Electrotechnics
ET

Course 8

Year I-ISA English

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= Course 8 =

ELECTRICAL CIRCUITS IN HARMONIC REGIME

Linear Electric Circuits in Permanent Sinusoidal Regime

Power in harmonic regime

Complex characterization of linear circuits

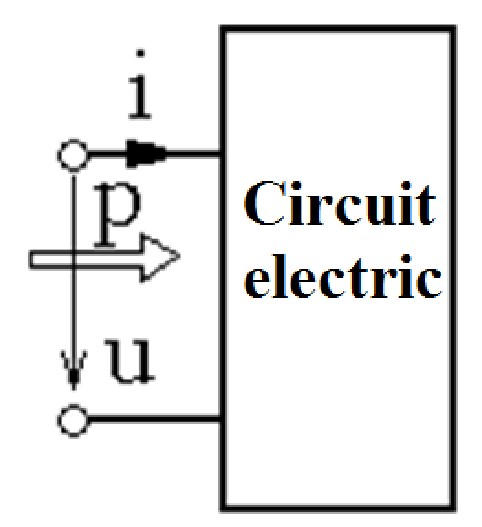
Specific laws and theorems under complex form

Complex equivalent impedances

Power in Harmonic (Sinusoidal) Regime

1. Instantaneous Power

- **Suppose we have a passive circuit with R, L, C and a sinusoidal voltage is applied to its terminals:**



$$u(t) = \sqrt{2}\,U \sin(\omega t + \gamma_u)$$

$$i(t) = \sqrt{2}\,I \sin(\omega t + \gamma_i)$$

- **Instantaneous Power:** $p(t) = u(t) \cdot i(t)$

$$\Rightarrow \quad p(t) = 2UI \sin(\omega t + \gamma_u) \sin(\omega t + \gamma_i)$$

$$\sin a \sin b = \frac{\cos(a-b) - \cos(a+b)}{2}$$

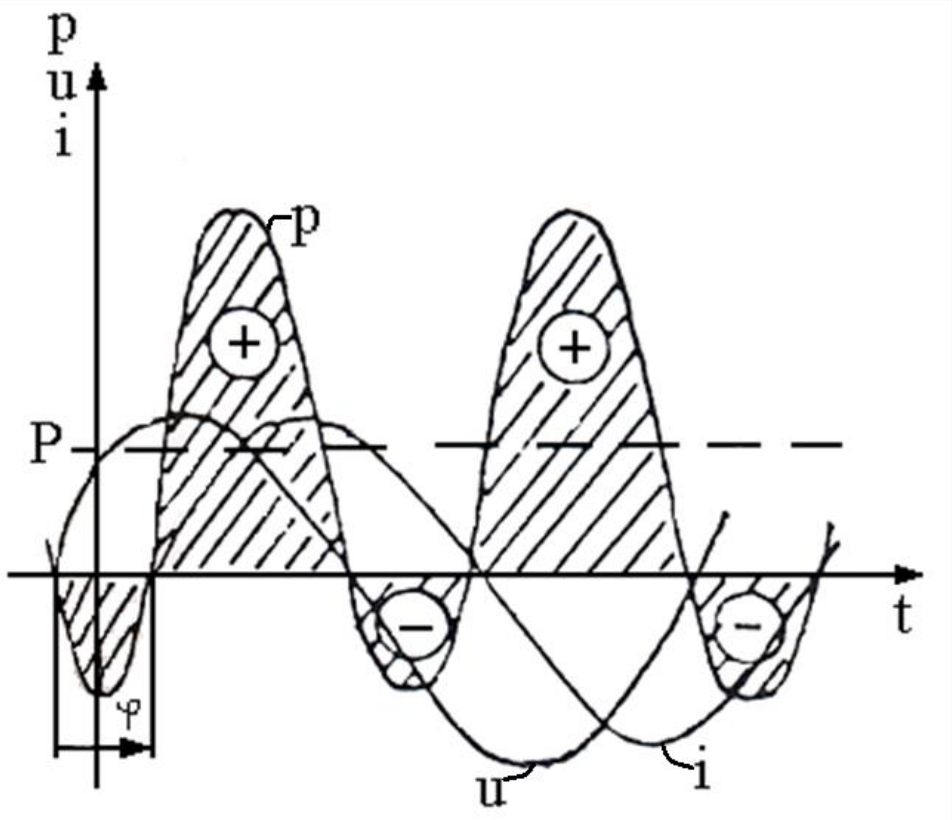
$$\Rightarrow \quad p(t) = UI\,[\cos(\gamma_u - \gamma_i) - \cos(2\omega t + \gamma_u + \gamma_i)]$$

