Mircea Ivan

CALCULUS with *MATHEMATICA*

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Preface

Most of this textbook is based on the Romanian edition of the mathematical analysis course offered for many years at the Technical University of Cluj–Napoca. The material included in the text has been tested and refined as part of lectures addressing students in the Faculty of Automation and Computer Science.

This book is intended as a text-book for a one semester mathematical analysis course for first-year graduate students.

As a prerequisite, the reader is assumed to be familiar with the basic concepts of set theory. To establish an unambiguous language, the first two chapters briefly review terms and symbols in logic and set theory. The subsequent chapters of the book cover elements of general topology, metric spaces, continuity of functions, sequences and series of numbers and functions, and differential calculus for functions of one and several variables.

The book includes worked examples that illustrate how Mathematical Analysis problems can be approached in Mathematica, a computer algebra system originally conceived by Stephen Wolfram. Their focus is the Mathematical problems themselves, rather than the built in functions of Mathematica.

I wish to extend my gratitude to my outstanding former students Dan Grecu and Marian Ursu for their sustained support at various stages of this undertaking.

Mircea Ivan

Cluj-Napoca

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Contents

1

Elements of Logic

Contents

1.1 The Statement Calculus. Sentential Connectives

A `statement` (or `proposition`) is a declarative sentence which has the quality that it can be classified as either true or false, but not both.

We denote the `truth value` of a statement p by $v(p)$. The value $v(p)$ is 1, if p is true and 0, if p is false.

In mathematical discourse and elsewhere one constantly encounters declarative sentences formed by modifying a sentence with the word `not` or connecting sentences with the combinations of words `and`, `or`, `implies` and `if and only if`. The corresponding logical operators for these five `sentential connectives` are called `logical connectors` and are denoted respectively by the symbols:

$$\neg, \quad \wedge, \quad \vee, \quad \to, \quad \leftrightarrow .$$

Thus, if p and q are statements, using the logical connectors, we can obtain the following statements:

The `negation` of the statement p is a statement denoted by

$$\neg p,$$

read *"non p"*, which is true if and only if p is false.

The `conjunction` of the statements p and q is a statement denoted by

$$p \wedge q,$$

read *"p and q"*. The conjunction statement is true if and only if the statements p and q are both true.

The `disjunction` of the statements p and q is a statement denoted by

$$p \vee q,$$

read *"p or q"*. The disjunction statement is true if and only if at least one of the statements p and q is true.

The `conditional` of the statements p and q (in this order) is a statement denoted by

$$p \to q,$$

read *"p implies q"* or *"if p then q"*. The conditional statement is false if and only if p is true and q is false.

The **biconditional** of the statements p and q is a statement denoted by

$$p \leftrightarrow q,$$

read "*p equivalent to q*" or "*p if and only if q*". The biconditional statement is true if and only if the statements are both true or false.

Below is the truth table for the composite statements mentioned above.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	1	0	0	1	1
0	1	1	0	1	1	0
1	0	0	0	1	0	0
1	1	0	1	1	1	1

A composite statement whose truth value is 1 regardless of the truth values of its components is called a **tautology**.

If the statement $(p \rightarrow q)$ is a tautology, then we shall write

$$p \Longrightarrow q.$$

We say that p is a **sufficient condition** for q and q is a **necessary condition** for p.

If the statement $(p \leftrightarrow q)$ is a tautology, then we shall write

$$p \Longleftrightarrow q.$$

We say that p is a **necessary and sufficient condition** for q and vice versa.

The following tautologies and relations are often used:

$\neg\neg p \Longleftrightarrow p;$
$p \rightarrow q \Longleftrightarrow \neg q \rightarrow \neg p;$
$\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q;$
$\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q.$

$v(\neg p) = 1 - v(p);$
$v(p \wedge q) = v(p)v(q);$
$v(p \vee q) = v(p) + v(q) - v(p)v(q).$

1.1.1 REMARK. *It is possible to eliminate the operators* $\vee$, $\rightarrow$ *and* $\leftrightarrow$ *from any statement; that is, any statement is equivalent to one obtained by using the negation* $\neg$ *and the conjunction* $\wedge$. *For example, we have:*

$$p \vee q \iff \neg(\neg p \wedge \neg q),$$
$$p \rightarrow q \iff \neg(p \wedge \neg q),$$
$$p \leftrightarrow q \iff \neg(p \wedge \neg q) \wedge \neg(q \wedge \neg p).$$

1.2 The Predicate Calculus

The symbols used in mathematical sentences can be `constants` or `variables`.

An expression containing a finite number of variables and becoming a statement when specific values are substituted for the variables is called a `predicate` or `statement function`.

1.2.1 EXAMPLE. The expression

$$p(x): \quad x \text{ is a real number}$$

is a predicate of the variable x. For $x = 7$ we obtain the statement

$$p(7): \quad 7 \text{ is a real number.}$$

If $p(x)$ and $q(x)$ are predicates, then $\neg p(x)$, $p(x) \wedge q(x)$, $p(x) \vee q(x)$, $p(x) \rightarrow q(x)$ and $p(x) \leftrightarrow q(x)$ are also predicates.

Let us introduce two new logical operators called `quantifiers`:

$\forall$ – `universal quantifier`;

$\exists$ – `existential quantifier`.

The quantifier $\forall$ assigns to a predicate $p(x)$ the statement

$$(\forall x)\ p(x),$$

which reads *"for every* x, $p(x)$*"* or *"for all* x, $p(x)$*"* or *"for each* x, $p(x)$*"*.

The quantifier $\exists$ assigns to a predicate $p(x)$ the statement

$$(\exists x)\ p(x),$$

which reads *"there exists an x such that $p(x)$"* or *"for some x, $p(x)$"* or *"for at least one x, $p(x)$"*.

Consequently we have the tautologies:

$$\neg((\forall x)p(x)) \iff (\exists x)\neg p(x),$$

$$\neg((\exists x)p(x)) \iff (\forall x)\neg p(x).$$

If $p(x, y)$ is a predicate of two variables, then the following properties hold:

$$(\forall x)(\forall y)\ p(x, y) \iff (\forall y)(\forall x)\ p(x, y),$$

$$(\exists x)(\exists y)\ p(x, y) \iff (\exists y)(\exists x)\ p(x, y),$$

but, in general,

$$(\exists x)(\forall y)\ p(x, y), \not\Leftrightarrow (\forall y)(\exists x)\ p(x, y).$$

If the statement

$$(\forall x)(p(x) \rightarrow q(x))$$

is true, then we write

$$p(x) \Longrightarrow q(x).$$

If the statement

$$(\forall x)(p(x) \leftrightarrow q(x))$$

is true, then we write

$$p(x) \iff q(x).$$

It is to be noted that, without violating the correctness of the mathematical reasoning, some of the logical symbols will be replaced by common words. For instance, the symbol $\wedge$ will be replaced by a comma and the quantifiers symbols will be omitted when there is no ambiguity.

2

Elements of Set Theory

Contents

2.1 Sets. Collections. Classes

According to Georg Cantor[1] a **set** is *any collection of definite, distinguishable objects of our intuition or of our intellect to be conceived as a whole.*

The objects of a set are called **elements** or **members** of the set.

If M denotes a given set of objects and x is one of these objects we say that x is an element of the set M (synonymously, x belongs to M) and we write $x \in M$, otherwise we write $x \notin M$. The notion of **membership**, denoted by the symbol "$\in$", is a relation between objects and sets.

