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Assignment 3

① $y'' - xy = 0 \qquad -\infty < x < \infty$

series solution in power of x.

let $y(0) = a$ & $y'(0) = b$

$\Rightarrow$ $y''(0) = 0$

$\Rightarrow$ $y''' - [y + y'x] = 0$

$y'''(0) - a = 0 \quad \Rightarrow \quad y'''(0) = a$

$\Rightarrow$ $y^{IV} - y' - (y''x + y') = 0$

$y^{IV}(0) - 2b = 0 \quad \Rightarrow \quad y^{IV}(0) = 2b$

$\Rightarrow$ $y^{V} - 2y'' - (y'''x + y'') = 0$

$y^{V}(0) = 0$

$$y(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{y'''(0)x^3}{3!} + \cdots$$

$$y(x) = a + bx + \frac{1}{2}(0)x^2 + \frac{ax^3}{3!} + \frac{2bx^4}{4!} \cdots$$

$$y(x) = a + bx + \frac{ax^3}{3!} + \frac{2bx^4}{4!} + \cdots$$

series solution in power of $x-1$

let $y(1) = a$ and $y'(1) = b$

$y''(1) = a$

$\Rightarrow$ $y''' - [y + y'x] = 0$

$y'''(1) - a - b = 0 \quad \Rightarrow \quad y'''(1) = a+b$

$\Rightarrow$ $y^{IV} - y' - [y''x + y'] = 0$

$y^{IV}(1) = 2b + a$

$\Rightarrow$ $y^{V} - 2y'' - [y'''x + y''] = 0$

$y^{V}(1) = 3a + a + b \quad \Rightarrow \quad y^{V}(1) = 4a + b$

$\Rightarrow \quad y(x) = y(1) + y'(1)\dfrac{(x-1)}{1!} + y''(1)\dfrac{(x-1)^2}{2!} + y'''(1)\dfrac{(x-1)^3}{3!}$

$$y(x) = a + b(x-1) + \frac{a}{2!}(x-1)^2 + (a+b)\frac{(x-1)^3}{3!}$$

$$+ (a+2b)\frac{(x-1)^4}{4!} + (4a+b)\frac{(x-1)^5}{5!} \ldots$$

(2) classify the singular point

a) $(1-x^2)y'' + 2xy' + n(n+1)y = 0$

Ans has singular point at $x = \pm 1$

$p(x) = \dfrac{2x}{1-x^2}$; $q(x) = \dfrac{n(n+1)}{1-x^2}$

For $x = 1$

$(x-1)p(x) = \dfrac{-2x}{1+x}$; $(x-1)^2 q(x) = -\dfrac{(x-1)n(n+1)}{1+x}$

$= -1 \qquad\qquad = 0$

it is analytic at $x=1$. Hence $x=1$ is a regular singular point.

For $x = -1$

$(x+1)p(x) = \dfrac{2x}{1-x}$; $(x+1)^2 q(x) = \dfrac{(x+1)n(n+1)}{1-x}$

$= -1 \qquad\qquad 0$

it is also analytic at $x=-1$. Hence $x=-1$ is also a regular singular point.

b) $x^3(x-2)y'' + x^3y' + 6y = 0$

Ans has singular point at $x = 0, 2$

$p(x) = \dfrac{1}{x-2} \qquad q(x) = \dfrac{6}{x^3(x-2)}$

At $x=0$

$x\,p(x) = \frac{x}{x-2}$ $\qquad x^2 q(x) = \frac{6}{x(x-2)}$

$= 0$

$\Rightarrow x^2 q(x)$ is not analytic at $x=0$. hence it is irregular singular point.

At $x=2$

$(x-2)\,p(x) = 1$ $\qquad (x-2)^2 q(x) = \frac{6(x-2)}{x^3}$

$= 0$

$\Rightarrow$ it is analytic at $x=2$. Hence $x=2$ is regular singular point.

c) $\left(x-\frac{\pi}{2}\right)^2 y'' + \cos x\, y' + \sin x\, y = 0$

Ans has singular point at $x=\pi/2$.