2. Active Power

- **The medium power absorbed in a period, called active power, is:**

$$P = \frac{1}{T}\int_0^T p(t)\,dt$$

$$\boxed{P = U\,I\cos\varphi \geq 0, \quad [W]} \qquad \textbf{(1)}$$



- **cosφ – power factor;**

Note

- **in resistive circuits φ=o** → **cosφ=1**

  → $P = U\,I$

- **in pure reactive circuits (L, C) φ = ± 90°** → **cosφ=0**

  → $P = 0$

Corresponding to this power, the **resistance** is defined as:

$$R = \frac{P}{I^2} = \frac{U}{I}\cos\varphi, \quad [\Omega]$$

and similarly, the **conductance**:

$$G = \frac{P}{U^2} = \frac{I}{U}\cos\varphi, \quad [S]$$

the active power :

$$P = R I^2$$

$$P = G U^2$$

$$P \geq 0, \quad [W]$$

3. The Apparent Power. Power Factor

- the **apparent power, S**, is the product of the current and voltage effective values:

$$S = U\,I,\ [VA] \qquad (2)$$

- the ratio between the apparent power and the and the square of the effective current value is called **impedance:**

$$Z = \frac{S}{I^2} = \frac{U}{I},\ [\Omega]$$

- the inverse value of the impedance is called **admitance:**

$$Y = \frac{S}{U^2} = \frac{I}{U} = \frac{1}{Z},\ [S]$$

- the **power factor** is the positive ratio between the active power, *P, and* the apparent power, *S*:

$$k_p = \frac{P}{S} \geq 0,\ [-]$$

Note

- in sinusoidal regime: $k_p = \cos\varphi$

4. Reactive Power

- by analogy with the active power the **reactive power, Q**, is defined with the relation:

$$Q = U I \sin\varphi, \quad [VAr] \qquad (3)$$

- from (1), (2), (3) ⇒

$$Q = \sqrt{S^2 - P^2} \qquad (4)$$

- the ratio between the reactive power and the square of the effective current value is called **reactance:**

$$X = \frac{Q}{I^2} = \frac{U}{I}\sin\varphi$$

- similar, the ratio between the reactive power and the square of the effective voltage value is called **susceptance:**

$$B = \frac{Q}{U^2} = \frac{I}{U}\sin\varphi$$

⇒ **Reactive power:**

$$Q = X I^2$$

$$Q = B U^2$$

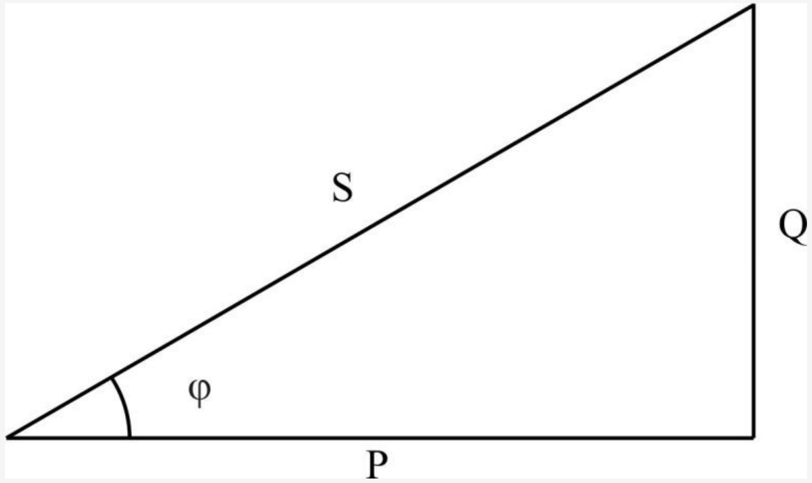
$$Q \geq 0 \, sau < 0$$

5. The Triangle of Powers

- **From relation (4):**

$$Q = \sqrt{S^2 - P^2}$$

➡ **that to the powers from an electrical circuit a right-angle triangle can be attached in which the hypotenuse is equal to the apparent power *S*, and the two legs of the right triangle are the active power P and the reactive power *Q*.**



