DELHI TECHNOLOGICAL UNIVERSITY
Applied Mathematics
Assignment - 4

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Q1 a), b) $\int_0^\infty e^{-st}(at^2+bt+c)\,dt$

$= \int_0^\infty e^{-st}at^2\,dt + \int_0^\infty e^{-st}bt\,dt + \int_0^\infty ce^{-st}\,dt$

$\int_0^\infty ae^{-st}t^2\,dt = a\int_0^\infty e^{-st}t^2\,dt$ — ①

$= a\left[t^2\frac{e^{-st}}{-s} - 2\left(\frac{te^{-st}}{s^2} - \frac{e^{-st}}{-s^3}\right)\right]_0^\infty$

$= a\left[0-2(0-0)-\left(0-2\left(0+\frac{1}{s^3}\right)\right)\right] = a\left[\frac{2}{s^3}\right] = \frac{2a}{s^3}$ — ①

$\int_0^\infty be^{-st}t\,dt = b\int_0^\infty e^{-st}t\,dt = b\left[t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^\infty = \frac{b}{s^2}$ — ②

$\int_0^\infty ce^{-st}\,dt = c\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{c}{s}$ — ③

Ans = ① + ② + ③ $= \frac{2a}{s^3} + \frac{b}{s^2} + \frac{c}{s}$

c) $\int e^{-st}\cos(at+b)\,dt = \cos(at+b)\frac{e^{-st}}{-s} + \int\left(a\sin(at+b)\frac{e^{-st}}{-s}\right)dt$

$I = \cos(at+b)\frac{e^{-st}}{-s} + a\left(\sin(at+b)\frac{e^{-st}}{s^2} - \int a\cos(at+b)\frac{e^{-st}}{s^2}\right)dt$

$I = \cos(at+b)\frac{e^{-st}}{-s} + a\sin(at+b)\frac{e^{-st}}{s^2} - \frac{a^2 I}{s^2}$

$$I\left(1+\frac{a^2}{s^2}\right) = \cos(at+b)\frac{e^{-st}}{-s} + a\sin(at+b)\frac{e^{-st}}{s^2}$$

$$I = \left[\frac{-s\cos(at+b)e^{-st} + a\sin(at+b)e^{-st}}{(s^2+a^2)}\right]_0^\infty$$

$$= 0+0-\left(\frac{-s\cos b+a\sin b}{s^2+a^2}\right) = \frac{s\cos b - a\sin b}{s^2+a^2}$$

d) $$\int_0^\infty e^{-st}te^t\,dt = \int_0^\infty te^{(1-s)t}\,dt$$

$$= \left[\frac{te^{(1-s)t}}{(1-s)} - \frac{e^{(1-s)t}}{(1-s)^2}\right]_0^\infty = 0+0-\left(0-\frac{1}{(1-s)^2}\right) = \frac{1}{(1-}$$

e) $$\int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^\pi 0\,dt + \int_\pi^\infty e^{-st}\sin t\,dt.$$

$$= 0 + I$$

$$I = \int e^{-st}\sin t\,dt = \sin t\frac{e^{-st}}{-s} - \int \cos t\frac{e^{-st}}{-s}\,dt$$

$$= \sin t\frac{e^{-st}}{-s} - \left(\cos t\frac{e^{-st}}{s^2} + \int \sin t\frac{e^{-st}}{s^2}\,dt\right)$$

$$= \sin t\frac{e^{-st}}{-s} - \cos t\frac{e^{-st}}{s^2} - \frac{I}{s^2}$$

$$[s^2+1]I = -s\sin t\,e^{-st} - \cos e^{-st}$$

$$I = \left[-\left(\frac{s\sin t\,e^{-st} + \cos t\,e^{-st}}{s^2+1}\right)\right]$$

$$= \frac{-0-0+(0+\cos\pi\,e^{-s\pi})}{s^2+1} = \frac{-e^{-s\pi}}{s^2+1}$$

2a) $L(t\sin 4t)$

$= (-1)^1 \frac{d}{ds}\left(L(\sin 4t)\right)$

$= (-1)\frac{d}{ds}\left(\frac{4}{s^2+16}\right)$

$= (-1)\left(\frac{-4.2s}{(s^2+16)^2}\right) = \frac{8s}{(s^2+16)^2}$

b). $L(t^2\cos 3t) = (-1)^2 \frac{d^2}{ds^2} L(\cos 3t) = \frac{d^2}{ds^2}\left(\frac{s}{s^2+9}\right)$

