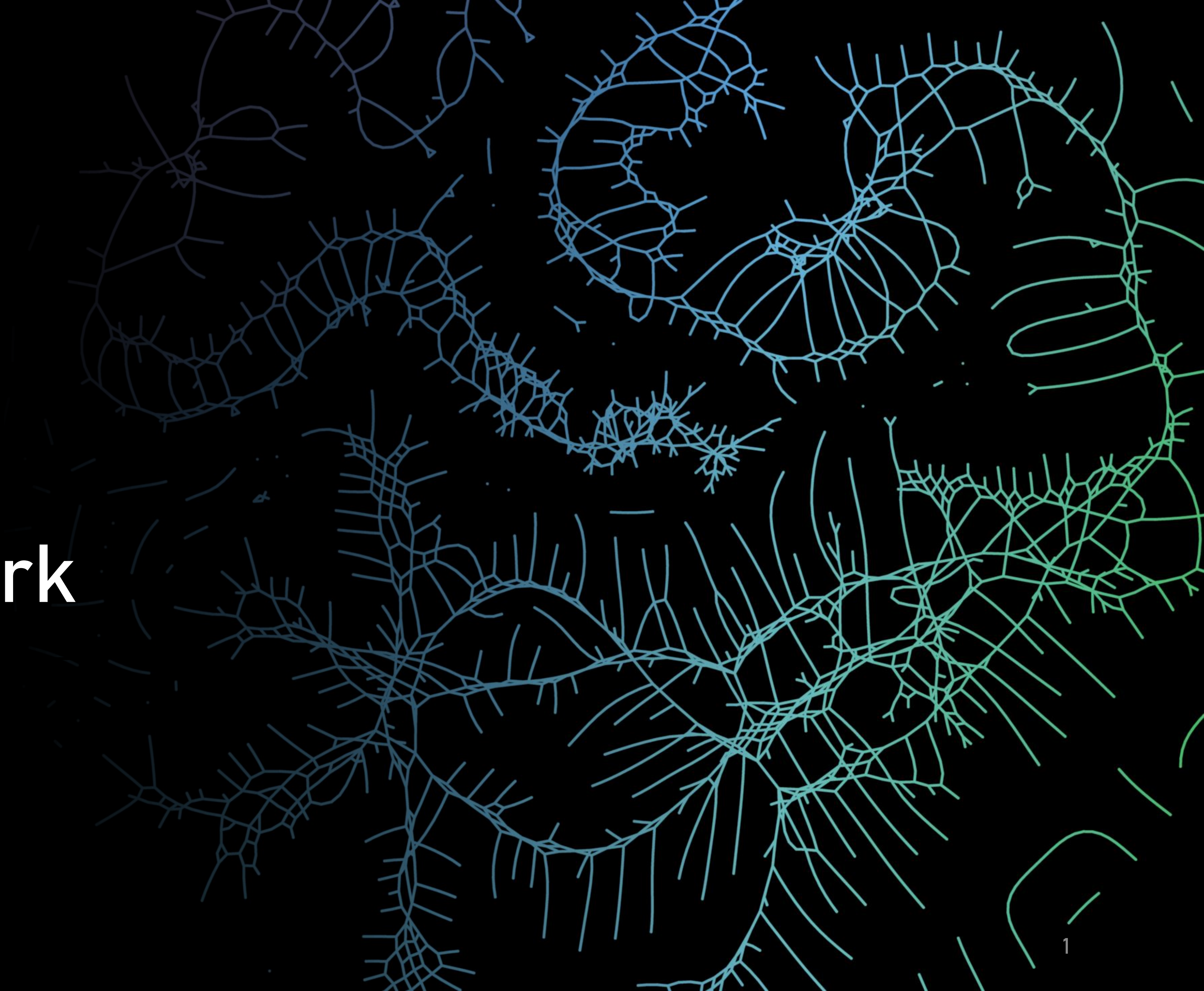

The Nuclear Decay Network

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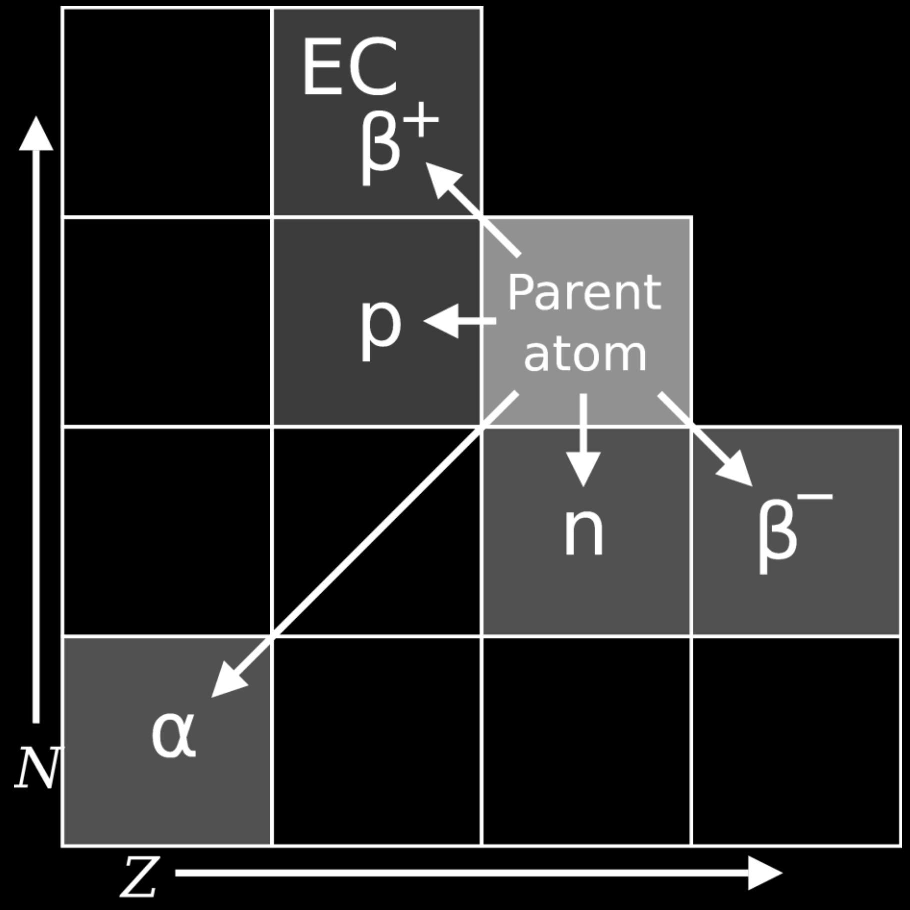
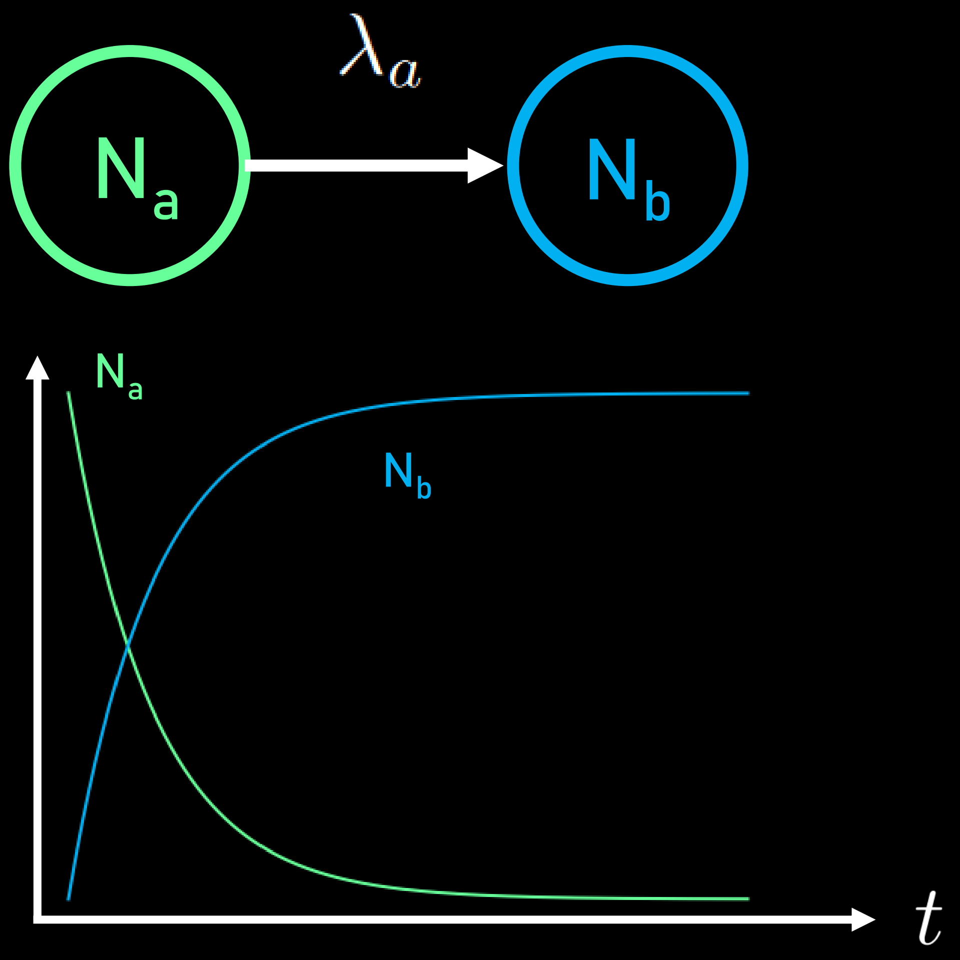
Created for Dr. Mailyan's PHY4035 course
at the Florida Institute of Technology

