Algorithmic Challenges: From Suffix Array to Suffix Tree

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Algorithms on Strings
Algorithms and Data Structures at edX

Outline

Construct suffix Tree

Input: String S

Output: Suffix tree of S

- You already know how to construct suffix tree
- But $O(|S|^2)$ will only work for short strings
- You will learn to build it in $O(|S| \log |S|)$ which enables very long texts!

General Plan

- Construct suffix array in $O(|S| \log |S|)$
- Compute additional information in $O(|S|)$
- Construct suffix tree from suffix array and additional information in $O(|S|)$

Suffix array and suffix tree

S = ababaa$

0 $

1 a$

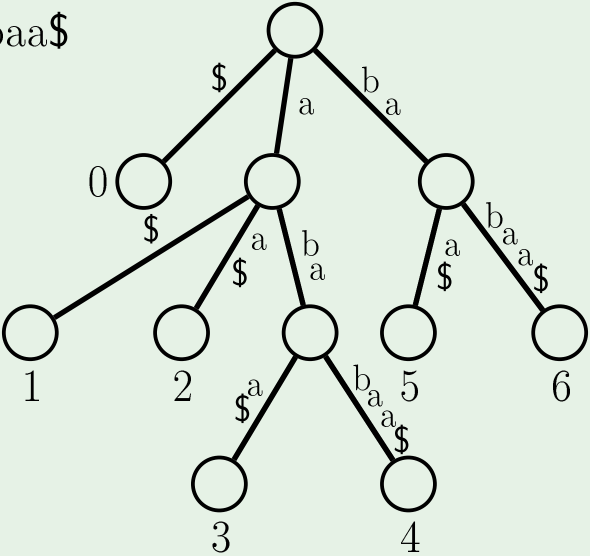
2 aa$

3 abaa$

4 ababaa$

5 baa$

6 babaa$



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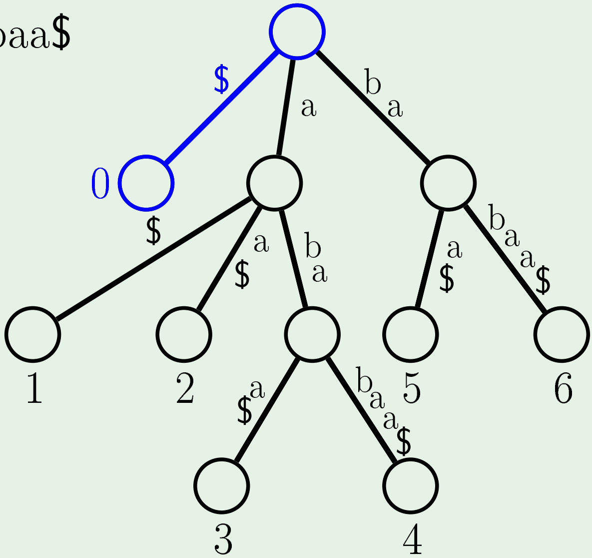

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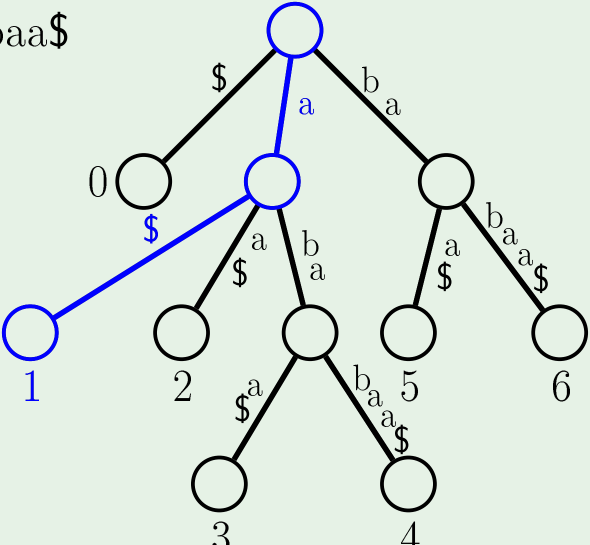