$= \frac{d}{ds}\left(\frac{(s^2+9) - s(2s)}{(s^2+9)^2}\right) = \frac{d}{ds}\left(\frac{s^2+9-2s^2}{(s^2+9)^2}\right)$

$= \frac{d}{ds}\left(\frac{9-s^2}{(s^2+9)^2}\right) = \frac{d}{ds}\left(\frac{9}{(s^2+9)^2}\right) - \frac{d}{ds}\left(\frac{s^2}{(s^2+9)^2}\right)$

$= \frac{9\times -2\times 2s}{(s^2+9)^3} - \left(\frac{(s^2+9)^2 2s - s^2(2(s^2+9))\times 2s}{(s^2+9)^4}\right)$

$= \frac{36s}{(s^2+9)^3} - \left(\frac{(s^2+9)^2 \times 2s - 4s^3(s^2+9)}{(s^2+9)^4}\right)$

$= \frac{36s}{(s^2+9)^3} - \left(\frac{\cancel{(s^2+9)}\left[(s^2+9)\times 2s - 4s^3\right]}{(s^2+9)^{\cancel{4}\,3}}\right)$

$= \frac{36s}{(s^2+9)^3} - \left[\frac{-2s^3+18s}{(s^2+9)^3}\right] = \cancel{\frac{36s}{(s^2+9)^3} + \left[2s(s^2\right.}$

$= \frac{36s + 2s^2 - 18s}{(s^2+9)^3} = \frac{2s^3+18s}{(s^2+9)^3} = \frac{2s\cancel{(s^2+9)}}{(s^2+9)^{\cancel{3}\,2}} = \frac{2s}{(s^2+9)^2}$

c). $L(t^2e^{-2t}) = (-1)^2 \frac{d}{ds^2} L(e^{-2t}) = \frac{d}{ds^2} \frac{1}{(s+2)}$

$= \frac{d}{ds} \frac{-1}{(s+2)^2} = \frac{2}{(s+2)^3}$

Q3. a) $L^{-1}\left(\frac{3}{s-5}\right) = 3\, L^{-1}\left(\frac{1}{s-5}\right) = 3 \times e^{5t}$

b). $L^{-1}\left(\frac{\pi}{s^2+\pi^2}\right) = \sin \pi t$

c) $\frac{s+3}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$

$(s+3) = A(s+2) + B(s-1)$

Put $s=-2$
$1 = B(-3),\ B = -\frac{1}{3}$

Put $s = 1$
$4 = 3A,\ A = \frac{4}{3}$

$L^{-1}\left(\frac{4}{3(s-1)} - \frac{1}{3(s+2)}\right) = \frac{4}{3}e^{t} - \frac{1}{3}e^{-2t}$

Q4. $\int_0^\infty e^{-st} f(t)\, dt.$

$= \int_0^2 e^{-st} k\, dt + \int_2^4 0\, dt + \int_4^\infty e^{-st} k\, dt$

$= \frac{k\left[e^{-st}\right]_0^2}{-s} + 0 + \frac{k\left[e^{-st}\right]_4^\infty}{-s} = \frac{k}{-s}\left[e^{-2s}-1\right] + \frac{k}{-s}\left[0-e^{-4s}\right]$

$= \frac{ke^{-4s}}{s} - \frac{ke^{-2s}}{s} + \frac{k}{s}$

Q5 $\int_0^\infty e^{-st} f(t)\, dt = \int_0^3 0\, dt + \int_3^\infty e^{-st} (t-3)^2\, dt$

$= (t-3)^2 \frac{e^{-st}}{-s} - \int 2(t-3)\frac{e^{-st}}{-s}\, dt$

$$= (t-3)^2 \frac{e^{-st}}{-s} - 2\left((t-3)\frac{e^{-st}}{s^2} - \int \frac{e^{-st}}{s^2} dt \right)$$ (3.)

$$= \left[(t-3)^2 \frac{e^{-st}}{-s} - 2(t-3)\frac{e^{-st}}{s^2} \mp \frac{2e^{-st}}{s^3} \right]_3^{\infty}$$

$$= 0 - 0 - 0 - 0 + 0 + \frac{2e^{-3s}}{27} = \frac{2e^{-3s}}{27} \text{ (Ans)}$$

