Paper Summary: On the evolution of random graphs

Original work by
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1 Introduction

Erdos and Renyi initiated the theory of random graphs in an earlier paper. In the paper they concentrated on the graphs being connected and the size of the components. In this paper they make an observation that there is a drastic change in the structure of the graph with respect to the number of vertices in the largest component. This change occurs when the number of edges are around half the number of vertices. First they define what it means to be a random graph and the various models that are considered and then the observe the evolution of the random graph as they transition over various phases.

2 Random Graph

Erdos and Renyi described what it means be a random graph. They concentrate their study on graphs that are not oriented, without parallel edges and slings. A random graph

is defined such that the edges are chosen randomly from all possible edges and that the probability of choossing each edge configuration is same and equal to $\frac{1}{\binom{\binom{n}{2}}{N}}$ for a graph with n vertices and N edges. There is also another view to this random graph process. At time t=1 we can unbiasely choose one out of all the possible $\binom{n}{2}$ edges. At time t=2 we can do the same on remaining edges and so on. The two definitions are equivalent and the second definition interprets the random graph as time. Thus we can look at how this step by step unravelling of the random graphs occur. The study of this process is what they call evolution of a graph.
The main aim of the paper is to show that random graphs exebit very clear cut features. The theorems proved can be categorized into two classes. The theorems of the first class deal with the appearance of certain subgraphs (e.g. trees, cycles of a given order etc.) or components, or other local structural properties, and show that for many types of local structural properties a definite threshold $A(n)$ can be given. The theorems of the second class are of similar type, only the properties considered are not of a local character, but global properties of the graph such as connectivity, total number of components, etc.

3 Evolution of random graph

The paper categorizes the evolution as we increase N(the number of edges) in terms of five clearly distinguishable phases.

3.1 Phase 1

Corresponds to the range $N = o(n)$. For this phase we can see that the components are trees. Trees of order k appear only when N reaches the magnitude of $n^{\frac{k-2}{k-1}}$. The authors make more observations to the distribution of the number of such trees of order k.

3.2 Phase 2

Corresponds to the range $N \approx cn$ with $0 < c < 1/2$. This phase contains cycles of any fixed order. The distribution of number of cycles of order k is a poisson distribution with mean dependent on k. Here either vertices belong to either trees or components containing exactly one cycle. Still most of the components are trees.

3.3 Phase 3

Corresponds to the range $N \approx cn$ with $c \geq 1/2$. The phase transition to this phase causes an abrupt change in the graph. While for the previous phase components were mostly trees. This phase has a greatest components of approximately $n^{\frac{2}{3}}$ vertices and has a rather complex structure. Except this giant component all other compenents are relatively small.

The evolution in this phase may be explained by the intuition that the small components (most of which are trees) melt, each after another, into the giant component, the smaller components having the larger chance of survival.

3.4 Phase 4

Corresponds to the range $N \approx cn \log n$ with $c \leq 1/2$. In this phase the graph surely becomes connected. There are only trees of small size outside the connected component here. Here the distribuition of the number of components follows a poisson distribution.

3.5 Phase 5

Corresponds to the range $N \approx n \log n\omega(n)$ where $w(n) \rightarrow +\inf$. In this range the whole graph is not only almost surely connected, but the orders of all points are almost surely asymptotically equal. Thus the graph becomes in this phase asymptotically regular.

4 Remarks

In short, the order of the largest component in the Erdos-Renyi random graph changes from logarithmic to sublinear and then to linear order in the number of vertices when the average vertex degree passes through one (for example from 0.99 to 1.01) as more edges are added. Erdos and Renyi described this phenomenon where rapid changes are observed to be one of the most striking facts concerning random graphs.