Review : Radioactive Decay

- Common Decay Modes:
  - Alpha
  - Beta (+/-)
  - Electron Capture
  - Gamma
- …There are many more (>50)!
- Each decay mode is associated with a direction (ΔN, ΔZ) on the nuclear chart
  - E.g. for alpha decay, (ΔN, ΔZ) = (-2, -2)



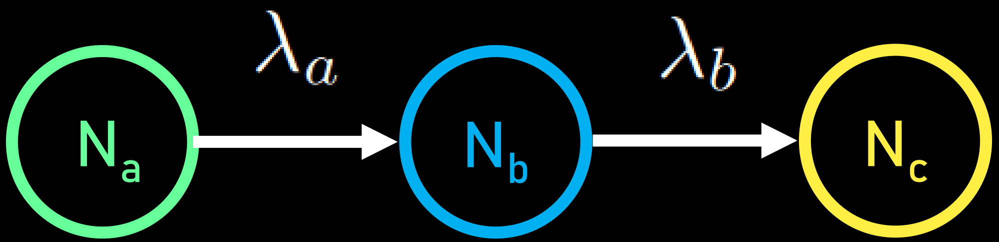
Two-Isotope Decay

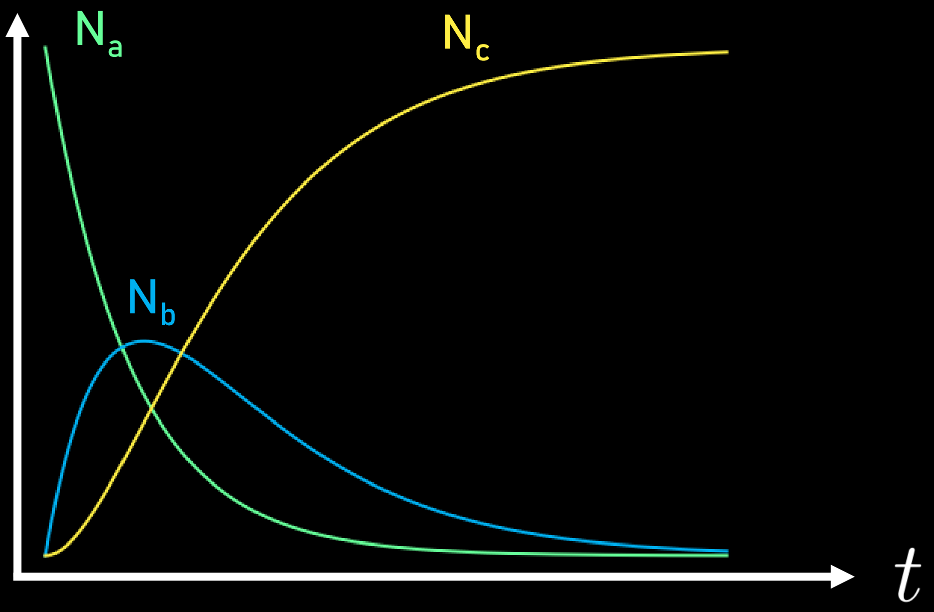


$$\frac{dN_a}{dt} = -\lambda_a N_a, \quad \boxed{\lambda_a = \frac{\ln 2}{t_{1/2}}}$$

