MA-102

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2K19/A14/35

# ASSIGNMENT-4

$\boxed{1}$ a) $L\left(2e^{-3t}\cos 5t - 3e^{-3t}\sin 5t\right)$

$2L\left(e^{-3t}\cos 5t\right) - 3L\left(e^{-3t}\sin 5t\right)$

$$= \frac{2(s+3)}{(s+3)^2+25} - \frac{3\cdot 5}{(s+3)^2+25} = \frac{2s-9}{s^2+6s+39}.$$

b) $L\left(\sqrt{t}\,e^{3t}\right) = \int_0^\infty \sqrt{t}\,e^{(3-s)t}\,dt$

$z = (s-3)t, \quad dz = (s-3)\,dz$

$$= \int_0^\infty \sqrt{\frac{z}{s-3}}\,e^{-z}\,\frac{dz}{s-3} = \frac{1}{(s-3)^{3/2}}\int_0^\infty z^{1/2}e^{-z}\,dz$$

$$= \frac{\sqrt{\pi}}{2(s-3)^{3/2}}.$$

c) $L(t\sin at)$

$$= \text{Im}\,L\left(t e^{iat}\right) = \frac{1!}{(s-ia)^2}$$

$$= \text{Im of } L\left(\frac{(s+ia)^2}{(s^2+a^2)^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

d) $L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right)$ $\quad \cos t = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$= L\left(\frac{1}{\sqrt{t}}\right) - \frac{1}{2!}L(\sqrt{t}) + \frac{1}{4!}L\left(t^{3/2}\right) \cdots$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\left(1 - \frac{1}{4s} + \frac{1}{2!(4s)^2} - \frac{1}{3!(4s)^3} + \cdots\right)$$

$$= \frac{\pi}{\sqrt{s}}e^{-1/4s}$$

e) $|t-1| + |t+1|$ , $t \geq 0$

$$f(t) = \begin{cases} 2 & , 0 \leq t \leq 1 \\ 2t & , t \geq 1 \end{cases}$$

$$L(f(t)) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^1 2e^{-st}\, dt + \int_1^\infty 2t\, e^{-st}\, dt$$

$$= \left.\frac{2e^{-st}}{-s}\right]_0^1 + 2\left[\frac{te^{-st}}{-s} + \int \frac{e^{-st}}{s}\, dt\right]$$

$$= \frac{2}{s}\left[1 + \frac{e^{-s}}{s}\right].$$

$\boxed{3}$

$f(t) = 0$ , $0 < t < 1$

$t^2 - 2t + 2$ , $t \geq 1$

$$L(f(t)) = \int_0^1 0 . e^{-st} dt + \int_1^\infty \frac{2te^{-st}}{-s} dt - 2\left\{\frac{te^{-st}}{-s} + \frac{e^{-st}}{s^2}\right) + \left.\frac{2e^{-st}}{-s}\right]_1^\infty$$

$$= \frac{t^3 e^{-st}}{-s} + \frac{2}{s}\left[\frac{te^{-st}}{-s} - \int \frac{e^{-st} dt}{-s}\right] + \frac{2te^{-st}}{s} + \frac{2e^{-st}}{s^2} - \frac{2e^{-st}}{s}$$

$$= \frac{e^{-s}}{s} + \cancel{\frac{2e^{-s}}{s^2}} + \frac{2e^{-s}}{s^3} - \cancel{\frac{2e^{-s}}{s}} - \cancel{\frac{2e^{-s}}{s^2}} + \frac{2e^{-s}}{s}$$

$$= (2+s^2)\frac{e^{-s}}{s^3}.$$

$\boxed{4}$

$$L(f(t)) = \frac{1}{1-e^{-s\tau}} \int_0^\tau e^{-st} f(t)\, dt . \qquad \tau = \frac{2\pi}{\omega}$$

$$= \frac{1}{1-e^{-s\tau}}\left[\int_0^{\pi/\omega} e^{-st} \sin\omega t\, dt + \int_{\pi/\omega}^{2\pi/\omega} 0 . e^{-st}\, dt\right.$$

$$I = \int e^{-st} \sin wt \, dt = \frac{e^{-st} \sin wt}{-s} + \int e^{-st} \frac{w \cos wt}{s} dt$$

$$= -\frac{e^{-st} \sin wt}{s} + \frac{w}{s}\left[ \frac{e^{-st} \cos wt}{-s} - \int \frac{w e^{-st} \sin wt}{s} dt \right]$$

$$I\left( \frac{s^2+w^2}{\cancel{s^2}} \right) = \frac{s e^{-st} \sin wt}{\cancel{s^2}} - \frac{w}{\cancel{s^2}} e^{-st} \cos wt$$

$$I = \frac{-e^{-st}}{s^2+w^2} \left( \cancel{s} \sin wt \; \cos wt \right) \Big]_{w}^{\pi}$$

$$= \frac{w}{s^2+w^2} \left( 1 - e^{-\frac{\pi s}{w}} \right)$$

$$L(f(t)) = \frac{1}{\left( 1 - e^{-\frac{\pi s}{w}} \right)} \frac{w}{s^2+w^2}$$

