MA-102

ASSIGNMENT-3

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2K19/A14/35

$\boxed{1}$ Let $y(0)=a$, $y''(0)=b$, $y''(0)=0$

$y''' - (y + y'x) = 0$, for power of x,

$y''' - a = 0$, $y'''(0) = a$.

$y^{iv} - y' - (y''x + y') = 0$

$y^{v}.(0) = 0$

$$y(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + y'''(0)\frac{x^3}{3!}$$

$$\Rightarrow y(x) = a + bx + \frac{ax^3}{3!} + \frac{2bx^4}{4!} + \cdots$$

Let $y(1)=a$, $y'(1)=b$, $y''(1)=a$, for power of $(x-1)$.

$y''' - (y + y'x) = 0$

$y'''(1) - a - b = 0$ $\quad y'''(1) = a+b$

$y^{iv} - y' - (y''x + y') = 0$

$y^{iv}(1) = 2b + a$.

$y^{v} - 2y'' - (y'''x + y'') = 0$

$y^{v}(1) = 3a + a + b = 4a + b$

$$y(x) = y(1) + y'(1)\frac{(x-1)}{1!} + y''(1)\frac{(x-1)^2}{2!} + y'''(1)\frac{(x-1)^3}{3!} + \cdots$$

$$\Rightarrow y(x) = a + b(x-1) + \frac{a}{2!}(x-1)^2 + \frac{(a+b)(x-1)^3}{3!} + \frac{(a+2b)(x-1)^4}{4!} + \frac{(4a+b)(x-1)^5}{5!} + \cdots$$

$\boxed{2}$ a) $(1-x^2)y'' + 2xy' + n(n+1)y = 0$

singular point $\Rightarrow$ $x = \pm 1$.

$p(x) = \dfrac{2x}{1-x^2}$, $q(x) = \dfrac{n(n+1)}{1-x^2}$

for $x=1$,

$(x-1)\,p(x) = \dfrac{-2x}{1+x}$, $(x-1)^2 q(x) = \dfrac{-(x-1)n(n+1)}{1+x} = 0$.

$= -1$

analytic at $x=0$,

hence $x=1$ is regular singular point.

for $x=-1$,

$\lim\limits_{x\to -1} (x+1)\dfrac{2x}{(1-x^2)} = -1$ finite.

$\lim\limits_{x\to -1} (x+1)^2 \dfrac{n(n+1)}{(1-x^2)} = 0$ finite.

hence $x=-1$ is regular singular point.

b) $x^3(x-2)y'' + x^3 y' + 6y = 0$

$x = 0, 2$ are singular points.

$x=0$, $\lim\limits_{x\to 0} \dfrac{x\,\cancel{x^3}}{\cancel{x^3}(x-2)} = 0$ finite.

$\lim\limits_{x\to 0} \dfrac{\cancel{x^2}\,6}{x^{\cancel{3}}(x-2)} = \infty$

$\therefore$ $x=0$ is irregular point.

$\boxed{3}$

a) $y'' + (x-1)y' + y = 0$ about $x=2$

$$\rightarrow \quad \frac{-a_0 - a_1 - 3a_1 + 3a_2}{4 \times 6} = \frac{2a_0 - 4a_1}{4!}$$

$$y(x) = a_0 + a_1(x-2) + \frac{(a_1 - a_0)}{2!}(x-2)^2 + \frac{(a_0+a_1)}{3!}(x-2)^3 + \left(\frac{2a_0 - 4a_1}{4!}\right)(x-2)^4 .$$

$$y(x) = a_0\left(1 - \frac{(x-2)^2}{2!} - \frac{(x-2)^3}{3!} - \frac{2(x-2)^4}{4!} + \cdots\right)$$
$$+ a_1\left((x-2) + \frac{(x-2)^2}{2!} - \frac{(x-2)^3}{3!} - \frac{4(x-2)^4}{4!} + \cdots\right)$$

b) $(1-x^2)y'' + ay'x + y = 0$ about $x=0$

$$y(x) = \sum_{r=0}^{\infty} a_r x^r$$
$$y'(x) = \sum r a_r x^{r-1}, \quad y''(x) = \sum r(r-1) a_r x^{r-2}$$

$$\sum r(r-1) a_r x^{r-2} - \sum r(r-1) a_r x^r + 2\sum r a_r x^r + \sum a_r x^r = 0$$

$$2a_2 + 6a_3 x + 2a_1 x + a_0 + a_1 x + \sum (r+1)(r+2) a_{r+2}$$
$$+ a_r(2r + 1 - r(r-1))] = 0$$

