# Assignment - 5

## Fourier Series and Fourier Transforms

### Fouries Series

Q1. Find the fourier series of the following function.

$$f(x) = x^2, \quad 0 \le x \le \pi$$
$$= -x^2, \quad -\pi \le x \le 0$$

Ans. $f(x) = 2\left(\pi - \frac{4}{\pi}\right)\sin x - \pi \sin 2x + \frac{2}{3}\left(\pi - \frac{4}{9\pi}\right)\sin 3x - \frac{\pi}{2}\sin 4x + \ldots$

Q2. An alternating current, after passing through a rectifier, has the form

$$i = I_0 \sin x, \quad 0 \le x \le \pi$$
$$= 0, \quad \pi \le x \le 2\pi$$

where $I_0$ is maximum current and the period is $2\pi$. Express $i$ as a fourier series.

Ans $i = \frac{I_0}{\pi}\left(1 + \frac{\pi}{2}\sin\theta) - \frac{2}{1\cdot 3}\cos 2\theta - \frac{2}{3\cdot 5}\cos 4\theta - \frac{2}{5\cdot 7}\cos 6\theta + \ldots\right)$.

Q3. Discuss the convergence of fourier series.

Q4. Find the fourier series to represent $f(x) = x\sin x$ for $0 < x < 2\pi$.

Ans. $f(x) = -1 + \pi\sin x - \frac{1}{2}\cos x + 2\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} + \right.$

Q5. Prove that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}, \quad -\pi < x < \pi$.

Hence show that

i) $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$ ii) $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Q6. Obtain a fourier series to represent the function $f(x) = |\sin x|$ for $-\pi < x < \pi$

Ans. $f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x + \ldots\right]$

Q7. Find half range cosine series for $f(x) = e^x, \quad 0 < x < \pi$

Ans $f(x) = \frac{e^{\pi}-1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n e^{\pi}}{\pi^2+1}\cos nx$

Q8. Find the fourier series to represent $f(x)$, where

$$f(x) = -a \quad , -c < x < 0$$
$$= a \quad , 0 < x < c$$

Ans $f(x) = \frac{4a}{\pi}\left[\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \cdots\right]$

Q9. If $-\pi < x < \pi$, prove that

$$x\sin x = 1 - \frac{1}{2}\cos x - \frac{2\cos 2x}{1\cdot 3} + \frac{2\cos 3x}{2\cdot 4} - \frac{2\cos 4x}{3\cdot 5} + \cdots$$

and hence show that

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$$

Q10. Find the fourier series for the function $f(x) = 2x - x^2$, $0 < x < 3$ and deduce that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Q11. Obtain the half range sine series of $f(x) = lx - x^2$ in $(0, l)$ and hence show that

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.$$

## Fourier Transform

Q1. Do fourier sine and cosine transform of $e^x$ exist? Explain.

Q2. Find the fourier cosine transform of $f(x) = \frac{1}{1+x^2}$.

Ans. $F_c[f(x)] = \frac{\pi}{2}e^{-\lambda}$.

Q3. Find the fourier sine transform of $e^{-x}$ $(x > 0)$ and show that

$$\int_0^{\infty}\frac{x\sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}, \quad m > 0.$$

Ans. $F_s[f(x)] = \sqrt{\frac{2}{\pi}}\,\frac{\lambda}{1+\lambda^2}$.

Q4. Find the fourier transform of the function $f(x) = e^{-ax^2}$, $a>0$ and hence find the fourier transform of $e^{-x^2/2}$.

Ans. $F[f(x)] = \frac{1}{\sqrt{2a}} e^{-s^2/4a}$

$F(e^{-x^2/2}) = e^{-s^2/2}$.

Q5. Use fourier integral to prove that

$$\int_0^\infty \frac{\sin \pi\lambda \sin \lambda x}{1-\lambda^2} d\lambda = \frac{\pi}{2}\sin x, \quad 0<x<\pi$$
$$= 0, \quad x>\pi.$$

Ans. $F_s[f(x)] = \frac{\pi \sin \pi\lambda}{2(1-\lambda^2)}$

Q6. solve the integral equation

$$\int_0^\infty f(x) \cos sx \, dx = 1-s, \quad 0 \le s \le 1$$
$$= 0, \quad s>1$$

and hence show that

$$\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}.$$

Ans. $f(x) = \frac{2}{\pi x^2}(1-\cos x)$

Q7. If $F(s)$ is the fourier transform of $f(x)$, then prove that $F[f(x) e^{-iax}] = F(s-a)$.

Q8. Define convolution of two functions $f(x)$ and $g(x)$ and hence prove that fourier transform of convolution of two functions is equal to the product of their fourier transform.

Q9. By applying an integral transform, solve the boundary value problem

$$f''(x) - f(x) = 3e^{-2x}, \quad (0<x<\infty), \quad f(0) = x_0$$

$f(\infty)$ is bounded.