Q6.a). $y'' + 2y' - 3y = 3$ $\quad y(0) = 4 \quad y'(0) = 7$

$$L(y'' + 2y' - 3y) = 3$$

$$s^2 L(y) - s(4) + 1(-7)] + 2(sLy - 4) - 3Ly = 3$$

$$Ly(s^2 + 2s - 3) = \frac{18 + 4s}{s}$$

$$Ly = \frac{18+4s}{s^2 - 3s - s - 3} = \frac{18+4s}{(s+3)(s-1)}$$

$$\frac{18+4s}{(s+3)(s-1)} = \frac{A}{(s+3)} + \frac{B}{(s-1)}$$

$$(18+4s) = A(s-1) + B(s+3)$$

$s = 1$

$22 = 4B, \quad B = 11/2$

$s = -3, \quad 6 = -4A, \quad A = -3/2.$

$$Ly = \frac{-3}{2(s+3)} + \frac{11}{2(s-1)}$$

Taking Invers Laplace transformation

$$y = -\frac{3}{2}e^{-3t} + \frac{11}{2}e^t$$

b. $y'' - 5y' + 4y = e^{2t}$ $\quad f(0) = \frac{19}{2}, \quad y'(0) = 8/3$

Taking Laplace Transformation.

$$(s^2 Ly - s f(0) + f'(0)) - 5(sLy - f(0)) + 4Ly = \frac{1}{s-2}$$

$$Ly(s^2 - 5s + 4) - \frac{19}{2}s + \frac{8}{3} + \frac{95}{2} = \frac{1}{s-2}$$

$$Ly(s^2 - 5s + 4) = \frac{1}{s-2} + \frac{19s}{2} - \frac{301}{6}$$

$$Ly = \frac{1}{(s-2)(s-4)(s-1)} + \frac{19s}{2(s-4)(s-1)} - \frac{301}{6(s-4)(s-1)}$$

$$Ly = \frac{1}{2}\frac{1}{(s-2)} + \frac{1}{6(s-4)} + \frac{1}{3(s-1)} + \frac{19}{2}\left(\frac{4}{3(s-4)} - \frac{1}{3(s-1)}\right)$$

$$- \frac{301}{6}\left(\frac{1}{3(s-4)} - \frac{1}{3(s-1)}\right)\Big)\Big)$$

Taking inverse Laplace Transformations

$$y = \frac{1}{2}e^{2t} + \frac{1}{6}e^{4t} + \frac{1}{3}e^{t} + \frac{19}{2} \times \frac{4}{3}^{2} e^{4t} - \frac{19}{2}\times\frac{1}{3}e^{t} - \frac{301}{18}e^{4t} + \frac{301}{18}e^{t}$$

$$= \frac{125}{9}e^{t} + \frac{1}{2}e^{2t} - \frac{35}{9}e^{4t} \text{ (Ans).}$$

c) $L\left(y' + 3y + 2\int_0^t (y(\tau)d\tau\right) = L(t)$

$$sLy - y(0) + 3Ly + 2\int_0^\infty e^{-st}\int_0^t y(\tau)d\tau = \frac{1}{s^2}$$

$$sLy + 3Ly + 2\left[\int_0^t y(\tau)\,d\tau\,\frac{e^{-st}}{-s}\right]_0^\infty - 2\int_0^\infty y(\tau)\frac{e^{-st}}{-s}dt = \frac{1}{s^2}$$

$$(s+3)Ly + 2[0] + \frac{2}{s}Ly = \frac{1}{s^2}$$

$$(s + 3 + \tfrac{2}{s})Ly = \frac{1}{s^2} \qquad (s^2 + 2s + s + 2)Ly = \frac{1}{s}$$

$$Ly = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Solving $A = 1/2$ $B = -1$ $C = \frac{1}{2}$

$$Ly = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}$$

Taking Inverse L.T. $y = \frac{1}{2}e^{0} - e^{-t} + \frac{1}{2}e^{-2t}$

9 a). $L^{-1}\left(\frac{1}{s-a} \times \frac{1}{s-b}\right)$

$= \int_0^t e^{at} \cdot e^{bt-bx}\, dx = \int_0^t e^{(a+b)t} \cdot e^{-bx}\, dx$

$= e^{(a+b)t} \times \frac{\left[e^{-bx}\right]_0^t}{-b} = \frac{e^{(a+b)t}}{-b}\left[e^{-bt} - 1\right]$