$$\frac{dN_b}{dt} = \lambda_a N_a$$

Three-Isotope Decay





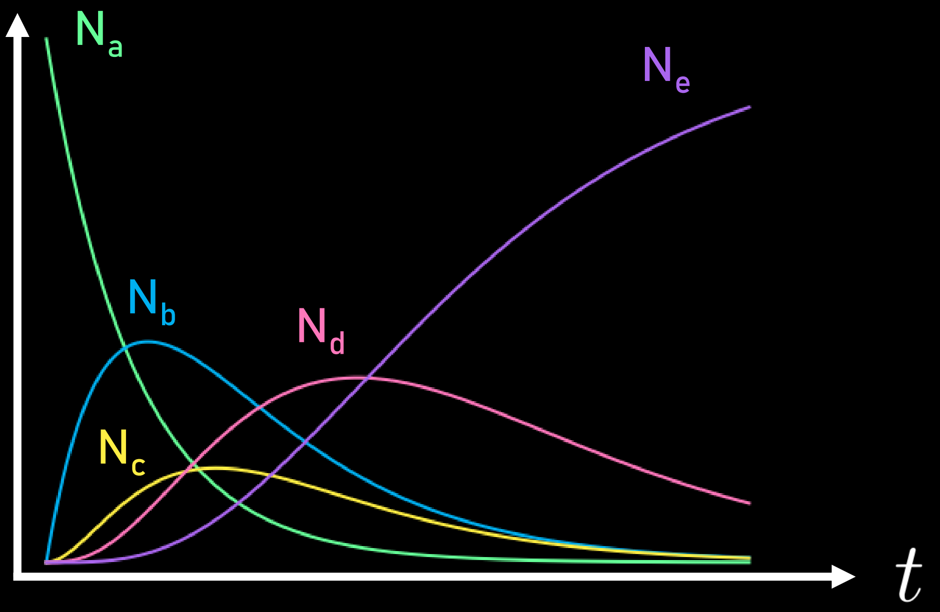
$$\frac{dN_a}{dt} = -\lambda_a N_a$$

$$\frac{dN_b}{dt} = \lambda_a N_a - \lambda_b N_b$$

$$\frac{dN_c}{dt} = \lambda_b N_b$$

N-Isotope Decay

$$N_a \xrightarrow{\lambda_a} N_b \xrightarrow{\lambda_b} N_c \xrightarrow{\lambda_c} N_d \xrightarrow{\lambda_d} N_e$$

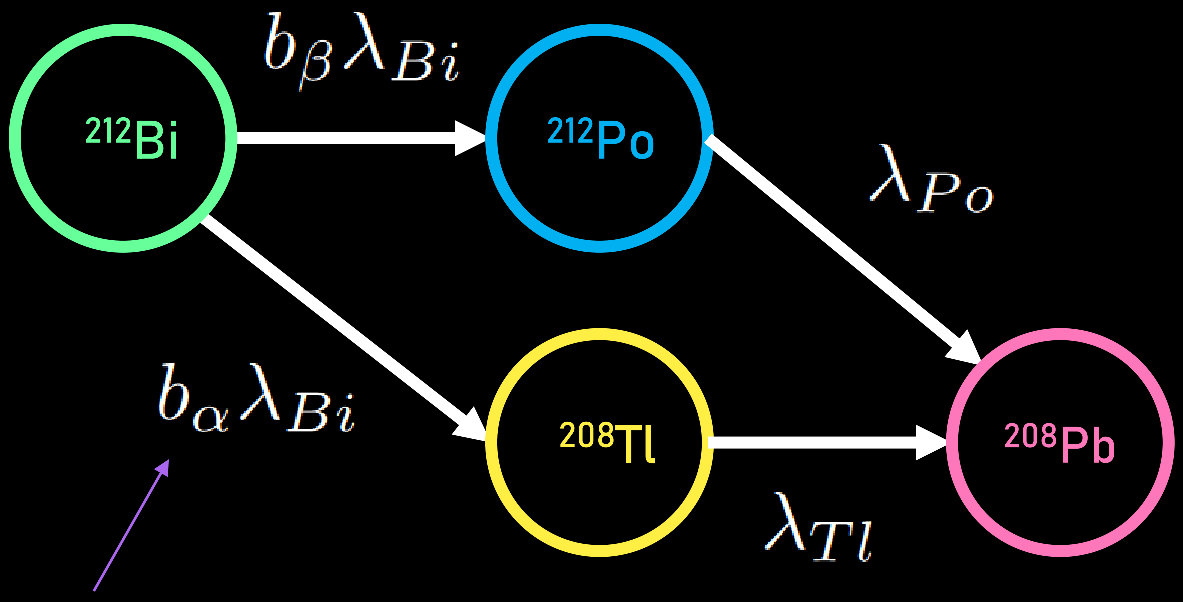


The Bateman Equation

$$N_n(t) = N_1(0) \times \left(\prod_{i=1}^{n-1} \lambda_i \right) \times \sum_{i=1}^{n} \frac{e^{-\lambda_i t}}{\prod\limits_{j=1, j \neq i}^{n} (\lambda_j - \lambda_i)}$$

Nontrivial Topologies

$$N_n(t) = N_1(0) \times \left(\prod_{i=1}^{n-1} \lambda_i \right) \times \sum_{i=1}^{n} \frac{e^{-\lambda_i t}}{\prod_{j=1, j\neq i}^{n} (\lambda_j - \lambda_i)}$$



"Branching Ratio"

$b_\alpha \approx 64\%$

$b_\beta \approx 36\%$

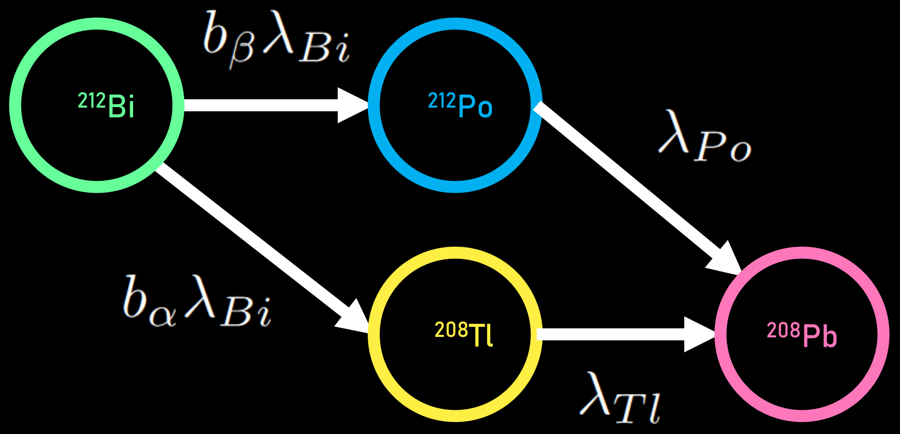
Example: a section of the ^{232}Th decay chain.

- The Bateman equation needs to be modified to describe this system.
- There are other issues…
    - Unable to account for branching without modification
    - Fails if any two decay constants are equal
    - Prone to numerical error
    - Only expresses solution for zero initial conditions
- J. Cetnar, *General solution of Bateman equations for nuclear transmutations*, Annals of Nuclear Energy, Volume 33, Issue 7

$$A(t) = \frac{N_n(t)}{N_1(0)} \lambda_n = \sum_{i=1}^{n} \lim_{\Delta_i \to 0} \sum_{m=0}^{\mu_i} \exp\left[-(\lambda_i + m\Delta_i)t\right] \left(\prod_{\substack{l=0 \\ l\neq m}}^{\mu_i} \frac{\lambda_i + l\Delta_i}{(l-m)\Delta_i} \right) \cdot \prod_{\substack{j=1 \\ j\neq i}}^{n} \prod_{k=0}^{\mu_j} \lim_{\Delta_j \to 0} \frac{\lambda_j + k\Delta_j}{\lambda_j + k\Delta_j - \lambda_i - m\Delta_i} = \sum_{i=1}^{n} \lambda_i^{\mu_i} \exp\left[-\lambda_i t\right] \cdot \lim_{\Delta_i \to 0} \Delta_i^{-\mu_i} \sum_{m=0}^{\mu_i} \exp\left[-m\Delta_i t\right] \cdot \left(\prod_{\substack{l=0 \\ l\neq m}}^{\mu_i} \frac{1}{(l-m)} \right) \prod_{\substack{j=1 \\ j\neq i}}^{n} \prod_{k=0}^{\mu_j} \lim_{\Delta_j \to 0} \frac{\lambda_j + k\Delta_j}{\lambda_j - \lambda_i + k\Delta_j - m\Delta_i}$$

