(1)

Assignment - 4

Q. Find the Laplace transform.

a) $e^{-3t}(2\cos 5t - 3\sin 5t)$

Ans = $\mathcal{L}[e^{-3t}(2\cos 5t - 3\sin 5t)]$

$= 2\mathcal{L}[e^{-3t}\cos 5t] - 3\mathcal{L}[e^{-3t}\sin 5t]$

$\Rightarrow$ Since $\mathcal{L}[\cos 5t] = \frac{s}{s^2+25}$ and $\mathcal{L}[\sin 5t] = \frac{5}{s^2+25}$

$\Rightarrow$ By First shifting theorem:-

$\Rightarrow \mathcal{L}[e^{at} f(t)] = F(s-a)$

$\Rightarrow 2\left[\frac{s+3}{(s+3)^2+25}\right] - 3\left[\frac{5}{(s+3)^2+25}\right]$

$\Rightarrow \frac{2s+6-15}{s^2+9+25+6s} = \frac{2s-9}{s^2+6s+34}$

(ii) $\sqrt{t}\, e^{3t}$

Ans $\mathcal{L}[\sqrt{t}\, e^{3t}]$

$\mathcal{L}[\sqrt{t}] = \int_0^\infty e^{-st}\sqrt{t}\, dt$

$st = p \Rightarrow dt = \frac{dp}{s}$

$= \int_0^\infty e^{-p}\left(\frac{p}{s}\right)^{1/2}\frac{dp}{s} =$

$= \frac{1}{s^{3/2}}\int_0^\infty e^{-p} p^{1/2}\, dp$ [Gamma Function]

$= \frac{1}{s^{3/2}}\Gamma(3/2) = \frac{\sqrt{\pi}}{2s^{3/2}}$

By First shifting theorem

$\mathcal{L}[e^{3t}\sqrt{t}] = \frac{\sqrt{\pi}}{2(s-3)^{3/2}}$

(iii) $t\sin(at)$

$\mathcal{L}[t\sin at]$

Since we know that $\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$

By Theorem of differentiation of transform

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$\mathcal{L}[t\sin at] = (-1)\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]$$

$$= (-1)(-1)\frac{2sa}{(s^2+a^2)^2} = \frac{2as}{(s^2+a^2)^2}$$

(iv) $\mathcal{L}\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]$

Ans $\mathcal{L}\left[\frac{1}{\sqrt{t}}\left[1-\frac{(\sqrt{t})^2}{2!}+\frac{(\sqrt{t})^4}{4!}-\frac{(\sqrt{t})^6}{6!}\dots\right]\right]$

$\mathcal{L}\left[\frac{1}{\sqrt{t}}\left[1-\frac{t}{2!}+\frac{t^2}{4!}-\frac{t^3}{6!}+\frac{t^4}{8!}\dots\right]\right]$

$\mathcal{L}\left[t^{-1/2}-\frac{t^{1/2}}{2!}+\frac{t^{3/2}}{4!}-\frac{t^{5/2}}{6!}+\frac{t^{7/2}}{8!}\dots\right]$

By linearity of Laplace transform.

$\Rightarrow \mathcal{L}[t^{-1/2}]-\frac{1}{2!}\mathcal{L}[t^{1/2}]+\frac{1}{4!}\mathcal{L}[t^{3/2}]-\frac{1}{6!}\mathcal{L}[t^{5/2}]+\dots$

$\Rightarrow \frac{\Gamma(\frac{1}{2})}{s^{1/2}}-\frac{1}{2!}\frac{\Gamma(3/2)}{s^{3/2}}+\frac{1}{4!}\frac{\Gamma(5/2)}{s^{5/2}}-\frac{1}{6!}\frac{\Gamma(7/2)}{s^{7/2}}\dots$

$\Rightarrow \frac{\Gamma(1/2)}{s^{1/2}}\left(1-\frac{1}{4s}+\frac{1}{2!(4s)^2}-\frac{1}{3!(4s)^3}+\dots\right)$