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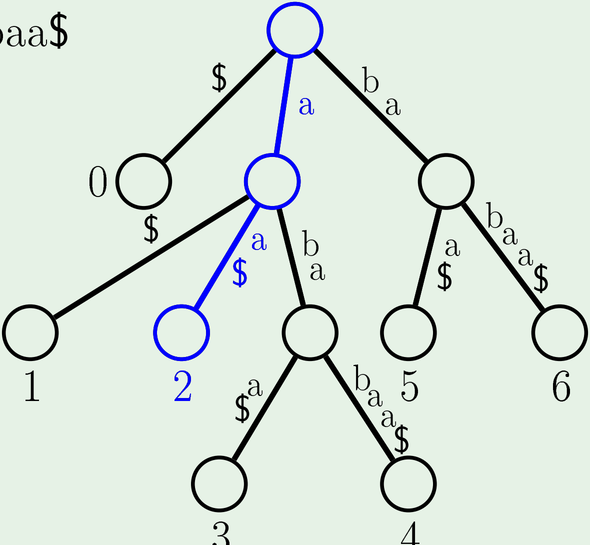

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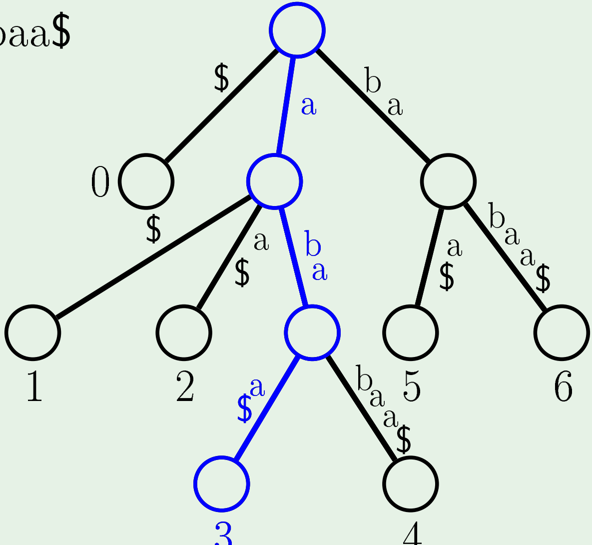

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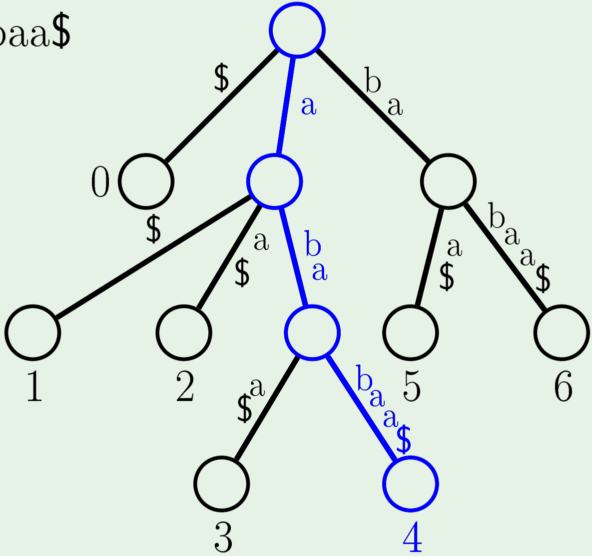

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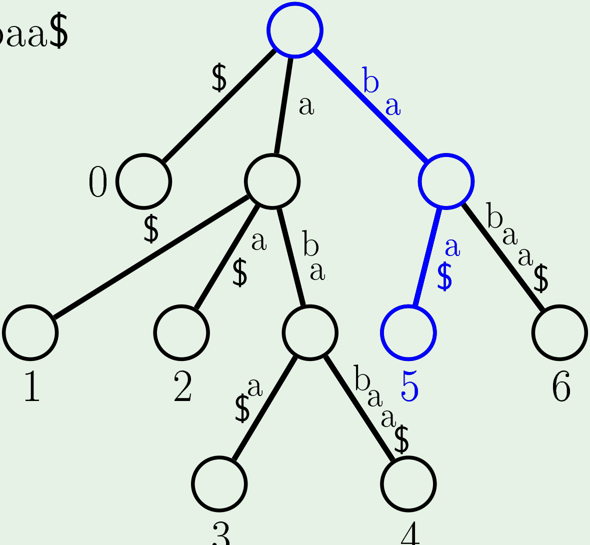

