Int. J. Appl. Math. Comput. Sci., 2020, Vol. 30, No. 1, 83–98
DOI: 10.34768/amcs-2020-0007

ANTI–PERIODIC SOLUTIONS FOR CLIFFORD–VALUED HIGH–ORDER HOPFIELD NEURAL NETWORKS WITH STATE–DEPENDENT AND LEAKAGE DELAYS

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A class of Clifford-valued high-order Hopfield neural networks (HHNNs) with state-dependent and leakage delays is considered. First, by using a continuation theorem of coincidence degree theory and the Wirtinger inequality, we obtain the existence of anti-periodic solutions of the networks considered. Then, by using the proof by contradiction, we obtain the global exponential stability of the anti-periodic solutions. Finally, two numerical examples are given to illustrate the feasibility of our results.

Keywords: Clifford-valued high-order Hopfield neural network, anti-periodic solution, coincidence degree, time-varying delay.

1. Introduction

It is well known that, in the design, implementation and application of neural networks, the dynamics of neural networks are a primary problem that must be considered. In the past decades, the dynamics of various neural networks have been extensively studied (Sakthivel *et al.*, 2013; Selvaraj *et al.*, 2018; He *et al.*, 2007; Li and Wang, 2013). In recent years, high-order Hopfield neural networks (HHNNs) have become the topic of in-depth analysis (Li and Yang, 2014; Li *et al.*, 2017; 2019c; Aouiti *et al.*, 2017; Aouiti, 2018; Zhao *et al.*, 2018; Xu and Li, 2017; Alimi *et al.*, 2018; Xu *et al.*, 2006; 2003; Ou, 2008; Lou and Cui, 2007; Xiang *et al.*, 2006). This is due to the fact that high-order neural networks have a stronger approximation property, a faster convergence rate, a greater storage capacity, and higher fault tolerance than lower-order neural networks. The applicability of HHNNs lies in their dynamical properties to a great extent. Therefore, many researchers have investigated their dynamical behaviour. For example, Lou and Cui (2007) studied the problem of global stability of HHNNs by using the Lyapunov method, linear matrix inequalities and analytic techniques; Xiang *et al.* (2006), by using coincidence degree theory as well as *a priori* estimates and a Lyapunov functional, derived some sufficient conditions for the existence and global exponential stability of periodic solutions for delayed HHNNs.

On the one hand, it is known that, as a generalization of real-valued neural networks, the research of complex-valued and quaternion-valued neural networks has attracted more and more attention because they have more advantages than real-valued neural networks in many aspects (Hu and Wang, 2012; Kan *et al.*, 2019; Wang *et al.*, 2019; Li and Qin, 2018; Li *et al.*, 2019a; 2019b; Liu *et al.*, 2020; 2018).

On the other hand, Clifford algebra, which is a unital associative algebra, was invented by William K. Clifford. It has been applied to different areas such as neural computing (Kuroe, 2011; Corrochano *et al.*, 1996), computer and robot vision (Hitzer *et al.*, 2013), image and signal processing (Rivera-Rovelo and

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Bayro-Corrochano, 2006), and others due to its practical and powerful framework for the representation and solution of geometrical problems; see the works of Hitzer *et al*. (2013), Rivera-Rovelo and Bayro-Corrochano (2006), Dorst *et al*. (2007), Bayro-Corrochano and Scheuermann (2010), Liu *et al*. (2016), Pearson and Bisset (1992) and the references therein. Recently, as the extension of real-valued neural networks, Clifford-valued neural networks, which include complex-valued and quaternion-valued neural networks as their special cases, have been an active research field (Liu *et al.*, 2016; Pearson and Bisset, 1992; 2007; Buchholz, 2005; Brackx *et al.*, 1982). For example, in the work of Buchholz *et al*. (2007), neural computation in the Clifford-valued domain was studied, while Kuroe (2011) proposed several models of fully connected Clifford-valued recurrent neural networks. In general, the dynamical properties of Clifford-valued neural networks are more complicated than those of real-valued and complex-valued ones. Up to now, very few dynamical properties of Clifford-valued neural networks have been explored.

In addition, anti-periodic oscillations are a very important dynamic aspect of neural networks because the signal transmission process of neural networks can often be described as an anti-periodic one. Moreover, anti-periodic functions are periodic ones because T-anti-periodic functions are $2T$-periodic functions, but not vice versa. Therefore, in recent years, many authors have studied the problem of anti-periodic oscillation for various neural networks (Ke and Miao, 2017; Şaylı and Yılmaz, 2017; Shi and Dong, 2010). For example, Ke and Miao (2017), by using the Lyapunov method, studied the existence and exponential stability of anti-periodic solutions of inertial neural networks with time delays; Şaylı and Yılmaz (2017) investigated the existence and exponential stability of anti-periodic solutions for state-dependent impulsive recurrent neural networks by employing the method of coincide degree theory and constructing an appropriate Lyapunov function. However, anti-periodic solutions for Clifford-valued neural networks with state-dependent delays have not been reported yet.

Moreover, as we all know, time delay is inevitable, and its existence may change the dynamical behavior of the system. Therefore, the delayed neural network system can better reflect the real signal transmission and information processing in the network. In particular, time delay in the leakage term can greatly change the dynamical performance of the system to make it complex or poor. At the same time, a change in time delay may depend not only on the running time of the system, but also on its immediate state. Therefore, it is of practical significance to consider neural network systems with state-dependent delays and leakage delays.

Motivated by the considerations mentioned above, in the present work, we study the existence and global exponential stability of anti-periodic solutions for Clifford-valued HHNNs with state-dependent and leakage delays.

We organize the paper as follows. In Section 2, we introduce the concept of Clifford algebra and model formulation. In Section 3, we obtain sufficient conditions for the existence of anti-periodic solutions of Clifford-valued HHNNs with state-dependent and leakage delays. In Section 4, the global exponential stability of anti-periodic solutions of Clifford-valued HHNNs with state-dependent and leakage delays is studied. In Section 5, we give two numerical examples to illustrate the feasibility of the obtained results. We draw a brief conclusion in Section 6.

Remark 1. It has been proved that Clifford-valued neural network models can use multi-state activation functions to process multi-level information, and require fewer network connection weight parameters (Kuroe, 2011; Buchholz and Sommer, 2008). Therefore, the study of Clifford-valued neural networks is of great theoretical significance and practical value.

2. Preliminaries and the model description

The real Clifford algebra over $\mathbb{R}^m$ is defined as

$$\mathcal{A} = \Big\{ \sum_{A\subseteq\{1,2,\dots,m\}} a^A e_A, a^A \in \mathbb{R} \Big\},$$

where

$$e_A = e_{h_1} e_{h_2} \cdots e_{h_\nu}$$

with

$$A = \{h_1, h_2, \dots, h_\nu\},$$
$$1 \le h_1 < h_2 < \dots < h_\nu \le m.$$

Moreover, $e_\emptyset = e_0 = 1$ and $e_{\{h\}} = e_h, h = 1, 2, \dots, m$ are called Clifford generators which satisfy the relations

$$e_i^2 = -1, \quad i = 1, 2, \dots, m,$$

and

$$e_i e_j + e_j e_i = 0, \quad i \ne j, \quad i, j = 1, 2, \dots, m.$$

For simplicity, when one element is the product of multiple Clifford generators, we will write its subscripts together. For example, $e_1 e_2 = e_{12}$ and $e_3 e_7 e_4 e_5 = e_{3745}$. We define $\Delta = \{\emptyset, 1, 2, \dots, A, \dots, 1 \cdot 2 \cdots m\}$; then it is easy to see that

$$\mathcal{A} = \Big\{ \sum_A a^A e_A, a^A \in \mathbb{R} \Big\},$$

where $\sum_A$ is a brief form of $\sum_{A\in\Delta}$ and $\mathcal{A}$ is isomorphic to $\mathbb{R}^{2m}$.

For any $x = \sum_A x^A e_A \in \mathcal{A}$, the involution of x is defined as

$$\overline{x} = \sum_A x^A \overline{e}_A,$$

where $\overline{e}_A = (-1)^{\frac{\chi[A](\chi[A]+1)}{2}} e_A$; if $A = \emptyset$, then $\chi[A] = 0$ and if $A = h_1 h_2 \dots h_\nu \in \Delta$, then $\chi[A] = \nu$. From the definition, it is directly deduced that $e_A \overline{e}_A = \overline{e}_A e_A = 1$. Moreover, for any Clifford number $x = \sum_A x^A e_A$, its involution can be denoted by $\overline{x} = \sum_A x^A \overline{e}_A$. In addition to this, the involution also satisfies $\overline{xy} = \overline{y}\,\overline{x}, \forall x, y \in \mathcal{A}$.

For a Clifford-valued function $z = \sum_A z^A e_A : \mathbb{R} \to \mathcal{A}$, where $z^A : \mathbb{R} \to \mathbb{R}, A \in \Delta$, its derivative is given by

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \sum_A \frac{\mathrm{d}z^A(t)}{\mathrm{d}t} e_A.$$

Since $e_B \overline{e}_A = (-1)^{\frac{\chi[A](\chi[A]+1)}{2}} e_B e_A$, we can simplify and express $e_B \overline{e}_A = e_C$ or $e_B \overline{e}_A = -e_C$, with e_C being some basis of the Clifford algebra $\mathcal{A}$. For example, $e_{12}\overline{e}_{27} = -e_{12}e_{17} = -e_1 e_2 e_2 e_7 = e_1 e_7 = e_{17}$. Hence it is possible to find a unique corresponding basis e_C for the given $e_B \overline{e}_A$. Define

$$\chi[B \cdot \overline{A}] = \begin{cases} 0 & \text{if } e_B \overline{e}_A = e_C, \\ 1 & \text{if } e_B \overline{e}_A = -e_C. \end{cases}$$

Then $e_B \overline{e}_A = (-1)^{\chi[B \cdot \overline{A}]} e_C$. In addition, for any $E \in \mathcal{A}$, we can find a unique E^C satisfying $E^{B \cdot \overline{A}} = (-1)^{\chi[B \cdot \overline{A}]} E^C$ for $e_B \overline{e}_A = (-1)^{\chi[B \cdot \overline{A}]} e_C$. Therefore,

$$\begin{aligned} E^{B \cdot \overline{A}} e_B \overline{e}_A &= E^{B \cdot \overline{A}} (-1)^{\chi[B \cdot \overline{A}]} e_C \\ &= (-1)^{\chi[B \cdot \overline{A}]} E^C (-1)^{\chi[B \cdot \overline{A}]} e_C \\ &= E^C e_C \end{aligned}$$

and

$$E = \sum_C E^C e_C \in \mathcal{A}.$$

In this paper, we consider the following Clifford-valued high-order Hopfield neural network with state-dependent and leakage delays:

$$\begin{aligned} \dot{X}_i(t) = {} & -c_i(t) X_i(t - \delta_i(t)) \\ & + \sum_{j=1}^{n} a_{ij}(t) f_j(X_j(\breve{\tau}_{ij}(t, X_j(t)))) \\ & + \sum_{j=1}^{n} \sum_{l=1}^{n} b_{ijl}(t) g_j(X_j(\breve{\sigma}_{ijl}(t, X_j(t)))) \\ & \times g_l(X_l(\breve{\nu}_{ijl}(t, X_l(t)))) + I_i(t), \end{aligned} \tag{1}$$

where $i = 1, 2, \dots, n$ and n is the number of units in the neural network, $X_i(t) \in \mathcal{A}$ corresponds to the state vector of the i-th unit at time t, $c_i(t) > 0$ represents the rate with which the i-th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $a_{ij}(t) \in \mathcal{A}$ and $b_{ijl}(t) \in \mathbb{R}$ are the first-order and second-order connection weights of the neural network, $\delta_i(t) \geq 0$, $\breve{\tau}_{ij}(t, X_j(t))) \geq 0$, $\breve{\sigma}_{ijl}(t, X_j(t))) \geq 0$ and $\breve{\nu}_{ijl}(t, X_l(t))) \geq 0$ correspond to the leakage and transmission delays, respectively, $I_i : \mathbb{R} \to \mathcal{A}$ denotes the external inputs and $f_j, g_j : \mathcal{A} \to \mathcal{A}$ are the activation functions of signal transmission.

