a control law that, based only on the information obtained through communication, achieves asymptotic synchronization in the sense of equation (8.3), that is, for all agents i and j and all initial conditions,

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\|_2 = 0.$$

Consider the *(state-dependent) diffusive coupling law*:

$$u_i(t) = cK \sum_{j=1}^{n} a_{ij}\big(x_j(t) - x_i(t)\big), \quad \text{for } i \in \{1,\dots,n\}, \tag{8.16}$$

where the scalar $c > 0$ is a coupling gain and $K \in \mathbb{R}^{p\times d}$ is a control gain matrix. Note that this interconnection law amounts to a static feedback controller.

Before solving this problem we generalize the transcription and synchronization Theorems 8.2 and 8.3 to this setting. The instructive proof of the following result is instructive and postponed to Section 8.4.5.

Theorem 8.5 (Synchronization of diffusively-coupled linear systems). *Consider n identical continuous-time linear control systems described by the couple $(\mathscr{A},\mathscr{B})$ and a digraph with Laplacian L and with eigenvalues $0 = \lambda_1, \lambda_2, \dots, \lambda_n$. Let w denote the left eigenvector of L with eigenvalue 0 satisfying $\mathbb{1}_n^\top w = 1$. The following statements hold:*

(i) the open-loop system obeys $\dot{\mathbf{x}} = (I_n \otimes \mathscr{A})\mathbf{x} + (I_n \otimes \mathscr{B})\mathbf{u}$, the diffusive coupling law is

$$\mathbf{u}(t) = -c(I_n \otimes K)(L \otimes I_d)\mathbf{x}(t),$$

and closed-loop system obeys

$$\dot{\mathbf{x}} = \big((I_n \otimes \mathscr{A}) - c(L \otimes \mathscr{B}K)\big)\mathbf{x}; \tag{8.17}$$

where we adopt the notation $\mathbf{x} = \big[x_1^\top,\dots,x_n^\top\big]^\top \in \mathbb{R}^{nd}$ and $\mathbf{u} = \big[u_1^\top,\dots,u_n^\top\big]^\top \in \mathbb{R}^{np}$;

(ii) if each matrix $\mathscr{A} - c\lambda_i \mathscr{B} K$, $i \in \{2,\dots,n\}$, is Hurwitz, then the n control systems (8.15) in closed loop with the diffusive coupling law (8.16) achieve asymptotic synchronization and each state trajectory satisfies

$$x_i(t) = \exp(\mathscr{A} t)\Big(\sum_{j=1}^{n} w_i x_j(0)\Big) + h_i(t), \qquad \textit{where } \lim_{t\to\infty} h_i(t) = \mathbb{0}_d.$$

Note: Assume that G contains a globally reachable node. Then one can show the following converse result: if $\mathbf{x}$ achieves asymptotic consensus for all initial conditions, then each matrix $\mathscr{A} + c\lambda_i \mathscr{B} K$, $i \in \{2,\dots,n\}$, is Hurwitz.

8.4.2 Stabilizability of linear control systems

We now review from linear control theory the notion of stabilizability and stabilizing feedback gain design.

Given matrices $\mathscr{A} \in \mathbb{R}^{d\times d}$ and $\mathscr{B} \in \mathbb{R}^{d\times p}$, a *continuous-time linear control systems* with d-dimensional state and p-dimensional input is

$$\dot{x} = \mathscr{A} x + \mathscr{B} u, \tag{8.18}$$

where $x_i \in \mathbb{R}^d$ is the state and $u_i \in \mathbb{R}^p$ is the control input.

Given a feedback gain matrix $\mathscr{K} \in \mathbb{R}^{d\times n}$, the feedback control signal $u = -\mathscr{K} x$ gives rise to the closed-loop linear system

$$\dot{x} = \mathscr{A} x + \mathscr{B}(-\mathscr{K} x) = (\mathscr{A} - \mathscr{B}\mathscr{K})x.$$

Definition 8.6. *The linear control system $(\mathscr{A}, \mathscr{B})$ is* stabilizable *if there exists a matrix $\mathscr{K}$ such that $\mathscr{A} - \mathscr{B}\mathscr{K}$ is Hurwitz.*

In other words, the closed-loop system is exponentially stable.

