Assignment-5
Fourier Series

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Ques-1- Find the fourier series of the following function

$$f(x) = \begin{cases} x^2, & 0 \le x \le \pi \\ -x^2, & -\pi \le x \le 0 \end{cases}$$

Ans- The fourier series can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

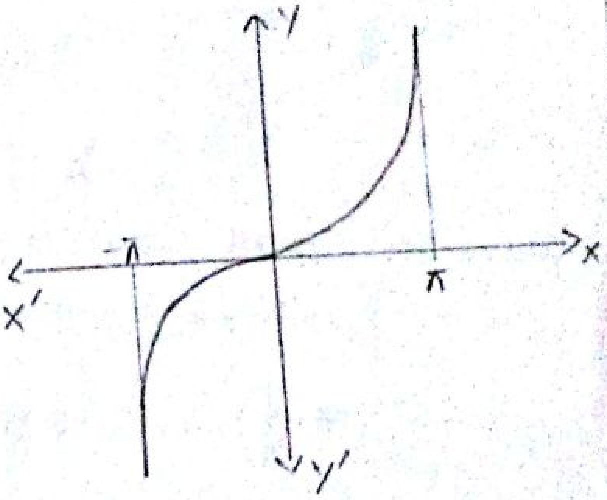
The fourier coefficients are

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} -x^2 dx + \int_0^{\pi} x^2 dx\right]$$

$$= \frac{1}{2\pi}\left[-\left[\frac{x^3}{3}\right]_{-\pi}^{0} + \left[\frac{x^3}{3}\right]_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[-\left[0+\frac{\pi^3}{3}\right] + \frac{\pi^3}{3}\right] = 0$$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x^2\cos nx \, dx + \int_0^{\pi} x^2 \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left\{-\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{0}\right.$$

$$\left. + \left[x^2\left[\frac{\sin nx}{n}\right] - 2x\left[-\frac{\cos nx}{n^2}\right] + 2\left[\frac{-\sin nx}{n^3}\right]\right]_0^{\pi}\right\}$$

$= \frac{1}{\pi}\left\{\left[\frac{2(-\pi)(-1)^n}{n^2}\right] + \frac{2\pi}{n^2}(-1)^n\right] = 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x^2 \sin nx + \int_0^{\pi} x^2 \sin nx\,dx\right]$

$= \frac{1}{\pi}\left[-\left[\frac{-x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_{-\pi}^{0}\right.$

$\left. + \left[x^2\left[-\frac{\cos nx}{n}\right] + 2x\left[\frac{\sin nx}{n^2}\right] + \frac{2\cos nx}{n^3}\right]_0^{\pi}\right.$

$= \frac{1}{\pi}\left[\frac{-4}{n^3} + (-1)^n\frac{4}{n^3} - \frac{2\pi^2}{n}(-1)^n\right]$

$b_1 = \frac{1}{\pi}\left[-4-4+2\pi^2\right] = \frac{1}{\pi}\left[2\pi^2 - 8\right] = 2\left\{\pi - \frac{4}{\pi}\right\}$

$b_2 = \frac{1}{\pi}\left[-\pi^2\right] = -\pi$

$b_3 = \frac{2}{\pi}\left(\frac{9\pi^2-4}{27}\right) = \frac{2}{3}\left(\pi - \frac{4}{9\pi}\right)$

$b_4 = \frac{2}{\pi}\left(-\frac{\pi^2}{4}\right) = -\pi/2$

Hence the fourier series becomes

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = b_1\sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$

$= 2\left(\pi - \frac{4}{\pi}\right)\sin x - \pi\sin 2x + \frac{2}{3}\left(\pi - \frac{4}{9\pi}\right)\sin 3x - \frac{\pi}{2}\sin 4x + \dots$

Que. 2 An alternating current, after passing through a rectifier, has the form

$i = I_0 \sin x \quad , \quad 0 \le x \le \pi$

$\;\; = 0 \quad , \quad \pi \le x \le 2\pi$

where I_0 is maximum current and the period is 2π.
Express i as fourier series.