$\Rightarrow \frac{\sqrt{\pi}}{s^{1/2}}e^{-1/4s}$

(v) $f(t) = |t-1|+|t+1|$, $t \geq 0$

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$$f(t) = \begin{cases} 2 & 0 < t \leq 1 \\ 2t & t > 1 \end{cases}$$

$$\mathcal{L}[f(t)] = \int_0^1 e^{-st}\, 2\, dt + \int_1^\infty e^{-st}\, 2t\, dt$$

$$= 2 \frac{e^{-st}}{-s}\Big|_0^1 + 2\left[t \frac{e^{-st}}{-s}\Big|_1^\infty - \int_1^\infty \frac{e^{-st}}{-s} dt \right]$$

$$= \frac{-2}{s}(e^{-s} - 1) + 2\left[\frac{1}{s}(e^{-s}) + \frac{1}{s} \frac{e^{-st}}{-s}\Big|_1^\infty \right]$$

$$= \frac{-2e^{-s}}{s} + \frac{2}{s} + \frac{2e^{-s}}{s} + \frac{-2}{s^2}(-e^{-s})$$

$$= \frac{2}{s}\left(1 + \frac{e^{-s}}{s}\right)$$

2. Find the Laplace transform of periodic functions

(i) $f(t) = \begin{cases} \sin \omega t & 0 < t < \pi/\omega \\ 0 & \pi/\omega < t < 2\pi/\omega \end{cases}$

The function is periodic with period $\frac{2\pi}{\omega}$

we know that $\mathcal{L}[f(t)] = \dfrac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}$

$$\Rightarrow \int_0^T e^{-st} f(t)\, dt = \int_0^{\pi/\omega} e^{-st} \sin \omega t\, dt + \underbrace{\int_{\pi/\omega}^{2\pi/\omega} e^{-st} (0)\, dt}_{=0}$$

$$I = \int_0^{\pi/\omega} e^{-st} \sin \omega t\, dt \quad -$$

$$I = \sin \omega t \frac{e^{-st}}{-s}\Big|_0^{\pi/\omega} - \int_0^{\pi/\omega} \omega \cos \omega t \frac{e^{-st}}{-s} dt$$

$$I = \frac{\omega}{s} \int_0^{\pi/\omega} \cos \omega t\, e^{-st} dt$$

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$$I = \frac{\omega}{s}\left[\cos\omega t \frac{e^{-st}}{-s}\Big|_0^{\pi/\omega} - \int_0^{\pi/\omega} -\omega\sin\omega t \frac{e^{-st}}{-s}\,dt\right]$$

$$I = \frac{\omega}{s}\left[\frac{e^{-s\pi/\omega}+1}{s} - \frac{\omega}{s} I\right]$$

$$I\left(\frac{s^2+\omega^2}{s\omega}\right) = \frac{e^{-s\pi/\omega}+1}{s}$$

$$I = (1+e^{-s\pi/\omega})\left[\frac{\omega}{s^2+\omega^2}\right]$$

$$\mathcal{L}[f(t)] = \frac{(1+e^{-s\pi/\omega})\left(\frac{\omega}{s^2+\omega^2}\right)}{1-(e^{-s\pi/\omega})^2}$$

$$= \frac{(1+e^{-s\pi/\omega})\left(\frac{\omega}{s^2+\omega^2}\right)}{(1-e^{-s\pi/\omega})(1+e^{-s\pi/\omega})}$$

$$= \left(\frac{\omega}{s^2+\omega^2}\right)\frac{1}{(-e^{-s\pi/\omega}+1)}$$

(ii) $f(t) = \begin{cases} t & 0 < t < \pi \\ \pi - t & \pi < t < 2\pi \end{cases}$

Sol: The function is periodic with period 2π

$$\mathcal{L}[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

$$\Rightarrow \int_0^T e^{-st} f(t)\,dt = \int_0^{\pi} e^{-st}\, t\,dt + \int_{\pi}^{2\pi} e^{-st}(\pi - t)\,dt$$

