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- RoLL No → 66
- BATCH → B-5
- MATHS ASSIGNMENT - 3

LTA®

Q1. If $u \to \log(x^2+y^2) + \tan^{-1}\frac{y}{x}$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \to 0$$

$\to$ $u \to \log(x^2+y^2)$

$$\frac{\partial u}{\partial x} \to \frac{2x}{x^2+y^2} + \frac{x^2}{x^2+y^2} \times \frac{-y}{x^2}$$

$$\frac{\partial u}{\partial x} \to \frac{2x-y}{(x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial x^2} \to \left[\frac{(x^2+y^2)\times 2 - (2x-y)\times 2x}{(x^2+y^2)^2}\right]$$

$$\to \frac{2x^2+2y^2-4x^2+2xy}{(x^2+y^2)^2} \quad \ldots\ldots (i)$$

and $$\frac{\partial u}{\partial y} \to \frac{2y}{x^2+y^2} + \frac{1}{1+y^2/x^2} \times \frac{1}{x}.$$

$$\to \frac{2y+x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} \to \frac{(x^2+y^2)\times 2 - (2y+x)2y}{(x^2+y^2)^2} \Rightarrow \frac{2x^2+2y^2-4y^2-2xy}{(x^2+y^2)^2} \ldots (ii)$$

adding eq (i) & (ii)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{4x^2-4x^2+4y^2-4y^2+2xy-2xy}{(x^2+y^2)^2} \to 0$$

Q2. If $z \Rightarrow f(x,y)$, and $x = e^u + e^{-v}$, $y = e^{-u} + e^v$
prove that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \to \frac{x\partial z}{\partial x} - \frac{y\partial z}{\partial y}.$$

$\to$ $$\frac{\partial z}{\partial u} \to \frac{\partial z}{\partial x}\cdot\frac{dx}{du} + \frac{\partial z}{\partial y}\cdot\frac{dy}{du}.$$

$$\frac{\partial z}{\partial u} \to \frac{\partial z}{\partial x}e^u - \frac{\partial z}{\partial y}e^{-u} \quad \ldots\ldots (i)$$

$\therefore \frac{\partial z}{\partial v} \longrightarrow \frac{\partial z}{\partial x} \cdot \frac{dx}{dv} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dv}$

$\therefore \frac{\partial z}{\partial v} \longrightarrow -\frac{\partial z}{\partial x} e^{-v} + \frac{\partial z}{\partial y} e^{v} \ldots$ (ii)

subtracting eq (ii) from eq (i) $\longrightarrow$

$\therefore \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \longrightarrow \frac{\partial z}{\partial x}(e^{u}+e^{-v}) - \frac{\partial z}{\partial y}(e^{-u}+e^{v})$

$\therefore \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \longrightarrow x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$

3 If $x^2+y^2+z^2-2xyz=1.$

show that

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} \Rightarrow 0.$$

→ Let $F(x,y) \Rightarrow x^2+y^2+z^2-2xyz-1.$

$\therefore \frac{\partial F}{\partial t} \longrightarrow \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt}.$

$\therefore \frac{\partial F}{\partial t} \Rightarrow (2x-2yz)\frac{dx}{dt} + (2y-2xz)\frac{dy}{dt} + (2z-2xy)\frac{dz}{dt}.$

$\therefore 0 \Rightarrow (2x-2yz)\,dx + (2y-2xz)\,dy + (2z-2xy)\,dz$

$0 \Rightarrow (x-yz)\,dx + (y-xz)\,dy + (z-xy)\,dz$

$\ldots$ (i)

$\rightarrow 2(x-yz)\,dx + 2(y-xz)\,dy + 2(z-xy)\,dz \Rightarrow 0.$ .... (i)

$(x-yz)\,dx + (y-xz)\,dy + (z-xy)\,dz \Rightarrow 0$ ... (ii)

dividing by $(x-yz)(y-xz)(z-xy)$, ... eq (ii).

