

NP-complete Problems: Search Problems

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Outline

1 Brute Force Search

2 Search Problems

3 Easy and Hard Problems

Traveling Salesman Problem

Hamiltonian Cycle Problem

Longest Path Problem

Integer Linear Programming Problem

Independent Set Problem

4 P and NP

Polynomial vs Exponential

running time:	n	n^2	n^3	2^n
less than 10^9 :	10^9	$10^{4.5}$	10^3	29

Improving Brute Force Search

Usually, an efficient (polynomial) algorithm searches for a solution among an exponential number of candidates:

- there are $n!$ permutations of n objects

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- there are $n!$ permutations of n objects
- there are 2^n ways to partition n objects into two sets
- there are n^{n-2} spanning trees in a complete graph on n vertices

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- For thousands of practically important problems we don't have an efficient algorithm yet
- An efficient algorithm for one such problem automatically gives efficient algorithms for all these problems!
- \$1M prize for constructing such an algorithm or proving that this is impossible!

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Boolean Formulas

Formula in conjunctive normal form

$$(x \vee y \vee z)(x \vee \bar{y})(y \vee \bar{z})(z \vee \bar{x})(\bar{x} \vee \bar{y} \vee \bar{z})$$

- x, y, z are Boolean variables (values: true/false or 1/0)
- literals are variables (x, y, z) and their negations ($\bar{x}, \bar{y}, \bar{z}$)
- clauses are disjunctions (logical or) of literals

Satisfiability (SAT)

Input: Formula F in conjunctive normal form (CNF).

Output: An assignment of Boolean values to the variables of F satisfying all clauses, if exists.

Examples

- The formula $(x \vee \bar{y})(\bar{x} \vee \bar{y})(x \vee y)$ is satisfiable: set $x = 1, y = 0$.
- The formula $(x \vee y \vee z)(x \vee \bar{y})(y \vee \bar{z})$ is satisfiable: set $x = 1, y = 1, z = 1$ or $x = 1, y = 0, z = 0$.
- The formula $(x \vee y \vee z)(x \vee \bar{y})(y \vee \bar{z})(z \vee \bar{x})(\bar{x} \vee \bar{y} \vee \bar{z})$ is unsatisfiable.

Satisfiability

- Classical hard problem
- Many applications: e.g., hardware/software verification, planning, scheduling
- Many hard problems are stated in terms of SAT naturally
- SAT solvers (will see later), SAT competition

- SAT is a typical search problem
- Search problem: given an instance I , find a solution S or report that none exists
- Main property: one must be able to check quickly whether S is indeed a solution for I
- By saying quickly, we mean in time polynomial in the length of I . In particular, the length of S should be polynomial in the length of I

Definition

A **search problem** is defined by an algorithm \mathcal{C} that takes an instance I and a candidate solution S , and runs in time polynomial in the length of I . We say that S is a solution to I iff $\mathcal{C}(S, I) = \text{true}$.

Example

For SAT, I is a Boolean formula, S is an assignment of Boolean constants to its variables. The corresponding algorithm \mathcal{C} checks whether S satisfies all clauses of I .

Next part

A few practical search problems for which polynomial algorithms remain unknown

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Traveling salesman problem (TSP)

Input: Pairwise distances between n cities and a budget b .

Output: A cycle that visits each vertex **exactly once** and has total length at most b .

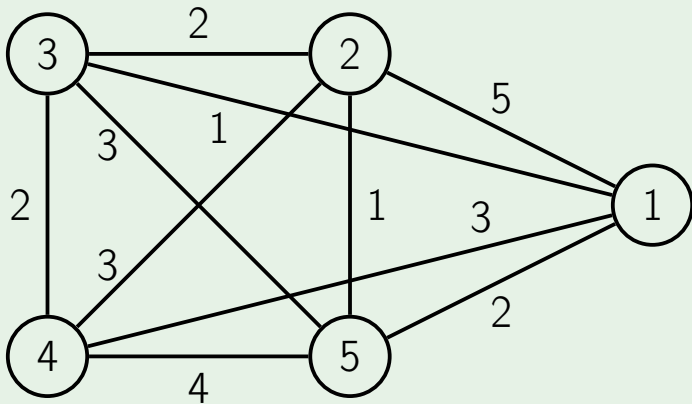
Delivery Company

[https://simple.wikipedia.org/wiki/
Travelling_salesman_problem](https://simple.wikipedia.org/wiki/Travelling_salesman_problem)