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$$\Rightarrow \left.\frac{t e^{-st}}{-s}\right|_0^{\pi} - \int_0^{\pi} \frac{e^{-st}}{-s}\,dt + \pi \left.\frac{e^{-st}}{-s}\right|_{\pi}^{2\pi} - \int_{\pi}^{2\pi} t e^{-st}\,dt$$

$$\Rightarrow -\frac{\pi}{s} e^{-s\pi} - \frac{1}{s^2}(e^{-s\pi}-1) - \frac{\pi}{s} e^{-2\pi s} + \frac{\pi}{s} e^{-\pi s}$$

$$-\left[\frac{-1}{s}(2\pi e^{-2\pi s} - \pi e^{-\pi s}) + \frac{1}{s}\left.\frac{e^{-st}}{-s}\right|_{\pi}^{2\pi}\right]$$

$$\Rightarrow -\frac{\pi}{s} e^{-s\pi} - \frac{e^{-s\pi}}{s^2} + \frac{1}{s^2} - \frac{\pi}{s} e^{-2\pi s} + \frac{\pi}{s} e^{-\pi s}$$

$$+ \frac{2\pi}{s} e^{-2\pi s} - \frac{\pi}{s} e^{-\pi s} + \frac{1}{s^2}(e^{-2\pi s} - e^{-\pi s})$$

$$\Rightarrow \frac{\pi}{s}(e^{-2\pi s} - e^{-\pi s}) + \frac{1}{s^2}(1 + e^{-2\pi s} - 2e^{-\pi s})$$

$$\mathcal{L}[f(t)] = \frac{\frac{\pi}{s}(e^{-2\pi s} - e^{-\pi s}) + \frac{1}{s^2}(1+e^{-2\pi s} - 2e^{-\pi s})}{(1-e^{-2\pi s})}$$

3. Using multiplicative property, Find Laplace transform

a) $t^3 e^{-3t}$

Using property

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$\mathcal{L}[t^3 e^{-3t}] = (-1)^3 \frac{d^3}{ds^3} \frac{1}{(s+3)}$$

$$= -\left[\frac{d^2}{ds^2} \frac{-1}{(s+3)^2}\right]$$

$$= -\left[\frac{d}{ds} \frac{2}{(s+3)^3}\right]$$

$$= \frac{6}{(s+3)^4}$$

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b) $te^{-t}\sin 3t$

$$\mathcal{L}[e^{-t}\sin 3t] = \frac{3}{(s+1)^2+9} = \frac{3}{(s+1)^2+9}$$

using property

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$\mathcal{L}[te^{-t}\sin 3t] = (-1)\frac{d}{ds}\left[\frac{3}{(s+1)^2+9}\right]$$

$$= -3\left[\frac{-1}{[(s+1)^2+9]^2}\right] 2(s+1)$$

$$= \frac{6(s+1)}{(s^2+2s+10)^2}$$

4) Using division property. find laplace transform (5)

a) $\frac{1-e^t}{t}$

$$\mathcal{L}[1-e^{+t}] = \mathcal{L}[1] - \mathcal{L}[e^{+t}]$$

$$= \frac{1}{s} - \frac{1}{s-1}$$

$$\mathcal{L}\left[\frac{1-e^{+t}}{t}\right] = \int_s^\infty \frac{1}{s}ds - \int_s^\infty \frac{1}{s-1}ds \quad \left[\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds\right]$$

$$= \ln s - \ln|s-1| \Big|_s^\infty$$

$$= \ln\left(\frac{s}{s-1}\right)\Big|_s^\infty$$

$$= -\ln\left(\frac{s}{s-1}\right) = \ln\left(\frac{s-1}{s}\right)$$

b) $\frac{\cos(at) - \cos(bt)}{t}$

$$\mathcal{L}[\cos(at) - \cos(bt)] = \mathcal{L}[\cos at] - \mathcal{L}[\cos bt]$$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}\int_s^\infty \frac{2s}{s^2+a^2}\,ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+b^2}\,ds$$