$$\frac{dx}{(y-xz)(z-xy)} + \frac{dy}{(x-yz)(z-xy)} + \frac{dz}{(x-yz)(y-xz)} \Rightarrow 0 \quad \text{... (iii)}$$

since $[(y-xz)(z-xy)]^2 \longrightarrow (y-xz)^2(z-xy)^2$

$\longrightarrow (y^2+x^2z^2-2xyz)(z^2+x^2y^2-2xyz)$

... eq. (iv)

since, $x^2+y^2+z^2-2xyz=1.$

$\therefore \quad y^2-2xyz \Rightarrow 1-x^2-z^2$

and $\quad z^2-2xyz \Rightarrow 1-x^2-y^2.$

$\rightarrow$ putting in eq (iv)

$\therefore (y-xz)^2(z-xy)^2 \longrightarrow (1-x^2-z^2+x^2z^2)$
$(1-x^2-y^2+x^2y^2)$

$\longrightarrow (1-x^2)(1-z^2) \times (1-x^2-y^2(1-x^2))$

$\longrightarrow \underline{(1-x^2)^2(1-z^2)(1-y^2)}.$

$\therefore (y-xz)(z-xy) \longrightarrow (1-x^2)\sqrt{(1-z^2)}\sqrt{1-y^2}$

similarly for

$(x-yz)(z-xy) \longrightarrow (1-y^2)\sqrt{(1-z^2)(1-x^2)}$

$(x-yz)(y-xz) \longrightarrow (1-z^2)\sqrt{(1-x^2)(1-y^2)}$

putting all these in eq (iii)

$$\frac{dx}{(1-x^2)\sqrt{1-y^2}\sqrt{1-z^2}} + \frac{dy}{(1-y^2)\sqrt{(1-z^2)(1-x^2)}} + \frac{dz}{(1-z^2)\sqrt{(1-x^2)(1-y^2)}} \Rightarrow 0$$

multiplying by $\sqrt{(1-x^2)}\sqrt{1-y^2}\sqrt{1-z^2}$

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} \Rightarrow 0.$$

4. If $u = f(x,y)$ and $x \Rightarrow r\cos\theta$, $y \Rightarrow r\sin\theta$, Prove that.

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

→ Here, $x^2+y^2 \Rightarrow r^2$.

and $\tan\theta \Rightarrow \frac{y}{x}$.

$$\therefore \frac{\partial u}{\partial r} \longrightarrow \frac{\partial u}{\partial x}\times\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\times\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial r} \longrightarrow +\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \quad \ldots (i)$$

$$\text{and } \frac{\partial u}{\partial \theta} \Rightarrow \frac{\partial u}{\partial x}\times\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\times\frac{\partial y}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} \longrightarrow \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \quad \ldots (ii)$$

$$\left(\frac{\partial u}{\partial \theta}\right)^2 = r^2\left[-\frac{\partial u}{\partial x}\sin\theta + \frac{\partial u}{\partial y}\cos\theta\right] \quad \ldots (ii)$$

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squaring eq (i) and adding to (ii)

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 \Rightarrow \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 .$$

5. verify Euler's Theorem, when

(a) $f(x,y) \rightarrow ax^2 + 2hxy + by^2$

$$\longrightarrow x^2\left[a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2\right]$$

Function of degree 2.

$$\therefore \frac{\partial f(x,y)}{\partial x} \longrightarrow 2ax + 2hy .$$

$$\frac{\partial f(x,y)}{\partial y} \longrightarrow 2hx + 2by .$$

$$\therefore x(2ax + 2hy) + (2hx + 2by)y .$$

$$\longrightarrow 2ax^2 + 2hxy + 2hxy + 2by^2$$

$$\longrightarrow 2\left[ax^2 + 2hxy + by^2\right]$$

$$\longrightarrow \underline{n \times f(x,y)}.$$

verified .

(b) $f(x,y) \longrightarrow \dfrac{x^3+y^3}{x+y}$

$$\longrightarrow \frac{x^{\cancel{3}\,2}}{\cancel{x}} \times \frac{\left[1 + \left(\frac{y}{x}\right)^3\right]}{\left[1 + \frac{y}{x}\right]} \longrightarrow x^2\left[\frac{1 + (y/x)^3}{1 + y/x}\right]$$

Function of degree 2.

