2K19/A14/35

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MA-102

Mathematics Assignment - II

I a) $y'' - 8y' + 16y = 0$

Auxillary eqn $\rightarrow$ $(D^2 - 8D + 16)y = 0$

$(D-4)(D-4) = 0$ $\quad D = 4, 4$

General solutn : $(c_1 + c_2 x)e^{4x}$

b) $y'''' - 4y''' + 8y'' - 8y' + 4y = 0$

$D^4 + 4D^2 + 4 - 4D^3 + 4D^2 - 8D = 0$

$(D^2)^2 + (-2D)^2 + (2)^2 - 4D^3 + 4D^2 - 8D = 0$

$(D^2 - 2D + 2)^2 = 0$

$D = 1 \pm i, 1 \pm i$

Soln :

$y = e^x [(c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x]$

c) $4y'''' - 4y''' - 23y'' + 12y' + 3y = 0$

$(4D^4 - 4D^3 - 23D^2 + 12D + 36)y = 0$

$(-2D)^2 + (6)^2 + (D)^2 - 24D^2 + 12D - 4D^3 = 0$

$(-2D^2 + D + 6)^2 = 0$

$(D-2)^2 (2D+3)^2 = 0$ $\quad D = 2, 2, -3/2, -3/2$

Solutn: $(c_1 + c_2 x)e^{2x} + (c_3 + c_4 x)e^{-3/2 x}$

$\boxed{2}$ a) $(D^2+a^2)y = \cot ax$ $\quad D = \pm ia$

C.F. $c_1 \cos ax + c_2 \sin ax$

$$W = a\begin{bmatrix} \cos ax & \sin ax \\ -\sin ax & \cos ax \end{bmatrix} = a.$$

$$y(x) = -y_1 \int \frac{y_2 X}{W}\,dx + y_2 \int \frac{y_1 X}{W}\,dx$$

$$= -\frac{\cos ax}{a} \int \cot ax \sin ax\,dx + \frac{\sin ax}{a} \int \cot ax \cos ax\,dx$$

$$= -\frac{\cos ax}{a} \int \cos ax\,dx + \frac{\sin ax}{a} \int \frac{1-\sin^2 ax}{\sin ax}$$

$$= \cancel{-\frac{\cos ax \sin ax}{a^2}} + \frac{\sin ax}{a^2} \log\left(\operatorname{cosec} ax - \cot ax\right) + \cancel{\frac{\sin ax \cos ax}{a^2}}$$

$$= \frac{\sin ax}{a^2} \log\left(\operatorname{cosec} ax - \cot ax\right)$$

Gen solution:

$$c_1 \cos ax + c_2 \sin ax + \frac{\sin ax}{a^2} \log\left|\operatorname{cosec} ax - \cot ax\right|.$$

b) $(D^3 - D^2 - 6D)y = x^2 + 1 + 3^x$

$D(D^2 - D - 6) = 0$

$D(D-3)(D+2) = 0$ $\quad D = 0, 3, -2.$

$$C.F = c_1 e^0 + c_2 e^{3x} + c_3^{-2x}$$

PI

$$y = \underset{①}{\frac{x^2}{(D^3-D^2-6D)}} + \underset{②}{\frac{1}{(D^3-D^2-6D)}} + \underset{③}{\frac{3^x}{(D^3-D^2-6D)}}$$

① $y = \dfrac{x^2}{D(D^2-D-6)} = \dfrac{1}{D}x^2\left(1-\left(\dfrac{D^2-D}{6}\right)\right)^{-1}\cdot\dfrac{-1}{6}$

$= \dfrac{-1}{6D}x^2\left(1+(D^2-D)+(D^2-D)^2+\cdots\right)$

$= -\dfrac{1}{6D}x^2\left(1+\dfrac{D^2-D}{6}+\dfrac{\cancel{D^4}+D^2-2\cancel{D^3}}{36}\right)$

$= -\dfrac{1}{6D}\left(x^2+\dfrac{7}{18}-\dfrac{2x}{6}\right)$

$= -\dfrac{1}{6}\left(\dfrac{x^3}{3}+\dfrac{7x}{18}-\dfrac{x^2}{6}\right).$

② $y = \dfrac{1}{D(D^2-D-6)} = \dfrac{e^{0x}}{D(D^2-D-6)}$

$\underset{(D=0)}{=} \dfrac{x}{3D^2-2D-6} = -\dfrac{x}{6}.$

③ $y = \dfrac{3^x}{D^3-D^2-6D} = \dfrac{e^{x\ln 3}}{D^3-D^2-6D}$

$= \dfrac{3^x}{(\ln 3)^3-(\ln 3)^2-6\ln 3}.$

So. General solution is,

$c_1e^{-2x}+c_2+c_3e^{3x}+\dfrac{-7x}{108}-\dfrac{x}{6}+\dfrac{x^2}{36}-\dfrac{x^3}{18}+\cancel{e^{x\ln 3}}$