Theorem 8.7 (Stabilizability of linear control systems). *Given matrices $\mathscr{A} \in \mathbb{R}^{d\times d}$ and $\mathscr{B} \in \mathbb{R}^{d\times p}$, the following statements are equivalent*

(i) the linear control system (8.18) is stabilizable,

(ii) there exists a $d \times d$ matrix $P \succ 0$ solving the Lyapunov matrix inequality

$$\mathscr{A} P + P \mathscr{A}^\top - 2\mathscr{B}\mathscr{B}^\top \prec 0. \tag{8.19}$$

Moreover, for any $P \succ 0$ satisfying the inequality (8.19), a stabilizing feedback gain matrix is $\mathscr{K} = \mathscr{B}^\top P^{-1}$.

We refer for example to (Hespanha, 2009) for a complete treatment of linear systems theory, including a detailed discussion of stabilizability. We recall that the Lyapunov matrix inequality can be solved easily as a *linear matrix inequality* (LMI) problem.

8.4.3 High-gain LMI design

Consider now the following algorithm to design the control gain matrix K and the coupling gain c. Recall the Lyapunov matrix equation (8.19) and the fact that it can be solved via an LMI solver.

Note: the design of K depends upon only the dynamics of each agent and the design of c depends upon only the communication graph.

High-gain LMI design

Input: the stabilizable pair $(\mathscr{A}, \mathscr{B})$
Output: a control gain matrix K and coupling gain c

set $P \leftarrow$ solution to the linear matrix equality $\mathscr{A}P + P\mathscr{A}^\mathsf{T} - 2\mathscr{B}\mathscr{B}^\mathsf{T} = -I_d$
set $K \leftarrow \mathscr{B}^\mathsf{T} P^{-1}$
set $c \leftarrow 1/\min\{\Re(\lambda_i) \mid i \in \{2, \dots, n\}\}$

Theorem 8.8 (High-gain LMI design for stabilizable linear control systems). *Consider n identical continuous-time linear control systems described by the couple $(\mathscr{A}, \mathscr{B})$ and a digraph G with Laplacian L and with eigenvalues $0 = \lambda_1, \lambda_2, \dots, \lambda_n$. If the pair $(\mathscr{A}, \mathscr{B})$ is stabilizable and the digraph G contains a globally reachable node, then*

(i) the high-gain LMI design algorithm is well posed in the sense that the solution matrix P exists unique and positive definite and the scalar c is well defined, and

(ii) the resulting pair (K, c) ensures that each matrix $\mathscr{A} - c\lambda_i\mathscr{B}K$, $i \in \{2, \dots, n\}$, is Hurwitz.

Note: any P solving the linear matrix inequality $\mathscr{A}P + P\mathscr{A}^\mathsf{T} - 2\mathscr{B}\mathscr{B}^\mathsf{T} \prec 0$ and any c larger than the stated value would satisfy the properties in Theorem 8.8.

Note: the last two theorems reduce the problem of analyzing a dynamical system of dimension nd to the analysis of objects of dimensions n (the Laplacian L) and d (the linear matrix equality in P).

Proof. Fact (i) is a direct consequence of Theorem 8.7 about the stabilizability of linear control systems.

Regarding fact (ii), we provide a proof only for the case when $\lambda_i \in \mathbb{R}$. Let P be the positive definite matrix computed in the high-gain LMI design algorithm. As Lyapunov equation for the ith subsystem, for $i \in \{2, \dots, n\}$, as:

$$\begin{aligned}
&(\mathscr{A} - c\lambda_i\mathscr{B}K)P + P(\mathscr{A} - c\lambda_i\mathscr{B}K)^\mathsf{T} \\
&\qquad = \mathscr{A}P + P\mathscr{A}^\mathsf{T} - c\lambda_i\Big(\mathscr{B}KP + PK^\mathsf{T}\mathscr{B}^\mathsf{T}\Big) \\
&\qquad = -I_d + 2\mathscr{B}\mathscr{B}^\mathsf{T} - c\lambda_i\Big(\mathscr{B}(\mathscr{B}^\mathsf{T}P^{-1})P + P(\mathscr{B}^\mathsf{T}P^{-1})^\mathsf{T}\mathscr{B}^\mathsf{T}\Big),
\end{aligned}$$

where we used both equalities from the high-gain LMI design: $\mathscr{A}P + P\mathscr{A}^\mathsf{T} - 2\mathscr{B}\mathscr{B}^\mathsf{T} = -I_d$ and $K = \mathscr{B}^\mathsf{T}P^{-1}$. We then continue to obtain

$$\begin{aligned}
(\mathscr{A} - c\lambda_i\mathscr{B}K)P + P(\mathscr{A} - c\lambda_i\mathscr{B}K)^\mathsf{T} &= -I_d + 2\mathscr{B}\mathscr{B}^\mathsf{T} - 2c\lambda_i\mathscr{B}\mathscr{B}^\mathsf{T} \\
&= -I_d + 2(1 - c\lambda_i)\mathscr{B}\mathscr{B}^\mathsf{T}.
\end{aligned}$$