$\boxed{5}$ a) $L(t^3 e^{-3t}) = (-1)^3 \frac{d^3}{ds^3}\left( \frac{1}{s+3} \right) = \frac{6}{(s+3)^4}$

b) $f(t) = e^{-t} \sin 3t$

$$f(s) = \frac{3}{(s+1)^2+9} = \frac{3}{s^2+2s+10}$$

$$L(t f(t)) = (-1) \frac{d}{ds}\left( \frac{3}{s^2+2s+10} \right) = \frac{3(2s+2)}{(s^2+2s+10)^2}$$

$$= \frac{6(s+1)}{(s^2+2s+10)^2}.$$

$\boxed{6}$ a) $L\left( \frac{f(t)}{t} \right) = \int_s^{\infty} f(s) \, ds$

$$f(t) = 1 - e^t = F(s) = \frac{1}{s} - \frac{1}{s-1}.$$

$$= \int_s^{\infty} \left( \frac{1}{s} - \frac{1}{s-1} \right) ds$$

$$= \ln\left( \frac{s}{s-1} \right) \Big]_s^{\infty} = \ln\left( \frac{s-1}{s} \right).$$

b) $f(t) = \cos at - \cos bt$

$$F(s) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$L\left(\frac{\cos at - \cos bt}{t}\right) = \frac{1}{2}\int_s^\infty \frac{2s}{s^2+a^2}\,ds - \frac{1}{2}\int_s^\infty \frac{2s\,ds}{s^2+b^2}$$

$$= \frac{1}{2}\ln\left|\frac{s^2+a^2}{s^2+b^2}\right|_s^\infty = -\frac{1}{2}\ln\left|\frac{s^2+a^2}{s^2+b^2}\right|.$$

$\boxed{7}$ a) $f(t) = \frac{\sin t}{t}$

$$L(f(t)) = L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= L\left(\frac{e^{-t}\sin t}{t}\right) = F(s) = \frac{1}{(s+1)^2+1}$$

$$\int_s^\infty \frac{ds}{(s+1)^2+1} = \tan^{-1}(s+1)\Big]_s^\infty = \pi/2 - \tan^{-1}(s+1).$$

$$L\left(\frac{\int_0^t e^{-t}\sin t\,dt}{t}\right) = \frac{1}{s}\cot^{-1}(s+1).$$

b) $L(t\,f(t)) = (-1)\frac{d}{ds}F(s)$

$$F(s) = L(f(t)) = \int_s^\infty L(e^{-t}\sin t)\,ds$$

$$= \int_s^\infty \frac{1}{(s+1)^2+1}\,ds = \cot^{-1}(s+1).$$

$$L(t\,f(t)) = \frac{-d}{ds}\left(\frac{\cot^{-1}(s+1)}{s}\right)$$

$$= t\left[\frac{t(s+1)}{1+(s+1)^2} + \frac{\cot^{-1}(s+1)}{s^2}\right]$$

$$= \frac{s + \cot^{-1}(s+1)\,(2+s^2+2s)}{s^2(1+(1+2s+s^2))}.$$

c) $L\left(\frac{\sin at}{t}\right) = \int_s^{\infty} \frac{a}{s^2+a^2}\, ds$ (at $s=0$).

$= \cancel{\frac{a}{a}} \tan^{-1}\left(\frac{s}{a}\right)\Big]_{\infty}^{s} = \pi/2$

d) $\int_0^{\infty} e^{-t}\left(\frac{\cos at - \cos bt}{t}\right) dt$

$= \cancel{f}\; L\left(\frac{\cos at}{t}\right)\Big]_{s=1} - L\left(\frac{\cos bt}{t}\right)\Big]_{s=1}$

$= \frac{1}{2}\int_1^{\infty} \frac{2s}{s^2+a^2}\, ds - \frac{1}{2}\int_1^{\infty} \frac{2s}{s^2+b^2}\, ds$

$= \frac{1}{2}\log\left|\frac{s^2+a^2}{s^2+b^2}\right|_1^{\infty} = \frac{1}{2}\ln\left|\frac{b^2+1}{a^2+1}\right|.$

$\boxed{8}$ a) $\frac{s^3-3s+4}{s^3}$ $L^{-1}\left(1-\frac{3}{s^2}+\frac{4}{s^3}\right)$

$= L^{-1}(1) - 3L^{-1}\left(\frac{1}{s^2}\right) + 4L^{-1}\left(\frac{1}{s^3}\right)$