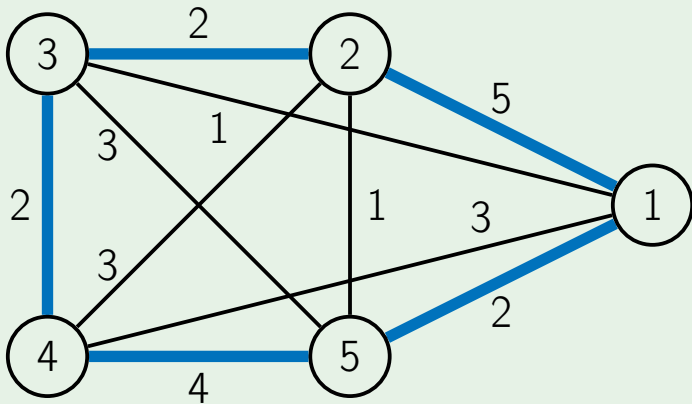
Drilling Holes in a Circuit Board

<https://developers.google.com/optimization/routing/tsp>

Example

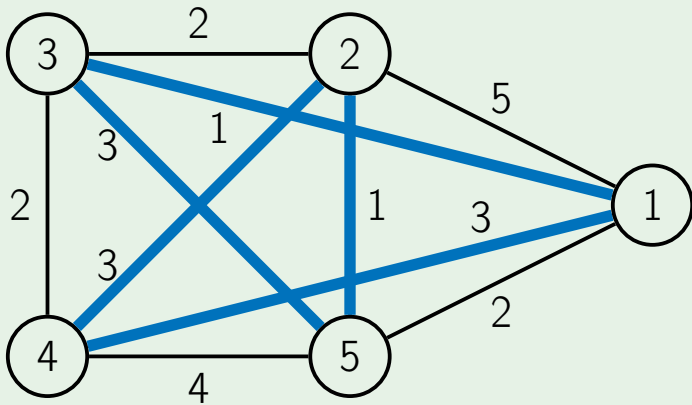


Example



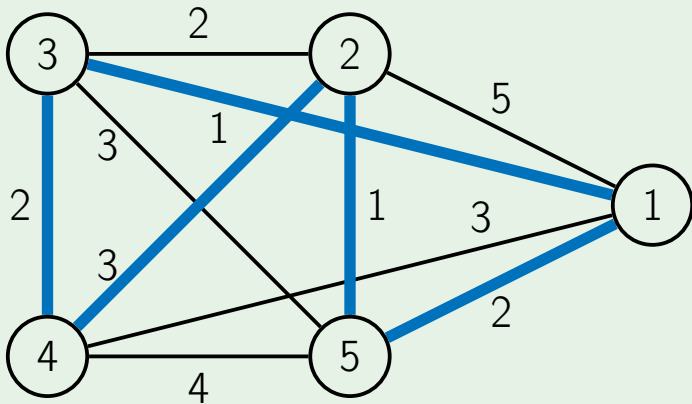
length: 15

Example



length: 13

Example



length: 9

Search Problem

- TSP is a search problem: given a sequence of vertices, it is easy to check whether it is a cycle visiting all the vertices of total length at most b
- TSP is usually stated as an optimization problem; we stated its decision version to guarantee that a candidate solution can be efficiently checked for correctness

Algorithms

- Check all permutations: about $O(n!)$, extremely slow
- Dynamic programming: $O(n^2 2^n)$
- No significantly better upper bound is known
- There are heuristic algorithms and approximation algorithms

Comparing to MST

MST

Decision version:

given n cities, connect them by $(n - 1)$ roads of minimal total length

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TSP

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No polynomial
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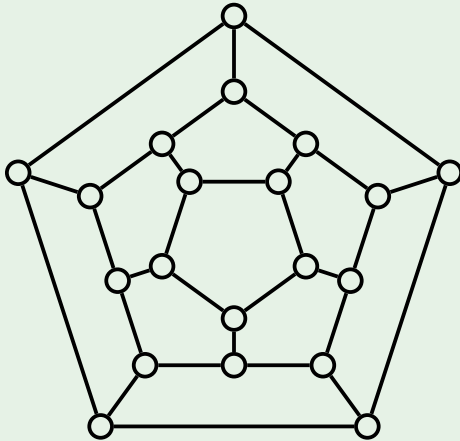
④ P and NP

