Assignment - 4

MA - 102

Department of Applied Mathematics.

By

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1. Using the definition, find the Laplace tranform of following functions :-

(a) $at^2 + bt + c$

Ans $\Rightarrow$ $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$, using definition.

$$\therefore L\{at^2+bt+c\} = \int_0^\infty (at^2+bt+c)e^{-st}\, dt.$$

$$= a\int_0^\infty e^{-st} t^2\, dt + b\int_0^\infty e^{-st} t\, dt + c\int_0^\infty e^{-st}\, dt$$

$$= a\left[\left. t^2 \frac{e^{-st}}{-s} \right|_0^\infty - \int_0^\infty 2t \cdot \frac{e^{-st}}{-s}\, dt \right] + b\left[\left. \frac{te^{-st}}{-s} \right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, dt \right]$$

$$+ c \left. \frac{e^{-st}}{(-s)} \right|_0^\infty$$

$$= \frac{2a}{s} \int_0^\infty te^{-st}\, dt + \frac{b}{s}\int_0^\infty \frac{e^{-st}}{} + \frac{c}{s}$$

$$= \frac{2a}{s}\left[\left. \frac{te^{-st}}{s} \right|_0^\infty + \frac{1}{+s}\int_0^\infty e^{-st}\, dt \right] + \frac{b}{s} \left. \frac{e^{-st}}{-s} \right|_0^\infty + \frac{c}{s}$$

$$= \frac{2a}{s}\left[\frac{1}{s} \left. \frac{e^{-st}}{-s} \right|_0^\infty \right. + \frac{b}{s^2} + \frac{c}{s}$$

$$= \frac{2a}{s^3} + \frac{b}{s^2} + \frac{c}{s}$$

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b) → same as part 'a'

c) $\cos(at+b)$

By definition of laplace transforms

$$L\{\cos(at+b)\} = \int_0^{\infty} e^{-st} \cdot \cos(at+b)\, dt$$

$$\cos(at+b) = Re\left[e^{i(at+b)}\right]$$

$$L\{\cos(at+b)\} = Re\left[\int_0^{\infty} e^{-st} e^{i(at+b)}\, dt\right]$$

$$= Re\left[e^{ib}\int_0^{\infty} e^{(ia-s)t}\, dt\right]$$

$$= Re\left[e^{ib} \left.\frac{e^{(ia-s)t}}{ia-s}\right|_0^{\infty}\right]$$

$$= Re\left[e^{ib} \left.\frac{e^{-(s-ia)t}}{-(s-ia)}\right|_0^{\infty}\right]$$

$$= Re\left[e^{ib}\left[\frac{1}{s-ia}\right]\right]$$

$$= Re\left[\frac{e^{ib}(s+ia)}{s^2+a^2}\right]$$

$$= Re\left[\cos b + i\sin b\, \frac{(s+ia)}{s^2+a^2}\right]$$

$$= Re\left[\frac{s\cos b - a\sin b + i[s\sin b + a\cos b]}{s^2+a^2}\right]$$

$$= \frac{s\cos b - a\sin b}{s^2+a^2}$$

d) te^{t}

L

e) $f($

By

$L\{f(t)\} =$

$=$

d) te^t

$$L\{te^t\} = \int_0^\infty e^{-st} . te^t \, dt .$$ [By definition]

$$= \int_0^\infty t e^{-(s-1)t} \, dt$$

$$= \left. \frac{te^{-(s-1)t}}{-(s-1)} \right|_0^\infty + \frac{1}{s-1} \int_0^\infty e^{-(s-1)t} \, dt$$

$$= 0 + \frac{1}{s-1} \left. \frac{e^{-(s-1)t}}{-(s-1)} \right|_0^\infty$$

$$= \frac{1}{(s-1)^2}$$

e) $$f(t) = \begin{cases} 0 & 0 \le t < \pi \\ \sin t & t \ge \pi \end{cases}$$

By definition of laplace transform

$$L\{f(t)\} = \int_0^\infty e^{-st} . f(t) \, dt.$$

$$= \int_0^\pi e^{-st} . f(t) \, dt + \int_\pi^\infty e^{-st} . f(t) . dt .$$

$$= 0 + \int_\pi^\infty e^{-st} \sin t .$$

$$= \text{img} \left(\int_\pi^\infty e^{-st} . e^{it} \, dt \right)$$

$$= \text{img} \left(\int_\pi^\infty e^{-(s-i)t} \, dt \right)$$

$$= \text{img} \left. \frac{e^{-(s-i)t}}{-(s-i)} \right|_\pi^\infty$$

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$$= \operatorname{img}\left(\frac{e^{-(s-i)\pi}}{-(s-i)} \right)$$