$a_2 = -a_0/2$

$a_3 = -a_1/2$ $\qquad a_4 = \frac{-3}{12} a_2 = \frac{a_0}{8}$

$$a_{r+2} = \frac{r^2 - 3r - 1}{(r+1)(r+2)}.$$

$$a_5 = \frac{-1}{20} a_3 = \frac{a_1}{40},$$

$$y(x) = a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \cdots\right) + a_1\left(x - \frac{x^3}{2} + \frac{x^5}{40} + \cdots\right)$$

(c) $y'' + \cos x\, y = 0$ about $x=0$

$y(0) = a$ $\quad y'(0) = b$

$y''(0) = -a$ $\quad y'''(0) = -b$

$y'''(0) - \sin x\, y + y' \cos x = 0$

$y^{iv} - [\cos x\, y + y' \sin x] + y'' \cos x - \sin x\, y' = 0$

$y^{iv}(0) = 2a.$

$y^{v} - 2[y'' \sin x + \cos x\, y'] - [y \cos x - \sin x\, y] + y''' \cos x - \sin x\, y'' = 0.$

$$y(x) = y(0) + y'(0) + y''(0)\frac{x^2}{2!} + y'''(0)\frac{x^3}{3!} + \cdots$$

$$y(x) = a + bx - \frac{ax^2}{2!} - \frac{bx^3}{3!} + \frac{2ax^4}{4!} + \frac{4bx^5}{5!}$$

$\boxed{4}$ $\quad (x^2-1)\, y''(x) + 3x\, y'(x) + x y(x) = 0$

at $y(0) = 4$, $y'(0) = 6$.

→ $y(x) = \sum_{r=0}^{\infty} a_r x^r$

$a_{r+2} = \dfrac{(r^2+2r)a_r + a_{r-1}}{(r+1)(r+2)}$

$a_2 = 0$

$a_3 = \left(\dfrac{a_0 + 3a_1}{6}\right)$

$y(x) = \sum_{r=0}^{\infty} a_r (x-2)^r$

$y'(x) = \sum a_r r (x^{r-1}) = a_1 = 6.$

$y(2) = a_0 \Rightarrow 4.$

$a_3 = 11/3$, $\quad a_4 = 1/2$, $\quad a_5 = 11/4.$

$$y(x) = 4 + 6(x-2)^2 + \frac{11}{3}(x-2)^3 + \frac{1}{2}(x-2)^4 + {}^{11}/_{4}(x-2)^5 + - - - -$$

$\boxed{5}$ a) $2x^2y'' - xy' + (x-5)y = 0$

$$y(x) = \sum_{r=0}^{\infty} C_n x^{n+r} \quad , \quad C_0 \neq 0$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n x^{n+r-2}$$

for $r = 5/2$,
$$C_n = \frac{-C_{n-1}}{(n+5/2)(2n+2)-5}$$

$$\rightarrow C_1 = \frac{-C_0}{9} \quad , \quad C_2 = -\frac{C_1}{22} = \frac{C_0}{198}$$

$$C_3 = \frac{-C_2}{39} = -\frac{C_0}{7722}$$

$$y_2(x) = C_0 x^{5/2}\left(1 - x/9 + \frac{x^2}{198} - \frac{x^3}{7722} + \cdots\right)$$

$$y(x) = A x^{-1}\left(1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \cdots\right)$$

for $C_0 = 1$
$$+ B x^{5/2}\left(1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \cdots\right)$$

b) $2x^2y'' + xy' + (x^2-3)y = 0$

$$y(x) = \sum_{n=0}^{\infty} C_n x^{n+r}$$

$$y''(x) = \sum (n+r)(n+r-1) C_n x^{n+r-2}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1}$$

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \cdots$$

$$= \left(2r(r-1)+r-3\right)c_0 x^{r} + 2(r+1)r\,c_1 x^{r+1} + (r+1)c_1 x^{r+1}$$

$$-3c_1 x^{r+1} + \sum_{n=2}^{\infty} x^{n+r}\left[2(n+r)(n+r-1)c_n + (n+r)c_n + c_{n-2} - 3c_n\right] = 0$$

$$y_3(x) = c_0 x^{3/2}\left(1 - \frac{x^2}{18} + \frac{x^4}{936} - \cdots\right)$$

$$y(x) = Ax^{-1}\left(1 + \frac{x^2}{4} - \frac{x^4}{48} - \cdots\right) + Bx^{3/2}\left(1 - \frac{x^2}{18} + \frac{x^4}{936} + \cdots\right)$$

