Bessel's function :- The differential Equation of the form

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

is called Bessel's differential Eq^n of order n. Its particular solutions are called Bessel functions

General sol^n of Bessel's Eqution :-

$$Y = A J_n(x) + B J_{-n}(x)$$

where

$$J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$J_{-n}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

Recurrence formula for $J_n(x)$

(1) $\frac{d}{dx}[x^nJ_n(x)]=x^nJ_{n-1}(x)$

Proof:- $\frac{d}{dx}$ $\quad J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

Multiply by x^n on both side and differentiate

$$\frac{d}{dx}[x^nJ_n(x)]=\frac{d}{dx}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\frac{x^{n+2r}}{2^{n+2r}}\cdot x^n\right)$$

$$=\frac{d}{dx}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\frac{x^{2n+2r}}{2^{n+2r}}\right)$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\frac{(2n+2r)\cdot x^{2n+2r-1}}{2^{n+2r}}$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r\,2\cancel{(n+r)}\,x^{2n+2r-1}}{r!\,\cancel{(n+r)}\,\Gamma(n+r)\,2^{n+2r}}$$

Gamma function

$\left[\because \Gamma n = n(n-1)(n-2)\ldots 2\cdot 1\right]$

In general $\Gamma(n+1)=n!$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot 2 \quad x^{2n+2r-1}}{r!\,\Gamma(n+r) \quad 2^{n+2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \cancel{2} \quad x^n \cdot x^{n+2r-1}}{r!\,\Gamma(n+r) \quad 2^{n+2r-1}}$$

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r)} \frac{x^{n+2r-1}}{2^{n+2r-1}}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma((n-1)+r+1)} \cdot \left(\frac{x}{2}\right)^{(n-1)+2r}$$

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$$

(2) $\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$, $n \neq 0$.

Proof:-

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \quad x^{-n}}{r!\,\Gamma(n+r+1)} \frac{x^{n+2r}}{2^{n+2r}}$$

$$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = \frac{d}{dx}\left[\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \frac{x^{2r}}{2^{n+2r}}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot 2r}{r!\,\Gamma(n+r+1)} \frac{x^{2r-1}}{2^{n+2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cancel{r}}{\cancel{r}(r-1)!\,\Gamma(n+r+1)} \frac{x^{2r-1}}{2^{n+2r-1}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \quad x^{-n} \cdot x^n \cdot x^{2r-1}}{(r-1)!\,\Gamma(n+r+1) \quad 2^{n+2r-1}}$$

$$= x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r}{(r-1)!\,\Gamma(n+r+1)} \frac{x^{n+2r-1}}{2^{n+2r-1}}$$

$$= x^{-n} \sum_{r=0}^{\infty} \frac{(-1)(-1)^{r-1}}{(r-1)!\,\Gamma((n+1)+(r-1)+1)} \cdot \left(\frac{x}{2}\right)^{(n+1)+2(r-1)}$$

$$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$$

(3) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$.

Proof:

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \frac{(n+2r)\cdot x^{n+2r-1}}{2^{n+2r}}$$

$$x J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} (n+2r) \cdot \frac{x^{n+2r}}{2^{n+2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, n}{r!\,\Gamma(n+r+1)} \cdot \frac{x^{n+2r}}{2^{n+2r}} + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\, 2r \cdot \frac{x^{n+2r}}{2^{n+2r}}$$

$$= n J_n(x) + \sum_{r=0}^{\infty} \frac{(-1)^r\, 2}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= n J_n(x) + \sum_{r=0}^{\infty} \frac{(-1)^r \cdot 2\, x^1 \cdot x^{-1}}{(r-1)!\,\Gamma(n+1)+(r-1)+1} \left(\frac{x}{2}\right)^{n+2r}$$

$$= n J_n(x) + \sum_{r=0}^{\infty} \frac{(-1)(-1)^{r-1}\, x}{(r-1)!\,\Gamma(n+1)+(r-1)+1} \cdot \left(\frac{x}{2}\right)^{(n+1)+2(r-1)}$$

$$= n J_n(x) - x J_{n+1}(x).$$

(4) $x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$.