In order to determine whether a collection is a set, we must be able to decide without any doubt whether an object belongs to the set or not. That is, either $x \in M$ or $x \notin M$ is true but not both.

Two sets are equal if and only if they have the same elements. The equality of two sets A and B will be denoted by

$$A = B,$$

otherwise we shall write

$$A \neq B.$$

In order to prove that two sets A and B are equal one should show that if $x \in A$, then $x \in B$ and if $x \in B$, then $x \in A$.

[1] 

Georg Ferdinand Cantor (1845–1918), a German mathematician; the founder of theory of sets.

The set whose members are $x_1, \ldots, x_n$ will be written

$$\{x_1, \ldots, x_n\}.$$

In particular, $\{x\}$ is called a **unit set**. It consists of one member namely x.

The **membership relation** of sets, rather than the concept of set itself, plays the principal role of set theory.

A special set in set theory is the **empty set**, also called the **null set** or the **void set**, which contains no elements. The notation for the empty set is the symbol $\emptyset$. Accordingly, $x \notin \emptyset$ for any x.

Another important relation for sets is the **inclusion** relation. A set A is a **subset** of a set B if each member of A is also a member of B. In this case we write

$$A \subseteq B$$

and we read *"A is included in B"*. This is synonymous with *"B includes A"*, symbolized by

$$B \supseteq A.$$

Consequently, the empty set is a subset of every set.

A set A is said to be a **proper subset** of a set B, denoted by $A \subset B$, iff $A \subseteq B$ and $A \neq B$.

The following property of the inclusion relation is obvious

$$A \subseteq B \quad \text{and} \quad B \subseteq A \quad \text{implies} \quad A = B.$$

The statements $1 \in \{1\}$, $\emptyset \in \{\emptyset\}$, $\emptyset \subseteq \{\emptyset\}$ are true. The statements $1 \subseteq \{1\}$, $\emptyset \in \emptyset$ are false.

Let M be a set and let p be a predicate defined on M.

The totality of all elements x belonging to M for which the sentence $p(x)$ is true represents a set denoted by

$$\{x \mid x \in M \wedge p(x)\}.$$

The objects defined by the symbol

$$\{x \mid p(x)\},$$

without any reference to a set M, are also called **classes**.

A set whose elements are themselves sets is also called a **collection**, a **family** or a **system**.

The empty set can be defined as

$$\{x \mid x \neq x\}.$$

2.1.1 THEOREM. *If M is a set and $A = \{x \mid x \in M \wedge x \notin x\}$, then $A \notin M$.*

Proof. Suppose that $A \in M$. By the definition of the set A, the relation $A \in A$, implies $A \notin A$. Similarly, $A \notin A$ implies $A \in A$. So, the relation $A \in M$ implies the relation $A \in A \iff A \notin A$, which is impossible. Hence, the assumption $A \in M$ is false. ◆◆◆

2.1.2 COROLLARY. *The class of all sets is not a set.*

Proof. Suppose that *"the class of all sets"* is a set. Denote it by M. Then the set A defined in theorem (2.1.1) does not belong to M. It follows that M can not be "the set of all sets". ◆◆◆

Therefore, we say **the class of all sets**.

2.2 Set Operations

Let A and B be sets, I be a nonempty **index set**, $\{A_i \mid i \in I\}$ be a collection of sets and $(B_n)_{n\in\mathbb{N}}$ be a sequence of sets. We define the following set operations:

- The **union** (sum) of A and B,

  $A \cup B \overset{def}{=} \{x \mid (x \in A) \vee (x \in B)\}$ (read *"A cup B"*);

- The **intersection** (product) of A and B,

  $A \cap B \overset{def}{=} \{x \mid (x \in A) \wedge (x \in B)\}$ (read *"A cap B"*);

  If $A \cap B = \emptyset$, we say that A and B are **disjoint**, otherwise A and B **intersect**.

- The **difference** between A and B,

  $A \setminus B \overset{def}{=} \{x \mid (x \in A) \wedge (x \notin B)\}$ (read *"A minus B"*);

In order to denote the difference $A \setminus B$, also called the `complement` of B relative to A, we shall use the notation $\complement_A(B)$ or, in the case when B is a subset of a universal set T, then the set $T \setminus B$ is denoted by the symbol

$$\complement(B).$$

- The `symmetric difference` of A and B,

$$A \Delta B \stackrel{def}{=} (A \setminus B) \cup (B \setminus A);$$

- The `Cartesian product` of A and B,

$$A \times B \stackrel{def}{=} \{(a,b) \mid (a \in A) \wedge (b \in B)\} \quad \text{(read “}A \textit{ times } B\text{”)};$$

- The `union` of a collection of sets,

$$\bigcup_{i \in I} A_i \stackrel{def}{=} \{x \mid (\exists i)(i \in I \wedge x \in A_i)\};$$

- The `intersection` of a collection of sets,

$$\bigcap_{i \in I} A_i \stackrel{def}{=} \{x \mid (\forall i)(i \in I \to x \in A_i)\};$$

- The `Cartesian product` of a collection of sets,

$$\prod_{i \in I} A_i \stackrel{def}{=} \{f \mid f : I \to \bigcup_{i \in I} A_i,\ f(i) \in A_i\};$$

- The `power set` of a set A as the collection of all the subsets of A,

$$\mathcal{P}(A) \stackrel{def}{=} \{X \mid X \subseteq A\};$$

- The `limit inferior` of a sequence of sets,

$$\liminf(B_n) \stackrel{def}{=} \bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} B_n;$$

- The `limit superior` of a sequence of sets,

$$\limsup(B_n) \stackrel{def}{=} \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} B_n.$$

2.3 Relations

In mathematics the word *relation* is used in the sense of relationship. Let X and Y be two arbitrary sets.

2.3.1 DEFINITION. *A subset $\mathcal{R}$ of the Cartesian product $X \times Y$ is called a* **binary relation** *defined from X to Y.*

If $(x, y) \in \mathcal{R}$, then we write $x\mathcal{R}y$ and we say that "*x is $\mathcal{R}$–related to y*". The sets:

$$\mathrm{Dom}(\mathcal{R}) = \{x \mid x \in X \wedge (\exists y)(y \in Y \wedge x\mathcal{R}y)\},$$

$$\mathrm{Ran}(\mathcal{R}) = \{y \mid y \in Y \wedge (\exists x)(x \in X \wedge x\mathcal{R}y)\}$$

are called the **domain** and the **range** of the relation $\mathcal{R}$, respectively. A relation from Z to Z is referred to as a **relation in** Z. The relation $Z \times Z$ is a relation in Z which we shall call the **universal relation** in Z. At the other extreme is the **void relation** in Z, consisting of the empty set. Intermediate is the **identity relation** in Z defined by

$$I_Z = \{(z, z) \mid z \in Z\}.$$

2.3.2 DEFINITION. *The relation $\mathcal{R}^{-1} \subseteq Y \times X$ defined by*

$$\mathcal{R}^{-1} = \{(y, x) \mid (y, x) \in Y \times X \wedge x\mathcal{R}y\}$$

is called the **symmetric** *or* **inverse** *of $\mathcal{R}$.*

2.3.3 DEFINITION. *If $\mathcal{R} \subseteq X \times Y$ and $\mathcal{S} \subseteq Y \times Z$, then the relation*

$$\mathcal{S} \circ \mathcal{R} = \{(x, z) \mid (\exists y)(y \in Y \wedge x\mathcal{R}y \wedge y\mathcal{S}z\}$$

is called the **composite** *relation of $\mathcal{R}$ and $\mathcal{S}$.*

2.3.4 DEFINITION. *The relation $\mathcal{R} \subseteq X \times X$ is said to be:*

- **reflexive,** *if $I_X \subseteq \mathcal{R}$, that is $x\mathcal{R}x$ for each $x \in X$;*
- **transitive,** *if $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$, that is $x\mathcal{R}y$ and $y\mathcal{R}z$ imply $x\mathcal{R}z$;*

- **symmetric**, *if* $\mathcal{R}^{-1} \subseteq \mathcal{R}$, *that is* $x\mathcal{R}y$ *implies* $y\mathcal{R}x$;
- **antisymmetric**, *if* $\mathcal{R} \cap \mathcal{R}^{-1} \subseteq I_X$, *that is* $x\mathcal{R}y$ *and* $y\mathcal{R}x$ *imply* $x = y$.

