Curs 1

- Differential equations
- Positive and linear functionals
- Reimann - Stieltjes integral
- Improper integrals
- Integrals depending on parameters
- Special functions
- Line integrals
- Multiple integrals
- Surface integrals

Elements of Differential Equations

A differential equation is an equation involving unknown functions, independent variables and derivatives of unknown functions.
Differentials equations are ubiquitous in science and engineering.
A solution to a differential equation is a function whose derivatives

D_1 Let n be a positive integer. A relation of the form

$$F(x, y, y', \dots, y^{(n)}) = 0 \qquad (\diamond)$$

between an independent variable x, an unknown function $y = y(x)$ and it's derivatives $y', y'', \dots, y^{(n)}$, is called an ORDINARY DIFFERENTI EQUATION (ODE) of order n

A function $y = f(x)$, is said to be a solution to the equation $(\diamond)$ if when substituted into $(\diamond)$ it reduces the eq.

the curve $y = f(x)$ is an integral

D_2 By an integral of the differential equation $(\diamond)$ is primaly meant a relation of the form $\Phi(x, y) = 0$ defining a solution y of $(\diamond)$ as an implicit function of x

For the purpose of simplicity, we restrict ourselve (temporarily) to the case $n = 2$

The general integral of equation $(\diamond)$ $(F(x, y, y', y'') = 0)$ is a relation $\Phi(x, y, C_1, C_2)$

from which one can obtain, by an appropiate choice of constants, any integral curve of $(\diamond)$ lying in some given domain

Finding the integrals of a differential equation is called its integration

The Cauchy problem (initial value problem)

Given the numbers x_0, y_0, z_0, find the solutions y of $(\diamond)$ satisfying the initial condition

$$y(x_0) = y_0, \quad y'(x_0) = y_1$$

The general solution

Let $y = \varphi(x, C_1, C_2)$ be a solution of eq $(\diamond)$ where C_1, C_2 are arbitrary constants.

If, for any point (x_0, y_0, y_1) in a domain D, the constants C_1, C_2 can be determined uniquely to solve the Cauchy problem with the initial conditions

$$y(x_0) = y_0 \quad , \quad y'(x_0) = y_1$$

then y is called the general solution of $(\diamond)$ in D

A singular solution of a differential equation is a solution that is not generic, that is, not obtainable from the general solution

Ex: Find the unknown function $y = y(x)$ satisfying

$(\clubsuit)$ $y' + y = 0$ (diff. eq. of order 1)

$$(y' + y)e^x = 0$$

$$(ye^x)' = 0$$

$$ye^x = C_1 \quad \text{(the general integral of } (\clubsuit) \text{)}$$

$$y = C_1 e^{-x} \quad \text{(the general solution of } (\clubsuit) \text{)}$$

A Cauchy problem (initial value problem) for $(\clubsuit)$

Find the solution y of $(\clubsuit)$ satisfying the initial condition

$y(0) = e$

The solution of the Cauchy problem is:

$$y = e^{1-x}$$

WARNING Most differential equations cannot be integrated analytically and it may be necessary to integrate them numerically, e.g

$$y' + xy^3 = 1$$

A numerical solution is given by

MATHEMATICA

IN: DSolve[{y'[x] + y[x] == 0 , y[0] == e }, y[x], x] //FullSimplify

OUT: $y[x] \to e^{1-x}$

The differential Equation $\frac{dy}{dx} = f(x)$

Suppose that the function $f: (a,b) \to \mathbb{R}$ is continous
Starting from the equality

$$y'(x) = f(x) \quad , x \in (a,b)$$

we write

$$y'(s) = f(s) \Big/ \int_{x_0}^{x} () \, ds$$

and one can verify that

$$y(x) = y_0 + \int_{x_0}^{x} f(s) ds \quad , x \in (a,b)$$

is the solution of the Cauchy problem with the initial data $y(x_0) = y_0$

The differential equation $\frac{dy}{dx} = g(y)$

Suppose that the function $g: (a,b) \to \mathbb{R}^*$ is continous
Starting from the equation

$$dx = \frac{dy}{g(y)} \quad , x \in (a,b)$$

$$x'(y) dy = \frac{dy}{g(y)} \Big| \int_{y_0}^{y}$$

we obtain a function $y \mapsto x(y)$

$$x(y) - x(y_0) = \int_{y_0}^{y} \frac{1}{g(y)} dy$$

E3 Solve the initial value problem

$$y' = y > 0 \qquad y(1) = 2$$

We write $\frac{dy}{y} = dx \;\Big|\; \int$

$$\log y = x + C \quad \text{(general integral)}$$

$$y = e^{x+C} \quad \text{(general solution)}$$

For $x = 1$ and $y = 2$, we get

$$2 = e^{1+C}, \quad e^{C} = 2e^{-1}$$

It follows that the solution to the initial value problem is $y = 2e^{x-1}$

USING MATHEMATICA

```
DSolveValue[y'[x] == y[x], y[x], x]
```
$e^{x}C_1$
```
DSolveValue[{y'[x] == y[x], y[1] == 2}, y[x], x]
```
$2e^{x-1}$ ✓

PT. EXAMEN

The Separable Equation $\frac{dy}{dx} = f(x)g(y)$

CURS 2

25.02.2022

The Separable Equation

$$\frac{dy}{dx} = f(x)g(y)$$

"Separating" the variables, we obtain $\frac{dy}{g(y)} = f(x)\,dx$, hence, using $dy(x) = y'(x)\,dx$,

$$\frac{y'(x)\,dx}{g(y(x))} = f(x)\,dx \quad \Big| \int_{x_0}^{x}$$

With the change of variable $Y = y(x)$ $(y_0 := y(x_0))$ we obtain

$$\int_{y_0}^{y} \frac{dY}{g(Y)} = \int_{x_0}^{x} f(x)\,dx$$

E4 Find the general integral and the general solution of the separable eq.

$$xy'\cos y + \sin y = 0, \quad x > 0, \; y \in (0, \pi)$$

Since $y' = \frac{dy}{dx}$, we obtain

$$\frac{\cos y}{\sin y}\,dy = -\frac{dx}{x} \quad \Big| \int \quad \Rightarrow \quad \log \sin y = -\log x + C$$

The general integral is $x \sin y = e^{C}$

The general solution is $y = \arcsin \frac{e^{C}}{x}$

The homogenous equation $y' = f\left(\frac{y}{x}\right)$

This type of equation is solved via a substitution. Indeed let $y = ux$
Then, easy calculation give separable eq.

$$y' = u'x + u = f(u), \quad \frac{dx}{x} = \frac{du}{f(u)-u}$$

We obtain $\displaystyle\int \frac{dx}{x} = \int \frac{du}{f(u)-u} + C$

With the notation $\displaystyle F(u) = \int \frac{du}{f(u)-u}$, we get

$$\ln x = F(u) + C, \quad \text{i.e.,} \quad x = Ce^{F(u)}$$

For $u = y/x$, we get the general integral

$$x = Ce^{F(y/x)}$$

Simultaneously, we get a parametric representation of the integral curves

$$\begin{cases} x = Ce^{F(u)} \\ y = Cue^{F(u)} \end{cases}$$

E5 Find the sol. of the initial value problem of the homogenous eq.

