

# NP-complete Problems: Reductions

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Advanced Algorithms and Complexity  
Data Structures and Algorithms

# Outline

- 1 Reductions
- 2 Showing **NP**-completeness
- 3 Independent Set  $\rightarrow$  Vertex Cover
- 4 3-SAT  $\rightarrow$  Independent Set
- 5 SAT  $\rightarrow$  3-SAT
- 6 All of **NP**  $\rightarrow$  SAT
- 7 Using SAT-solvers

# Informally

We say that a search problem  $A$  is reduced to a search problem  $B$  and write  $A \rightarrow B$ , if a polynomial time algorithm for  $B$  can be used (as a black box) to solve  $A$  in polynomial time.

Reduction:  $A \rightarrow B$

instance  $I$  of  $A$

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Algorithm for  $A$

Algorithm for  $B$

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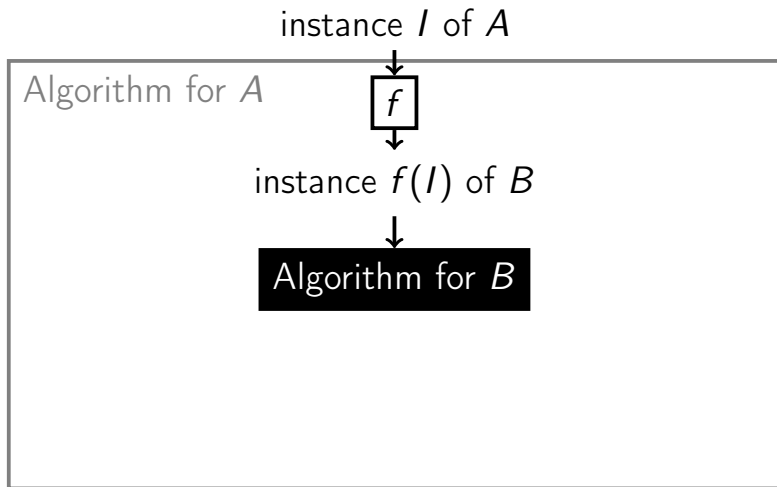
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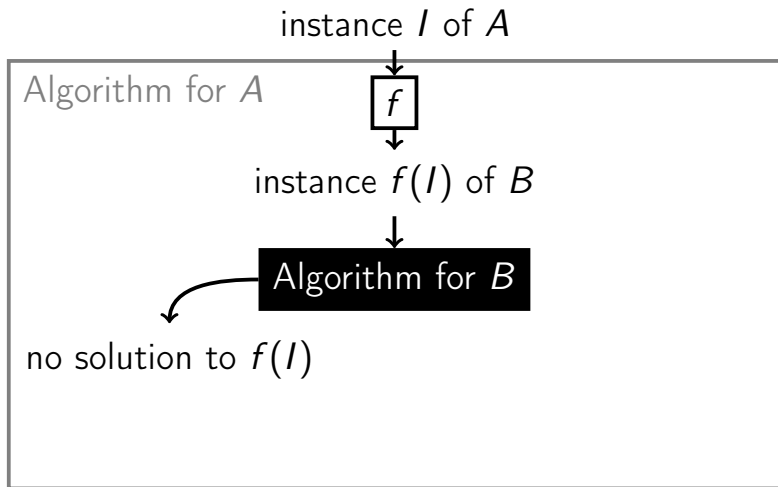
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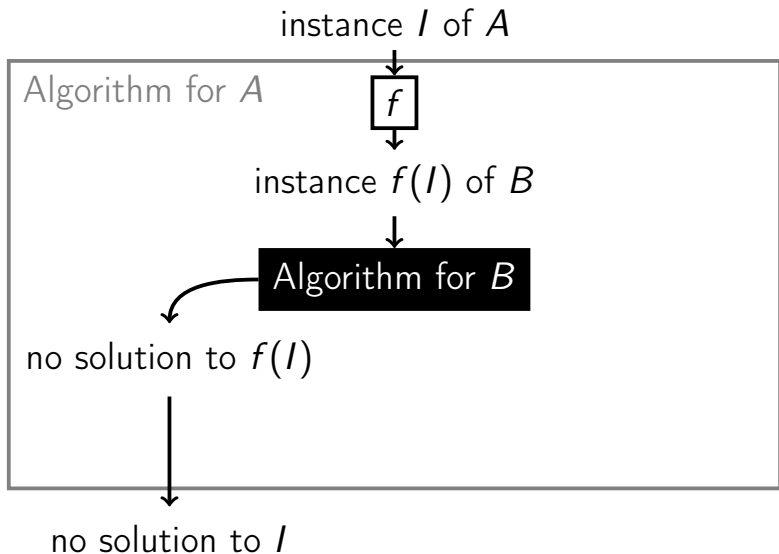