$$= \frac{1}{2}\ln|s^2+a^2| - \frac{1}{2}\ln|s^2+b^2| \Big|_s^\infty$$

$$= \frac{1}{2}\ln\left|\frac{s^2+a^2}{s^2+b^2}\right| \Big|_s^\infty$$

$$= \frac{1}{2}\left[-\ln\left|\frac{s^2+a^2}{s^2+b^2}\right|\right]$$

$$= \ln\left|\frac{s^2+b^2}{s^2+a^2}\right|^{1/2}$$

5) Evaluate

a) $L\left[e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right]$

$\Rightarrow$ $L[\sin at] = \frac{a}{s^2+a^2}$

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$= a \times \frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$

$$L\left[\int_0^t \frac{\sin at}{t}\right] = \frac{1}{s}\cot^{-1}\left(\frac{s}{a}\right)$$

$$L\left[e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{1}{s+1}\cot^{-1}\left(\frac{s+1}{a}\right)$$

$$= \frac{1}{s+1}\cot^{-1}(s+1) \quad [\text{for } a=1]$$

b) $\mathcal{L}\left\{t \int_0^t \frac{e^{-t}\sin t}{t}\, dt\right\}$

$\mathcal{L}[\sin t] = \frac{1}{s^2+1}$

$\mathcal{L}[e^{-t}\sin t] = \frac{1}{(s+1)^2+1}$

$\mathcal{L}\left[\frac{e^{-t}\sin t}{t}\right] = \int_s^\infty \frac{ds}{(s+1)^2+1}$

$= \cancel{\cot}\ \tan^{-1}(s+1)\Big|_s^\infty$

$= \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$

$\mathcal{L}\left[\int_0^t \frac{e^{-t}\sin t}{t}\, dt\right] = \frac{\cot^{-1}(s+1)}{s}$

$\mathcal{L}\left[t \int_0^t \frac{e^{-t}\sin t}{t}\, dt\right] = (-1)\frac{d}{ds}\left[\frac{\cot^{-1}(s+1)}{s}\right]$

$= -1 \frac{\left[\frac{-1}{(s+1)^2+1}\, s - \cot^{-1}(s+1)\right]}{s^2}$

$= \frac{s + (s^2+2s+2)\cot^{-1}(s+1)}{s^2[s^2+2s+2]}$

c) $\int_0^\infty \frac{\sin at}{t}\, dt \quad a > 0$

$\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$

$\mathcal{L}\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a\, ds}{s^2+a^2} = a \times \frac{1}{a} \tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty$

$= \cot^{-1}\left(\frac{s}{a}\right)$

$$\int_0^\infty \frac{\sin at}{t} = \mathcal{L}\left[\frac{\sin at}{t}\right]_{s=0} = \cot^{-1}(0) = \pi/2$$

$$\int_0^\infty e^{-t}\left(\frac{\cos at - \cos bt}{t}\right)$$

$$\mathcal{L}[\cos at - \cos bt] = \mathcal{L}[\cos at] - \mathcal{L}[\cos bt]$$

$$= \frac{a}{s^2+a^2} - \frac{b}{s^2+b^2}$$

$$\mathcal{L}\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}\int_s^\infty \frac{2a\, ds}{s^2+a^2} - \frac{1}{2}\int_s^\infty \frac{2b\, ds}{s^2+b^2}$$

$$= \frac{1}{2} \ln\left|\frac{s^2+a^2}{s^2+b^2}\right|_s^\infty$$

$$= \frac{1}{2}\ln\left|\frac{s^2+b^2}{s^2+a^2}\right|$$

$$\mathcal{L}\left[e^{-t}\left(\frac{\cos at - \cos bt}{t}\right)\right] = \frac{1}{2}\ln\left|\frac{(s+1)^2+b^2}{(s+1)^2+a^2}\right|$$

$$\Rightarrow \int_0^\infty e^{-t}\left(\frac{\cos at - \cos bt}{t}\right)dt = \mathcal{L}\left[e^{-t}\left(\frac{\cos at - \cos bt}{t}\right)\right]_{s=0}$$

$$= \frac{1}{2}\ln\left|\frac{1+b^2}{1+a^2}\right|$$

5) Find the inverse Laplace transform.