$$y' = \frac{y}{x} + \frac{y^2}{x^2}, \quad y(1) = 1 \qquad 0 < x < e, \quad y > 0$$

Consider the change of function $y = ux$. We obtain

$$y' = u'x + u$$

and

$$u'x = u^2, \quad \frac{du}{u^2} = \frac{dx}{x}, \quad -\frac{1}{u} = \ln x + C, \quad y = \frac{x}{-C - \ln x}$$

For $x = 1$ and $y = 1$, we obtain $C = -1$ and $y = \frac{x}{1-\ln x}$

The Linear Differential Equation

$$y' + P(x)y = Q(x)$$

Multiplying both sides with the integrating factor $e^{\int_{x_0}^{x} P(s)ds}$, we obtain

$$(y' + P(x)y)\, e^{\int_{x_0}^{x} P(s)ds} = Q(x)\, e^{\int_{x_0}^{x} P(s)ds},$$

i.e.

$$\left(y\, e^{\int_{x_0}^{x} P(s)ds}\right)' = Q(x)\, e^{\int_{x_0}^{x} P(s)ds}$$

therefore

$$y\, e^{\int_{x_0}^{x} P(s)ds} - y_0 = \int_{x_0}^{x} Q(t)\, e^{\int_{x_0}^{t} P(s)ds}\, dt$$

$$y = y_0 e^{-\int_{x_0}^{x} P(s)\,ds} + e^{-\int_{x_0}^{x} P(s)\,ds} \int_{x_0}^{x} Q(t)\, e^{\int_{x_0}^{t} P(s)ds}\, dt$$

Finally, we obtain the general solution of the linear differential equations

$$y = y_0 e^{-\int_{x_0}^{x} P(s)ds} + \int_{x_0}^{x} Q(t)\, e^{\int_{x}^{t} P(s)ds}\, dt$$

Eg Solve the initial value problem for the linear differential eq.

$$y' + \frac{y}{x} = x^2, \quad y(1) = 1, \quad x>0, \; y>0$$

The integrating factor $e^{\int P(x)dx}$, is $e^{\int \frac{1}{x}dx} = x$

We obtain $\left(y e^{\int \frac{dx}{x}}\right)' = x^2 e^{\int \frac{dx}{x}}$, i.e. $(yx)' = x^3$

Integrating both sides we get a general integral

$$yx = \frac{x^4}{4} + C$$

We use the initial condition $y(1) = 1$ to determine the value of the constant C. For $x=1$ and $y=1$ we get $C = \frac{3}{4}$, and

The Bernoulli Equation

$$y' + P(x)y = Q(x)y^a$$

For $a \neq 0$ or $a \neq 1$ with the chang function $y = z^{\frac{1}{1-a}}$, we obtain the linear eq.

$$z' + (1-a)P(x)z = (1-a)Q(x)$$

The Riccati Equation

$$y' + P(x)y^2 + Q(x)y + R(x) = 0$$

In order to solve a Ricati equation, one will need a particular sol.
Without knowing at least one solution, there is absolutely no chance to find any solution to such an eq.
Indeed, let y_1 be a particular solution. With the chance of function $y = z + y_1$, the Riccati eq. transforms to the Bernoulli one

$$z' + (2P(x)y_1 + Q(x))z + P(x)z^2 = 0$$

- $y' = \arctan y$
- $y' + y^3 = x$
- $y^{(5)} + y' = y$

The exact differential equation

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

Positive Linear Functions (PLF) CURS 3 04.03.2022

D1 A functional $A: C[a,b] \longrightarrow \mathbb{R}$ is said to be linear and positive if the following properties are satisfied

1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ $\quad \forall f,g \in C[a,b], \forall \alpha,\beta \in \mathbb{R}$

2) $h \geq 0 \Rightarrow A(h) \geq 0$ $\quad \forall h \in C[a,b]$

E2 $A(f) = 0$, $\quad A(f) = f(a)$,

$$A(f) = \int_a^b f(x)\,dx$$

Properties of PLFs

R3 Any PLF is "nondecreasing"

$$f \geq g \iff f-g \geq 0 \Rightarrow A(f-g) \geq 0 \iff A(f) \geq A(g)$$

$\left(f \geq g \overset{def}{\iff} f(x) \geq g(x), \forall x \in [a,b] \right)$
(i.e. the graph of f is above that of g)

T4 If $A: C[a,b] \to \mathbb{R}$ is a PLF, then
$$|A(f)| \leq A(|f|), \quad \forall f \in C[a,b]$$

We will use: $|x| \leq y \iff -y \leq x \leq y$, $\forall x,y \in \mathbb{R}$

For all $f \in C[a,b]$, we have

$$-|f| \leq f \leq |f| \quad \Big| \, A$$

It follows that
$$A(-|f|) \leq A(f) \leq A(|f|)$$

i.e

$$-A(|f|) \leq A(f) \leq A(|f|)$$

i.e

$$|A(f)| \leq A(|f|)$$

Mean-value Theorem

T5 If $A: C[a,b] \to \mathbb{R}$ is a pLF, then for any $f \in C[a,b]$ there exists a point $c \in [a,b]$ such that

$$A(f) = f(c) \cdot A(1)$$

Let $M = \sup_{x \in [a,b]} f(x)$, $m = \inf_{x\in[a,b]} f(x)$. we have

$$m \cdot 1 \leq f \leq M \cdot 1$$
$$\Rightarrow A(m \cdot 1) \leq A(f) \leq A(M \cdot 1)$$
$$\Rightarrow m \cdot A(1) \leq A(f) \leq M \cdot A(1)$$

If $A(1) = 0$, then $A(f) = 0$, and T5 is satisfied for all $c \in [a,b]$. If $A(1) \neq 0$, then $m \leq \frac{A(f)}{A(1)} \leq M$, hence, (cf. $A(1) > 0$) hence, using the "Darboux property" of f, we deduce the existance

E6 For any $f \in C[a,b]$ there exists $c \in [a,b]$ such that

$$\underbrace{\int_a^b f(x)\,dx}_{A(f)} = f(c)\underbrace{(b-a)}_{A(1)}$$

E7 (Yuan et al. 2019) If $f, g : [a,b] \to \mathbb{R}$ are integrable $g \geq 0$ and f is a Darboux function on (a,b), then there exists $c \in (a,b)$ such that

$$\int_a^b f(x) g(x)\,dx = f(c) \cdot \int_a^b g(x)\,dx$$

E8 If $f \in C[a,b]$, $0<a<b$, there exists $c \in [a,b]$ such that

$$\int_b^a \frac{f(x)}{x}\,dx = f(c)\cdot \log\frac{a}{b}$$

The Cauchy-Buniakovsky-Schwarz Inequality

T9 If $A: C[a,b] \to \mathbb{R}$ is a PLF, then for all $f, g \in C[a,b]$ the following inequality is satisfied

$$(A(fg))^2 \leq A(f^2)A(g^2) \qquad (*)$$

The following relations are satisfied for all $t \in \mathbb{R}$:

