Flows in Networks: The Edmonds-Karp Algorithm

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Advanced Algorithms and Complexity
Data Structures and Algorithms

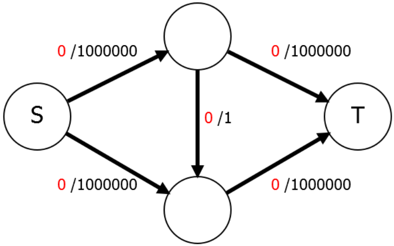
Learning Objectives

- Implement the Edmonds-Karp algorithm.
- Understand the runtime bound.

Last Time

- Ford-Fulkerson algorithm for Maxflow.
- Runtime $O(|E||f|)$.
- Sometimes very slow if graph has large capacities.

Bad Example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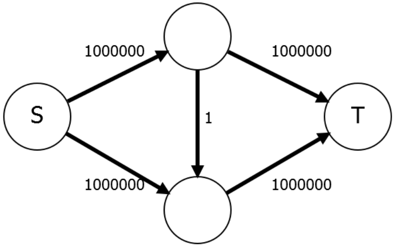
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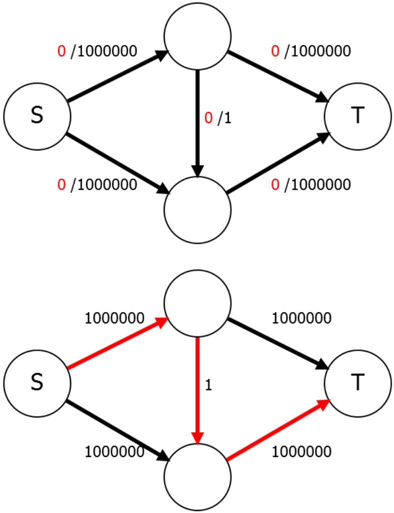

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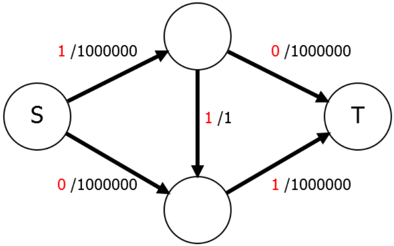

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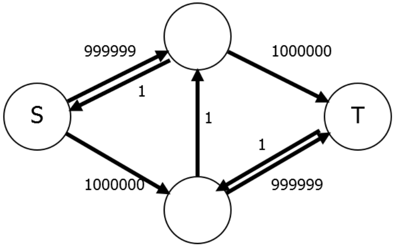

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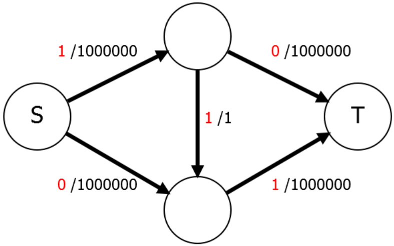
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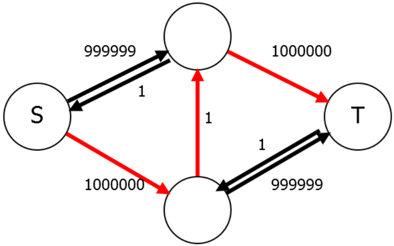
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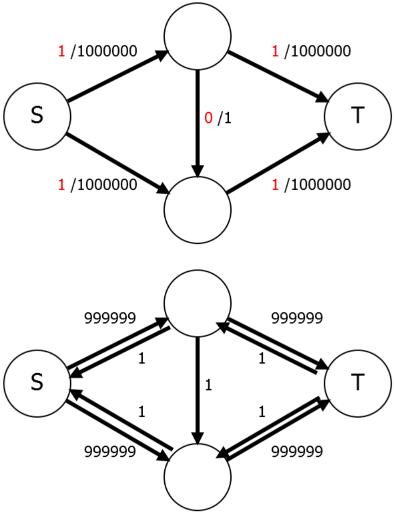

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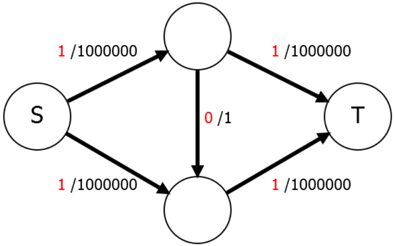

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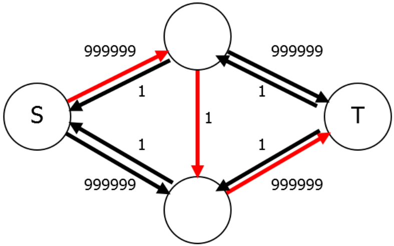

