Decomposition of Graphs: Strongly Connected Components

Daniel Kane

Department of Computer Science and Engineering
University of California, San Diego

Graph Algorithms
Data Structures and Algorithms

Learning Objectives

- Understand the definition of a strongly connected component of a directed graph.
- Give some other notions of connectivity within a directed graph.

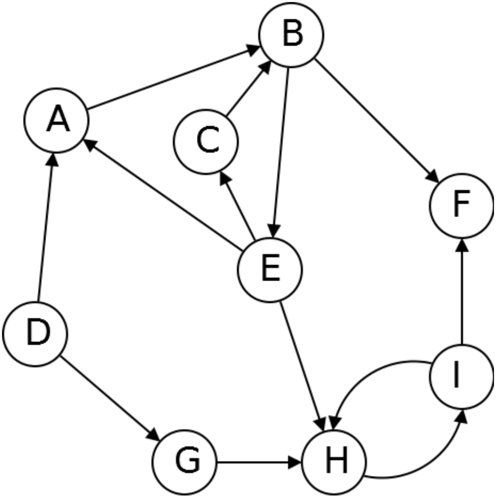
Outline

1. Motivation
2. Definition

Connectivity in Digraphs

In undirected graphs, have connected components.
Directed graphs are more complicated.

Example



Possible Notions

- Connected by edges in any direction.
- One vertex reachable from another.
- Two vertices both reachable from the other.

Outline

Strongly Connected Components

Definition

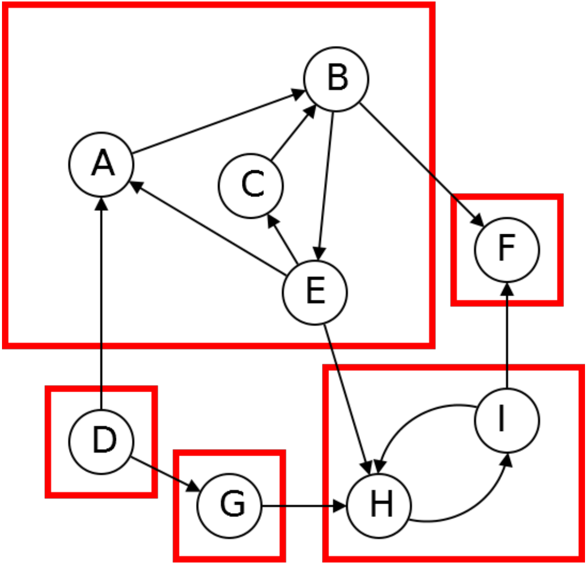
Two vertices v, w in a directed graph are connected if you can reach v from w and can reach w from v.

Result

Theorem

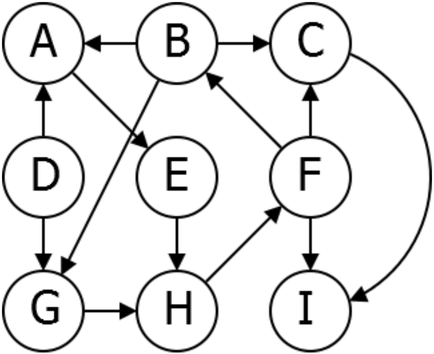
A directed graph can be partitioned into strongly connected components where two vertices are connected if and only if they are in the same component.

Example



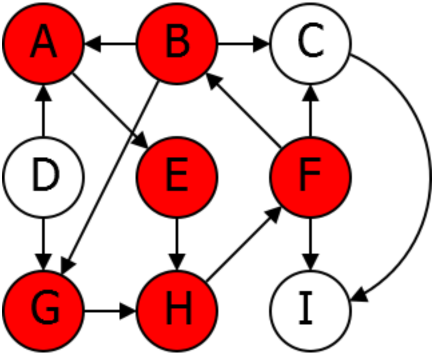
Problem

What is the SCC of A?



Solution

$A, B, E, F, G, H.$



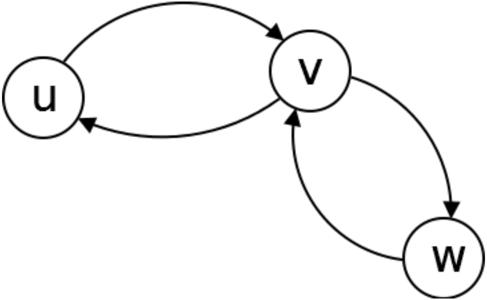
Result

Theorem

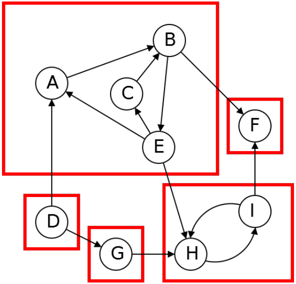
A directed graph can be partitioned into strongly connected components where two vertices are connected if and only if they are in the same component.

Proof

Need to show an equivalence relation.

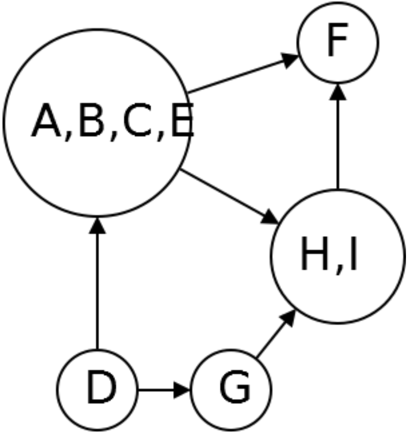


Metagraph



We can also draw a metagraph showing how the strongly connected components connect to one another.

Example



Note: It's a DAG.

DAG

Theorem

The metagraph of a graph G is always a DAG.

Proof

Proof.

Suppose not. Must be a cycle $\mathcal{C}$. Any nodes in cycle can reach any others. Should all be in same SCC. Contradiction. $\square$

Summary

- Can partition vertices into strongly connected components.
- Metagraph describes how strongly connected components connect to each other.
- Metagraph always a DAG.

Next Time

How to compute the strongly connected components of a graph.