Q1 Do fourier sine and cosine transforms of e^x exist? Explain.

Solution For existence of fourier sine and cosine transforms, $f(x)$ should be absolutely integratable in $(-\infty, \infty)$

i.e

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty \quad \text{[or should be finite]}.$$

For $f(x) = e^x$, the integral blows up at $+\infty$, hence its sine and cosine fourier transforms does not exist.

Q2 Find the fourier cosine Transform of $f(x) = \frac{1}{1+x^2}$

Q3 Find the fourier sine transform of e^{-x} $(x>0)$ and show that $\int_0^\infty \frac{x\sin(mx)}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$, $m>0$.

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(sx)\,dx$$

s = parameter.

$$\therefore F_s(e^{-x}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin(sx)\,dx$$

consider

$$I_1 = \int e^{-x}\sin(sx)\,dx$$

$$= -e^{-x}\sin(sx) + s\int e^{-x}\cos(sx)\,dx$$

$$I_1 = -e^{-x}\sin(sx) + s\left[-e^{-x}\cos(sx) - s\int e^{-x}\sin(sx)\,dx\right]$$

$$I_1 = -e^{-x}\sin(sx) = se^{-x}\cos(sx) - s^2 I_1$$

$$\therefore I_1 = \frac{-1}{1+s^2}[\sin sx + s\cos(sx)]e^{-x}$$

$$\therefore F_s(e^{-x}) = \sqrt{\frac{2}{\pi}}\left[\frac{-1}{1+s^2}(\sin sx + s\cos(sx))e^{-x}\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s}{1+s^2}\right]$$

Now

$$F_s(e^{-x}) = \sqrt{\frac{2}{\pi}}\left[\frac{s}{1+s^2}\right]$$

Taking inverse fourier sine transform, we get

$$F_s^{-1}(F_s(e^{-x})) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{s}{1+s^2}\sin(sx)\,dx$$

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}\sin(sx)\,dx$$

substitute

$x = m$

and $s = x$

$$\therefore \quad e^{-m} = \frac{2}{\pi}\int_0^{\infty} \frac{x}{1+x^2} \sin(mx)\, dx$$

$$\frac{\pi}{2} e^{-m} = \int_0^{\infty} \frac{x}{1+x^2} \sin(mx)\, dx$$

Hence, proved.

Q4 Find the fourier transform of the function $f(x) = e^{-ax^2}$ $a > 0$, and hence find the fourier transfor of $e^{-x^2/2}$

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\omega x}$$

$$\therefore F\{e^{-ax^2}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(ax^2 - isx)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax^2 - isx + \frac{i^2 s^2}{4a} - \frac{i^2 s^2}{4a}\right)}$$

$$= \frac{e^{-s^2/4a}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\sqrt{a}x - is/2\sqrt{a}\right]^2} dx$$

Let $\sqrt{a}x - \frac{is}{2\sqrt{a}} = t$

$$\therefore dx = \frac{dt}{\sqrt{a}}$$

$$= \frac{e^{-s^2/4a}}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

Since e^{-t^2} is a even function.

$$\therefore \; = \frac{e^{-s^2/4a}}{\sqrt{2\pi a}} \cdot 2\int_0^\infty e^{-t^2} dt$$

$$= \frac{e^{-s^2/4a}}{\sqrt{2\pi a}} \cdot \Gamma(1/2)$$

$$= \frac{e^{-s^2/4a}}{\sqrt{2\pi a}} \cdot \sqrt{\pi}.$$

$$= \frac{e^{-s^2/4a}}{\sqrt{2a}}$$

$$\therefore \; F\{e^{-ax^2}\} = \frac{e^{-s^2/4a}}{\sqrt{2a}}$$

Put $a = 1/2$

$$\therefore \; F\{e^{-x^2/2}\} = \frac{e^{-s^2/2}}{\sqrt{1}}$$

$\therefore$ Required transform is $e^{-s^2/2}$

Q5 Use fourier integral to prove that.