$+\dfrac{3^x}{(\ln 3)^3-(\ln 3)^2-6\ln 3}.$

c) $(D^4+D^2+1)\,y = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$

$(D^2+1)^2 - D^2 = 0$

$(D^2+D^2+1)(D^2-D^2+1) = 0$

$D = \frac{-1 \pm \sqrt{3}i}{2}, \quad \frac{1 \pm \sqrt{3}i}{2}$

$CF = e^{-x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + e^{x/2}\left(c_3\cos\frac{\sqrt{3}}{2}x + c_4\sin\frac{\sqrt{3}}{2}x\right)$

PI

$$\frac{e^{-x/2}\cos(\sqrt{3}/2\, x)}{(D^4+D^2+1)}$$

$$= e^{-x/2}\frac{\cos\left(\frac{\sqrt{3}}{2}x\right)}{(D-1/2)^4+\left(D-\frac{1}{2}\right)^2+1} = \frac{e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)}{(D^2+3/4)(D^2-2D+7/4)}$$

$$= e^{-x/2}\frac{\cos(\sqrt{3}/2\, x)\,x}{2D(1-2D)} = \frac{e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)\,x}{2D+3} \times \frac{2D-3}{2D-3}$$

$$= \frac{e^{-x/2}\cos(\sqrt{3}/2\, x)\,x(2D-3)}{4D^2-9} = \frac{e^{-x/2}}{-12}(2D-3)\left(x\cos\left(\frac{\sqrt{3}}{2}x\right)\right)$$

$$= \frac{e^{-x/2}}{-12}\left[\left(2\cos\frac{\sqrt{3}x}{2} - \frac{\sqrt{3}x}{2}\sin\left(\frac{\sqrt{3}x}{2}\right)\right) - 3x\cos\frac{\sqrt{3}}{2}x\right]$$

$$= \frac{e^{-x/2}}{-6}\cos\frac{\sqrt{3}}{2}x + \frac{e^{-x/2}\sqrt{3}x}{2}\sin\frac{\sqrt{3}}{2}x + \frac{e^{-x/2}x}{4}\cos\frac{\sqrt{3}x}{2}$$

General solution $=$ CF + PI

d) $(D^4+2D^2+1)y = x^2\cos x$

$(D^2+1)^2 = 0 \qquad D = \pm i, \pm i.$

$CF = (c_1+c_2x)\cos x + (c_3+c_4x)\sin x.$

PI $\quad y = \dfrac{x^2\cos x}{(D^2+1)^2}$

$= (\text{R.P of } e^{ix}) \dfrac{x^2}{(D^2+1)^2}$

$= (\text{R.P}) \text{ of } e^{ix} \dfrac{x^2}{((D+i)^2+1)^2}$

$= (\text{R.P of } e^{ix}) \dfrac{x^2}{(D^2+2iD)^2}$

$= (\text{R.P of } e^{ix}) \dfrac{x^2}{-4D^2}\left(1+\dfrac{D}{2i}\right)^{-2}$

$= (\text{R.P of}) \; e^{ix} \dfrac{x^2}{-4D^2}\left(1 + iD + 3\left(\dfrac{iD}{2}\right)^2 + \cdots\right)$

$= (\text{R.P of} -\dfrac{e^{ix}}{4}) \left[\dfrac{1}{D^2}\left(x^2+2ix-\dfrac{3}{2}\right)\right]$

$= \text{R.P of} -\dfrac{e^{ix}}{4}\left(\dfrac{x^4}{12} + \dfrac{ix^3}{3} - \dfrac{3}{4}x^2\right]$

$= \text{R.P of} \left(-\dfrac{x^4 e^{ix}}{48} - \dfrac{ix^3e^{ix}}{12} + \dfrac{3}{16}x^2e^{ix}\right)$

$$= \frac{1}{48}\left(4x^3 \sin x - (x^4 - 9x^2)\cos x\right)$$

General soluth;

$$(c_1 x + c_2)\cos x + (c_3 x + c_4)\sin x + \frac{1}{48}\left(4x^3 \sin x\right)$$
$$- \frac{1}{48}\left(x^4 - 9x^2\right)\cos x .$$