Suffix array and suffix tree

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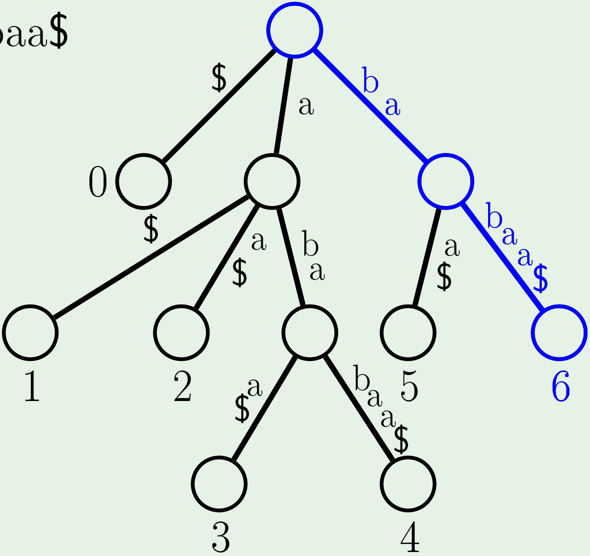

Definition

The longest common prefix (or just “lcp”) of two strings S and T is the longest such string u that u is both a prefix of S and T. We denote by LCP(S, T) the length of the “lcp” of S and T.

Example

$\mathrm{LCP}(\text{“ababc”}, \text{“abc”}) = 2$

$\mathrm{LCP}(\text{“a”}, \text{“b”}) = 0$

Suffix array, suffix tree and lcp

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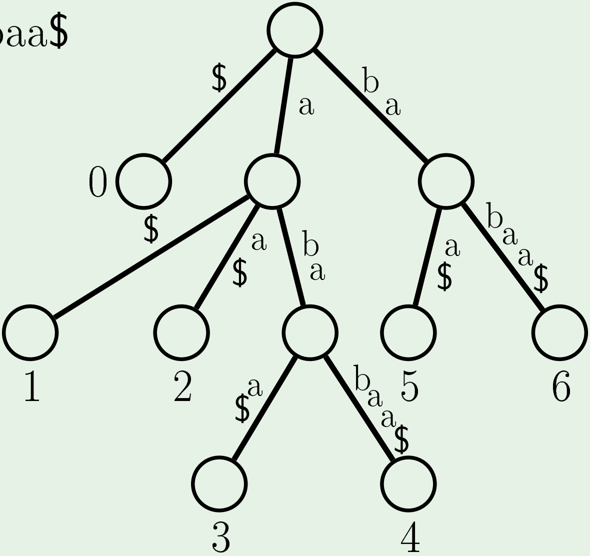

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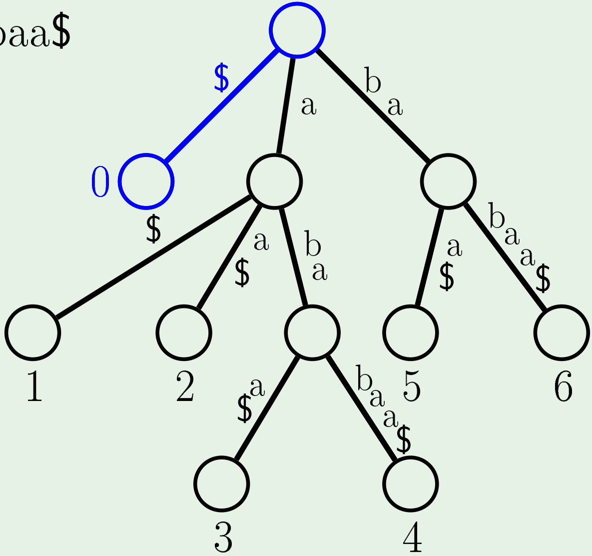

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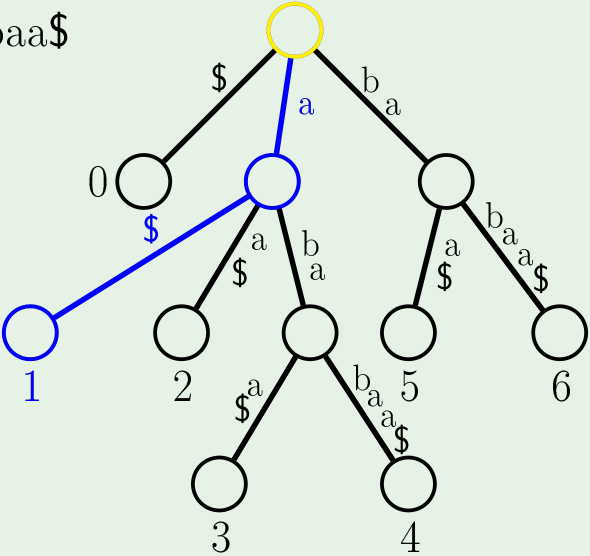