a) $\frac{s^2 - 3s + 4}{s^3}$

$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}\right]$$

$\Rightarrow$ According to linearity of inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] - 3\,\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + 4\,\mathcal{L}^{-1}\left(\frac{1}{s^3}\right)$$

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$f(t) = 1 - 3t + 4t^2$ $\left[\text{using } \frac{1}{s^{\alpha+1}} = \frac{t^{\alpha}}{\Gamma(\alpha+1)} \right.$

(b) $\frac{s+2}{s^2-4s+13}$

$\Rightarrow \frac{s+2}{s^2-4s+4+9} = \frac{s-2+4}{(s-2)^2+3^2}$

$= \frac{(s-2)}{(s-2)^2+3^2} + \frac{4}{3} \times \frac{3}{(s-2)^2+3^2}$

$= \mathcal{L}^{-1}\left[\frac{(s-2)^2}{(s-2)^2+3^2} + \frac{4}{3} \times \frac{3}{(s-2)^2+3^2} \right]$

$= \mathcal{L}^{-1}\left[\frac{(s-2)^2}{(s-2)^2+3^2} \right] + \frac{4}{3} \mathcal{L}^{-1}\left[\frac{3}{(s-2)^2+3^2} \right]$

$= f(t) = e^{2t}\cos 3t + \frac{4}{3} e^{2t} \sin 3t$

(c) $\frac{4s+5}{(s-1)^2(s+2)}$

$\frac{4s+5}{(s-1)^2(s+2)} = \frac{As+B}{(s-1)^2} + \frac{C}{s+2}$

$= \frac{(As+B)(s+2) + C(s-1)^2}{(s-1)^2(s+2)}$

$\frac{4s+5}{(s-1)^2(s+2)} = \frac{s^2(A+C) + s(2A+B-2C) + 2B+C}{(s-1)^2(s+2)}$

comparing $A+C=0 \Rightarrow A=-C$

$2A+B-2C = 4$

$2B+C = 5$ — (I)

$B-4C = 4$ — (II)

Solving (I) and (II)

we get $B = \frac{24}{9}$; $C = \frac{-3}{9}$; $A = \frac{3}{9}$

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$$\frac{3(s-1+1)}{9(s-1)^2} + \frac{24}{9(s-1)^2} - \frac{3}{9(s+2)}$$

$$\frac{3}{9(s-1)} + \frac{3}{9(s-1)^2} + \frac{24}{9(s-1)^2} - \frac{3}{9(s+2)}$$

$$\frac{3}{9(s-1)} + \frac{\cancel{27}\,3}{\cancel{9}(s-1)^2} - \frac{3}{4(s+2)}$$

$$\frac{3}{9}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \frac{\cancel{27}\,3}{\cancel{9}}\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) - \frac{3}{9}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$$

$$\Rightarrow \frac{3}{9}e^{t} + \frac{\cancel{27}\,3te^{t}}{\cancel{9}} - \frac{3}{9}e^{-2t}$$