For any $c \geq 1/\min\{\Re(\lambda_i) \mid i \in \{2, \dots, n\}\} = 1/\min\{\lambda_i \mid i \in \{2, \dots, n\}\}$, we know that $c\lambda_i \geq 1$ and therefore $1 - c\lambda_i \leq 0$. In summary we have proved that

$$(\mathscr{A} - c\lambda_i\mathscr{B}K)P + P(\mathscr{A} - c\lambda_i\mathscr{B}K)^\mathsf{T} \prec 0.$$

Therefore the linear system $\dot{x} = (\mathscr{A} - c\lambda_i\mathscr{B}K)x$ has quadratic Lyapunov function $x \mapsto x^\mathsf{T}P^{-1}x$. ■

8.4.4 Extension to output feedback design

We now present the basic concepts about the problem of output feedback synchronization. As in equation (8.1), the agent is now an input/output control system described by

$$\begin{aligned}
\dot{x}_i(t) &= \mathscr{A}x_i(t) + \mathscr{B}u_i(t), \\
y_i(t) &= \mathscr{C}x_i(t).
\end{aligned} \tag{8.20}$$

Here $x_i \in \mathbb{R}^d$ is the state, $u_i \in \mathbb{R}^p$ is the control input, and $y_i \in \mathbb{R}^q$ is the output signal. Each agent receives the signal

$$\zeta_i = c \sum_{j=1}^{n} a_{ij}(y_i - y_j), \tag{8.21}$$

and executes the following *observer-based diffusive coupling law*

$$\begin{aligned} \dot{v}_i &= (\mathscr{A} + \mathscr{B}K)v_i + F\Big(c \sum\nolimits_{j=1}^{n} a_{ij}\mathscr{C}(v_i - v_j) - \zeta_i\Big), \\ u_i &= Kv_i. \end{aligned} \tag{8.22}$$

Here c is a coupling gain, v_i is the protocol state, and K and F are control and observer gain matrices to be designed.

One can show the following generalization of Theorem 8.5: if each matrix $\mathscr{A} + \mathscr{B}K$ and $\mathscr{A} \pm c\lambda_i F\mathscr{C}$, $i \in \{2, \dots, n\}$, is Hurwitz, then the n input/output control systems (8.20) in closed loop with the observer-based diffusive coupling law (8.21)-(8.21) achieve asymptotic synchronization in the state and protocol state variables. We refer the interested reader to (Li et al., 2010; Li and Duan, 2014) for design methods to compute appropriate gain parameters c, K and F.

8.4.5 Proof of synchronization over directed graphs

Proof of Theorem 8.5. To prove statement (i), we proceed as in the proof of Theorem 8.2(i). We stack the n dynamical systems (8.15) to obtain $\dot{\mathbf{x}} = (I_n \otimes \mathscr{A})\mathbf{x} + (I_n \otimes \mathscr{B})\mathbf{u}$. We write the diffusive coupling law (8.16) as

$$\mathbf{u}(t) = -c(I_n \otimes K)\mathbf{z}(t), \text{ where } \quad z_i(t) = \sum_{j=1}^{n} a_{ij}\big(x_i(t) - x_j(t)\big) = \sum_{j=1}^{n} \ell_{ij}x_j(t). \tag{8.23}$$

where ℓ_{ij} is the (ij) entry of the Laplacian L. The last equality is equivalent to $\mathbf{z}(t) = (L \otimes I_d)\mathbf{x}(t)$, so that $\mathbf{u}(t) = -c(I_n \otimes K)(L \otimes I_d)\mathbf{x}(t)$. Finally, the mixed product property (8.10d) implies

$$\dot{\mathbf{x}} = (I_n \otimes \mathscr{A})\mathbf{x} - (I_n \otimes \mathscr{B})c(I_n \otimes K)(L \otimes I_d)\mathbf{x} = \big((I_n \otimes \mathscr{A}) - c(L \otimes \mathscr{B}K)\big)\mathbf{x}. \tag{8.24}$$

Note that $L \in \mathbb{R}^{n\times n}$, $\mathscr{B} \in \mathbb{R}^{d\times p}$ and $K \in \mathbb{R}^{p\times d}$ together imply that $\mathscr{B}K$ has dimensions $d \times d$ and that $L \otimes \mathscr{B}K$ has dimensions $nd \times nd$ so that equation (8.24) is dimensionally correct. This concludes the proof of statement (i).