$$\int_0^\infty \frac{\sin \pi\lambda \; \sin \lambda x}{1-\lambda^2} \, d\lambda = \frac{\pi}{2}\sin x \,, \quad 0<x<\pi$$

$$= 0 \,, \quad x > \pi.$$

Consider the function,

$$f(x) = \begin{cases} \frac{\pi}{2}\sin x \,, & 0<x<\pi \\ 0 \,, & x>\pi \end{cases}$$

Taking fourier sine transform, we get

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\lambda x)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\pi \frac{\pi}{2} \sin x \sin(\lambda x)\, dx + \int_\pi^\infty 0 \sin(\lambda x)\, dx \right]$$

$$= \sqrt{\frac{\pi}{2}} \int_0^\pi \sin x \sin(\lambda x)\, dx$$

$$= \sqrt{\frac{\pi}{2}} \int_0^\pi \frac{\cos(\lambda-1)x - \cos(\lambda+1)x}{2}\, dx$$

$$= \sqrt{\frac{\pi}{2}} \frac{1}{2} \left[\frac{\sin(\lambda-1)x}{\lambda-1} - \frac{\sin(\lambda+1)x}{\lambda+1} \right]_0^\pi$$

$$= \sqrt{\frac{\pi}{2}} \left[\frac{\sin(\lambda-1)\pi}{2(\lambda-1)} - \frac{\sin(\lambda+1)\pi}{2(\lambda+1)} \right]$$

$$= \sqrt{\frac{\pi}{2}} \left[\frac{-\sin(\pi-\lambda\pi)}{2(\lambda-1)} - \frac{\sin(\pi+\lambda\pi)}{2(\lambda+1)} \right]$$

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{\sin\lambda\pi}{2} \left[\frac{1}{\lambda+1} - \frac{1}{\lambda-1} \right]$$

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{\sin\lambda\pi}{2} \left[\frac{-2}{\lambda^2-1} \right]$$

$$F_s\{f(x)\} = \sqrt{\pi/2} \cdot \frac{\sin\lambda\pi}{\lambda^2-1}$$

Taking inverse sin transform, we get

$$f(x) = \cancel{\sqrt{\frac{2}{\pi}}} \int_0^\infty \cancel{\sqrt{\frac{\pi}{2}}} \cdot \frac{\sin\lambda\pi}{\lambda^2-1} \cdot \sin\lambda x\, d\lambda$$

$$\therefore \int_0^\infty \frac{\sin\lambda\pi}{\lambda^2-1} \cdot \sin\lambda x\, d\lambda = \begin{cases} \frac{\pi}{2}\sin x, & 0 \le x < \pi \\ 0, & x > \pi \end{cases}$$

Hence proved

Q6 Solve the integral equation

$$\int_0^\infty f(x)\cos(sx)\,dx = 1-s \quad , \quad 0 \le s \le 1$$
$$= 0 \quad , \quad s > 1 .$$

and hence show that

$$\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \pi/2 .$$

Consider $f(s) = \begin{cases} 1-s , & 0 \le s \le 1 \\ 0 , & s > 1 \end{cases}$

Taking inverse cosine transform ,

∴ we get ,

$$F_c^{-1}\{f(s)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(s)\cos sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 (1-s)\cos(sx)\,ds + \int_1^\infty 0\,ds$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin(sx)}{x}\right]_0^1 + \left[\frac{\sin sx}{x}\,s\right]_0^1 - \int_0^1 \frac{\sin sx}{x}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\cos sx}{-x^2}\right]_0^1$$

$$F_c^{-1}\{f(s)\} = \sqrt{\frac{2}{\pi}}\,\frac{1-\cos x}{x^2}$$

Taking fourier cosine transform we get

$$f(s) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2}\cos(sx)\,dx$$

By comparing $f(x) = 2/\pi\left(\frac{1-\cos x}{x^2}\right)$.

Putting $\boxed{s=0}$

$$1 = \frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2}\,dx$$

Put $x = 2x$, we get

$$\frac{\pi}{2} = \int_0^\infty \frac{2\sin^2 x}{4x^2}\,2\,dx$$

$$\therefore \boxed{\int_0^\infty \frac{\sin^2 x}{x^2} = \pi/2}$$

Q7 If $F(\bar{s})$ is the fourier transform of $f(x)$, then prove that $F\{f(x)e^{-iax}\} = F(\overline{s-a})$

By definition of fouriere transform

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$\therefore F\{f(x)e^{-iax}\} = \int_{-\infty}^{\infty} f(x)\, e^{-iax}\, e^{isx}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{i(s-a)x}\, dx$$

$$= F(\overline{s-a})$$

since $(s-a)$ becomes parameter.