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Ans -

The fourier series for the given alternating current can be written as

$$i(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx$

$$= \frac{1}{2\pi} \int_0^{2\pi} I_0 \sin x\,dx = \frac{I_0}{2\pi} [-\cos x]_0^{\pi}$$

$$= -\frac{I_0}{2\pi} [-1-1] = \frac{I_0}{\pi}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} I_0 \sin x \cos nx\,dx = \frac{I_0}{2\pi} \left[\int_0^{\pi} (\sin(n+1)x - \sin(n-1)x)\,dx \right]$$

$$= \frac{I_0}{2\pi} \left[-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)} \right]_0^{\pi}$$

$$= \frac{I_0}{2\pi} \left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} - \left(\frac{-1}{n+1} + \frac{1}{n-1} \right) \right]$$

$$= \frac{I_0}{2\pi} \left[(-1)^{n-1} \left[\frac{-1}{n+1} + \frac{1}{n-1} \right] - \left[\frac{-n+1+n+1}{n^2-1} \right] \right]$$

$$= \frac{I_0}{2\pi} \left[(-1)^{n-1} \left[\frac{-n+1+n+1}{n^2-1} \right] - \frac{2}{n^2-1} \right]$$

$$= \frac{I_0}{2\pi} \left[(-1)^{n-1} \left(\frac{2}{n^2-1} \right) - \frac{2}{n^2-1} \right] = \frac{I_0}{(n^2-1)\pi} \left((-1)^{n-1} - 1 \right)$$

$$= \frac{-I_0}{\pi(n^2-1)} (1 + (-1)^n) \qquad n \neq 1$$

$$a_1 = \frac{1}{2\pi} \int_0^{\pi} 2 I_0 \sin x \cos x\,dx = \frac{I_0}{2\pi} \int_0^{\pi} \sin 2x\,dx$$

$$= \frac{-I_0}{2\pi} \left[\frac{\cos 2x}{2} \right]_0^{\pi} = -\frac{I_0}{4\pi} [1-1] = 0$$

$a_2 = \frac{-2I_0}{3\pi}$; $a_4 = \frac{-2I_0}{15\pi}$; $a_6 = \frac{-2I_0}{35\pi}$

$a_3 = 0$; $a_5 = 0$; $a_7 = 0$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 2I_0 \sin x \sin nx\,dx = \frac{I_0}{2\pi}\int_0^{2\pi}(\cos(n-1)x - \cos(n+1)x)\,dx$$

$$= \frac{I_0}{2\pi}\left\{\left[\frac{\sin(n-1)x}{(n-1)}\right]_0^{2\pi} - \left[\frac{\sin(n+1)x}{(n+1)}\right]_0^{2\pi}\right\} \quad n \neq 1$$

$$= \frac{I_0}{2\pi}\{0-0\} = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} I_0 \sin x \sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} I_0 \sin^2 x\,dx \quad , \quad \frac{I_0}{\pi}\int_0^{2\pi}\sin^2 x\,dx$$

$$= \frac{I_0}{\pi}\int_0^{2\pi}\frac{(1-\cos 2x)}{2}\,dx = \frac{I_0}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{2\pi} = \frac{I_0}{2\pi}[2\pi - 0] = I_0$$

Hence the fourier series can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$$

$$= \frac{I_0}{\pi} + I_0 \sin x - \frac{2I_0}{3\pi}\cos 2x - \frac{2I_0}{15\pi}\cos 4x - \frac{2I_0}{35\pi}\cos 6x - \dots$$

Let $x = \theta$

$$I = \frac{I_0}{\pi}\left(1 + \pi\sin\theta\right) - \frac{I_0}{\pi}\left(\frac{2}{3}\cos 2\theta + \frac{2}{15}\cos 4\theta + \frac{2}{35}\cos 6\theta + \dots\right]$$

$$= \frac{I_0}{\pi}\left[1 + \pi\sin\theta - \frac{2}{1.3}\cos 2\theta - \frac{2}{3.5}\cos 4\theta - \frac{2}{5.7}\cos 6\theta - \dots\right]$$

ques-4 Find the fourier series to express $f(x) = x\sin x$ for $0 < x < 2\pi$ ③

Ans - The function $f(x)$ is periodic in interval $[0, 2\pi]$ with a period of 2π. using fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

The fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin x\,dx = \frac{1}{2\pi}\left[-x\cos x + \sin x\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}[-2\pi] = -1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\sin(n+1)x + \int_0^{2\pi} x\sin(1-n)x\right]$$

$$= \frac{1}{2\pi}\left[-x\frac{\cos(n+1)}{(n+1)} + \frac{\sin(n+1)x}{(n+1)^2} - x\frac{\cos(1-n)x}{(1-n)} + \frac{\sin(1-n)x}{(1-n)}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-2\pi}{(n+1)}(-1)^n - \frac{2\pi(-1)^n}{(1-n)}\right] = (-1)^n\left[\frac{-1}{n+1} - \frac{1}{1-n}\right]$$

$$= (-1)^n\frac{2}{n^2-1} \qquad n \neq 1$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 2x\sin x\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left(\cos(n-1)x - \cos(n+1)x\right)dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\cos(n-1)x\,dx - \int_0^{2\pi} x\cos(n+1)x\,dx\right.$$

$$= \frac{1}{2\pi}\left[x\frac{\sin(n-1)x}{(n-1)} + \frac{\cos(n-1)x}{(n-1)^2} - \left[x\frac{\sin(n+1)x}{n+1} + \frac{\cos(n+1)x}{(n+1)^2}\right]\right]_0^{2\pi}$$

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$$\frac{1}{2\pi}\left[\frac{x\sin(n-1)x}{(n-1)} + \frac{\cos(n-1)x}{(n-1)^2} - \frac{x\sin(n+1)x}{(n+1)} - \frac{\cos(n+1)x}{(n+1)^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^n}{(n-1)^2} - \frac{(-1)^n}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[(-1)^n\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right] - \left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right]\right]$$

$$= \frac{1}{2\pi}\left[(-1)^n\left[\frac{(n+1)^2-(n-1)^2}{(n^2-1)^2}\right] - \left[\frac{(n+1)^2-(n-1)^2}{(n^2-1)^2}\right]\right]$$

$$= \frac{1}{2\pi(n^2-1)^2}\left[(-1)^n[4n] - 4n\right]$$

$$= \frac{4n}{2\pi(n^2-1)^2}\left[(-1)^n - 1\right] = \frac{2n}{\pi(n^2-1)^2}\left[(-1)^n - 1\right] \qquad n \neq 1$$

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} 2x\sin^2 x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x - \int_0^{2\pi} x\cos 2x\,dx\right] = \frac{1}{2\pi}\left[\left[\frac{x^2}{2}\right]_0^{2\pi} - \left[\frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left(2\pi^2 - 0 - \left(\frac{1}{4} - \frac{1}{4}\right)\right) = \pi$$

$$b_2 = \frac{2.2}{\pi(9)}[1-1] = 0$$

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} 2x\sin x\cos x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx = \frac{1}{2\pi}\left[x\left[-\frac{\cos 2x}{2}\right] - \left[\frac{-\sin 2x}{4}\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi}{2} + 0 - 0\right] = \frac{-1}{2}$$

$$a_2 = \frac{2}{3}, \quad a_3 = -\frac{2}{8}, \quad a_4 = \frac{2}{15}$$

Hence the fourier series can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= -1 + a_1\cos x + a_2\cos 2x + \ldots + b_1\sin x + b_2\sin 2x + \ldots$$

$$= -1 + \pi\sin x - \frac{1}{2}\cos x + 2\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} + \ldots\right]$$

Que 5 Prove that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$, $-\pi < x < \pi$

Hence show that

i) $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$ (ii) $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

Ans - The fourier series can be evaluated as

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Since $f(x)$ is an even function

so $b_n = 0$

Hence the fourier series becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{1}{3\pi}\left[\pi^3 - 0\right] = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}(-1)^n - 0\right] = (-1)^n \frac{4}{n^2}$$

Hence the fourier series can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4}{n^2} \cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$$

(i) Put $x = \pi$

Since $x = \pi$ is a continuous point so substituting in the fourier series we get

$$\pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\frac{2\pi^2}{3} = 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$\frac{2\pi^2}{12} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

(11)

<u>Ques-6</u> Obtain a fourier series to represent this function

$f(x) = |\sin x|$ for $-\pi < x < \pi$

<u>Ans</u>- The function $f(x)$ is periodic in interval $[-\pi, \pi]$ with a period of 2π. Using fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

The fourier coefficients are

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

The function can be written as

$$f(x) = \begin{cases} -\sin x & -\pi \le x < 0 \\ \sin x & 0 \le x < \pi \end{cases}$$