$$= -\operatorname{img}\left(e^{-s\pi} \cdot \frac{e^{i\pi}}{s-i} \right)$$

$$= -\operatorname{img}\left(e^{-s\pi} \cdot \frac{(s+i)[-1+i0]}{s^2+1} \right)$$

$$= +\operatorname{img}\left(e^{-s\pi} \cdot \frac{s+i}{s^2+1} \right)$$

$$= \frac{e^{-s\pi}}{s^2+1}$$

ii) t^2 ce

Q2 Find laplace transformations of following functions :-

a) $t \sin 4t$.

$$\sin 4t = \operatorname{img}(e^{4it})$$

$$\therefore (t \sin 4t) = \operatorname{img}(t e^{4it})$$

By first shifting thm,

$$\text{if } L\{f(t)\} = \bar{f}(s)$$

$$L\{e^{at} \cdot f(t)\} = \bar{f}(s-a)$$

Now,

Let $f(t) = t$

$$\therefore \bar{f}(s) = \frac{1}{s^2}$$

$$\therefore L\{te^{i4t}\} = \frac{1}{(s-4i)^2}$$

$$L\{t \sin 4t\} = \operatorname{img}\left(L\{te^{i4t}\} \right)$$

$$= \operatorname{img}\left(\frac{1}{(s-4i)^2} \right)$$

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$$= \text{img}\left(\frac{(s+4i)^2}{(s^2+16)^2} \right)$$

$$= \text{img}\left[\frac{s^2-16+8is}{(s^2+16)^2} \right]$$

$$= \frac{8s}{(s^2+16)^2}.$$

ii) $t^2 \cos 3t$.

$$t^2\cos 3t = \text{real}\left[t^2 e^{i3t} \right]$$

By first shifting th^m

if $L\{f(t)\} = \bar{f}(s)$

then $L\{e^{at}.f(t)\} = \bar{f}(s-a)$

Now, if $f(t) = t^2$

$$L\{t^2\} = \frac{2}{s^3}$$

$$\therefore L\{e^{i3t}.t^2\} = \frac{2}{(s-3i)^3}$$

$$\therefore \mathcal{L}\{t^2\cos 3t\} = Re\left[\frac{2}{(s-3i)^3} \right].$$

$$= Re\left[\frac{(s+3i)^3\, 2}{(s^2+9)^3} \right]$$

$$= 2\,Re\left[\frac{s^3+27i^3+3i^2.9.s+3s^2 3i}{(s^2+9)^3} \right]$$

$$= \frac{2}{(s^2+9)^3}\left[s^3-27s \right]$$

$$= \frac{2s}{(s^2+9)^3}\left[s^2-27 \right].$$

c) $t^2 e^{-2t}$.

By first shift thm,

if $L\{f(t)\} = \bar{f}(s)$

then $L\{e^{at} f(t)\} = \bar{f}(s-a)$

if $t^2 = f(t)$

then $\bar{f}(s) = \frac{2}{s^3}$

$\therefore L\{e^{-2t} \cdot t^2\} = \frac{2}{[s-(-2)]^3} = \frac{2}{(s+2)^3}$.

Q3 In the following problems, the laplace transform $F(s) = L\{f(t)\}$ is given. Find the inverse laplace transform, $f(t)$.

Ans3 i) $\bar{f}(s) = \frac{3}{s-5}$

$= 3\left[\frac{1}{s-5}\right]$.

$L\{f(t)\} = \bar{f}(s)$.

