Lecture 13: Complex Eigenvalues & Factorization

Consider the rotation matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{1}$$

To find eigenvalues, we write

$$A - \lambda I = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix},$$

and calculate its determinant

$$\det(A - \lambda I) = \lambda^2 + 1 = 0.$$

We see that A has only complex eigenvalues

$$\lambda = \pm\sqrt{-1} = \pm i.$$

Therefore, it is impossible to diagonalize the rotation matrix. In general, if a matrix has complex eigenvalues, it is not diagonalizable. In this lecture, we shall study matrices with complex eigenvalues.

Since eigenvalues are roots of characteristic polynomials with real coefficients, complex eigenvalues always appear in pairs: If

$$\lambda_0 = a + bi$$

is a complex eigenvalue, so is its conjugate

$$\bar{\lambda}_0 = a - bi.$$

For any complex eigenvalue, we can proceed to find its (complex) eigenvectors in the same way as we did for real eigenvalues.

Example 13.1. For the matrix A in (1) above, find eigenvectors.

Solution. We already know its eigenvalues: $\lambda_0 = i$, $\bar{\lambda}_0 = -i$. For $\lambda_0 = i$, we solve $(A - iI)\vec{x} = \vec{0}$ as follows: performing (complex) row operations to get (noting that $i^2 = -1$)

$$A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \overset{-iR_1+R_2 \to R_2}{\longrightarrow} \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix}$$

$$\Longrightarrow -ix_1 - x_2 = 0 \Longrightarrow x_2 = -ix_1$$

$$\text{take } x_1 = 1, \quad \vec{u} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is an eigenvalue for $\lambda = i$.

For $\lambda = -i$, we proceed similarly as follows:

$$A + iI = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \overset{iR_1+R2 \to R_2}{\longrightarrow} \begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix} \Longrightarrow ix_1 - x_2 = 0$$

$$\text{take } x_1 = 1,\ \vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is an eigenvalue for $\lambda = -i$. We note that

$$\bar{\lambda}_0 = -i \text{ is the conjugate of } \lambda_0 = i$$
$$\vec{v} = \overline{\vec{u}} \text{ (the conjugate vector of eigenvector } \vec{u}\text{) is an eigenvector for } \bar{\lambda}_0.$$

This observation can be justified as a general rule as follows.

$$(A + \lambda_0 I)\,\vec{u} = 0 \overset{by\ taking\ conjugate\ on\ equation}{\Longrightarrow} \overline{(A + \lambda_0 I)\,\vec{u}} = 0$$
$$\Longrightarrow \overline{(A + \lambda_0 I)}\left(\overline{\vec{u}}\right) = 0 \Longrightarrow \left(\bar{A} - \overline{(\lambda_0 I)}\right)\left(\overline{\vec{u}}\right) = 0 \Longrightarrow \left(A - \bar{\lambda}_0 I\right)\overline{\vec{u}} = 0.$$

Remark. In general, if $\vec{u}$ is an eigenvector associated with λ, then $\overline{\vec{u}}$, the conjugate of $\vec{u}$, is an eigenvector associated with $\bar{\lambda}$, the conjugate of λ. Therefore, for each pair, $\left(\lambda, \bar{\lambda}\right)$, of eigenvalue, we only need to compute eigenvectors for λ. The eigenvectors for $\bar{\lambda}$ can be obtained easily by taking conjugates.

Though A is not diagonalizable in the classic sense, we can still simplify it by introducing a term called "block-diagonal" matrix.

Example 13.2. For the matrix A in (1) above that has complex eigenvalues, we proceed to choose P and D as follows: pick one complex eigenvalue and its eigenvector

$$\lambda_0 = i,\ \vec{u} = \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

separate their real part and imaginary part $(\operatorname{Re}(a + bi) = a,\ \operatorname{Im}(a + bi) = b)$:

$$\operatorname{Re}(\lambda_0) = 0,\ \operatorname{Im}(\lambda_0) = 1,\ \operatorname{Re}(\vec{u}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \operatorname{Im}(\vec{u}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