$$\therefore x\frac{\partial f(x,y)}{\partial x} . \longrightarrow x \times \left[\frac{(x+y)3x^2 - (x^3+y^3)\times 1}{(x+y)^2}\right]$$

$$\rightarrow x \times \left[\frac{3x^3 + 3x^2y - x^3 - y^3}{(x+y)^2}\right]$$

$$\rightarrow x \times \left[\frac{2x^3 + 3x^2y - y^3}{(x+y)^2}\right]$$

$$\rightarrow \frac{2x^4 + 3x^3y - xy^3}{(x+y)^2} \quad \ldots \text{(i)}$$

and $y\frac{\partial f}{\partial y} \Rightarrow y \times \left[\frac{(x+y)3y^2 - (x^3+y^3)}{(x+y)^2}\right]$

$$\rightarrow y \times \left(\frac{[3y^2x + 3y^3 - x^3 - y^3]}{(x+y)^2}\right)$$

$$\rightarrow \frac{3y^3x + 3y^4 - x^3y - y^4}{(x+y)^2} \quad \ldots \text{(ii)}$$

adding eq (i) & (ii)

$$\rightarrow \frac{2x^4 + 3x^3y - xy^3 + 3y^3x + 2y^4 - x^3y}{(x+y)^2}$$

$$\rightarrow \frac{2x^4 + 2x^3y + 2y^3x + 2y^4}{(x+y)^2}$$

$$\rightarrow \frac{2[x^4 + x^3y + y^3x + y^4]}{(x+y)^2}$$

$$\rightarrow \frac{2 \times [x^3(x+y) + y^3(x+y)]}{(x+y)^2}$$

$$\rightarrow \frac{2(x^3+y^3)}{(x+y)} \quad \text{(verified)}.$$

6 If $u = \tan^{-1}\frac{x^3+y^3}{x+y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

and $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u$.

→ By Euler,

Let

$f(x) = \tan u = \frac{x^3+y^3}{x+y}$.

→ $f(x)$ is a function of degree $\frac{x^3(1+y^3/x^3)}{x(1+y/x)}$ =

$= x^2\frac{\left[1+(y/x)^3\right]}{1+y/x} = (2)$.

$\therefore x\frac{\partial f'(x)}{\partial x} + y\frac{\partial f'(y)}{\partial y} = nf(x)$.

→ $x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} \rightarrow 2\times\tan u$.

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \rightarrow 2\sin u\cos u$.

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \rightarrow \sin 2u$ .... (i)

Differentiating eq (i) with respect to x.

$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} \rightarrow 2\cos 2u\frac{\partial u}{\partial x}$ ... (ii)

$\therefore$

Differentiating eq(i) with respect to y.

$$x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2} \longrightarrow 2\cos 2u \frac{\partial u}{\partial y} \quad \ldots (iii)$$

multiplying eq(ii) with x & eq(iii) with y.
and adding.

$$x\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} \longrightarrow (2\cos 2u - 1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\longrightarrow (2\cos 2u - 1) \times 2\sin u \cos u.$$

$$\longrightarrow [2(2\cos^2 x) - 3] \times 2\sin u \cos u.$$

$$\rightarrow (4\cos^2 x - 3\cos x)\sin u \times 2.$$

$$\rightarrow 2\cos 3u \sin u.$$

7. If $u \Rightarrow \tan^{-1}\frac{y^2}{x}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} \longrightarrow -\sin^2 u \sin 2u.$$

→ Let $y \rightarrow \lambda y$ & $x \rightarrow \lambda x$. in $f(x,y) \Rightarrow y^2/x$

$\therefore \quad f(\lambda x, \lambda y) \rightarrow \frac{\lambda^2 y^2}{\lambda x} \rightarrow \lambda^{1}\left(\frac{y^2}{x}\right)$

So, it is one degree equation.

Let $f(x) = \tan u = \frac{y^2}{x}$.

By Euler's theorem :→

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) \longrightarrow \tan u.$$

$$x\sec^2 u \frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} \longrightarrow \tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \Rightarrow \frac{\sin 2u}{2} \quad \ldots\ldots (i)$$

Differentiating eq (i) with respect to x

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} \longrightarrow \cos 2u \frac{\partial u}{\partial x} \quad \text{(iii)}$$

Differentiating eq (i) with respect to y.

