1. The following statements are equivalent –

(a) A is diagonalizable

(b) $\sum m_g(\lambda) = n$; where $m_g(\lambda)$ is the geometric multiplicity of λ & n is number of var

(c) $m_g(\lambda) = m_a(\lambda)$ for every λ; $m_a(\lambda)$ is the algebraic multiplicity.

Consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\therefore$ $\lambda = 1, 1$ or $m_a(\lambda) = 2$ — (1)

Also $(A - \lambda I)X = (A - I)X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} X = 0 \Rightarrow X = k\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ i.e. $m_g(\lambda) = 1$ — (2)

$\therefore$ $m_a(1) \neq m_g(1) \Rightarrow A$ is not diagonalizable.

Next we consider $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ so that $|A - \lambda I| = (1-\lambda)(\lambda - 1)(\lambda + 1) = 0 \Rightarrow \lambda = 1, 1, -1$

or $m_a(1) = 2$ and $m_a(-1) = 1$. — (3)

$(A - 1I)X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} X = 0 \Rightarrow X = k_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow m_g(1) = 2$ — (4)

$\therefore$ from (3) & (4) $m_a(1) = m_g(1) = 2$.

$\Rightarrow$ Given matrix is diagonalizable.

2. $\left[\begin{array}{cccc|c} 2 & -1 & 3 & 0 & 3 \\ 1 & 2 & -1 & -5 & 4 \\ 1 & 3 & -2 & -7 & 5 \end{array}\right] \xrightarrow{R_{21}} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -5 & 4 \\ 2 & -1 & 3 & 0 & 3 \\ 1 & 3 & -2 & -7 & 5 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -5 & 4 \\ 0 & -5 & 5 & 10 & -5 \\ 0 & 1 & -1 & -2 & 1 \end{array}\right] \xrightarrow{-R_2/5} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -5 & 4 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 1 & -1 & -2 & 1 \end{array}\right]$ *

* $\xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -5 & 4 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\rho(A) = \rho(A:B) = 2 < 4 = n \Rightarrow \infty$ many solutions

Put $x_3 = k_1$ and $x_4 = k_2$

we obtain $x_2 = 1 + x_3 + 2x_4 = 1 + k_1 + 2k_2$ and $x_1 = 2 - k_1 + k_2$

3. Put $x = e^z$ to obtain $[D(D-1) - D - 3]y = ze^{2z}$ or $[(D-3)(D+1)]y = ze^{2z}$

$\therefore$ $y_{CF} = c_1 e^{3z} + c_2 e^{-z} = c_1 x^3 + c_2/x$

$y_{P.I} = e^{2z}\dfrac{1}{D^2 + 2D - 3} z = -\frac{1}{3}e^{2z}\left[1 - \left(\frac{2D}{3} + \frac{D^2}{3}\right)\right]^{-1} z = -\frac{1}{3}e^{2z}\left[1 + \left(\frac{2D}{3} + \frac{D^2}{3}\right) + \cdots\right] z$

$= -\frac{1}{3}e^{2z}\left(z + 2/3\right) = -\frac{x^2}{3}\left(\log x + 2/3\right)$.

4. $P.I = -y_1\int \dfrac{y_2 X}{W[y_1, y_2]}dx + y_2\int \dfrac{y_1 X}{W[y_1, y_2]}dx = -2\cos 2x\int \sec 2x \tan 2x\, dx + 2\sin 2x \int \sec 2x\, dx$

$= -1 + \sin 2x \log(\sec 2x + \tan 2x)$.

$\left(W[y_1, y_2] = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\right)$

5. Put $y = \sum_{n=0}^{\infty} a_n x^n$ into $(1-x^2)y'' - xy' + a^2 y = 0$ to obtain

$\sum\left[(n+2)(n+1)a_{n+2} - n(n-1)a_n - na_n + a^2 a_n\right] = 0$ so that (**)

$a_{n+2} = \dfrac{n^2 - a^2}{(n+2)(n+1)} a_n$; $n = 0, 1, \ldots$

$n = 0$; $a_2 = \dfrac{-a^2}{2 \cdot 1} a_0$ $n = 1$; $a_3 = \dfrac{1 - a^2}{3 \cdot 2} a_1$

$n = 2$; $a_4 = -\dfrac{(2^2 - a^2)a^2 a_0}{\lfloor 4}$ $n = 3$; $a_5 = \dfrac{(3^2 - a^2)(1^2 - a^2)}{\lfloor 5} a_1$

$\vdots$ $\vdots$

—x—x—x— (**)

$\therefore$ $y = a_0\left[1 - \dfrac{a^2}{\lfloor 2}x^2 - \dfrac{(2^2 - a^2)a^2}{\lfloor 4}x^4 \ldots\right]$

$+ a_1\left[x + \dfrac{1 - a^2}{\lfloor 3}x^3 + \dfrac{(3^2 - a^2)(1 - a^2)}{\lfloor 5}x^5 + \ldots\right]$

$= a_0 y_1 + a_1 y_2$

clearly y_1 & y_2 are L.I