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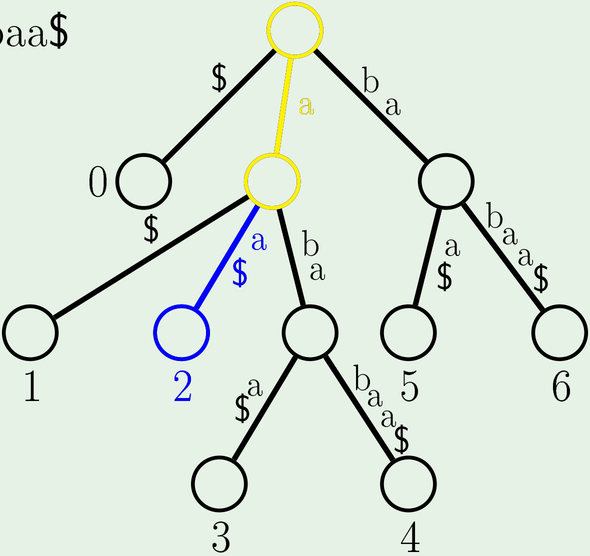

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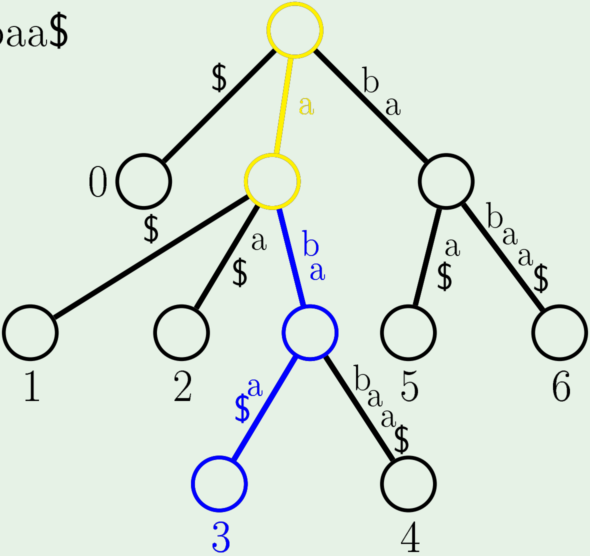

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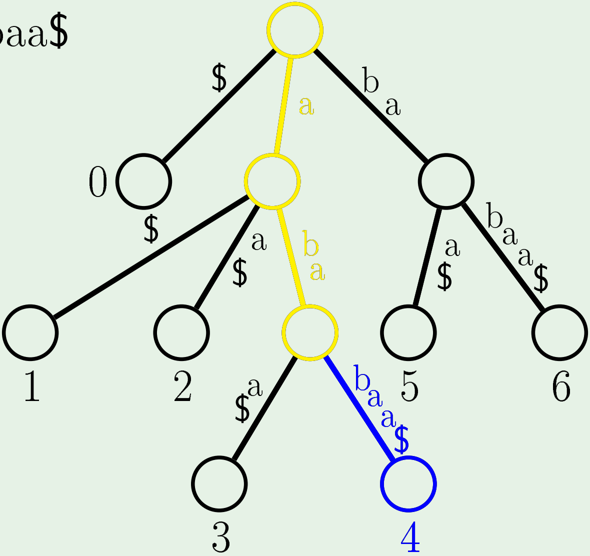

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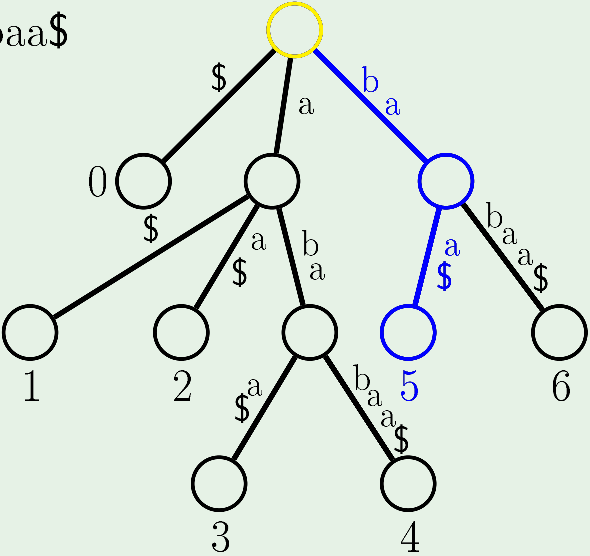