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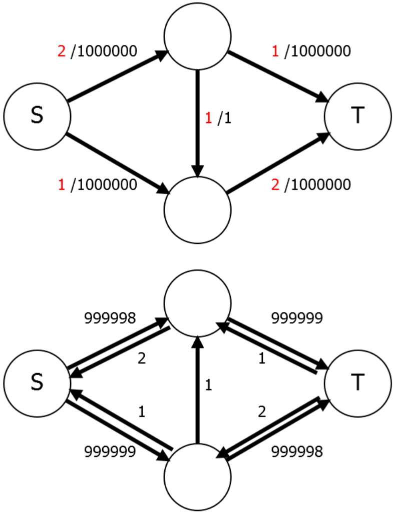

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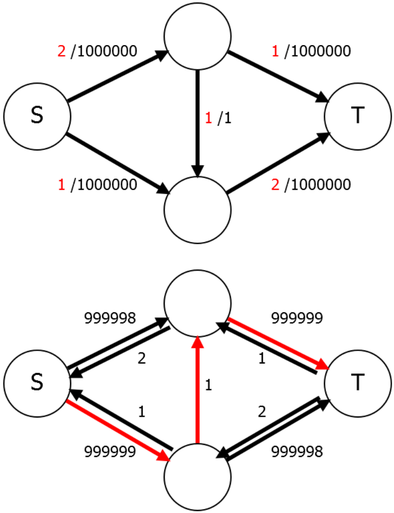

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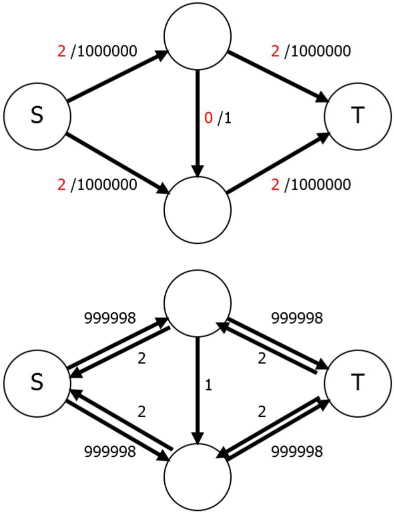

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Fix

Fortunately, the Ford-Fulkerson algorithm gives us a choice as to which augmenting path to use.

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Is there a way to choose augmenting paths to avoid this kind of runtime problem?

Edmonds-Karp Algorithm

Use the Ford-Fulkerson algorithm, always choosing the shortest (in terms of number of edges) augmenting path.

Edmonds-Karp Algorithm

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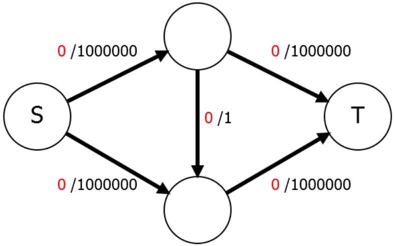
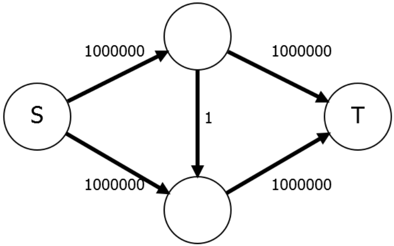
Find augmenting paths using BFS instead of DFS.

Edmonds-Karp Execution





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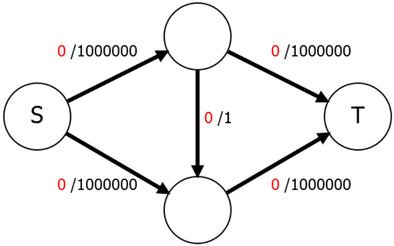

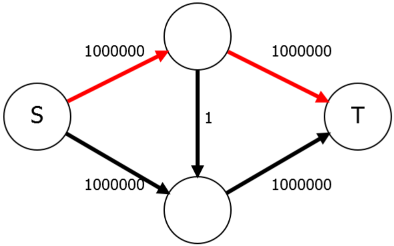

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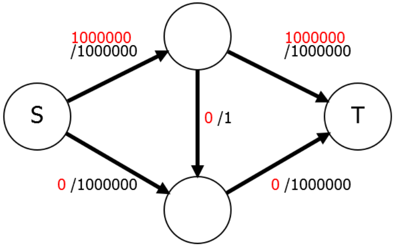