Pythagorean Theorem:

$$S^2 = P^2 + Q^2$$

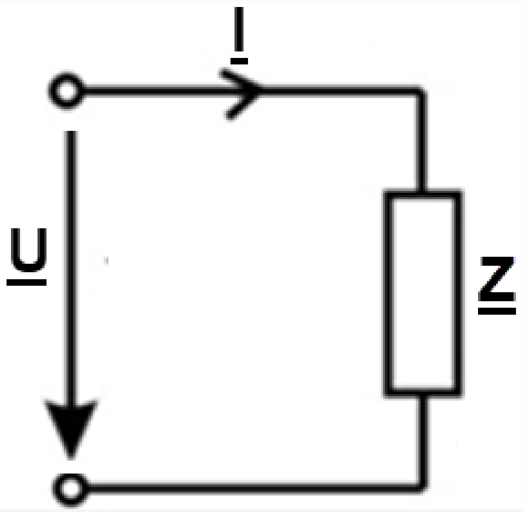
$$P = S\cos\varphi$$

$$Q = S\sin\varphi$$

Complex Characterization of Linear Circuits

1. Complex Impedance and Admittance

- the ratio between the complex voltage and the complex current (simplified or non-simplified) defines a complex quantity characteristic to a circuit branch called **complex impedance:**



$$\underline{Z} = \frac{\underline{U}}{\underline{I}} = \frac{U e^{j\gamma_u}}{I e^{j\gamma_i}} = \frac{U}{I} e^{j(\gamma_u - \gamma_i)} = \frac{U}{I} e^{j\varphi}$$

$$\Rightarrow \underline{Z} = Z e^{j\varphi}, \qquad e^{j\varphi} = \cos\varphi + j\sin\varphi$$

$$\Rightarrow \underline{Z} = \underbrace{\frac{U}{I}\cos\varphi}_{R} + j\underbrace{\frac{U}{I}\sin\varphi}_{X}$$

$$\Rightarrow \boxed{\underline{Z} = R + jX}$$

$$\Rightarrow \boxed{Z = \sqrt{R^2 + X^2}}$$

complex admittance represents the ratio between the complex current and complex voltage from a circuit branch:

$$\underline{Y} = \frac{\underline{I}}{\underline{U}} = \frac{1}{\underline{Z}} = \frac{I}{U} e^{-j\varphi}, \quad [S]$$

$$\underline{Y} = \underbrace{\frac{I}{U}\cos\varphi}_{G} - j\underbrace{\frac{I}{U}\sin\varphi}_{B} \quad \Rightarrow \quad \boxed{\underline{Y} = G - jB}$$

$$\boxed{Y = \sqrt{G^2 + B^2}}$$

where:

- G – conductance;
- B – susceptance.

Attention:

$$\underline{Z} = \frac{1}{\underline{Y}}$$

$$R = \frac{1}{G}; \quad X \neq \frac{1}{B}$$

2. Complex Power

- because the instantaneous power is not a sinusoidal parameter a complex symbol can not be attached to it; however, in order to write under a compact form, the three types of power (P, S, Q) the complex writing of the apparent power is used like:

$$\underline{S} = \underline{U}\,\underline{I}^*$$

where:

- **U** –complex value of the voltage;
- **I*** – conjugated complex value of the current.

$$\underline{I} = Ie^{j\gamma} \Leftrightarrow \underline{I}^* = I\,e^{-j\gamma_i}$$

$$\underline{S} = Ue^{j\gamma_u} I\,e^{-j\gamma_i} = U\,I\,e^{j(\gamma_u - \gamma_i)} = U\,I\,e^{j\varphi}$$