$$(f+tg)^2 \geq 0$$

f, g - vectors
t - scalar

$$\Rightarrow f^2 + 2tfg + t^2g^2 \geq 0 \quad | A$$

$$\Rightarrow A(f^2) + 2tA(fg) + t^2A(g^2) \geq 0$$

If $A(g^2) = 0$, then $A(f^2) + 2tA(fg) \geq 0 \quad \forall t \in \mathbb{R}$
therefore, $A(fg) = 0$, and inequality $(*)$ becomes an equality.
If $A(g^2) > 0$, then $\Delta\ 4A(fg)^2 - 4A(f^2)A(g^2) \leq 0$, hence

$$(A(fg))^2 \leq A(f^2)A(g^2)$$

T10 If $A: C[a,b] \to \mathbb{R}$ is a PLF, $A(1) = 1$, and the func. f and $g \in C[a,b]$ are synchronously monotonic (have the same monotony on each subset of $[a,b]$),

$$A(fg) \geq A(f)\cdot A(g)$$

We have:

$$(f(x) - f(y))(g(x) - g(y)) \geq 0, \quad \forall x, y \in [a,b]$$

$$\Rightarrow (f(x) - f)(g(x) - g) \geq 0 \quad , \forall x \in [a,b],$$

$$\Rightarrow f(x)g(x)\cdot 1 - f(x)g - g(x)\cdot f + fg \geq 0, \forall x \in [a,b]$$

$$\Rightarrow f(x)g(x)A(1) - f(x)A(g) - g(x)A(f) + A(fg) \geq 0 , \forall x \in [a,b]$$

$$\Leftrightarrow \quad fg - fA(g) - gA(f) + A(fg)\cdot 1 \geq 0$$

$$\Rightarrow \quad A(fg) - A(f)A(g) - A(f)A(g) + A(fg) \geq 0$$

$$\Rightarrow \quad A(fg) \geq A(f)A(g)$$

Ex For $f, g : [a,b] \to \mathbb{R}$ non-increasing or non-decreasing we have:

$$\int_a^b f(x)g(x)\,dx \geq \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$$

We apply the Chebyshev inequality (T.10) for

$$A(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The Riemann-Stieltjes Integral

CURS 5

18. 03. 2022

E_6 Investigate the integral $\int_1^{\infty} x^{\alpha}\, dx$ for convergence

We have

$$\int_1^b x^{\alpha}\, dx = \begin{cases} \dfrac{b^{\alpha+1}-1}{} \end{cases}$$

Examen sigur
Integrale improprii

E_7 Investigate the convergence of the integral $\int_0^1 x^{\beta}\, dx$

We have $\int_0^1 x^{\beta}\, dx \overset{x=1/t}{=\!=\!=} \int_1^{\infty} t^{-\beta-2}\, dt$

hence, by the previous example we get

$$\int_0^1 x^{\beta} \sim \begin{cases} \text{convergent}, & \beta > -1 \\ \text{divergent}, & \beta \le -1 \end{cases}$$

Convergence Tests for Improper Integrals

T_8 [Cauchy's General Convergence test]

If the function $f : [\alpha, \infty) \to \mathbb{R}$ is integrable on each interval of the form $[a, b]$, then $\int_a^\infty f(x)\,dx$ is convergent iff:

$$\lim_{\alpha, \beta \to \infty} \int_\alpha^\beta f(x)\,dx = 0$$

R_9 If $f : [a, \infty) \to [0, \infty)$ is integrable on every interval $[a, b]$ and $\int_a^\infty f(x)\,dx \sim$ CONV, then the limit $\lim_{x \to \infty} f(x)$ is NOT necessarily zero

Indeed, consider the function

$$g(x) = \begin{cases} 0, & x \in [1, \infty) \setminus \mathbb{N} \\ 1, & \text{otherwise} \end{cases}$$

T_{10} [Dirichlet's Test]

If $f : [a, \infty) \to \mathbb{R}$ is integrable on each interval of the form $[a, b]$, If there exists a constant M such that

$$\left| \int_\alpha^\beta f(x)\,dx \right| < M, \quad \forall \alpha, \beta \geq a$$

and the function $g : [a, \infty) \to \mathbb{R}$ decreases to zero when $x \to \infty$, then the integral $\int_a^\infty f(x) g(x)\,dx$ is convergent

E11 Study the int.

a) $\int_0^{\infty} \frac{\sin x}{x} dx$,

b) $\int_0^{\infty} \frac{|\sin x|}{x} dx$,

Tema: posibil Examen

for convergence

E13 Determine if the following int. is conv or div.

$$\int_0^{\infty} \sin x^2 dx$$

We have: $\int_1^{\infty} \sin x^2 dx \overset{x=\sqrt{t}}{=\!=\!=} \frac{1}{2}\int_1^{\infty} \frac{\sin t}{\sqrt{t}} dt$

$$\left| \int_{\alpha}^{\beta} \sin t \, dt \right| \leq 2, \quad (\forall \alpha, \beta \geq 1)$$

and the function $g: [1, \infty) \to \mathbb{R}$, $g(t) = \frac{1}{\sqrt{t}}$, decreases to zero

By Dirichlet's Test, the integral $\int_1^{\infty} \sin x^2 dx$ is convergent, therefore, the integral $\int_0^{\infty} \sin x^2 dx$ is also conv.

T14 The Comparison Test

If $f: [a, \infty) \to \mathbb{R}$ is integrable on each interval $[a, b]$ $0 < a < b$ and the limit

$$\lim_{x\to\infty} \frac{f(x)}{x}$$

is ~~positive~~ and finite, we have

$$\int_a^\infty f(x)\,dx = \begin{cases} \text{convergent} & , \alpha < -1 \\ \text{divergent} & , \alpha \geq -1 \end{cases}$$

Remark For $f: [0,\infty) \to [0,\infty)$ and $\alpha < -1$, the following relation is satisfied

$$\lim_{x\to\infty} \frac{f(x)}{x^\alpha} = 0 \Rightarrow \int_0^\infty f(x)\,dx \text{ convergent}$$

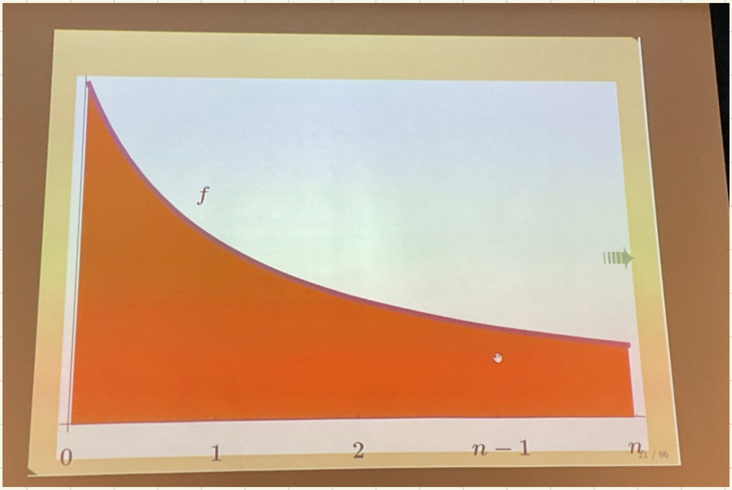
T_{16} The Cauchy's Test

If the function $f: [0,\infty) \to [0,\infty)$ is non-increasing, then

$$\int_0^\infty f(x)\,dx \sim \sum_{n=0}^\infty f(n)$$

→ since the function $t \mapsto \int_0^t f(x)\,dx$ is non-decreasing (f positive) the integral $\int_0^\infty f(x)\,dx$ is the limit of the sequence