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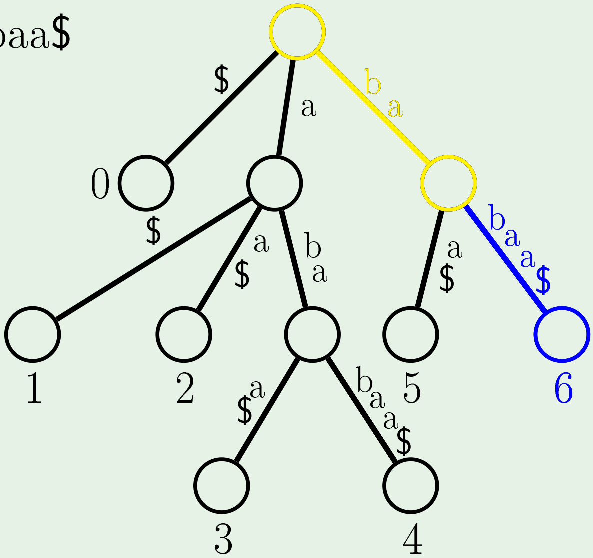

LCP array

Definition

Consider suffix array A of string S in the raw form, that is $A[0] < A[1] < A[2] < \cdots < A[|S| - 1]$ are all the suffixes of S in lexicographic order. LCP array of string S is the array lcp of size $|S| - 1$ such that for each i such that $0 \leq i \leq |S| - 2$,

$$lcp[i] = LCP(A[i], A[i + 1])$$

LCP array

$S = \text{ababaa\$}$

0 \$

1 a\$

2 aa\$

3 abaa\$

4 ababaa\$

5 baa\$

6 babaa\$

$\text{lcp} = [\ ,\ ,\ ,\ ,\ ,\ ]$

LCP array

S = ababaa$

0 $

1 a$

2 aa$

3 abaa$

4 ababaa$

5 baa$

6 babaa$

lcp = [, , , , ,]

LCP array

$S = ababaa\$$

0 $

1 a$

2 aa$

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$lcp = [0, \ , \ , \ , \ , \ ]$

LCP array

$S = \text{ababaa\$}$

0 \$

1 a\$

2 aa\$

3 abaa\$

4 ababaa\$

5 baa\$

6 babaa\$

$\text{lcp} = [0, 1, \ , \ , \ , \ ]$

LCP array

$S = ababaa\$$

0 $\$$

1 a$\$$ $\qquad$ $lcp = [0, 1, 1, \ , \ , \ ]$

2 aa$\$$

3 abaa$\$$

4 ababaa$\$$

5 baa$\$$

6 babaa$\$$

LCP array

$S = \text{ababaa\$}$

0 \$

1 a\$

2 aa\$

3 abaa\$

4 ababaa\$

5 baa\$

6 babaa\$

$$\text{lcp} = [0, 1, 1, 3, \ , \ ]$$

LCP array

$S = ababaa\$$

0 $

1 a$

2 aa$

3 abaa$

4 ababaa$

5 baa$

6 babaa$

$\text{lcp} = [0, 1, 1, 3, 0, \quad]$

LCP array

S = ababaa$

0 $

1 a$

2 aa$

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4 ababaa$

5 baa$

6 babaa$

lcp = [0, 1, 1, 3, 0, 2]

LCP array property

Lemma

For any $i < j$, $LCP(A[i], A[j]) \leq lcp[i]$ and $LCP(A[i], A[j]) \leq lcp[j-1]$.