By shift thm

$L\{e^{5t} f(t)\} = \bar{f}(s-5)$

$\therefore \bar{f}(s) = \frac{3}{s}$

$\therefore f(t) = L^{-1}\left[\frac{3}{s}\right]$

$= 3[1]$

$\therefore \boxed{f(t) = 3e^{5t}}$

ii) $\dfrac{\pi}{s^2+\pi^2}$

Given $\bar{f}(s) = \dfrac{\pi}{s^2+\pi^2} = L\{f(t)\}$

we know that $L\{\sin at\} = \dfrac{a}{s^2+a^2}$

$\therefore \quad f(t) = L^{-1}\left\{\dfrac{\pi}{s^2+\pi^2}\right\}$

$= \sin(\pi t).$

c) $\dfrac{s+3}{(s+2)(s-1)}$

$f(t) = L^{-1}\left\{\dfrac{s+3}{(s+2)(s-1)}\right\}.$

$\dfrac{s+3}{(s+2)(s-1)} = \dfrac{A}{(s+2)} + \dfrac{B}{(s-1)}$

$\therefore \quad A = -1/3 \quad , \quad B = +4/3$

$f(t) = L^{-1}\left\{\dfrac{-1}{3(s+2)} + \dfrac{4}{3}\dfrac{1}{(s-1)}\right\}$

$= \dfrac{-1}{3}e^{-2t} + \dfrac{4}{3}e^{t}$

$= \dfrac{1}{3}\left[4e^{t} - e^{-2t}\right].$

Q4 Find the laplace transform of func.

$$f(t) = \begin{cases} k, & 0 \le t < 2 \\ 0, & 2 \le t < 4 \\ k, & t \ge 4. \end{cases}$$

By definition of laplace transformation

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \int_0^2 e^{-st} f(t)\, dt + \int_2^4 e^{-st} f(t)\, dt + \int_4^\infty e^{-st} f(t)\, dt$$

$$= k \int_0^2 e^{-st}\, dt + k \int_4^\infty e^{-st}\, dt$$

$$= k \left. \frac{e^{-st}}{-s} \right|_0^2 + k \left. \frac{e^{-st}}{-s} \right|_4^\infty$$

$$= k \left[\frac{e^{-2s}}{-s} + \frac{1}{s} + \frac{e^{-4s}}{s} \right]$$

$$= \frac{k}{s} \left[e^{-4s} + 1 - e^{-2s} \right]$$

$$= \frac{k}{s} \left[1 + e^{4s} - e^{2s} \right]$$

Q5 Find the laplace transform of functions.

$$f(t) = \begin{cases} 0 & 0 \le t < 3 \\ (t-3)^2 & t \ge 3 \end{cases}$$

By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \int_3^\infty e^{-st} \cdot (t-3)^2\, dt.$$

Put

$t - 3 = u$

$$\therefore = \int_0^\infty e^{-s(3+u)} \cdot u^2 \, du.$$

$$= \int_0^\infty e^{-3s} e^{-su} \cdot u^2 \, du$$

$$= e^{-3s} \int_0^\infty e^{-su} u^2 \, du$$

$$= e^{-3s} \cdot L\{t^2\}$$

$$= e^{-3s} \cdot \frac{2}{s^3}$$

$$= \frac{2e^{-3s}}{s^3}.$$

Q6 Solve the initial value problem using laplace transformation.

a) $y'' + 2y' - 3y = 3$, $\quad y(0) = 4$

$y'(0) = -7$.

Considering

$$y'' + 2y' - 3y = 3$$

Taking laplace of each term.

$$L\{y''\} + 2L\{y'\} - 3L\{y\} = L\{3\}$$

Let $L\{y\} = \bar{y}$.

$$\therefore \quad (s^2 + 2s - 3)\bar{y} = \frac{3}{s} + y(0)[s+2] + y'(0)[1].$$

$$(s^2 + 2s - 3)\bar{y} = \frac{3}{s} + 4(s+2) - 7$$

$$\bar{y} = \frac{3 + 4(s)(s+2) - 7(s)}{s(s^2+2s-3)}$$

$$= \frac{4s^2 + s + 3}{s(s-1)(s+3)}$$

$$\frac{4s^2+s+3}{s(s-1)(s+3)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{s+3}$$

$\therefore$ $\left.\begin{aligned} A &= -1 \\ B &= 2 \\ C &= 3 \end{aligned}\right]$ By comparing.

$$\bar{y} = \frac{-1}{s} + \frac{2}{(s-1)} + \frac{3}{s+3}$$

$$\therefore y = L^{-1}\left\{ \frac{-1}{s} + \frac{2}{s-1} + \frac{3}{s+3} \right\}$$

$$y = -1 + 2e^{t} + 3e^{-3t}$$

b) $y'' - 5y' + 4y = e^{2t}$, $y(0) = \frac{19}{2}$, $y'(0) = \frac{8}{3}$.

$$(D^2 - 5D + 4)y = e^{2t}$$

Taking laplace of each term ; we get

$$L\{y\} = \bar{y}$$

$$(s^2 - 5s + 4)\bar{y} = \frac{1}{s-2} + y(0)[s-5] + y'(0).$$

$$\bar{y} = \frac{1}{s^2+4-5s}\left[\frac{1}{(s-2)} + \frac{19}{2}(s-5) + \frac{8}{3} \right)$$