To prove statement (ii), let $\Pi_n = I_n - \mathbb{1}_n w^\top$ denote a projection matrix on the subspace of zero-average vectors; note that $\Pi_n^2 = \Pi_n$ and $w^\top \Pi_n = \mathbb{0}_n^\top$. As in Exercise E2.13(iii) (where A is row-stochastic), one can easily see

$$\Pi_n L = L\Pi_n = L. \tag{8.25}$$

Define the *consensus error* $\mathbf{e} \in \mathbb{R}^{nd}$ by

$$\mathbf{e} = (\Pi_n \otimes I_d)\mathbf{x}. \tag{8.26}$$

Note that $\mathbf{e} = \mathbb{0}_{nd}$ if and only if $x_1 = \dots = x_n$. Using the mixed product property and the fact that L and Π_n commute, we compute

$$\begin{aligned} \dot{\mathbf{e}} &= (\Pi_n \otimes I_d)\big((I_n \otimes \mathscr{A}) - c(L \otimes \mathscr{B}K)\big)\mathbf{x} \\ &= \big((I_n \otimes \mathscr{A}) - c(L \otimes \mathscr{B}K)\big)(\Pi_n \otimes I_d)\mathbf{x} = \big((I_n \otimes \mathscr{A}) - c(L \otimes \mathscr{B}K)\big)\mathbf{e}. \end{aligned}$$

Let J be the Jordan normal form of L and let T satisfy $L = TJT^{-1}$. Recall that the first column of T is $\mathbb{1}_n$ and the first row of T^{-1} is w. We define the *transformed consensus error* $\tilde{\mathbf{e}} = (T^{-1} \otimes I_d)\mathbf{e} \in \mathbb{R}^{nd}$ and, noting $(T^{-1} \otimes I_d)^{-1} = (T \otimes I_d)$, we compute

$$\begin{aligned} \dot{\tilde{\mathbf{e}}} &= (T^{-1} \otimes I_d)\Big((I_n \otimes \mathscr{A}) - c(L \otimes \mathscr{B}K)\Big)(T \otimes I_d)\tilde{\mathbf{e}} \\ &= \Big((I_n \otimes \mathscr{A}) - c(J \otimes \mathscr{B}K)\Big)\tilde{\mathbf{e}}. \end{aligned} \tag{8.27}$$

The first d entries of the vector $\tilde{\mathbf{e}}(t)$ are identically zero at all times t, because one can show $\tilde{\mathbf{e}}_1(t) = (w^\mathsf{T} \otimes I_d)\mathbf{e}(t) = \mathbb{0}_d$. Next, since the Jordan normal form J is block diagonal, say with blocks $J_1, \dots, J_m$ (with $J_1 = 0$), we can write the dynamics (8.27) as decoupled equations. If J_i corresponds to a simple eigenvalue λ_i and is a one dimensional block, then we have

$$\dot{\tilde{\mathbf{e}}}_i = (\mathscr{A} - c\lambda_i \mathscr{B}K)\tilde{\mathbf{e}}_i.$$

One can show that, for arbitrary dimensional Jordan blocks corresponding to eigenvalues λ_i, $i \in \{2, \dots, n\}$, the asymptotic stability condition is that $\mathscr{A} - c\lambda_i\mathscr{B}K$ is Hurwitz. In other words, if each matrix is Hurwitz, then $\tilde{\mathbf{e}}(t)$ and $\mathbf{e}(t)$ vanish asymptotically so that $\mathbf{x}$ achieves asymptotic consensus. This concludes the proof of statement (ii). ■

8.5 Historical notes and further reading

The Kronecker formalism is related to the early work (Wu and Chua, 1995) and the textbook (Wu, 2007). Theorems 8.5 and 8.8 and the abbreviated treatment in Section 8.4.4 are due to (Li et al., 2010), see also (Xia and Scardovi, 2016, Theorem 1), (Li and Duan, 2014, Theorem 1). An early reference on the observability problem is (Tuna, 2012). A comprehensive treatment is in the text (Li and Duan, 2014).