Nontrivial Topologies

$$\frac{dN(Bi)}{dt} = -\lambda_{Bi}N(Bi)$$
$$\frac{dN(Po)}{dt} = b_\beta\lambda_{Bi}N(Bi) - \lambda_{Po}N(Po)$$
$$\frac{dN(Tl)}{dt} = b_\alpha\lambda_{Bi}N(Bi) - \lambda_{Tl}N(Tl)$$
$$\frac{dN(Pb)}{dt} = \lambda_{Po}N(Po) + \lambda_{Tl}N(Tl)$$



$$\underbrace{\frac{d}{dt}\begin{bmatrix} N(Bi) \\ N(Po) \\ N(Tl) \\ N(Pb) \end{bmatrix}}_{\vec{x}'} = \underbrace{\begin{bmatrix} \lambda_{Bi} & 0 & 0 & 0 \\ b_\beta\lambda_{Bi} & -\lambda_{Po} & 0 & 0 \\ b_\alpha\lambda_{Bi} & 0 & -\lambda_{Tl} & 0 \\ 0 & \lambda_{Po} & \lambda_{Tl} & 0 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} N(Bi) \\ N(Po) \\ N(Tl) \\ N(Pb) \end{bmatrix}}_{\vec{x}}$$

Solving the System of ODEs

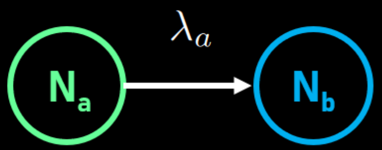
- Eigenvectors, like we did in DiffEQ?
  - Sounds good, but we run into issues with numerical precision.
  - $O(n^3)$ – Slow for large chains
- Numerical integration?
  - What timescale do we use?
  - Integration errors
- Solution: matrix exponential?
  - Still working out some overflow errors…

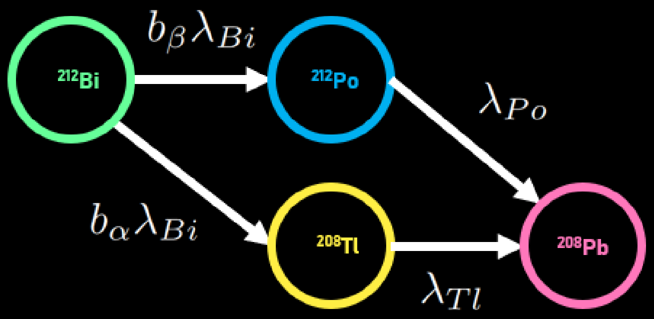
$$\vec{x}' = \mathbf{M}\vec{x}$$

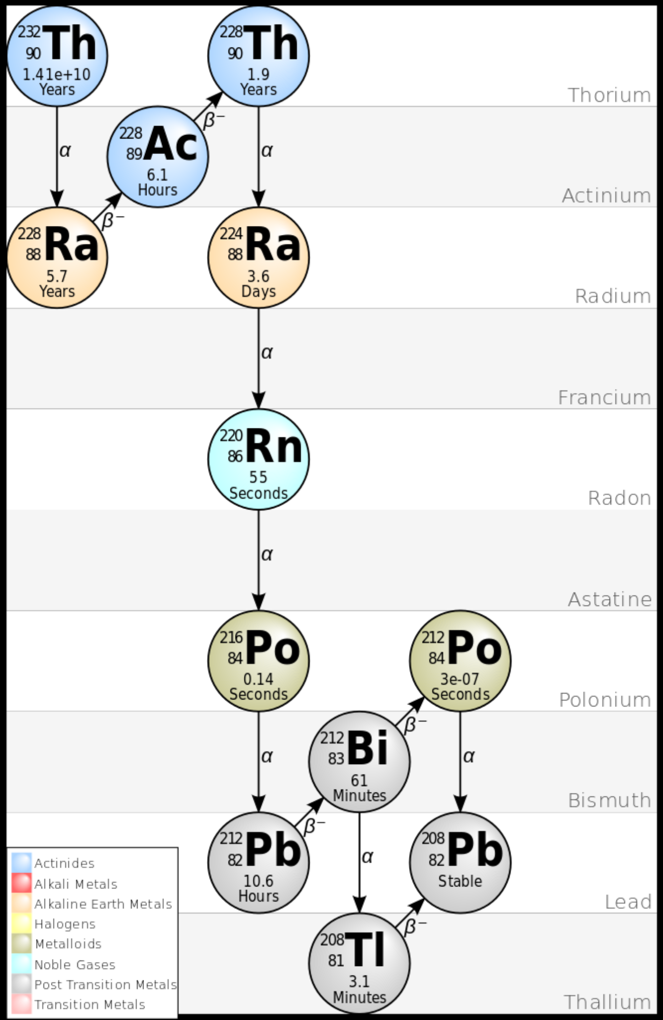
$$\implies \vec{x}(t) = \vec{x}(0)e^{\mathbf{M}t}$$

