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Assignment 5

Q1 $f(x) = \pi + x, \quad -\pi < x < \pi$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi + x\,dx = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi+x)\cos nx\,dx = \frac{1}{\pi}\left(\pi\int_{-\pi}^{\pi}\cos nx\,dx + \int_{-\pi}^{\pi}\underbrace{x\cos nx}_{\text{odd } f^n}\,dx\right) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi+x)\sin nx\,dx = \frac{1}{\pi}\left(\pi\underbrace{\int_{-\pi}^{\pi}\sin nx\,dx}_{=0} + \int_{-\pi}^{\pi} x\sin nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} + \underbrace{\int_{-\pi}^{\pi}\frac{\cos nx\,dx}{n}}_{=0}\right)$$

$$= \frac{-2\cos n\pi}{n} = \frac{2(-1)^{n+1}}{n}$$

$\therefore$

$$f(x) = \pi + x = \pi + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx)$$

$$\frac{x}{2} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx) = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4}\cdots$$

for $x = \pi/2$

$$\frac{\pi}{4} = 1 - 0 + \left(\frac{-1}{3}\right) - 0 + \left(\frac{+1}{5}\right)\cdots = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots$$

Q2 $f(x) = |\cos x|$ for $-\pi < x < \pi$

$\therefore f(x) = \cos x$ for $\frac{-\pi}{2} < x < \frac{\pi}{2}$

$$a_0 = \frac{1}{2\times\frac{\pi}{2}}\int_{-\pi/2}^{\pi/2}\cos x\,dx = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi/2}\int_{-\pi/2}^{\pi/2}\cos x\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi/2} 2\cos x\cos nx\,dx$$

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$$a_n = \frac{2}{\pi}\left(\int_0^{\pi/2} \cos(n+1)x + \cos(n-1)x\,dx\right) = \frac{2}{\pi}\left(\left[\frac{\sin(n+1)x}{(n+1)} + \frac{\sin(n-1)x}{(n-1)}\right]_0^{\pi/2}\right)$$

$$a_n = \frac{-4}{\pi}\frac{\cos n\pi/2}{n^2-1} = \begin{cases} 0, & n \text{ is odd} \\ \frac{-4\cos n\pi/2}{\pi(n^2-1)}, & n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi/2}\int_{-\pi/2}^{\pi/2} \cos x \sin nx\,dx = 0$$

$\therefore$

$$f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty}\left(\frac{-4\cos n\pi/2}{\pi(n^2-1)}\cos nx\right)$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \frac{\cos 6x}{35}\cdots\right)$$

Q3 $f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ 2, & \pi/2 < x < \pi \end{cases}$

In sine series,

$a_0 = a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\left(\int_0^{\pi/2}\sin nx\,dx + 2\int_{\pi/2}^{\pi}\sin nx\,dx\right)$$

$$= \frac{2}{\pi}\left(\left.\frac{-\cos nx}{n}\right|_0^{\pi/2} + 2\left.\frac{-\cos nx}{n}\right|_{\pi/2}^{\pi}\right) = \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} + 1 - 2\underbrace{\cos n\pi}_{(-1)^n}\right)$$

$\therefore$

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} + 1 - 2(-1)^n\right)\sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos\frac{n\pi}{2} + 1 - 2(-1)^n\right)\sin nx$$

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Q4 $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \end{cases}$

i) Series of cosine

$$a_0 = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\int_0^1 x\,dx + \frac{1}{2}\int_1^2 2-x\,dx = \frac{1}{2}$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 x\cos\frac{n\pi x}{2}\,dx + 2\int_1^2 \cos\frac{n\pi x}{2}\,dx - \int_1^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \left[\frac{x}{n\pi/2}\sin\frac{n\pi x}{2} + \frac{\cos\frac{n\pi x}{2}}{(n\pi/2)^2}\right]_0^1 + \left[\frac{2\sin\frac{n\pi x}{2}}{n\pi/2}\right]_1^2$$

$$- \left[\frac{x}{n\pi/2}\sin\frac{n\pi x}{2} + \frac{\cos\frac{n\pi x}{2}}{(n\pi/2)^2}\right]_1^2$$

$$a_n = \frac{2\cos n\pi/2}{(n\pi/2)^2} + \frac{(-1)^{n+1}}{(n\pi/2)^2} - \frac{1}{(n\pi/2)^2}$$

for $n = 4k,\ 4k+1,\ 4k+3,\quad a_n = 0$

for $n = 4k+2$

$$a_n = \frac{-4}{(n\pi/2)^2}$$

$$\therefore f(x) = \frac{1}{2} - \frac{16}{\pi^2}\sum_0^\infty \frac{\cos(4n+2)\frac{\pi}{2}x}{(4n+2)^2}$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\sum_0^\infty \frac{\cos(2n+1)\pi x}{(2n+1)^2}$$

ii) $a_0 = a_n = 0$

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \int_0^1 x\sin\frac{n\pi x}{2}\,dx + 2\int_1^2 \sin\frac{n\pi x}{2}\,dx - \int_1^2 x\sin\frac{n\pi x}{2}\,dx$$