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$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-\sin x)\,dx + \int_{0}^{\pi}(\sin x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left[\cos x\right]_{-\pi}^{0} + (-\cos x)_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[1-(-1)-(-1-1)\right] = \frac{1}{2\pi}[2+2] = \frac{4}{2\pi} = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}-2\sin x\cos nx\,dx + \int_{0}^{\pi}2\sin x\cos nx\right]$$

$$= \frac{1}{2\pi}\left[-\int_{-\pi}^{0}(\sin(n+1)x+\sin(1-n)x)\,dx + \int_{0}^{\pi}(\sin(n+1)x+\sin(1-n)x)\,dx\right]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(1-n)x}{(1-n)}\right]_{-\pi}^{0} - \left[\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(1-n)x}{(1-n)}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{1}{n+1} + \frac{1}{1-n} - \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{1-n}\right] - \left[\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{1-n} - \frac{1}{n+1} - \frac{1}{1-n}\right]\right\}$$

$$= \frac{1}{2\pi}\left[2\left[\frac{2}{1-n^2} + (-1)^n\frac{2}{1-n^2}\right]\right] = \frac{2}{\pi(1-n^2)}(1+(-1)^n) \quad n \neq 1$$

$b_n = 0$ {since $f(x)$ is an even function}

$$a_1 = \frac{1}{2\pi}\left[\int_{-\pi}^{0}-\sin 2x\,dx + \int_{0}^{\pi}\sin 2x\,dx\right]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{\cos 2x}{2}\right]_{-\pi}^{0} + \left[-\frac{\cos 2x}{2}\right]_{0}^{\pi}\right\} = \frac{1}{2\pi}\left[\frac{1}{2}(1-(+1)) - \frac{1}{2}(1-1)\right] = 0$$

$a_2 = \frac{-2}{3\pi}\cdot 2 = \frac{-4}{3\pi}$; $a_4 = \frac{-2}{15\pi}(2) = \frac{-4}{15\pi}$; $a_6 = \frac{-4}{35\pi}$

$a_3 = \frac{-2}{8\pi}(0) = 0$; $a_5 = 0$;

Hence the fourier series for the given function in the interval $(-\pi, \pi)$ can be written as

$$f(x) = \frac{2}{\pi} - \frac{4}{3\pi}\cos 2x - \frac{4}{15\pi}\cos 4x - \frac{4}{35\pi}\cos 6x \cdots$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x + \cdots\right]$$

$\frac{2\pi}{12}$ Ques 7 Find half range cosine series for $f(x) = e^x$, $0 < x < \pi$

$\frac{\pi}{6}$ Sol) Assuming the function $f(x)$ in the interval $(-\pi, 0)$, such that it becomes even function in the interval

(31) $[-\pi, \pi]$. Therefore, half range fourier cosine series will be

$$x = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad -(1)$$

$$a_0 = \frac{2}{2\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} e^x\,dx = \frac{1}{\pi}(e^{\pi}-1) \qquad -(2)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^x \cos nx\,dx$$

$$\text{Let } I = \int e^x \cos nx\,dx = \cos nx\, e^x + n\int e^x \sin nx\,dx$$

$$= \cos nx\, e^x + n\left[\sin nx\, e^x - \int n \cos nx\, e^x\,dx\right]$$

$$= e^x \cos nx + e^x n \sin nx - n^2 \int e^x \cos nx\,dx$$

$$= e^x \cos nx + n e^x \sin nx - n^2 I$$

$$I = \frac{e^x(\cos nx + n \sin nx)}{(n^2+1)}$$

Ques

Thus a_n can be written as

$$a_n = \frac{2}{\pi}\left\{\left[e^x\left[\frac{\cos nx + n\sin nx}{n^2+1}\right]\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left[\frac{e^{\pi}(-1)^n - 1}{n^2+1}\right]$$

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Thus the half range fourier cosine series can be written as

$$f(x) = \frac{(e^{\pi}-1)}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1 - e^{\pi}(-1)^n}{n^2+1}\cos nx \qquad \{\text{using } (1)\}$$

$$= \frac{e^{\pi}-1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n e^{\pi}}{n^2+1}\cos nx$$