Proof:

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\,(n+2r)}{r!\,\Gamma(n+r+1)} \cdot \frac{x^{n+2r-1}}{2^{n+2r}}$$

$$x J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\,(-n+2n+2r)}{r!\,\Gamma(n+r+1)} \frac{x^{n+2r}}{2^{n+2r}}$$

$$x J_n'(x) = -n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} + \sum_{r=0}^{\infty} \frac{(-1)^r\, 2(n+r)}{r!\,\Gamma(n+r+1)} \cdot \frac{x^{n+2r}}{2^{n+2r}}$$

$$= -nJ_n(x) + \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \cancel{(n+r)}}{r!\, \cancel{(n+r)}\, \Gamma(n+r)} \cdot \frac{x^{n+2r}}{2^{n+2r-1}}$$

$$= -nJ_n(x) + \sum_{r=0}^{\infty} \frac{(-1)^r \cdot x \cdot x^{-1}}{r!\, \Gamma(n+r)} \frac{x^{n+2r}}{2^{n+2r-1}}$$

$$= -nJ_n(x) + \sum_{r=0}^{\infty} \frac{(-1)^r \cdot x}{r!\, \Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= -nJ_n(x) + x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \Gamma(n-1)+(r+1)} \left(\frac{x}{2}\right)^{(n-1)+2r}$$

$$= -nJ_n(x) + xJ_{n-1}(x).$$

5) $2J_n'(x) = J_n(x) - J_{n+1}(x).$

Proof:- from Ist two recurrence formula.

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \quad \text{——(1)}$$

$$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x) \quad \text{——(2)}.$$

Add (1) & (2)

$$2\cancel{x}J_n'(x) = \cancel{x}[J_{n-1}(x) - J_{n+1}(x)].$$

$$\boxed{2J_n'(x) = J_{n-1}(x) - J_{n+1}(x).}$$

6) $2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$

Proof: from 3rd and 4.th recurrence formula,

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x). \quad \text{——(1)}$$

$$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x). \quad \text{——(2)}$$

Subtract (1) & (2)

$$0 = 2nJ_n(x) - x[J_{n-1}(x) + J_{n+1}(x)].$$

$$2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)].$$

Expansions for J_0 and J_1.

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}.$$

Put $n=0$

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+1)} \cdot \left(\frac{x}{2}\right)^{2r}$$

$$= 1 - \frac{1}{1!\,\Gamma 2} \cdot \left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(3)} \cdot \left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma(4)} \cdot \left(\frac{x}{2}\right)^6 + \cdots$$

$$= 1 - \frac{1}{1!\,1!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,2!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,3!}\left(\frac{x}{2}\right)^6 + \cdots$$

$$\left[\because\ \Gamma(2) = 1\,\Gamma(1) = 1,\ \Gamma(3) = 2\,\Gamma(2) = 2\cdot 1 = 2 = 2! \quad \Gamma(4) = 3\,\Gamma(3) = 3\cdot 2 = 3!\right]$$

$$J_0(x) = 1 - \frac{1}{1!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots$$

& for $J_1(x)$,

Put $n=1$

$$J_1(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+2)} \cdot \left(\frac{x}{2}\right)^{1+2r}.$$

$$= \frac{x}{2} - \frac{1}{1!\,\Gamma(3)} \cdot \left(\frac{x}{2}\right)^3 + \frac{1}{2!\,\Gamma(4)}\left(\frac{x}{2}\right)^5 - \frac{1}{3!\,\Gamma(5)} \cdot \left(\frac{x}{2}\right)^7 + \cdots$$

$$= \frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5 - \frac{1}{3!\,4!}\left(\frac{x}{2}\right)^7 + \cdots$$

$$J_1(x) = \frac{x}{2}\left[1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,4!}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

Ques: Show that (Some Examples)

(i) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin x$

(ii) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

(iii) $\left(J_{1/2}(x)\right)^2 + \left(J_{-1/2}(x)\right)^2 = \frac{2}{\pi x}$.

Sol^n:- (i) $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}$.