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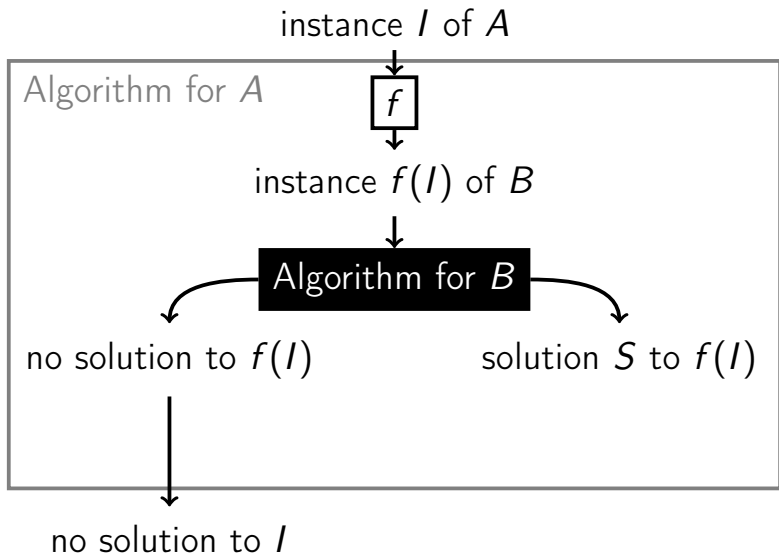




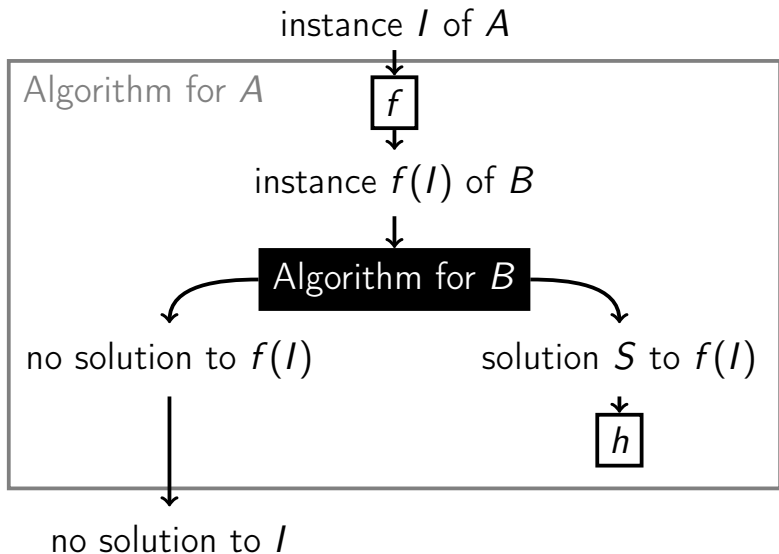
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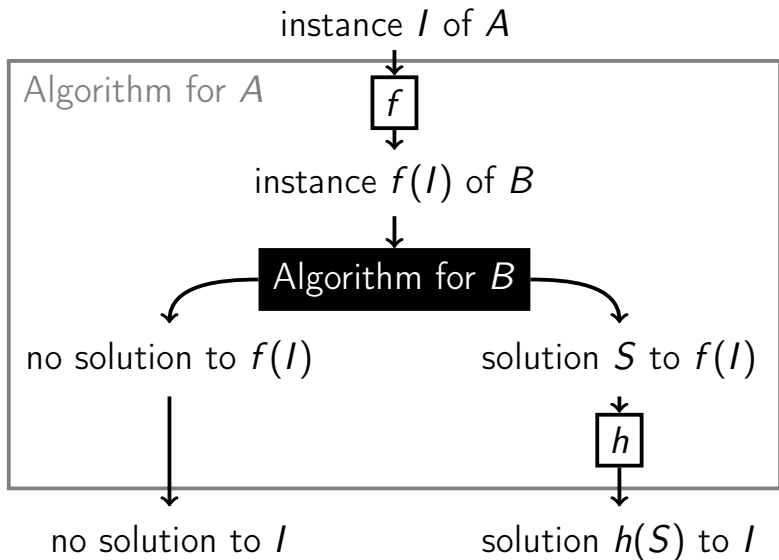
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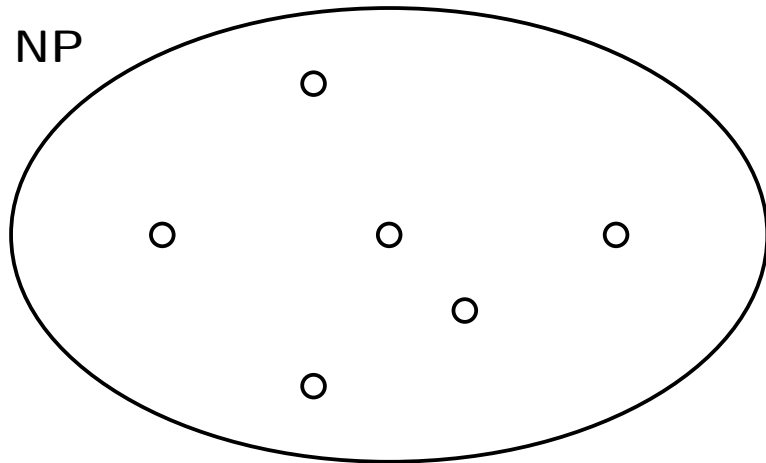