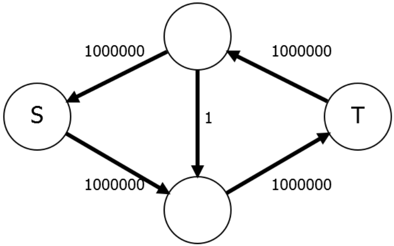

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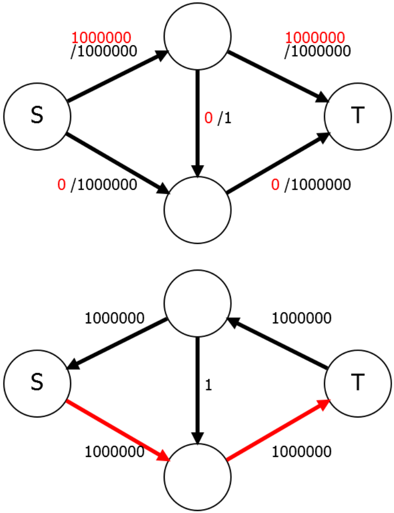

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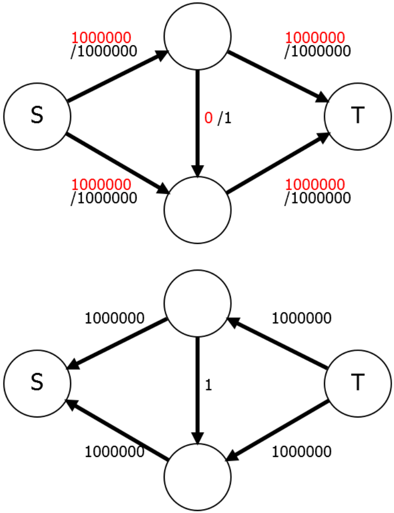

Augmenting Paths

Need to analyze augmenting paths.

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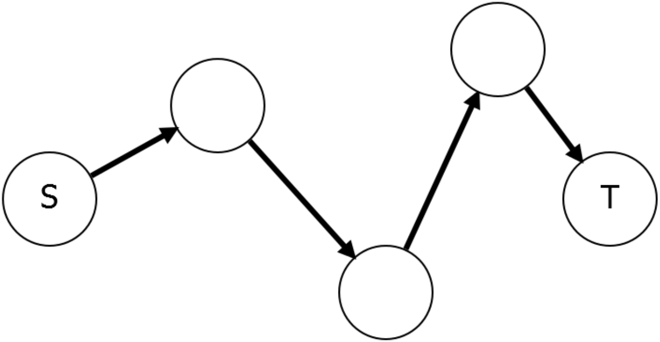
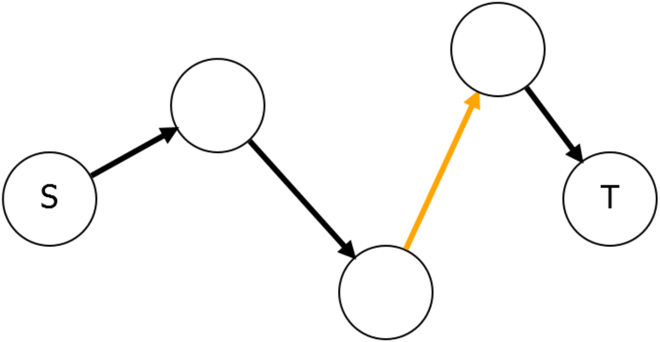


Augmenting Paths

Augmenting flow always saturates (uses all the available flow from) an edge.



Augmenting Paths

Changes to residual network.

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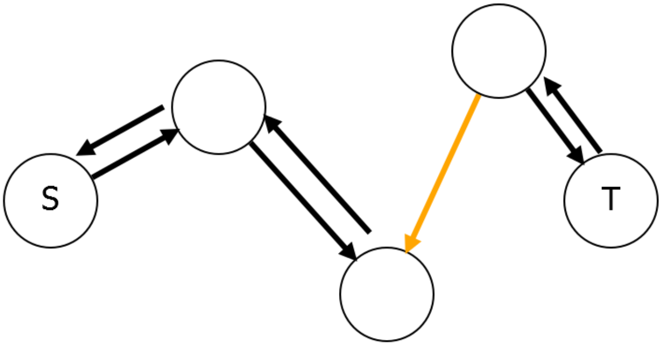


Analysis Idea

- Show that no edge is saturated too many times.
- Fails to hold in the bad case, where the middle edge is repeatedly saturated.

Increasing Distances

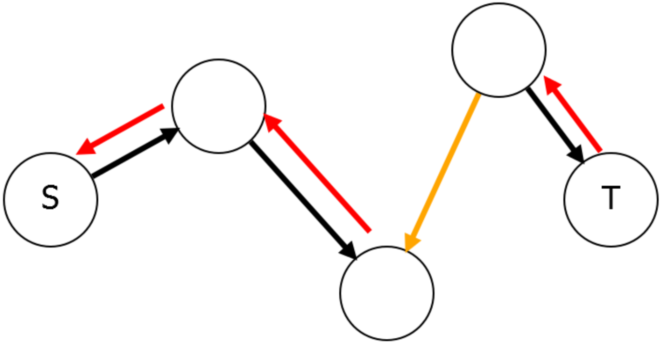
We will need this critical Lemma:

Lemma

As the Edmonds-Karp algorithm executes, for any vertex $v \in V$ the distance $d_{G_f}(s, v)$ only increases.
Similarly, $d_{G_f}(v, t)$ and $d_{G_f}(s, t)$ can only increase.