Throughout this paper, we make the following assumptions:

(A_1) Let $X_i = \sum_A x_i^A e^A$, where $x_i^A : \mathbb{R} \to \mathbb{R}, i = 1, 2, \dots, n, A \in \Delta$. We assume that $a_{ij}(t), f_j(X_j), g_j(X_j)$ can be expressed similarly as

$$a_{ij}(t) = \sum_A a_{ij}^A(t) e^A,$$

$$f_j(X_j) = \sum_A f_j^A(x_j^0, x_j^1, \dots, x_j^{12\cdots m}) e^A,$$

$$g_j(X_j) = \sum_A g_j^A(x_j^0, x_j^1, \dots, x_j^{12\cdots m}) e^A,$$

where $a_{ij}^A : \mathbb{R} \to \mathbb{R}$, $f_j^A, g_j^A : \mathbb{R}^{2^m} \to \mathbb{R}$, $i, j = 1, 2, \dots, n$, $A \in \Delta$.

(A_2) For $i, j, l = 1, 2, \dots, n$, $b_{ijl} \in C(\mathbb{R}, \mathbb{R}), c_i \in C(\mathbb{R}, \mathbb{R}^+), \delta_i \in C(\mathbb{R}, \mathbb{R}^+) \cap C^1(\mathbb{R}, \mathbb{R}), \breve{\tau}_{ij}, \breve{\sigma}_{ijl}, \breve{\nu}_{ijl} \in BC(\mathbb{R} \times \mathcal{A}, \mathbb{R}^+)$ satisfying $\breve{\tau}_{ij}(t, \cdot) \leq t, \breve{\sigma}_{ijl}(t, \cdot) \leq t, \breve{\nu}_{ijl}(t, \cdot) \leq t$ for $t \in \mathbb{R}$, $f_j, g_j \in C(\mathcal{A}, \mathcal{A})$, $a_{ij}, I_i \in C(\mathbb{R}, \mathcal{A})$, and there exists $\omega > 0$ such that

$$a_{ij}\Big(t + \frac{\omega}{2}\Big) f_j(u) = -a_{ij}(t) f_j(-u),$$

$$b_{ijl}\Big(t + \frac{\omega}{2}\Big) g_j(u) g_l(v) = -b_{ijl}(t) g_j(-u) g_l(-v),$$

$$\delta_i\Big(t + \frac{\omega}{2}\Big) = \delta_i(t), \quad \breve{\tau}_{ij}\Big(t + \frac{\omega}{2}, \cdot\Big) = \breve{\tau}_{ij}(t, \cdot),$$

$$c_i\Big(t + \frac{\omega}{2}\Big) = c_i(t), \quad \breve{\sigma}_{ijl}\Big(t + \frac{\omega}{2}, \cdot\Big) = \breve{\sigma}_{ijl}(t, \cdot),$$

$$\breve{\nu}_{ijl}\Big(t + \frac{\omega}{2}, \cdot\Big) = \breve{\nu}_{ijl}(t, \cdot), \quad I_i\Big(t + \frac{\omega}{2}\Big) = -I_i(t),$$

for $t \in \mathbb{R}, u, v \in \mathcal{A}$.

(A_3) There exist positive constants M_f and M_g such that

$$|f_j^A(u)| \leq M_f, \quad |g_j^A(u)| \leq M_g, \quad \forall u \in \mathbb{R}^{2^m},$$

where $j = 1, 2, \dots, n$, $A \in \Delta$.

(A_4) There exist constants $L_f > 0$ and $L_g > 0$ such that, $\forall x = (x_1, x_2, \dots, x_{2^m})^T, y = (y_1, y_2, \dots, y_{2^m})^T \in \mathbb{R}^{2^m}$, $j = 1, 2, \dots, n$,

$$|f_j^A(x) - f_j^A(y)| \leq \sum_{\nu=1}^{2^m} L_f |x_\nu - y_\nu|,$$

$$|g_j^A(x) - g_j^A(y)| \leq \sum_{\nu=1}^{2^m} L_g |x_\nu - y_\nu|, \quad A \in \Delta.$$

For convenience, we introduce the following notation:

$$c_i^+ = \sup_{t\in[0,\omega]} c_i(t), \quad c_i^- = \inf_{t\in[0,\omega]} c_i(t),$$

$$a_{ij}^+ = \max_{A\in\Delta}\Big\{\sup_{t\in[0,\omega]} |a_{ij}^A(t)|\Big\}, \quad b_{ijl}^+ = \sup_{t\in[0,\omega]} |b_{ijl}(t)|,$$

$$\dot{\delta}_i^+ = \sup_{t\in[0,\omega]} \dot{\delta}_i(t), \quad \delta_i^+ = \sup_{t\in[0,\omega]} \delta_i(t),$$

$$\delta^+ = \max_{1\le i\le n}\Big\{\sup_{t\in[0,\omega]} \delta_i(t)\Big\},$$

$$\tau^+ = \max_{1\le i,j\le n}\Big\{\sup_{t\in[0,\omega],x\in\mathcal{A}} \breve{\tau}_{ij}(t,x)\Big\},$$

$$\sigma^+ = \max_{1\le i,j,l\le n}\Big\{\sup_{t\in[0,\omega],x\in\mathcal{A}} \breve{\sigma}_{ijl}(t,x)\Big\},$$

$$\nu^+ = \max_{1\le i,j,l\le n}\Big\{\sup_{t\in[0,\omega],x\in\mathcal{A}} \breve{\nu}_{ijl}(t,x)\Big\},$$

$$\rho = \max\{\delta^+,\tau^+,\sigma^+,\nu^+\},$$

$$\bar{I}_i^A = \sup_{t\in[0,\omega]} |I_i^A(t)|, \quad I_i^+ = \max_{A\in\Delta} \bar{I}_i^A,$$

where $i,j,l = 1,2,\dots,n$ and $A\in\Delta$.

The initial value of the system (1) is given by

$$X_i(s) = \phi_i(s), \quad \dot{X}_i(s) = \dot{\phi}_i(s), \quad s\in[-\rho,0],$$

where $i = 1,2,\dots,n$, $\phi_i \in C^1([-\rho,0],\mathcal{A})$, $A\in\Delta$.

According to the previous discussion, $e_A\bar{e}_A = \bar{e}_A e_A = 1$ and $e_B\bar{e}_A e_A = e_B$, and for any $E\in\mathcal{A}$ one can find a unique E^C satisfying the following property:

$$E^C e_C f^A e_A = (-1)^{\chi[B\cdot\bar{A}]} E^C f^A e_B = E^{B\cdot\bar{A}} f^A e_B.$$

Thus, we can decompose the system (1) into the following real-valued one:

$$\begin{aligned}\dot{x}_i^A(t) &= -c_i(t)x_i^A(t-\delta_i(t)) \\ &\quad + \sum_B\sum_{j=1}^n a_{ij}^{A\cdot\bar{B}}(t) f_j^B(x_j(\tilde{\tau}_{ij}(t,x_j(t)))) \\ &\quad + \sum_B\sum_{j=1}^n\sum_{l=1}^n b_{ijl}(t) g_j^{A\cdot\bar{B}}(x_j(\tilde{\sigma}_{ijl}(t,x_j(t)))) \\ &\quad \times g_l^B(x_l(\tilde{\nu}_{ijl}(t,x_l(t)))) + I_i^A(t) \\ &\triangleq F_i^A(t,x(t)),\end{aligned} \tag{2}$$

where $i = 1,2,\dots,n$, $A\in\Delta$,

$$\sum_A x_i^A(t)e_A = X_i(t),$$

$$a_{ij}(t) = \sum_C a_{ij}^C(t)e_C, \quad b_{ijl}(t) = \sum_C b_{ijl}^C(t)e_C,$$

$$f_j(X_j(\breve{\tau}_{ij}(t,X_j(t)))) = \sum_C f_j^C(x_j(\tilde{\tau}_{ij}(t,x_j(t))))e_C,$$

$$g_j(X_j(\breve{\sigma}_{ijl}(t,X_j(t)))) = \sum_C g_j^C(x_j(\tilde{\sigma}_{ijl}(t,x_j(t))))e_C,$$

$$\tilde{\tau}_{ij}, \tilde{\sigma}_{ijl} : \mathbb{R}\times\mathbb{R}^{2^m} \to \mathbb{R}^+,$$

for $e_A\bar{e}_B = (-1)^{\chi[A\cdot\bar{B}]}e_C$, $a_{ij}^{A\cdot\bar{B}}(t) \triangleq (-1)^{\chi[A\cdot\bar{B}]}a_{ij}^C(t)$,

$$\begin{aligned}&g_j^{A\cdot\bar{B}}(x_j(\tilde{\sigma}_{ijl}(t,x_j(t)))) \\ &\quad \triangleq (-1)^{\chi[A\cdot\bar{B}]} g_j^C(x_j(\tilde{\sigma}_{ijl}(t,x_j(t)))), \\ &f_j^B(x_j(\tilde{\tau}_{ij}(t,x_j(t)))) \\ &\quad \triangleq f_j^B(x_j^0(\tilde{\tau}_{ij}(t,x_j(t))), x_j^1(\tilde{\tau}_{ij}(t,x_j(t))),\dots, \\ &\qquad x_j^{12\cdots m}(\tilde{\tau}_{ij}(t,x_j(t)))), \\ &g_j^{A\cdot\bar{B}}(x_j(\tilde{\sigma}_{ijl}(t,x_j(t)))) \\ &\quad \triangleq g_j^{A\cdot\bar{B}}(x_j^0(\tilde{\sigma}_{ijl}(t,x_j(t))), x_j^1(\tilde{\sigma}_{ijl}(t,x_j(t))),\dots, \\ &\qquad x_j^{12\cdots m}(\tilde{\sigma}_{ijl}(t,x_j(t)))), \\ &g_l^B(x_l(\tilde{\nu}_{ijl}(t,x_l(t)))) \\ &\quad \triangleq g_l^B(x_l^0(\tilde{\nu}_{ijl}(t,x_l(t))), x_l^1(\tilde{\nu}_{ijl}(t,x_l(t))),\dots, \\ &\qquad x_l^{12\cdots m}(\tilde{\nu}_{ijl}(t,x_l(t)))),\end{aligned}$$

$i,j,l = 1,2,\dots,n$, $A,B,C\in\Delta$.