$$e^{\mathbf{M}} = \sum_{k=0}^{\infty} \frac{\mathbf{M}^k}{k!}$$

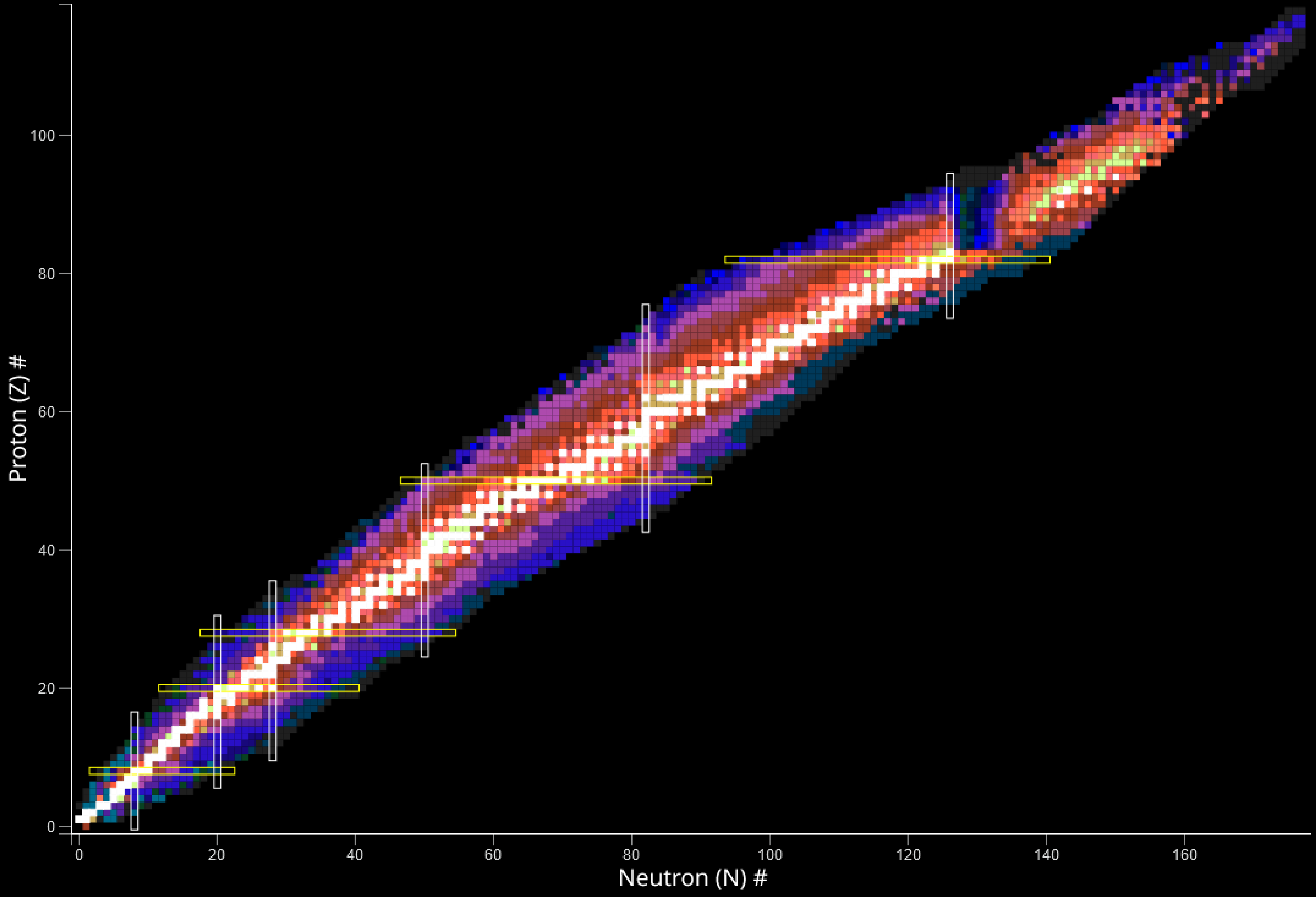
How big can we get?







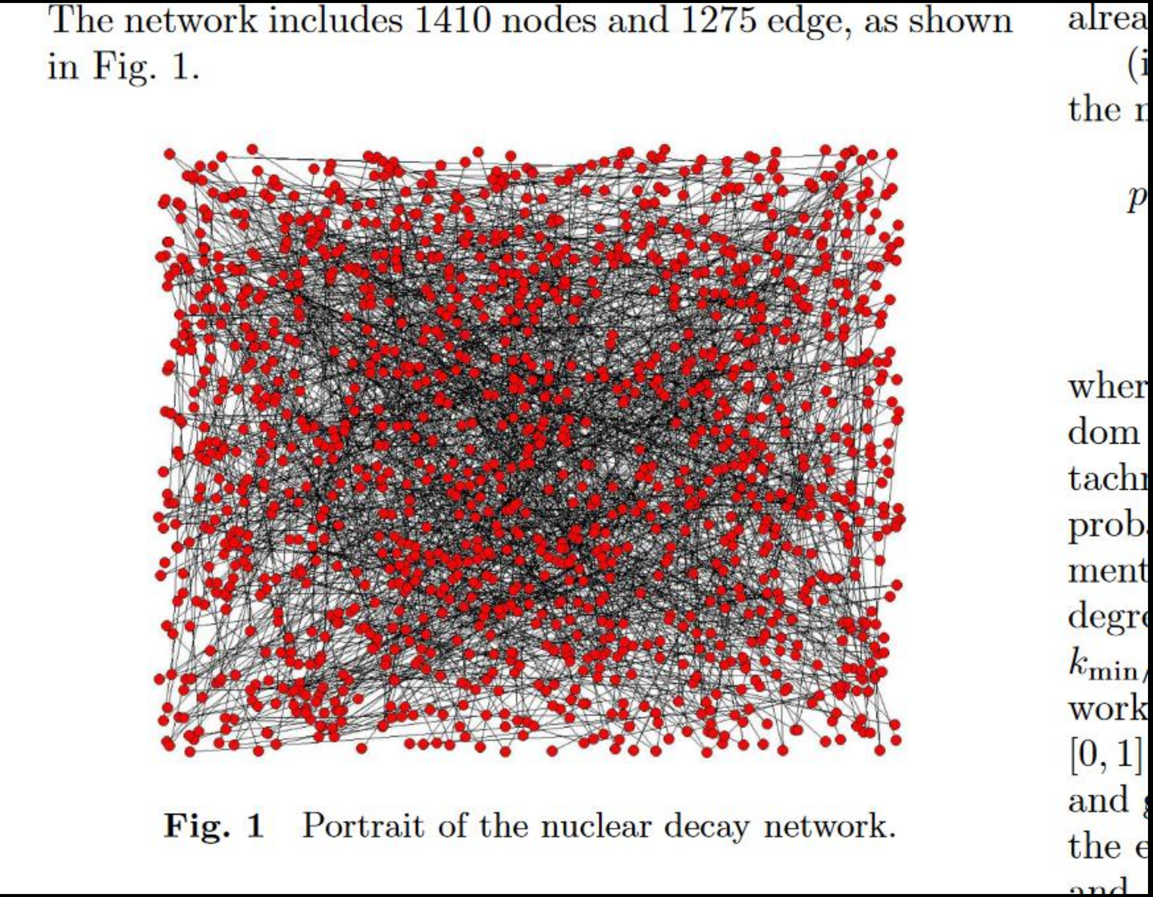
commons.wikimedia.org/wiki/File:Decay_Chain_Thorium.svg



Proton (Z) #
Neutron (N) #
0
20
40
60
80
100
120
140
160

Has anyone done this before?