(c) $x^2y'' - xy' - (x^2 + 5/4)y = 0$

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad (c_0 \neq 0)$$

$$y'(x) = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} \quad \cdots$$

$$\left(r(r-1)c_0 - rc_0 - \frac{5}{4}c_0\right)x^{r} + \left((r+1)rc_1 - (r+1)c_1 - \frac{5}{4}c_1\right)x^{r+1}$$
$$+ \sum_{n=2}^{\infty} x^{n+r}\left((n+r)(n+r-1)c_n - (n+r)c_n - c_{n-2} - 5/4\, c_n\right) = 0$$

$$\left[r^2 - r - r - 5/4\right]c_0 = 0$$

$$4r^2 - 8r - 5 = 0$$

$$r = 5/2, \ -1/2$$

$$c_1\left[r^2 - 9/4\right] = 0$$

for $r = 5/4, -1/2$

$\Rightarrow c_1 = 0$

$$c_n = \frac{c_{n-2}}{(n+r)(n+r-2) - 5/4}$$

for $r = -1/2$

$$C_n = \frac{C_{n-2}}{(n-1/2)(n-5/2)-5/4}$$

$$C_2 = \frac{-C_0}{2}, \quad C_4 = \frac{-C_0}{8}, \quad C_6 = \frac{-C_0}{144}$$

$\rightarrow C_1 = C_3 = C_5 = C_7 = 0.$

$$y(x) = C_0 x^{-1/2}\left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{144} - \frac{x^8}{5760} + \cdots\right)$$

||ly $\cancel{y(x)} \cancel{=} \cancel{x}$ for $r = 5/2$.

$\boxed{6}$ $$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

$$(1-x^2)u_1 + 2nx u = 0$$

Differentiating $(n+1)$ times using Leibnitz rule,

$$^{n+1}C_0\, u_{n+2}(1-x^2) + {}^{n+1}C_1\, u_{n+1}(-2x) + {}^{n+1}C_2\, u_n(-2)$$
$$+ 2n\left[{}^{n+1}C_0\, u_{n+1}\, x + {}^{n+1}C_1\, u_n(1)\right] = 0$$

$\nearrow$ Legendre eqn with solutn $y(x) = C u_n$

$P_n(1) = 1$ $\quad P_n(x) = C\frac{d^n}{dx^n}(x^2-1)^n = C\frac{d^n}{dx^n}(1-x)^n(1+x)^n$

$$P_n(1) = C\left[{}^nC_0 (1+x)^n n!\right].$$

$$1 = C 2^n n!$$

$$C = \frac{1}{2^n n!}.$$ Hence Proved

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$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

$$y_1(x) = 1 - \alpha(\alpha+1)\frac{x^2}{2!} + \alpha(\alpha+1)(\alpha-2)(\alpha+3)\frac{x^4}{4!} - - -$$

$$y_2(x) = x - (\alpha-1)(\alpha+2)\frac{x^3}{3!} + (\alpha-1)(\alpha+2)(\alpha-3)(\alpha+4)\frac{x^5}{5!} - - -$$

If $\alpha = n$ (non-negative integer) when $n =$ even, then $y_1(x)$ is a polynomial in even powers of x.

$$\int_{-1}^{1} P_n(x)\, P_m(x) = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

$$(1-x^2)y''(x) - 2xy'(x) + \alpha(\alpha+1)y(x) = 0$$

for $y(x) = P_n(x)$

$$(1-x^2)P_n''(x) - 2x P_n'(x) + n(n+1)P_n(x) = 0 \quad -①$$

for $y(x) = P_m(x)$

$$(1-x^2)P_m''(x) - 2x P_m'(x) + m(m+1)P_m(x) = 0 \quad -②$$

Multiply ① & ②,

$$(1-x^2)\left(P_n''(x)P_m(x) - P_m''(x)P_n(x)\right) - 2x\left(P_n'(x)P_m(x) - P_m'(x)P_n(x)\right)$$
$$- P_m(x)P_n(x)\Big]\left[n(n+1) - m(m+1)\right] = 0$$

$$\underbrace{\int_{-1}^{1} \frac{d}{dx}(1-x^2)\left(P_n'(x)P_m(x) - P_m'(x)P_n(x)\right)}_{0} + \int_{-1}^{1} P_m(x)P_n(x)\,\left(n(n+1) - m(m+1)\right) = 0$$

$\int_{-1}^{1} P_m P_n \, dx \Rightarrow 0$ for $(n = m)$.