$$\underline{S} = S\cos\varphi + jS\sin\varphi = P + jQ$$

$$\underline{S} = P + jQ \qquad S = \sqrt{P^2 + Q^2}$$

3. Passive Circuit Elements in Complex

❑ Ideal Resistor

$$u = Ri \quad \leftrightarrow \quad \underline{U} = R\,\underline{I}$$

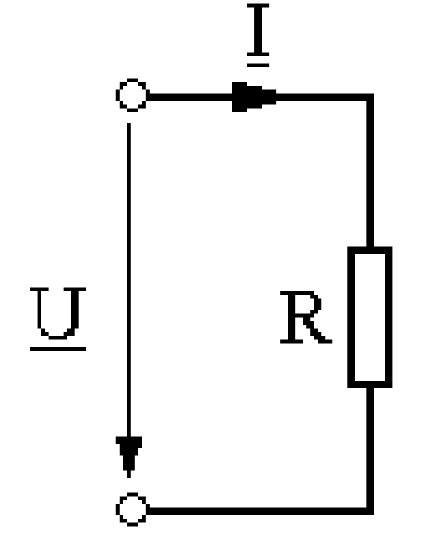
$$\underline{Z} = \frac{\underline{U}}{\underline{I}} = R; \qquad X = 0; \qquad \varphi = arctg\,\frac{X}{R} = 0$$

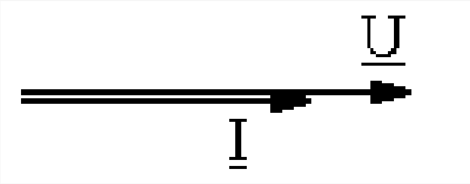
$$\underline{Y} = \frac{1}{\underline{Z}} = \frac{1}{R} = G - jB; \quad G = \frac{1}{R}; \quad B = 0$$

$$\underline{S} = P + jQ = \underline{Z}\,I^2 = R\,I^2 = \underline{Y}^* U^2 = \frac{U^2}{R}$$

$$P = R\,I^2 = G\,U^2;$$

$$Q = 0$$





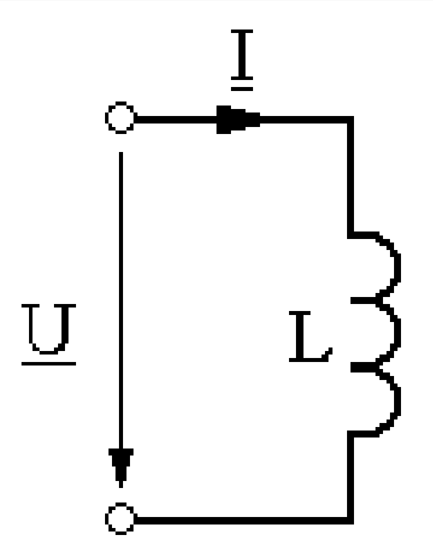
❑ Ideal Inductor

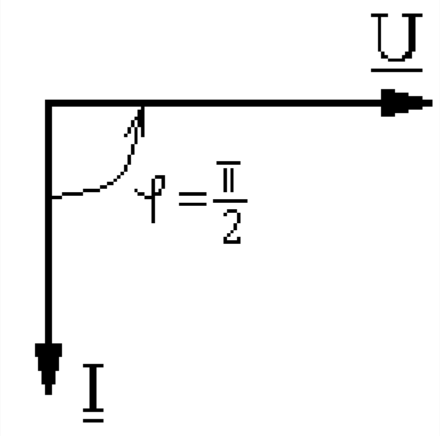
$R = 0$

$$u = L\frac{di}{dt} \leftrightarrow \underline{U} = j\omega L\,\underline{I}$$

$$\underline{Z} = \frac{\underline{U}}{\underline{I}} = j\omega L$$

$$R = 0;\quad X = \omega L;\quad \varphi = \frac{\pi}{2}$$





$$\underline{Y} = \frac{1}{\underline{Z}} = \frac{1}{j\omega L} = -j\frac{1}{\omega L} = G - jB;\quad G = 0;\quad B = \frac{1}{\omega L}$$

$$\underline{S} = P + jQ = \underline{Z}\,I^2 = j\omega L\,I^2 = \underline{Y}^* U^2 = j\frac{U^2}{\omega L}$$