The initial value of the system (2) is given by

$$x_i^A(s) = \phi_i^A(s), \quad \dot{x}_i^A(s) = \dot{\phi}_i^A(s), \quad s\in[-\rho,0].$$

Remark 2. If $x = (x_1^0, x_1^1,\dots, x_1^{1\cdot2\cdots m}, x_2^0, x_2^1,\dots, x_2^{1\cdot2\cdot m},\dots, x_n^0, x_n^1,\dots, x_n^{1\cdot2\cdot m})^T$ is a solution to the system (2), then $X = (X_1,\dots,X_n)^T$ must be a solution to the system (1), where $X_i = \sum_A x_i^A e_A$, $i = 1,2,\dots,n$ and $A\in\Delta$. Thus, the problem of finding an $(\omega/2)$-anti-periodic solution for the system (1) reduces to finding one for the system (2). For considering the stability of solution of the system (1), we just need to consider the stability of solutions of the system (2).

Remark 3. The Clifford-valued system (1) includes real-valued ($m = 0$), complex-valued ($m = 1$) and quaternion-valued ($m = 2$) systems as its special cases.

Definition 1. Let $x = (x_1^0, x_1^1,\dots, x_1^{1\cdot2\cdots m}, x_2^0, x_2^1,\dots, x_2^{1\cdot2\cdots m},\dots, x_n^0, x_n^1, \dots, x_n^{1\cdot2\cdots m})^T$ be an anti-periodic solution of the system (2) with the initial value

$$\begin{aligned}\varphi = (&\varphi^0, \varphi^1,\dots,\varphi^{1\cdot2\cdots m}, \varphi_2^0, \varphi_2^1,\dots,\varphi_2^{1\cdot2\cdots m}, \\ &\dots,\varphi_n^0,\varphi_n^1,\dots,\varphi_n^{1\cdot2\cdots m})^T,\end{aligned}$$

and let

$$\begin{aligned}y = (&y_1^0, y_1^1,\dots, y_1^{1\cdot2\cdots m}, y_2^0, y_2^1,\dots, y_2^{1\cdot2\cdots m}, \\ &\dots, y_n^0, y_n^1,\dots, y_n^{1\cdot2\cdots m})^T\end{aligned}$$

be an arbitrary solution of the system (2) with the initial value

$$\psi = (\psi^0, \dots, \psi^{1\cdot 2\cdots m}, \psi_2^0, \psi_2^1, \dots, \psi_2^{1\cdot 2\cdots m}, \dots, \psi_n^0, \psi_n^1, \dots, \psi_n^{1\cdot 2\cdots m})^T,$$

where $\varphi, \psi \in C^1([-\rho, 0], \mathbb{R}^{2^m n})$. If there exist constants $\lambda > 0$ and $M > 0$ such that

$$\|x(t) - y(t)\| \le M\|\varphi - \psi\|_1 e^{-\lambda t}, \quad \forall t > 0,$$

then the anti-periodic solution of the system (2) is said to be globally exponentially stable, where

$$\|x(t) - y(t)\| = \max\Big\{ \max_{1\le i\le n} \max_{A\in\Delta} |x_i^A(t) - y_i^A(t)|, \max_{1\le i\le n} \max_{A\in\Delta} |\dot{x}_i^A(t) - \dot{y}_i^A(t)| \Big\},$$

$$\|\varphi - \psi\|_1 = \max\Big\{ \max_{1\le i\le n} \max_{A\in\Delta} \sup_{s\in[-\rho,0]} |\varphi_i^A(s) - \psi_i^A(s)|, \max_{1\le i\le n} \max_{A\in\Delta} \sup_{s\in[-\rho,0]} |\dot{\varphi}_i^A(s) - \dot{\psi}_i^A(s)| \Big\}.$$

Lemma 1. (Amster, 2013) (Wirtinger inequality) *If u is a C^1 function such that $u(0) = u(T)$, then*

$$\|u - \bar{u}\|_{L_2} \le \frac{T}{2\pi}\|u'\|_{L_2},$$

where

$$\|u'\|_{L_2} := \Big(\int_0^T |u(t)|^2 \,\mathrm{d}t \Big)^{\frac{1}{2}}$$

and

$$\bar{u} = \frac{1}{T}\int_0^T u(t)\,\mathrm{d}t.$$

Lemma 2. (Amster, 2013) *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces, and let $L : \operatorname{Dom} L \subset \mathbb{X} \to \mathbb{Y}$ be linear and $N : \mathbb{X} \to \mathbb{Y}$ continuous. Assume that L is one-to-one and $H := L^{-1}N$ is compact. Furthermore, assume there exists a bounded and open subset $\Omega \subset \mathbb{X}$ with $0 \in \Omega$ such that the equation $Lx = \lambda Nx$ has no solutions in $\partial\Omega \cap \operatorname{Dom} L$ for any $\lambda \in (0, 1)$. Then the problem $Lx = Nx$ has at least one solution in $\overline{\Omega}$.*

3. Existence of anti-periodic solutions

In this section, based on a new continuation theorem of coincidence degree theory, we shall study the existence of anti-periodic solutions of the system (1).

Let

$$\mathbb{X} = \Big\{ x : x = (x_1^0, x_1^1, \dots, x_1^{1\cdot 2\cdots m}, x_2^0, \dots, x_2^{1\cdot 2\cdots m}, \dots, x_n^0, \dots, x_n^{1\cdot 2\cdots m})^T \in C(\mathbb{R}, \mathbb{R}^{2^m n}), x\Big(t + \frac{\omega}{2}\Big) = -x(t),\ \forall t \in \mathbb{R} \Big\}.$$

Then $\mathbb{X}$ is a Banach space with the norm

$$\|x\|_{\mathbb{X}} = \max_{A\in\Delta} \max_{1\le i\le n} |x_i^A|_0,$$

where

$$|x_i^A|_0 = \sup_{t\in[0,\omega]} |x_i^A(t)|, \quad i = 1, 2, \dots, n, \quad A \in \Delta.$$

Define a linear operator $L : \operatorname{Dom}(L) \subset \mathbb{X} \to \mathbb{X}$ by

$$Lx = \dot{x},$$

where $\operatorname{Dom} L = \{x : x \in \mathbb{X}, \dot{x} \in \mathbb{X}\}$, and a continuous operator $N : \mathbb{X} \to \mathbb{X}$ by

$$Nx = ((Nx)_1^0, (Nx)_1^1, \dots, (Nx)_1^{1\cdot 2\cdots m}, (Nx)_2^0, \dots, (Nx)_2^{1\cdot 2\cdots m}, \dots, (Nx)_n^0, \dots, (Nx)_n^{1\cdot 2\cdots m})^T,$$

where

$$(Nx)_i^A(t) = F_i^A(t, x(t)), \quad i = 1, 2, \dots, n, \quad A \in \Delta.$$

It is easy to see that $\ker L = \{0\}$ and $\operatorname{Im} L = \mathbb{X}$, so L is reversible. Let $H := L^{-1}N$. By applying the Arzela–Ascoli theorem, we know that H is compact.

Theorem 1. *Let (A_1)–(A_3) hold, and assume that*

(A_5) $1 - \dot{\delta}_i^+ > 0$ *and*

$$\frac{1}{\omega} - \frac{c_i^+}{2\pi(1 - \dot{\delta}_i^+)} > 0, \quad i = 1, 2, \dots, n.$$

Then the system (1) *has at least one* $(\omega/2)$*-anti-periodic solution.*

Proof. Let $x \in \operatorname{Dom} L \subset \mathbb{X}$ be an arbitrary solution of $Lx = \lambda Nx$ for a certain $\lambda \in (0, 1)$; then we have

$$\dot{x}_i^A(t) = \lambda F_i^A(t, x(t)), \quad i = 1, 2, \dots, n, \quad A \in \Delta. \tag{3}$$

Multiplying both the sides of (3) by $\dot{x}_i^A(t)$ and then integrating the result over the interval $[0,\omega]$, we obtain

$$\begin{aligned}
&\int_0^\omega (\dot{x}_i^A(t))^2\,\mathrm{d}t\\
&\le \int_0^\omega \Big| -c_i(t)x_i^A(t-\delta_i(t))\dot{x}_i^A(t)\\
&\quad+\sum_B\sum_{j=1}^n a_{ij}^{A\cdot\bar{B}}(t)f_j^B(x_j(\tilde{\tau}_{ij}(t,x_j(t))))\dot{x}_i^A(t)\\
&\quad+\sum_B\sum_{j=1}^n\sum_{l=1}^n b_{ijl}(t)g_j^{A\cdot\bar{B}}(x_j(\tilde{\sigma}_{ijl}(t,x_j(t))))\\
&\quad\times g_l^B(x_l(\tilde{\nu}_{ijl}(t,x_l(t))))\dot{x}_i^A(t)+I_i^A(t)\dot{x}_i^A(t)\Big|\,\mathrm{d}t\\
&\le c_i^+\int_0^\omega |x_i^A(t-\delta_i(t))\dot{x}_i^A(t)|\,\mathrm{d}t+\Big(2^m\sum_{j=1}^n a_{ij}^+M_f\\
&\quad+2^m\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+{M_g}^2+\bar{I}_i^A\Big)\int_0^\omega |\dot{x}_i^A(t)|\,\mathrm{d}t\\
&\le c_i^+\Big(\int_0^\omega |x_i^A(t-\delta_i(t))|^2\,\mathrm{d}t\Big)^{\frac12}\Big(\int_0^\omega |\dot{x}_i^A(t)|^2\,\mathrm{d}t\Big)^{\frac12}\\
&\quad+\sqrt{\omega}\Big(2^m\sum_{j=1}^n a_{ij}^+M_f\\
&\quad+2^m\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+{M_g}^2+\bar{I}_i^A\Big)\Big(\int_0^\omega |\dot{x}_i^A(t)|^2\,\mathrm{d}t\Big)^{\frac12},
\end{aligned}$$

that is, for $i=1,2,\dots,n$, $A\in\Delta$,

$$\begin{aligned}
&\Big(\int_0^\omega |\dot{x}_i^A(t)|^2\mathrm{d}t\Big)^{\frac12}\\
&\le c_i^+\Big(\int_0^\omega |x_i^A(t-\delta_i(t))|^2\,\mathrm{d}t\Big)^{\frac12}+\sqrt{\omega}\Big(2^m\sum_{j=1}^n a_{ij}^+M_f\\
&\quad+2^m\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+{M_g}^2+\bar{I}_i^A\Big)\\
&\le \frac{c_i^+}{1-\dot{\delta}_i^+}\Big(\int_0^\omega |x_i^A(t)|^2\,\mathrm{d}t\Big)^{\frac12}+\sqrt{\omega}\Big(2^m\sum_{j=1}^n a_{ij}^+M_f\\
&\quad+2^m\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+{M_g}^2+\bar{I}_i^A\Big). \qquad (4)
\end{aligned}$$