2.3.5 DEFINITION. *A reflexive, transitive and antisymmetric relation is referred to as an* **ordering relation**.

2.3.6 EXAMPLE. The relation $\mathcal{R} = \{(x, y) \mid x, y \in \mathbb{R},\ x \leq y\}$ is an ordering relation.

2.3.7 DEFINITION. *A reflexive, transitive and symmetric relation is referred to as an* **equivalence relation**.

If the relation $\mathcal{R}$ in X is an equivalence relation, then it is obvious that $\mathrm{Dom}(\mathcal{R}) = X$ and hence we use the terminology *"an equivalence relation on X"*.

2.3.8 EXAMPLE. The relation $\mathcal{R} = \{(x, y) \mid x, y \in \mathbb{R},\ x = y\}$ is an equivalence relation.

Let $\mathcal{R}$ be an equivalence relation on X and $a \in X$. We define C_a as the set of all $x \in X$ such that $a\mathcal{R}x$. Then, C_a is called an **equivalence class** relative to $\mathcal{R}$. One can easily show that, for any two elements $a, b \in X$, either $C_a = C_b$ or $C_a \cap C_b = \emptyset$, that is the equivalence classes are mutually exclusive.

2.3.9 REMARK. *We note that the formation of equivalence classes is an important device for introducing new concepts and entities in mathematics.*

2.4 Functions

We shall define the concept of **function** in terms of notions already introduced.

2.4.1 DEFINITION. *A relation $f \subseteq X \times Y$ that satisfies the property*

$$xfy \ \wedge \ xfz \Longrightarrow y = z$$

*is called a **functional relation**.*

2.4.2 DEFINITION. *If $f \subseteq X \times Y$ is a functional relation and* $\mathrm{Dom}(f) = X$, *then f is called a **function**.*

Synonyms for the word function are numerous and include: **transformation**, **map**, **mapping**, **correspondence**, **operator**, etc. We express the fact that f is a **function from** X **to** Y by the symbol

$$f\colon X \to Y.$$

The relation xfy is usually written in the form $y = f(x)$.

2.4.3 DEFINITION. *The function $f : X \to Y$ is:*

- ***one-to-one** or **injective**, (an **injection**) if*

  $$(\forall x_1)(\forall x_2)(x_1, x_2 \in X,\ (x_1 \neq x_2) \to (f(x_1) \neq f(x_2)))$$

- ***onto** or **surjective**, (a **surjection**), if* $\mathrm{Ran}(f) = Y$;

- ***one-to-one and onto** or **bijective**, (a **bijection**), if it is simultaneously injective and surjective.*

If $f : A \to B$ is a bijection, then we can find for each $b \in B$ exactly one $a \in A$ such that $f(a) = b$. The function $g : B \to A$ which assigns this a to b is called the **inverse function** of f and is denoted by f^{-1}. Of course, the inverse of a bijection is again a bijection. The inverse f^{-1} satisfies the following properties:

$$(\forall b)(b \in B \to f \circ f^{-1}(b) = b);$$

$$(\forall a)(a \in A \to f^{-1} \circ f(a) = a).$$

We consider the sets A, B, $A_0 \subseteq A$, $B_0 \subseteq B$ and a function $f : A \to B$. We define the following sets:

- `the graph of function` f

$$G_f \stackrel{def}{=} \{(a, f(a)) \mid a \in A\} \subseteq A \times B;$$

- the `image of the set` A_0 `under` f

$$f(A_0) \stackrel{def}{=} \{b \mid b \in B \wedge (\exists a)(a \in A_0 \wedge b = f(a))\} \subseteq B;$$

- the `inverse image of the set` B_0 `under` f

$$f^{-1}(B_0) \stackrel{def}{=} \{a \mid a \in A \wedge (\exists b)(b \in B_0 \wedge f(a) = b)\} \subseteq A;$$

- the `set of all functions from` A `to` B

$$B^A \stackrel{def}{=} \{f \mid f : A \to B\}.$$

The following statements can be easily proved:

- $A_1 \subseteq A_2 \to f(A_1) \subseteq f(A_2)$;
- $B_1 \subseteq B_2 \to f^{-1}(B_1) \subseteq f^{-1}(B_2)$;
- $f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i)$;
- $f\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} f(A_i)$;
- $f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i)$;
- $f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$;
- $A_0 \subseteq f^{-1}(f(A_0))$;
- $f(f^{-1}(B_0)) \subseteq B_0$.

2.5 Cardinal Numbers

Let $\mathcal{F}$ be the class of all sets. We consider a relation "$\sim$", where

$$A \sim B \quad \overset{def}{\Leftrightarrow} \quad \exists f : A \to B, \quad \text{bijection.}$$

2.5.1 THEOREM. *The relation "$\sim$" is an equivalence relation.*

Proof. Let $A \in \mathcal{F}$. The function $f\colon A \to A$, defined by $f(x) = x$, is a bijection. Hence $A \sim A$ and so the relation "$\sim$" is reflexive.

Let $A, B, C \in \mathcal{F}$ such that $A \sim B$ and $B \sim C$. Then, there exist the bijections $f : A \to B$, $g : B \to C$. The function $g \circ f$ is a bijection. Hence, $A \sim C$ and so the relation "$\sim$" is transitive.

Let $A \sim B$. Then there exists a bijection $f : A \to B$. Its inverse, $f^{-1} : B \to A$ is again a bijection. Hence $B \sim A$ and so the relation "$\sim$" is symmetric.

In what follows we will assign a symbol to a given set A which is called the `cardinal number` of A. The symbol is the same for all sets which are equivalent with respect to the relation "$\sim$". Consequently, the cardinal number of a set A can be defined as the equivalence class of all sets having the same cardinal number as A and is denoted by $\mathsf{card}(A)$ or $|A|$.

If the set A is finite, then $\mathsf{card}(A)$ can be identified with the number of its elements.

2.5.2 EXAMPLE.

$$\begin{aligned}
|\{0,1,2\}| &= 3;\\
\mathsf{card}(\emptyset) &= 0;\\
\mathsf{card}(\mathbb{N}) &= \mathsf{card}(\mathbb{Z}) = \mathsf{card}(\mathbb{Q}) = \aleph_0 \text{ (Aleph-zero)};\\
\mathsf{card}(\mathbb{R}) &= \mathsf{card}(\mathbb{C}) = \aleph \text{ (Aleph)}.
\end{aligned}$$

The sets with cardinal number $\aleph_0$ are said to be `countable`.