Proof

	$\ldots$
i	abababa ba
i + 1	ababab c
	$\ldots$
j	abbcabab

Proof

	$\dots$
i	ababababa
i + 1	abababc
	$\dots$
j	abbcabab

Proof

	$\ldots$
i	ababababa
$i+1$	xxxxxxxxx
	$\ldots$
j	abbcabab

If $\mathrm{LCP}(A[i], A[j]) > \mathrm{LCP}(A[i], A[i+1])$

Proof

$$\begin{array}{ll} & \ldots \\ \mathrm{i} & \mathrm{abababab a} \\ \mathrm{i}+1 & \mathrm{xxxxxxxxx} \quad k = 1 \\ & \ldots \\ \mathrm{j} & \mathrm{abbcabab} \end{array}$$

If $\mathrm{LCP}(A[i], A[j]) > \mathrm{LCP}(A[i], A[i+1])$
Consider $k = \mathrm{LCP}(A[i], A[i+1])$

Proof

$\dots$

i	ababababa	
$i+1$	a_	$k = 1$
	$\dots$	
j	abbcabab	

If $k = |A[i+1]|$, then $A[i+1] < A[i]$ – contradiction

Proof

$\dots$

i	abababab a
i + 1	axxxxxxxx $k = 1$
	$\dots$
j	abbcabab

Otherwise $A[j][k] = A[i][k] \neq A[i+1][k]$

Proof

$$\begin{array}{lll} & \ldots & \\ i & \text{abababab} & \\ i+1 & \text{acxxxxxxx} & k=1 \\ & \ldots \bigvee & \\ j & \text{abbcabab} & \end{array}$$

If $A[j][k] = A[i][k] < A[i+1][k]$, then $A[j] < A[i+1]$ — contradiction

Proof

$\dots$

i ababababa

$\vee$

$i+1$ aaxxxxxxx $k = 1$

$\dots$

j abbcabab

If $A[i][k] > A[i+1][k]$, then $A[i] > A[i+1]$ — contradiction $\square$

Computing LCP array

- For each i, compute $LCP(A[i], A[i+1])$ via comparing $A[i]$ and $A[i+1]$ character-by-character
- $O(|S|)$ for each i, $O(|S|)$ different i — total time $O(|S|^2)$
- How to do this faster?

Outline

Idea

Lemma

Let h be the longest common prefix between S_{i-1} and its adjacent (next) suffix in the suffix array of string S. Then the longest common prefix between S_i and its adjacent (next) suffix in the suffix array is at least $h - 1$.

$$S = \text{abracadabra\$}$$

index	sorted suffix	LCP
$\ldots$	$\ldots$	$\ldots$
$i = 10$	a$	
7	abra$	
$\ldots$	$\ldots$	$\ldots$
$j = 3$	acadabra$	
$\ldots$	$\ldots$	$\ldots$
$i - 1 = 9$	ra$	
$j - 1 = 2$	racadabra$	

$$S = \text{abracadabra\$}$$

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$$S = \text{abracadabra\$}$$

index	sorted suffix	LCP
$\dots$	$\dots$	$\dots$
$i = 10$	a\$	$1 \geq h - 1$
7	abra\$	
$\dots$	$\dots$	$\dots$
$j = 3$	acadabra\$	
$\dots$	$\dots$	$\dots$
$i - 1 = 9$	ra\$	$h = 2$
$j - 1 = 2$	racadabra\$	

Idea

- Start by computing $\mathrm{LCP}(A[0], A[1])$ directly
- Instead of computing to $\mathrm{LCP}(A[1], A[2])$, move $A[0]$ one position to the right in the string, get some $A[k]$ and compute $\mathrm{LCP}(A[k], A[k+1])$
- Repeat this until LCP array is fully computed
- Length of the LCP never decreases by more than one each iteration

Notation

- Let $A_{n(i)}$ be the suffix starting in the next position in the string after A[i]

Example

a	b	a	b	d	a	b	c

- $A[0] =$ "ababdabc", $A[1] =$ "abc"
- Compute $\mathrm{LCP}(A[0], A[1]) = 2$ directly
- $\mathrm{LCP}(A_{n(0)}, A_{n(1)}) \geq \mathrm{LCP}(A[0], A[1]) - 1$
- $A[0] < A[1] \Rightarrow A_{n(0)} < A_{n(1)}$
- LCP of $A_{n(0)}$ with the next in order $A[j]$ is also at least $\mathrm{LCP}(A[0], A[1]) - 1$

Example

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- $A[0] < A[1] \Rightarrow A_{n(0)} < A_{n(1)}$
- LCP of $A_{n(0)}$ with the next in order $A[j]$ is also at least $\mathrm{LCP}(A[0], A[1]) - 1$
- Compute $\mathrm{LCP}(A_{n(0)}, A[j])$ directly,