# Formally

## Definition

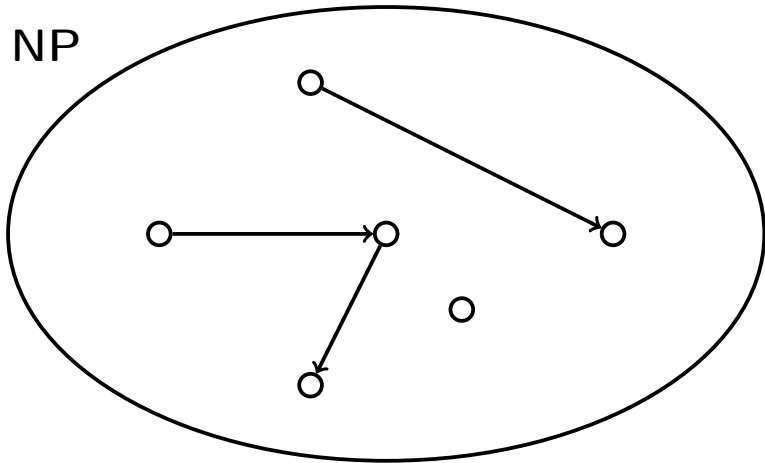
We say that a search problem  $A$  is reduced to a search problem  $B$  and write  $A \rightarrow B$ , if there exists a polynomial time algorithm  $f$  that converts any instance  $I$  of  $A$  into an instance  $f(I)$  of  $B$ , together with a polynomial time algorithm  $h$  that converts any solution  $S$  to  $f(I)$  back to a solution  $h(S)$  to  $A$ . If there is no solution to  $f(I)$ , then there is no solution to  $I$ .

# Graph of Search Problems



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NP



# NP-complete Problems

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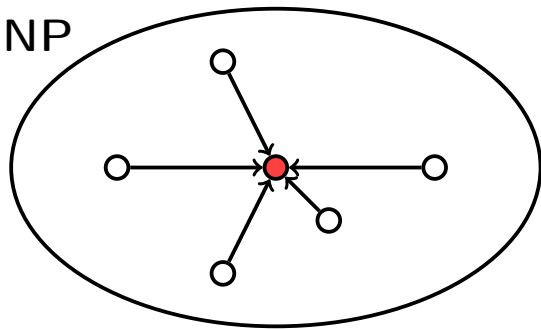
A search problem is called **NP-complete** if all other search problems reduce to it.



# NP-complete Problems

## Definition

A search problem is called **NP-complete** if all other search problems reduce to it.



## Do they exist?

It's not at all immediate that **NP**-complete problems even exist. We'll see later that all hard problems that we've seen in the previous part are in fact **NP**-complete!

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Two ways of using  $A \rightarrow B$ :

- 1 if  $B$  is easy (can be solved in polynomial time), then so is  $A$
- 2 if  $A$  is hard (cannot be solved in polynomial time), then so is  $B$

# Reductions Compose

## Lemma

If  $A \rightarrow B$  and  $B \rightarrow C$ , then  $A \rightarrow C$ .

# Proof

- The reductions  $A \rightarrow B$  and  $B \rightarrow C$  are given by pairs of polytime algorithms  $(f_{AB}, h_{AB})$  and  $(f_{BC}, h_{BC})$ .

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- To transform an instance  $I_A$  of  $A$  to an instance  $I_C$  of  $C$  we apply a polytime algorithm  $f_{BC} \circ f_{AB}$ :  $I_C = f_{BC}(f_{AB}(I_A))$ .

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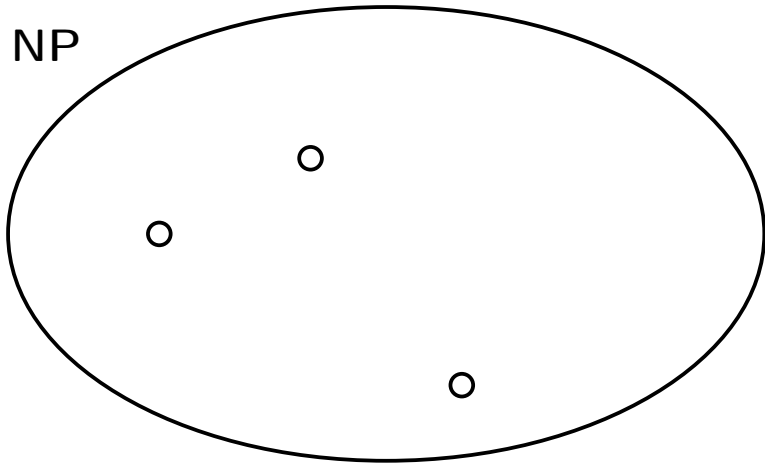
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- To transform a solution  $S_C$  to  $I_C$  to a solution  $S_A$  to  $I_A$  we apply a polytime algorithm  $h_{AB} \circ h_{BC}$ :  
 $S_A = h_{AB}(h_{BC}(S_C))$ .