The most important operations with cardinal numbers are defined as follows:

- $|A| + |B| \overset{def}{=} |A \cup B|$, where $A \cap B = \emptyset$;
- $|A| \cdot |B| \overset{def}{=} |A \times B|$;

- $|A|^{|B|} \stackrel{def}{=} |A^B|;$
- $|A| \leq |B| \stackrel{def}{=} \exists f : A \to B$ an injection;
- $|A| < |B| \stackrel{def}{=} \nexists f : A \to B$ a surjection.

It is customary to denote the set of all functions from A to any set of n elements by n^A. Thus, 2^A denotes the set of all functions from A to a set of two elements, which we will ordinarily take to be $\{0, 1\}$, i.e.,

$$2^A = \{f \mid f : A \to \{0, 1\}\}.$$

We will prove that the cardinal number of 2^A is always larger than the cardinal number of A, a classic result due to Cantor.

2.5.3 THEOREM. *If A is a set, then* $|A| < 2^{|A|}$.

Proof. We assume that there exists a surjection $F : A \to 2^A$. Consider the element $f \in 2^A$, defined by

$$f(x) = \begin{cases} 0, & \text{if } F(x)(x) = 1, \\ 1, & \text{if } F(x)(x) = 0. \end{cases}$$

Note that $(\forall x)(x \in A \to F(x) \neq f)$, hence the function F is not a surjection. We have proved that there exists no surjection defined from A onto the set 2^A, hence $|A| < 2^{|A|}$. ❖❖❖

We have obtained the inequalities:

$$|A| < 2^{|A|} < 2^{2^{|A|}} \ldots$$

Therefore, there exists an infinite set of cardinal numbers. On the other hand it is not too difficult to prove that $\aleph = 2^{\aleph_0}$.

The most famous problem in the field of set theory is the `continuum hypothesis`, proposed by Cantor:

"No cardinal lies between $\aleph_0$ and $\aleph$".

He was unable to prove this as a theorem of set theory. Work by Gödel[2] in 1938, and Cohen[3] in 1963 demonstrated the independence of the continuum hypothesis by showing that the axioms of the set theory would remain consistent, assuming they were initially consistent, if either the continuum hypothesis or its negation were added [6].

2 

Kurt Gödel
(1906–1978),
an American
mathematician

3 

Paul Joseph Cohen
(1934–),
an American
mathematician

2.6 Exercises: Sets. Functions

P
2.6.1 Find the intersection and the limit inferior of the sequence of sets $(A_n)_{n\in\mathbb{N}}$,

$$A_n = [0, n].$$

P
2.6.2 Find the limit inferior and the limit superior of the sequence of sets $(B_n)_{n\in\mathbb{N}}$,

$$B_{2n} = [-2n, 0],\ \ B_{2n+1} = [0, 2n+1].$$

P
2.6.3 Find the limit superior and the union of the sequence of sets $(C_n)_{n\in\mathbb{N}}$,

$$C_n = \left[\frac{-1}{n+1},\ n\right].$$

P
2.6.4 Prove that for any sequence of sets $(A_n)_{n\in\mathbb{N}}$ the following relations hold:

$$\bigcap_{n\in\mathbb{N}} A_n \subseteq \liminf A_n \subseteq \limsup A_n \subseteq \bigcup_{n\in\mathbb{N}} A_n.$$

P
2.6.5

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets. Define the sequence $(B_n)_{n\in\mathbb{N}}$, by

$$B_n = \bigcup_{k=0}^{n} A_k.$$

Prove that:

$$B_n \subseteq B_{n+1}, \qquad n \in \mathbb{N}; \tag{a}$$

$$\bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} B_n. \tag{b}$$

P
2.6.6

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets. Define the sequence $(B_n)_{n\in\mathbb{N}}$, by

$$B_0 = A_0, \qquad B_n = A_n \setminus \bigcup_{k=0}^{n-1} A_k, \quad n \in \mathbb{N}^*.$$

Prove that:

(a) $B_i \cap B_j = \emptyset$, for all $i \neq j$, $\quad i, j \in \mathbb{N}$;

(b) $\displaystyle\bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} B_n.$

P
2.6.7

Let $A_1, \ldots, A_{n+1}$ be arbitrary sets such that $A_{n+1} = A_1$. Prove that the following statements are equivalent:

(a) $\displaystyle\bigcup_{k=1}^{n} (A_k \setminus A_{k+1}) = \bigcup_{k=1}^{n} A_k;$

(b) $\displaystyle\bigcap_{k=1}^{n} A_k = \emptyset.$

P
2.6.8

Let $\{A_i\}_{i\in I}$ be a family of sets. Prove that the following relations hold:

(a) $\displaystyle\bigcap_{i\in I} \mathcal{P}(A_i) = \mathcal{P}\left(\bigcap_{i\in I} A_i\right);$

(b) $\displaystyle\bigcup_{i\in I} \mathcal{P}(A_i) \subseteq \mathcal{P}\left(\bigcup_{i\in I} A_i\right).$

P
2.6.9 Let $\{A_i\}_{i\in I}$ be a family of sets. Prove that the following statements are equivalent:

(a) $\mathcal{P}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \mathcal{P}(A_i)$;

(b) $(\exists i_0)(i_0 \in I, \ (\forall i)(i \in I \to A_i \subseteq A_{i_0}))$.

P
2.6.10 Let $\{A_i\}_{i\in I}$, $\{B_i\}_{i\in I}$ be two families of sets such that $A_i \cap B_j = \emptyset$, for all $i, j \in I$, $i \neq j$. Prove that

$$\bigcap_{i\in I}(A_i \cup B_i) = \left(\bigcap_{i\in I} A_i\right) \cup \left(\bigcap_{i\in I} B_i\right).$$

P
2.6.11 Let $(A_n)_{n\in\mathbb{N}}$, be a sequence of sets and A be a set. Prove that

$$\limsup(A_n \Delta A) = (A \setminus (\liminf A_n)) \cup ((\limsup A_n) \setminus A).$$

P
2.6.12 Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets. Prove that the following statements are equivalent:

(a) $\lim A_n = A$;

(b) $\limsup(A \Delta A_n) = \emptyset$.

P 2.6.13 Let $I = \{1, \ldots, n\}$ and $\{X_i\}_{i \in I}$ be a family of arbitrary sets. For $k \in I$ consider the set

$$A_k = \{H \mid H \subseteq I,\ \mathsf{card}(H) = k\}.$$

Prove that the following relations hold:

(a) $$\bigcap_{H \in A_k} \bigcup_{i \in H} X_i \subseteq \bigcup_{H \in A_k} \bigcap_{i \in H} X_i, \quad \text{if} \quad k \leq \frac{n+1}{2};$$

(b) $$\bigcup_{H \in A_k} \bigcap_{i \in H} X_i \subseteq \bigcap_{H \in A_k} \bigcup_{i \in H} X_i, \quad \text{if} \quad k \geq \frac{n+1}{2}.$$

3

Elements of General Topology

Contents

3.1 Topologies. Topological spaces. Open Sets

Topology as a science was, as it is generally believed, formed through the works of the great French mathematician Henri Poincaré[1] at the end of the 19-th century.

Poincaré defined topology as the science which describes the quantitative properties of geometric figures not only in the ordinary space, but also in spaces of more than three dimensions.

Nowadays topology has become a powerful instrument of mathematical research, and its language acquired a universal importance.