Put $n = 1/2$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(1/2+r+1)} \left(\frac{x}{2}\right)^{1/2+2r}.$$

$$= \frac{1}{\Gamma(3/2)}\left(\frac{x}{2}\right)^{1/2} - \frac{1}{\Gamma(5/2)}\left(\frac{x}{2}\right)^{5/2} + \frac{1}{2!\,\Gamma(7/2)} \cdot \left(\frac{x}{2}\right)^{7/2} - \cdots$$

$$= \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\Gamma(3/2)} - \frac{1}{\Gamma(5/2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(7/2)} \cdot \left(\frac{x}{2}\right)^4 - \cdots\right].$$

$$= \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\frac{1}{2}\Gamma(1/2)} - \frac{1}{\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)} \cdot \left(\frac{x}{2}\right)^4 - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\cdot\Gamma(1/2)}\left[\frac{2}{1!} - \frac{\cancel{4}}{3}\frac{x^2}{\cancel{4}} + \frac{\cancel{8}}{5\cdot 3\cdot 2}\frac{x^4}{\cancel{16}_{2}} - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\,\Gamma(1/2)}\left[\frac{2}{1!} - \frac{2x^2}{3\cdot 2} + \frac{2x^4}{5\cdot 4\cdot 3\cdot 2} - \cdots\right].$$

$$= \frac{\sqrt{x}}{\sqrt{2}\,\Gamma(1/2)}\left[\frac{2}{1!} - \frac{2}{3!}x^2 + \frac{2}{5!}x^4 - \cdots\right]$$

Now multiplying the series by $x/2$ and outside by $2/x$, we get

$$J_{1/2}(x) = \frac{2}{x} \times \frac{\sqrt{x}}{\sqrt{2}\,\Gamma(1/2)}\left[\frac{x}{2}\cdot\frac{2}{1!} - \frac{2}{3!} \times \frac{x}{2} \cdot x^2 + \frac{2}{5!}\cdot\frac{x}{2}\cdot x^4 - \cdots\right]$$

$$J_{1/2}(x) = \sqrt{\frac{2}{x}}\,\frac{1}{\Gamma(1/2)}\left[\frac{x}{1!} - \frac{1}{3!}x^3 + \frac{x^5}{5!} - \cdots\right].$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{x}{1!} - \frac{1}{3!}x^3 + \frac{x^5}{5!} - \cdots\right] \qquad \left(\because \Gamma(1/2) = \sqrt{\pi}\right)$$

(ii) Similarly taking $n = -\frac{1}{2}$, it can be shown that

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x.$$

(iii) $$\left[J_{\frac{1}{2}}(x)\right]^2 = \frac{2}{\pi x} \cdot \sin^2 x$$

$$\& \left[J_{-\frac{1}{2}}(x)\right]^2 = \frac{2}{\pi x} \cdot \cos^2 x.$$

Now, $$\left(J_{\frac{1}{2}}(x)\right)^2 + \left(J_{-\frac{1}{2}}(x)\right)^2 = \frac{2}{\pi x}\left[\sin^2 x + \cos^2 x\right].$$

$$\boxed{\left(J_{\frac{1}{2}}(x)\right)^2 + \left(J_{-\frac{1}{2}}(x)\right)^2 = \frac{2}{\pi x}}$$

<u>Ques.</u> Prove that $\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$.

<u>Solⁿ.</u> L.H.S. $$\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = J_n J_{n+1} + x J_n' J_{n+1} + x J_n J_{n+1}'$$

$$= J_n J_{n+1} + (x J_n') J_{n+1} + J_n (x J_{n+1}').$$

By Recurrence relation, 3rd and 4th

$$x J_n' = n J_n - x J_{n+1} \quad \text{---(1)}$$

$$x J_n' = -n J_n(x) + x J_{n-1}(x) \quad \text{---(2)}$$

replace n by $n+1$ in (2), we get

$$x J_{n+1}' = -(n+1) J_{n+1}(x) + x J_n(x).$$

Now, put value of $x J_n'$ and $x J_{n+1}'$

$$\frac{d}{dx}\left[x J_n + J_{n+1}\right] = J_n \cdot J_{n+1} + \left(n J_n - x J_{n+1}\right) J_{n+1} + J_n\left[-(n+1) J_{n+1}(x) + x J_n(x)\right]$$

$$= \cancel{J_n J_{n+1}} + \cancel{n J_n J_{n+1}} - x J_{n+1}^2 - \cancel{n J_n J_{n+1}} - \cancel{J_n J_{n+1}} + x J_n^2$$

$$\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right].$$

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Ques: Prove that $\int x^3 J_0(x)dx = x^3 J_1(x) - 2x^2 J_2(x) + C.$