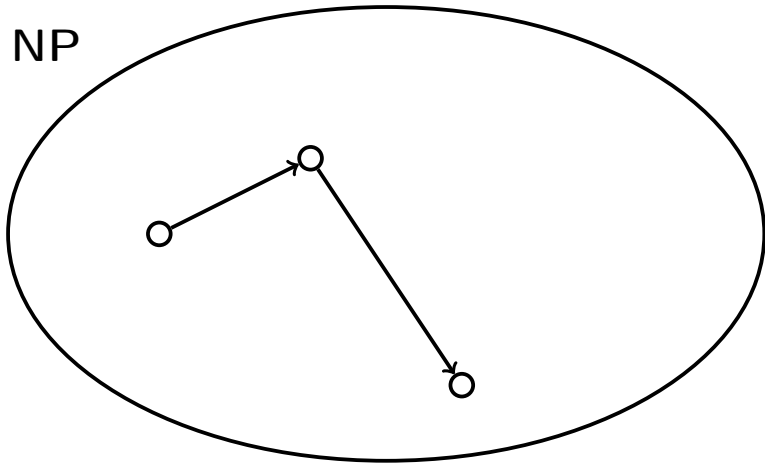
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NP



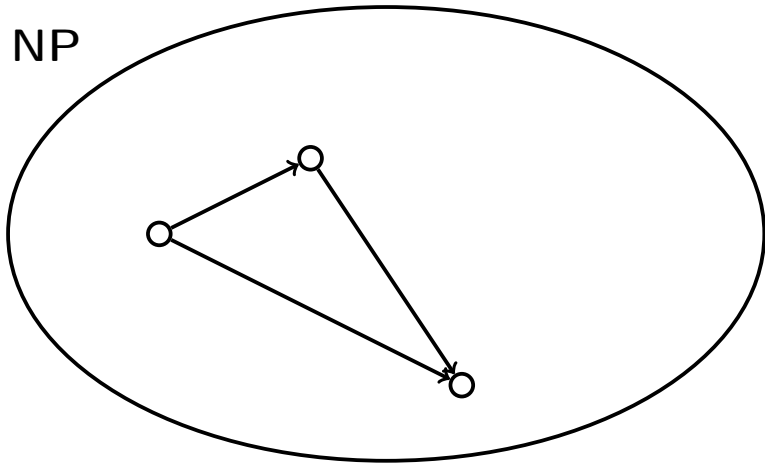
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# Showing **NP**-completeness

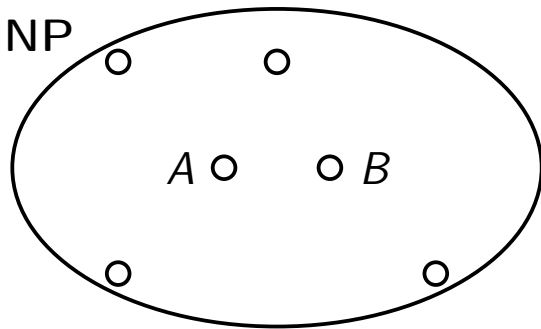
## Corollary

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# Showing **NP**-completeness

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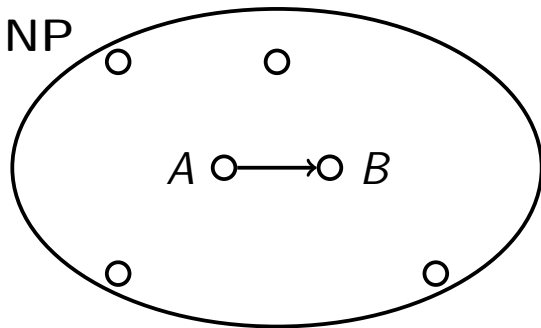
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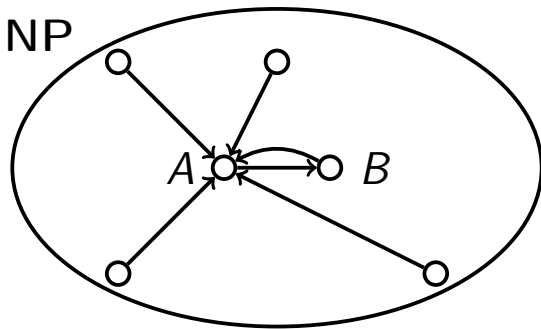
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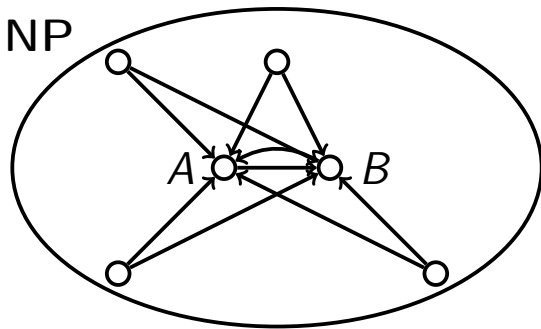
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# Showing **NP**-completeness

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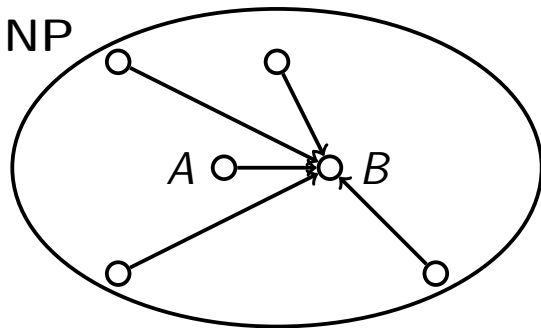