- *Mapping Nuclear Decay to a Complex Network*, L. Yong et al., Communications in Theoretical Physics 57(3), 390-492 (2012).
    - Unsatisfactory analysis
    - Terrible visualization
- Let's do it right…

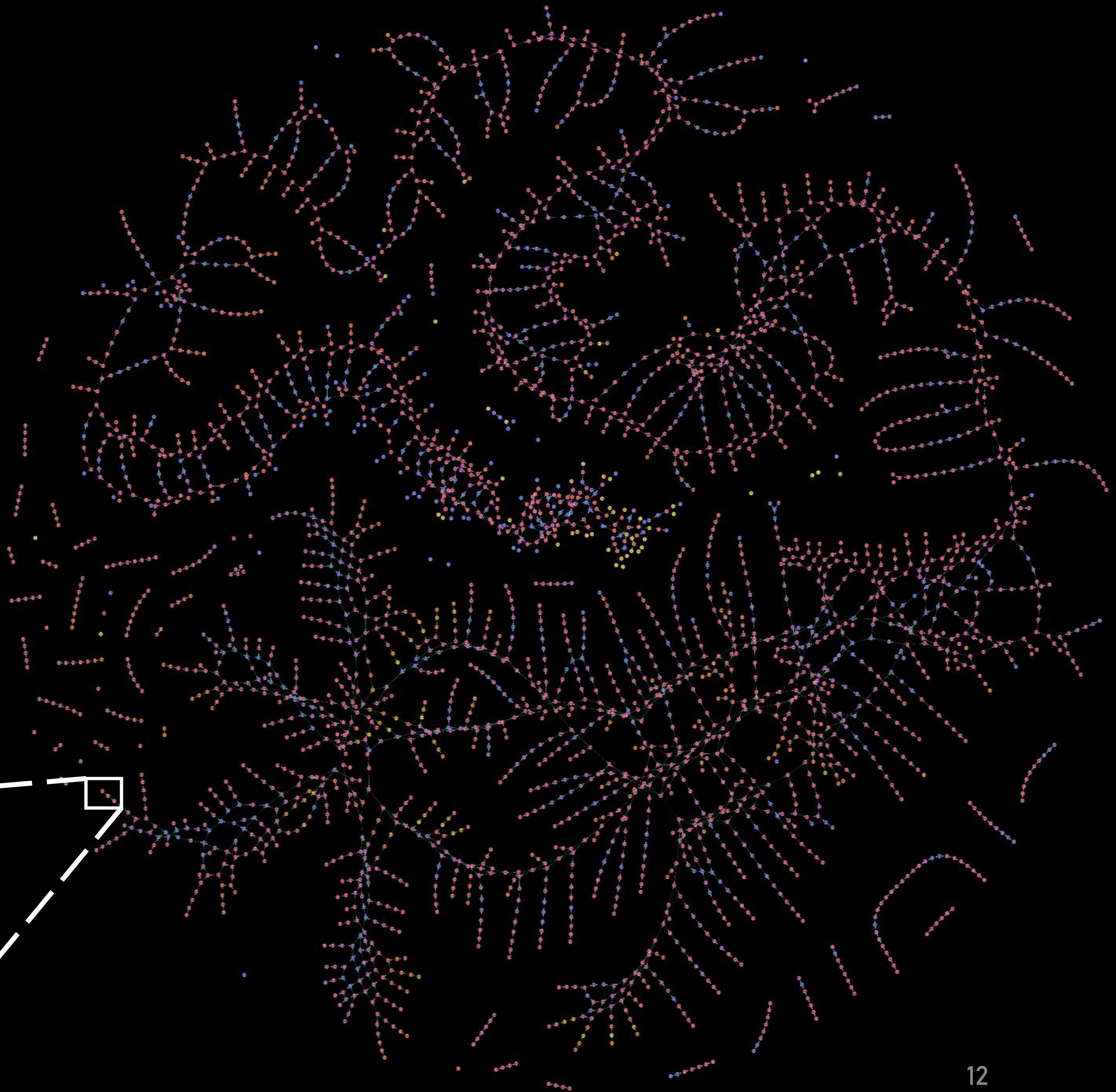


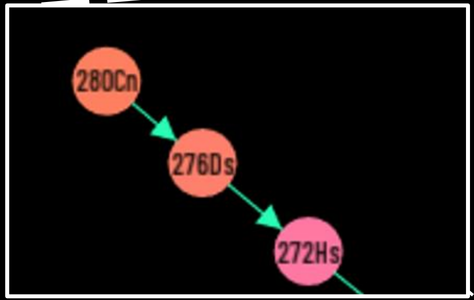
The network includes 1410 nodes and 1275 edge, as shown in Fig. 1.

Fig. 1 Portrait of the nuclear decay network.

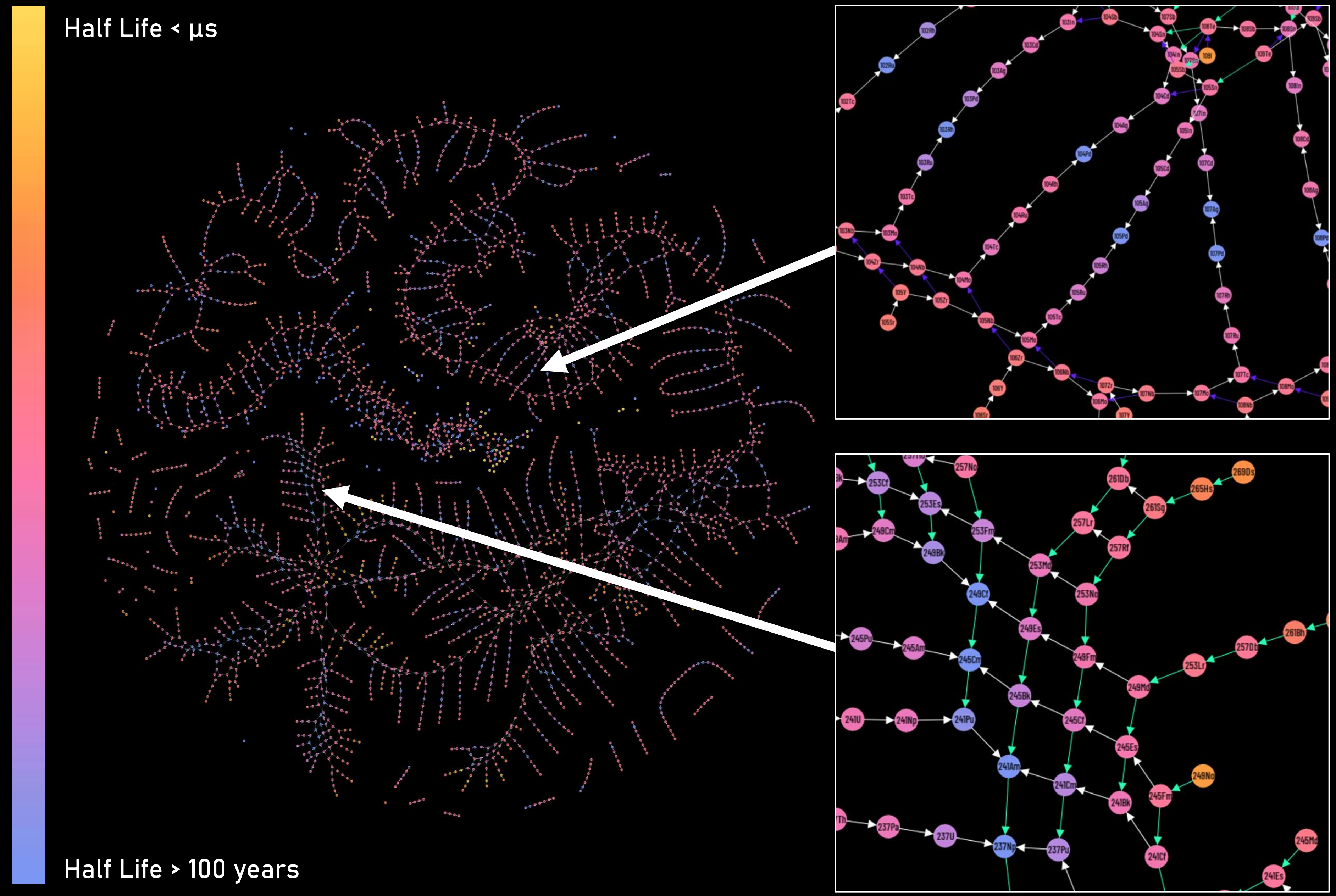
A Better Portrait

- (of the nuclear decay network)
- Created using the NUBASE2020 Evaluation of Nuclear Physics Properties, Chinese Phys. C 45, 030001 (2021)
- >3500 Nuclei (nodes)
- Network edges are weighted by decay constants.
- Edges with branching ratios < 0.1% are omitted from the visualization.
- Visualized/embedded using a homemade force-directed layout combined with simulated annealing.
- Full image is 8192 pixels on each side.