Sol^n: By 1st Recurrence Relation, $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x).$

Put $n=1$, $\frac{d}{dx}\left[x J_1(x)\right] = x J_0(x)$. —(1)

Multiplying the eq^n(1) by x^2 & integrating w.r.t. x.

$$\int x^3 J_0(x) = \int x^2 \frac{d}{dx}(x J_1(x))\, dx$$

$$= x^2 \int \frac{d}{dx}[x J_1(x)] - \int \frac{d}{dx}(x^2) \int \frac{d}{dx}(x J_1(x))\, dx + C$$

$$= x^2 (x J_1(x)) - \int 2x \cdot x \cdot J_1(x) \cdot dx + C$$

$$= x^3 J_1(x) - 2\int x^2 J_1(x) dx + C.$$

$\left[\because x^2 J_2(x) = \int x^2 J_1(x) dx \right.$, By putting $\left. n=2\right]$.

$$\int x^3 J_0(x)dx = x^3 J_1(x) - 2x^2 J_2(x) + C.$$

Orthogonal Properties of Bessel function

Statement: If α & β are two distinct roots of $J_n(x) = 0$

Then $$\int_0^1 x J_n(\alpha x) J_n(\beta x) dx = \begin{cases} 0 & \text{if } \alpha \neq \beta. \\ \frac{1}{2}[J_n'(\alpha)]^2 = \frac{1}{2}[J_{n+1}(\alpha)] & \text{if } \alpha = \beta. \end{cases}$$

Proof: We Know that $J_n(\lambda x)$ is the sol^n of eq^n.

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - n^2) y = 0.$$

if $u = J_n(\alpha x)$ & $v = J_n(\beta x)$ then

$$x^2 \frac{d^2 u}{dx^2} + x\frac{du}{dx} + (\alpha^2 x^2 - n^2) u = 0 \quad \text{—— (1)}$$

$$x^2 \frac{d^2 v}{dx^2} + x\frac{dv}{dx} + (\beta^2 x^2 - n^2) v = 0. \quad \text{—— (2)}$$

Multiplying (1) by $\frac{v}{x}$ & (2) by $\frac{u}{x}$ & subtracting, we get

$$\frac{v}{x}\left[x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+(\alpha^2x^2-n^2)u\right]-\left\{\frac{u}{x}\left[x^2\frac{d^2v}{dx^2}+x\frac{dv}{dx}+(\beta^2x^2-n^2)v\right]\right\}=0$$

$$x\left[v\frac{d^2u}{dx^2}-u\frac{d^2v}{dx^2}\right]+\left[v\frac{du}{dx}-u\frac{dv}{dx}\right]+\frac{uv}{x}\left[\alpha^2x^2-\cancel{n^2}-\beta^2x^2+\cancel{n^2}\right]=0$$

$$\frac{d}{dx}\left[x\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\right]+(\alpha^2-\beta^2)uvx=0.$$

$$(\alpha^2-\beta^2)uvx=-\frac{d}{dx}\left[x\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\right].$$

Integrating both sides from 0 to 1.

$$(\alpha^2-\beta^2)\int_0^1 uvx\,dx=-\int_0^1\frac{d}{dx}\left[x\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\right]\cdot du.$$

$$\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx=\frac{-1}{\alpha^2-\beta^2}\left[x\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\right]_0^1$$

$$\left[\begin{array}{l}\because\ u=J_n(\alpha x),\ \text{then}\ \frac{du}{dx}=\alpha J_n'(\alpha x).\\ \ \ \&\ v=J_n(\beta x)\ \text{then}\ \frac{dv}{dx}=\beta J_n'(\beta x)\end{array}\right].$$

$$\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx=\frac{-1}{\alpha^2-\beta^2}\left[x\left(J_n(\beta x)\,\alpha J_n'(\alpha x)-J_n(\alpha x)\,\beta J_n'(\beta x)\right)\right]_0^1$$

$$\int_0^1 x\,J_n(\alpha x)J_n(\beta x)\,dx=\frac{-1}{\alpha^2-\beta^2}\left[\alpha J_n(\beta)J_n'(\alpha)-\beta J_n(\alpha)J_n'(\beta)\right].$$

since α & β are roots of $J_n(x)$

then $J_n(\alpha)=J_n(\beta)=0.$

Case I :- if $\alpha\neq\beta$.

