Paper Summary: On the size of kakeya sets in finite fields

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1 Introduction

A Kakeya set is a subset of $\mathbb{F}^n$, where $\mathbb{F}$ is a finite field of q elements, that contains a line in every direction.
This problem follows from the study of the Kakeya needle problem(1917). The problem is as follows: What is the least area in the plane required to continuously rotate a needle of unit length and zero thickness around completely (i.e. by 360°)? Besicovitch in 1919 showed that there exists Kakeya sets $\mathbb{R}^2$ of arbitrarily small area. This paper studies the version in the setting of finite fields. It answers a question from Wolff who in 1999 stated the conjecture that the size of every Kakeya set is at least $C_n \cdot q^n$, where C_n depends only on n. The lower bound showed by Wolff was of the form $C_n \cdot q^{(n+2)/2}$. This bound was further improved both for general n and for specific small values of n (e.g for n = 3, 4). For general n, the previous best lower bound was $C_n \cdot q^{4n/7}$. Dvir proved the conjecture by using the polynmial method. We briefly summarize and comment on his proof.

2 Using the Polynomial Method

We first define the appropriate terms and lemmas that will be useful in the proof.

Definition 2.1. ((δ, γ)-Kakeya Set). A set $K \subset \mathbb{F}^n$ is a (δ, γ)-Kakeya Set if there exists a set $L \subset \mathbb{F}^n$ of size at least $\delta \cdot q^n$ such that for every $x \in L$ there is a line in direction x that intersects K in at least $\gamma \cdot q$ points.

We mention the schwartz zippel lemma that bounds the number of roots of a n variate degree d polynomial.

Lemma 2.2. *(Schwartz-Zippel) Let $f \in \mathbb{F}[x_1, \ldots, x_n]$ be a non zero polynomial with $deg(f) \leq d$. Then*

$$|\{x \in \mathbb{F}^n | f(x) = 0\}| \leq d \cdot q^{n-1}$$

We now state the theorem:

Theorem 2.3. *Let $K \subset \mathbb{F}^n$ be a (δ, γ)-Kakeya Set. Then*

$$|K| \geq \binom{d+n-1}{n-1}$$

where

$$d = \lfloor q \cdot \min\{\delta, \gamma\} \rfloor - 2$$

Suppose that the theorem is false then the number of monomials of degree d is larger than the size K. Therefore, there exists a homogenous n variate degree d polynomial g such that g is not the zero polynomial and $\forall x \in K, g(x) = 0$. Note that this is a consequence of solving a system of linear equations. We now show that g has too many zeros and hence must be identically zero.
Since K is a (δ, γ)-Kakeya set, there exists a set $L \subset \mathbb{F}^n$ of size at least $\delta \cdot q^n$ such that for every $y \in L$ there exists a line with direction y that intersects K in at least $\gamma \cdot q$ points. We now claim that g is zero on L as well. The main idea behind this claim is that if a degree d univariate polynomial has $(d+1)$ zeros then it must be identically zero on the entire space.
Since size of L is $\delta \cdot q^n$, we have that a δ fraction of points are roots. But as $\delta > d/q$ we have a contradiction from the schwartz zippel lemma.
The paper then presented a similar extension to show that $|K| \geq \binom{q+n-1}{n}$. This gives an answer to the question answered by Wolff.

3 Remarks

We observe that the polynomial method used mainly works along the lines of two observations:-

- A polynomial of degree d has at most d roots. (Schwartz zippel lemma is a generalization of this).

- Given a set S such that $|S| \leq d$, there exists a non zero univariate polynomial of degree at most d that has set S as its roots. (Used to show existence of a g that has the kakeya set as its roots).