Q8 Define convulation of two functions $f(x)$ and $g(x)$ and hence prove that fourier transform of convulation of two functions is equal to the product of their fourier transforms

Convulation refers to a mathematical operations on two function $f(x)$ and $g(x)$, and giving the integral of the pointwise multiplication of the two functions as a function of the amount that one of the original function is translated.

And in case of fourier transforms it is called convulation th^m ~~thm~~ for fourier transforms which states that fourier transform of a convulation is the pointwise function product of fourier transform i.e

$$F\{f^* g\} = F\{f\} . F\{g\} . \underbrace{\sqrt{2\pi}}_{\text{constant facto}}$$

Proof, → By definition of fourier transform :

$$F\{(f^*g)(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t)\ g(x-t)\ e^{+iwx} \right] dx\ dt$$

Put $x-t = s$, then $x = t+s$, this becomes

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, g(s)\ e^{iw(t+s)}\ ds\ dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{iwt} dt \int_{-\infty}^{\infty} g(s)\ e^{iws}\ ds$$

$$= \sqrt{2\pi}\ F\{f(x)\} . F\{g(x)\}$$

Hence, proved.

Q7 By applying an integral transform, solve the boundary value problem.

$$f''(x) - f(x) = 3e^{-2x} , \quad (0 < x < \infty)$$

$f(\infty)$ is bounded. $f(0) = x_0$

Scanned by CamScanner

Solution

Applying fourier sine transform on both sides

$$f_s[f''(x)] - f_s[f(x)] = 3 f_s[e^{-2x}]$$

$$-s^2 F_s(s) + s\sqrt{\frac{2}{\pi}} f(0) - F_s(s) = 3 f_s[e^{-2x}]$$

where

$F_s(s)$ is fourier sine transform of $f(x)$.

Consider

$$\Rightarrow f_s[e^{-2x}]$$

$$I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x} \cdot \sin(sx)\, dx$$

$$I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x} \cdot \sin(sx)\, dx$$

$$I = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-2x}}{-2} \sin(sx) + \frac{s}{2} \int e^{-2x} \cos sx \right]$$

$$I = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-2x}}{-2} \sin(sx) + \frac{s}{2} \left[\frac{e^{-2x}}{-2} \cos sx - \frac{s}{2} \int e^{-2x} \sin(sx)\, dx \right] \right]$$

$$I\left[1 + \frac{s^2}{4}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{-1}{2}\right] \left[e^{-2x} \sin sx + \frac{s}{2} e^{-2x} \cos(sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2}$$

$$-s^2 F_s(s) + s\sqrt{\frac{2}{\pi}} x_0 - F_s(s) = 3\sqrt{\frac{2}{\pi}} \frac{s}{s^2+4}.$$

$$\therefore F_s(s) = \sqrt{\frac{2}{\pi}} \frac{s x_0}{s^2+1} - 3\sqrt{\frac{2}{\pi}} \frac{s}{(s^2+4)(s^2+1)}$$

$$= \sqrt{\frac{2}{\pi}}\left[(x_0-1)\left(\frac{s}{s^2+1}\right) + \frac{s}{s^2+4}\right]$$

Taking inverse fourier sin transform we get

$$f(x) = F_s^{-1}\left\{\sqrt{\frac{2}{\pi}}\left[(x_0-1)\left(\frac{s}{s^2+1}\right) + \frac{s}{s^2+4}\right]\right\}$$

$$= F_c^{-1}\left\{\sqrt{\frac{2}{\pi}}\,\frac{s^2}{s^2+4}\right\} + (x_0-1)\,F_s\left\{\sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+1}\right\}.$$

$$= e^{-2x} + (x_0-1)\,e^{-x}$$

calculated just above.

$$\therefore\ f(x) = (x_0-1)\,e^{-x} + e^{-2x}.$$