$$x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} \longrightarrow \cos 2u \frac{\partial u}{\partial y} \quad \text{(iv)}$$

multiplying eq (iii) with x and (iv) with y and adding :-

$$\frac{x\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} \longrightarrow \frac{x\partial u}{\partial x}\cos 2u$$

$$+ xy\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} \longrightarrow y\frac{\partial u}{\partial y}\cos 2u.$$

$$\rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} \Rightarrow (\cos 2u - 1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\longrightarrow (\cos 2u - 1) \times \frac{\sin 2u}{2} \quad \text{(From eq (i))}$$

$$\longrightarrow [2\cos^2 u - 2] \times \frac{\sin 2u}{2}$$

$$\longrightarrow -\sin^2 u \sin 2u$$

8. Expand the function $f(x,y) = e^x \log(1+y)$ in powers of x and y upto terms of third degree.

$\rightarrow$ $f(x,y) = e^x \ln(1+y)$ $\qquad f(0,0) = 0$

$f_x \Rightarrow e^x \ln(1+y)$ $\qquad f_x(0,0) \rightarrow 0$

$f_y = \dfrac{e^x}{1+y} \longrightarrow$ $\qquad f_y(0,0) \rightarrow 1.$

$f_{xx} \Rightarrow e^x \ln(1+y)$ $\qquad f_{xx}(0,0) \Rightarrow 0$

$f_{xy} \Rightarrow \frac{e^x}{1+y}$ $\qquad f_{xy}(0,0) \to 1.$

$f_{yy} \Rightarrow -\frac{e^x}{(1+y)^2}$ $\qquad f_{yy}(0,0) \Rightarrow -1$

$f_{xxx} \Rightarrow e^x \ln(1+y)$ $\qquad f_{xxx}(0,0) \to 0.$

$f_{xxy} \longrightarrow \frac{e^x}{1+y}$ $\qquad f_{xxy}(0,0) \Rightarrow 1.$

$f_{xyy} \Rightarrow \frac{-e^x}{(1+y)^2}$ $\qquad f_{xyy}(0) \Rightarrow -1.$

$f_{yyy} \Rightarrow \frac{2e^x}{(1+y)^3}$ $\qquad f_{yyy}(0) = 2.$

$\therefore f(x,y) \Rightarrow f(0,0) + xf_x + yf_y + \frac{1}{2!}\left[x^2 f_{xx} + 2f_{xy}xy + y^2 f_{yy}\right]$

$+ \frac{1}{3!}\left[x^3 f_{xxx} + 3x^2y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right]$

$\longrightarrow y + \frac{1}{2!}\left[2x\,xy\right] \frac{-y^2}{2} + \frac{1}{6}\left[3x^2y - 3xy^2 + 2y^3\right]$

$\longrightarrow y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3}.$

9. Expand the function $f(x,y) = x^y$ in power of $(x-1)$ & $(y-1)$ upto terms of third degree.

$\rightarrow \quad f(x,y) \longrightarrow x^y.$

$f(1,1) \longrightarrow 1.$

$f_x \rightarrow y x^{y-1}$ $\rightarrow f(1,1) \rightarrow (1)$.
$f_y \rightarrow x^y \ln x$ $\rightarrow f(1,1) \rightarrow 0$.

$f_{xx} \rightarrow y(y-1) x^{y-2}$ $f(1,1) \rightarrow 0$.
$f_{xy} \rightarrow y x^{y-1} \cancel{\ln x} + x^{y-1}$ $f(1,1) \rightarrow 0+1 \rightarrow 1$.
$f_{yy} \rightarrow (\ln x) x^y (\ln x)$ $f(1,1) \rightarrow 0$

$\therefore f_{xxx} \rightarrow y(y-1)(y-2) x^{y-3}$ $f(1,1) \rightarrow 0$.
$f_{xxy} \rightarrow (2y-1) x^{y-2} + (y^2-y) x^{y-2} \ln x$ $f(1,1) \rightarrow 1$.
$f_{yyx} \rightarrow (2\ln x) x^y + y x^{y-1} (\ln x)^3$ $f(1,1) \rightarrow 0$.
$f_{yyy} \rightarrow (\ln x)^3 x^y$. $f(1,1) \rightarrow 0$