$p(x) = \frac{\cos x}{(x-\pi/2)^2}$ $\qquad q(x) = \frac{\sin x}{(x-\pi/2)^2}$

At $x=\pi/2$

$(x-\pi/2)\,p(x) = \frac{\cos x}{x-\pi/2}$ $\qquad (x-\pi/2)^2 q(x) = \sin x$

$= -1$ $\qquad = 1$

it is analytic at $x=\pi/2$. Hence $x=\pi/2$ is a regular singular point.

(3) Find power series

a) $y'' + (x-1)y' + y = 0$ about $x=2$.

$$= \frac{\frac{-a_0 - a_1 - 3a_1 + 3a_0}{6}}{4} = \frac{2a_0 - 4a_1}{4!}$$

so ;

$$\Rightarrow y(x) = a_0 + a_1(x-2) + \left(\frac{a_1 - a_0}{2!}\right)(x-2)^2 - \frac{1}{3!}(a_0 + a_1)(x-2)^3$$

$$+ \left(\frac{2a_0 - 4a_1}{4!}\right)(x-2)^4 \ldots$$

$$\Rightarrow y(x) = a_0\left(1 - \frac{(x-2)^2}{2!} - \frac{(x-2)^3}{3!} + \frac{2(x-2)^4}{4!} \ldots\right)$$

$$+ a_1\left((x-2) + \frac{(x-2)^2}{2!} - \frac{(x-2)^3}{3!} - \frac{4(x-2)^4}{4!} \ldots\right)$$

(b) $(1-x^2)y'' + 2y'x + y = 0$ about $(x=0)$

Ans

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \; ; \; y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n + 2\sum_{n=1}^{\infty} n a_n x^n$$

$$+ \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n + 2\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow 2a_2 + 6a_3 x + 2a_1 x + a_0 + a_1 x + \sum_{n=2}^{\infty} x^n \left[(n+1)(n+2)a_{n+2} + a_n(2n+1 - n(n-1))\right] = 0$$

$$a_2 = -a_0/2$$

$$a_3 = -a_1/2$$

$$a_{n+2} = \frac{n^2 - 3n - 1}{(n+1)(n+2)} a_n \qquad n \geq 2$$

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$$a_4 = \frac{-3}{12} a_2 = \frac{a_0}{8} \qquad (n=2)$$

For $n=3$; $a_5 = \frac{-1}{20} a_3 \Rightarrow a_5 = \frac{a_1}{40}$

$$\Rightarrow y(x) = a_0 + a_1 x + \frac{-a_0}{2} x^2 - \frac{a_1}{2} x^3 + \frac{a_0}{8} x^4 + \frac{a_1}{40} x^5 + \cdots$$

$$\Rightarrow y(x) = a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \cdots\right) + a_1\left(x - \frac{x^3}{2} + \frac{x^5}{40} \cdots\right)$$

(c) $y'' + \cos x\, y = 0$ about $x=0$

Ans Let $y(0) = a$ $\quad y'(0) = b$

$\Rightarrow y''(0) = -a$

$\Rightarrow y''' - \sin x\, y + y' \cos x = 0$

$y'''(0) = -b$

$\Rightarrow y^{iv} - [\cos x\, y + y' \sin x] + y'' \cos x - \sin x\, y' = 0$

$y^{iv} - 2y' \sin x - y \cos x + y'' \cos x = 0$

$y^{iv}(0) = a + a = 2a$

$\Rightarrow y^{v} - 2[y'' \sin x + \cos x\, y'] - [y' \cos x - \sin x\, y] + y''' \cos x$
$- \sin x\, y'' = 0$

$\Rightarrow y^{v} - 2b - b - b = 0 \Rightarrow y^{v} = 4b$

We know that

$$y(x) = y(0) + y'(0)x + \frac{y''(0)x^2}{2!} + \frac{y'''(0)x^3}{3!} + \cdots$$

$$y(x) = a + bx - \frac{ax^2}{2!} - \frac{bx^3}{3!} + \frac{2ax^4}{4!} + \frac{4bx^5}{6!} \cdots$$

(4) Find power series solution

a) $(x^2-1)y''(x) + 3xy'(x) + xy(x) = 0$ at $y(0)=4$ $y'(0)=6$

Ans $\quad y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y(x) = 4 + 6x + \frac{11}{3}x^3 + \frac{1}{2}x^4 + \frac{11}{4}x^5 + \cdots$$

b) $(x^2-1)y''(x) + 3xy'(x) + xy(x) = 0$; $y(2)=4$ $y'(2)=6$

a. Using recurrence relation.