3.1.1 DEFINITION. *A family $\mathcal{T}$ of subsets of a non-empty set X satisfying the properties:*

(T_1) *the union of any collection of sets from $\mathcal{T}$ belongs to $\mathcal{T}$;*

(T_2) *the intersection of any two sets from $\mathcal{T}$ belongs to $\mathcal{T}$;*

(T_3) *X and $\emptyset$ belong to $\mathcal{T}$,*

is called a **topology** *on X.*

3.1.2 EXAMPLE. The family

$$\mathcal{T} = \{\emptyset, \{1\}, \{3\}, \{1,3\}, \{1,2,3\}\}$$

is a topology on $X = \{1, 2, 3\}$.

1 

Jules Henri Poincaré
(1854–1912)
A great French
matematician

3.1.3 DEFINITION. *The pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X is called a* **topological space**.

The elements of X are called **points** of the topological space. The elements of $\mathcal{T}$ are called **open sets**.

The axioms of definition (3.1.1) can be stated as follows:

(T_1) the union of any collection of open sets is an open set;
(T_2) the intersection of any two open sets is an open set;
(T_3) X and $\emptyset$ are open sets.

We can easily verify that the following families are topologies:

$\mathcal{T}_0 = \{\emptyset, X\}$ – the **indiscrete** (trivial, minimal, non-discrete) topology on X;

$\mathcal{T}_1 = \mathcal{P}(X)$ – the **discrete** (maximal) topology on X;

$\mathcal{T}_n = \{G, \emptyset \mid G \subseteq \mathbb{R} \wedge (\forall x)(\exists I)(x \in G, I\text{–open interval} \wedge x \in I \subseteq G)\}$ – the **natural topology** of the real axis.

Analogously, we define the natural topology on the space $\mathbb{R}^n$.

Let $(X, \mathcal{T})$ be a topological space and A be a subset of X.

3.1.4 DEFINITION. *The family*

$$\mathcal{T}_A = \{G \cap A \mid G \in \mathcal{T}\}$$

is called the **induced topology** *by $\mathcal{T}$ on A and the pair $(A, \mathcal{T}_A)$ is called a* **topological subspace** *of $(X, \mathcal{T})$.*

3.2 Closed Sets

3.2.1 DEFINITION. *In a topological space a set is said to be* **closed** *if and only if its complement is open.*

We shall denote the family of all closed sets in the topological space $(X, \mathcal{T})$ by $\mathcal{F}$, i.e.,

$$\mathcal{F} = \{F \mid \complement(F) \in \mathcal{T}\} = \{\complement(G) \mid G \in \mathcal{T}\}.$$

3.2.2 REMARK. *The notions of closed set and open set are dual notions. They are neither exclusive (some sets can be simultaneously open and closed) nor exhaustive (some sets are neither open nor closed).*

3.2.3 EXAMPLE. In case of the trivial topology $\mathcal{T}_0 = \{X, \emptyset\}$ we have $\mathcal{F}_0 = \{X, \emptyset\}$.

3.2.4 EXAMPLE. In case of the discrete topology $\mathcal{T}_1 = \mathcal{P}(X)$ we have $\mathcal{F}_1 = \mathcal{P}(X)$.

3.2.5 EXAMPLE. In case of the topology $\mathcal{T} = \{\emptyset, \{1\}, \{1,2\}\}$ we have $\mathcal{F} = \{\emptyset, \{2\}, \{1,2\}\}$.

3.3 Neighborhoods

Let $(X, \mathcal{T})$ be a topological space and $x \in X$ be an arbitrary point.

3.3.1 DEFINITION. *A set $V \subseteq X$ is called a* `neighborhood` *of x if there exists a set $G \in \mathcal{T}$ such that*

$$x \in G \subseteq V.$$

The family of neighborhoods of a point $x \in X$ is denoted by $\mathcal{V}_x$ and is called a `system of neighborhoods` of x. Hence

$$\mathcal{V}_x = \{V \mid V \subseteq X \wedge (\exists G)(G \in \mathcal{T} \wedge x \in G \subseteq V)\}.$$

3.3.2 EXAMPLE. In case of the trivial topology $\mathcal{T}_0 = \{\emptyset, X\}$ we have

$$\mathcal{V}_x = \{X\},$$

for all $x \in X$.

3.3.3 EXAMPLE. In case of the discrete topology $\mathcal{T}_1 = \mathcal{P}(X)$ we have

$$\mathcal{V}_x = \{V \mid (V \subseteq X) \wedge (x \in V)\},$$

for all $x \in X$.

3.3.4 EXAMPLE. In case of the topology $\mathcal{T} = \{\emptyset, \{1\}, \{1,2\}\}$ we have

$$\mathcal{V}_1 = \{\{1\}, \{1,2\}\}, \ \mathcal{V}_2 = \{\{1,2\}\}.$$

The following theorem gives a characterization of the open sets.

3.3.5 THEOREM. *In a topological space a set is open if and only if it is neighborhood for all its points.*

Proof. Necessity. Let G be an open set in the topological space $(X, \mathcal{T})$ and $x \in G$. We can write

$$G \in \mathcal{T} \wedge x \in G \subseteq G.$$

Using the definition of a neighborhood, it follows that

$$G \in \mathcal{V}_x.$$

Sufficiency. Suppose that G is a neighborhood for all its points. Since $x \in G$, there exists an open set G_x such that

$$x \in G_x \subseteq G.$$

We deduce

$$(\forall x)(\exists G_x)(x \in G \to x \in G_x \subseteq \bigcup_{x \in G} G_x \subseteq G),$$

therefore

$$G \subseteq \bigcup_{x \in G} G_x \subseteq G,$$

that is

$$G = \bigcup_{x \in G} G_x,$$

which shows that G is open (as a union of open sets).

3.3.6 DEFINITION. *A function $\mathcal{V} : X \to \mathcal{P}(\mathcal{P}(X))$ which assigns to each point $x \in X$ the system of neighborhoods $\mathcal{V}_x$ is called a* `complete system of neighborhoods`.

3.3.7 THEOREM. *The complete system of neighborhoods has the following properties:*

($\mathcal{V}_1$) $V \in \mathcal{V}_x \to x \in V$;

($\mathcal{V}_2$) $(V \in \mathcal{V}_x \wedge V \subseteq A) \to A \in \mathcal{V}_x$;

($\mathcal{V}_3$) $V, U \in \mathcal{V}_x \to V \cap U \in \mathcal{V}_x$;

$$(\mathcal{V}_4) \qquad V \in \mathcal{V}_x \to (\exists W)(\forall y)(W \in \mathcal{V}_x \wedge (y \in W \to V \in \mathcal{V}_y)).$$

Proof. The first three properties are obvious.

In order to prove the last property, let us consider a neighborhood $V \in \mathcal{V}_x$. Then, there exists $G \in \mathcal{T}$ such that $x \in G \subseteq V$. Choosing $W = G$, one can see that for all $y \in W$, $W \in \mathcal{V}_y$ (cf. theorem (3.3.5)). The inclusion $W \subseteq V$ and property $(\mathcal{V}_2)$ imply $V \in \mathcal{V}_y$. ❖

3.4 Interior of a Set

Let $(X, \mathcal{T})$ be a topological space and $x \in X$.

3.4.1 DEFINITION. *A point x is said to be an* **interior point** *of $A \subseteq X$ if there exists a neighborhood $V \in \mathcal{V}_x$ such that $V \subseteq A$.*

The set of all interior points of A is called the **interior** of A and denoted by $\mathrm{int}(A)$. Therefore,

$$\mathrm{int}(A) = \{x \mid x \in X \wedge (\exists V)(V \in \mathcal{V}_x \wedge V \subseteq A)\}.$$

3.4.2 DEFINITION. *A function which assigns to each set its interior is called the* **interior operator** *and is denoted by* int.