$$\left(\int_0^n f(x)\,dx \right)$$


f
0
1
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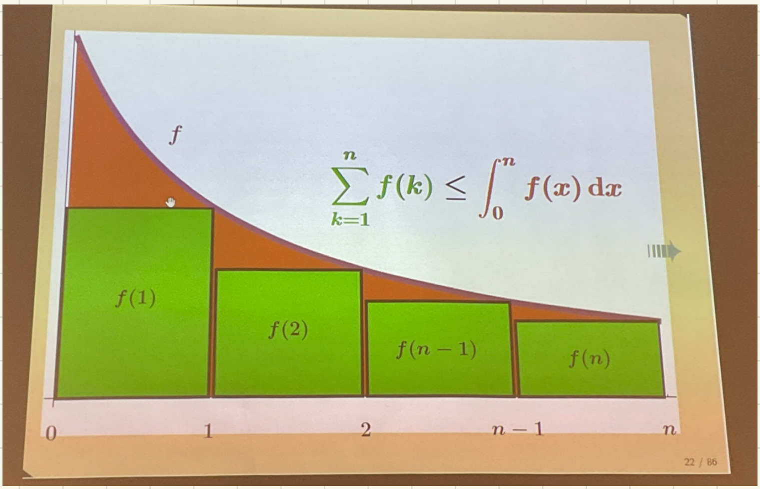
f
$\sum_{k=1}^{n} f(k) \leq \int_0^n f(x)\,dx$
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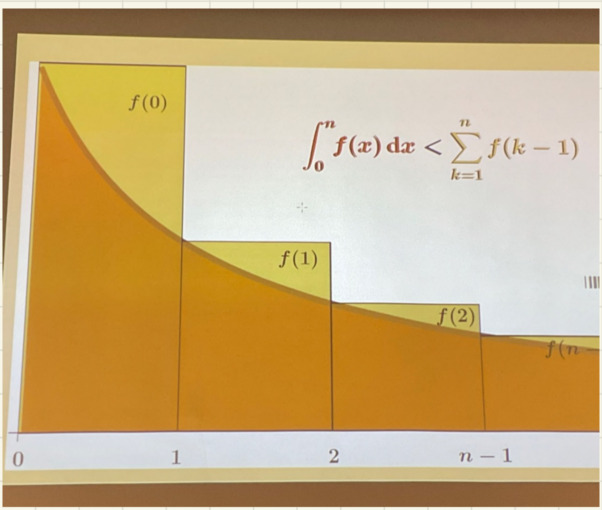
22 / 86


$f(0)$
$\int_0^n f(x)\,dx < \sum_{k=1}^{n} f(k-1)$
$f(1)$
$f(2)$
0
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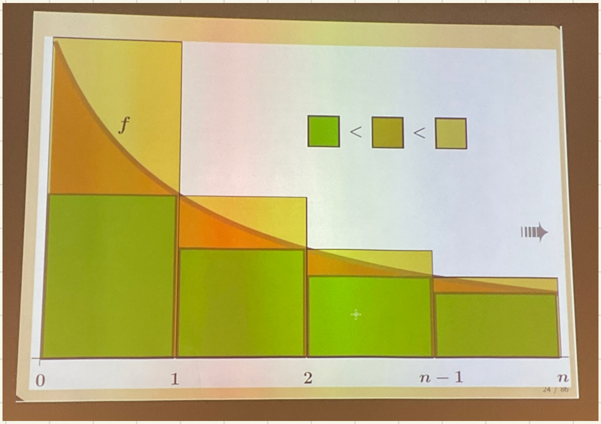


From
$$\int_0^n f(x)\,dx = \sum_{k=1}^{n} \int_{k-1}^{k} f(x)\,dx$$

we deduce
$$\sum_{k=1}^{n} f(k) \le \int_0^n f(x)\,dx \le \sum_{k=1}^{n} f(k-1)$$

hence
$$\sum_{n=0}^{\infty} f(n) \sim \int_0^{\infty} f(x)\,dx$$

E17 Using the Cauchy integral test
$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n+1} \sim \int_0^{\infty} \frac{dx}{x+1} = \log(x+1)\Big|_0^{\infty} = \infty$$

E18 Using Cauchy integral test
$$\sum_{n=2}^{\infty} \frac{1}{n\log n} \sim \int_2^{\infty} \frac{dx}{x\log x} = \log\log x\Big|_2^{\infty} = \infty$$

Remark By the change of variable
$$x = \frac{bt+a}{t+1}$$
the integral $\int_a^{b-0} f(x)\,dx$ becomes $\int_0^{\infty} g(t)\,dt$

Ex. Euler's Integral Examen Maybe
$$\int_0^{\infty} \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin \pi a}, \quad a \in (0,1)$$

Solution #1 We use the expansion of $\frac{1}{\sin \pi a}$, $a \in (0,1)$ in partial fractions

Taking $x=0$ in
$$\cos ax = \frac{2 \sin a\pi}{\pi}\left(\frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^n a}{a^2-n^2}\cos nx\right)$$
we obtain
$$\frac{\pi}{\sin a\pi} = \frac{1}{a} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{a+n} + \frac{1}{a-n}\right) \Rightarrow$$
$$\int_0^{\infty} \frac{x^{a-1}}{1+x}\,dx = \int_0^1 + \int_1^{\infty} \overset{x=\frac{1}{t}}{=} \int_0^1 \frac{x^{a-1}}{1+x}\,dx + \int_0^1 \frac{t^{-a}}{1+t}\,dt$$
$$= \int_0^1 x^{a-1} \sum_{n=0}^{\infty} (-1)^n x^n\,dx + \int_0^1 t^{-a} \sum_{n=0}^{\infty} (-1)^n t^n\,dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{a-1+n}\, dx + \sum_{n=0}^{\infty} (-1)^n \int_0^1 t^{-a+n}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{a+n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1-a}$$

$$= \frac{1}{a} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{a+n} + \frac{1}{a-n} \right) = \frac{\pi}{\sin \pi a}$$

CURS 6

Integrals Depending on a parameter

Let $X \subset \mathbb{R}$ be an interval. Let $Y \subset \mathbb{R}$ and $f : X \times Y \to \mathbb{R}$ such that the application $x \mapsto f(x,y) : X \to \mathbb{R}$ be integrable for all $y \in Y$

D21 The function $I : Y \to \mathbb{R}$,

$$I(y) = \int_X f(x,y)\,dx$$

is an integral depending on a parameter

E22

$$\int_0^1 \cos(xy)\,dx = \begin{cases} \dfrac{\sin y}{y}, & y \in \mathbb{R}^* \\ 1, & y = 0 \end{cases}$$

E23

$$F(y) = \int_0^1 \frac{\sin(xy)}{x}\,dx, \quad y \in \mathbb{R}$$

Study convergence

is a non-elementary function

(* MATHEMATICA *)

$$\int_0^1 \frac{\mathrm{Sin}[x\,y]}{x}\,dx$$

```
SinIntegral[y]
TraditionalForm[%]
```