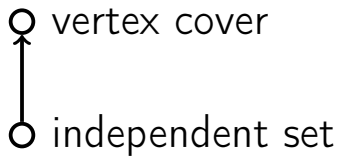
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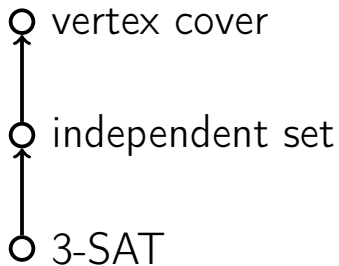
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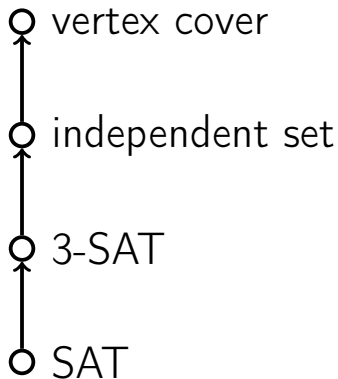
# Plan



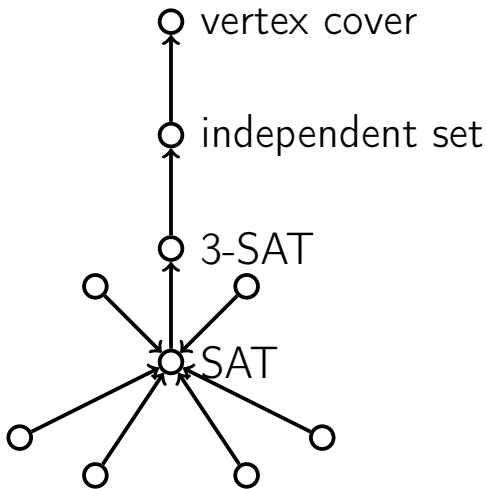
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## Independent set

**Input:** A graph and a budget  $b$ .

**Output:** A subset of at least  $b$  vertices such that no two of them are adjacent.

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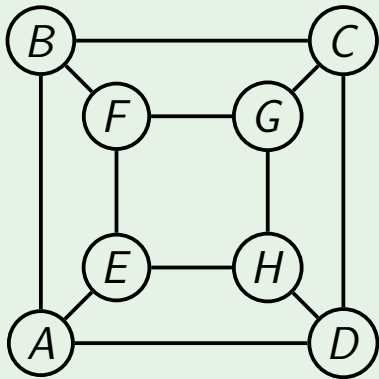
## Vertex cover

**Input:** A graph and a budget  $b$ .

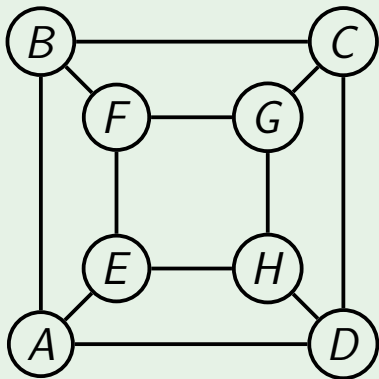
**Output:** A subset of at most  $b$  vertices that touches every edge.



# Example

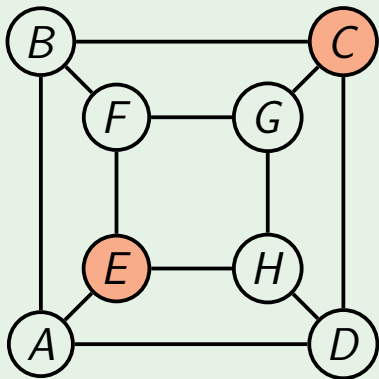


## Example



Independent sets:

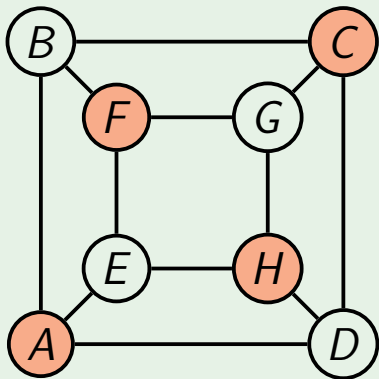
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Independent sets:

$\{E, C\}$

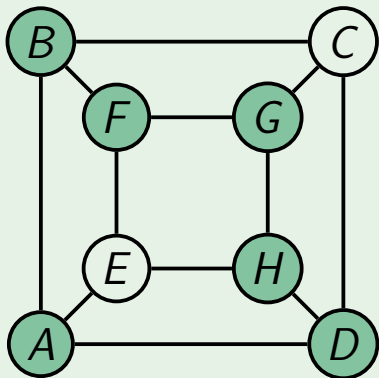
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Independent sets:

$\{E, C\}$   $\{A, C, F, H\}$

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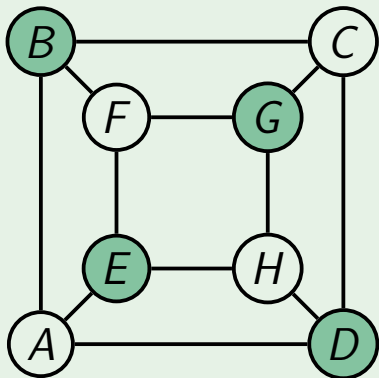
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Vertex covers:

$\{A, B, D, F, G, H\}$

## Example



Independent sets:

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Vertex covers:

$\{A, B, D, F, G, H\}$

$\{B, D, E, G\}$

## Lemma

$I$  is an independent set of  $G(V, E)$ , if and only if  $V - I$  is a vertex cover of  $G$ .