3.4.3 THEOREM. *The set $\mathrm{int}(A)$ is the largest open set contained in A, i.e.,*

$$\mathrm{int}(A) = \bigcup_{A \supseteq G \in \mathcal{T}} G.$$

Proof. Let $x \in \mathrm{int}(A)$. Then, there exists a neighborhood $V \in \mathcal{V}_x$ such that $V \subseteq A$. Therefore there exists a neighborhood $G \in \mathcal{T}$ such that $x \in G \subseteq V \subseteq A$.

The relations

$$(\forall x)(\exists G)(x \in \mathrm{int}(A) \to x \in G \subseteq \bigcup_{A \subseteq G \in \mathcal{T}} G)$$

give

$$\mathsf{int}(A) \subseteq \bigcup_{A \supseteq G \in \mathcal{T}} G. \qquad (\diamond)$$

Let $x \in \bigcup_{A \supseteq G \in \mathcal{T}} G$. Then, there exists a set $G \in \mathcal{T}$ such that $x \in G \subseteq A$. Since $G \in \mathcal{V}_x$, we deduce that $x \in \mathsf{int}(A)$, and

$$\bigcup_{A \supseteq G \in \mathcal{T}} G \subseteq \mathsf{int}(A). \qquad (\diamond\diamond)$$

Relations $(\diamond)$ and $(\diamond\diamond)$ conclude the proof. ❖❖❖

From theorem (3.4.3) one can deduce that the interior operator is a mapping from $\mathcal{P}(X)$ onto $\mathcal{T}$.

3.4.4 EXAMPLE. In case of the topology $\mathcal{T} = \{\emptyset, \{1\}, \{1,2\}\}$ we have:

$$\mathsf{int}(\{1\}) = \{1\}, \quad \mathsf{int}(\{2\}) = \emptyset.$$

3.4.5 EXAMPLE. In case of the natural topology we have:

$$\mathsf{int}([0,1]) = (0,1), \quad \mathsf{int}([0,1)) = (0,1), \quad \mathsf{int}(\{0\}) = \emptyset.$$

3.4.6 THEOREM. *In a topological space a set M is open iff $M = \mathsf{int}(M)$.*

Proof. Necessity. Let M be an open set of the topological space $(X, \mathcal{T})$. We have:

$$\bigcup_{M \supseteq G \in \mathcal{T}} G \subseteq M; \qquad (\diamond)$$

$$M \subseteq \bigcup_{M \supseteq G \in \mathcal{T}} G \qquad (\diamond\diamond)$$

($M \supseteq M \in \mathcal{T}$, hence M is a term of the union).

From $(\diamond)$ and $(\diamond\diamond)$ we deduce

$$M = \bigcup_{M \supseteq G \in \mathcal{T}} G = \mathsf{int}(M).$$

Sufficiency. Suppose that

$$M = \mathsf{int}(M).$$

Since $\mathsf{int}(M)$ is open (as a union of open sets), M is open. ⁂

3.4.7 REMARK. *From theorem (3.4.6) we deduce that a set G is open iff*

$$G \subseteq \mathsf{int}(G).$$

3.4.8 THEOREM. *The interior operator, $\mathsf{int}\colon \mathcal{P}(X) \to \mathcal{T}$, has the following properties:*

(I_1) $\mathsf{int}(A) \subseteq A$;
(I_2) $\mathsf{int}(A \cap B) = \mathsf{int}(A) \cap \mathsf{int}(B)$;
(I_3) $\mathsf{int}(\mathsf{int}(A)) = \mathsf{int}(A)$;
(I_4) $\mathsf{int}(X) = X$.

Proof. (I_1) The inclusion $\mathsf{int}(A) \subseteq A$ can be obtained from the relation

$$\mathsf{int}(A) = \bigcup_{A \supseteq G \in \mathcal{T}} G \subseteq A.$$

(I_2) It is obvious that

$$A \subseteq B \Longrightarrow \mathsf{int}(A) \subseteq \mathsf{int}(B).$$

From the relations:

$$A \cap B \subseteq A \Longrightarrow \mathsf{int}(A \cap B) \subseteq \mathsf{int}(A),$$

$$A \cap B \subseteq B \Longrightarrow \mathsf{int}(A \cap B) \subseteq \mathsf{int}(B)$$

it follows that

$$\mathsf{int}(A \cap B) \subseteq \mathsf{int}(A) \cap \mathsf{int}(B). \qquad (\diamond)$$

From the relations:

$$\mathsf{int}(A) \subseteq A,$$

$$\mathsf{int}(B) \subseteq B,$$

we obtain

$$\mathsf{int}(A) \cap \mathsf{int}(B) \subseteq A \cap B,$$

therefore

$$\mathsf{int}(\mathsf{int}(A) \cap \mathsf{int}(B)) \subseteq \mathsf{int}(A \cap B),$$

that is

$$\mathsf{int}(A) \cap \mathsf{int}(B) \subseteq \mathsf{int}(A \cap B). \qquad (\diamond\diamond)$$

From $(\diamond)$ and $(\diamond\diamond)$ it follows

$$\mathsf{int}(A) \cap \mathsf{int}(B) = \mathsf{int}(A \cap B).$$

(I_3) Using the fact that $\mathsf{int}(A) \in \mathcal{T}$, together with theorem (3.4.6), we obtain

$$\mathsf{int}(\mathsf{int}(A)) = \mathsf{int}(A).$$

(I_4) X is an open set, therefore $X = \mathsf{int}(X)$. ◆◆◆

Properties I_1—I_4 are specific axioms for the interior operator as one can see from the following theorem.

3.4.9 THEOREM. *If the function $f : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies the conditions:*

(i_1) $f(A) \subseteq A$;
(i_2) $f(A \cap B) = f(A) \cap f(B)$;
(i_3) $f(f(A)) = f(A)$;
(i_4) $f(X) = X$,

for all $A, B \subseteq X$, then

(1) *the family $\mathcal{T} = \{G \mid (G \subseteq X) \wedge (f(G) = G)\}$ is a topology on X;*

(2) *in the topological space $(X, \mathcal{T})$ the function f is the interior operator.*

◆◆◆

Proof. It is clear that any function $f : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the condition

$$f(A \cap B) = f(A) \cap f(B),$$

for all $A, B \subseteq X$, is increasing, i.e.,

$$A \subseteq B \to f(A) \subseteq f(B).$$

- Let $\{G_i\}_{i \in I}$ be a collection of sets in $\mathcal{T}$. Relation (i_1) implies

$$f\Big(\bigcup_{i\in I} G_i\Big) \subseteq \bigcup_{i\in I} G_i.$$

For all $i \in I$ we have

$$G_i \subseteq \bigcup_{i\in I} G_i,$$

therefore,

$$(\forall i)(i \in I \to f(G_i) \subseteq f(\bigcup_{i\in I} G_i)),$$

that is

$$(\forall i)(i \in I \to G_i \subseteq f(\bigcup_{i\in I} G_i)),$$

and we obtain the converse inclusion

$$\bigcup_{i\in I} G_i \subseteq f(\bigcup_{i\in I} G_i).$$

Consequently

$$f(\bigcup_{i\in I} G_i) = \bigcup_{i\in I} G_i,$$

i.e.,

$$\bigcup_{i\in I} G_i \in \mathcal{T}.$$

- Let $G_1, G_2 \in \mathcal{T}$. Then, from (i_2) we obtain

$$G_1 \cap G_2 = f(G_1) \cap f(G_2) = f(G_1 \cap G_2),$$

therefore $G_1 \cap G_2 \in \mathcal{T}$.