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$$b_n = \left[\frac{-x\cos\left(\frac{n\pi x}{2}\right)}{n\pi/2} + \frac{\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^2} \right]_0^1 + \frac{2}{n\pi/2}\left[-\cos\frac{n\pi x}{2} \right]_1^2$$

$$- \left[\frac{-x\cos(n\pi/2\, x)}{n\pi/2} + \frac{\sin\left(\frac{n\pi}{2}x\right)}{(n\pi/2)^2} \right]_1^2$$

$$= 2\frac{\sin(n\pi/2)}{(n\pi/2)^2} + \cancel{\frac{2(-1)^n}{n\pi/2}} + \cancel{\frac{2(-1)^{n+1}}{n\pi/2}} = \frac{2\sin(n\pi/2)}{(n\pi/2)^2}$$

for $n = 4k, 4k+2$; $a_n = 0$

for $n = 4k+1$, $a_n = \frac{2}{(n\pi/2)^2}$

for $n = 4k+3$, $a_n = \frac{-2}{(n\pi/2)^2}$

$\therefore$

$$f(x) = \frac{8}{\pi^2}\sum_1^\infty (-1)^{n-1}\frac{\sin(2n-1)\pi x/2}{(2n-1)^2}$$

96 $f(x) = e^x$, $-\pi < x < \pi$ & $f(x+2\pi) = f(x)$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi}\left(\frac{e^\pi - e^{-\pi}}{2}\right) = \frac{\sinh\pi}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos nx\, dx = \frac{1}{\pi}\left(e^x\cos nx + n\int \sin nx\, e^x dx \right)$$

$$I = \int_{-\pi}^{\pi} e^x\cos nx\, dx = e^x\cos nx + n\int \sin nx\, e^x dx$$

$$= 2\cos n\pi\sinh\pi + n\left(\sin nx\, e^x - n\underbrace{\int \cos nx\, e^x dx}_{=I} \right)$$

$$\therefore I = \frac{2\cos n\pi\sinh\pi}{1+n^2}$$

$$\therefore a_n = \frac{2\cos n\pi\sinh\pi}{\pi(1+n^2)}$$

$$b_n = \int_{-\pi}^{\pi} e^{x}\sin nx\,dx = e^{x}\sin nx - n\underbrace{\int_{-\pi}^{\pi} e^{x}\cos nx\,dx}_{a_n}$$

$\therefore$

$$b_n = 0 - na_n = -na_n = -n \times \frac{2\cos n\pi \sinh \pi}{\pi(1+n^2)}$$

$\therefore$

$$f(x) = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi}\sum \frac{(-1)^n}{1+n^2}\left(\cos nx - n\sin nx\right)$$

Q7

x	0	1	2	3	4	5
y	9	18	24	28	26	20

$$a_0 = \frac{1}{k}\sum_{1}^{k} y = \frac{1}{6}(9+18+24+28+26+20) = 20.83$$

$$a_1 = \frac{2}{k}\sum_{1}^{k} y\cos\left(1\times\frac{\pi}{3}x\right)$$

$$= \frac{2}{6}\left(9 + 18\cos\frac{\pi}{3} + 24\cos\frac{2\pi}{3} + 28\cos\pi + 26\cos\frac{4\pi}{3} + 20\cos\frac{5\pi}{3}\right)$$

$$= \frac{1}{3}(-25) = -8.33$$

$$b_1 = \frac{2}{k}\sum_{1}^{k} y\sin\left(1\times\frac{\pi}{3}\cdot x\right)$$

$$= \frac{2}{6}\left(9\sin 0 + 18\sin\frac{\pi}{3} + 24\sin\frac{2\pi}{3} + 28\sin\pi + 26\sin\frac{4\pi}{3} + 20\sin\frac{5\pi}{3}\right)$$

$$= -1.156$$

$\therefore$ fourier series upto 1^{st} harmonic will be

$$f(x) = 20.83 - 8.33\cos\left(\frac{\pi x}{3}\right) - 1.156\sin\left(\frac{\pi x}{3}\right)$$