$\therefore f(x-1+1, y-1+1)$

$$\rightarrow f(1,1) + \frac{1}{1!}\left((x-1) f_x + (y-1) f_y\right)$$

$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx} + 2(x-1)(y-1) f_{xy} + (y-1)^2 f_{yy}\right]$$

$$+ \frac{1}{3!}\left[(x-1)^3 f_{xxx} + 3(x-1)^2(y-1) f_{xxy} + 3(x-1)(y-1)^2 f_{xyy} + (y-1)^3 f_{yyy}\right]$$

$$\rightarrow 1 + \frac{1}{1!}\left[(x-1)\times 1 + (y-1)\times 0\right] + \frac{1}{2!}\left[0 + (x-1)(y-1)\times 1 + 0\right]$$

$$+ \frac{1}{3!}\left[(x-1)^3 \times 0 + 3(x-1)^2(y-1)1 + 3(x-1)(y-1)^2 \times 0 + 0\right]$$

$$\rightarrow 1 + (x-1) + \frac{(x-1)(y-1)}{2} + \frac{(x-1)^3}{6}$$

$$+ \frac{(x-1)^2(y-1)}{2} + \frac{(x-1)(y-1)^2}{2}.$$

10. Examine the function $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ for extreme values.

$\rightarrow$ $f(x) \rightarrow x^4 + y^4 - 2x^2 + 4xy - 2y^2.$

$\frac{\partial f}{\partial x} \longrightarrow 4x^3 - 4x + 4y \Rightarrow 0 \quad \ldots (i)$

$\frac{\partial f}{\partial y} \longrightarrow 4y^3 + 4x - 4y \Rightarrow 0. \quad \ldots (ii)$

~~at~~ $\boxed{\cancel{x = y^3}}$

$\boxed{\cancel{-4y^9 + 4y^3 + 4y = 0} \\ \cancel{-y^9 + y^3 + y \Rightarrow 0}}$

adding both the equation (i) and (ii)

$\underline{x^3 + y^3 = 0}$

$\rightarrow$ always positive.

$\therefore (x+y)(x^2 + y^2 - xy) = 0$

$\boxed{x = -y}$

putting $x = -y$ in eq (i)

$4x^3 - 4x - 4x = 0$

$x^3 - 2x = 0$

$x \Rightarrow 0, (y \Rightarrow \sqrt{2}, y = -\sqrt{2}$ at $x \Rightarrow -\sqrt{2}$ and $x = \sqrt{2}$).

$\frac{\partial^2 f}{\partial x^2} \longrightarrow 12x^2 - 4.$

$\frac{\partial^2 f}{\partial x \partial y} \Rightarrow 4$

$\frac{\partial^2 f}{\partial y^2} \Rightarrow 12y^2 - 4.$

$\therefore$ at $x \Rightarrow -\sqrt{2}$ and $\sqrt{2}$.

$\frac{\partial^2 f}{\partial x^2} \rightarrow 12(2) - 4 \rightarrow 20 > 0$.

and at $x = -\sqrt{2}$.,	at $x = \sqrt{2}$
$\frac{\partial^2 f}{\partial x \partial y} = 4$	$\frac{\partial^2 f}{\partial x \partial y} = 4$
$\frac{\partial^2 f}{\partial y^2} \Rightarrow 20$.	$\frac{\partial^2 f}{\partial y^2} = 20$

$\therefore 20 \times 20 - 16 \Rightarrow 400 - 16 \rightarrow 384 > 0$.

Hence $x \Rightarrow -\sqrt{2}$ and $\sqrt{2}$ is point of local minima.

$\therefore$ at $\underline{x=0}$, $\underline{y=0}$.

$\Rightarrow 4 \times 4 - (4)^2$.

$\Rightarrow 0$. $(so\ rt - s^2 = 0)$

Further investigation at $x=0, y=0$.

at $f(0,0) = 0$ and for points along the x-axis, where $y=0$, $f(x,y) = x^4 - 2x^2 = x^2(x^2-2)$ which is negative for points in the neighbourhood of origin.

Again for point along the line $y = x$, $f(x,y) = 2x^4 > 0$.

$\therefore$ In the neighbourhood of $(0,0)$, there are points where $f(x,y) < f(0,0)$ and there are points where $f(x,y) > f(0,0)$

Hence, $f(0,0)$ is not an extremum value.