$\boxed{3}$

$y_1 = e^x \qquad y_2 = xe^x$

$y_1' = e^x \qquad y_2' = e^x + xe^x$

a) $W(x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} \neq 0$

$\therefore$ e^x and xe^x are linearly independent.

b) $y = c_1 e^x + c_2 x e^x = (c_1 + c_2 x)e^x$

c) $y(0) = 1$

$(c_1 + c_2 \times 0)e^0 = 1 \Rightarrow \boxed{c_1 = 1}$

$y'(0) = 4$

$y' = (c_1 + c_2 x)e^x + c_2 e^x$

$y(0) = c_1 + c_2 = 4 \qquad \boxed{c_2 = 3}$

Particular soln. $\rightarrow y = (1 + 3x)e^x .$

$\boxed{4}$ $(D-m_1)y = t$.

$\therefore$ $(D-m_1)(D-m_2)y = 0$

$\Rightarrow (D-m_1)t = 0$

$\Rightarrow \frac{dt}{dx} - m_1 t = 0 \Rightarrow \frac{1}{t}\frac{dt}{dx} = m_1$

$\log t = m_1 x + \log c$

$\underline{t = c_1 e^{m_1 x}}$.

$(D-m_1)y = c_1 e^{m_1 x}$

$\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$.

$y e^{-m_1 x} = \int c_1 e^{m_1 x - m_1 x}$

$y = (c_1 x + c_2) e^{m_1 x}$.

$\boxed{5}$ a) $(D^2 - 3D + 2)y = \frac{e^x}{1+e^x}$

$(D-2)(D-1) = 0$ $D = 2, 1,$

$CF = c_1 e^x + c_2 e^{2x}$

$$W(x) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$$

$$y(x) = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= -e^x \int \frac{e^{2x} \cdot e^x}{(1+e^x) e^{3x}} dx + e^{2x} \int \frac{e^x e^x}{(1+e^x) e^{3x}} dx$$

$$= -e^x \int \frac{dx}{1+e^x} + e^{2x} \int \frac{dx}{e^x(1+e^x)}$$

$$= e^{x} \int \frac{e^{-x}}{1+e^{-x}}\,dx + e^{2x}\int \left(\frac{1}{e^{x}} - \frac{1}{e^{x}+1}\right) dx$$

$$= e^{x} \ln(e^{x}+1) + e^{2x}\left[(-e^{-x}) - \ln|e^{-x}+1|\right)$$

$$= e^{x} \ln|e^{-x}+1| - e^{x} + e^{2x} \ln|e^{-x}+1| .$$

CF + PI

$$\Rightarrow \quad c_1 e^{x} + c_2 e^{2x} + e^{x} \ln|e^{-x}+1| - e^{x} + e^{2x} \ln|e^{x}+1| .$$

b) $(D^2-1)y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$

$D = \pm 1.$

$CF = c_1 e^{x} + c_2 e^{-x}$

$$w(x) = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -2$$

PI

$$-e^{x}\int \frac{e^{-x}}{-2}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)dx \quad ①$$

$$+ \; e^{-x}\int \frac{e^{x}}{-2}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)dx \quad ②$$

① $e^{-x} = t, \; -e^{-x}dx = dt$

$$\frac{e^{x}}{2}\int (t\sin t + \cos t)\,dt = -\frac{e^{-x}}{2}\left[(-t\cos t) + 2\int \cos t\,dt\right]$$

$$= \frac{-e^{x}}{2}\left[-e^{-x}\cos e^{-x} + 2\sin e^{-x}\right].$$

② $$\frac{e^{-x}}{-2}\int e^{x}\left[\cos e^{-x} + e^{-x}\sin e^{-x}\right]dx$$

$$= \frac{e^{-x}}{-2}\cdot e^{x}\cos e^{-x} = -\frac{\cos(e^{-x})}{2}$$

Ans = PI + CF .

(c)

$$y'' + y = \frac{1}{1+\sin x}$$

$(D^2+1)y = 0$, $D = \pm i$

$CF = c_1 \sin x + c_2 \cos x$

$$W(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1$$

PI

$$= \sin x \int \frac{\cos x}{1+\sin x}\,dx - \cos x \int \frac{\sin x}{1+\sin x}\,dx$$

$$= \sin x \ln|1+\sin x| - \cos x \int \frac{\sin x\,(1-\sin x)}{1+\sin x\,(1-\sin x)}\,dx$$

$$= \quad '' \quad -\cos x \int \sec x \tan x - \tan^2 x \,dx$$

$$= \quad '' \quad -\cos x \left[\sec x \tan x - \sec^2 x + 1\right]$$

$$= \sin x \ln|1+\sin x| - 1 + \sin x - x\cos x .$$

$\rightarrow$ CF + PI

$c_1 \sin x + c_2 \cos x + \sin x \ln|\sin x + 1| - 1 + \sin x - x\cos x .$

6 a) $x^3 \frac{d^3y}{dx^3} - 4x^2 \frac{d^2y}{dx^2} + 8x\frac{dy}{dx} - 8y = 4 \ln x$