We then assemble D and P as

$$D = \begin{bmatrix} \operatorname{Re}(\lambda) & \operatorname{Im}(\lambda) \\ -\operatorname{Im}(\lambda) & \operatorname{Re}(\lambda) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$
$$P = [\operatorname{Re}(\vec{u}), \operatorname{Im}(\vec{u})] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that D is not really a diagonal matrix. We can verify $AP = PD$:

$$AP = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$PD = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In general, if a matrix A has complex eigenvalues, it may be similar to a block-diagonal matrix B, i.e., there exists an invertible matrix P such that $AP = PB$, where B has the form

$$B = \begin{bmatrix} C_1 & 0 & \dots & 0 \\ 0 & C_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & C_r \end{bmatrix}, \quad P = [P_1, P_2, ..., P_r] \tag{2}$$

C_k is either a real eigenvalue λ_k (1×1 matrix), or C_k is a 2×2 matrix (called a block), corresponding to a complex eigenvalue $\lambda_k = \operatorname{Re}(\lambda_k) + i \operatorname{Im}(\lambda_k)$, in the form

$$C_k = \begin{bmatrix} \operatorname{Re}(\lambda_k) & \operatorname{Im}(\lambda_k) \\ -\operatorname{Im}(\lambda_k) & \operatorname{Re}(\lambda_k) \end{bmatrix}, \tag{3}$$

and P is assembled accordingly: if C_k is an real eigenvalue, the corresponding column P_k in P is an eigenvector (real); if C_k is a 2×2 block as in (3) associated with a complex eigenvalue λ_k, then P_k consists of two columns: $P_k = [\operatorname{Re}(\vec{u}_k), \operatorname{Im}(\vec{u}_k)]$ where $\vec{u}_k$ is a complex eigenvector for λ_k (note that both $\operatorname{Re}(\vec{u}_k)$ and $\operatorname{Im}(\vec{u}_k)$ are real vectors).

Example 13.3. Some examples of block diagonal matrices (2): block matrix from v.s. normal matrix form.

Example 1 *1. 3×3 matrix with one diagonal entry and one 2×2 block:*

$$\begin{bmatrix} 2 & 0 \\ 0 & \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & -3 & 2 \end{bmatrix}, \ (\lambda_1 = 2,\ \lambda_2 = 2 + 3i,\ \lambda_3 = 2 - 3i)$$

2. 4×4 matrix with two diagonal entries and one 2×2 block:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \begin{bmatrix} -4 & 2 \\ -2 & -4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & -2 & -4 \end{bmatrix}, \ (\lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = -4 + 2i,\ \lambda_4 = -4 - 2i)$$

3. 4×4 matrix with two 2×2 blocks:

$$\begin{bmatrix} \begin{bmatrix} -4 & 2 \\ -2 & -4 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -4 & 2 & 0 & 0 \\ -2 & -4 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}, \ (\lambda_1 = -4+2i,\ \lambda_2 = -4-2i,\ \lambda_3 = 2+3i,\ \lambda_4 = 2 - 3i).$$

The process of finding a block diagonal matrix B in the form (2) and P such as $P^{-1}AP = B$ is called factorization. Note that if all eigenvalues are real, factorization = diagonalization.

Factorization process:

Step #1. Find all eigenvalues.

Step #2. For real eigenvalues, find a basis in each eigenspace $Null\,(A-\lambda)$. For each pair of complex eigenvalues λ and $\bar{\lambda}$, find a (complex) basis in $Null\,(A-\lambda)$. Then, write complex eigenvectors in the basis in the form $\vec{u} = \operatorname{Re}(\vec{u}) + i \operatorname{Im}(\vec{u})$.

Note that, the total number of such vectors must be equal to the dimension. Otherwise, it is not factorizable.

Step #3. Assemble P and B as in (2): if C_1 is a real eigenvalue, the corresponding column P_1 is one of its eigenvectors in the basis; if C_1 is a 2×2 block matrix as in (3), the corresponding P_1 consists of two columns $(\operatorname{Re}(\vec{u}), i \operatorname{Im}(\vec{u}))$.

Remark. Suppose that A is a upper (or lower) block-triangle matrix in the form

$$A = \begin{bmatrix} B_1 & * & ... & * \\ 0 & B_2 & ... & * \\ ... & ... & ... & ... \\ 0 & 0 & ... & B_r \end{bmatrix},$$

where each B_i is a square matrix (with same or different sizes). Then

$$\det(A) = \det(B_1)\det(B_2)\cdots\det(B_r).$$
$$\det(A-\lambda) = \det(B_1-\lambda)\det(B_2-\lambda)\cdots\det(B_r-\lambda).$$

Therefore, eigenvalues of A = collection of all eigenvalues of $B_1, ..., B_r$.