1) $$\frac{(s+2)^2}{(s^2+4s+8)^2}$$

$$\frac{(s+2)^2}{(s^2+4s+8)^2} = \frac{1}{2}\left[\frac{1}{(s+2)^2+2^2} + \frac{(s+2)^2-2^2}{[(s+2)^2+2^2]^2}\right]$$

$$= \frac{1}{2}\left[\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+2^2}\right] + \mathcal{L}^{-1}\left[\frac{(s+2)^2-2^2}{((s+2)^2+2^2)^2}\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s+2)^2+2^2}\right] + \mathcal{L}^{-1}\left[\frac{(s+2)^2-2^2}{((s+2)^2+2^2)^2}\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}e^{-2t}\sin 2t + e^{-2t}t\cos 2t\right]$$

$$= e^{-2t}\left(\frac{\sin 2t}{4} + \frac{t\cos 2t}{2}\right)$$

2) $$\frac{s}{(s^2+a^2)^2}$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$\frac{f(t)}{t} = \frac{1}{2}\mathcal{L}^{-1}\left[\int_s^\infty \frac{2s\,ds}{(s^2+a^2)^2}\right]$$ put $s^2+a^2 = p$, $2s\,ds = dp$

$$\frac{f(t)}{t} = \frac{1}{2}\mathcal{L}^{-1}\left[\int_{s^2+a^2}^{\infty}\frac{dp}{p^2}\right]$$

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$$\frac{f(t)}{t} = \frac{1}{2}\, \mathcal{L}^{-1}\left[\frac{-1}{p} \Big| \frac{\infty}{s^2+a^2} \right]$$

$$\frac{f(t)}{t} = \frac{1}{2a}\, \mathcal{L}^{-1}\left[\frac{a}{s^2+a^2} \right]$$

$$\frac{f(t)}{t} = \frac{1}{2a} \sin at \Rightarrow f(t) = \frac{t}{2a} \sin at$$

(f) $\log \frac{s+1}{s-1}$

$\Rightarrow$ $t f(t) = \mathcal{L}^{-1}\left[-\frac{d}{ds} \log\left[\frac{s+1}{s-1}\right]\right]$

$\Rightarrow$ $t f(t) = \mathcal{L}^{-1}\left[-\left[\frac{s-1}{s+1} \left[\frac{s-1-(s+1)}{(s-1)^2} \right]\right]\right]$

$\Rightarrow$ $t f(t) = \mathcal{L}^{-1}\left[-\left(\frac{s-1}{s+1}\right)\left[\frac{-2}{(s-1)^2}\right]\right]$

$\Rightarrow$ $t f(t) = \mathcal{L}^{-1}\left[\frac{2}{(s-1)(s+1)} \right]$

$\Rightarrow$ $t f(t) = \mathcal{L}^{-1}\left[\frac{2}{2} \left[\frac{1}{s-1} - \frac{1}{s+1} \right]\right]$

$\Rightarrow$ $t f(t) = \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$

$$f(t) = \frac{e^{t} - e^{-t}}{t} \times \frac{2}{2}$$

$$f(t) = \frac{2}{t} \sin ht$$

2) Use convolution theorem to evaluate

a) $\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2} \right]$

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Using convolution theorem

$$\mathcal{L}^{-1}[F(s)\,G(s)] = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right] = \frac{1}{a}\int_0^t \sin a(t-\tau)\cos a\tau\,d\tau$$

$$\Rightarrow \frac{1}{a}\int_0^t (\sin at\cos a\tau - \cos at\sin a\tau)\cos a\tau\,d\tau$$

$$\Rightarrow \frac{1}{a}\left[\sin at\int_0^t \cancel{\sin}\cos^2 a\tau\,d\tau - \cos at\int_0^t \sin a\tau\cos a\tau\,d\tau\right]$$

$$\Rightarrow \frac{1}{a}\left[\frac{\sin at}{2}\int_0^t (1+\cos 2a\tau)\,d\tau - \frac{\cos at}{a}\,\frac{\sin^2 at}{2}\right]$$

$$\Rightarrow \frac{1}{a}\left[\frac{\sin at}{2}\left[t+\frac{\sin 2a\tau}{2a}\Big|_0^t\right] - \frac{\cos at}{a}\,\frac{\sin^2 at}{2}\right]$$