Less Massive
More Massive

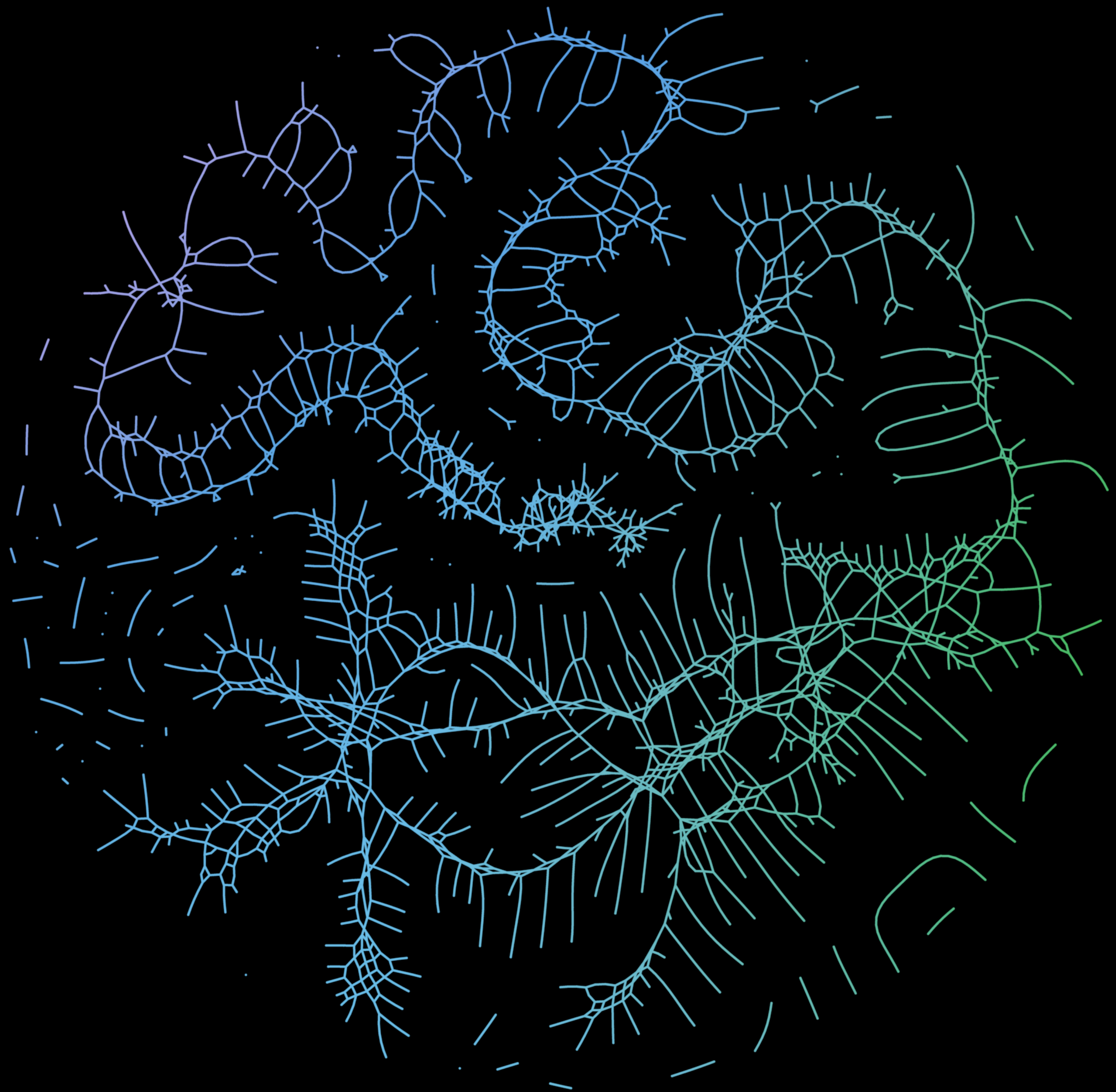


- Decays become slower as isotopes approach the valley of stability
- Unstable isotopes usually have more possible decay modes
- Driplines form the 'bones' of the network

- High-A elements exhibit an α-β mesh topology

Thanks!

…Questions?