Example 13.4. Factorize

$$A = \begin{bmatrix} 2 & 9 & 0 & 2 \\ -1 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Solution. Step 1. Find eigenvalues. We notice that

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 9 & 0 & 2 \\ -1 & 2-\lambda & 1 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 1 & -1-\lambda \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2-\lambda & 9 \\ -1 & 2-\lambda \end{bmatrix} & * \\ 0 & \begin{bmatrix} 3-\lambda & 0 \\ 1 & -1-\lambda \end{bmatrix} \end{bmatrix}$$

is an upper block-triangle. Thus,

$$\det(A) = \det\left(\begin{bmatrix} 2-\lambda & 9 \\ -1 & 2-\lambda \end{bmatrix}\right) \cdot \det\begin{bmatrix} 3-\lambda & 0 \\ 1 & -1-\lambda \end{bmatrix}$$
$$= [(2-\lambda)(2-\lambda)+9]\,[(3-\lambda)(-1-\lambda)] = 0.$$

Hence

$$[(2-\lambda)(2-\lambda)+9] = 0 \implies \lambda^2 - 4\lambda + 13 = 0 \implies \lambda = 2 \pm 3i$$

or

$$(3-\lambda)(-1-\lambda) = 0 \implies \lambda = 3, \ \lambda = -1.$$

There are two real eigenvalues $\lambda_1 = 3,\ \lambda_2 = -1$, and one pair of complex eigenvalue $\mu = 2+3i$, and $2-3i$, with real part and imaginary part

$$\mathrm{Re}\,(\mu) = 2, \quad \mathrm{Im}\,(\mu) = 3.$$

Step 2. We next find eigenvectors. For $\lambda_1 = 3$, we carry out row operation:

$$A - 3I = \begin{bmatrix} 2-3 & 9 & 0 & 2 \\ -1 & 2-3 & 1 & 0 \\ 0 & 0 & 3-3 & 0 \\ 0 & 0 & 1 & -1-3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 9 & 0 & 2 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{bmatrix} \underset{\longrightarrow}{\substack{R_2 - R_1 \to R_2 \\ R_3 \to R_4}} \begin{bmatrix} -1 & 9 & 0 & 2 \\ 0 & -10 & 1 & -2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{19}{5} \\ 0 & 1 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

System became

$$\begin{aligned} x_1 - \frac{19}{5}x_4 &= 0 \\ x_2 - \frac{1}{5}x_4 &= 0 \\ x_3 - 4x_4 &= 0. \end{aligned}$$

The eigenvalue is, by taking $x_4 = 5$,

$$\vec{v}_1 = \begin{bmatrix} 19 \\ 1 \\ 20 \\ 5 \end{bmatrix}, \quad \lambda_1 = 3.$$

For $\lambda_2 = -1$, we proceed as follows:

$$A + I = \begin{bmatrix} 3 & 9 & 0 & 2 \\ -1 & 3 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 9 & 0 & 2 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\longrightarrow}{\scriptstyle 3R_2 + R_1 \to R_1} \begin{bmatrix} 0 & 18 & 0 & 2 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\longrightarrow}{\substack{R_1/18 \to R_1 \\ (-3)\,R_1 + R_2 \to R_2}} \begin{bmatrix} 0 & 1 & 0 & 1/9 \\ -1 & 0 & 0 & -1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1/9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We obtain that

$$
\begin{aligned}
x_1 &= -\frac{1}{3}x_4 \\
x_2 &= -\frac{1}{9}x_4 \\
x_3 &= 0.
\end{aligned}
$$

By taking $x_4 = -9$,

$$
\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -9 \end{bmatrix}, \ \lambda_2 = -1.
$$

For $\mu = 2 + 3i$, we proceed analogously:

$$
A - \mu I = \begin{bmatrix} 2-(2+3i) & 9 & 0 & 2 \\ -1 & 2-(2+3i) & 1 & 0 \\ 0 & 0 & 3-(2+3i) & 0 \\ 0 & 0 & 1 & -1-(2+3i) \end{bmatrix}
$$

$$
= \begin{bmatrix} -3i & 9 & 0 & 2 \\ -1 & -3i & 1 & 0 \\ 0 & 0 & 1-3i & 0 \\ 0 & 0 & 1 & -3-3i \end{bmatrix} \underset{\longrightarrow}{\begin{matrix} R_1 - 3iR_2 \to R_1 \\ R_4 - R_3/(1-3i) \to R_4 \\ R_2 - R_3/(1-3i) \to R_2 \end{matrix}} \begin{bmatrix} 0 & 0 & 0 & 2 \\ -1 & -3i & 0 & 0 \\ 0 & 0 & 1-3i & 0 \\ 0 & 0 & 0 & -3-3i \end{bmatrix}
$$

$$
\underset{\longrightarrow}{\begin{matrix} R_3/(1-3i) - R_3 \\ R_1 - R_4(-3-3i) \to R_1 \\ R_4/(-3-3i) - R_4 \end{matrix}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -3i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

