Delhi Technological University
Department of Applied Mathematics
Assignment-III

Course: Mathematics-II **Code: MA-102**

1. Find the power series solution of $\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+(x^2+2)y=0$ in powers of x (that is, about $x=0$).

   Ans. $y = c_0\left(1-x^2+\frac{1}{4}x^4+\dots\right)+c_1\left(x-\frac{1}{2}x^3+\frac{3}{40}x^5+\dots\right).$

2. Find the power series solution of the initial-value problem

$$(x^2-1)\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+xy=0, \quad y(0)=4, \quad y'(0)=6.$$

   Ans. $y=4+6x+\frac{11}{3}x^3+\frac{1}{2}x^4+\frac{11}{4}x^5+\dots.$

3. Find the power series solution in powers of $(x-1)$ of the initial-value problem

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+2y=0, \quad y(1)=1, \quad y'(1)=2.$$

   Ans. $y=1+2(x-1)-2(x-1)^2+\frac{2}{3}(x-1)^3-\frac{1}{6}(x-1)^4+\frac{1}{15}(x-1)^5+\dots$

4. Determine whether the equation $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+(x^2-8)y=0$ has two linearly independent Frobenius series solutions.

5. Find the exponents in the possible Frobenius series solutions of the equation

$$2x^2(x+1)\frac{d^2y}{dx^2}+3x(x+1)^3\frac{dy}{dx}+(x^2-1)y=0.$$

6. Use the method of Frobenius to find solution of

$$2x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+(x-5)y=0$$

   in some interval $0<x<R$.

   Ans. $y=C_1x^{5/2}\left(1-\frac{1}{9}x+\frac{1}{198}x^2-\frac{1}{7722}x^3+\dots\right)$

$$+C_2x^{-1}\left(1+\frac{1}{5}x+\frac{1}{30}x^2+\frac{1}{90}x^3+\dots\right).$$

7. Find the series solution of the Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \alpha(\alpha+1)y = 0,$$

where α is a constant.

Ans. $$y = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \dots\right] + c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 + \dots\right].$$

8. Define Legendre's polynomial $P_n(x)$. Prove that

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n\,, \\ \dfrac{2}{2n+1} & \text{if } m = n\,. \end{cases}$$

9. Express $x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Legendre polynomial.
Ans. $$x^4 + 2x^3 + 2x^2 - x - 3 = \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{40}{21}P_2(x) + \frac{1}{5}P_1(x) - \frac{224}{105}P_0(x).$$

10. Find the series solutions of Bessel's equation of order α, $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2-\alpha^2)y = 0$, where α is a parameter.
Ans. $$y = C_1\,x^{\alpha}\left[1 - \frac{x^2}{2(2\alpha+2)} + \frac{x^4}{2.4(2\alpha+2)(2\alpha+4)} - \dots\right] + C_2\,x^{-\alpha}\left[1 - \frac{x^2}{2(2-2\alpha)} + \frac{x^4}{2.4(2-2\alpha)(4-2\alpha)} - \dots\right],\ \alpha \neq 0, 1, 2, \dots.$$

11. Prove that
   (a) $P_n(1) = 1$
   (b) $P_n(-1) = (-1)^n$
   (c) $P_n'(1) = \dfrac{n(n+1)}{2}$
   (d) $P_n'(-1) = (-1)^{n-1}\dfrac{n(n+1)}{2}$
   (e) $P_n(-x) = (-1)^n P_n(x)$.

12. Define Bessel function $J_n(x)$ of the first kind of order n. Prove that $J_{-n}(x) = (-1)^n J_n(x)$ for $n = 1, 2, 3, \dots$.

13. Prove that

$$\int_0^1 x J_n(ax)\,J_n(bx)\,dx = \begin{cases} 0, & \text{if } a \neq b \\ \dfrac{J_{n+1}^2(x)}{2}, & \text{if } a = b \end{cases},$$

where a and b are the roots of $J_n(x) = 0$.

14. Prove that $\displaystyle\int_0^1 xJ_n^2(ax)\,dx = \frac{1}{2}\left[J_n'^2(a) + \left(1 - \frac{n^2}{a^2}\right)J_n^2(a)\right].$

15. Show that

(a) $\displaystyle\int_0^x x^{-n}J_{n+1}(x)\,dx = \frac{1}{2^n\Gamma(n+1)} - x^{-n}J_n(x),\ n > 1.$

(b) $\displaystyle\int_0^\infty x^{-n}J_{n+1}(x)\,dx = \frac{1}{2^n\Gamma(n+1)},\ n > -\frac{1}{2}.$

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