Hamiltonian cycle

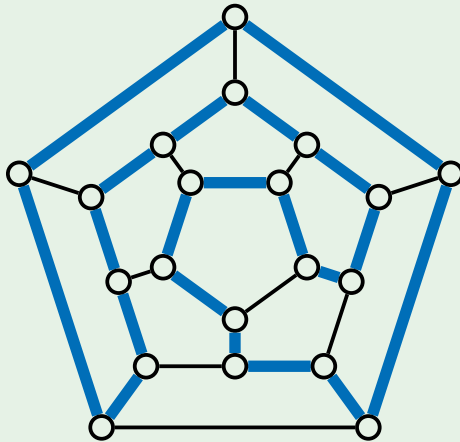
Input: A graph.

Output: A cycle that visits each vertex of the graph exactly once.

Example



Example



Eulerian cycle

Input: A graph.

Output: A cycle that visits each edge of the graph exactly once.

Eulerian cycle

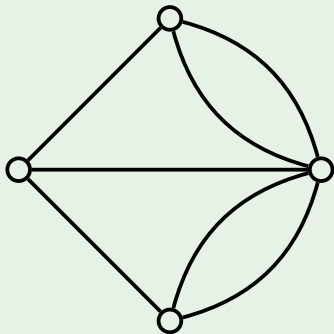
Input: A graph.

Output: A cycle that visits each edge of the graph exactly once.

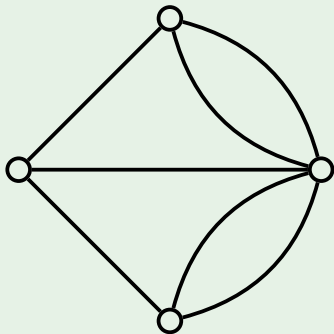
Theorem

A graph has an Eulerian cycle if and only if it is connected and the degree of each vertex is even.

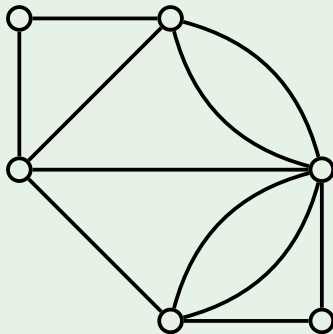
Non-Eulerian graph



Non-Eulerian graph



Eulerian graph



Eulerian cycle

Find a cycle visiting
each **edge** exactly
once

Eulerian cycle

Find a cycle visiting each **edge** exactly once

Can be solved efficiently

Eulerian cycle

Find a cycle visiting each **edge** exactly once

Hamiltonian cycle

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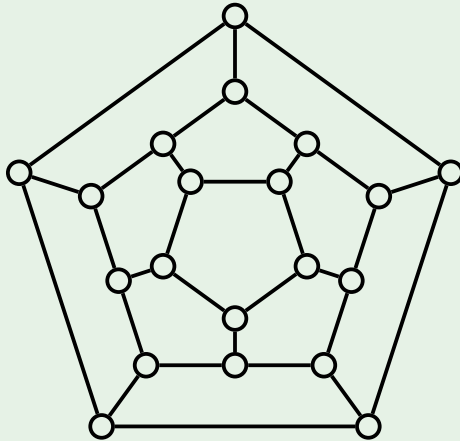
④ P and NP

Longest path

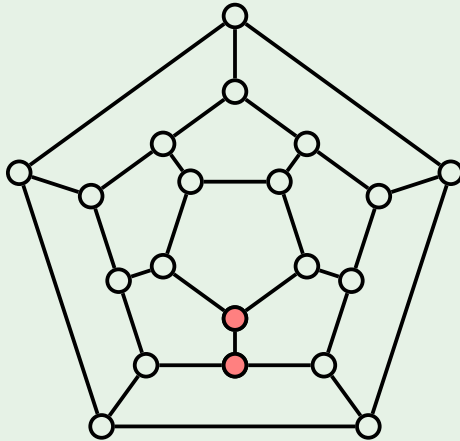
Input: A weighted graph, two vertices s , t , and a budget b .

Output: A simple path (containing no repeated vertices) of total length at least b .