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$$\bar{y} = \frac{1}{(s-1)(s-4)}\left[\frac{6 + 57(s-5)(s-2) + 16(s-2)}{6(s-2)}\right]$$

$$\bar{y} = \frac{57s^2 - 383s + 544}{6(s-2)(s-4)(s-1)}.$$

Let

$$\frac{57s^2 - 383s + 544}{6(s-2)(s-4)(s-1)} = \frac{A}{s-2} + \frac{B}{(s-4)} + \frac{C}{(s-1)}$$

By comparing co-efficients:

$$A = -\frac{1}{2}, \quad B = \frac{-38}{18}, \quad C = \frac{109}{9}$$

$$y = L^{-1}\left\{\frac{-1}{2(s-2)}\right\} - L^{-1}\left\{\frac{38}{18(s-4)}\right\} + L^{-1}\left\{\frac{109}{9(s-1)}\right\}.$$

$$y = -\frac{e^{2t}}{2} - \frac{38}{18}e^{4t} + \frac{109}{9}e^{t}.$$

c) $y' + 3y + 2\int_0^t y(z)\,dz\,dt = t, \quad y(0) = 0$

by Taking Laplace transform of each other.

$$L\{y'\} + 3L\{y\} + 2L\left\{\int_0^t y(z)\,dz\,dt\right\} = L\{t\}$$

Let $L\{y\} = \bar{y}$

$$\therefore s\bar{y} - y(0) + 3\bar{y} + \frac{2\bar{y}}{s} = \frac{1}{s^2}$$

$$\bar{y}\left[s + 3 + \frac{2}{s}\right] = \frac{1}{s^2}$$

$$\bar{y} = \frac{1}{s[s^2+3s+2]}$$

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Let

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$A = \frac{1}{2}$

$B = -1,$ $\quad C = 2$

$$\therefore \quad y = L^{-1}\left\{ \frac{1}{2s} - \frac{1}{s-1} + \frac{2}{s-2} \right\}.$$

$$y = \frac{1}{2}L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s-1}\right\} + 2L^{-1}\left\{\frac{1}{s-2}\right\}.$$

$$y = \frac{1}{2} - e^{t} + 2e^{2t}.$$

Q7 Find the solution of the initial value problem, using transformation .

$$ty'' + 2ty' + 2y = 2$$

$y(0) = 1$

$y'(0)$ is

Let $y'(0) = a$.

Taking laplace of each term

$$L\{ty''\} + 2\{ty'\} + 2L\{y\} = L\{2\}.$$

Let $L\{y\} = \bar{y}$

$$-\frac{d}{ds}\left[s^2\bar{y} - s - a\right] + 2\left[-\frac{d}{ds}(s\bar{y} - 1)\right] + 2\bar{y} =$$

$$-\bar{y}'\left[s^2+2s\right] - \bar{y}\left[2s + 2 + -2\right] + 1 = \frac{2}{s}.$$

$$\bar{y}'(s^2+2s) + \bar{y}(2s) = 1 - 2/s.$$

$$\bar{y}' + \frac{2s}{s^2+2s}\bar{y} = \frac{s-2}{s(s)(s+2)}$$

$$\bar{y}' + \frac{2}{s+2}\bar{y} = \frac{s-2}{s+2}\,\frac{1}{s^2}.$$

Solving above linear differential eqn.

$$IF = e^{\int \frac{2}{s+2}ds}$$

$$= e^{2\log(s+2)}$$

$$= (s+2)^2$$

$\therefore$

$$\bar{y}(s+2)^2 = \int \frac{s-2}{\cancel{(s+2)}}\,\frac{1}{s^2}\,(s+2)^{\cancel{2}}\,ds + C$$

$$\bar{y}(s+2)^2 = \int \frac{s^2-4}{s^2} + c$$

$$\bar{y}(s+2)^2 = s + \frac{4}{s} + c$$

$$\bar{y} = \frac{s}{(s+2)^2} + \frac{4}{s(s+2)^2} + \frac{c}{(s+2)^2}$$

$$\bar{y} = \frac{\cancel{s+2}}{(s+2)^{\cancel{2}}} - \frac{2}{(s+2)^2} + \frac{c}{(s+2)^2} + \frac{4}{(s+2)^2}\cdot\frac{1}{s}$$

$$y = e^{-2t} + (c-2)(t)(e^{-2t}) + 4\int_0^t e^{-2u}\cdot u\,du$$

$$= e^{-2t} + (c-2)(te^{-2t}) + 4\left[-e^{-2t}\left[\frac{t}{2}+\frac{1}{4}\right] + \frac{1}{4}\right]$$