$$\Rightarrow \frac{1}{a}\left[\frac{\sin at}{2}\left(t+\frac{\sin 2at}{2a}\right) - \frac{\cos at\,\sin^2 at}{2a}\right.$$

$$\Rightarrow \frac{t\sin at}{2a} + \frac{2\sin^2 at\cos at}{4a} - \frac{\cos at\sin^2 at}{2a}$$

$$\Rightarrow \frac{t\sin at}{2a}$$

b) $\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)(s^2+9)}\right]$

Ans $\mathcal{L}^{-1}[F(s)\,G(s)] = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\cdot\frac{1}{s^2+9}\right] = \int_0^t \sin\tau\,\frac{\sin 3(t-\tau)}{3}\,d\tau$$

$$\Rightarrow \frac{1}{3}\int_0^t \sin\tau\,[\sin 3t\cos 3\tau - \cos 3t\sin 3\tau]\,d\tau$$

$$\Rightarrow \frac{1}{3}\left[\sin 3t\int_0^t \sin\tau\cos 3\tau\,d\tau - \cos 3t\int_0^t \sin\tau\sin 3\tau\,d\tau\right]$$

$$\Rightarrow \frac{1}{3}\left[\frac{\sin 3t}{2}\int_0^t [\sin 4\tau + \sin(-2\tau)]\,d\tau - \frac{\cos 3t}{2}\int_0^t (\cos 2\tau - \cos 4\tau)\,d\tau\right.$$

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$\Rightarrow \frac{1}{3}\left[\frac{\sin 3t}{2}\left(\frac{-\cos 4t+1}{4} - \frac{1}{2}(-\cos 2t+1)\right) - \frac{\cos 3t}{2}\left(\frac{\sin 2t}{2} - \right.\right.$

$\Rightarrow \frac{1}{3}\left[\frac{\sin 3t}{2}\left[\frac{-\cos 4t+1}{4} + \frac{\cos 2t-1}{2}\right] - \frac{\cos 3t}{2}\left[\frac{\sin 2t}{2} - \right.\right.$

8. Use Laplace transform method to solve the proble

a) $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$; $x(0)=2$ $x'(0)=-1$

Taking Laplace transform both sides

$\mathcal{L}[x''(t)] - 2\mathcal{L}[x'(t)] + \mathcal{L}[x(t)] = \mathcal{L}[e^t]$

$\mathcal{L}[x(t)] = X(s)$

$\Rightarrow \mathcal{L}[x''(t)] = s^2X(s) - sx(0) - x'(0)$

$s^2X(s) - 2s + 1$

$\mathcal{L}[x'(t)] = sX(s) - x(0)$

$sX(s) - 2$

$\Rightarrow s^2X(s) - 2s + 1 - 2sX(s) + 4 + X(s) = \frac{1}{s-1}$

$\Rightarrow X(s)[s^2-2s+1] = \frac{1}{s-1} + 2s - 5$

$\Rightarrow X(s)(s-1)^2 = \frac{1+(2s-5)(s-1)}{(s-1)}$

$\Rightarrow X(s) = \frac{1+2s^2-2s-5s+5}{(s-1)^3}$

$\Rightarrow X(s) = \frac{2s^2-7s+6}{(s-1)^3}$

$\Rightarrow X(s) = \frac{2[s^2-2s+1]}{(s-1)^3} - \frac{3(s-1)}{(s-1)^3} + \frac{1}{(s-1)^3}$

$\Rightarrow X(s) = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$

$\Rightarrow x(t) = 2\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - 3\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s-1)^3}\right]$

$x(t) = 2e^{t} - 3\frac{te^{t}}{\lfloor 2} + \frac{t^2 e^t}{\lfloor 3}$

$x(t) = 2e^{t} - 3te^{t} + \frac{1}{2}t^2e^{t}$

$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0 \quad [y(0)=0, y'(0)=0, y''(0)=6]$