## Proof

- $\Rightarrow$  If  $I$  is an independent set, then there is no edge with both endpoints in  $I$ .  
Hence  $V - I$  touches every edge.
- $\Leftarrow$  If  $V - I$  touches every edge, then each edge has at least one endpoint in  $V - I$ .  
Hence  $I$  is an independent set. □

# Reduction

Independent set  $\rightarrow$  vertex cover: to check whether  $G(V, E)$  has an independent set of size at least  $b$ , check whether  $G$  has a vertex cover of size at most  $|V| - b$ :

- $f(G(V, E), b) = (G(V, E), |V| - b)$
- $h(S) = V - S$



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## 3-SAT

**Input:** Formula  $F$  in 3-CNF (a collection of clauses each having at most three literals).

**Output:** An assignment of Boolean values to the variables of  $F$  satisfying all clauses, if exists.

# Goal

Design a polynomial time algorithm that, given a 3-CNF formula  $F$ , outputs a graph  $G$  and an integer  $b$ , such that:

*$F$  is satisfiable, if and only if  $G$  has an independent set of size at least  $b$ .*

We need to find an assignment of Boolean values to variables, such that each clause contains at least one satisfied literal.

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## Example

- Setting  $x = 1, y = 1, z = 1$  satisfies a formula  $(x \vee y \vee z)(x \vee \bar{y})(y \vee \bar{z})$ .

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- Setting  $x = 1, y = 1, z = 1$  satisfies a formula  $(x \vee y \vee z)(x \vee \bar{y})(y \vee \bar{z})$ .
- Setting  $x = 1, y = 0, z = 0$  also satisfies it:  $(x \vee y \vee z)(x \vee \bar{y})(y \vee \bar{z})$ .

Alternatively, we need to select at least one literal from each clause, such that the set of selected literals is consistent: it does not contain a literal  $\ell$  together with its negation  $\bar{\ell}$ .

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- Consistent:  $\{x, x, \bar{z}\}$ ,  $\{x, x, y\}$ ,  $\{x, \bar{y}, \bar{z}\}$ .



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- Consistent:  $\{x, x, \bar{z}\}$ ,  $\{x, x, y\}$ ,  $\{x, \bar{y}, \bar{z}\}$ .
- Inconsistent:  $\{y, \bar{y}, \bar{z}\}$ ,  $\{z, x, \bar{z}\}$ .

# Using Alternative Statement

$$(x \vee y \vee z)(x \vee \bar{y})(y \vee \bar{z})(z \vee \bar{x})(\bar{x} \vee \bar{y} \vee \bar{z})$$

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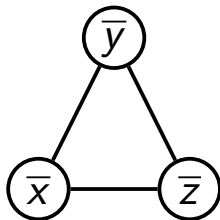
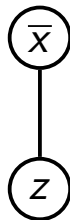
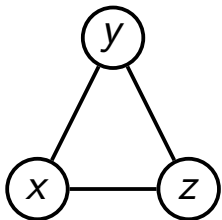
$$(z)$$

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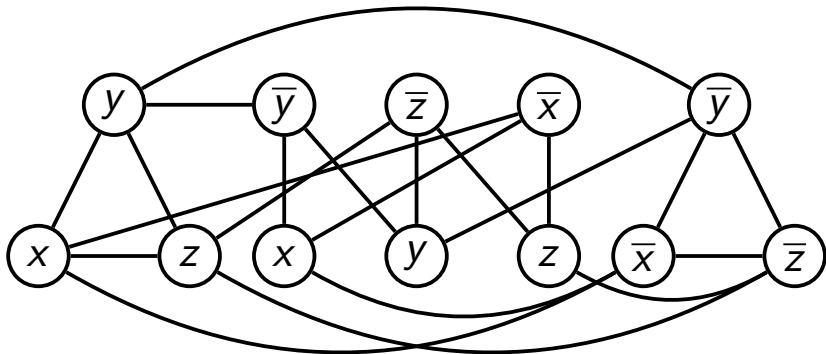
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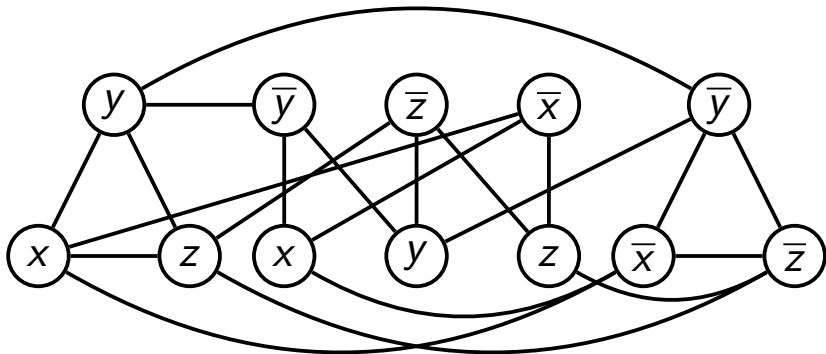
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the formula is satisfiable iff the resulting graph has independent set of size 5

# Transforming an Instance

- For each clause of the input formula  $F$ , introduce three (or two, or one) vertices in  $G$  labeled with the literals of this clause. Join every two of them.