System $(A - \mu I)\,x = 0$ becomes

$$
\begin{aligned}
-x_1 - 3ix_2 &= 0 \\
x_3 &= 0 \\
x_4 &= 0
\end{aligned}
$$

We obtain the complex eigenvector, by taking $x_2 = 1$,

$$
\vec{u} = \begin{bmatrix} -3i \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mu = 2 + 3i.
$$

$$\mathrm{Re}\,(\mu) = 2,\ \mathrm{Re}\,(\vec{u}) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathrm{Im}\,(\mu) = 3,\ \mathrm{Im}\,(\vec{u}) = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Step #3. We finally assemble P and B as follows:

$$B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 19 & 3 & 0 & -3 \\ 1 & 1 & 1 & 0 \\ 20 & 0 & 0 & 0 \\ 5 & -9 & 0 & 0 \end{bmatrix}.$$

We may verify $AP = PB$ by a direct computation:

$$AP = \begin{bmatrix} 2 & 9 & 0 & 2 \\ -1 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 19 & 3 & 0 & -3 \\ 1 & 1 & 1 & 0 \\ 20 & 0 & 0 & 0 \\ 5 & -9 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 57 & -3 & 9 & -6 \\ 3 & -1 & 2 & 3 \\ 60 & 0 & 0 & 0 \\ 15 & 9 & 0 & 0 \end{bmatrix}$$

$$PB = \begin{bmatrix} 19 & 3 & 0 & -3 \\ 1 & 1 & 1 & 0 \\ 20 & 0 & 0 & 0 \\ 5 & -9 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 57 & -3 & 9 & -6 \\ 3 & -1 & 2 & 3 \\ 60 & 0 & 0 & 0 \\ 15 & 9 & 0 & 0 \end{bmatrix}.$$

We conclude our discussion by stating (without proof) the following general diagonalization theorem.

Theorem (Jordan Canonical Form) Let C be a $n \times n$ complex matrix (i.e., entries of C could be either real numbers or complex numbers). Then there are a complex invertible matrix P and a complex block diagonal matrix J such that

$$P^{-1}CP = J.$$

Furthermore, the block matrix J has the following form

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_p \end{bmatrix},$$

where for each i, $i = 1, ..., p$, J_i is either a diagonal matrix of dimension $p_i \times p_i$

$$J_i = \begin{bmatrix} c_i & 0 & \cdots & 0 \\ 0 & c_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_i \end{bmatrix}_{p_i \times p_i}, \quad c_i \text{ is an eigenvalue (real or complex)}$$

or

$$J_i = \begin{bmatrix} c_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & c_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & c_i & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & c_i \end{bmatrix}_{p_i \times p_i}, \quad c_i \text{ is an eigenvalue (real or complex).}$$

We call such matrix J Jordan Canonical Form. The block

$$J_i = \begin{bmatrix} c_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & c_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & c_i & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & c_i \end{bmatrix}_{p_i \times p_i}$$

occurs when c_i is an eigenvalue (real or complex) with multiplicity of at least p_i.

- **Homework #13**

1. Factorize matrices.

   (a) $A == \begin{bmatrix} 1 & -2 & -1 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

   (b) $A == \begin{bmatrix} 1 & 5 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & -3 & 4 \end{bmatrix}$

2. Let A be a $n \times n$ symmetric matrix whose entries are all real numbers.

   (a) Show that for any real or complex vector $X_{n\times 1}$, the quantity

   $$q = \bar{X}^T AX \quad \text{is a real number.}$$

   (Hint: show $\bar{q} = q = q^T$.)

   (b) Show that if λ is an eigenvalue of A, then λ must be a real number. (Hint: using part (a) with X being an eigenvector.)

   (c) A has exactly n real eigenvalues (counting multiplicities).