$$P = 0;$$

$$Q = \omega L\,I^2 = \frac{U^2}{\omega L}$$

❑ Ideal Capacitor

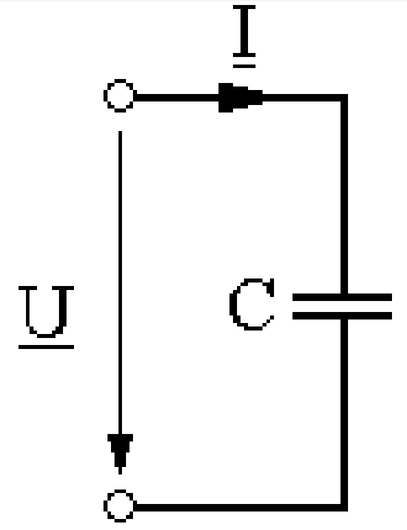
$$i=\frac{dq}{dt}=C\frac{du}{dt} \quad \leftrightarrow \quad \underline{I}=j\omega C\underline{U}$$

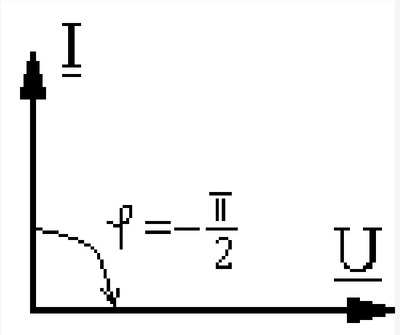
$$\underline{Z}=\frac{\underline{U}}{\underline{I}}=\frac{1}{j\omega C}=-j\frac{1}{\omega C}$$

$$R=0;\quad X=-\frac{1}{\omega C};\quad \varphi=-\frac{\pi}{2}$$

$$\underline{Y}=\frac{1}{\underline{Z}}=j\omega C=G-jB;\quad G=0;\quad B=-\omega C$$

$$\underline{S}=P+jQ=\underline{Z}I^2=-j\frac{I^2}{\omega C}=\underline{Y}^{*}U^2=-j\omega CU^2$$





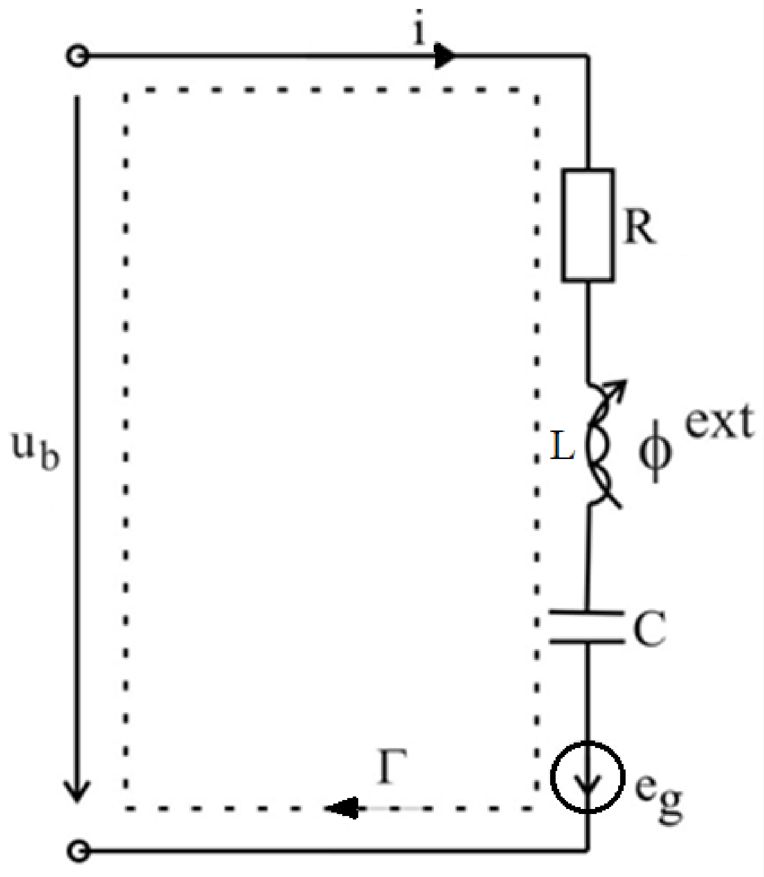
$$P=0;$$

$$Q=-\frac{I^2}{\omega C}=-\omega CU^2$$

Specific Laws and Theorems under Complex Form

1. Ohm's Law Under Complex Form

- an active branch in the circuit is considered, characterized by the parameters ***R, L, C*** and having a sinusoidal electromotive voltage generator e_g