$a_2 = 0$

$a_3 = \frac{(a_0 + 3a_1)}{6} \Rightarrow a_{n+2} = \frac{(n^2+2n)a_n + a_{n-1}}{(n+1)(n+2)}$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-2)^n$$

$y(2) = a_0 \Rightarrow a_0 = 4$

$y'(x) = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1} \Rightarrow a_1 = 6$

$a_3 = \frac{4+18}{6} \Rightarrow \frac{11}{3} \Rightarrow a_4 = 1/2$

$a_5 = 11/4$

$$y(x) = 4 + 6(x-2) + \frac{11}{3}(x-2)^3 + \frac{1}{2}(x-2)^4 + \frac{11}{4}(x-2)^5 + \cdots$$

5) Find solution of differential equation by Frobenius method.

a) $2x^2y'' - xy' + (x-5)y = 0$

$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ $\quad c_0 \neq 0$

$y'(x) = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$; $y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$

For $r = 5/2$

$$c_n = \frac{-c_{n-1}}{(n+5/2)(2n+2)-5}$$

$\Rightarrow c_1 = \frac{-c_0}{(7/2)(4)-5} \Rightarrow c_1 = \frac{-c_0}{9}$ $(n=1)$

$\Rightarrow c_2 = \frac{-c_1}{(9/2)(6)-5} \Rightarrow c_2 = \frac{-c_1}{22} \Rightarrow c_2 = \frac{c_0}{198}$ $(n=2)$

$\Rightarrow c_3 = \frac{-c_2}{(11/2)(8)-5} \Rightarrow c_3 = \frac{-c_2}{39} \Rightarrow c_3 = \frac{-c_0}{7722}$ $(n=3)$

$$y_2(x) = c_0 x^{5/2}\left(1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \cdots\right)$$

$$y(x) = A x^{-1}\left(1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} \cdots\right) + B x^{5/2}\left(1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} \cdots\right)$$

[For $c_0 = 1$]

(b) $2x^2 y'' + xy' + (x^2-3)y = 0$

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad [c_0 \neq 0]$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \;;\; y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$$\Rightarrow 2\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - 3\sum_{n=0} c_n x^{n+r} = 0$$

$$\Rightarrow [2r(r-1) + r - 3] c_0 x^r + 2\sum_{n=1}^{\infty} (n+r)(n+r-1) c_n x^{n+r} + \sum_{n=1}^{\infty} (n+r) c_n x^{n+r}$$

$$+ \sum_{n=2}^{\infty} c_{n-2} x^{n+r} - 3\sum_{n=1}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow [2r(r-1) + r - 3] c_0 x^r + 2(r+1) r c_1 x^{r+1} + (r+1) c_1 x^{r+1} - 3c_1 x^{r+1}$$

$$+ \sum_{n=2}^{\infty} x^{n+r}\left[2(n+r)(n+r-1) c_n + (n+r) c_n + c_{n-2} - 3c_n\right] = 0$$

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$$y_2(x) = c_0 x^{3/2}\left(1 - \frac{x^2}{18} + \frac{x^4}{936} \cdots\cdots\right)$$

$$\Rightarrow y(x) = A x^{-1}\left(1 + \frac{x^2}{4} - \frac{x^4}{48} \cdots\right) + B x^{3/2}\left(1 - \frac{x^2}{18} + \frac{x^4}{936} \cdots\right)$$