$$\int_0^1 x\,J_n(\alpha x)J_n(\beta x)\,dx=\frac{-1}{\alpha^2-\beta^2}\left[\alpha\cdot 0\cdot\cancel{J_n(\beta)}-\beta\cdot 0\right]$$

$$\boxed{\int_0^1 x\,J_n(\alpha x)J_n(\beta x)\,dx=0}$$

Case II :- if $\alpha = \beta$.

$$\int_0^1 x J_n(\alpha x) J_n(\beta x) dx = \frac{-1}{\alpha^2-\beta^2}\left[\alpha J_n(\beta) J_n'(\alpha) - \beta J_n(\alpha) J_n'(\beta)\right].$$

when $\alpha = \beta$,

$\frac{0}{0}$ form then apply L'hospital rule, then

$$= \lim_{\alpha \to \beta} -\left[\frac{0 - \beta J_n'(\alpha) J_n'(\beta)}{2\alpha}\right].$$

($\because$ Its value can be found by considering β as a root of $J_n(x)=0$ & α as a variable approaching α.)

$$= \frac{\cancel{\beta} J_n'(\beta) J_n'(\beta)}{2\cancel{\beta}}.$$

$$= \frac{[J_n'(\beta)]^2}{2} = \frac{[J_n'(\alpha)]^2}{2}$$

Since, $J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$.

$J_n'(\alpha) = \frac{n}{\alpha} J_n(\alpha) - J_{n+1}(\alpha)$.

$J_n'(\alpha) = -J_{n+1}(\alpha)$. $\quad (\because J_n(\alpha) = 0)$.

$$\int_0^1 x J_n(\alpha x) J_n(\beta x) dx = \frac{[J_{n+1}(\alpha)]^2}{2}, \text{ if } \alpha = \beta.$$

Frobenius Method:- In power series method, $x=0$ should be ordinary point but in frobenius method $x=0$ should be a regular singular point.

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0. \quad \text{---} (1)$$

(i) Assume the solⁿ to be $y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots)$

(ii) Substitute values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in (1).

(iii) Equate to zero the coefficient of the lowest degree term in x. It gives a quadratic eqⁿ known as the indicial equation.

(iv) Equating to zero the coefficients of the other powers of x, find the values of $a_1, a_2, a_3 \dots$ in terms of a_0.

The complete solⁿ depends on the nature of roots of the indicial eqⁿ.

Case 1:- when roots of the indicial eqⁿ are distinct & do not differ by an integer, the complete solⁿ is

$$y = c_1(y)_{m_1} + c_2(y)_{m_2}$$

where m_1, m_2 are roots.

Example:- Solve in series the eqⁿ $9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$.

Solⁿ:- Here. $x=0$ is singular pt since coefficient of $y''=0$ at $x=0$.

Substitute $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots = \sum_{r=0}^{\infty} a_r x^{m+r}$

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \dots$$

$$\& \frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \dots$$

in the given eqⁿ, we obtain

$$9x(1-x)\left[m(m-1)a_0x^{m-2}+(m+1)ma_1x^{m-1}+(m+2)(m+1)a_2x^m+\cdots\right]$$
$$-12\left[ma_0x^{m-1}+(m+1)a_1x^m+(m+2)a_2x^{m+1}+\cdots\right]+4\left[a_0x^m+a_1x^{m+1}+a_2x^{m+2}+\cdots\right]=0.$$

The lowest power of x is x^{m-1}. Its coefficient equated to zero gives

$$a_0\left[9m(m-1)-12m\right]=0 \quad \text{i.e. } m(3m-7)=0 \quad \text{as } a_0\neq 0.$$

Thus roots of indicial eq^n are $m=0,\ 7/3$ i.e. Roots are distinct & do not differ by an integer.