$$= e^{-2t} + (c-2)(te^{-2t}) + -2te^{-2t} - e^{-2t} + 1$$

$$= 1 + (c-4)\,e^{-2t}.\,t.$$

$$y' = (c-4)\left[e^{-2t} - 2te^{-2t}\right]$$

At $t=0$

$y' = a$

$$a = (c-4)[1-0]$$

$$\boxed{c-4 = a}$$

$$\therefore \boxed{y = 1 + ate^{-2t}}$$

Q8 Using convulation, solve the initial value problem

$$y'' + 9y = \sin 3t$$

$$y(0) = 0$$
$$y'(0) = 0$$

Taking laplace $\qquad$ Let $L\{y\} = \bar{y}$

$$(s^2+9)\bar{y} = L\{\sin 3t\} + 0[s] + 0[1]$$

$$\bar{y} = \frac{3}{(s^2+9)^2}$$

$$y = 3L^{-1}\left\{\frac{1}{(s^2+9)^2}\right\}.$$

Consider

$$L^{-1}\left\{\frac{1}{(s^2+9)} \cdot \frac{1}{(s^2+9)}\right\}$$

By convulation thm.

$$L^{-1}\left\{\bar{f}_1(s)\,\bar{f}_2(s)\right\} = \int_0^t f_1(t-u)\,f_2(u)\,du.$$

$$\therefore \bar{f}_1(s) = \bar{f}_2(s) = \frac{1}{s^2+9}$$

$$\therefore f_1(t) = f_2(t) = L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= L^{-1}\left\{\frac{3}{s^2+9} \cdot \frac{1}{3}\right\}$$

$$= \frac{\sin 3t}{3}$$

$$\therefore L^{-1}\left\{\frac{1}{(s^2+9)^2}\right\} = \int_0^t \frac{\sin 3u}{3} \cdot \frac{\sin 3(t-u)}{3}\, du$$

$$= \frac{1}{9}\int_0^t \sin 3u \cdot \sin(3t-3u)\, du$$

$$= \frac{1}{18}\left[\int_0^t \cos(3u+3u-3t) - \cos(3t)\right] du$$

$$= \frac{1}{18}\int_0^t \cos(6u-3t) - \cos 3t\, dt$$

$$= \frac{1}{18}\left[\frac{\sin(6u-3t)}{6}\Big|_0^t - (\cos 3t)\cdot t\right]$$

$$= \frac{1}{18}\left[\frac{\sin 3t}{6} - \frac{\sin(-3t)}{6} - \cos(3t)\cdot t\right]$$

$$= \frac{1}{18}\left[\frac{\sin 3t}{3} - \cos(3t)\cdot t\right]$$

$$\boxed{\frac{y}{3} = \frac{1}{54}\left[\sin 3t - 3t\cos 3t\right]}$$

$$\boxed{y = \frac{1}{18}\left[\sin 3t - 3t\cos 3t\right]}$$

Q9 In the following problems use the convulation th^m to find the inverse laplace transformation :→

a) $\dfrac{1}{(s-a)(s-b)}$

$\bar{f}_1(s) = \dfrac{1}{s-a}$ $\therefore f_1(t) = e^{at}$

$f_2(s) = \dfrac{1}{s-b}$ $f_2(t) = e^{bt}$

By convulation th^m

$$L^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\} = \int_0^t e^{a(t-u)} e^{bu}\, du$$

$$= e^{at}.\int_0^t e^{(b-a)u}\, du$$

$$= \frac{e^{at}}{(b-a)}\left[e^{(b-a)u}\right]_0^t$$

$$= \frac{e^{at}}{b-a}\left[e^{(b-a)t} - 1\right]$$

$$= \frac{1}{b-a}\left[e^{bt} - e^{at}\right]$$

b) $\dfrac{1}{s^2(s^2+16)}$

$\bar{f}_1(s) = \dfrac{1}{s^2}$ $f_1(t) = 1/t$

$\bar{f}_2(s) = \dfrac{1}{s^2+16}$ $f_2(t) = L^{-1}\left\{\dfrac{4}{s^2+16}\cdot\dfrac{1}{4}\right\}$

$$= \frac{\sin 4t}{4}.$$

by convolution thm,

$$L^{-1}\left\{\bar{f}_1(s)\ \bar{f}_2(s)\right\} = \int_0^t f_1(t-u)\ f_2(u)\ du$$