(c) $x^2 y'' - x y' - \left(x^2 + \frac{5}{4}\right) y = 0$

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad [c_0 \neq 0]$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \; ; \quad y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+2} - \frac{5}{4}\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \left[r(r-1)c_0 - r c_0 - \frac{5}{4}c_0\right] x^r + \left[(r+1) r c_1 - (r+1) c_1 - \frac{5}{4} c_1\right] x^{r+1}$$
$$+ \sum_{n=2}^{\infty} x^{n+r}\left[(n+r)(n+r-1)c_n - (n+r)c_n - c_{n-2} - 5/4\, c_n\right] = 0$$

equating to zero various coefficients

$\left[r^2 - r - r - \frac{5}{4}\right] c_0 = 0$

$4r^2 - 8r - 5 = 0$

$r = 5/2, -1/2$

$\left[r^2 - \frac{9}{4}\right] c_1 = 0$

For $r = 5/2, -1/2$

$\Rightarrow c_1 = 0$

$$c_n = \frac{c_{n-2}}{(n+r)(n+r-2) - 5/4}$$

For $r = -1/2$

$$c_n = \frac{c_{n-2}}{(n-1/2)(n-5/2) - 5/4}$$

$c_2 = \frac{-c_0}{2} \quad (n=2)$ | $c_4 = \frac{-c_0}{8} \quad (n=4)$ | $c_6 = \frac{-c_0}{144} \quad (n=6)$

$c_1 = c_3 = c_5 = c_7 = 0$

$$y(x) = c_0 x^{-1/2}\left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{144} - \frac{x^8}{5760} \cdots\right)$$

The solution with $r = 5/2$ is also contained in solution for $r = -1/2$.

Hence the general solution is given by

$$y(x) = x^{-1/2} - \frac{x^{3/2}}{2} - \frac{x^{7/2}}{8} - \frac{x^{11/2}}{144} \cdots \quad [\text{For } c_0 = 1]$$

) $x^2 y'' + (x^2 - 3x)y' + y = 0$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad [c_0 \neq 0]$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \; ; \; y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r+1} - 3\sum_{n=0}^{\infty} c_n (n+r) x^{n+r}$$

$$+ \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r) c_n x^{n+r+1} = \sum_{n=1}^{\infty} (n+r-1) c_{n-1} x^{n+r}$$

$$\Rightarrow [r(r-1) - 3r + 1] c_0 x^r + \sum_{n=1}^{\infty} x^{n+r} [(n+r)(n+r-1) c_n + (n+r-1) c_{n-1}$$

$$- 3 c_n (n+r) + c_n] = 0$$

$\Rightarrow$ equating to zero various coefficients

$$r^2 - 4r + 1 = 0$$

$$r = \frac{4 \pm \sqrt{16-4}}{2} \qquad r_1 - r_2 \text{ is not an integer}$$

$$r = 2 \pm \sqrt{3} \qquad \text{hence } y(x) = A y_1(x) + B y_2(x)$$

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$$y_2(x) = c_0 x^{2-\sqrt{3}}\left(1 + \frac{(2-\sqrt{3})x}{6+2\sqrt{3}} + \frac{9-5\sqrt{3}}{16\sqrt{3}}x^2 + \cdots\right)$$

General solution :-

$$y(x) = A y_1(x) + B y_2(x)$$

$$= A x^{2+\sqrt{3}}\left(1 - \frac{(2+\sqrt{3})}{2\sqrt{3}+1}x + \frac{(9+5\sqrt{3})}{28+12\sqrt{3}}x^2 \cdots\right) + B x^{2-\sqrt{3}}\left(1 + \frac{(2-\sqrt{3})}{6+2\sqrt{3}}x + \frac{(9-5\sqrt{3})}{16\sqrt{3}}x^2 \cdots\right)$$

⑥ Prove $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

Let $u = (x^2-1)^n$

$u_1 = 2xn(x^2-1)^{n-1}$

$\Rightarrow \quad u_1 = \frac{2nxu}{(x^2-1)}$

$\Rightarrow \quad (1-x^2)u_1 + 2nxu = 0$

Differentiating $(n+1)$ times using Leibnitz rule

$\Rightarrow \; {}^{n+1}C_0 u_{n+2}(1-x^2) + {}^{n+1}C_1 u_{n+1}(-2x) + {}^{n+1}C_2 u_n(-2)$
$+ 2n[{}^{n+1}C_0 u_{n+1} x + {}^{n+1}C_1 u_n (1)] = 0$

$\Rightarrow \; u_{n+2}(1-x^2) - 2(n+1)x u_{n+1} - (n+1)n u_n + 2nx u_{n+1}$
$+ 2n(n+1)u_n = 0$