Proof

- New edges all point backwards along augmenting path.



Proof

- Augmenting path is a shortest path.

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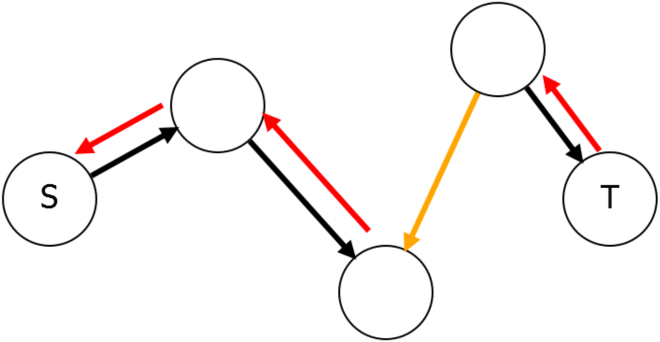
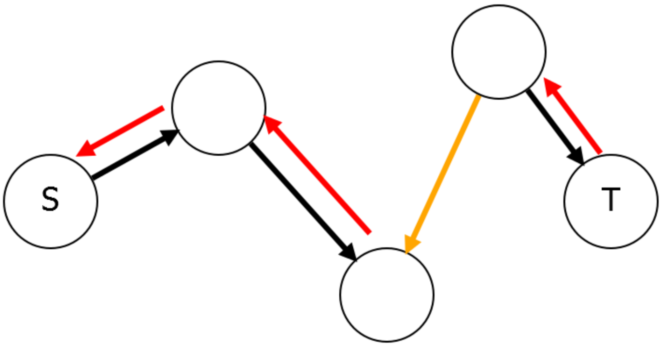


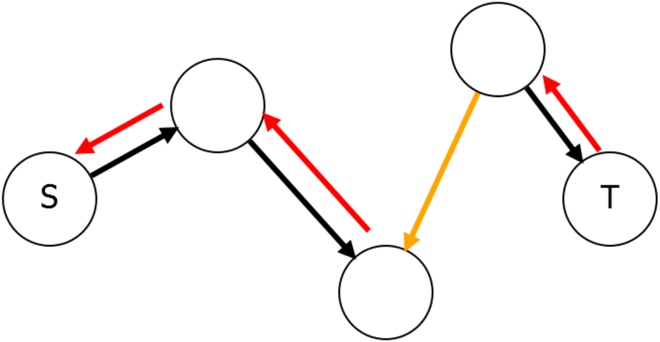
Proof

- All new edges point from vertices further from s to vertices closer.



Proof

- New edges do not help you get from s to v faster.



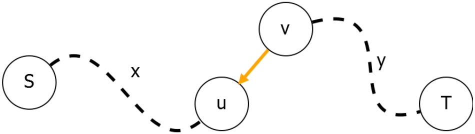
Reuse Limit

Lemma

When running the Edmonds-Karp algorithm, if an edge e is saturated, it will not be used in an augmenting path again, until $d_{G_f}(s, t)$ increases.

Proof

- Initially, $d(s, u) = x, d(v, t) = y, d(s, t) = x + y + 1$.
- When used again, $d(s, v) \geq x + 1$, $d(u, t) \geq y + 1$.
- Therefore, when used again, $d(s, t) \geq (x+1)+(y+1)+1 = x+y+3$.



Analysis

- $d_{G_f}(s, t)$ can only increase $|V|$ times.
- Each time can only have $O(|E|)$ many saturated edges.
- Therefore, only $O(|V||E|)$ many augmenting path.
- Each path takes $O(|E|)$ time.
- Total runtime: $O(|V||E|^2)$.

Problem

Which of the following is true about the Edmonds-Karp algorithm:

1. No edge is saturated more than $|V|$ times.
2. The lengths of the augmenting paths decrease as the algorithm progresses.
3. Changing the capacities of edges will not affect the runtime.

Solution

Which of the following is true about the Edmonds-Karp algorithm:

1. No edge is saturated more than $|V|$ times.
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Summary

- Choose augmenting paths based on path length.
- Removes runtime dependence on numerical sizes of capacities.

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- Choose augmenting paths based on path length.
- Removes runtime dependence on numerical sizes of capacities.
- There are better, much more complicated algorithms.
- State of the art $O(|V||E|)$.

Next Time

Applications of Maxflow algorithms.