- From (i_4) it follows that $X \in \mathcal{T}$.
- From (i_1) we deduce $f(\emptyset) \subseteq \emptyset$ and hence $f(\emptyset) = \emptyset$. This means that $\emptyset \in \mathcal{T}$, which proves that $\mathcal{T}$ is a topology on X.

(2) From the equality $f(f(A)) = f(A)$ we deduce $f(A) \in \mathcal{T}$. From $f(A) \subseteq A$, we obtain $\operatorname{int} f(A) \subseteq \operatorname{int}(A)$, that is $f(A) \subseteq \operatorname{int}(A)$. From $\operatorname{int}(A) \subseteq A$ we obtain $f(\operatorname{int}(A)) \subseteq f(A)$, that is $\operatorname{int}(A) \subseteq f(A)$, hence

$$f(A) = \operatorname{int}(A).$$

Consequently, the function f is the interior operator.

3.5 Closure of a Set

3.5.1 DEFINITION. *A point x in the topological space $(X, \mathcal{T})$ is said to be an* `adherent point` *of a set $A \subseteq X$ if all neighborhoods of x intersect A.*

The set of all adherent points of A is called the `closure` of A and is denoted by $\mathsf{ad}(A)$ or $\overline{A}$, i.e.,

$$\overline{A} = \{x \mid (x \in X) \wedge (\forall V)((V \in \mathcal{V}_x) \rightarrow (V \cap A \neq \emptyset))\}.$$

3.5.2 DEFINITION. *The function which assigns to each set its closure is called the* `closure operator`*, and is denoted by* ad.

3.5.3 REMARK. *From definition* (3.5.1) *one can deduce that*

$$A \subseteq \overline{A},$$

for all subsets $A \subseteq X$.

3.5.4 THEOREM. *In a topological space a set M is closed iff $M = \overline{M}$.*

Proof. Necessity. Let M be a closed set in the topological space $(X, \mathcal{T})$ and let $x \in X$. We have

$$x \notin M \iff x \in \complement(M).$$

Then, taking into account the fact that $\complement(M) \in \mathcal{T}$ it follows that $\complement(M)$ is a neighborhood of the point x.

From the relations

$$\complement(M) \in \mathcal{V}_x, \quad M \cap \complement(M) = \emptyset,$$

using the definition of the closure, we obtain

$$x \notin \overline{M},$$

therefore

$$\overline{M} \subseteq M.$$

Hence, using remark (3.5.3), we deduce

$$M = \overline{M}.$$

Sufficiency. Suppose that the following equality is true

$$\overline{M} = M.$$

We have:

$$\begin{aligned} x \in \complement(M) = \complement(\overline{M}) &\iff x \notin \overline{M} \\ &\iff (\exists V)(V \in \mathcal{V}_x \wedge V \cap M = \emptyset) \\ &\iff (\exists V)(V \in \mathcal{V}_x \wedge V \subseteq \complement(M)) \\ &\iff x \in \mathsf{int}(\complement(M)), \end{aligned}$$

therefore

$$\complement(M) = \mathsf{int}(\complement(M)),$$

that is

$$\complement(M) \in \mathcal{T} \Leftrightarrow M \in \mathcal{F}.$$

3.5.5 THEOREM. *Let $(X, \mathcal{T})$ be a topological space and $M \subseteq X$. Then, the following equalities are true:*

1. $\mathsf{int}(M) = \complement(\overline{\complement(M)});$
2. $\overline{M} = \complement(\mathsf{int}(\complement(M))).$

Proof. (1) Let $x \in X$. From the sequence of equivalent statements

$$x \notin \mathsf{int}(M)$$

$$\begin{aligned} &\iff (\forall V)(V \in \mathcal{V}_x \to V \nsubseteq M) \\ &\iff (\forall V)(V \in \mathcal{V}_x \to V \cap \complement(M) \neq \emptyset) \\ &\iff x \in \overline{\complement(M)} \iff x \notin \complement(\overline{\complement(M)}), \end{aligned}$$

we deduce that $\mathsf{int}(M) = \complement(\overline{\complement(M)})$.

(2) Choosing $A = \complement(M)$ in the equality

$$\mathsf{int}(A) = \complement(\overline{\complement(A)}),$$

we obtain

$$\mathsf{int}(\complement(M)) = \complement(\overline{\complement(\complement(M))}),$$

that is

$$\mathsf{int}(\complement(M)) = \complement(\overline{M}),$$

therefore

$$\complement(\mathsf{int}(\complement(M))) = \overline{M}.$$

1.5.6 THEOREM. *In a topological space the closure of a set M is the smallest closed set containing M, i.e.,*

$$\overline{M} = \bigcap_{M \subseteq F \in \mathcal{F}} F.$$

Proof. The theorem follows from the equalities:

$$\begin{aligned}\overline{M} &= \complement(\mathsf{int}(\complement(M))) = \complement\left(\bigcup_{\complement(M) \supseteq G \in \mathcal{T}} G\right) \\ &= \bigcap_{\complement(M) \supseteq G \in \mathcal{T}} \complement(G) = \bigcap_{M \subseteq \complement(G) \in \mathcal{F}} \complement(G) \\ &= \bigcap_{M \subseteq F \in \mathcal{F}} F.\end{aligned}$$

1.5.7 REMARK. *Note that the closure of a set is a closed set. Hence the closure operator is a mapping from $\mathcal{P}(X)$ onto $\mathcal{F}$,*

$$\mathsf{ad} : \mathcal{P}(X) \to \mathcal{F}.$$

3.5.8 THEOREM. *In a topological space $(X, \mathcal{T})$ the closure operator has the following properties:*

(A_1) $A \subseteq \overline{A}$;
(A_2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$;
(A_3) $\overline{\overline{A}} = \overline{A}$;
(A_4) $\overline{\emptyset} = \emptyset$.

Proof. (A_1) The inclusion $A \subseteq \overline{A}$ is easily obtained from the definition of the closure operator. It is also obvious that

$$A \subseteq B \Longrightarrow \overline{A} \subseteq \overline{B}.$$

(A_2) From the inclusions

$$A \subseteq A \cup B,$$

$$B \subseteq A \cup B,$$

we deduce that

$$\overline{A} \subseteq \overline{A \cup B},$$

$$\overline{B} \subseteq \overline{A \cup B},$$

therefore

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}. \qquad (\diamond)$$

From the inclusions

$$A \subseteq \overline{A},$$

$$B \subseteq \overline{B},$$

it follows that

$$A \cup B \subseteq \overline{A} \cup \overline{B}$$

and thus

$$\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}. \qquad (\diamond\diamond)$$

Relations $(\diamond)$ and $(\diamond\diamond)$ imply

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

(A_3) The equality $\overline{\overline{A}} = \overline{A}$ results from the fact that the set $\overline{A}$ is closed.

(A_4) The equality $\overline{\emptyset} = \emptyset$ is deduced from the fact that $\emptyset$ is a closed set.

Properties A_1–A_4 are specific for the closure operator as one can see from the following theorem.