$$\oint_{\Gamma} \overline{E}\,\overline{ds} = -\frac{d\Phi}{dt} + e_g$$

$$\oint_{\Gamma} \overline{E}\,\overline{ds} = u_R + u_C - u_b$$

$$\Phi = L_{jj} \cdot i_j + \Phi^{(ext)}$$

$$e_g - \frac{d\Phi^{(ext)}}{dt} + u_b = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt$$

$$\underline{E}_g - j\omega\underline{\Phi}^{(ext)} + \underline{U}_b = R\underline{I} + j\omega L\underline{I} + \frac{1}{j\omega C}\underline{I}$$

$$\underline{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right) \Rightarrow \underline{E}_g - j\omega\underline{\Phi}^{(ext)} + \underline{U}_b = \underline{Z}\cdot\underline{I}$$

Ohm's Law in complex

- **for uncoupled branches in inductive**

$$\Phi^{(ext)} = 0 \Rightarrow \underline{E}_g + \underline{U}_b = \underline{Z}\,\underline{I}$$

- **another way to write it**

$$\underline{E}_{gk} + \underline{U}_k = \underline{Z}_{kk}\underline{I}_k + \sum_{\substack{k=1\\k\neq j}}^{\ell}\underline{Z}_{kj}\underline{I}_j$$

$$\underline{Z}_{kk} = R_k + j\left(\omega L_{kk} - \frac{1}{\omega C_k}\right) \qquad \underline{Z}_{kj} = j\omega L_{kj}$$

$$\underline{Z}_{kj} = 0 \Rightarrow \underline{E}_{gk} + \underline{U}_k = \underline{Z}_{kk}\underline{I}_k$$

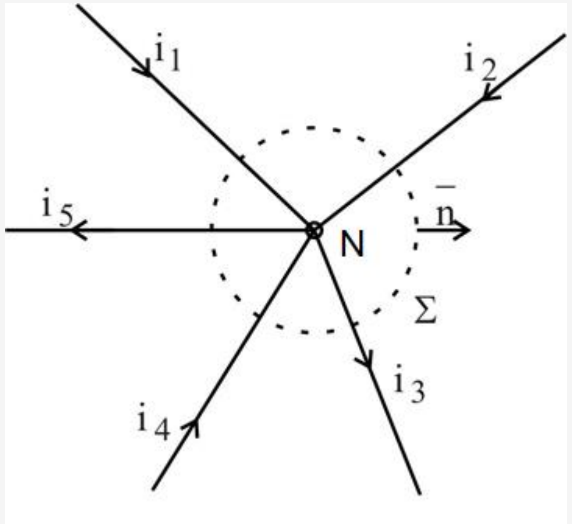
2. Kirchhoff Theorems in Complex

A) Kirchhoff's First Theorem

- refers to the network nodes;
- the algebraic sum of the instantaneous values of the currents meeting in a node is null:

$$\sum_{k \in N} i_k = 0 \qquad (1)$$

Example



- relation (1) transcribed in complex:

$$\sum_{k \in N} \underline{I}_k = 0 \qquad (2)$$

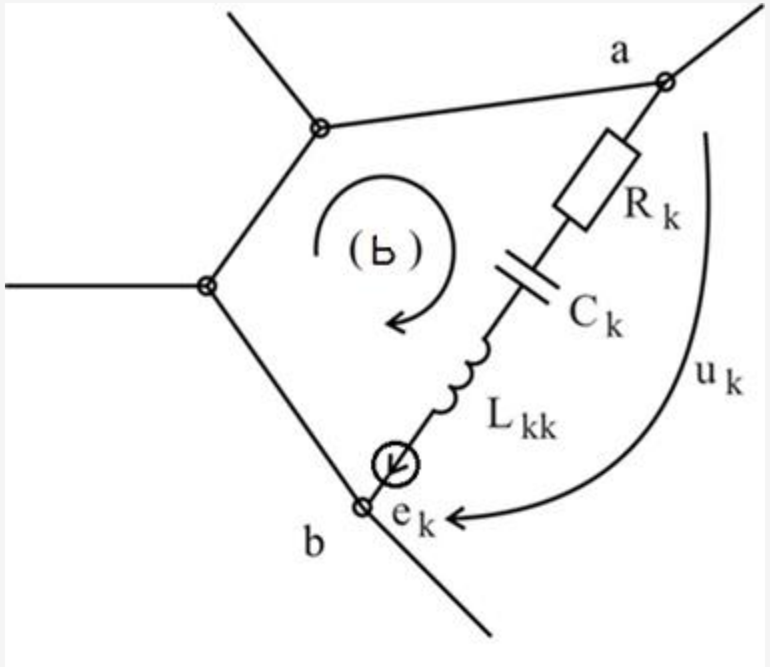
The algebraic sum of the complex images of the currents from the adjacent branches of a node is null.