$\Rightarrow \; (1-x^2)u_n'' - 2x u_{n+1} + n(n+1)u_n = 0$

This is a Legendre equation whose solution $y(x) = c\, u_n$

We know that $P_n(1) = 1$

$P_n(x) = c\frac{d^n}{dx^n}(x^2-1)^n = c\frac{d^n (1-x)^n(1+x)^n}{dx^n}$

$P_n(1) = c\left[{}^nC_0 (1+x)^n n!\right]$

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$$1 = c\, 2^n n!$$

$$\frac{1}{2^n n!} = c$$

Hence, $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$ for $n = 0, 1, 2$

(8) $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

soln of Legendre equation:

$$y_1(x) = 1 - \alpha(\alpha+1)\frac{x^2}{2!} + \alpha(\alpha+1)(\alpha-2)(\alpha+3)\frac{x^4}{4!} \cdots$$

$$y_2(x) = x - (\alpha-1)(\alpha+2)\frac{x^3}{3!} + (\alpha-1)(\alpha+2)(\alpha-3)(\alpha+4)\frac{x^5}{5!} \cdots$$

$\Rightarrow$ if $\alpha = n$ (non negative integer). when n = even then $y_1(x)$ is a polynomial in even powers of x and if n = odd then $y_2(x)$ is a polynomial in odd power of x. These polynomials when multiplied by some suitable constant are called Legendre Polynomial.

$$\int_{-1}^{1} P_n(x) P_m(x) = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}$$

We know that

$$(1-x^2)y''(x) - 2xy'(x) + \alpha(\alpha+1)y(x) = 0$$

For $y(x) = P_n(x)$

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \quad \text{--- (I)}$$

For $y(x) = P_m(x)$

$$(1-x^2)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0 \quad \text{--- (II)}$$

Multiplying (I) with $P_m(x)$ and (II) with $P_n(x)$ and subtracting

$\Rightarrow$ $(1-x^2)\left[P_n''(x)P_m(x) - P_m''(x)P_n(x)\right] - 2x\left[P_n'(x)P_m(x) - P_m'(x)P_n(x)\right]$
$+ P_m(x)P_n(x)\left[n(n+1) - m(m+1)\right] = 0$

$\Rightarrow$ $\dfrac{d}{dx}\left[(1-x^2)\left[P_n'(x)P_m(x) - P_m'(x)P_n(x)\right]\right] + P_m(x)P_n(x)\left[n(n+1)-m(m+1)\right] = 0$

Integrating b/s

$\Rightarrow$ $\left[(1-x^2)P_n'(x)P_m(x) - P_m'(x)P_n(x)\right]_{-1}^{1} + \left[n(n+1)-m(m+1)\right]\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$

$\Rightarrow$ $\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0 \qquad [m \neq n]$

$\Rightarrow$ For $m = n$

$\alpha(x,t) = \dfrac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\,t^n$

Squaring b/s

$$\frac{1}{1-2xt+t^2} = \left[\sum_{n=0}^{\infty} P_n(x)\,t^n\right]^2$$

integrating b/s.

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \sum_{n=0}^{\infty}\left[\int_{-1}^{1} P_n^2(x)\,dx\right] t^{2n}$$

$$\left.\frac{\ln(1-2xt+t^2)}{-2t}\right|_{-1}^{1} = \sum_{n=0}^{\infty}\left[\int_{-1}^{1} P_n^2(x)\,dx\right] t^{2n}$$

$$\frac{-1}{2t}\left[\ln(1-t)^2 - \ln(1+t)^2\right] = \sum_{n=0}^{\infty}\left[\int_{-1}^{1} P_n^2(x)\,dx\right] t^{2n}$$

$$2\left[1 + \frac{t^2}{3} + \frac{t^4}{5} + \frac{t^6}{7} \cdots\right] = \sum_{n=0}^{\infty}\left[\int_{-1}^{1} P_n^2(x)\,dx\right] t^{2n}$$

equating coefficient of t^{2n}

$$\frac{2}{2n+1} = \sum_{n=0}^{\infty} \int_{-1}^{1} P_n^2 dx$$

$$\frac{2}{2n+1} = \int_{-1}^{1} P_n(x) P_m(x) dx \qquad [m=n]$$

(9) $x(x-1)y'' + (3x-1)y' + y = 0$
has singular point at $x=0$ and $x=1$

$p(x) = \dfrac{3x-1}{x(x-1)}$ $\qquad q(x) = \dfrac{1}{x(x-1)}$

$x\,p(x) = \dfrac{(3x-1)}{x-1}$ $\qquad x^2 q(x) = \dfrac{x}{x-1}$

Clearly $x p(x)$ and $x^2 q(x)$ are analytic at $x=0$. Hence $x=0$ is regular singular point.