Q8

$$f(t) = \begin{cases} e^{t}, & -\infty < t < 0 \\ e^{-t}, & 0 < t < \infty \end{cases}$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt$$

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$$= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} e^{(1-i\omega)t}\,dt + \int_{0}^{\infty} e^{-(1+i\omega)t}\,dt\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1-i\omega} + \frac{1}{1+i\omega}\right) = \sqrt{\frac{2}{\pi}}\;\frac{1}{1+\omega^2}$$

Q9 $$f(x) = \begin{cases} 1-x^2 , & |x| \le 1 \\ 0 , & |x| > 1 \end{cases}$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)\,e^{-i\omega x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1} e^{-i\omega x}\,dx - \left[\frac{x^2 e^{-i\omega x}}{-i\omega} + 2\int_{-1}^{1}\frac{x\,e^{-i\omega x}}{i\omega}\,dx\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-i\omega x}}{-i\omega} - \left[\frac{x^2 e^{-i\omega x}}{-i\omega} + \frac{2}{i\omega}\left(\frac{x\,e^{-i\omega x}}{-i\omega} - \frac{e^{-i\omega x}}{(i\omega)^2}\right)\right]\right]_{-1}^{1}$$

$$= \frac{-2\sqrt{2}}{\sqrt{\pi}\,\omega^3}\left(\omega\cosh\omega + i\sinh\omega\right) = -2\sqrt{\frac{2}{\pi}}\left(\frac{\omega\cos\omega - \sin\omega}{\omega^3}\right)$$

$f(x)$ is an even f^n $\therefore$ $F(\omega) = F_c(\omega)$

$$f(x) = F_c^{-1}(\omega) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\left(\frac{-2\sqrt{2}}{\sqrt{\pi}}\left[\frac{\omega\cos\omega - \sin\omega}{\omega^3}\right]\right)\cos\omega x\,d\omega$$

put $x = 1/2$

$$1-\left(\frac{1}{2}\right)^2 = -\frac{4}{\pi}\int_{0}^{\infty}\frac{(t\cos t - \sin t)}{t^3}\cos t/2\,dt$$

$\therefore$

$$\int_{0}^{\infty}\frac{(t\cos t - \sin t)\cos(t/2)}{t^3}\,dt = \frac{-3\pi}{16}$$

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Q10 $F(w) = e^{-|w|a}$, $a > 0$, $F(w)$ is an even f^n

$$f(x) = F^{-1}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(w) e^{iwx} dw = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-wa} \cos xw \, dw$$

$$= \sqrt{\frac{2}{\pi}} \mathcal{L}\{f(w)\} = \sqrt{\frac{2}{\pi}} \mathcal{L}\{\cos(xw)\}_{s=a} = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + x^2}$$

Q11 $f(x) = e^{-x}$

$$F_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \sin wx \, dx = \sqrt{\frac{2}{\pi}} \mathcal{L}\{\sin wx\}_{s=1} = \frac{w}{1+w^2} \times \sqrt{\frac{2}{\pi}}$$

$$F_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \cos wx \, dx = \sqrt{\frac{2}{\pi}} \mathcal{L}\{\cos wx\}_{s=1} = \frac{1}{1+w^2} \times \sqrt{\frac{2}{\pi}}$$

Q12 $y'(x) - y(x) = u_0(x) e^{-x}$, $-\infty < x < \infty$

taking fourier transform

$$iw F(w) - F(w) = F\left[u_0(x) e^{-x} \right]$$

$$F(w)(iw - 1) = \frac{1}{\sqrt{2\pi}} \times \frac{1}{1+iw}$$

$$F(w) = \frac{-1}{\sqrt{2\pi}} \frac{1}{1+w^2}$$

$$F(f(x)) = \frac{-1}{\sqrt{2\pi}} \frac{1}{1+w^2}$$

$$f(x) = \frac{-1}{\sqrt{2\pi}} F^{-1}\left(\frac{1}{1+w^2}\right) = \frac{-1}{\sqrt{2\pi}} \times \sqrt{\frac{\pi}{2}} \times e^{-|x|} = -\frac{e^{-|x|}}{2}$$

$\therefore$

$$f(x) = \begin{cases} -\frac{e^{x}}{2}, & x \le 0 \\ -\frac{e^{-x}}{2}, & x > 0 \end{cases}$$