$\mathrm{Si}(y)$

T24 If $f:[a,b]\times[c,d]\to\mathbb{R}$ is continous, then the integral $I:[c,d]\to\mathbb{R}$

$$I(y)=\int_a^b f(x,y)\,dx \quad \text{is also continous}$$

T25 If (f_n) is a sequence of integrable functions on the interval $[a,b]$ that converges uniformly to an integrable function $[a,b]$, then

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx \stackrel{\leftrightarrows}{=} \int_a^b \lim_{n\to\infty} f_n(x)\,dx$$

Consider the functions: $f:[a,b]\times[c,d]\to\mathbb{R}$

$\alpha,\beta:[c,d]\to[a,b]$

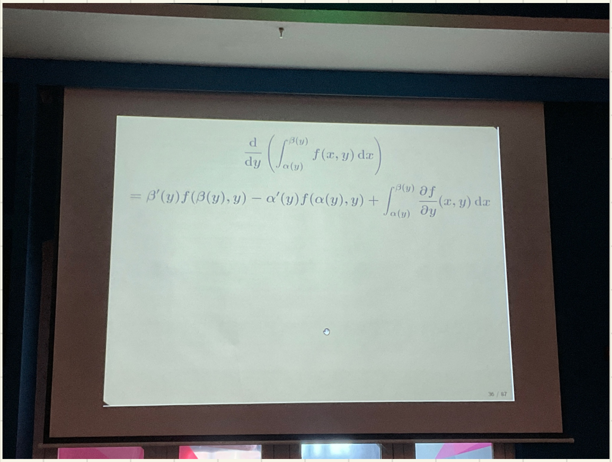
T26 If f is continous, there exists $\frac{\partial f}{\partial y}$ continous and the functions α si β possess derivatives, then the function

$I:[c,d]\to\mathbb{R}$

$$I(y)=\int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx$$

is differentiable and

$$I'(y)=\beta'(y)f(\beta(y),y)-\alpha'(y)f(\alpha(y),y)+\int_{\alpha(y)}^{\beta(y)}\frac{\partial f}{\partial y}(x,y)\,dx$$



$$\frac{d}{dy}\left(\int_{\alpha}^{\beta} f(x,y)\,dx\right) = \int_{\alpha}^{\beta} \frac{\partial}{\partial y}\left(f(x,y)\right) dx$$

$$\frac{d}{dy}\left(\int_{\alpha(y)}^{\beta(y)} g(x)\,dx\right) = \beta'(y) g(\beta(y)) - \alpha'(y) g(\alpha(y))$$

E27 Evaluate the integral

$$I(y) = \int_0^y \frac{\ln(1+xy)}{1+x^2}\,dx \quad , y \geq 0$$

We have:

$$I'(y) = \frac{\ln(1+y^2)}{1+y^2} + \int_0^y \frac{x}{(1+x^2)(1+xy)}\,dx$$

$$= \frac{\ln(1+y^2)}{1+y^2} + \int_0^y \frac{1}{1+y^2}\left(\frac{x+y}{x^2+1} - \frac{y}{xy+1}\right) dx$$

$$= \frac{\ln(1+y^2)}{1+y^2} + \frac{1}{1+y^2}\left(\frac{1}{2}\ln(x^2+1) + y\arctan x - \ln(1+xy)\right)\Bigg|_{x=0}^{x=y} =$$

$$= \frac{1}{2}\frac{\ln(1+y^2)}{1+y^2} + \frac{y\arctan y}{1+y^2} = \frac{1}{2}\left(\ln(1+y^2)\arctan y\right)'.$$

We deduce: $I(y) = \frac{1}{2}\ln(1+y^2)\arctan y + C$

i.e.,

$$\int_0^y \frac{\ln(1+xy)}{1+x^2}\,dx = \frac{1}{2}\ln(1+y^2)\arctan y + C$$

For $y=0$, we obtain

$$0 = \int_0^0 \frac{\ln(1+0)}{1+x^2}\,dx = \frac{1}{2}\ln(1+0^2)\arctan 0 + C$$

hence $C = 0$ and

$$\int_0^y \frac{\ln(1+xy)}{1+x^2}\,dx = \frac{1}{2}\ln(1+y^2)\arctan y$$

HW 28 A simpler method to evaluate the integral

$$I(y) = \int_0^y \frac{\ln(1+xy)}{1+x^2}\,dx, \quad y \geq 0$$

E29 Calculate the integral (see also Frullani Integrals)

$$F(a,b) := \int_0^\infty \frac{e^{-ax^2} - e^{-bx^2}}{x}\,dx, \quad a,b>0$$

même pdu 0

We have :

$$\frac{\partial F}{\partial a}(a,b) = -\int_0^\infty x e^{-ax^2}\,dx = \frac{e^{-ax^2}}{2a}\Bigg|_0^\infty = 0 - \frac{1}{2a}$$

hence $F(a,b) = -\frac{1}{2}\log a + C(b)$

For $a=b$, we get

$$0 = F(b,b) = -\frac{1}{2}\log b + C(b)$$

hence $C(b) = \frac{1}{2}\log b$, and $F(a,b) = \frac{1}{2}\log\frac{b}{a}$

CURS 7

Integration under the integral sign

T30 If the function $f:[a,b]\times[c,d]\to\mathbb{R}$ is continous, then:

$$\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx \;\overset{\leftarrow}{\rightleftharpoons}\; \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

Consider the auxiliary function $H:[a,b]\to\mathbb{R}$

$$H(t)=\int_a^t\left(\int_c^d f(x,y)\,dy\right)dx-\int_c^d\left(\int_a^t f(x,y)\,dx\right)dy$$

negru ♡ galben

alb ♡ midnight blue

We can see that

$$H(a)=\int_a^a\left(\int_c^d f(x,y)\,dy\right)dx-\int_c^d\left(\int_a^a f(x,y)\,dx\right)dy=0$$

We have to prove that $H(b)=0$

We have:

$$H'(t)=\int_c^d f(t,y)\,dy-\int_c^d\frac{d}{dt}\left(\int_a^t f(x,y)\,dx\right)dy$$

$$=\int_c^d f(t,y)\,dy-\int_c^d f(t,y)\,dy=0\,,\quad \forall t\in[a,b]$$

It follows that H is constant, hence $H(b) = H(a) = 0$

E31 $\displaystyle\int_0^1 \frac{x^b - x^a}{\ln x}\,dx$ $\quad (a, b > -1)$

$$\int_0^1 \frac{x^b - x^a}{\ln x}\,dx = \int_0^1 \frac{x^y}{\ln x}\Big|_{y=a}^{y=b}\,dx =$$

$$= \int_0^1 \left(\int_a^b x^y\,dy\right)dx \; \overset{\leftrightarrows}{\equiv} \; \int_a^b \left(\int_0^1 x^y\,dx\right)dy =$$

$$= \int_a^b \frac{x^{y+1}}{y+1}\Big|_{x=0}^{x=1}\,dy = \int_a^b \frac{1}{y+1}\,dy =$$

$$= \ln(y+1)\Big|_{y=a}^{y=b} = \ln\frac{b+1}{a+1}$$

R32 Note that

$$\int_0^1 \frac{x^b - x^a}{\ln x}\,dx = \int_0^\infty \frac{e^{-(b+1)t} - e^{-(a+1)t}}{t}\,dt \quad \text{(Frullani)}$$