Since $x\in X$ is $(\omega/2)$-anti-periodic and $x\in C^1$, by Lemma 1 we have

$$\Big(\int_0^\omega |x_i^A(t)|^2\,\mathrm{d}t\Big)^{\frac12}\le\frac{\omega}{2\pi}\Big(\int_0^\omega |\dot{x}_i^A(t)|^2\,\mathrm{d}t\Big)^{\frac12}, \qquad (5)$$

where $i=1,2,\dots,n$ and $A\in\Delta$. From (4) and (5), for $i=1,2,\dots,n$ and $A\in\Delta$, we can get

$$\begin{aligned}
&\Big(\int_0^\omega |\dot{x}_i^A(t)|^2\,\mathrm{d}t\Big)^{\frac12}\\
&\le \frac{c_i^+\omega}{2\pi(1-\dot{\delta}_i^+)}\Big(\int_0^\omega |\dot{x}_i^A(t)|^2\mathrm{d}t\Big)^{\frac12}\\
&\quad+2^m\sqrt{\omega}\Big(\sum_{j=1}^n a_{ij}^+M_f+\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+{M_g}^2+\bar{I}_i^A\Big),
\end{aligned}$$

that is,

$$\Big(\int_0^\omega |\dot{x}_i^A(t)|^2\,\mathrm{d}t\Big)^{\frac12}\le\frac{2^m\sqrt{\omega}\Big(\sum_{j=1}^n a_{ij}^+M_f+\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+{M_g}^2+\bar{I}_i^A\Big)}{1-\frac{c_i^+\omega}{2\pi(1-\dot{\delta}_i^+)}}. \qquad (6)$$

Again, since $x\in X$ is $(\omega/2)$-anti-periodic, there exist $\xi_i^A\in[0,\omega]$ such that

$$x_i^A(\xi_i^A)=0,\quad i=1,2,\dots,n,\quad A\in\Delta. \qquad (7)$$

By (7), for $i=1,2,\dots,n$, $\quad A\in\Delta$, we have

$$\begin{aligned}
|x_i^A|_0 &\le |x_i^A(\xi_i^A)|+\int_0^\omega |\dot{x}_i^A(t)|\,\mathrm{d}t\\
&\le\sqrt{\omega}\Big(\int_0^\omega |\dot{x}_i^A(t)|^2\,\mathrm{d}t\Big)^{\frac12}. \qquad (8)
\end{aligned}$$

Then, from (6) and (8) it follows that

$$\begin{aligned}
&\|x\|_{\mathbb{X}}\\
&\le \max_{1\le i\le n}\left\{\frac{2^m\Big(\sum_{j=1}^n a_{ij}^+M_f+\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+{M_g}^2\Big)+I_i^+}{\frac{1}{\omega}-\frac{c_i^+}{2\pi(1-\dot{\delta}_i^+)}}\right\}\\
&\triangleq M.
\end{aligned}$$

Take $\Omega=\{x\in\mathbb{X}:\|x\|_{\mathbb{X}}<M+1\}$; then it is easy to see that all of the requirements of Lemma 2 are fulfilled. Thus, by Lemma 2, we have that $Lx=Nx$ has at least one $(\omega/2)$-anti-periodic solution in $\mathbb{X}$, that is, (2) has at least one $(\omega/2)$-anti-periodic solution. In view of Remark 2, (1) has at least one $(\omega/2)$-anti-periodic solution. The proof is complete. ■

Remark 4. From the conditions of Theorem 1, one can see that Theorem 1 is still valid for judging the existence of anti-periodic solutions of the Clifford-valued system (1), although it is proved by decomposing the system (1) into the real-valued systems (2). That is to say, under the assumption (A_1), as long as the system (1) is anti-periodic and the activation functions and state-dependent delays are bounded, the system (1) has anti-periodic solutions.

4. Global exponential stability of the anti-periodic solution

In this section, we write

$$\|x(t)\| = \max_{1\le i\le n}\max_{A\in\Delta}\{|x_i^A(t)|, |\dot{x}_i^A(t)|\},$$

for $x \in C^1(\mathbb{R}, \mathbb{R}^{2^m n})$ and

$$\|\varphi\|_1 = \max_{1\le i\le n}\max_{A\in\Delta}\left\{\sup_{s\in[-\rho,0]}|\varphi_i^A(s)|, \sup_{s\in[-\rho,0]}|\dot{\varphi}_i^A(s)|\right\}$$

for $\varphi \in C([-\rho,0], \mathbb{R}^{2^m n})$.

Theorem 2. *Let (A_1)–(A_4) hold. Furthermore, assume that*

(A_7) *For $i, j, l = 1, 2, \ldots, n$,*

$$\breve{\tau}_{ij}(t,\cdot) \equiv t-\tau_{ij}(t), \breve{\sigma}_{ijl}(t,\cdot) \equiv t-\sigma_{ijl}(t),$$

$$\breve{\nu}_{ijl}(t,\cdot) \equiv t-\nu_{ijl}(t).$$

(A_8) $\max_{1\le i\le n}\left\{\frac{\varpi_i}{c_i^-}, (1+\frac{c_i^+}{c_i^-})\varpi_i\right\} < 1$, *where*

$$\varpi_i = c_i^+\delta_i^+ + 2^{2m}\sum_{j=1}^n a_{ij}^+ L_f + 2^{2m+1}\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+ M_g L_g.$$

Then the system (1) *has a unique $(\omega/2)$-anti-periodic solution that is globally exponentially stable.*

Proof. By Theorem 1, we know that system (2) has an $\omega/2$-periodic solution x. Consider it with the initial value φ. Let y be an arbitrary solution of the system (2) with the initial value ψ. Write $u = y - x$; by (2) we have

$$\begin{aligned}\dot{u}_i^A(t) &= -c_i(t)u_i^A(t-\delta_i(t)) \\ &\quad + \sum_B\sum_{j=1}^n a_{ij}^{A\cdot\bar{B}}(t)\big(f_j^B(y_j(t-\tau_{ij}(t))) \\ &\quad - f_j^B(x_j(t-\tau_{ij}(t)))\big) \\ &\quad + \sum_B\sum_{j=1}^n\sum_{l=1}^n b_{ijl}(t)\big(g_j^{A\cdot\bar{B}}(y_j(t-\sigma_{ijl}(t))) \\ &\quad \times g_l^B(y_l(t-\nu_{ijl}(t))) - g_j^{A\cdot\bar{B}}(x_j(t-\sigma_{ijl}(t))) \\ &\quad \times g_l^B(x_l(t-\nu_{ijl}(t)))\big)\end{aligned} \tag{9}$$

and the initial value of (9) is

$$u_i^A(s) = \psi_i^A(s) - \varphi_i^A(s), \quad s\in[-\rho,0], \tag{10}$$

where $i = 1, 2, \ldots, n$ and $A \in \Delta$.

Rewrite (9) as

$$\begin{aligned}&\dot{u}_i^A(t) \quad (11)\\ &= -c_i(t)u_i^A(t) + c_i(t)\int_{t-\delta_i(t)}^t \dot{u}_i^A(s)\,\mathrm{d}s \\ &\quad + \sum_B\sum_{j=1}^n a_{ij}^{A\cdot\bar{B}}(t)\big(f_j^B(y_j(t-\tau_{ij}(t))) \\ &\quad - f_j^B(x_j(t-\tau_{ij}(t)))\big) + \sum_B\sum_{j=1}^n\sum_{l=1}^n b_{ijl}(t) \\ &\quad \times\big(g_j^{A\cdot\bar{B}}(y_j(t-\sigma_{ijl}(t)))g_l^B(y_l(t-\nu_{ijl}(t))) \\ &\quad - g_j^{A\cdot\bar{B}}(x_j(t-\sigma_{ijl}(t)))g_l^B(x_l(t-\nu_{ijl}(t)))\big), \quad (12)\end{aligned}$$

where $i = 1, 2, \ldots, n$ and $A \in \Delta$. Multiplying both the sides of (9) by $e^{\int_0^s c_i(u)\,\mathrm{d}u}$ and integrating the result over the interval $[0, t]$, for $i = 1, 2, \ldots, n$ and $A \in \Delta$, we have

$$\begin{aligned}u_i^A(t) &= u_i^A(0)e^{-\int_0^t c_i(u)\,\mathrm{d}u} + \int_0^t e^{-\int_s^t c_i(u)\,\mathrm{d}u} \\ &\quad \times\Big[c_i(s)\int_{s-\delta_i(s)}^s \dot{u}_i^A(\theta)\,\mathrm{d}\theta + \sum_B\sum_{j=1}^n a_{ij}^{A\cdot\bar{B}}(s) \\ &\quad \times\big(f_j^B(y_j(s-\tau_{ij}(s))) - f_j^B(x_j(s-\tau_{ij}(s)))\big) \\ &\quad + \sum_B\sum_{j=1}^n\sum_{l=1}^n b_{ijl}(s) \\ &\quad \times\big(g_j^{A\cdot\bar{B}}(y_j(s-\sigma_{ijl}(s)))g_l^B(y_l(s-\nu_{ijl}(s))) \\ &\quad - g_j^{A\cdot\bar{B}}(x_j(s-\sigma_{ijl}(s))) \\ &\quad \times g_l^B(x_l(s-\nu_{ijl}(s)))\big)\Big]\mathrm{d}s.\end{aligned} \tag{13}$$

For $\vartheta \in \mathbb{R}$, $i = 1, 2, \ldots, n$, we define functions $\Gamma_i(\vartheta), \tilde{\Gamma}_i(\vartheta)$ as follows:

$$\begin{aligned}\Gamma_i(\vartheta) &= c_i^- - \vartheta - \Big(c_i^+\delta_i^+ e^{\vartheta\delta_i^+} + 2^{2m}\sum_{j=1}^n a_{ij}^+ L_f e^{\vartheta\tau_{ij}^+} \\ &\quad + 2^{2m}\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+ M_g L_g e^{\vartheta\sigma_{ijl}^+} \\ &\quad + 2^{2m}\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+ M_g L_g e^{\vartheta\nu_{ijl}^+}\Big),\end{aligned}$$

$$\begin{aligned}\tilde{\Gamma}_i(\vartheta) &= c_i^- - \vartheta - \big(c_i^+ + c_i^- - \vartheta\big)\Big(c_i^+\delta_i^+ e^{\vartheta\delta_i^+} \\ &\quad + 2^{2m}\sum_{j=1}^n a_{ij}^+ L_f e^{\vartheta\tau_{ij}^+} \\ &\quad + 2^{2m}\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+ M_g L_g e^{\vartheta\sigma_{ijl}^+} \\ &\quad + 2^{2m}\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+ M_g L_g e^{\vartheta\nu_{ijl}^+}\Big).\end{aligned}$$

According to (A_3), for $i = 1, 2, \ldots, n$, we have

$$\Gamma_i(0) = c_i^- - \Big(c_i^+\delta_i^+ + 2^{2m}\sum_{j=1}^n a_{ij}^+ L_f + 2^{2m+1}\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+ M_g L_g\Big) > 0,$$

$$\tilde{\Gamma}_i(0) = c_i^- - (c_i^+ + c_i^-)\Big(c_i^+\delta_i^+ + 2^{2m}\sum_{j=1}^n a_{ij}^+ L_f + 2^{2m+1}\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+ M_g L_g\Big) > 0.$$

Obviously, $\Gamma_i(\vartheta)$ and $\tilde{\Gamma}_i(\vartheta)$ are continuous on $[0, +\infty)$, $\Gamma_i(\vartheta), \tilde{\Gamma}_i(\vartheta) \to -\infty$ as $\vartheta \to +\infty$. Hence, there exist $\vartheta_i, \tilde{\vartheta}_i$ such that $\Gamma_i(\vartheta_i) = \tilde{\Gamma}_i(\tilde{\vartheta}_i) = 0$ and $\Gamma_i(\vartheta) > 0, \tilde{\Gamma}_i(\vartheta) > 0$, where $\vartheta \in (0, \xi_i)$, $\xi_i = \min\{\vartheta_i, \tilde{\vartheta}_i\}, i = 1, 2, \ldots, n$. Let $\xi = \min\{\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \tilde{\vartheta}_1, \tilde{\vartheta}_2, \ldots, \tilde{\vartheta}_n\}$; we have $\Gamma_i(\xi) \geq 0, \tilde{\Gamma}_i(\xi) \geq 0, i = 1, 2, \ldots, n$.