11. If the sides of a plane triangle ABC vary in such a way that its circumradius remains constant, then prove that :→

→ In a Euclidean planar triangle :→
Law of sign is

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

In Δ

$$A+B+C = \pi.$$
$$d(A+B+C) = 0$$
$$dA + dB + dC = 0 \quad \ldots \text{eq (i)}.$$

$$a = 2R\sin A.$$
$$da = 2R\cos A\, dA.$$
$$\therefore \frac{da}{\cos A} = 2R\, dA \ldots \text{(ii)}$$

similarly for b, c.

$$b = 2R\sin B$$
$$\frac{db}{\cos B} = 2R\, dB, \ldots \text{(iii)}$$
$$\frac{dc}{\cos C} \longrightarrow 2R\, dC \ldots \text{(iv)}$$

adding eq (ii), (iii) and (iv)

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R(dA + dB + dC)$$
$$\longrightarrow 0. \quad (\text{From eq (i)})$$

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12\. A ballon is in the form of right circular cylinder of radius 1.5 and length 4m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and length by 0.05 m. find the approximate percentage change in the volume of the ballon.

→ Volume → $\pi R^2 H + \frac{2}{3}\pi R^3$

By ; Total partial derivative :→

$$dV \Rightarrow \frac{\partial(\text{volume})}{\partial R}(\Delta R) + \frac{\partial(\text{volume})}{\partial H} \times (\Delta H).$$

$$\therefore \frac{dV}{V} \Rightarrow \frac{1}{V}\left[\frac{\partial(\text{volume})}{\partial R}\Delta R + \frac{\partial(\text{volume})}{\partial H}\Delta H\right]$$

$$\therefore \frac{dV}{V} \rightarrow \frac{1}{V}\left[(2\pi RH + 2\pi R^2)\Delta R + \pi R^2 \Delta H\right]$$

$$\therefore \frac{dV}{V} = \frac{2\cancel{\pi}R(R+H)\Delta R + \cancel{\pi}R^2\Delta H}{\cancel{\pi}R^2 H + \frac{2}{3}\cancel{\pi}R^3}$$

$$\frac{dV}{V} \rightarrow \frac{2 \times (5.5) \times 0.01}{(1.5)^2 \times 4 + \frac{2}{3} \times (1.5)^3} + \frac{\cancel{\Delta H}\ 0.05}{4 + \frac{2}{3} \times 1.5}$$

$$\frac{dV}{V} \rightarrow \frac{11 \times 0.01}{9 + 1.4} + \frac{0.05}{5}$$

$$\frac{dV}{V} \rightarrow 0.023$$

$$\therefore \frac{dV}{V} \times 100 \rightarrow 2.3\ \%.$$

13. Given $x+y+z = a$, find the maximum value of $x^m y^n z^p$.

$\rightarrow$ $f(x) \Rightarrow x^m y^n z^p$.

$z \Rightarrow a-x-y$.

$\therefore f(x) \Rightarrow x^m y^n [a-(x+y)]^p \rightarrow x^m y^n (a-x-y)^p$

$\therefore \frac{\partial f}{\partial x} \rightarrow m x^{m-1} y^n (a-x-y)^p + x^m y^n \times p(a-x-y)^{p-1}(-1)$.

$\rightarrow x^{m-1} y^n (a-x-y)^{p-1} [m(a-x-y) - xp] \Rightarrow 0$

and $\frac{\partial f}{\partial y} \rightarrow n y^{n-1} x^m (a-x-y)^p + x^m y^n \times p(a-x-y)^{p-1} \times (-1)$.

$\rightarrow y^{n-1} x^m (a-x-y)^{p-1} [n(a-x-y) - yp]$

$\rightarrow y^{n-1} x^m (a-x-y)^{p-1} [na - nx - (n+p)y] \Rightarrow 0$

This process is ~~becoming~~ so complex; so,

Applying Lagrange Method $\therefore \rightarrow$

$F(x) \Rightarrow x^m y^n z^p + \lambda(x+y+z-a)$.

$\frac{dF}{dx} \rightarrow m x^{m-1} y^n z^p + \lambda \Rightarrow 0$.