Let $\quad y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \begin{bmatrix} y(0) = a \\ y'(0) = b \end{bmatrix}$

$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$+ \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$
$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$\Rightarrow -2a_2 - a_1 + a_0 + (-6a_3 - 2a_2 + a_1 + 3a_1)x$$
$$+ \sum_{n=2}^{\infty} x^n \left[n(n-1)a_n - (n+2)(n+1)a_{n+2} + 3n a_n - (n+1)a_{n+1} + a_n \right] = 0$$

$$a_2 = -\frac{\{a_1 + a_0\}}{2}$$

$$a_3 = \frac{-\{a_2 + 2a_1\}}{3}$$

$$\Rightarrow \quad a_n[n(n-1) + 3n + 1] - (n+1)a_{n+1} = (n+2)(n+1)a_{n+2}$$

$$\Rightarrow \quad a_n[(n+1)^2] - (n+1)a_{n+1} = (n+2)(n+1)a_{n+2}$$

$$\Rightarrow \quad a_{n+2} = \frac{(n+1)a_n - a_{n+1}}{n+2}$$

$$a_3 = \frac{-\left(\frac{a_0 - a_1}{2}\right) + 2a_1}{3} = \frac{5a_1 - a_0}{6}$$

$$\Rightarrow \quad a_4 = \frac{3a_2 - a_3}{4} = \frac{3\left(\frac{a_0 - a_1}{2}\right) - \left(\frac{5a_1 - a_0}{6}\right)}{4}$$

$$= \frac{9a_0 - 9a_1 - 5a_1 + a_0}{24} = \frac{10a_0 - 14a_1}{a_0}$$

$$y(x) = a_0 + a_1 x + \left(\frac{a_0 - a_1}{2}\right)x^2 + \left(\frac{5a_1 - a_0}{6}\right)x^3 + \left(\frac{10a_0 - 14a_1}{4!}\right)x^4 + \cdots$$

$$y(x) = a_0\left(1 + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{10x^4}{4!} \cdots\right) + a_1\left(x - \frac{x^2}{2!} + \frac{5x^3}{3!} - \frac{14x^4}{4!} \cdots\right)$$

(10) $$(1-x^2)y''(x) - xy'(x) + a^2 y(x) = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad ; \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + a^2 \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

(11) f) $\frac{d}{dx}(x^v J_v) = x^v J_{v-1}(x)$

$\Rightarrow$ $$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n+v}}{2^{2n+v} \; n! \; \Gamma(n+v+1)}$$

$\Rightarrow$ $$x^v J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n+2v}}{2^{2n+v} \; n! \; \Gamma(n+v+1)}$$ [Multiply x^v b/s]

Differentiating w.r.t x

$\Rightarrow$ $$\frac{d}{dx}(x^v J_v(x)) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2v) \; x^{2n+2v-1}}{2^{2n+v} \; n! \; \Gamma(n+v+1)}$$

$\Rightarrow$ $$= \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n+2v-1} \; \cancel{(n+v)}}{2^{2n+v-1} \; n! \; \cancel{(n+v)} \; \Gamma(n+v)}$$

$\Rightarrow$ $$= x^v \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n+v-1}}{2^{2n+v-1} \; n! \; \Gamma(n+v)}$$

$$= x^v \; J_{v-1}(x)$$

Hence proved.

e) $\frac{d}{dx}(x^{-v} J_v) = -x^{-v} J_{v+1}(x)$

$\Rightarrow$ $$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n+v}}{2^{2n+v} \; n! \; \Gamma(n+v+1)}$$