$= 1-3t+t^2.$

b) $\frac{s+2}{(s-2)^2+9} = e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t.$

c) $\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$

$4s+5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$

$A = \frac{1}{3},\quad B = 3,\quad C = -1/3.$

$= \frac{1}{3(s-1)} + \frac{3}{(s-1)^2} - \frac{1}{3(s+2)}$

$= \frac{e^t}{3} + 3te^t - \frac{1}{3}e^{-2t}.$

f) $\log\left(\frac{s+1}{s-1}\right)$

$$L^{-1}\left(\int_s^\infty F(s)\,ds\right) = \frac{f(t)}{t}$$

$$L^{-1}\left(\int_s^\infty \frac{ds}{s+1} - \int_s^\infty \frac{ds}{s-1}\right) = \frac{e^{-t}}{t} - \frac{e^{t}}{t}$$

g) $$\frac{e^{-4s}}{s^2+9} = \frac{1}{3}L^{-1}\left(\frac{3}{s^2+9}\right) = \frac{1}{3}\sin 3t$$

$$= L^{-1}\left(e^{-as}F(s)\right) = u(t-a)f(t-a)$$

$$= L^{-1}\left(\frac{e^{-4s}}{s^2+9}\right) = \frac{u(t-4)\sin 3(t-4)}{3}.$$

$\boxed{9}$ a) $$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left(\frac{1}{a}\frac{s}{s^2+a^2} * \frac{a}{s^2+a^2}\right)$$

$$F(s) = \frac{s}{s^2+a^2} \Rightarrow f(t) = \cos at$$

$$= \frac{1}{a}L^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{a}{s^2+a^2}\right)$$

$$G(s) = \frac{a}{s^2+a^2} \Rightarrow g(t) = \sin at$$

$$= \frac{1}{2a}\int_0^t 2\cos a\tau \sin(at-a\tau)\,d\tau$$

$$= \frac{1}{2a}\int_0^t \left(\sin(a\tau-a\tau+at) + \sin(a\tau-a\tau+at)\right)d\tau$$

$$= \frac{1}{2a}\left[\frac{\cos(at-2a\tau)}{2a} + \frac{t\sin at}{a}\right].$$

b) $L^{-1}\left(\frac{1}{s^2+1}\cdot\frac{1}{s^2+9}\right)$

$f(t) = \sin t \quad , \quad F(s) = \frac{1}{s^2+1}$

$g(t) = \sin 3t \quad , \quad G(s) = \frac{3}{s^2+9}$ .

$= \frac{1}{2\cdot 3}\int_0^t 2\sin\tau \sin(3t-3\tau)\,d\tau$

$= \frac{-1}{6}\int_0^t \cos(\tau+3t-3\tau) + \cos(\tau-3t)\,d\tau$

$= \cancel{\frac{1}{6}}\ \frac{2\sin t}{24} - 2\sin 3t + \frac{\sin t}{24} + \frac{\sin 3t}{24}$ .

$= \frac{\sin t}{8} - \frac{\sin 3t}{24}$ .

$\boxed{10}$ $L(f''(x)) = s^2 L(f(x)) - s f(0) - f'(0))$

$L(f'(x)) = s L(f(x)) - f(0))$

$s^2 L(x(t)) - s(2) + 1 - 2(sL(x(t)) - 2) + L(x(t)) = \frac{1}{s-1}$

$L(x(t)) = \frac{1}{(s-1)^3} + \frac{2s-5}{(s-1)^2}$

$x(t) = \frac{1}{2} L^{-1}\left(\frac{2}{(s-1)^3}\right) + L^{-1}\left(\frac{2s-5}{(s-1)^2}\right)$

$= \frac{1}{2}t^2 e^t + L^{-1}\left(\frac{2}{(s-1)} - \frac{3}{(s-1)^2}\right)$

$= \frac{1}{2}t^2 e^t + 2e^t - 3te^t$ .

b) $s^3 L(f(x)) - s^2 f(0) - s f'(0) - f''(0)$

$+ 2\left(s^2 f(x)) - s f(0) - f'(0)\right) - (sL(f(x)) - f(0))$

$- 2L(f(x)) = 0$

$$L(f(x))\left[s^3 - 2s^2 - s - 2\right] - 6 = 0$$

$$L(f(x)) = \frac{6}{s^3 + 2s^2 - s - 2}$$

$$\frac{6}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$6 = A(s+2)(s+1) + B(s-1)(s+1) + C(s+2)(s-1)$$

$A = 1,\ C = -3,\ B = 2.$

$$f(x) = L^{-1}\left(\frac{1}{s-1}\right) + 2L^{-1}\left(\frac{1}{s+2}\right) - 3L^{-1}\left(\frac{1}{s+1}\right)$$

$$= e^{x} + 2e^{-2x} - 3e^{-x}.$$

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