By choosing

$$0 < \lambda < \min\{\xi, \min_{1\leq i\leq n}\{c_i^-\}\},$$

for $i = 1, 2, \ldots, n$, we have

$$\Gamma_i(\lambda) > 0, \quad \tilde{\Gamma}_i(\lambda) > 0,$$

that is,

$$\frac{1}{c_i^- - \lambda}\Big(c_i^+\delta_i^+ e^{\lambda\delta_i^+} + 2^{2m}\sum_{j=1}^n a_{ij}^+ L_f e^{\lambda\tau_{ij}^+} + 2^{2m}\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+ M_g L_g(e^{\lambda\sigma_{ijl}^+} + e^{\lambda\nu_{ijl}^+})\Big) < 1 \quad (14)$$

and

$$\Big(1 + \frac{c_i^+}{c_i^- - \lambda}\Big)\Big(c_i^+\delta_i^+ e^{\lambda\delta_i^+} + 2^{2m}\sum_{j=1}^n a_{ij}^+ L_f e^{\lambda\tau_{ij}^+} + 2^{2m}\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+ M_g L_g(e^{\lambda\sigma_{ijl}^+} + e^{\lambda\nu_{ijl}^+})\Big) < 1. \quad (15)$$

Let

$$M = \max_{1\leq i\leq n}\Big\{c_i^-\Big(c_i^+\delta_i^+ + 2^{2m}\sum_{j=1}^n a_{ij}^+ L_f + 2^{2m+1}\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+ M_g L_g\Big)^{-1}\Big\}.$$

According to (A_3), we have $M > 1$. It is obvious that, for $i = 1, 2, \ldots, n$,

$$\frac{1}{M} < \frac{1}{c_i^- - \lambda}\Big(c_i^+\delta_i^+ e^{\lambda\delta_i^+} + 2^{2m}\sum_{j=1}^n a_{ij}^+ L_f e^{\lambda\tau_{ij}^+} + 2^{2m}\sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+ M_g L_g(e^{\lambda\sigma_{ijl}^+} + e^{\lambda\nu_{ijl}^+})\Big). \quad (16)$$

In view of (10), we find

$$|u_i^A(t)| = |\psi_i^A(t) - \varphi_i^A(t)|, \quad t \in [-\rho, 0],$$

where $i = 1, 2, \ldots, n$ and $A \in \Delta$; then we have

$$\|u(t)\| \leq \|\varphi - \psi\|_1 \leq M\|\varphi - \psi\|_1 e^{-\lambda t}, \quad t \in [-\rho, 0].$$

We assert that

$$\|u(t)\| \leq M\|\varphi - \psi\|_1 e^{-\lambda t}, \quad t > 0. \quad (17)$$

To prove (17), we first prove that, for any $l > 1$,

$$\|u(t)\| < lM\|\varphi - \psi\|_1 e^{-\lambda t}, \quad t > 0. \quad (18)$$

On the contrary there must exist $t_1 > 0$, $i_1, i_2 \in \{1, 2, \ldots, n\}$ and $A_1, A_2 \in \Delta$ such that

$$\begin{aligned}\|u(t_1)\| &= \max\{|u_{i_1}^{A_1}(t_1)|,\ |\dot{u}_{i_2}^{A_2}(t_1)|\} \\ &= lM\|\varphi - \psi\|_1 e^{-\lambda t_1}\end{aligned} \quad (19)$$

and

$$\|u(t)\| < lM\|\varphi - \psi\|_1 e^{-\lambda t}, \quad t \in [-\rho, t_1). \quad (20)$$

By (13), (14), (16) and (20), we have

$$\begin{aligned}&|u_{i_1}^{A_1}(t_1)| \\ &= \Big|u_{i_1}^{A_1}(0)e^{-\int_0^{t_1} c_{i_1}(u)\,\mathrm{d}u} + \int_0^{t_1} e^{-\int_s^{t_1} c_{i_1}(u)\,\mathrm{d}u} \\ &\quad\times\Big[c_{i_1}(s)\int_{s-\delta_{i_1}(s)}^s \dot{u}_{i_1}^{A_1}(\theta)\,\mathrm{d}\theta \\ &\quad + \sum_B\sum_{j=1}^n a_{i_1 j}^{A_1\cdot\bar{B}}(s)\big(f_j^B(y_j(s - \tau_{i_1 j}(s))) \\ &\quad - f_j^B(x_j(s - \tau_{i_1 j}(s)))\big) \\ &\quad + \sum_B\sum_{j=1}^n\sum_{l=1}^n b_{i_1 jl}(s)\big(g_j^{A_1\cdot\bar{B}}(y_j(s - \sigma_{i_1 jl}(s)))\end{aligned}$$

$$
\begin{aligned}
&\times g_l^B(y_l(s-\nu_{i_1jl}(s))) - g_j^{A_1\cdot\bar{B}}(x_j(s-\sigma_{i_1jl}(s)))\\
&\times g_l^B(x_l(s-\nu_{i_1jl}(s))))\Big]\,\mathrm{d}s\Big|\\
\le{}& \|\varphi-\psi\|_1 e^{-t_1c_{i_1}^-} + \int_0^{t_1} e^{-(t_1-s)c_{i_1}^-}\\
&\times\Big[c_{i_1}^+\delta_{i_1}^+ lMe^{-\lambda(s-\delta_{i_1}(s))}\|\varphi-\psi\|_1\\
&+2^m\sum_{j=1}^n a_{i_1j}^+L_f\sum_D|y_j^D(s-\tau_{i_1j}(s))\\
&-x_j^D(s-\tau_{i_1j}(s))|\\
&+\sum_B\sum_{j=1}^n\sum_{l=1}^n b_{i_1jl}^+\big(\big|\big(g_j^{A_1\cdot\bar{B}}(x_j(s-\sigma_{i_1jl}(s)))\\
&-g_j^{A_1\cdot\bar{B}}(y_j(s-\sigma_{i_1jl}(s)))\big)g_l^B(x_l(s-\nu_{i_1jl}(s)))\big|\\
&+\big|\big(g_l^B(x_l(s-\nu_{i_1jl}(s)))-g_l^B(y_l(s-\nu_{i_1jl}(s)))\big)\\
&\times g_j^{A_1\cdot\bar{B}}(y_j(s-\sigma_{i_1jl}(s)))\big|\big)\Big]\mathrm{d}s\\
\le{}& \|\varphi-\psi\|_1 e^{-t_1c_{i_1}^-} + \int_0^{t_1} e^{-(t_1-s)c_{i_1}^-}\\
&\times\Big[c_{i_1}^+\delta_{i_1}^+ lMe^{-\lambda(s-\delta_{i_1}(s))}\|\varphi-\psi\|_1\\
&+2^m\sum_{j=1}^n a_{i_1j}^+L_f\sum_D|y_j^D(s-\tau_{i_1j}(s))\\
&-x_j^D(s-\tau_{i_1j}(s))|\\
&+2^m\sum_{j=1}^n\sum_{l=1}^n b_{i_1jl}^+M_gL_g\Big(\sum_D|y_j^D(s-\sigma_{i_1jl}(s))\\
&-x_j^D(s-\sigma_{i_1jl}(s))|\\
&+\sum_D|y_l^D(s-\nu_{i_1jl}(s))-x_l^D(s-\nu_{i_1jl}(s))|\Big)\Big]\mathrm{d}s\\
\le{}& \|\varphi-\psi\|_1 e^{-t_1c_{i_1}^-} + \int_0^{t_1} e^{-(t_1-s)c_{i_1}^-}\\
&\times\Big[c_{i_1}^+\delta_{i_1}^+ lMe^{-\lambda(s-\delta_{i_1}(s))}\|\varphi-\psi\|_1\\
&+2^{2m}\sum_{j=1}^n a_{i_1j}^+L_f lMe^{-\lambda(s-\tau_{i_1j}(s))}\|\varphi-\psi\|_1\\
&+2^{2m}\sum_{j=1}^n\sum_{l=1}^n b_{i_1jl}^+M_gL_g\big(lMe^{-\lambda(s-\sigma_{i_1jl}(s))}\\
&\times\|\varphi-\psi\|_1+lMe^{-\lambda(s-\nu_{i_1jl}(s))}\|\varphi-\psi\|_1\big)\Big]\mathrm{d}s\\
\le{}& \|\varphi-\psi\|_1 e^{-t_1c_{i_1}^-} + lM\|\varphi-\psi\|_1\\
&\times\Big(c_{i_1}^+\delta_{i_1}^+e^{\lambda\delta_{i_1}^+}+2^{2m}\sum_{j=1}^n a_{i_1j}^+L_fe^{\lambda\tau_{i_1j}^+}\\
&+2^{2m}\sum_{j=1}^n\sum_{l=1}^n b_{i_1jl}^+M_gL_g\big(e^{\lambda\sigma_{i_1jl}^+}+e^{\lambda\nu_{i_1jl}^+}\big)\Big)\\
&\times\int_0^{t_1} e^{-(t_1-s)c_{i_1}^-}e^{-\lambda s}\mathrm{d}s\\
={}& \|\varphi-\psi\|_1 e^{-t_1c_{i_1}^-} + lM\|\varphi-\psi\|_1\\
&\times\Big(c_{i_1}^+\delta_{i_1}^+e^{\lambda\delta_{i_1}^+}+2^{2m}\sum_{j=1}^n a_{i_1j}^+L_fe^{\lambda\tau_{i_1j}^+}\\
&+2^{2m}\sum_{j=1}^n\sum_{l=1}^n b_{i_1jl}^+M_gL_g\big(e^{\lambda\sigma_{i_1jl}^+}+e^{\lambda\nu_{i_1jl}^+}\big)\Big)\\
&\times\frac{e^{-\lambda t_1}}{c_{i_1}^- -\lambda}(1-e^{(\lambda-c_{i_1}^-)t_1})\\
={}& lM\|\varphi-\psi\|_1e^{-\lambda t_1}\Big\{\frac{1}{lM}e^{(\lambda-c_{i_1}^-)t_1}+\frac{1-e^{(\lambda-c_{i_1}^-)t_1}}{c_{i_1}^- -\lambda}\\
&\times\Big(c_{i_1}^+\delta_{i_1}^+e^{\lambda\delta_{i_1}^+}+2^{2m}\sum_{j=1}^n a_{i_1j}^+L_fe^{\lambda\tau_{i_1j}^+}\\
&+2^{2m}\sum_{j=1}^n\sum_{l=1}^n b_{i_1jl}^+M_gL_g\big(e^{\lambda\sigma_{i_1jl}^+}+e^{\lambda\nu_{i_1jl}^+}\big)\Big)\Big\}\\
<{}& lM\|\varphi-\psi\|_1e^{-\lambda t_1}\Big\{\frac{1}{M}e^{(\lambda-c_{i_1}^-)t_1}+\frac{1-e^{(\lambda-c_{i_1}^-)t_1}}{c_{i_1}^- -\lambda}\\
&\times\Big(c_{i_1}^+\delta_{i_1}^+e^{\lambda\delta_{i_1}^+}+2^{2m}\sum_{j=1}^n a_{i_1j}^+L_fe^{\lambda\tau_{i_1j}^+}\\
&+2^{2m}\sum_{j=1}^n\sum_{l=1}^n b_{i_1jl}^+M_gL_g\big(e^{\lambda\sigma_{i_1jl}^+}+e^{\lambda\nu_{i_1jl}^+}\big)\Big)\Big\}\\
={}& lM\|\varphi-\psi\|_1e^{-\lambda t_1}\Big\{\Big(\frac{1}{M}-\frac{1}{c_{i_1}^- -\lambda}\Big(c_{i_1}^+\delta_{i_1}^+e^{\lambda\delta_{i_1}^+}\\
&+2^{2m}\sum_{j=1}^n a_{i_1j}^+L_fe^{\lambda\tau_{i_1j}^+}+2^{2m}\\
&\times\sum_{j=1}^n\sum_{l=1}^n b_{i_1jl}^+M_gL_g\big(e^{\lambda\sigma_{i_1jl}^+}+e^{\lambda\nu_{i_1jl}^+}\big)\Big)\Big)e^{(\lambda-c_{i_1}^-)t_1}\\
&+\frac{1}{c_{i_1}^- -\lambda}\Big(c_{i_1}^+\delta_{i_1}^+e^{\lambda\delta_{i_1}^+}+2^{2m}\sum_{j=1}^n a_{i_1j}^+L_fe^{\lambda\tau_{i_1j}^+}\\
&+2^{2m}\sum_{j=1}^n\sum_{l=1}^n b_{i_1jl}^+M_gL_g\big(e^{\lambda\sigma_{i_1jl}^+}+e^{\lambda\nu_{i_1jl}^+}\big)\Big)\Big\}\\
\le{}& lM\|\varphi-\psi\|_1e^{-\lambda t_1}\frac{1}{c_{i_1}^- -\lambda}\Big(c_{i_1}^+\delta_{i_1}^+e^{\lambda\delta_{i_1}^+}\\
&+2^{2m}\sum_{j=1}^n a_{i_1j}^+L_fe^{\lambda\tau_{i_1j}^+}\\
&+2^{2m}\sum_{j=1}^n\sum_{l=1}^n b_{i_1jl}^+M_gL_g\big(e^{\lambda\sigma_{i_1jl}^+}+e^{\lambda\nu_{i_1jl}^+}\big)\Big)\\
<{}& lM\|\varphi-\psi\|_1e^{-\lambda t_1},\\
& i=1,2,\dots,n,\quad A\in\Delta.
\end{aligned}
\tag{21}
$$