Second-order Laplacian flows are widely studied. Early references are the works by Chow (1982) and Chow and Kokotović (1985) on slow coherency and area aggregation of power networks, modeled as first and second-order Laplacian flows; see also (Avramovic et al., 1980; Chow et al., 1984; Saksena et al., 1984) among others.

In the consensus literature, an early reference to second-order Laplacian flows is (Ren and Atkins, 2005). Relevant references include (Ren, 2008a,b; Zhu et al., 2009; Zhang and Tian, 2009; Yu et al., 2010); see also (Ren and Atkins, 2005; Ren, 2008b). We refer to (Zhu et al., 2009) for convergence results for general digraphs and gains with arbitrary signs, and to (Zhang and Tian, 2009) for the discrete-time setting.

8.6 Exercises

E8.1 **Properties of the Kronecker product.** Prove properties (8.10a)–(8.10d), (8.11) and (8.12) of the Kronecker product.

E8.2 **The vectorization operator, the Kronecker product, and linear equations in matrices.** Given a matrix $X \in \mathbb{R}^{m\times n}$, the *vectorization of* X is the vector of dimension mn obtained by stacking all columns of X, that is,

$$\operatorname{vec}(X) = [x_{11}, \dots, x_{m1}, x_{12}, \dots, x_{m2}, \dots, x_{1n}, \dots, x_{mn}]^\mathsf{T} \in \mathbb{R}^{mn}. \tag{E8.1}$$

Given matrices A, B, and C, show that

(i) the following equivalence between matrix equations hold:

$$AXB = C \qquad \Longleftrightarrow \qquad (B^\mathsf{T} \otimes A)\operatorname{vec}(X) = \operatorname{vec}(C); \tag{E8.2}$$

(ii) the equation $AXB = C$ has a unique solution in X if and only if both A and B are square and invertible, in which case the solution is

$$\operatorname{vec}(X) = ((B^\mathsf{T})^{-1} \otimes A^{-1})\operatorname{vec}(C);$$

(iii) equality (E8.2) implies the following equivalence for the so-called *generalized Sylvester equation*:

$$AX + YB = C \qquad \Longleftrightarrow \qquad (I \otimes A)\operatorname{vec}(X) + (B^\mathsf{T} \otimes I)\operatorname{vec}(Y) = \operatorname{vec}(C).$$

E8.3 **Second-order Laplacian matrices.** Given a Laplacian matrix L and non-negative coefficients $k_\mathrm{p}, k_\mathrm{d}, \gamma_\mathrm{p}, \gamma_\mathrm{d} \in \mathbb{R}$, define the *second-order Laplacian matrix* $\mathcal{L} \in \mathbb{R}^{2n\times 2n}$ by

$$\mathcal{L} = \begin{bmatrix} \mathbb{0}_{n\times n} & I_n \\ -k_\mathrm{p} I_n - \gamma_\mathrm{p} L & -k_\mathrm{d} I_n - \gamma_\mathrm{d} L \end{bmatrix}, \tag{E8.3}$$

and write the second-order Laplacian system (8.6) in first-order form as $\begin{bmatrix} \dot{q}(t) \\ \dot{v}(t) \end{bmatrix} = \mathcal{L}\begin{bmatrix} q(t) \\ v(t) \end{bmatrix}$. Show that

(i) the characteristic polynomial of $\mathcal{L}$ is

$$\det(\eta I_{2n} - \mathcal{L}) = \det\big(\eta^2 I_n + \eta(k_\mathrm{d} I_n + \gamma_\mathrm{d} L) + (k_\mathrm{p} I_n + \gamma_\mathrm{p} L)\big);$$