The coeff. of x^m equated to zero gives

$$a_1\left[9(m+1)m-12(m+1)\right]+a_0\left[4-9m(m-1)\right]=0.$$

$$\Rightarrow 3a_1(3m-4)(m+1)-a_0(3m-4)(3m+1)=0.$$

$$\Rightarrow 3a_1(m+1)=a_0(3m+1).$$

Similarly $3a_2(m+2)=a_1(3m+4)$, $3a_3(m+3)=a_2(5m+7)$ & so on.

$$\therefore\ a_1=\frac{(3m+1)a_0}{3(m+1)},\quad a_2=\frac{(3m+4)a_1}{3(m+2)}=\frac{(3m+4)}{3(m+2)}\times\frac{(3m+1)}{3(m+1)}.a_0.$$

$$a_3=\frac{(3m+7)(3m+4)(3m+1)}{3^3(m+3)(m+2)(m+1)}a_0 \quad \text{etc.}$$

When $m=0$, $a_1=\frac{1}{3}a_0$, $a_2=\frac{1\cdot4}{3\cdot6}a_0$, $a_3=\frac{1\cdot4\cdot7}{3\cdot6\cdot9}a_0$

$$y_1=a_0\left[1+\frac{1}{3}x+\frac{1}{3}\cdot\frac{4}{6}\cdot x^2+\frac{1}{3}\cdot\frac{4}{6}\cdot\frac{7}{9}\cdot x^3+\cdots\right]$$

When $m=7/3$, $a_1=\frac{(7+1)}{3(\frac{7}{3}+1)}a_0=\frac{8\times\cancel{3}}{\cancel{3}\times10}a_0=\frac{8}{10}a_0.$

$$a_2=\frac{(7+4)}{3(\frac{7}{3}+2)}\times\frac{(7+1)}{3(\frac{7}{3}+1)}a_0=\frac{11\times8}{13\times10}a_0.$$

$$a_3=\frac{(7+7)}{3^3(\frac{7}{3}+3)}\frac{(7+4)}{(\frac{7}{3}+2)}\frac{(7+1)}{(\frac{7}{3}+1)}a_0=\frac{8}{10}\cdot\frac{11}{13}\cdot\frac{14}{16}.a_0.$$

$$y_2 = a_0 x^{7/3}\left[1+\frac{8}{10}x+\frac{8\cdot 11}{10\cdot 13}x^2+\frac{8\cdot 11\cdot 14}{10\cdot 13\cdot 16}x^3+\cdots\right].$$

Thus the complete solⁿ is $y = C_1 y_1 + C_2 y_2$

$$y = C_1\left[1+\frac{1}{3}x+\frac{1\cdot 4}{3\cdot 6}x^2+\frac{1}{3}\cdot\frac{4}{6}\cdot\frac{7}{9}x^3+\cdots\right]$$
$$+C_2 x^{7/3}\left[1+\frac{8}{10}x+\frac{8\cdot 11}{10\cdot 13}x^2+\frac{8\cdot 11\cdot 14}{10\cdot 13\cdot 16}x^3+\cdots\right]$$

where $C_1 = c_1 a_0$, $C_2 = c_2 a_0$.

Case II:- when roots of the indicial eqⁿ are equal the complete solⁿ is

$$y = C_1 (y)_{m_1} + C_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

where m_1, m_1 are the roots.

Example:- Solve in series the eqⁿ $x\frac{d^2y}{dx^2}+\frac{dy}{dx}-xy=0$

Solⁿ:- Substituting $y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots$

$$\frac{dy}{dx} = m a_0 x^{m-1} + (m+1)a_1 x^m + (m+2)a_2 x^{m+1} + \cdots$$

$$\frac{d^2y}{dx^2} = m(m-1)a_0 x^{m-2} + (m+1)m a_1 x^{m-1} + (m+2)(m+1)a_2 x^m + \cdots$$

in the given eqⁿ, we obtain

$$x\left[m(m-1)a_0 x^{m-2} + (m+1)m a_1 x^{m-1} + (m+2)(m+1)a_2 x^m + \cdots\right]$$
$$+\left[m a_0 x^{m-1} + (m+1)a_1 x^m + (m+2)a_2 x^{m+1} + \cdots\right]$$
$$-x\left[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots\right] = 0.$$

The lowest power of x is x^{m-1}. Its coefficient equated to zero gives $a_0[m(m-1)+m] = 0$.

$$m^2 = 0 \text{ as } a_0 \neq 0$$

$$\Rightarrow m = 0, 0.$$

The coefficients of $x^m, x^{m+1}, \ldots$ equated to zero gives

$$a_1[(m+1)m + m+1] = 0 \quad \text{i.e. } a_1 = 0$$

$$a_2(m+2)^2 - a_0 = 0, \quad a_3(m+3)^2 + a_1 = 0, \quad a_4(m+4)^2 + a_2 = 0 \text{ \& so on}$$

Clearly $a_3 = a_5 = a_7 = \cdots = 0$.