In addition, by (13), for $i = 1, 2, \ldots, n$, $A \in \Delta$, we have

$$
\begin{aligned}
&\dot{u}_i^A(t)\\
&\quad = -c_i(t)u_i^A(0)e^{-\int_0^t c_i(u)\,\mathrm{d}u} + c_i(t)\int_{t-\delta_i(t)}^{t} \dot{u}_i^A(\theta)\,\mathrm{d}\theta\\
&\qquad + \sum_B \sum_{j=1}^{n} a_{ij}^{A\cdot\bar{B}}(t)\big(f_j^B(y_j(t-\tau_{ij}(t)))\\
&\qquad - f_j^B(x_j(t-\tau_{ij}(t)))\big) + \sum_B \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)\\
&\qquad \times\big(g_j^{A\cdot\bar{B}}(y_j(t-\sigma_{ijl}(t)))g_l^B(y_l(t-\nu_{ijl}(t)))\\
&\qquad - g_j^{A\cdot\bar{B}}(x_j(t-\sigma_{ijl}(t)))g_l^B(x_l(t-\nu_{ijl}(t)))\big)\\
&\qquad - \int_0^t c_i(t)e^{-\int_s^t c_i(u)\,\mathrm{d}u}\Big[c_i(s)\int_{s-\delta_i(s)}^{s}\dot{u}_i^A(\theta)\,\mathrm{d}\theta\\
&\qquad + \sum_B \sum_{j=1}^{n} a_{ij}^{A\cdot\bar{B}}(s)\big(f_j^B(y_j(s-\tau_{ij}(s)))\\
&\qquad - f_j^B(x_j(s-\tau_{ij}(s)))\big) + \sum_B \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s)\\
&\qquad \times\big(g_j^{A\cdot\bar{B}}(y_j(s-\sigma_{ijl}(s)))g_l^B(y_l(s-\nu_{ijl}(s)))\\
&\qquad - g_j^{A\cdot\bar{B}}(x_j(s-\sigma_{ijl}(s)))\\
&\qquad \times g_l^B(x_l(s-\nu_{ijl}(s)))\big)\Big]\,\mathrm{d}s.
\end{aligned}
\tag{22}
$$

From (15), (16), (20) and (22) it follows that

$$
\begin{aligned}
&|\dot{u}_{i_2}^{A_2}(t_1)|\\
&\quad \le c_{i_2}^{+}\|\varphi-\psi\|_1 e^{-t_1 c_{i_2}^{-}} + c_{i_2}^{+}\delta_{i_2}^{+} lM\|\varphi-\psi\|_1\\
&\qquad \times e^{-\lambda(t_1-\delta_{i2}(t_1))} + 2^m\sum_{j=1}^{n} a_{i_2 j}^{+}L_f\\
&\qquad \times \sum_D |y_j^D(t_1-\tau_{i_2 j}(t_1)) - x_j^D(t_1-\tau_{i_2 j}(t_1))|\\
&\qquad + 2^m\sum_{j=1}^{n}\sum_{l=1}^{n} b_{i_2 jl}^{+}M_g L_g\Big(\sum_D |y_j^D(t_1-\sigma_{i_2 jl}(t_1))\\
&\qquad - x_j^D(t_1-\sigma_{i_2 jl}(t_1))| + \sum_D |y_l^D(t_1-\nu_{i_2 jl}(t_1))\\
&\qquad - x_l^D(t_1-\nu_{i_2 jl}(t_1))|\Big) + \int_0^{t_1} c_{i_2}^{+}e^{-(t_1-s)c_{i_2}^{-}}\\
&\qquad \times\Big[c_{i_2}^{+}\delta_{i_2}^{+} lM\|\varphi-\psi\|_1 e^{-\lambda(s-\delta_{i_2}^{+})}\\
&\qquad + 2^m\sum_{j=1}^{n} a_{i_2 j}^{+}L_f\sum_D |y_j^D(s-\tau_{i_2 j}(s)))\\
&\qquad - x_j^D(s-\tau_{i_2 j}(s)))|\\
&\qquad + 2^m\sum_{j=1}^{n}\sum_{l=1}^{n} b_{i_2 jl}^{+}M_g L_g\Big(\sum_D |y_j^D(s-\sigma_{i_2 jl}(s))\\
&\qquad - x_j^D(s-\sigma_{i_2 jl}(s))|\\
&\qquad + \sum_D |y_l^D(s-\nu_{i_2 jl}(s))\\
&\qquad - x_l^D(s-\nu_{i_2 jl}(s))|\Big)\Big]\,\mathrm{d}s\\
&\quad \le lM\|\varphi-\psi\|_1 e^{-\lambda t_1}\Big\{\frac{c_{i_2}^{+}}{lM}e^{(\lambda-c_{i_2}^{-})t_1}\\
&\qquad + \Big(1 + c_{i_2}^{+}\int_0^{t_1} e^{(t_1-s)(\lambda-c_{i_2}^{-})}\mathrm{d}s\Big)\Big[c_{i_2}^{+}\delta_{i_2}^{+}e^{\lambda\delta_{i_2}^{+}}\\
&\qquad + 2^{2m}\sum_{j=1}^{n} a_{i_2 j}^{+}L_f e^{\lambda\tau_{i_2 j}^{+}}\\
&\qquad + 2^{2m}\sum_{j=1}^{n}\sum_{l=1}^{n} b_{i_2 jl}^{+}M_g L_g\big(e^{\lambda\sigma_{i_2 jl}^{+}} + e^{\lambda\nu_{i_2 jl}^{+}}\big)\Big]\Big\}\\
&\quad < lM\|\varphi-\psi\|_1 e^{-\lambda t_1}\Big\{\Big(\frac{1}{M} - \frac{1}{c_{i_2}^{-}-\lambda}\Big[c_{i_2}^{+}\delta_{i_2}^{+}e^{\lambda\delta_{i_2}^{+}}\\
&\qquad + 2^{2m}\sum_{j=1}^{n} a_{i_2 j}^{+}L_f e^{\lambda\tau_{i_2 j}^{+}} + 2^{2m}\sum_{j=1}^{n}\sum_{l=1}^{n} b_{i_2 jl}^{+}M_g L_g\\
&\qquad \times\big(e^{\lambda\sigma_{i_2 jl}^{+}} + e^{\lambda\nu_{i_2 jl}^{+}}\big)\Big]\Big)c_{i_2}^{+}e^{(\lambda-c_{i_2}^{-})t_1}\\
&\qquad + \Big(1 + \frac{c_{i_2}^{+}}{c_{i_2}^{-}-\lambda}\Big)\Big[c_{i_2}^{+}\delta_{i_2}^{+}e^{\lambda\delta_{i_2}^{+}}\\
&\qquad + 2^{2m}\sum_{j=1}^{n} a_{i_2 j}^{+}L_f e^{\lambda\tau_{i_2 j}^{+}}\\
&\qquad + 2^{2m}\sum_{j=1}^{n}\sum_{l=1}^{n} b_{i_2 jl}^{+}M_g L_g\big(e^{\lambda\sigma_{i_2 jl}^{+}} + e^{\lambda\nu_{i_2 jl}^{+}}\big)\Big]\Big\}\\
&\quad < lM\|\varphi-\psi\|_1 e^{-\lambda t_1}\Big(1 + \frac{c_{i_2}^{+}}{c_{i_2}^{-}-\lambda}\Big)\\
&\qquad \times\Big[c_{i_2}^{+}\delta_{i_2}^{+}e^{\lambda\delta_{i_2}^{+}} + 2^{2m}\sum_{j=1}^{n} a_{i_2 j}^{+}L_f e^{\lambda\tau_{i_2 j}^{+}}\\
&\qquad + 2^{2m}\sum_{j=1}^{n}\sum_{l=1}^{n} b_{i_2 jl}^{+}M_g L_g\big(e^{\lambda\sigma_{i_2 jl}^{+}} + e^{\lambda\nu_{i_2 jl}^{+}}\big)\Big]\\
&\quad < lM\|\varphi-\psi\|_1 e^{-\lambda t_1}.
\end{aligned}
\tag{23}
$$

In view of (21) and (23), we can get

$$\|u(t_1)\| < lM\|\varphi-\psi\|_1 e^{-\lambda t_1},$$

which contradicts (19). Hence, (18) holds. Letting $p \to 1$, we get that (17) holds. It follows that

$$\|x(t)-y(t)\| \le M\|\varphi-\psi\|_1 e^{-\lambda t}, \quad t > 0.$$

Therefore, the system (2) has a unique $(\omega/2)$-anti-periodic solution that is global exponentially stable. The uniqueness of the $(\omega/2)$-anti-periodic solution follows from its global exponential stability. In view of Remark 2, the system (1) also has a unique $(\omega/2)$-anti-periodic solution that is globally exponentially stable. The proof is complete. ■

5. Numerical examples

In this section, we give a numerical example to show the feasibility and effectiveness of the results obtained in this paper.