(ii) given the eigenvalues $\lambda_1, \dots, \lambda_n$, of L, the $2n$ eigenvalues $\eta_{1,+}, \eta_{1,-}, \dots, \eta_{n,+}, \eta_{n,-}$ of $\mathcal{L}$ are solutions to

$$\eta^2 + (k_\mathrm{d} + \gamma_\mathrm{d}\lambda_i)\eta + (k_\mathrm{p} + \gamma_\mathrm{p}\lambda_i) = 0, \quad i \in \{1, \dots, n\}, \tag{E8.4}$$

that is, $\eta_{1,\pm} = \frac{-k_\mathrm{d} \pm \sqrt{k_\mathrm{d}^2 - 4k_\mathrm{p}}}{2}$ corresponding to $\lambda_1 = 0$ and, for $i \in \{2, \dots, n\}$,

$$\eta_{i,\pm} = \frac{-(k_\mathrm{d} + \gamma_\mathrm{d}\lambda_i) \pm \sqrt{(k_\mathrm{d} + \gamma_\mathrm{d}\lambda_i)^2 - 4(k_\mathrm{p} + \gamma_\mathrm{p}\lambda_i)}}{2};$$

(iii) if the undirected graph associated to L is connected, $k_\mathrm{p} + \gamma_\mathrm{p} > 0$ and $k_\mathrm{d} + \gamma_\mathrm{d} > 0$, then each eigenvalue $\eta_{i,\pm}$, $i \in \{2, \dots, n\}$, has negative real part;

(iv) $\mathcal{L}$ is similar to the Kronecker product expression (8.13) in Theorem 8.2 with a permutation similarity transform (i.e., a simple reordering of rows and columns).

E8.4 **Eigenvectors of the second-order Laplacian matrix.** Consider a Laplacian matrix L, scalar coefficients $k_\mathrm{p}, k_\mathrm{d}, \gamma_\mathrm{p}, \gamma_\mathrm{d} \in \mathbb{R}$ and the induced second-order Laplacian matrix $\mathcal{L}$ (as in (E8.3)). Let $v_{\mathrm{l},i}$ and $v_{\mathrm{r},i}$ be the left and right eigenvectors of L corresponding to the eigenvalue λ_i, show that

(i) the right eigenvectors of $\mathcal{L}$ corresponding to the eigenvalues $\eta_{i,\pm}$ are

$$\begin{bmatrix} v_{\mathrm{r},i} \\ \eta_{i,\pm} v_{\mathrm{r},i} \end{bmatrix},$$

(ii) for $k_{\mathrm{p}} > 0$, the left eigenvectors of $\mathcal{L}$ corresponding to the eigenvalues $\eta_{i,\pm}$ are

$$\begin{bmatrix} v_{1,i} \\ \dfrac{-\eta_{i,\pm}}{k_{\mathrm{p}} + \gamma_{\mathrm{p}}\lambda_i} v_{1,i} \end{bmatrix}.$$

E8.5 **Synchronization of inductors/capacitors circuits.** Consider a circuit composed of n identical resonant inductor/-capacitor storage nodes (i.e., a parallel interconnection of a capacitor and an inductor) coupled through a connected and undirected graph whose edges are identical resistors; see Figure E8.1. The parameters ℓ, c, r take identical values on each inductor, capacitor and resistors, respectively.

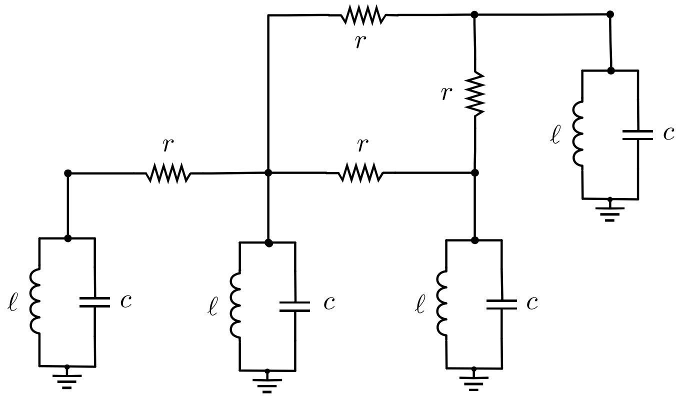


Figure E8.1: A circuit of identical inductor/capacitor storage nodes coupled through identical resistors.

(i) Write a state-space model of the resistively-coupled inductor/capacitor storage nodes in terms of the time constant $\tau = 1/rc$, the resonant frequency $\omega_0 = 1/\sqrt{\ell c}$, and the unweighted Laplacian matrix L of the resistive network.
(ii) Characterize the asymptotic behavior of this system.