(change of variabe $x = e^{-t}$)

The Gauss Integral – Method 1

E33 [Gauss (1)]

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

$$I = \int_0^\infty e^{-x^2}\,dx \overset{x=ty}{=\!=} t\int_0^\infty e^{-t^2y^2}\,dy \quad \Big|\cdot e^{-t^2}$$

Sumisag

$$e^{-t^2} I = t e^{-t^2}\int_0^\infty e^{-t^2y^2}\,dy \quad \Big| \int_0^\infty (\,)\,dt$$

Sumisag

$$I\cdot I = \int_0^\infty t e^{-t^2}\int_0^\infty e^{-t^2y^2}\,dy\,dt$$

$$\overset{\rightleftharpoons}{=\!=} \int_0^\infty\int_0^\infty t e^{-t^2(1+y^2)}\,dt\,dy$$

$$= \int_0^\infty \frac{e^{-t^2(1+y^2)}}{-2(1+y^2)}\Bigg|_{t=0}^{t=\infty} dy = \frac{1}{2}\int_0^\infty \frac{1}{1+y^2}\,dy = \frac{\pi}{4}$$

Frullani Integral Formulae

T_{34} [The First Frullani Integral Formula]

Let $f:(0,\infty)\to\mathbb{R}$ be a continous function, $a,b>0$
If the limits $f(0_+)$ and $f(\infty)$ exists finitely
then the following equality is satisfied

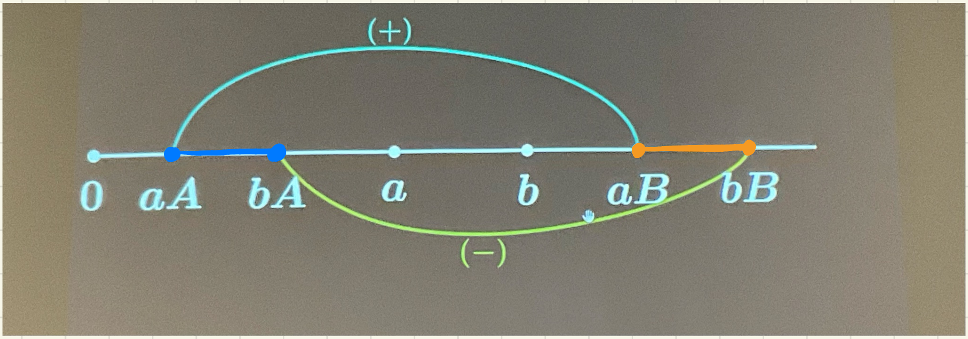
$$\int_0^{\infty}\frac{f(ax)-f(bx)}{x}\,dx=\big(f(0_+)-f(\infty)\big)\ln\frac{b}{a}$$

For definitenees, let $a<b$. We consider the numbers A and B such that $0<A<B$. We have:

$$\int_A^B\frac{f(ax)-f(bx)}{x}\,dx=\int_A^B\frac{f(ax)}{x}\,dx-\int_A^B\frac{f(bx)}{x}\,dx$$

$\underline{\underline{ax=y,\ bx=y}}$

$$\int_{aA}^{aB}\frac{f(y)}{y}\,dy-\int_{bA}^{bB}\frac{f(y)}{y}\,dy$$



$$= \int_{aA}^{aB} \frac{f(y)}{y}\,dy - \int_{bA}^{bB} \frac{f(y)}{y}\,dy \underset{\text{teorema de medie}}{=} f(\alpha)\int_{aA}^{bA} \frac{dy}{y} - f(\beta)\int_{bA}^{bB} \frac{dy}{y}$$

$$= \left(f(\alpha) - f(\beta)\right) \ln\frac{b}{a} \quad , \alpha \in (aA, bA) \quad , \beta \in (bA, bB)$$

We have used the mean-value formula

$$A(f) = \int_p^q \frac{f(x)}{x}\,dx = f(\varepsilon)A(1) = f(\varepsilon)\int_p^q \frac{1}{x}\,dx =$$

$$f(\varepsilon)\log\frac{p}{q}$$

For $A \to 0_+$, $B \to \infty$, we get

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\,dx = \left(f(0_+) - f(\infty)\right) \ln\frac{b}{a}$$

E35 Evaluate

$$\int_0^\infty \frac{e^{-2x} - e^{-3x}}{x}\,dx$$

We apply the First Frullani Formula T34

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\,dx = \left(f(0_+) - f(\infty)\right) \ln\frac{b}{a}$$

for $f(x) = e^{-x}$, $a=2$, $b=3$

$$\int_0^\infty \frac{e^{-2x} - e^{-3x}}{x} dx = \left(e^{-0} - e^{-\infty}\right) \ln \frac{3}{2} =$$

$$= (1-0) \ln \frac{3}{2} = \ln \frac{3}{2}$$

CURS 8

E36 Evaluate $\int_0^\infty \frac{e^{-ax^2} - e^{-bx^2}}{x}\, dx$, $a,b>0$

We write

$$\int_0^\infty \frac{e^{-(\sqrt{a}\,x)^2} - e^{-(\sqrt{b}\,x)^2}}{x}\, dx, \quad a,b>0$$

and we apply the First Frullani Theorem

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx = \left(f(0_+) - f(\infty)\right) \ln\frac{b}{a}$$

for $f(x) = e^{-x^2}$, $a := \sqrt{a}$, $b := \sqrt{b}$

We obtain:

$$\int_0^\infty \frac{e^{-ax^2} - e^{-bx^2}}{x}\, dx = \frac{1}{2}\log\frac{b}{a}$$

E37 Evaluate $\int_0^\infty \frac{e^{-ax^\lambda} - e^{-bx^\lambda}}{x}\, dx$, $a,b,\lambda>0$

We take $x^\lambda = y$, or we write

$$\int_0^\infty \frac{e^{-(a^{\frac{1}{\lambda}}x)^\lambda} - e^{-(b^{\frac{1}{\lambda}}x)^\lambda}}{x}\, dx,$$

and apply the First Frullani Theorem

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = \left(f(0_+)-f(\infty)\right)\ln\frac{b}{a}$$

for $f(x)=e^{-x^\lambda}$, $a := a^{\frac{1}{\lambda}}$, $b := b^{\frac{1}{\lambda}}$

We obtain

$$\int_0^\infty \frac{e^{-ax^\lambda}-e^{-bx^\lambda}}{x}\,dx = \frac{1}{\lambda}\log\frac{b}{a}$$

E38 $$\int_0^\infty \frac{\sin^4 x}{x^3}\,dx = \log 2$$

We obtain:

$$\int_0^\infty \frac{\sin^4 x}{x^3}\,dx = \int_0^\infty \frac{\left(\frac{\sin x}{x}\right)^2-\left(\frac{\sin 2x}{2x}\right)^2}{x}\,dx$$

$$= \left(\frac{\sin x}{x}\right)^2\Bigg|_0^\infty \log\frac{1}{2} = (0-1)\log\frac{1}{2} = \log 2$$