7 Ques-8 Find the Fourier series to represent $f(x)$, when

$$f(x) = -a, \quad -c < x < 0$$
$$= a, \quad 0 < x < c$$

Ans - The function $f(x)$ is periodic in the interval $[-c, c]$ with a period $2c$.
To make the function periodic in the interval $[-\pi, \pi]$
Consider a variable z such that $-\pi \le z \le \pi$

$-c \le x \le c \qquad -\pi \le z \le \pi$

$-1 \le \frac{x}{c} \le 1 \qquad -1 \le \frac{z}{\pi} \le 1$

$\frac{x}{c} = \frac{z}{\pi} \qquad x = \frac{zc}{\pi} \qquad z = \frac{x\pi}{c}$

$f(x) = f\left(\frac{zc}{\pi}\right) = \phi(z)$

Using the above relations

$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right]$

$a_0 = \frac{1}{2\pi}\int_{-c}^{c} f(x)\frac{\pi}{c}\,dx = \frac{1}{2c}\int_{-c}^{c} f(x)\,dx$

$= \frac{1}{2c}\left[\int_{-c}^{0} -a\,dx + \int_{0}^{c} a\,dx\right] = \frac{1}{2c}\left[-[ax]_{-c}^{0} + [ax]_{0}^{c}\right]$

$= \frac{a}{2c}[-c+c] = 0$

$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx$

$= \frac{1}{c}\left[\int_{-c}^{0} -a\cos\frac{n\pi x}{c} + \int_{0}^{c} a\cos\frac{n\pi x}{c}\right]$

$= \frac{a}{c}\left[-\left[\sin\frac{n\pi x}{c}\right]_{-c}^{0}\frac{c}{n\pi} + \frac{c}{n\pi}\left[\sin\frac{n\pi x}{c}\right]_{0}^{c}\right] = 0$

$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx$

$= \frac{1}{c}\left[\int_{-c}^{0} -a\sin\frac{n\pi x}{c}\,dx + \int_{0}^{c} a\sin\frac{n\pi x}{c}\,dx\right]$

$= \frac{a}{c}\left[\frac{c}{n\pi}\left[\cos\frac{n\pi x}{c}\right]_{-c}^{0} - \frac{c}{n\pi}\left[\cos\frac{n\pi x}{c}\right]_{0}^{c}\right]$

$= \frac{a}{\cancel{c}}\cdot\frac{\cancel{c}}{n\pi}\left[1-(-1)^n - [(-1)^n - 1]\right]$

$= \frac{a}{n\pi}\left[2 - 2(-1)^n\right] = \frac{2a}{n\pi}\left[1-(-1)^n\right]$

$b_1 = \frac{2a}{\pi}[2] = \frac{4a}{\pi}$

$b_2 = 0$

$b_3 = \frac{2a}{3\pi}\cdot 2 = \frac{4a}{3\pi}$

$b_4 = \frac{2a}{4\pi}[0] = 0$

$b_5 = \frac{2a}{5\pi}\cdot 2 = \frac{4a}{5\pi}$

$b_6 = \frac{4a}{7\pi}$

Hence the fourier series can be written as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

$$= b_1 \sin\frac{\pi x}{c} + b_2 \sin\frac{2\pi x}{c} + b_3 \sin\frac{3\pi x}{c} + \dots$$

$$= \frac{4a}{\pi}\left[\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \frac{1}{7}\sin\frac{7\pi x}{c} + \dots\right]$$

Ques-9 If $-\pi < x < \pi$ prove that

$$x\sin x = 1 - \frac{1}{2}\cos x - \frac{2\cos 2x}{1.3} + \frac{2\cos 3x}{2.4} - \frac{2\cos 4x}{3.5} + \dots$$

and hence show that

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots$$

Ans - The function $f(x)$ is periodic in interval $[-\pi, \pi]$ with a period of 2π. Using fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Since $f(x) = x\sin x$ is an even function

therefore $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = 0$

Hence the series becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin x\, dx = \frac{1}{\pi}\int_0^{\pi} x\sin x\, dx$$

$$= \frac{1}{\pi}\left[-x\cos x + \sin x\right]_0^{\pi} = \frac{1}{\pi}\left[-\pi(-1)\right] = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x(\sin(1+n)x + \sin(1-n)x)\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin(n+1)x\,dx + \int_0^{\pi} x\sin(1-n)x\,dx\right]$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos(n+1)x}{(n+1)}\right) - \left(-\frac{\sin(n+1)x}{(n+1)^2}\right) + x\left(-\frac{\cos(1-n)x}{(1-n)}\right) - \left(-\frac{\sin(1-n)x}{1-n}\right)\right]_0^{\pi}$$