Algorithm

- Compute $\text{LCP}(A[0], A[1])$ directly, save as lcp
- First suffix goes to the next in the string
- Second suffix is the next in the order
- Compute LCP knowing that first $\text{lcp} - 1$ characters are equal, save lcp
- Repeat

LCPOfSuffixes(S, i, j, equal)

```
lcp ← max(0, equal)
while i + lcp < |S| and j + lcp < |S|:
    if S[i + lcp] == S[j + lcp]:
        lcp ← lcp + 1
    else:
        break
return lcp
```

InvertSuffixArray(order)

```
pos ← array of size |order|
for i from 0 to |pos| − 1:
    pos[order[i]] ← i
return pos
```

ComputeLCPArray(S, order)

```
lcpArray ← array of size |S| − 1
lcp ← 0
posInOrder ← InvertSuffixArray(order)
suffix ← order[0]
for i from 0 to |S| − 1:
   orderIndex ← posInOrder[suffix]
   if orderIndex == |S| − 1:
      lcp ← 0
      suffix ← (suffix + 1) mod |S|
      continue
   nextSuffix ← order[orderIndex + 1]
   lcp ← LCPOfSuffixes(S, suffix, nextSuffix, lcp − 1)
   lcpArray[orderIndex] ← lcp
   suffix ← (suffix + 1) mod |S|
return lcpArray
```

Analysis

Lemma

This algorithm computes LCP array in $O(|S|)$

Proof

- Each comparison increases lcp
- lcp $\leq |S|$
- Each iteration lcp decreases by at most 1
- Number of comparisons is $O(|S|)$ $\square$

Outline

Building suffix tree

$S = \text{ababaa\$}$

6 \$

5 a\$

4 aa\$

2 abaa\$

0 ababaa\$

3 baa\$

1 babaa\$

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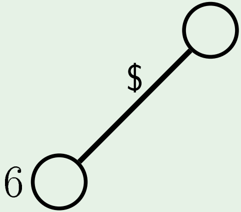

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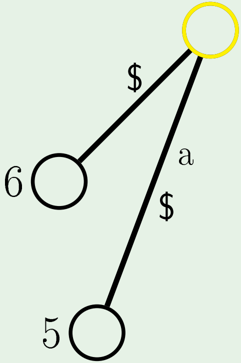

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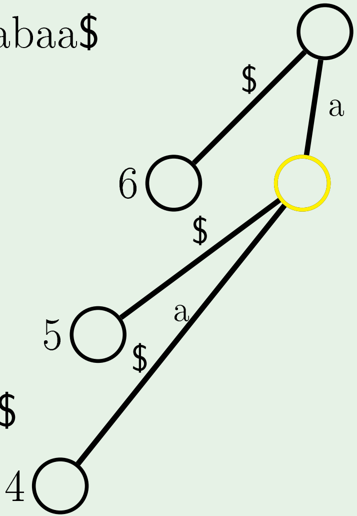

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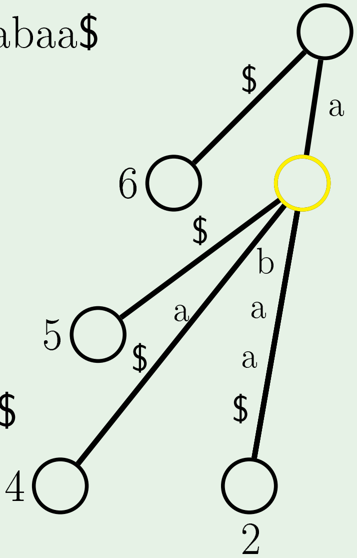

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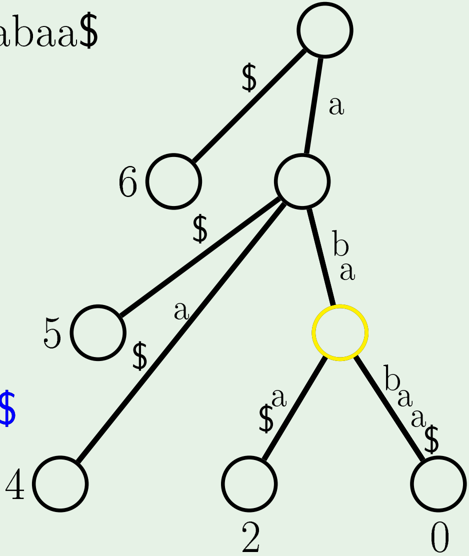