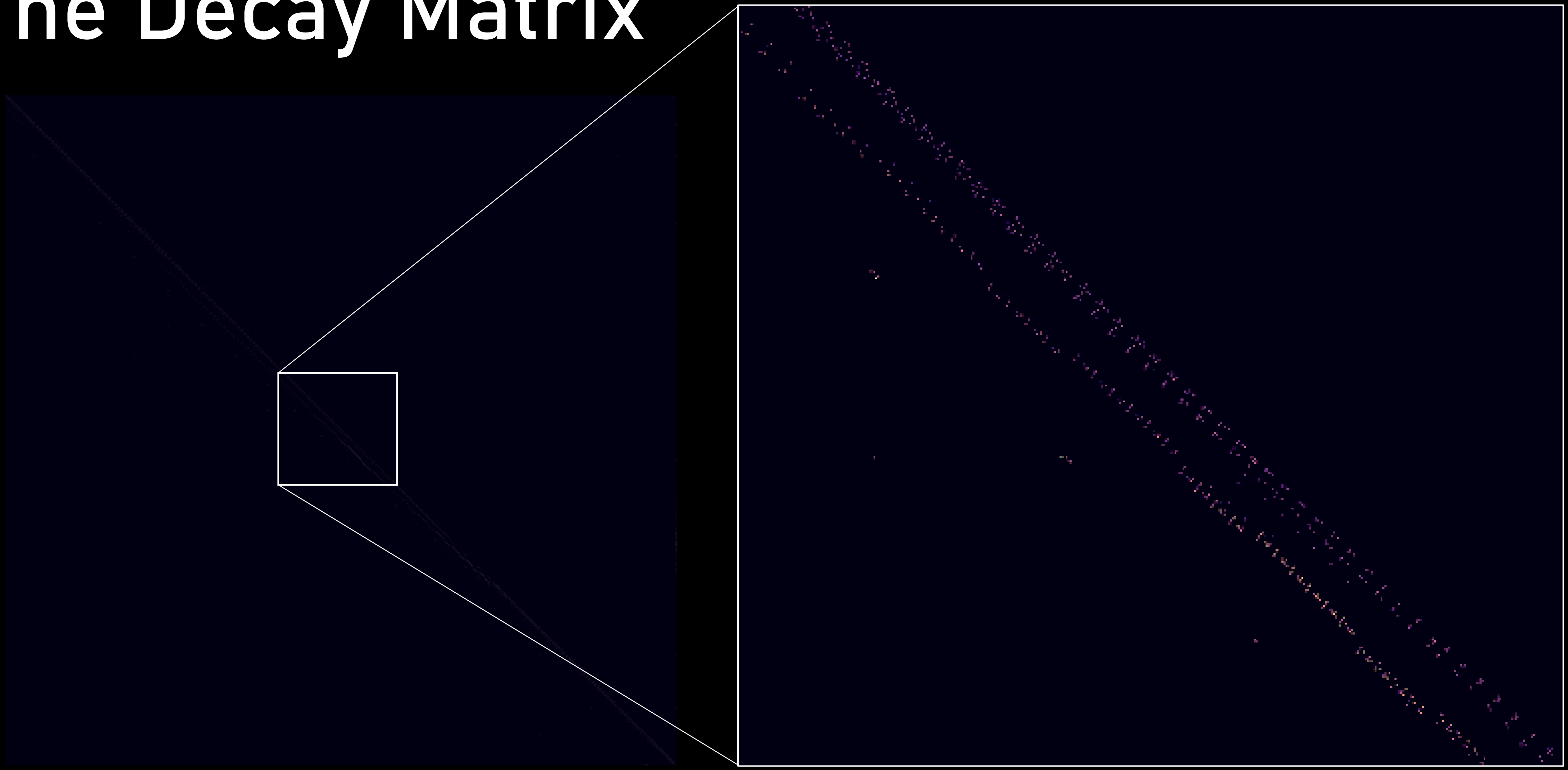


Backup Slides

Source Code

- https://github.com/Rachmanin0xFF/nuclear-decay-rates/

The Decay Matrix



line power-law instead of locating on it.

2 Construction Method of Nuclear Decay Network

According to the basic science data of international evaluation of nuclear structure and decay database, which is a part of National Scientific Data Sharing Project proposed by the Ministry of Science and Technology of the People's Republic of China in 2003, 1631 nuclear decay reactions are obtained. Since nuclear decay connects nuclei with different decay reaction, we can construct nuclear decay network by treating a reactant decay nucleus as a node and the decay reaction between two nuclei as a link. The network includes 1410 nodes and 1275 edge, as shown in Fig. 1.

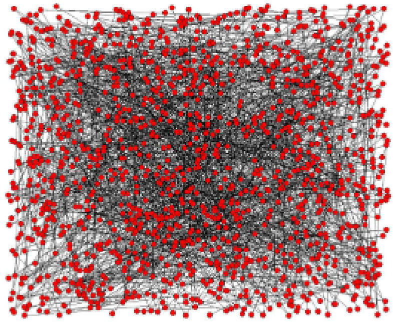


Fig. 1 Portrait of the nuclear decay network.

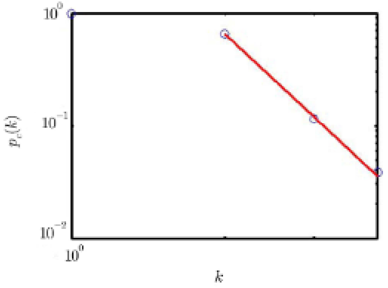


Fig. 2 Cumulative degree distribution of nuclear decay network.

The average degree of the network of nuclear decay is about 1.8. And the cumulative degree distribution of the network can be numerically calculated, as shown in Fig. 2. Although the degree of the network takes only a few values, $k = 1, 2, 3, 4$, the cumulative degree distribution ($p_c(k)$) still meets the typical power-law distribution, i.e., scale-free property, and the corresponding exponent is about 4.1. And similar with most of the actual network have been found, head point of the cumulative de-

3 Fitting of the LUHNM Theory

Large Unifying Hybrid Network model (LUHNM)[14−22] is a general network theory, which can be used to depict the diversity and complexity of the natural network. Three hybrid ratios (dr, fd, gr) are associated with the topological properties of the LUHNM. The corresponding generalization is as the following.

(i) Growth: Starting with isolated m_0 nodes, which are not linked to each other, at every time step, a new node with $m(\leq m_0)$ edges, which connect to m different nodes already present in the network, is added to the network.

(ii) Attachment: While choosing the nodes to which the new node i connects, attachment probability is[14]

$$p(k_i) \sim \frac{r}{d+r}\frac{(1-\mathrm{gr})k_i+\mathrm{gr}}{\sum_j[(1-\mathrm{gr})k_j+\mathrm{gr}]} + \frac{d}{d+r}\Big[(1-\mathrm{fd})\Big[\Big[\frac{k_i}{k_{\max}}\Big]\Big]+\mathrm{fd}\Big[\Big[\frac{k_{\min}}{k_i}\Big]\Big]\Big],$$

where k_i is degree of node i, r is time interval of the random attachment, d is time interval of the determinate attachment, gr controls proportion between general equal probability random attachment and preferential attachment of BA,[23] fd controls proportion between minimize-degree determinate attachment and maximum-degree one, $k_{\min}/k_{\max}$ is minimum/maximum degree in current network, $[[\bullet]]$ is round operator, $d \in [0,1]$, $r \in [0,1]$, $\mathrm{gr} \in [0,1]$, $\mathrm{fd} \in [0,1]$, $\mathrm{dr} = d + r$, and $\mathrm{dr} \in [0,1]$. When $\mathrm{dr} = 0$, and $\mathrm{gr} = 0$, LUHNM turns to BA[23] model. For unifying the expression, the parameters are rewritten as $\mathrm{dr} = 0/1$ and $\mathrm{gr} = 0/1$. When $\mathrm{dr} = 0/1$ and $\mathrm{gr} = 1/0$, LUHNM turns to ER[24] model.

Since LUHNM is well studied, we try to find its appropriate characterization related with nuclear decay network. After a lot of simulations, it is found that the suitable fitting parameters are as follows: $m = m_0 = 1$, $\mathrm{dr} = 4/1$, $\mathrm{fd} = 1/0$, $\mathrm{gr} = 1/1$ and corresponding LUHNM is a network of 1410 nodes and 2818 edges. Figure 3 shows the contrast of cumulative degree distribution between LUHNM theory[14−22] and nuclear decay network and they fit well.

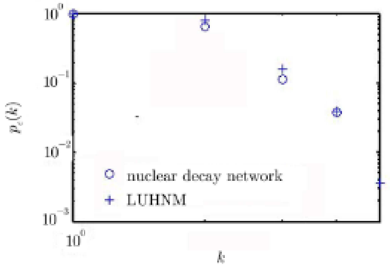


Fig. 3 Cumulative degree distribution comparison between LUHNM theory and simulation of nuclear decay network.