

# Flows in Networks: Maxflow-Mincut

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# Learning Objectives

- Understand the relationship between flows and cuts.
- Produce a cut with size matching that of a maximum flow.
- Identify when a flow is maximum.

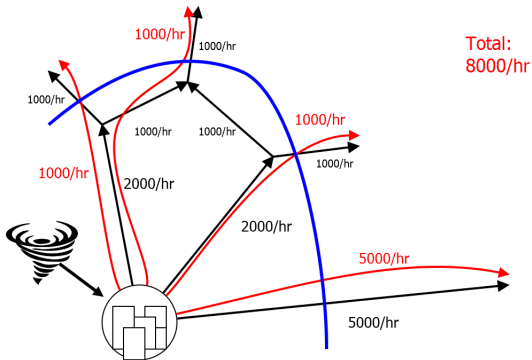
# Problem

In order to find maxflows, we need a way of verifying that they are optimal.

In particular, we need techniques for bounding the size of the maxflow.

# Idea

Recall our original example:



# Idea

Find a **bottleneck** for the flow. All flow needs to cross the bottleneck.

# Cuts

## Definition

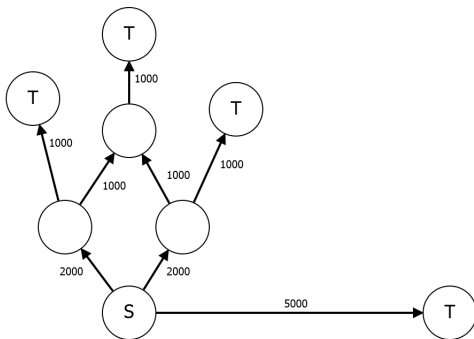
Given a network  $G$ , a **cut**  $\mathcal{C}$ , is a set of vertices of  $G$  so that  $\mathcal{C}$  contains all sources of  $G$  and no sinks of  $G$ .

The **size** of a cut is given by

$$|\mathcal{C}| := \sum_{e \text{ out of } \mathcal{C}} c_e$$

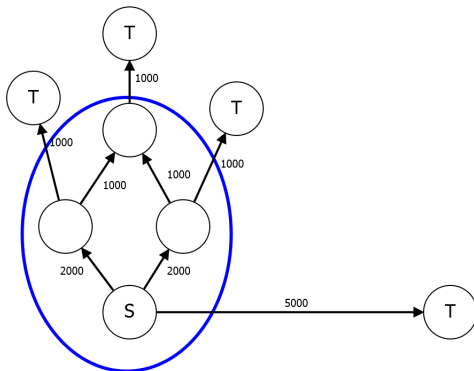
# Example

Network  $G$ .



# Example

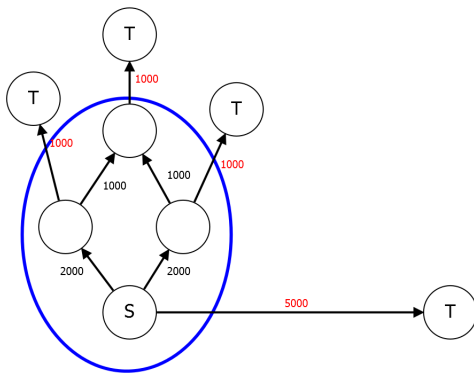
Cut  $\mathcal{C}$ .





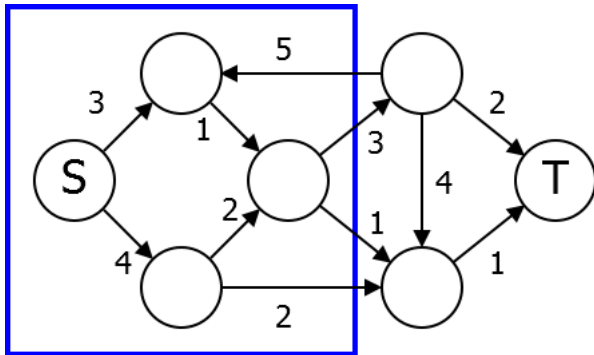
# Example

Edges cut. Total size 8000.



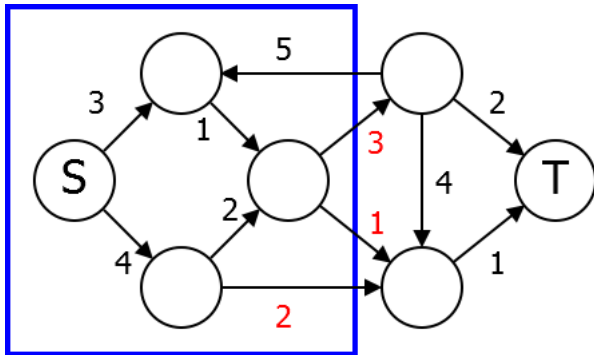
# Problem

What is the size of the cut below?



# Solution

$$1 + 2 + 3 = 6.$$



# Bound

## Lemma

Let  $G$  be a network. For any flow  $f$  and any cut  $\mathcal{C}$ ,

$$|f| \leq |\mathcal{C}|.$$

# Proof

$$\begin{aligned} |f| &= \sum_{v \text{ source}} \left( \sum_{e \text{ out of } v} f_e - \sum_{e \text{ into } v} f_e \right) \\ &= \sum_{v \in \mathcal{C}} \left( \sum_{e \text{ out of } v} f_e - \sum_{e \text{ into } v} f_e \right) \\ &= \sum_{e \text{ out of } \mathcal{C}} f_e - \sum_{e \text{ into } \mathcal{C}} f_e \\ &\leq \sum_{e \text{ out of } \mathcal{C}} C_e = |\mathcal{C}|. \end{aligned}$$

# Bounds

In other words, for any cut  $\mathcal{C}$ , we get an upper bound on the maxflow. In particular,

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Question: Is this bound good enough?  
Surprisingly, it is...



# Maxflow-Mincut

## Theorem

For any network  $G$ ,

$$\max_{\text{flows } f} |f| = \min_{\text{cuts } \mathcal{C}} |\mathcal{C}|.$$

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## Theorem

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In other words, there is always a cut small enough to give the correct upper bound.

# A Special Case

What happens when  $\text{Maxflow} = 0$ ?

- There is **no** path from source to sink.
- Let  $\mathcal{C}$  be the set of vertices reachable from sources.
- There are no edges out of  $\mathcal{C}$ .
- So  $|\mathcal{C}| = 0$ .

# The General Case

- Let  $f$  be a maxflow for  $G$ .
- Note that  $G_f$  has maxflow 0.
- There is a cut  $\mathcal{C}$  with size 0 in  $G_f$ .
- Claim:  $|\mathcal{C}| = |f|$ .

# Proof

$$\begin{aligned} |f| &= \sum_{e \text{ out of } \mathcal{C}} f_e - \sum_{e \text{ into } \mathcal{C}} f_e \\ &= \sum_{e \text{ out of } \mathcal{C}} C_e - \sum_{e \text{ into } \mathcal{C}} 0 \\ &= |\mathcal{C}|. \end{aligned}$$

# Conclusion

- We have found an  $f$  and  $\mathcal{C}$  with  $|f| = |\mathcal{C}|$ .
- By Lemma, cannot have larger  $|f|$  or smaller  $|\mathcal{C}|$ .
- So  $\max |f| = \min |\mathcal{C}|$ .

# Summary

- Can always check if flow is maximal by finding matching cut.
- $f$  a maxflow only if there is no source-sink path in  $G_f$ .
- We will use this in an algorithm next time.