Also, $a_2 = \dfrac{a_0}{(m+2)^2}$, $a_4 = \dfrac{a_2}{(m+4)^2} = \dfrac{a_0}{(m+2)^2(m+4)^2}$

$$\therefore \quad y = a_0 x^m\left[1 + \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} + \frac{x^6}{(m+2)^2(m+4)^2(m+6)^2} + \cdots\right] \quad (1)$$

Putting $m=0$, the first sol^n is

$$y_1 = a_0\left[1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\right] = (y)_{m=0}.$$

To get the second sol^n, diff. (1) partially w.r.t. m

$$\frac{\partial y}{\partial m} = a_0 \frac{\partial (x^m)}{\partial m}\left[1 + \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} + \cdots\right] +$$

$$a_0 x^m \frac{\partial}{\partial m}\left[1 + \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} + \cdots\right].$$

$$\left[\because \frac{\partial}{\partial m} x^m = x^m . \log x\right].$$

$$\Rightarrow \frac{\partial y}{\partial m} = a_0 x^m . \log x\left[1 + \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} + \cdots\right] + a_0 x^m\left[\frac{-x^2}{(m+2)^2} \frac{2}{m+2}\right.$$

$$\left. + \frac{x^4}{(m+2)^2(m+4)^2}\left(\frac{2}{m+2} + \frac{2}{m+4}\right) + \cdots\right].$$

$$\frac{\partial y}{\partial m} = y \log x + a_0 x^m\left[\frac{-x^2 \cdot 2}{(m+2)^2(m+2)} + \frac{x^4}{(m+2)^2(m+4)^2}\left(\frac{2}{m+2} + \frac{2}{m+4}\right) + \cdots\right].$$

2nd sol^n is $y_2 = \left(\dfrac{\partial y}{\partial m}\right)_{m=0}$

$$y_2 = (y)_{m=0} \log x + a_0\left[\frac{-1}{2^2} . x^2 + \frac{1}{2^2 \cdot 4^2}\left(1 + \frac{1}{2}\right)x^4 + \frac{1}{2^2 \cdot 4^2 \cdot 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 - \cdots\right]$$

Hence the complete solⁿ is $y = C_1 y_1 + C_2 y_2$.

$$y = C_1 \cdot a_0\left(1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right)+C_2\left[(y)_{m=0}\log x + a_0\left(\frac{-1}{2^2}x^2 + \frac{1}{2^2\cdot 4^2}\left(1+\frac{1}{2}\right)x^4+\cdots\right)\right].$$

$$= C_1 a_0\left(1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right)+C_2\log x\, a_0\left(1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right)$$

$$+C_2 a_0\left[\frac{-1}{2^2}x^2+\frac{1}{2^2\cdot 4^2}\left(1+\frac{1}{2}\right)x^4+\cdots\right].$$

$$y = (C_1 + C_2\log x)\, a_0\left(1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right)+C_2 a_0\left[\frac{-1}{2^2}x^2 + \frac{1}{2^2\cdot 4^2}\left(1+\frac{1}{2}\right)x^4+\cdots\right].$$

Case III :- when roots of indicial eqⁿ are distinct and differ by an integer, making a coefficient of y infinite.

Let m_1 & m_2 be the roots s.t. $m_1 < m_2$. If some of the coefficients of y series become infinite when $m = m_1$, we modify the form of y by replacing a_0 by $b_0(m-m_1)$. Then the complete solⁿ is

$$y = C_1 (y)_{m_2} + C_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

Example:- Series solⁿ of the eqⁿ. $x(1-x)\frac{d^2y}{dx^2} - (1+3x)\frac{dy}{dx} - y = 0$.

Solⁿ:- Here $x=0$ is a singular pt, since coeff. of y'' is zero at $x=0$.