E39 $$\int_0^\infty \frac{\sin^3 x}{x^2}\,dx = \frac{3}{4}\log 3$$

We have:

$$\frac{\sin^3 x}{x} = -\frac{3}{4}\left(\frac{\sin 3x}{3x}-\frac{\sin x}{x}\right)$$

and use the First Frullani Theorem

T_{40} **[The Second Frullani Integral Formula]**

Let $f:(0,\infty)\to\mathbb{R}$ be continous

If the limit $f(0_+)$ exists and, for any $\lambda>0$

$\int_\lambda^\infty \frac{f(x)}{x}dx$, is convergent, then the following equality is satisfied

$$\boxed{\int_0^\infty \frac{f(ax)-f(bx)}{x}dx = f(0_+)\ln\frac{b}{a}}$$

E_{41} $$\int_0^\infty \frac{\cos ax-\cos bx}{x}dx \quad , a,b\in\mathbb{R}^*$$

By applying the second Frullani formula

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}dx = f(0_+)\ln\frac{b}{a}$$

for $f(x)=\cos x$ we get

$$\int_0^\infty \frac{\cos ax-\cos bx}{x}dx = \int_0^\infty \frac{\cos|a|x-\cos|b|x}{x}dx$$

$$= \cos 0 \ln\left|\frac{b}{a}\right| = \ln\left|\frac{b}{a}\right|$$

E42 $\displaystyle\int_0^\infty \frac{(1-e^{-x})^2}{x^2}\,dx$

$$\int_0^\infty \frac{(1-e^{-x})^2}{x^2}\,dx = 0 + 2\int_0^\infty \frac{e^{-x}-e^{-2x}}{x}\,dx = \log 4$$

Lobachevsky's Formula for Integrals

T43 If the function $f: \mathbb{R} \to \mathbb{R}$ satisfies the conditions

$$f(x+\pi) = f(\pi - x) = f(x), \quad \forall x \in \mathbb{R}$$

then

$$\int_0^\infty f(x)\frac{\sin x}{x}\,dx = \int_0^{\frac{\pi}{2}} f(x)\,dx$$

provided that the integrals entering the previous formula are convergent

E44 For $f(x) = 1$ we get

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^{\frac{\pi}{2}} 1\,dx = \frac{\pi}{2}$$

E45 $\displaystyle\int_0^\infty \frac{\sin ax}{x}\,dx = \frac{\pi}{2}, \quad a > 0$

$$\int_0^\infty \frac{\sin ax}{x}\,dx \overset{x=\frac{t}{a}}{=\!=\!=} \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$$

E46 For $f(x) = \sin^2 x$ we get

$$\int_0^\infty \frac{\sin^3 x}{x}\,dx = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{\pi}{4}$$

E47

$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx =$$

$$= \frac{\cos bx - \cos ax}{x}\Big|_\infty^0 + \int_0^\infty \frac{b\sin bx - a\sin ax}{x}\,dx =$$

$$= 0 + \int_0^\infty \frac{|b|\sin|b|x - |a|\sin|a|x}{x}\,dx = \frac{\pi}{2}(|b| - |a|)$$

Lobachevsky's Formula for Integrals (II)

T48 If the function $f:\mathbb{R}\to\mathbb{R}$ satisfies the equalities

$$f(x+\pi) = -f(x) \text{ and } f(-x) = f(x),\ \forall x\in\mathbb{R}$$

then

$$\int_0^\infty f(x)\,\frac{\sin x}{x}\,dx = \int_0^{\frac{\pi}{2}} f(x)\cos x\,dx$$

provided that the integrals entering the previous formula are convergent

E49 $$\int_0^{\infty} \frac{\sin x \cos x}{x}\, dx$$

$$\int_0^{\infty} \frac{\sin x \cos x}{x}\, dx = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \frac{\pi}{4}$$

SPECIAL FUNCTIONS

- Special functions are just that: specialized functions beyond the familiar elementary functions. Usually we call a function "special" when it belongs to the toolbox of the applied mathematician, physicist or engineer.

A List of Special Functions

The Euler Gamma (Γ) function:

$$\Gamma(a) = \int_0^{\infty} e^{-t} t^{a-1}\, dt \quad , \quad a > 0$$

The Euler Beta (β) function

$$\beta(a,b) = \int_0^{1} t^{a-1} (1-t)^{b-1}\, dt \quad , \quad a,b > 0$$

The Integral Sine:

$$si(x) = -\int_x^\infty \frac{\sin t}{t}\,dt \quad , \; x \geq 0$$

The integral Cosine:

$$ci(x) = -\int_x^\infty \frac{\cos t}{t}\,dt \quad , \; x > 0$$

Fresnel Integral Sine:

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin t^2\,dt \quad , \; x \in \mathbb{R}$$

Fresnel Integral Cosine:

$$C(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos t^2\,dt \quad , \; x \in \mathbb{R}$$

The Error Function:

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt \quad , \; x \in \mathbb{R}$$

The Exponential Integral

$$Ei(x) = vp \int_{-\infty}^x \frac{e^t}{t}\,dt \quad , \; x \in \mathbb{R}^*$$

The Integral Logarithm

$$li(x) = \int_0^x \frac{dt}{\ln t} , \quad 0 \leq x < 1$$

The Lobachevsky Function

$$L(x) = -\int_0^x \ln \cos t \, dt , \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

The Elliptic Integral of the First Kind

$$F(\varphi, k) = \int_0^\varphi \frac{dt}{\sqrt{1-k^2\sin^2 t}} , \quad -1 < k < 1, \; \varphi \in \mathbb{R}$$

The Elliptic Integral of the Second Kind:

$$E(\varphi, k) = \int_0^\varphi \sqrt{1-k^2\sin^2 t} \, dt , \quad -1 < k < 1, \; \varphi \in \mathbb{R}$$

The Hypergeometric Function

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k , \quad a, b, c, z \in \mathbb{C}$$

Some properties of β and Γ functions

① $\beta(a,b) = \beta(b,a)$

In $\int_0^1 x^{a-1}(1-x)^{b-1} dx$, we take $x = 1-t$

② $\beta(a,b) = \displaystyle\int_0^\infty \frac{y^{a-1}}{(1+y)^{a+b}}\,dy$

In $\displaystyle\int_0^1 x^{a-1}(1-x)^{b-1}\,dx$, we make the substitution

$x = \dfrac{y}{1+y}$

③ $\displaystyle\int_0^\infty \frac{x^p}{(1+x)^q}\,dx = \beta(p+1, q-p-1) \qquad q > p+1 > 0$

E4 $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}(1+x)^2}\,dx = \beta\left(\frac{1}{2}, \frac{3}{2}\right) = \frac{\pi}{2}$

⑤ $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx = \int_0^1 \frac{dx}{\sqrt{x(1-x)}}$$

$$= 2\int_0^1 \frac{(\sqrt{x})'\,dx}{\sqrt{1-(\sqrt{x})^2}} = 2\arcsin\sqrt{x}\,\Big|_0^1 = \pi$$

⑥ $\Gamma(a+1) = a\Gamma(a), \quad a > 0$

$$\Gamma(a+1) = \int_0^\infty e^{-x}x^a\,dx = -\int_0^\infty (e^{-x})'x^a\,dx$$

$$= -e^{-x}x^a\Big|_0^\infty + \int_0^\infty e^{-x}(x^a)'dx = 0 + a\int_0^\infty e^{-x}x^{a-1}dx$$

$$= a\,\Gamma(a)$$

⑦ $\Gamma(n+1) = n!, \quad n \in \mathbb{N}$ Factorial generalizat

We use $\Gamma(a+1) = a\Gamma(a)$ and $\Gamma(1) = 1$

$\Gamma(2) = 1\,\Gamma(1), \quad \Gamma(3) = 2\,\Gamma(2), \ldots, \Gamma(n+1) = n\Gamma(n)$