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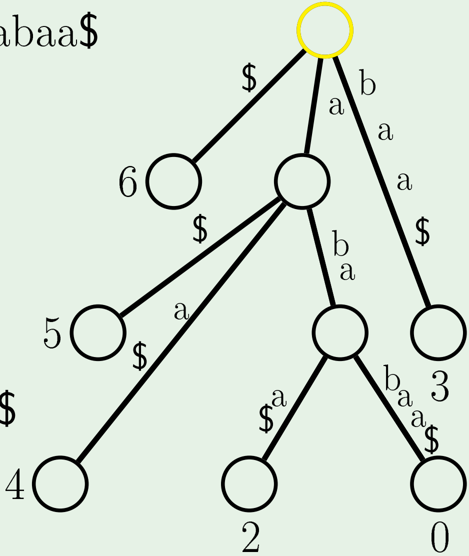

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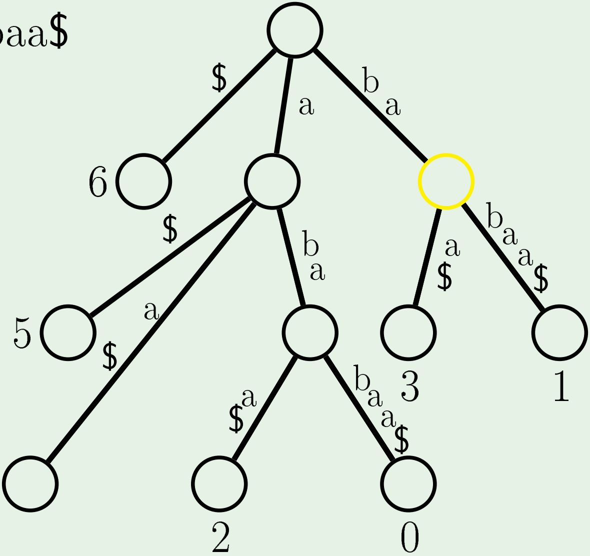

Algorithm

- Build suffix array and LCP array
- Start from only root vertex
- Grow first edge for the first suffix
- For each next suffix, go up from the leaf until LCP with previous is below
- Build a new edge for the new suffix

class SuffixTreeNode:

SuffixTreeNode parent
Map<char, SuffixTreeNode> children
integer stringDepth
integer edgeStart
integer edgeEnd

STFromSA(S, order, lcpArray)

```
root ← new SuffixTreeNode(
   children = {}, parent = nil, stringDepth = 0,
   edgeStart = −1, edgeEnd = −1)
lcpPrev ← 0
curNode ← root
for i from 0 to |S| − 1:
   suffix ← order[i]
   while curNode.stringDepth > lcpPrev:
      curNode ← curNode.parent
   if curNode.stringDepth == lcpPrev:
      curNode ← CreateNewLeaf(curNode, S, suffix)
   else:
      edgeStart ← order[i − 1] + curNode.stringDepth
      offset ← lcpPrev − curNode.stringDepth
      midNode ← BreakEdge(curNode, S, edgeStart, offset)
      curNode ← CreateNewLeaf(midNode, S, suffix)
   if i < |S| − 1:
      lcpPrev ← lcpArray[i]
return root
```

CreateNewLeaf(node, S, suffix)

```
leaf ← new SuffixTreeNode(
    children = {},
    parent = node,
    stringDepth = |S| − suffix,
    edgeStart = suffix + node.stringDepth,
    edgeEnd = |S| − 1)
node.children[S[leaf.edgeStart]] ← leaf
return leaf
```

BreakEdge(node, S, start, offset)

startChar ← S[start]
midChar ← S[start + offset]
midNode ← new SuffixTreeNode(
    children = {} ,
    parent = node,
    stringDepth = node.stringDepth + offset,
    edgeStart = start,
    edgeEnd = start + offset − 1)
midNode.children[midChar] ← node.children[startChar]
node.children[startChar].parent ← midNode
node.children[startChar].edgeStart+ = offset
node.children[startChar] ← midNode
return midNode

Analysis

Lemma

This algorithm runs in $O(|S|)$

Proof

- Total number of edges in suffix tree is $O(|S|)$
- For each edge, we go at most once down and at most once up
- Constant time to create a new edge and possibly a new node

Conclusion

- Can build suffix tree from suffix array in linear time
- Can build suffix tree from scratch in time $O(|S| \log |S|)$