$t = \ln x$, $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$,

$x^2 \frac{d^2y}{dx^2} = D(D-1)y$

$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y .$

$D(D-1)(D-2)y - 4D(D-1)y + 8Dy - 8 = 4t$

$D^3 - 7D^2 + 14D - 8$ $\qquad D = 1, 2, 4$

$= (D-1)(D-2)(D-4)$

$CF = c_1 e^t + c_2 e^{2t} + c_3 e^{4t}$

PI . $\dfrac{4t}{(D-1)(D-2)(D-4)}$

$= \left[\frac{1}{3}\left(\frac{1}{D-1}\right) - \frac{1}{2}\left(\frac{1}{D-2}\right) + \frac{1}{6}\left(\frac{1}{D-4}\right)\right] 4t$

$= \frac{4}{3} e^t \int t e^{-t} dt - 2e^{2t} \int t e^{-2t} dt + \frac{2}{3} e^{4t} \int t e^{-4t} dt$

$= -\frac{4}{3}t - \frac{4}{3} + \frac{2t}{2} + \frac{1}{2} - \frac{t}{6} - \frac{1}{24} = -\frac{t}{2} - \frac{7}{8}$

→ CF+PI

→ $c_1 x + c_2 x^2 + c_3 x^4 - \frac{\ln x}{2} - \frac{7}{8}$.

b) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \ln x \sin(\ln(x))$

$\ln x = t$.

$(D(D-1) + D + 1)y = t \sin t$

$(D^2+1)y = t \sin t$ $\qquad D = \pm i$.

$CF = c_1 \cos t + c_2 \sin t$.

$PI = \dfrac{t \sin t}{D^2+1} = \operatorname{Im}\left(\dfrac{t}{(D+i)^2+1}\right) e^{it}$

$= \operatorname{Im}\left(e^{it} \frac{1}{2iD}\left(\frac{t}{1+\frac{D}{2i}}\right)\right)$

$= \operatorname{Im}\left(\frac{e^{it}}{2iD}\left(t - \frac{1}{2i}\right)\right)$

$$= \operatorname{Im}\left(\frac{e^{it}}{2i}\left(\frac{t^2}{2} - \frac{1}{2i}\right)\right)$$

$$= \operatorname{Im}\left[-\left(\frac{\cos t + i\sin t}{2}\right)\left(\frac{t^2 i}{2} + \frac{t}{2}\right)\right]$$

$$= -\frac{t^2 \cos t}{4} + t \sin t.$$

$\boxed{9}$

$$\frac{d^2x}{dt^2} + w^2 x = 0 \qquad T = \frac{2\pi}{w}$$

$x = c_1 \sin wt + c_2 \cos wt$

$x_1 = c_1 \sin w + c_2 \cos w$

$x_2 = c_1 \sin 2w + c_2 \cos 2w$

$$\frac{x_1 + x_3}{2x_2} = \frac{c_1(\sin w + \sin 3w) + c_2(\cos w + \cos 3w)}{2(c_1 \sin 2w + c_2 \cos 2w)}$$

$$= \frac{2(c_1 \sin 2w + c_2 \cos 2w)\cos w}{2(c_1 \sin 2w + c_2 \cos 2w)}$$

$$\cos^{-1}\left(\frac{x_1 + x_2}{2x_2}\right) = w$$

$$T = \frac{2\pi}{w} = \frac{2\pi}{\cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)}$$

$\boxed{10}$ a)

$(D^2+1)y = 0 \qquad y = c_1 \sin x + c_2 \cos x$

at $x = 0$, $c_2 = 1$

at $x = \pi/2$, $c_1 = 1$

$y = \sin x + \cos x$ (unique solution)

c)

$$y = c_1 \sin x + c_2 \cos x$$

$\therefore 1 = c_2$ (at $x = 0$, $x = 2\pi$)

$\therefore$

c_1 is arbitrary,

$\therefore$ infinite solution.

b) $y = c_1 \sin x + c_2 \cos x$

at $x = 0$, $y = 1$ $\boxed{c_2 = 1}$

at $x = \pi$, $y = -1$

not possible as $y(\pi) = 1$.

$\therefore$ no solution

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