E8.6 **Laplacian oscillators.** Given the Laplacian matrix $L = L^{\mathsf{T}} \in \mathbb{R}^{n\times n}$ of an undirected, weighted, and connected graph with edge weights a_{ij}, $i, j \in \{1, \dots, n\}$, define the *Laplacian oscillator flow* by

$$\ddot{x}(t) + Lx(t) = \mathbb{0}_n. \tag{E8.5}$$

Recall that this equation arises for example as the discretization of the wave equation in Example 7.1.4. This flow is written as first-order differential equation as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} \mathbb{0}_{n\times n} & I_n \\ -L & \mathbb{0}_{n\times n} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} =: \mathcal{L} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}.$$

(i) Write the second-order Laplacian flow in components.
(ii) Write the characteristic polynomial of the matrix $\mathcal{L}$ using only the determinant of an $n \times n$ matrix.
(iii) Given the eigenvalues $\lambda_1 = 0, \lambda_2, \dots, \lambda_n$ of L, show that the eigenvalues $\eta_1, \dots, \eta_{2n}$ of $\mathcal{L}$ satisfy

$$\eta_1 = \eta_2 = 0, \qquad \eta_{2i,2i-1} = \pm\sqrt{\lambda_i}\mathrm{i}, \quad \text{for } i \in \{2, \dots, n\},$$

where i is the imaginary unit.
(iv) Show that the solution is the superposition of a ramp signal and of $n-1$ harmonics, that is,

$$x(t) = \big(\operatorname{average}(x(0)) + \operatorname{average}(\dot{x}(0))t\big)\mathbb{1}_n + \sum_{i=2}^{n} a_i \sin(\sqrt{\lambda_i}t + \phi_i)v_i,$$

where $\{\mathbb{1}_n/\sqrt{n}, v_2, \dots, v_n\}$ are the orthonormal eigenvectors of L and where the amplitudes a_i and phases ϕ_i are determined by the initial conditions $\big(x(0), \dot{x}(0)\big)$.

E8.7 **The Cartesian product of graphs.** The *Cartesian product* $F \square H$ of two graph $F = (V_F, E_F)$ and $H = (V_H, E_H)$ is a graph with vertex set $V_F \times V_H$ and an edge between nodes (f_1, h_1) and (f_2, h_2) if and only if either ($f_1 = f_2$ and $\{h_1, h_2\} \in E_H$) or ($\{f_1, f_2\} \in E_F$ and $h_1 = h_2$). Clearly, if $|V_F| = m$ and $|V_F| = n$, then the number of edges in $F \square H$ is mn; moreover, the number of edges in $F \square H$ is $m|E_H| + n|E_F|$.

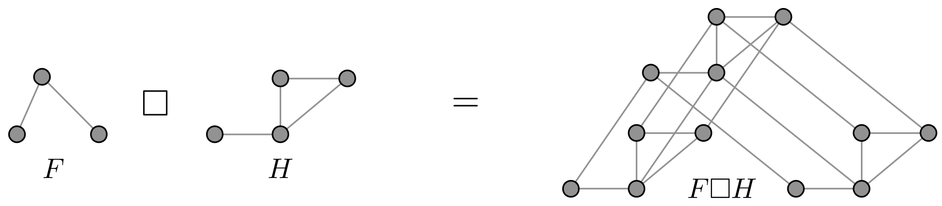


Figure E8.2: An example Cartesian product: The graph $F \square H$ is obtained from F by (i) replacing each of its vertices with a copy of H and (ii) each of its edges with $n = |V_H|$ edges connecting corresponding vertices of H in the two copies.

Let $L(F)$ and $L(G)$ denote the Laplacian matrices of F and G, respectively, with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_m$ and $\mu_1 \leq \cdots \leq \mu_n$, respectively. It is known that

$$L(F \square H) = L(F) \otimes I_n + I_m \otimes L(H). \tag{E8.6}$$

Show that

(i) if (λ_i, v_i) is an eigenpair for $L(F)$ and (μ_j, u_j) is an eigenpair for $L(H)$, for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, then $(\lambda_i + \mu_j, v_i \otimes u_j)$ is an eigenpair for $L(F \square H)$,

(ii) the second smallest eigenvalue of $L(F \square H)$ is $\min(\lambda_2, \mu_2)$,

(iii) $F \square H$ is connected if and only if both F and H are connected.

Hint: *The original work is by Fiedler (1973) and notable surveys include (Mohar, 1991; Merris, 1994).*