Taking Laplace transform b/s

$\mathcal{L}[x'''(t)] + 2\mathcal{L}[x''(t)] - \mathcal{L}[x'(t)] - 2\mathcal{L}[x(t)] = \mathcal{L}[0]$

$\Rightarrow \mathcal{L}[x'''(t)] = s^3X(s) - s^2(0) - s(0) - 6$

$= s^3X(s) - 6$

$\Rightarrow \mathcal{L}[x''(t)] = s^2X(s) - s(0) - (0)$

$= s^2X(s)$

$\Rightarrow \mathcal{L}[x'(t)] = sX(s) - 0$

$\Rightarrow s^3X(s) - 6 + 2s^2X(s) - sX(s) - 2X(s) = 0 \quad [\mathcal{L}[0]=0]$

$\Rightarrow X(s)[s^3 + 2s^2 - s - 2] = 6$

$\Rightarrow X(s)[s^2(s+2) - 1(s+2)] = 6$

$\Rightarrow X(s)(s+2)(s^2-1) = 6$

$\Rightarrow X(s) = \frac{6}{(s+2)(s+1)(s-1)}$

$\Rightarrow X(s) = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s-1}$

$\frac{A(s^2-1) + B(s^2+s-2) + C(s^2+3s+2)}{(s+2)(s+1)(s-1)}$

$\left.\begin{aligned} A + B + C &= 0 \\ B + 3C &= 0 \\ -A - 2B + 2C &= 6 \end{aligned}\right\}$ solving these $A = 2, B = -3, C = 1$

$\Rightarrow X(s) = \frac{2}{s+2} - \frac{3}{s+1} + \frac{1}{s-1}$

$\Rightarrow y(x) = 2\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - 3\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$

$= 2e^{-2x} - 3e^{-x} + e^{x}$

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c) $t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + ty = \cos t$; $y(0) = 1$

$\mathcal{L}[y(t)] = Y(s)$ and taking laplace transform b/s

$\Rightarrow \mathcal{L}[ty''(t)] + 2\mathcal{L}[y'(t)] + \mathcal{L}[ty(t)] = \mathcal{L}[\cos t]$

$\Rightarrow \mathcal{L}[ty''(t)] = -\frac{d}{ds}\mathcal{L}[y''(t)]$

$= -\frac{d}{ds}[s^2Y(s) - s - c] =$

$= -[2sY(s) + Y'(s)s^2 - 1]$

$\Rightarrow \mathcal{L}[y'(t)] = sY(s) - 1$

$\Rightarrow \mathcal{L}[ty(t)] = -\frac{d}{ds}[\mathcal{L}[y(t)]]$

$= -Y'(s)$

$\cancel{-2sY(s)} - Y'(s)s^2 + 1 + \cancel{2sY(s)} - 2 - Y'(s) = \frac{s}{s^2+1}$

$\Rightarrow -Y'(s)[s^2+1] = s^2+s+1 / s^2+1$

$\Rightarrow Y'(s) = \frac{-(s^2+s+1)}{(s^2+1)^2}$

$\Rightarrow Y'(s) = \frac{-1}{s^2+1} - \frac{s}{(s^2+1)^2}$

$Y(s) = -\tan^{-1}(s) + \frac{1}{2(s^2+1)}$

$\mathcal{L}^{-1}[Y(s)] = -\mathcal{L}^{-1}[\tan^{-1}(s)] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$

$y(t) = -\frac{1}{t}\mathcal{L}^{-1}\left[\frac{-d}{ds}\tan^{-1}(s)\right] + \frac{1}{2}\sin t$

$\Rightarrow y(t) = \frac{-1}{t}\mathcal{L}^{-1}\left[\frac{-1}{1+s^2}\right] + \frac{1}{2}\sin t$

$\Rightarrow y(t) = \sin t\left(\frac{1}{t} + \frac{1}{2}\right) \Rightarrow y(t) = \frac{1}{2}\sin t\left(1 + \frac{2}{t}\right)$

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