$\lambda \Rightarrow -m x^{m-1} y^n z^p$ ... (i)

$\frac{dF}{dy} \rightarrow n y^{n-1} x^m z^p + \lambda = 0$

$\lambda = -n y^{n-1} x^m z^p$ ... (ii)

$\frac{\partial F}{\partial z} \rightarrow p z^{p-1} x^m y^n + \lambda$

$\lambda = -p z^{p-1} x^m y^n$ ... (iii) equating (i) and (ii).

$$m x^{m-1} y^n \cancel{z^p} \rightarrow n y^{n-1} x^m \cancel{z^p}$$

$$\boxed{y \rightarrow \frac{n x}{m}} \quad \ldots \text{(iv)}$$

similarly through (ii) and (iii).

$$n y^{n-1} \cancel{x^m} z^p \rightarrow p z^{p-1} \cancel{x^m} y^n$$

$$\boxed{z \rightarrow \frac{p y}{n}} \quad \ldots \text{(v)}$$

putting eq (iv) and (v) in $x+y+z=a$.

$$\frac{my}{n} \cancel{x} + \cancel{x}y + \frac{py}{n} \Rightarrow a.$$

$$\frac{my + ny + p \cdot y}{n} = a.$$

$$\boxed{y \rightarrow \frac{an}{(m+n+p)}}$$

and $x = \dfrac{ma}{(m+n+p)}$

$$z \rightarrow \frac{pa}{m+n+p}$$

putting in $x^m y^n z^p$

$$\rightarrow \frac{(ma)^m \times (an)^n \times (pa)^p}{(m+n+p)^{m+n+p}}$$

$$\rightarrow \frac{m^m n^n p^p a^{m+n+p}}{(m+n+p)^{m+n+p}}.$$

Q14. Show that the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

→ Let x, y, z be the coordinate of point in the ellipsoid.
So, Because of symmetry → another point is symmetric, and dimensions are $2x, 2y, 2z$ respectively.
∴ Volume → $8xyz$.
Lagrange →

$$F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right).$$

$$\frac{\partial F}{\partial x} \rightarrow 8yz + \frac{2\lambda x}{a^2} = 0$$

$$-\frac{4yza^2}{x} = \lambda \quad \ldots (i)$$

$$\frac{\partial F}{\partial y} \rightarrow 8xz + \frac{2\lambda y}{b^2} = 0$$

$$-\frac{4xzb^2}{y} = \lambda \quad \ldots (ii)$$

equating eq(i) and (ii)

$$\frac{y\cancel{z}a^2}{x} = \frac{4x\cancel{z}b^2}{y}.$$

$$\frac{y}{x} = \frac{b}{a}.$$

$$\boxed{y = \frac{bx}{a}.}$$

similarly, $\boxed{z = \frac{cx}{a}.}$

$\therefore$ putting in ellipsoid.

$$\frac{3x^2}{a^2} = 1.$$

$$x^2 = \frac{a^2}{3}$$

$$\boxed{x = \frac{a}{\sqrt{3}}}$$

$$y = \frac{b}{\cancel{a}} \times \frac{\cancel{a}}{\sqrt{3}} = \frac{b}{\sqrt{3}},$$

$$z = \frac{c}{a} \times \frac{a}{\sqrt{3}} \rightarrow \frac{c}{\sqrt{3}}.$$

so, $\boxed{\text{volume} \rightarrow \frac{8abc}{3\sqrt{3}}}$ unit^3.

Q15. Prove that the rectangular solid of max^m volume which can be inscribed in a sphere is a cube.

→ Let the sphere be $x^2 + y^2 + z^2 = a^2$.

Let the side of rectangular solid be $2x$, $2y$, $2z$ because curve is symmetric.

$\therefore$ $V \rightarrow 8yxz$.

$\therefore$ $F(x,y,z) \rightarrow 8yxz + \lambda(x^2+y^2+z^2-a^2)$.

$$\frac{dF}{dx} \rightarrow 8yz + 2\lambda x = 0$$

$$\lambda = -\frac{4yz}{x}. \quad \ldots (i)$$

$$\frac{\partial F}{\partial y} \rightarrow 8xz + 2y\lambda = 0$$

$$\lambda = -\frac{4xz}{y} \quad \ldots (ii)$$

$\frac{\partial F}{\partial z} \Rightarrow 8yx + 2\lambda z$.

$\lambda = -\frac{4xy}{z}$. ... (iii)

from eq (i), (ii) and (iii)

$\boxed{x = y = z}$

Hence it is a cube.

and. Volume is $8x^3 \rightarrow \frac{8 \times a^3}{\sqrt{3}}$ unit^3.

$\rightarrow \frac{8a^3}{3\sqrt{3}}$ unit^3.