$\frac{e^{at} - e^{(a+b)t}}{-b} = \frac{e^{(a+b)t} - e^{at}}{b}.$

b. $L^{-1}\left(\frac{1}{s^2} \times \frac{1}{s^2+16}\right)$

$= \frac{1}{4}\, t \int_0^t t \times \sin(4t - 4x)\, dx$

$= \frac{t}{16}\left[\cos(4t-4x)\right]_0^t = \frac{t}{16}(1 - \cos 4t).$

c) $L^{-1}\left(\frac{1}{s^2+9} \times \frac{1}{s^2+9}\right)$

$= \int_0^t \frac{1}{3}\sin(3t) \times \frac{1}{3}\sin(3t-3x)\, dx = \frac{1}{9}\sin 3t \int_0^t \sin(3t - 3x)\, dx$

$= \frac{1}{9}\sin 3t \left[\frac{\cos(3t-3x)}{3}\right]_0^t = \frac{1}{27}\sin 3t\left[1 - \cos 3t\right]$

d) $L^{-1}\left(\frac{s}{s^2+4} \times \frac{1}{s^2+4}\right) = \int_0^t \cos 2t \times \frac{1}{4}\sin(2t-2x)\, dx$

$= \frac{\cos 2t}{4}\int_0^t \sin(2t-2x) = \frac{\cos 2t}{4}\left[\frac{\cos(2t-2x)}{2}\right]_0^t$

$= \frac{\cos 2t}{8}\left[1 - \cos 2t\right]$

Q7. $(-1)\frac{d}{ds}\left[s^2\bar{y} - s - c\right] + 2(-1)\frac{d}{ds}\left[s\bar{y} - 1\right] + 2\bar{y} = \frac{2}{s}$

$(s^2+2s)\bar{y}' + 2s\bar{y} = 1 - \frac{2}{s}$

$(s+2)\bar{y} + 2\bar{y} = \frac{s-2}{s+2}$

$\bar{y}' + \frac{2}{s+2}\bar{y} = \frac{s-2}{s^2(s+2)}$

Integrating factor: $e^{\int 2/s+2\, ds} = e^{2\ln(s+2)} = (s+2)^2$

$(s+2)^2\bar{y} = \int \frac{(s-2)(s+2)}{s^2} ds = \int 1 - \frac{4}{s^2} ds = s + \frac{4}{s} + c_1$

$\bar{y} = \frac{s}{(s+2)^2} + \frac{4}{s(s+2)^2} + \frac{c_1}{(s+2)^2}$

$= \frac{s+2-2}{(s+2)^2} + \frac{4}{s(s+2)^2} + \frac{c_1}{(s+2)^2}$

$= \frac{1}{s+2} - \frac{2}{(s+2)^2} + \frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+2)^2} + \frac{c_1}{(s+2)^2}$

$\bar{y} = \frac{c_1}{(s+2)^2} + \frac{1}{s}$. Taking Inverse.

$y = c_1 e^{-2t}\, t + 1$

Q 8

$y'' + 9y = \sin 3t$

$L(y'' + 9y) = L\sin 3t$

$[s^2 Ly + s(0) + (0)] + 9Ly = \frac{3}{s^2+9}$

$Ly(s^2+9) = \frac{3}{s^2+9}$

$Ly = \frac{3}{(s^2+9)^2}$

$$L^{-1}\left(\frac{3}{s^2+9}\times\frac{1}{s^2+9}\right) = \int_0^t \sin 3t \times \frac{1}{3}\sin(3t-3x)\,dx \quad (5)$$

$$= \frac{\sin 3t}{3}\int_0^t \sin(3t-3x)\,dx = \frac{\sin 3t}{3}\left[\frac{\cos(3t-3x)}{3}\right]_0^t$$

$$= \frac{\sin 3t}{9}(1-\cos 3t).$$

Q 10. Let $I = \int_0^\infty e^{-st}\sin\omega t\,dt$

$$I = \sin\omega t\,\frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty \omega\cos\omega t\,\frac{e^{-st}}{-s}\,dt$$

$$I = \sin\omega t\,\frac{e^{-st}}{-s}\Big]_0^\infty + \frac{\omega}{s}\left(\cos\omega t\,\frac{e^{-st}}{-s} + \int_0^\infty \omega\sin\omega t\,\frac{e^{-st}}{-s}\,dt\right)$$

$$I = 0 + \frac{\omega}{s}\left(\cos\omega t\,\frac{e^{-st}}{-s}\right)_0^\infty - \frac{\omega^2}{s^2}I$$

$$\frac{(s^2+\omega^2)}{\cancel{s^2}}\,I = \frac{\omega}{\cancel{s^2}}$$

$$I = \frac{\omega}{s^2+\omega^2} \quad \text{QED}$$