Example 1. Consider the following Clifford-valued HHNN with time-varying and leakage delays:

$$\begin{aligned}\dot{X}_i(t) &= -c_i(t)X_i(t-\delta_i(t))\\ &\quad + \sum_{j=1}^{2} a_{ij}(t)f_j(X_j(t-\tau_{ij}(t)))\\ &\quad + \sum_{j=1}^{2}\sum_{l=1}^{2} b_{ijl}(t)g_j(X_j(t-\sigma_{ijl}(t)))\\ &\quad \times g_l(X_l(t-\nu_{ijl}(t))) + I_i(t), \quad i=1,2, \quad (24)\end{aligned}$$

where $m=3$, the Clifford generators are e_1, e_2, e_3, $X_i = x_i^0 + x_i^1 e_1 + x_i^2 e_2 + x_i^3 e_3 + x_i^{12} e_{12} + x_i^{13} e_{13} + x_i^{23} e_{23} + x_i^{123} e_{123} \in \mathcal{A}$,

$$\begin{aligned} f_i(X_i) &= \sin^2 x_i^0 + 3\sin^2 x_i^1 e_1 + 2\sin^2 x_i^2 e_2\\ &\quad + 0.1\cos^2 x_i^3 e_3 + 5\cos^2 x_i^{12} e_{12}\\ &\quad + 2.3\sin^2 x_i^{13} e_{13} + 1.5\sin^2 x_i^{23} e_{23}\\ &\quad + 0.6\sin^2 x_i^{123} e_{123},\end{aligned}$$

$$\begin{aligned} g_i(X_i) &= 3.6\cos^2 x_i^0 + 2.8\cos^2 x_i^1 e_1 + 1.6\cos^2 x_i^2 e_2\\ &\quad + 6\sin^2 x_i^3 e_3 + 5.3\sin^2 x_i^{12} e_{12} + \sin^2 x_i^{13} e_{13}\\ &\quad + 1.5\sin^2 x_i^{23} e_{23} + 0.8\sin^2 x_i^{123} e_{123},\end{aligned}$$

$$c_i(t) = 10\cos^2(30t) + 2,$$

$$\begin{aligned} I_i(t) &= \sin(30t) + 0.1\cos(30t)e_1 + 0.6\sin(30t)e_2\\ &\quad + 0.6\sin(30t)e_3 + 0.6\sin(30t)e_{12}\\ &\quad + 0.6\sin(30t)e_{13} + 0.6\sin(30t)e_{23}\\ &\quad - \cos(30t)e_{123},\end{aligned}$$

$$\begin{aligned} a_{11}(t) &= a_{12}(t)\\ &= 2e^{-1}\cos(30t) - 0.6\sin(30t)e_1\\ &\quad + 0.01\sin(30t)e_2 + 0.1e^{-0.1}\sin(30t)e_3\\ &\quad - 1.5\cos(30t)e_{12} + \cos(30t)e_{13}\\ &\quad + 2\cos(30t)e_{123},\end{aligned}$$

$$\begin{aligned} a_{21}(t) &= a_{22}(t)\\ &= 3\cos(30t) + 2\sin(30t)e_1 - 0.3\sin(30t)e_2\\ &\quad - \sin(30t)e_3 - 0.1\sin(30t)e_{12}\\ &\quad - \cos(30t)e_{13} - 2\cos(30t)e_{23},\end{aligned}$$

$$\begin{aligned} b_{ijl}(t) &= 5e^{-2}\cos(30t) + 2e^{-3}\sin(30t)e_1\\ &\quad + 0.2\sin(30t)e_2 + 0.3\sin(30t)e_3\\ &\quad + \cos(30t)e_{12} + \sin(30t)e_{13} + \sin(30t)e_{23},\end{aligned}$$

$$\delta_i(t) = \frac{1}{120}\sin(60t), \quad \tau_{ij}(t) = \frac{1}{3}|\cos(30t)|,$$

$$\sigma_{ijl}(t) = \frac{1}{6}\cos^2(30t), \quad \nu_{ijl}(t) = \frac{1}{10}\sin^2(30t),$$

where $i, j, l = 1, 2$.

It is easy to see that

$$M_f = 5, \quad M_g = 6, \quad c_i^+ = 12, \quad \dot{\delta}_i^+ = 0.5, \quad \omega = \frac{\pi}{15}.$$

Then

$$1 - \dot{\delta}_i^+ = 0.5 > 0, \quad \frac{1}{\omega} - \frac{c_i^+}{2\pi(1-\dot{\delta}_i^+)} \approx 0.95 > 0.$$

Thus, the conditions (A_1)–(A_5) are satisfied. Therefore, according to Theorem 1, the system (24) has a $(\pi/15)$-anti-periodic solution. Setting the three different initial values, the transient states of eight parts of the system (24) are shown in Figs. 1–4. ♦

Example 2. Consider the following Clifford-valued HHNN with time-varying and leakage delays:

$$\begin{aligned}\dot{X}_i(t) &= -c_i(t)X_i(t-\delta_i(t))\\ &\quad + \sum_{j=1}^{2} a_{ij}(t)f_j(X_j(t-\tau_{ij}(t)))\\ &\quad + \sum_{j=1}^{2}\sum_{l=1}^{2} b_{ijl}(t)g_j(X_j(t-\sigma_{ijl}(t)))\\ &\quad \times g_l(X_l(t-\nu_{ijl}(t))) + I_i(t), \quad i=1,2,\end{aligned} \quad (25)$$

where $m=2$, the Clifford generators are e_1, e_2, $X_i = x_i^0 + x_i^1 e_1 + x_i^2 e_2 + x_i^{12} e_{12} \in \mathcal{A}$,

$$\begin{aligned} f_i(X_i) &= 0.03|\sin x_i^0| + 0.01|\sin x_i^1|e_1\\ &\quad + 0.02|\sin x_i^2|e_2 + 0.05|\sin x_i^{12}|e_{12},\\ g_i(X_i) &= 0.5|\sin x_i^0| + 0.36|\sin x_i^1|e_1\\ &\quad + 0.01|\sin x_i^2|e_2 + 0.26|\sin x_i^{12}|e_{12},\end{aligned}$$

$$c_i(t) = 0.01\cos t + 0.06,$$

$$I_i(t) = \sin\frac{1}{2}t + 0.1\cos\frac{1}{2}te_1 + 0.6\sin\frac{1}{2}te_2 - \cos\frac{1}{2}te_{12},$$

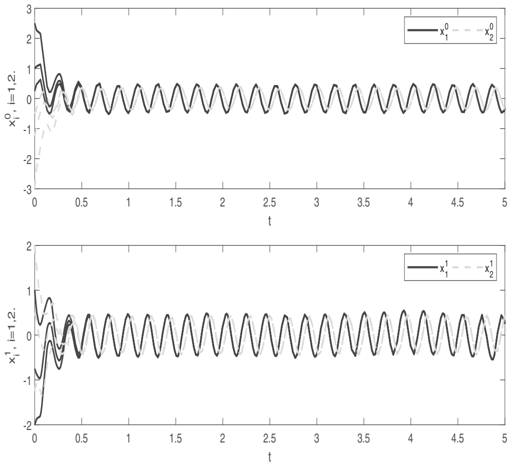

Fig. 1. State trajectories for the system (24) with the initial values $x_i^0(0) = (0.25, -0.5)^T, (1, -1.25)^T, (2.5, -3)^T$ and $x_i^1(0) = (1, 2)^T, (-0.75, -1.25)^T, (-2, 0.5)^T$, $i = 1, 2$.

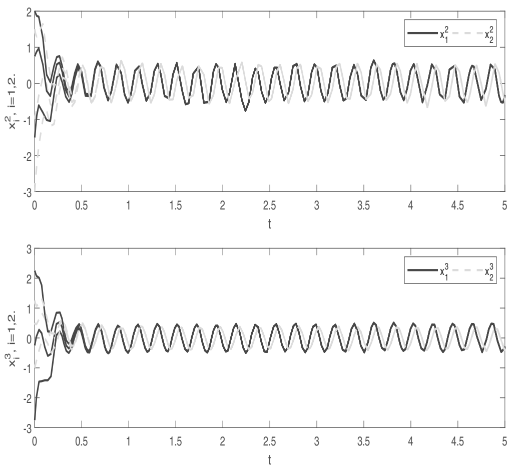

Fig. 2. State trajectories for the system (24) with the initial values $x_i^2(0) = (2, -1)^T, (-1.5, -3)^T, (0.75, 1.5)^T$ and $x_i^3(0) = (-0.25, 0.5)^T, (2.25, 1.25)^T, (-2.75, -1)^T$, $i = 1, 2$.

$$
\begin{aligned}
a_{11}(t) &= a_{12}(t) \\
&= 0.02e^{-1}\cos\frac{1}{2}t - 0.006\sin\frac{1}{2}te_1 \\
&\quad +0.001e^{-0.1}\sin\frac{1}{2}te_2 - 0.015e^{-0.3}\cos\frac{1}{2}te_{12}, \\
a_{21}(t) &= a_{22}(t) \\
&= 0.005\cos\frac{1}{2}t + 0.002e^{-2}\sin\frac{1}{2}te_1 \\
&\quad -0.003e^{-3}\sin\frac{1}{2}te_2 - 0.005e^{-0.1}\cos\frac{1}{2}te_{12}, \\
b_{ijl}(t) &= 0.001e^{-2}\cos\frac{1}{2}t + 0.002e^{-3}\sin\frac{1}{2}te_1 \\
&\quad +0.0003\sin\frac{1}{2}te_2 + 0.001e^{-3}\cos\frac{1}{2}te_{12},
\end{aligned}
$$

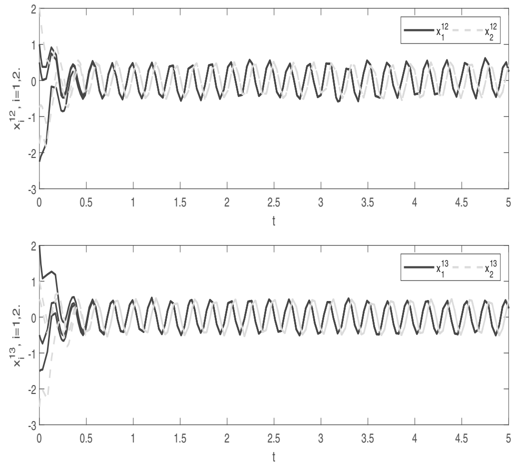

Fig. 3. State trajectories for the system (24) with the initial values $x_i^{12}(0) = (0.5, -0.75)^T, (1, -1.75)^T, (-2.25, 2)^T$ and $x_i^{13}(0) = (-1.5, 1)^T, (-0.5, 0.5)^T, (2, -2.5)^T$, $i = 1, 2$.