$$-i_1 - i_2 + i_3 - i_4 + i_5 = 0 \Leftrightarrow -\underline{I}_1 - \underline{I}_2 + \underline{I}_3 - \underline{I}_4 + \underline{I}_5 = 0$$

Attention: $\sum_{k \in N} I_k \neq 0$

B) Kirchhoff's Second Theorem

- refers to network loops;
- we consider the *k* branch of a network loop:



- **Ohm's Law for this branch:**

$$\underline{E}_k + \underline{U}_k = \underline{Z}_{kk}\underline{I}_k + j\omega\underline{\Phi}_k^{(ext)} \qquad \textbf{(1)}$$

where:

- $\underline{Z}_{kk}$ – self-impedance of branch *k*: $\underline{Z}_{kk} = R_k + j\left(\omega L_{kk} - \dfrac{1}{\omega C_k}\right)$ **(2)**

$\underline{\Phi}_k^{(ext)}$ – magnetic flux, due to the coupling existent between the branch *k* and other branches:

$$\Phi_k^{(ext)} = \sum_{k \neq j} L_{kj} \cdot i_j \Longrightarrow \underline{\Phi}_k^{(ext)} = \sum_{k \neq j} L_{kj} \cdot \underline{I}_j \qquad \textbf{(3)}$$

(3)→(1) $\Longrightarrow$

$$\underline{E}_k + \underline{U}_k = \underline{Z}_{kk}\underline{I}_k + j\omega\sum_{k \neq j} L_{kj}\underline{I}_j \qquad \textbf{(4)}$$

- If the coupling impedance is: $\underline{Z}_{kj} = j\omega L_{kj}$ **(5)**

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(5)→(4)

- **Ohm's Law is placed in this form:** $$\underline{E}_k + \underline{U}_k = \underline{Z}_{kk}\underline{I}_k + \sum_{k \neq j} \underline{Z}_{kj}\underline{I}_j \quad \textbf{(6)}$$

- **the sum of all the relations with the same form as (6) along the Γ contour of a network loop (b) and taking into account that:** $\sum_{k \in B} \underline{U}_k = 0$

$$\sum_{k \in B} \underline{E}_k = \sum_{k \in B} \left(\underline{Z}_{kk}\underline{I}_k + \sum_{k \neq j} \underline{Z}_{kj}\underline{I}_j \right)$$

Kirchhoff's Second Theorem

- **In uncoupled circuits:**

$$L_{kj} = 0 \Rightarrow \sum_{k \in B} \underline{E}_k = \sum_{k \in B} \underline{Z}_{kk}\underline{I}_k$$

In a network loop the algebraic sum of the complex images of the electromotive voltages is equal with the algebraic sum of the complex changes in potential from that loop.

❑ Voltage between Two Nodes

$$\underline{U}_{AB} = \sum_{A \to B} \underline{U}_k = \sum_{A \to B} \left(\sum_{j=1}^{l} \underline{Z}_{kj} \underline{I}_j - \underline{E}_k \right)$$

- **in uncoupled circuits:**

$$\underline{Z}_{kj} = 0 \quad \Rightarrow \quad \underline{U}_{AB} = \sum_{A \to B} \underline{U}_k = \sum_{A \to B} \left(\underline{Z}_{kk} \underline{I}_k - \underline{E}_k \right)$$

Thank you for your attention!!!

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Questions???