$\Rightarrow$ $$x^{-v} J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n}}{2^{2n+v} \; n! \; \Gamma(n+v+1)}$$

Differentiating b/s w.r.t x

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$$\frac{d}{dx}(x^{-v} J_v) = \sum_{n=1}^{\infty} \frac{(-1)^n \; 2n \; x^{2n-1}}{2^{2n+v} \; n! \; \Gamma(n+v+1)} \; \frac{x^v}{x^v}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)}{x^v} \frac{(-1)^n \; 2(n+1) \; x^{2n+1+v}}{2^{2n+v+2} (n+1)! \; \Gamma(n+v+2)}$$

$$= -x^{-v} \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n+v+1}}{2^{2n+v+1} \; n! \; \Gamma(n+v+2)}$$

$$= -x^{-v} \; J_{v+1}(x)$$

d) $2v J_v(x) = x\,[J_{v-1}(x) + J_{v+1}(x)]$

we know that $\frac{d}{dx}[x^v J_v(x)] = x^v J_{v-1}(x)$

$\Rightarrow v x^{v-1} J_v(x) + J_v'(x) x^v = x^v J_{v-1}(x)$ —— ①

Illy $\frac{d}{dx}[x^{-v} J_v(x)] = -x^{-v} J_{v+1}(x)$

$\Rightarrow -v x^{-v-1} J_v(x) + J_v'(x) x^{-v} = -x^{-v} J_{v+1}(x)$

Multiply b/s by x^{2v}

$\Rightarrow -v x^{v-1} J_v(x) + x^v J_v'(x) = -x^v J_{v+1}(x)$ —— (II)

Subtracting ① and (II)

$\Rightarrow 2v x^{v-1} J_v(x) = x^v J_{v-1}(x) + x^v J_{v+1}(x)$

$\Rightarrow 2v J_v(x) = x\,[J_{v-1}(x) + J_{v+1}(x)]$

Hence proved.

c) $2J_v'(x) = J_{v-1}(x) - J_{v+1}(x)$

$\frac{d}{dx}[x^v J_v(x)] = x^v J_{v-1}(x)$

$\rightarrow v x^{v-1} J_v(x) + J_v'(x) x^v = x^v J_{v-1}(x)$ —— ①

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$\Rightarrow \frac{d}{dx}[x^{-v} J_v(x)] = -x^{-v} J_{v+1}(x)$

$\Rightarrow -v x^{-v-1} J_v(x) + J_v'(x) x^{-v} = -x^{-v} J_{v+1}(x)$

Multiply b/s by x^{2v}

$\Rightarrow -v x^{v-1} J_v(x) + x^v J_v'(x) = -x^v J_{v+1}(x)$ — (ii)

Adding (i) and (ii)

$\Rightarrow 2x^v J_v'(x) = x^v (J_{v-1}(x) - J_{v+1}(x))$

$\Rightarrow 2 J_v'(x) = J_{v-1}(x) - J_{v+1}(x)$ [Hence proved]

(b) $x J_v'(x) = -v J_v(x) + x J_{v-1}(x)$

We know that $\frac{d}{dx}[x^v J_v(x)] = x^v J_{v-1}(x)$

$\Rightarrow v x^{v-1} J_v(x) + J_v'(x) x^v = x^v J_{v-1}(x)$

$\Rightarrow \frac{v}{x} J_v(x) + J_v'(x) = J_{v-1}(x)$

Multiply x b/s

$\Rightarrow v J_v(x) + x J_v'(x) = x J_{v-1}(x)$

Hence proved

(a) $x J_v'(x) = v J_v(x) - x J_{v+1}(x)$

we know that $\frac{d}{dx}[x^{-v} J_v(x)] = -x^{-v} J_{v+1}(x)$

$\Rightarrow -v x^{-v-1} J_v(x) + J_v'(x) x^{-v} = -x^{-v} J_{v+1}(x)$

$\Rightarrow -v x^{-1} J_v(x) + J_v'(x) = -J_{v+1}(x)$

Multiply x b/s

$\Rightarrow -v J_v(x) + x J_v'(x) = -x J_{v+1}(x)$

$\Rightarrow x J_v'(x) = v J_v(x) - x J_{v+1}(x)$.

Hence proved