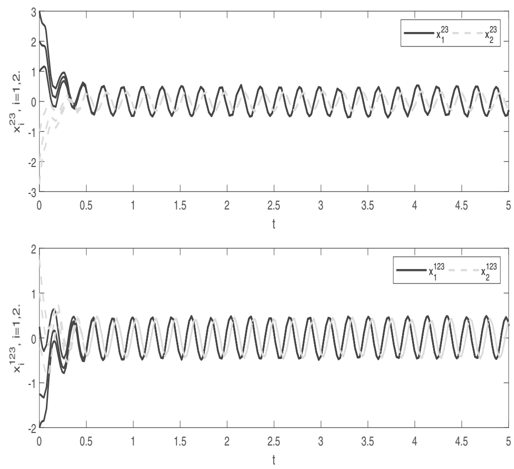

Fig. 4. State trajectories for the system (24) with the initial values $x_i^{23}(0) = (1, -1)^T, (2, -2)^T, (3, -2.75)^T$ and $x_i^{123}(0) = (0.25, 1)^T, (-1.25, 1.75)^T, (-2, -0.25)^T$, $i = 1, 2$.

$$
\delta_i(t) = \frac{1}{80}\sin t + \frac{1}{40}, \quad \tau_{ij}(t) = \frac{1}{5}|\sin t|,
$$

$$
\sigma_{ijl}(t) = \frac{1}{2}|\cos t|, \quad \nu_{ijl}(t) = \frac{1}{6}|\sin t| + \frac{1}{6},
$$

$$
i, j, l = 1, 2.
$$

It is easy to see that

$$
L_f = M_f = 0.05, \quad L_g = M_g = 0.5,
$$

$$
a_{11}^+ = a_{12}^+ = 0.006, \quad a_{21}^+ = a_{22}^+ = 0.005,
$$

$$
b_{ijl}^+ = 0.0003, \quad c_i^+ = 0.07, \quad c_i^- = 0.05,
$$

$$\dot{\delta}_i^+ = \frac{1}{80}, \quad \delta_i^+ = \frac{3}{80} \quad \omega = 4\pi, \quad i, j, l = 1, 2.$$

Then

$$1 - \dot{\delta}_i^+ \approx 0.99 > 0,$$

$$\frac{1}{\omega} - \frac{c_i^+}{2\pi(1 - \dot{\delta}_i^+)} \approx 0.07 > 0,$$

$$\varpi_i \approx 0.02,$$

$$\max_{1 \le i \le n} \left\{ \frac{\varpi_i}{c_i^-}, (1 + \frac{c_i^+}{c_i^-})\varpi_i \right\} \approx 0.436 < 1.$$

Thus, the conditions (A_1)–(A_8) are satisfied. Therefore, according to Theorem 2, the system (25) has a unique 2π-anti-periodic solution that is globally exponentially stable.

By using the Simulink toolbox in MATLAB, the fact is verified by the numerical simulation in Figs. 5 and 6 and there are five differential initial values which shows the state trajectories of the system (25). Figures 7–10 show simulation results of the system (25) with 2 random initial conditions. ♦

Remark 5. The results of Examples 1 and 2 cannot be obtained by the approaches outlined by Ou (2008), Ke and Miao (2017), Şaylı and Yılmaz (2017), Shi and Dong (2010), Li *et al.* (2019a; 2019b) and others.

6. Conclusion

In this paper, by using a continuation theorem of coincidence degree theory with inequality techniques, we established the existence of anti-periodic solutions for a class of Clifford-valued HHNNs with state-dependent and leakage delays. By using the proof by contradiction, we obtained the global exponential stability of the anti-periodic solution. This is the first paper to study the anti-periodic solutions for Clifford-valued HHNNs. The results of this paper are essentially new even when our system degenerates into real-valued, complex-valued and quaternion-valued systems. Our methods can be used to study other types of Clifford-valued neural network models with delays.

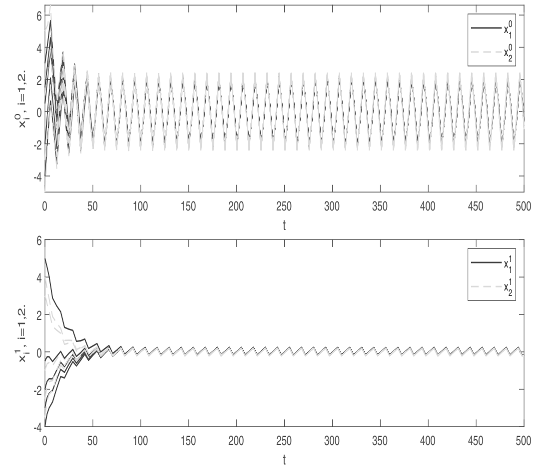

Fig. 5. Transient states of the solutions $x_i^0(t)$ and $x_i^1(t)$ $(i = 1, 2)$ of the system (25) with the initial values $x_i^0(0) = (0.5, 2)^T, (3, -3)^T, (-2, -5)^T, (-4, 1)^T, (1.5, 5)^T$ and $x_i^1(0) = (-2, -1)^T, (5, 4)^T, (-3, -2.5)^T, (-0.5, -3.5)^T, (-4, 3)^T$, $i = 1, 2$.

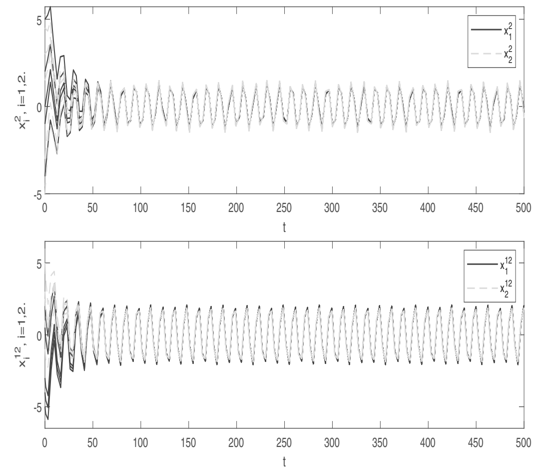

Fig. 6. Transient states of the solutions $x_i^2(t)$ and $x_i^{12}(t)$ $(i = 1, 2)$ of the system (25) with the initial values $x_i^2(0) = (0, 1.5)^T, (2, 4.5)^T, (-4, 3)^T, (-1, -5)^T, (5, -3)^T$ and $x_i^{12}(0) = (2, 1)^T, (-4, -1)^T, (-3, 5)^T, (-5, 2.5)^T, (0.5, 3)^T$, $i = 1, 2$.

Acknowledgment

This work is supported by the National Natural Sciences Foundation of the People's Republic of China under the grants no. 11861072 and no. 11361072.

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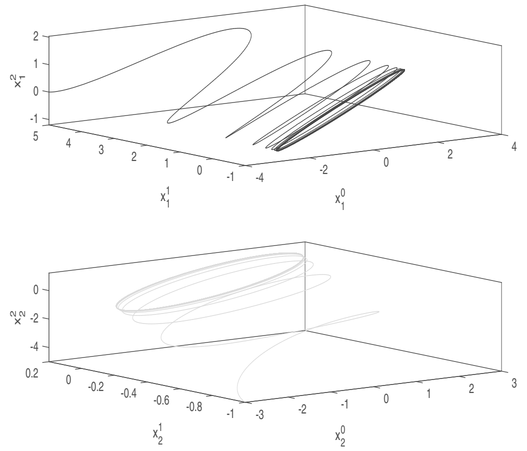

Fig. 7. Phrase trajectories of the solutions x_i^0, x_i^1 and x_i^2 $(i = 1, 2)$ of the system (25) in the 3-dimensional space.

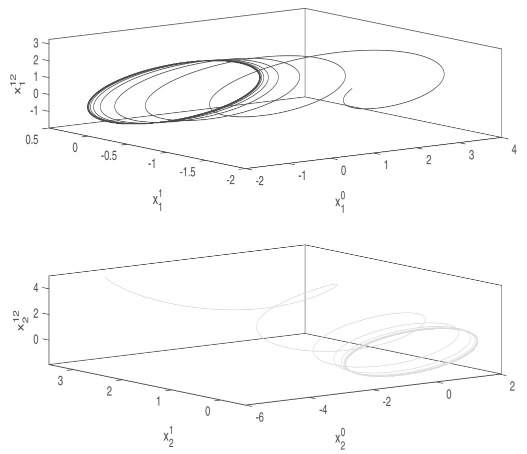

Fig. 8. Phrase trajectories of the solutions x_i^0, x_i^1 and x_i^{12} $(i = 1, 2)$ of the system (25) in the 3-dimensional space.

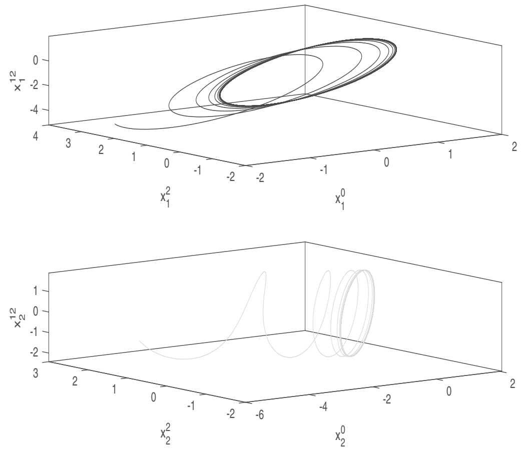

Fig. 9. Phrase trajectories of the solutions x_i^0, x_i^2 and x_i^{12} $(i = 1, 2)$ of the system (25) in the 3-dimensional space.

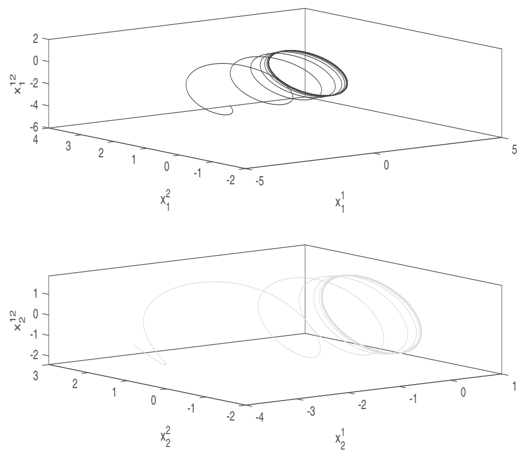

Fig. 10. Phrase trajectories of the solutions x_i^1, x_i^2 and x_i^{12} $(i = 1, 2)$ of the system (25) in the 3-dimensional space.

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Received: 14 March 2019
Revised: 27 September 2019
Accepted: 18 October 2019