The equation

⑧ $x! := \Gamma(x+1), \quad x > -1$

defines the generalized factorial. For example

$$\left(-\tfrac{1}{2}\right)! = \Gamma\left(\tfrac{1}{2}\right) \overset{(13)}{=\!=} \sqrt{\pi}$$

⑨ $\displaystyle\int_0^\infty e^{-xy}x^{a-1}dx = \frac{\Gamma(a)}{y^a}, \quad a, y > 0$

$$\int_0^\infty e^{-xy}x^{a-1}dx \overset{x=\frac{t}{y}}{=\!=} \int_0^\infty e^{-t}\frac{t^{a-1}}{y^{a-1}}\frac{dt}{y}$$

$$= \frac{1}{y^a}\int_0^\infty e^{-t}t^{a-1}dt = \frac{\Gamma(a)}{y^a}$$

(10) $\beta(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

In (9) we take $a := a+b$, $y := y+1$ and get:

$$\frac{\Gamma(a+b)}{(1+y)^{a+b}} = \int_0^\infty e^{-x(y+1)} x^{a+b-1}\,dx \quad \Big| y^{a-1} \Big| \int_0^\infty 0\,dy$$

It follows that:

$$\Gamma(a+b) \int_0^\infty \frac{y^{a-1}}{(1+y)^{a+b}}\,dy$$

$$= \int_0^\infty y^{a-1} \left(\int_0^\infty e^{-x(y+1)} x^{a+b-1}\,dx \right) dy$$

$$\underline{\underline{\int \rightleftarrows \int}} \quad \int_0^\infty e^{-x} x^{a+b-1} \underbrace{\left(\int_0^\infty e^{-xy} y^{a-1}\,dy \right)}_{\frac{\Gamma(a)}{x^a}} \quad \text{i.e.}$$

$$\Gamma(a+b)\beta(a,b) = \int_0^\infty e^{-x} x^{a+b-1} \frac{\Gamma(a)}{x^a}\,dx$$

$$= \Gamma(a) \int_0^\infty e^{-x} x^{b-1}\,dx$$

$$= \Gamma(a)\Gamma(b)$$

CURS 9

The Euler Reflection Formula

(12) $\Gamma(a)\Gamma(1-a) = \dfrac{\pi}{\sin \pi a}$, $0<a<1$

For $b = 1-a$ from (10), (2) using the Euler integral we deduce

$$\Gamma(a)\Gamma(1-a) \overset{(10)}{=} B(a, 1-a) \overset{(2)}{=} \int_0^\infty \frac{y^{a-1}}{1+y}\,dy$$

$$\underset{\text{Euler's Integral}}{=} \frac{\pi}{\sin \pi a}$$

Examen: Euler's Reflection Formula

(13) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\left(\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right) = \Gamma\left(\tfrac{1}{2}\right)\Gamma\left(1-\tfrac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}} \right) \text{ (12)}$$

E14 $B\left(\frac{1}{2}, \frac{3}{2}\right) = \frac{\pi}{2}$

$$B\left(\tfrac{1}{2}, \tfrac{3}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)} = \frac{\Gamma\left(\frac{1}{2}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{1!} = \frac{\pi}{2}$$

(15) $B(a,n) = \dfrac{(n-1)!}{a(a+1)\ldots(a+n-1)}$, $a>0$, $n \in \mathbb{N}^*$

$$B(a,n) = \frac{\Gamma(a)\Gamma(n)}{\Gamma(a+n)} = \frac{\Gamma(a)\Gamma(n)}{(a+n-1)\Gamma(a+n-1)}$$

$$= \frac{\Gamma(a)\Gamma(n)}{(a+n-1)(a+n-2)\dots(a+1)a\Gamma(a)} = \dots$$

$$= \frac{(n-1)!}{(a+n-1)(a+n-2)\dots(a+1)a}$$ Rising factorial

Euler - Weierstrauss Formula

(16) $$\Gamma(a) = \lim_{n\to\infty} n^a \frac{(n-1)!}{a(a+1)\dots(a+n-1)}, \quad a>0$$ EXAMEN ??

$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1}\, dx$$

$$= \int_0^\infty \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{-n-a} x^{a-1}\, dx$$ BASIC FORMULA

$$\overset{\lim \rightleftarrows \int}{=} \lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n-a} x^{a-1}\, dx$$

$$\overset{x=ny}{=} \lim_{n\to\infty} n^a \int_0^\infty \frac{y^{a-1}}{(1+y)^{n+a}}\, dy$$

$$\overset{(2)}{=} \lim_{n\to\infty} n^a B(a,n)$$

$$\overset{(15)}{=} \lim_{n\to\infty} n^a \frac{(n-1)!}{a(a+1)\dots(a+n-1)}$$

The Weierstrauss Formula for Γ

(17) $$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}}$$ analytic on $\mathbb{C}$

By (16) $\left(\Gamma(z) = \lim_{n\to\infty} (n+1)^z \frac{n!}{z(z+1)\dots(z+n)} \right)$ we obtain

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} z(n+1)^{-z} \prod_{k=1}^{n} \frac{k+z}{k}$$

$$= \lim_{n\to\infty} z e^{-z\ln(n+1)} \prod_{k=1}^{n} \frac{k+z}{k}$$

$$= \lim_{n\to\infty} z e^{-z\ln(n+1) + z\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)} \prod_{k=1}^{n} \frac{k+z}{k} e^{-\frac{z}{k}}$$

$$= z e^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}}$$

Γ for complex numbers

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt,$$ analytic for $\mathcal{R}(z) > 0$

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$$

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{e^{\frac{z}{n}}}{1+\frac{z}{n}}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$$

Where γ is Euler's constant

E18 $\Gamma(1+i) = \int_0^\infty t^i e^{-t}\, dt$

$$= \int_0^\infty \left(\cos(\log t) + i\sin(\log t)\right) e^{-t}\, dt \approx 0{,}498 - 0{,}154i$$

The Wallis Formula

(19) $\lim_{n\to\infty} \frac{(2n)!!}{(2n-1)!!} \frac{1}{\sqrt{n}} = \sqrt{\pi}$ (Wallis)

$$\sqrt{\pi} = \Gamma\left(\tfrac{1}{2}\right) \overset{(16)}{=} \lim_{n\to\infty} \frac{n^{\frac{1}{2}}(n-1)!}{\frac{1}{2}\left(\frac{1}{2}+1\right)\dots\left(\frac{1}{2}+n-1\right)}$$

$$= \lim_{n\to\infty} \sqrt{n}\, \frac{2^n (n-1)!}{(2n-1)!!} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \frac{2^n n!}{(2n-1)!!}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \frac{(2n)!!}{(2n-1)!!}$$

(20) $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\dots\Gamma\left(\frac{n-1}{n}\right) = \sqrt{\frac{2^{n-1}\pi^{n-1}}{n}}$