Let solⁿ is $y = \sum_{r=0}^{\infty} a_r x^{m+r} = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots$ —— (1)

$$\frac{dy}{dx} = m a_0 x^{m-1} + (m+1)a_1 x^m + (m+2)a_2 x^{m+1} + \cdots$$

& $$\frac{d^2y}{dx^2} = m(m-1)a_0 x^{m-2} + (m+1)m a_1 x^{m-1} + (m+2)(m+1)a_2 x^m + \cdots$$

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in given eq^n, we obtain

$$x(1-x)[m(m-1)a_0x^{m-2}+(m+1)ma_1x^{m-1}+(m+2)(m+1)a_2x^{m}+\cdots]-$$
$$-(1+3x)[ma_0x^{m-1}+(m+1)a_1x^{m}+(m+2)a_2x^{m+1}+\cdots]-$$
$$[a_0x^{m}+a_1x^{m+1}+a_2x^{m+2}+\cdots]=0$$

Equating to zero the coefficients of the lowest power of x, we get $a_0[m(m-1)-m]=0$

$\Rightarrow m(m-2)=0$ as $a_0\neq 0$.

$\Rightarrow m=0,2$, two roots are distinct & differ by an integer.

Equating to zero the coefficients of successive powers of x, we get

$(m-1)a_1=(m+1)a_0$, $ma_2=(m+2)a_1$, $(m+1)a_3=(m+3)a_2$ & so on

$$a_1=\frac{m+1}{m-1}a_0,\quad a_2=\frac{(m+1)(m+2)}{(m-1)m}a_0,\quad a_3=\frac{(m+1)(m+2)(m+3)}{(m-1)m(m+1)}a_0$$

Thus,

$$y=a_0x^m\left[1+\frac{m+1}{m-1}x+\frac{(m+1)(m+2)}{(m-1)m}x^2+\frac{(m+1)(m+2)(m+3)}{(m-1)m(m+1)}x^3+\cdots\right] \quad (2)$$

Putting $m=2$, Ist sol^n is

$$y_1=a_0x^2\left[1+3x+\frac{3\cdot 4}{2}x^2+\frac{4\cdot 5}{2}x^3+\cdots\right]$$

If we put $m=0$ in (2), the coefficient become infinite.

To avoid this difficulty, put $a_0=b_0(m-0)$,

$$y=b_0x^m\left[m+\frac{m(m+1)}{m-1}x+\frac{(m+1)(m+2)}{m-1}x^2+\frac{(m+1)(m+2)(m+3)}{(m-1)(m+1)}x^3+\cdots\right]$$

$$\therefore \frac{\partial y}{\partial m}=b_0x^m\log x\left[m+\frac{m(m+1)}{m-1}x+\frac{(m+1)(m+2)}{m-1}x^2+\frac{(m+1)(m+2)(m+3)}{(m-1)(m+1)}x^3+\cdots\right]$$
$$+b_0x^m\left[1+\frac{m^2-2m-1}{(m-1)^2}x+\frac{m^2-m-5}{(m-1)^2}x^2+\frac{m^2-2m-11}{(m-1)^2}x^3+\cdots\right]$$

$\therefore$ 2nd sol^n is, $y_2=\left(\frac{\partial y}{\partial m}\right)_{m=0}$.

$$=b_0\log x[-1\cdot 2x^2-2\cdot 3x^3-3\cdot 4x^4-\cdots]+b_0[1-x-5x^2-11x^3-\cdots]$$

Hence the complete sol^n is $y = c_1 y_1 + c_2 y_2$

$$y = \frac{1}{2} c_1 a_0 \left[1.2x^2 + 2.3x^3 + 3.4x^4 + \cdots\right] - b_0 c_2 \log x \left[1.2x^2 + 2.3x^3 + 3.4x^4 + \cdots\right] - b_0 c_2 \left[-1 + x + 5x^2 + 11x^3 + \cdots\right]$$

$$y = (C_1 + C_2 \log x)(1.2x^2 + 2.3x^3 + 3.4x^4 + \cdots) + C_2(-1 + x + 5x^2 + 11x^3 + \cdots)$$

where $C_1 = \frac{1}{2} c_1 a_0$, $C_2 = -b_0 c_2$.

3rd Unit is Complete.