# Paper Summary: Weight Choosability of Graphs

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## 1 Introduction

Suppose the edges of a graph G are assigned 3-element lists of real weights. Is it possible to choose a weight for each edge from its list so that the sums of weights around adjacent vertices were different?
Let $S$ be a subset of the field of real numbers $\mathbb{R}$ and let $G$ be a simple graph. $G$ is weight colorable by $S$ if $\exists w : E \to S$ such that for any two adjacent vertices $u, v \in V(G)$, the sum of weights of the edges incident with $u$ is different than the sum of weights of the edges incident with $v$.
This paper explored the idea of proving weight coloring results using algebraic methods. They use the method of combinatorial nullstellensatz by Alon. The basic idea is as follows: associate a multivariable polynomial $P_G$ with a graph $G$, so that a non-zero substitution for variables of $P_G$ gives a desired weighting of $G$. Then by observing the exponents

of variables in the expansion of $P_G$ into a linear combination of monomials. If there is a non-vanishing monomial with the highest exponent of a variable less than 3, then by Combinatorial Nullstellensatz, $G$ is weight colorable by any set of three real weights.
They prove the statement that every graph without an isolated edge is 3-weight choosable for several classes of graphs, including cliques, complete bipartite graphs, and trees, by providing general recursive constructions preserving the desired algebraic properties.

## 2 Main Theorem

The main theorem used in nullstellensatz is as follows:

**Theorem 2.1.** *Let $F$ be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. Suppose the degree $deg(f)$ of $f$ is coefficient of $\sum_{i=1}^{n} t_i$, where each $t_i$ is a nonnegative integer, and suppose the coefficient of $\Pi_{i=1}^{n} {x_i}^{t_i}$ in $f$ is nonzero. Then, if $S_1, \ldots, S_n$ are subsets of $F$ with $|S_i| > t_i$ , there are $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ so that $f(s_1, \ldots, s_n) \neq 0$.*

We consider the following polynomial for weight colorable.
Let $E(G) = \{e_1, \ldots, e_m\}$ be the set of edges of a simple graph $G$. Let $x_i$ be variables assigned to the edges $e_i$ . For each vertex $u \in V$ , let $E_u$ be the set of edges incident to $u$, and let $X_u = \sum_{e_j \in E_u} x_j$ . Now fix any orientation of $G$ and define a polynomial $P_G$ in variables $x_1, \ldots, x_m$ by

$$P_G(x_1, \ldots, x_m) = \prod_{(u,v) \in E(G)} (X_u - X_v)$$

If polynomial is not equal to zero then there exists a valid weight coloring.
Suppose $M = c x_1^{k_1} \ldots x_m^{k_m}$ is a monomial in the expansion of $P$ with $c \neq 0$. Let $h(M)$ be the highest exponent of a variable in $M$. Define the monomial index of $P$, denoted by $mind(P)$, as the minimum of $h(M)$ taken over all non-vanishing monomials $M$ in $P$.

## 3 Permanent

Combinatorial nullstellensatz implies that if $mind(G) \leq k$ then $G$ is $(k + 1)$-weight choosable. Hence to prove that certain class of graphs are 3 weight colorable, we need to show that $mind(G) \leq 2$. To study the monomials in the expansion of $P_G$ we look at the permanent of matrices.
The permanent rank of a matrix $A$ is the size of a largest square submatrix with non-zero permanent. Let $A^{(k)} = [A, \ldots, A]$ be a matrix formed of $k$ copies of a matrix $A$. $pind(A)$ is the minimum $k$ for which $A(k)$ has the permanent rank equal to the size of $A$.

**Lemma 3.1.** *Let $A = (a_{ij})$ be a square matrix of size $m$ and finite permanent index. Let $P(x_1, \ldots, x_m) = \prod_{i=1}^{m} (a_{i1}x_1 + \ldots + a_{im}x_m)$. Then $mind(P) = pind(A)$.*

Hence we can use $pind(A)$ to estimate $mind(P)$ and hence the weight color.

**Lemma 3.2.** *Let $A$ and $L$ be square matrices of size $n$ such that each column of $L$ is a linear combination of columns of $A$. Let $n_j$ be the number of those columns of $L$ in which the $j^{th}$ column of $A$ appears with a non-zero coefficient. If $n_j \leq r$ and $per(L) \neq 0$, then $pind(A) \leq r$.*

## 4 Main Result

The main result of the paper is as follows:

**Theorem 4.1.** *Let $G = (V, E)$ be a simple graph with $mind(G) \leq 2$. Let $U$ be a nonempty subset of $V(G)$. Let $F$ be a graph obtained by adding two new vertices $u, v$ to $G$ and joining them to each vertex of $U$. Let $H$ be a graph obtained from $F$ by joining $u$ and $v$ by an edge. Then $mind(F) \leq 2$ and $mind(H) \leq 2$.*

The theorem allows for recursive constructions of many graphs with low monomial index.
From the main theorem the following observation follows: if $G \neq K_2$ is a clique, complete bipartite graph, or a tree, then $mind(G) \leq 2$. This proves that these classes of graphs are three weight colorable.
Other properties that follow are:
Every graph $G$ is an induced subgraph of a graph $H$ such that $mind(H) \leq 2$ and $\chi(H) \leq \chi(G) + 1$.
Every graph $G$ can be transformed into a graph $H$ with $mind(H) \leq 2$ by subdividing each edge of $G$ with at most three vertices. Every graph $G$ can be transformed into a graph $H$ with $mind(H) \leq 2$ by subdividing each edge of $G$ with at most three vertices.
These results bring the hope that the conjecture that every graph except $K_2$ is three weight colorable. The main question that still remains open is whether there is a finite bound on the monomial index for graphs.

## 5 Remarks

Along with the undirected case, the authors also studied this problem in the directed case. In the directed case they were able to prove that every digraph is 2-weight choosable. But the conjecture still remains open for every connected undirected graph.
The paper showed yet another novel application of the combinatorial nullstellensatz method.