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- $F$  is satisfiable if and only if  $G$  has independent set of size equal to the number of clauses in  $F$ .
- Transformation takes polynomial time.

# Transforming a Solution

- Given a solution  $S$  for  $G$ , just read the labels of the vertices from  $S$  to get a satisfying assignment of  $F$  (takes polynomial time).

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- Given a solution  $S$  for  $G$ , just read the labels of the vertices from  $S$  to get a satisfying assignment of  $F$  (takes polynomial time).
- If there is no solution for  $G$ , then  $F$  is unsatisfiable: indeed, a satisfying assignment for  $F$  would give a required independent set in  $G$ .

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## Goal

Transform a CNF formula into an equisatisfiable 3-CNF formula. That is, reduce a problem to its special case.

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- The second clause is shorter than  $C$
- Repeat while there is a long clause

# Running time

The running time of the transformation is polynomial: at each iteration we replace a clause with a shorter clause and a 3-clause. Hence the total number of iterations is at most the total number of literals of the initial formula.

# Correctness

## Lemma

The formulas  $F = (\ell_1 \vee \ell_2 \vee A) \dots$  and  $F' = (\ell_1 \vee \ell_2 \vee y)(\bar{y} \vee A) \dots$  are equisatisfiable.

## Proof

$$F = (\ell_1 \vee \ell_2 \vee A) \dots$$

$$F' = (\ell_1 \vee \ell_2 \vee y)(\bar{y} \vee A) \dots$$

$\Rightarrow$  If either  $\ell_1$  or  $\ell_2$  is satisfied, set  $y = 0$ .  
Otherwise  $A$  must be satisfied. Then set  $y = 1$ .

$\Leftarrow$  If  $(\ell_1 \vee \ell_2 \vee y)(\bar{y} \vee A)$  are satisfied, then  
so is  $(\ell_1 \vee \ell_2 \vee A)$ . □

# Transforming a Solution

Given a satisfying assignment for  $F'$ , just throw away the values of all new variables ( $y$ 's) to get a satisfying assignment of the initial formula.

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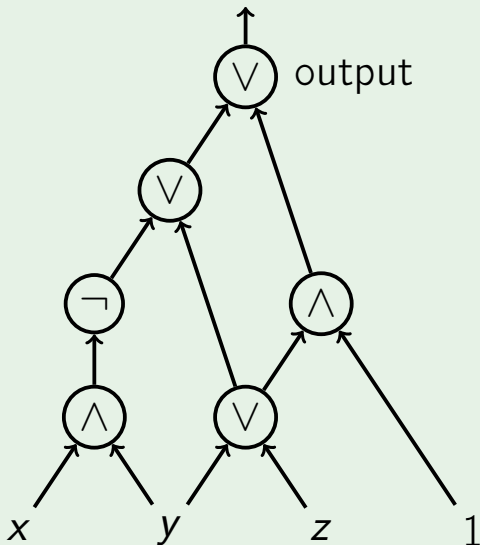
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## Goal

Show that **every** search problem reduces to SAT.

Instead, we show that any problem reduces to Circuit SAT problem, which, in turn, reduces to SAT.

# Circuit



## Definition

A **circuit** is a directed acyclic graph of in-degree at most 2. Nodes of in-degree 0 are called **inputs** and are marked by Boolean variables and constants. Nodes of in-degree 1 and 2 are called **gates**: gates of in-degree 1 are labeled with NOT, gates of in-degree 2 are labeled with AND or OR. One of the sinks is marked as **output**.

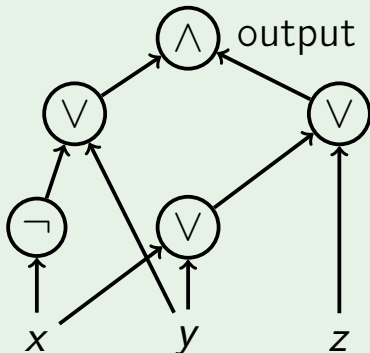
## Circuit-SAT

Input: A circuit.

Output: An assignment of Boolean values to the input variables of the circuit that makes the output true.

SAT is a special case of Circuit-SAT as a CNF formula can be represented as a circuit:

Example:  $(x \vee y \vee z)(y \vee \bar{x})$



# Circuit-SAT $\rightarrow$ SAT

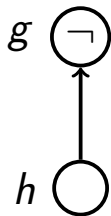
To reduce Circuit-SAT to SAT, we need to design a polynomial time algorithm that for a given circuit outputs a CNF formula which is satisfiable, if and only if the circuit is satisfiable

# Idea

- Introduce a Boolean variable for each gate
- For each gate, write down a few clauses that describe the relationship between this gate and its direct predecessors