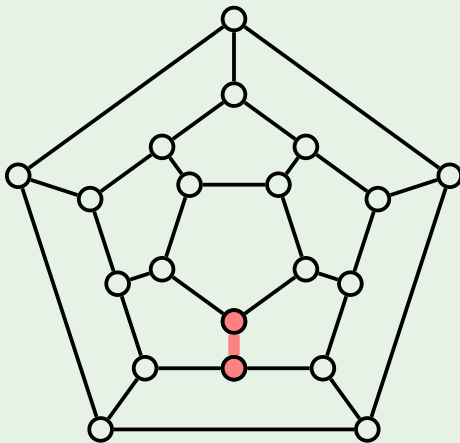
Example



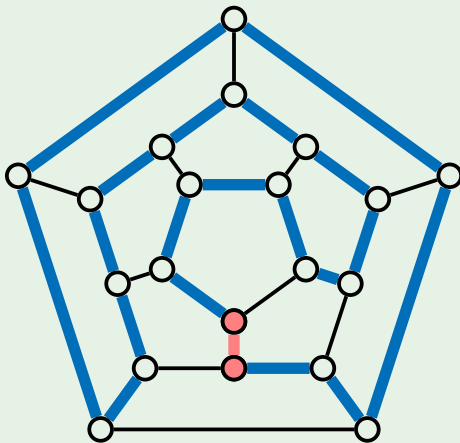
Example



Example



Example



Shortest path

Find a simple path
from s to t of total
length **at most** b

Shortest path

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Can be solved
efficiently

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Integer linear programming

Input: A set of linear inequalities $\mathbf{Ax} \leq \mathbf{b}$.

Output: Integer solution.

Example

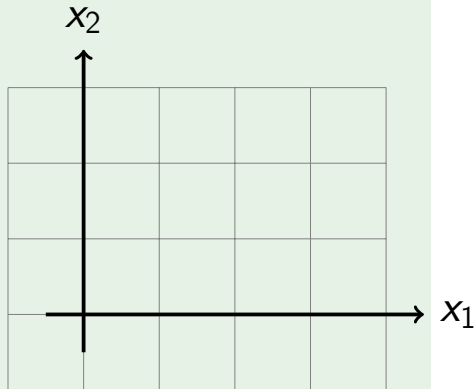
$$x_1 \geq 0.5$$

$$-x_1 + 8x_2 \geq 0$$

$$-x_1 - 8x_2 \geq -8$$

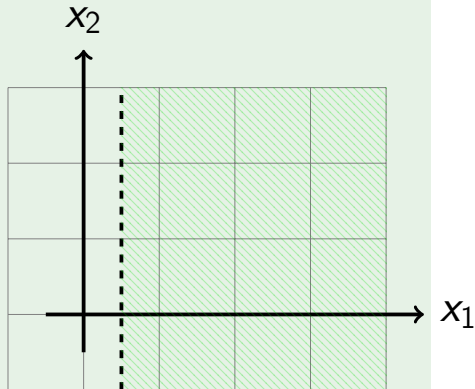
Example

$$\begin{aligned}x_1 &\geq 0.5 \\ -x_1 + 8x_2 &\geq 0 \\ -x_1 - 8x_2 &\geq -8\end{aligned}$$



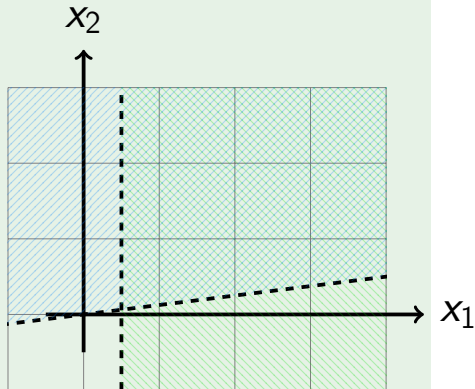
Example

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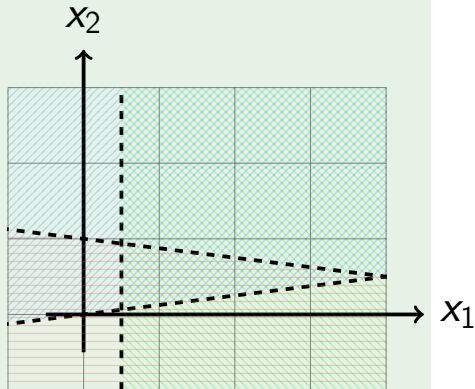
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LP (decision version)

Find a **real**
solution of a system of
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LP

(decision version)

Find a **real**
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Can be solved
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LP (decision version)