$$= \int_0^t \frac{(t-u)\sin 4u}{4}\ du$$

$$= \frac{1}{4}\left[t\int_0^t \sin 4u\, du - \int_0^t u\sin 4u\ du\right]$$

$$= \frac{1}{4}\left[t\left(-\frac{\cos 4u}{4}\right)\Big|_0^t - \frac{u}{4}\left[-\frac{\cos 4u}{4}\Big|_0^t + \frac{1}{4}\int_0^t -\cos 4u\ du\right]\right]$$

$$= \frac{1}{4}\left[t\left(\frac{1-\cos 4t}{4}\right) + \frac{t\cos 4t}{4} + \quad - \frac{\sin 4t}{4\times 4}\right]$$

$$= \frac{1}{4}\left[\frac{t}{4} - \frac{\sin 4t}{4\times 4}\right.$$

$$= \frac{1}{64}\left[4t - \sin 4t\right].$$

c) $\dfrac{1}{(s^2+9)^2}$

done in ques 8

d) $\dfrac{s}{(s^2+4)}$

$\bar{f}_1(s) = \dfrac{s}{s^2+4}$ $\quad f_1(t) = \cos 2t$

$\bar{f}_2(s) = \dfrac{1}{s^2+4}$ $\quad f_2(t) = \dfrac{\sin 2t}{2}.$

By convolution th^m

$$L\left\{ \bar{f}_1(s)\ \bar{f}_2(s) \right\} = \int_0^t f_1(t-u)\ f_2(u)\ du.$$

$$= \frac{1}{2} \int_0^t \sin(2u) \cdot \cos(2t-2u)\ du$$

$$= \frac{1}{2} \cdot \frac{1}{2} \int_0^t \sin 2t + \sin(4u-2t)\ du$$

$$= \frac{1}{4} \int_0^t \sin 2t + \sin(4u-2t)\ du$$

$$= \frac{1}{4} \left[t \sin 2t - \frac{\cos(4u-2t)}{4} \Big|_0^t \right]$$

$$= \frac{1}{4} \left[t \sin 2t - \cancel{\frac{\cos 2t}{4}} + \cancel{\frac{\cos 2t}{4}} \right]$$

$$\boxed{y = \frac{t \sin 2t}{4}.}$$

Q10

$$L\{\sin \omega t\} = \frac{\omega}{\omega^2 + s^2}$$

$\sin \omega t$ is an periodic function with period 2π.

$a = 2\pi$

For periodic function

$$L\{f(t)\} = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\ dt.$$

where a is period of $f(t)$

$$\therefore a = \frac{2\pi}{\omega}$$

$$L\{\sin \omega t\} = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{2\pi/\omega} e^{-st} \sin \omega t \, dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \operatorname{img} \int_0^{2\pi/\omega} e^{-st} . e^{i\omega t} \, dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \operatorname{img} \left[\int_0^{\frac{2\pi}{\omega}} e^{-t[s-i\omega]} \, dt \right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \operatorname{img} \left. \frac{e^{-t[s-i\omega]}}{-[s-i\omega]} \right|_0^{2\pi/\omega}$$

$$= \frac{-1}{1-e^{-2\pi s}} \cdot \operatorname{img} \left[\frac{e^{-\frac{2\pi}{\omega}[s-i\omega]} - 1}{s-i\omega} \right]$$

$$= \frac{-1}{1-e^{-2\pi s}} \cdot \operatorname{img} \left[\frac{\left[e^{-\frac{2\pi s}{\omega}} . e^{\frac{2\pi\omega i}{\omega}} - 1\right]}{s-i\omega} \right]$$

$$= \frac{-1}{1-e^{-\frac{2\pi s}{\omega}}} \operatorname{img} \left[\frac{e^{-2\pi s}[\cos 2\pi + i \sin 2\pi) - 1}{s-i\omega} \right]$$

$$= \frac{-1}{1-e^{-\frac{2\pi s}{\omega}}} \operatorname{img} \left[\frac{e^{-\frac{2\pi s}{\omega}} - 1}{s-i\omega} \right]$$

$$= \frac{1-e^{-\frac{2\pi s}{\omega}}}{1-e^{-\frac{2\pi s}{\omega}}} \operatorname{img} \left[\frac{1}{s-i\omega} \right]$$

$$= \operatorname{img} \left[\frac{s+i\omega}{s^2+\omega^2} \right]$$

$$= \frac{\omega}{s^2+\omega^2}$$

HP$_3$