$n \neq 1$

$= \frac{1}{\pi}\left[\frac{-\pi(-1)^{n+1}}{(n+1)} - \frac{\pi(-1)^{n-1}}{1-n}\right] = \frac{\pi}{\pi}\left[(-1)^{n-1}\left[\frac{-1}{n+1} + \frac{1}{n-1}\right]\right]$

$= \left[(-1)^{n+1}\left[\frac{1-n+1+n}{n^2-1}\right]\right]$

$= (-1)^{n-1}\frac{2}{n^2-1} \qquad n \neq 1$

$a_2 = -\frac{2}{3}$, $a_5 = \frac{2}{24}$

$a_3 = \frac{2}{8}$, $a_6 = \frac{-2}{35}$

$a_4 = \frac{-2}{15}$, $a_7 = \frac{2}{48}$

$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$

$= \frac{1}{\pi}\left[x\left[\frac{-\cos 2x}{2}\right]_0^{\pi} - \left[-\frac{\sin 2x}{4}\right]_0^{\pi}\right]$

$= \frac{1}{\pi}\left[-\frac{\pi\cos 2\pi}{2}\right] = \frac{-1}{2}$

Hence the fourier series can be written as

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$x\sin x = 1 - \frac{1}{2}\cos x - \frac{2}{1.3}\cos 2x + \frac{2}{2.4}\cos 3x - \frac{2}{3.5}\cos 4x + \dots$

Put $x = \pi/2$

Since $x = \pi/2$ is a continuous point

putting $x = \pi/2$ in the fourier series, we get

$\frac{\pi}{2} = 1 - 0 - \frac{2}{1.3}(-1) + \frac{2}{2.4}(0) - \frac{2}{3.5}(1) + \dots$

$\frac{\pi}{2.2} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \dots$

$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \dots$

Ques-10 Find the fourier series for the function

$f(x) = 2x - x^2$, $0 < x < 3$, and deduce that

$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

Ans - The fourier series for the function periodic in the interval $(0,3)$ can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{c}\right) + b_n \sin\left(\frac{n\pi x}{c}\right)\right)$$

Here $c = 3/2$

$$a_0 = \frac{1}{3}\int_0^3 (2x - x^2)\,dx = \frac{1}{3}\left|x^2 - \frac{x^3}{3}\right|_0^3 = \frac{1}{3}[9-9] = 0$$

$$a_n = \frac{1}{(3/2)}\int_0^3 (2x-x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[\left[(2x-x^2)\left(\frac{3}{2n\pi}\right)\sin\left(\frac{2n\pi x}{3}\right)\right]_0^3 - \int_0^3 (2-2x)\left(\frac{3}{2n\pi}\right)\sin\left[\frac{2n\pi x}{3}\right]dx\right]$$

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{3}{2n\pi}\right)\sin\frac{2n\pi x}{3} + (2-2x)\left(\frac{3}{2n\pi}\right)^2\cos\frac{2n\pi x}{3} - 2\left(\frac{3}{2n\pi}\right)^2\sin\left(\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[(-6)\left(\frac{3}{2n\pi}\right)^2\right] = \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{1}{3/2}\int_0^3 (2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= 2/3\left[-(2x-x^2)\left(\frac{3}{2n\pi}\right)\cos\left(\frac{2n\pi x}{3}\right) + (2-2x)\left(\frac{3}{2n\pi}\right)^2\sin\left(\frac{2n\pi x}{3}\right) + (-2)\cos\left(\frac{2n\pi x}{3}\right)\left(\frac{3}{2n\pi}\right)^3\right]_0^3$$

$$= \frac{\cancel{2}}{\cancel{3}}\left[\cancel{3}\cdot\frac{3}{\cancel{2}n\pi}\right] = \frac{3}{n\pi}$$