Find a **real**
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ILP

Find an **integer**
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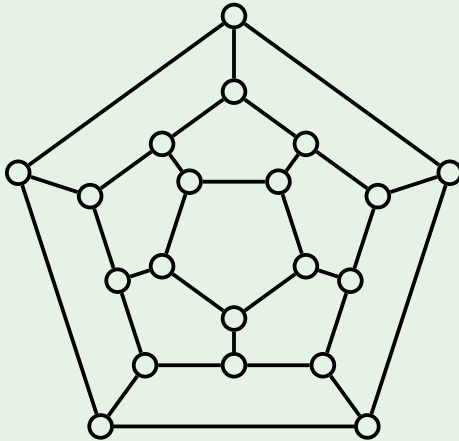
4 P and NP

Independent set

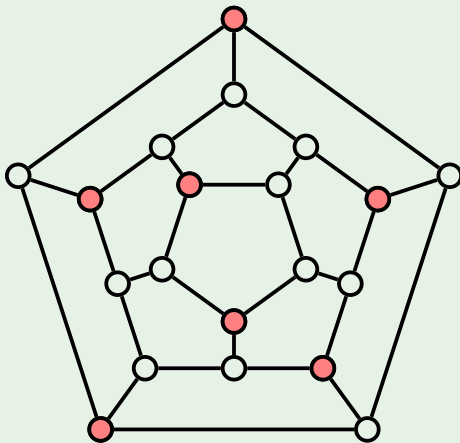
Input: A graph and a budget b .

Output: A subset of vertices of size at least b such that no two of them are adjacent.

Example

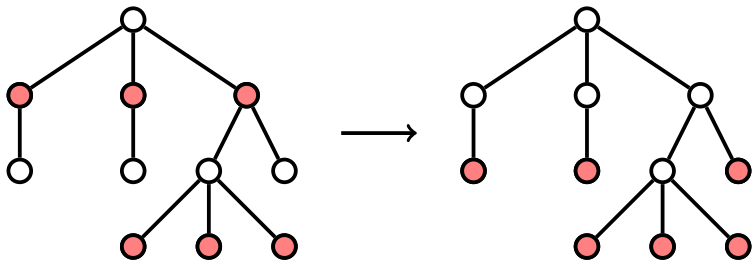


Example



Independent Sets in a Tree

A maximal independent set in a tree can be found by a simple greedy algorithm: it is safe to take into a solution all the leaves.



Independent set in a tree

Find an independent
set of size at least b in
a given tree

Independent set in a tree

Find an independent
set of size at least b in
a given tree

Can be solved
efficiently

Independent set in a tree

Find an independent
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a given **tree**

Independent set in a graph

Find an independent
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a given **graph**

Can be solved
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Independent set in a tree

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Independent set in a graph

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No polynomial
algorithm known!

Next part

It turns out that all these hard problems are in a sense a single hard problem:
a polynomial time algorithm for any of these problems can be used to solve all of them in polynomial time!

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Class NP

Definition

A **search problem** is defined by an algorithm \mathcal{C} that takes an instance I and a candidate solution S , and runs in time polynomial in the length of I . We say that S is a solution to I iff $\mathcal{C}(S, I) = \text{true}$.

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Definition

NP is the class of all search problems.

- **NP** stands for “non-deterministic polynomial time”: one can guess a solution, and then verify its correctness in polynomial time
- In other words, the class **NP** contains all problems whose solutions can be efficiently verified

Class P

Definition

P is the class of all search problems that can be solved in polynomial time.

Class P

Problems whose
solution can be
found efficiently

Class P

Problems whose solution can be found efficiently

- MST
- Shortest path
- LP
- IS on trees

Class P

Problems whose solution can be **found** efficiently

Class NP

Problems whose solution can be **verified** efficiently

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Class P

Problems whose solution can be **found** efficiently

- MST
- Shortest path
- LP
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Class NP

Problems whose solution can be **verified** efficiently

- TSP
- Longest path
- ILP
- IS on graphs

The main open problem in Computer Science

Is P equal to NP ?

The main open problem in Computer Science

Is P equal to NP ?

Millenium Prize Problem

Clay Mathematics Institute: \$1M prize for solving the problem

- If $P=NP$, then all search problems can be solved in polynomial time.

- If $P=NP$, then all search problems can be solved in polynomial time.
- If $P \neq NP$, then there exist search problems that cannot be solved in polynomial time.

Next part

We'll show that the satisfiability problem, the traveling salesman problem, the independent set problem, the integer linear programming are the hardest problems in **NP**.