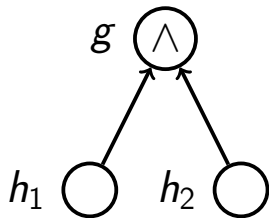


# NOT Gates



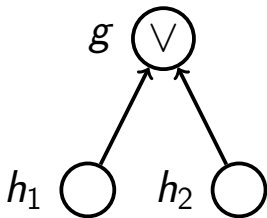
$$(h \vee g)(\bar{h} \vee \bar{g})$$

# AND Gates



$$(h_1 \vee \bar{g})(h_2 \vee \bar{g})(\bar{h}_1 \vee \bar{h}_2 \vee g)$$

# OR Gates



$$(\overline{h_1} \vee g)(\overline{h_2} \vee g)(h_1 \vee h_2 \vee \overline{g})$$

# Output Gate

$$g \bigcirc \text{output} \quad (g)$$

- The resulting CNF formula is consistent with the initial circuit: in any satisfying assignment of the formula, the value of  $g$  is equal to the value of the gate labeled with  $g$  in the circuit

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- Therefore, the CNF formula is equisatisfiable to the circuit
- The reduction takes polynomial time

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Reduce every search problem to Circuit-SAT.



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- In particular,  $|S|$  is polynomial in  $|I|$

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- Note that a computer is in fact a circuit (of constant size!) implemented on a chip
- Each step of the algorithm  $\mathcal{C}(I, S)$  is performed by this computer's circuit
- This gives a circuit of size polynomial in  $|I|$  that has input bits for  $I$  and  $S$  and outputs whether  $S$  is a solution for  $I$  (a separate circuit for each input length)

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To solve an instance  $I$  of the problem  $A$ :

- take a circuit corresponding to  $\mathcal{C}(I, \cdot)$

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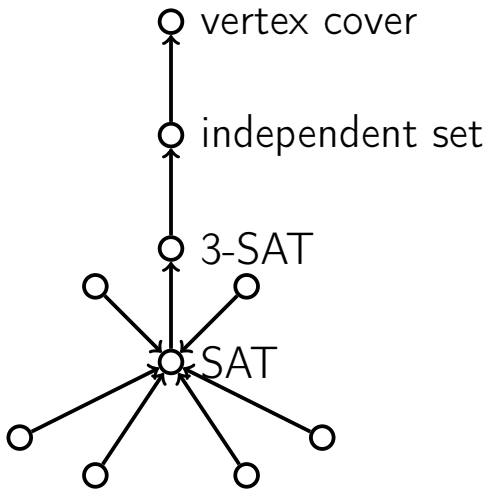


# Reduction

To solve an instance  $I$  of the problem  $A$ :

- take a circuit corresponding to  $\mathcal{C}(I, \cdot)$
- the inputs to this circuit encode candidate solutions
- use a Circuit-SAT algorithm for this circuit to find a solution (if exists)

# Summary



# Outline

- 1 Reductions
- 2 Showing **NP**-completeness
- 3 Independent Set  $\rightarrow$  Vertex Cover
- 4 3-SAT  $\rightarrow$  Independent Set
- 5 SAT  $\rightarrow$  3-SAT
- 6 All of **NP**  $\rightarrow$  SAT
- 7 Using SAT-solvers

# Sudoku Puzzle

This part

A simple and efficient Sudoku solver

# SAT: Theory and Practice

Theory: we have no algorithm checking the satisfiability of a CNF formula  $F$  with  $n$  variables in time  $\text{poly}(|F|) \cdot 1.99^n$

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**Practice:** SAT-solvers routinely solve instances with thousands of variables

# Solving Hard Problems in Practice

An easy way to solve a hard combinatorial problem in practice:

- Reduce the problem to SAT (many problems are reduced to SAT in a natural way)

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An easy way to solve a hard combinatorial problem in practice:

- Reduce the problem to SAT (many problems are reduced to SAT in a natural way)
- Use a SAT solver



# Sudoku Puzzle

Goal: fill in with digits the partially completed  $9 \times 9$  grid so that each row, each column, and each of the nine  $3 \times 3$  subgrids contains all the digits from 1 to 9.

Example

# Variables

There will be  $9 \times 9 \times 9 = 729$  Boolean variables: for  $1 \leq i, j, k \leq 9$ ,  $x_{ijk} = 1$ , if and only if the cell  $[i, j]$  contains the digit  $k$

# Exactly One Is True

Clauses expressing the fact that exactly one of the literals  $\ell_1, \ell_2, \ell_3$  is equal to 1:

$$(\ell_1 \vee \ell_2 \vee \ell_3)(\bar{\ell}_1 \vee \bar{\ell}_2)(\bar{\ell}_1 \vee \bar{\ell}_3)(\bar{\ell}_2 \vee \bar{\ell}_3)$$

# Constraints

- Cell  $[i, j]$  contains exactly one digit:  
 $\text{ExactlyOneOf}(x_{ij1}, x_{ij2}, \dots, x_{ij9})$

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- $k$  appears exactly once in row  $i$ :  
 $\text{ExactlyOneOf}(x_{i1k}, x_{i2k}, \dots, x_{i9k})$

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- $k$  appears exactly once in a  $3 \times 3$  block:  
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- $[i, j]$  already contains  $k$ :  $(x_{ijk})$

# Resulting Formula

State-of-the-art SAT-solvers find a satisfying assignment for the resulting formula in blink of an eye, though the corresponding search space has size about  $2^{729} \approx 10^{220}$