Hence the fourier series can be written as

$$f(x) = \sum_{n=1}^{\infty}\left[\left(\frac{-9}{n^2\pi^2}\right)\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)\right]$$

$$2x - x^2 = \frac{-9}{\pi^2}\left(\frac{1}{1^2}\cos\left(\frac{2\pi x}{3}\right) + \frac{1}{2^2}\cos\left(\frac{4\pi x}{3}\right) + \frac{1}{3^2}\cos\left(\frac{6\pi x}{3}\right) + \ldots\right)$$
$$+ \frac{3}{\pi}\left(1.\sin\left(\frac{2\pi x}{3}\right) + \frac{1}{2}\sin\left(\frac{4\pi x}{3}\right) + \frac{1}{3}\sin\left(\frac{6\pi x}{3}\right) + \ldots\right)$$

Since $f(x)$ is not continuous at $x=3$ so the value of $f(x)$ at $x=3$ is not calculated directly by substitution. {Diricklet cond^n}

$$f(3) = \frac{1}{2}[f(3+x) + f(3-x)]$$

$$= \frac{1}{2}[0 + 2.3 - 9] = -\frac{3}{2}$$

$$-\frac{3}{2} = -\frac{9}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \sum\frac{1}{n^2}$$

Hence proved

Ques 11 obtain the half range sine series of $f(x) = lx - x^2$ in $(0,l)$ and hence show that

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$$

Ans - Assuming the function $f(x)$ in the interval $(-l,0)$, such that it becomes odd function in the interval $(-l,l)$ Therefore half range fourier sine series will be

$$lx - x^2 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^l lx\sin\frac{n\pi x}{l}dx - \int_0^l x^2\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2}{l}\left[l\left[\frac{-l}{n\pi}x\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^l - \left[x^2\left(\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)\right.\right.$$

$$\left.\left.+ 2x\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} + \frac{2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l\right]$$

$$= \frac{2}{l}\left[\frac{-l^3}{n\pi}(-1)^n - \left[-\frac{l^3}{n\pi}(-1)^n + \frac{2l^3}{n^3\pi^3}(-1)^n - \frac{2l^3}{n^3\pi^3}\right]\right]$$

$$= \frac{2}{l}\left[\frac{-2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{2}{l}\cdot\frac{2l^3}{n^3\pi^3}\left[1 - (-1)^n\right] = \frac{4l^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

Hence the half range fourier sine series in the interval $(0,l)$ can be written as

$$lx - x^2 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty}\frac{4l^2}{n^3\pi^3}\left[1 - (-1)^n\right]\sin\frac{n\pi x}{l}$$

$$= b_1\sin\pi x + b_2\sin 2\pi x + b_3\sin 3\pi x + \cdots$$

$$b_1 = \frac{4l^2}{\pi^3}(2) \qquad b_3 = \frac{4l^2}{27\pi^3}(2)$$

$$= \frac{8l^2}{27\pi^3}$$

$$b_2 = 0$$

Thus $lx - x^2 = \frac{8l^2}{\pi^3}\sin\frac{x\pi}{l} + \frac{8l^2}{27\pi^3}\sin\frac{3x\pi}{l} + \frac{8l^2}{125\pi^3}\sin\frac{5x\pi}{l} + \dots$

$$= \frac{8l^2}{\pi^3}\sin\frac{\pi x}{l} + \frac{8l^2}{27\pi^3}\sin\frac{3\pi x}{l} + \frac{8l^2}{125\pi^3}\sin\frac{5\pi x}{l} + \dots$$

Since $x = l/2$ is a continuous pt. so substituting in the fourier series we get

$$\frac{l^2}{2} - \frac{l^2}{4} = \frac{8l^2}{\pi^3}\sin\frac{\pi}{2} + \frac{8l^2}{27\pi^3}\sin\frac{3\pi}{2} + \frac{8l^2}{125\pi^3}\sin\frac{5\pi}{2} + \dots$$

$$\frac{l^2}{4} = \frac{8l^2}{\pi^3}\left[1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots\right]$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots$$

Ques-3 Discuss the convergence of fourier series.

Ans - If $f(x)$ is a periodic function with period 2π and if $f(x)$ and $f'(x)$ both are piecewise continuous in the interval $-\pi \le x \le \pi$, then the fourier series of $f(x)$ is convergent. It converges to $f(x)$ at every point x at which $f(x)$ is continuous, and to the mean value $[f(x+) + f(x-)]/2$ at every point x at which $f(x)$ is discontinuous, where $f(x+)$ and $f(x-)$ are the right and left hand limits respectively.