$$\left(1 - 2xt + t^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$$

$$\int_{-1}^{1} \frac{1}{1-2xt+t^2} = \int_{-1}^{1} \sum_{n=0}^{\infty} P_n^2(x) t^{2n}$$

$$\left[\frac{\ln |1-2xt+t^2|}{-2t}\right]_{-1}^{1} = \sum_{n=0}^{\infty} \int_{-1}^{1} P_n^2(x)\, dx \; t^{2n}$$

$$\frac{\ln|1+t| - \ln|1-t|}{t} = \sum_{n=0}^{\infty} \int_{-1}^{1} P_n^2(x)\, dx \; t^{2n}$$

$$\frac{\left(t - \frac{t^2}{2} + \cdots \frac{t^n}{n}\right) + \left(t + \frac{t^2}{2} + \cdots\right)}{t} = 2 + \frac{2t^2}{3} + \cdots \frac{2t^{2n}}{2n+1}.$$

coefficient of t^{2n} is $\frac{2}{2n+1}$.

$$\therefore \int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}.$$

$\boxed{9}$ at $x = 0$, $P_0(x) = 0$,

↳ Singular point. (regular).

$$\lim_{x \to 0} \frac{3x-1}{x(x-1)} x = 1$$

$$\lim_{x \to 0} \frac{x^2}{x(x-1)} = 0.$$

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$x^2 y'' - xy'' + 3xy' - y' + y = 0$$

↳ $\sum (n+r)(n+r-1) a_n x^{n+r} - \sum (n+r)(n+r-1) a_n x^{n+r-1}$

$+ 3 \sum (n+r) a_n x^{n+r} - \sum (n+r) a_n x^{n+r-1} + \sum a_n x^{n+r} = 0.$

Coefficient of x^{r-1},

$$[r(r-1) - r]\,a_0 = 0 \qquad \underline{r = 0, 2}.$$

$$r(r-2) = 0$$

coefficient of x^r,

$$r(r-1)\,a_0 - (r+1)r\,a_1 + 3r\,a_0 - a_1(r+1) + a_0 = 0$$

$$[r(r+1) + 3r + 1]\,a_0 = (r+1)^2 a_1 \qquad \underline{a_0 = a_1}.$$

coefficient x^{n+r}.

$$(n+r)(n+r-1)\,a_n - (n+r+1)(n+r)\,a_{n+1} + 3(n+r)\,a_n$$
$$- (n+r+1)\,a_{n+1} + a_n = 0$$

$$\underline{a_n = a_{n+1}}.$$

for $r=2$, $\quad y_1 = x^2[a_0 + a_0 x + \cdots a_0 x^n]$

for $r=0$, $\quad y_2 = \left(\frac{\partial y}{\partial r}\right)_{r=0} = \frac{\partial}{\partial r}\left[x^r(a_0 + a_0 x \cdots a_0 x^n)\right]$

$$= x^r \log x\, a_0 [1 + x + \cdots x^n].$$

$$y = c_1 y_1 + c_2 y_2$$
$$= (c_1 x^2 + c_2 \log x)[1 + x + \cdots x^n].$$

$\boxed{10}$ $\quad x=0$, ordinary point.

Let $y = \sum_{n=0}^{\infty} a_n x^n$.

$$\Rightarrow \sum_{n=0}^{\infty} n(n-1)\,a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)\,a_n x^n - \sum n\,a_n x^n$$
$$+ a^2 \sum a_n x^n = 0.$$

$\underline{x^0}$ $\quad 2a_2 + a^2 a_0 = 0 \qquad a_2 = \dfrac{-a^2 a_0}{2}.$

$\underline{x^n}$ $\quad (n+2)(n+1)\, a_{n+2} - n(n-1)\, a_n - n a_n + a^2 a_n = 0$

$$a_{n+2} = \frac{(n^2 - a^2)\, a_n}{(n+2)(n+1)}$$

$$a_2 = \frac{-a^2 a_0}{2}, \qquad a_4 = \frac{(2^2 - a^2)\, a_2}{4 \cdot 3}$$

$$a_{2n} = \frac{((2n-2)^2 - a^2) \cdots\cdots -a^2}{(2n)!}$$

|ly $\qquad a_3 = \dfrac{(1-a^2)\, a_1}{3 \cdot 2}, \qquad a_5 = \dfrac{(3-a^2)\, a_2}{5 \cdot 4}$

$$a_{2n+1} = \frac{(2n-1)^2 - a^2\,[(n-3)^2 - a^2 \cdots\cdots (1-a^2)\, a_1}{(2n+1)! \times \cdots}$$

$$y = c_1 y_1 + c_2 y_2.$$

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{2n!} \left[\prod_{k=0}^{n-1} (4k^2 - a^2) \right] x^{2n}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} \left[\prod_{k=0}^{n-1} (4k^2 + 4k + 1 - a^2) \right] x^{2n+1}.$$

END

x x x