3.5.9 THEOREM. *If for all the sets $A, B \subseteq X$ the function $f : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies the conditions:*

(a_1) $A \subseteq f(A)$;
(a_2) $f(A \cup B) = f(A) \cup f(B)$;
(a_3) $f(f(A)) = f(A)$;
(a_4) $f(\emptyset) = \emptyset$,

then:

(1) *the family $\mathcal{T} = \{\complement(M) \mid (M \subseteq X) \wedge (f(M) = M)\}$ is a topology on X;*

(2) *in the topological space $(X, \mathcal{T})$ the function f is the closure operator.*

Proof. (1) We observe that any function satisfying the condition

$$f(A \cup B) = f(A) \cup f(B)$$

is increasing, i.e.,

$$A \subseteq B \Longrightarrow f(A) \subseteq f(B).$$

- Let $\{G_i\}_{i \in I}$ be a collection of sets in $\mathcal{T}$. Therefore there exist $F_i \subseteq X$, $i \in I$ such that

$$(\forall i)(i \in I \to (G_i = \complement(F_i) \ \wedge \ f(F_i) = F_i)).$$

From the relations

$$(\forall i)(i \in I \to \bigcap_{i \in I} F_i \subseteq F_i),$$

we deduce

$$(\forall i)(i \in I \to f(\bigcap_{i \in I} F_i) \subseteq f(F_i)),$$

hence

$$(\forall i)(i \in I \rightarrow f(\bigcap_{i\in I} F_i) \subseteq F_i),$$

so

$$f(\bigcap_{i\in I} F_i) \subseteq \bigcap_{i\in I} F_i.$$

By virtue of (a_1) we have the converse inclusion

$$\bigcap_{i\in I} F_i \subseteq f(\bigcap_{i\in I} F_i),$$

and so

$$f(\bigcap_{i\in I} F_i) = \bigcap_{i\in I} F_i.$$

Consequently

$$\bigcup_{i\in I} G_i = \complement(\bigcap_{i\in I} F_i) \in \mathcal{T}.$$

- Let $G_1, G_2 \in \mathcal{T}$. Then there exist $F_1, F_2 \subseteq X$ such that

$$G_1 = \complement(F_1), \quad G_2 = \complement(F_2), \quad f(F_1) = F_1, \quad f(F_2) = F_2.$$

By virtue of condition (a_2) we have

$$f(F_1 \cup F_2) = F_1 \cup F_2,$$

therefore

$$G_1 \cap G_2 = \complement(F_1) \cap \complement(F_2) = \complement(F_1 \cup F_2)$$

and

$$G_1 \cap G_2 \in \mathcal{T}.$$

- From (a_1) we deduce $X \subseteq f(X) \subseteq X$, hence $f(X) = X$, and $\emptyset \in \mathcal{T}$.

By virtue of condition (a_4) we have $f(\emptyset) = \emptyset$, hence $X \in \mathcal{T}$.

We have proved that $\mathcal{T}$ is a topology on X.

(2) From the equality $f(f(A)) = f(A)$ we deduce that $\complement(f(A)) \in \mathcal{T}$. It follows that $f(A)$ is closed, therefore

$$\overline{f(A)} = f(A).$$

From condition (a_1) we deduce that $\overline{A} \subseteq \overline{f(A)}$ and consequently

$$\overline{A} \subseteq f(A).$$

On the other hand, from $A \subseteq \overline{A}$ it follows that

$$f(A) \subseteq f(\overline{A}) = \overline{A},$$

hence

$$f(A) = \overline{A}.$$

Thus, f is the closure operator.

3.5.10 EXAMPLE. In case of the natural topology of the real axis we have:

$$\overline{[0,1]} = [0,1], \quad \overline{(0,1)} = [0,1], \quad \overline{\{0\}} = \{0\}.$$

3.5.11 EXAMPLE. In case of the topology $\mathcal{T} = \{\emptyset, \{1\}, \{1,2\}\}$ we have:

$$\mathcal{F} = \{\emptyset, \{2\}, \{1,2\}\},$$

$$\overline{\{1\}} = \{1,2\}, \quad \overline{\{2\}} = \{2\}.$$

3.5.12 EXAMPLE. In case of the topology

$$\mathcal{T} = \{\emptyset, \{n, n+1, n+2, \ldots\} \mid n \in \mathbb{N}\}$$

we have

$$\mathcal{F} = \{\emptyset, \mathbb{N}, \{0, 1, \ldots, n\} \mid n \in \mathbb{N}\},$$

$$\overline{\{0\}} = \{0\}, \quad \overline{\{1\}} = \{0,1\}, \quad \overline{\{0, 2, 4, \ldots, 2n, \ldots\}} = \mathbb{N}.$$

3.6 Limit Points. The Derived Set

3.6.1 DEFINITION. *A point x in the topological space $(X, \mathcal{T})$ is called a* **limit point** *(also called a* **cluster point** *or an* **accumulation point** *[20]) of a given set $A \subseteq X$ if there is at least one point in A, other than x, in each neighborhood of the point x.*

Observe that, if x is a limit point of A, then, for all neighborhoods V of x, $V \setminus \{x\}$ intersects A.

The set of all limit points of A is called the **derived set** of A and is denoted by

$$A'.$$

From the equality

$$V \cap (A \setminus \{x\}) = (V \setminus \{x\}) \cap A,$$

we deduce

$$A' = \{x \mid (x \in X) \wedge (\forall V)(V \in \mathcal{V}_x \to V \cap (A \setminus \{x\}) \neq \emptyset)\},$$

that is

$$A' = \{x \mid (x \in X) \wedge (x \in \overline{A \setminus \{x\}})\}.$$

3.6.2 REMARK. *The previous equality implies the inclusion*

$$A' \subseteq \overline{A}.$$

3.6.3 DEFINITION. *A point of $A \setminus A'$ is said to be an* **isolated point** *of A.*

3.6.4 DEFINITION. *A set is said to be* **discrete** *if each of its points is isolated.*

3.6.5 EXAMPLE. In case of the natural topology of the real axis we have:

$$[0,1)' = [0,1]; \quad \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots \mid n \in \mathbb{N}^*\}' = \{0\}.$$

1.6.6 EXAMPLE. In case of the discrete topology $\mathcal{T} = \mathcal{P}(X)$, for all $A \subseteq X$, we have

$$A' = \emptyset.$$

1.6.7 EXAMPLE. In case of the topology $\mathcal{T} = \{\emptyset, \{1\}, \{1,2\}\}$ we have:

$$\{1\}' = \{2\}, \quad \{2\}' = \emptyset.$$

1.6.8 THEOREM. *In a topological space $(X, \mathcal{T})$ the derivation operator has the following properties:*

(D_1) $(\forall x)(x \in X \to x \notin \{x\}')$;
(D_2) $(A \cup B)' = A' \cup B'$;
(D_3) $A'' \subseteq A \cup A'$;
(D_4) $\emptyset' = \emptyset$.

Proof. (D_1) Let $x \in X$. From the relations

$$x \in \{x\}' \iff x \in \overline{\{x\} \setminus \{x\}} \iff x \in \overline{\emptyset} = \emptyset,$$

we deduce

$$x \notin \{x\}'.$$

(D_2) Let $x \in X$. From the relations

$$x \in (A \cup B)' \iff x \in \overline{(A \cup B) \setminus \{x\}}$$
$$\iff x \in \overline{(A \setminus \{x\}) \cup (B \setminus \{x\})} = \overline{A \setminus \{x\}} \cup \overline{B \setminus \{x\}}$$
$$\iff x \in \overline{A \setminus \{x\}} \vee x \in \overline{B \setminus \{x\}} \iff x \in A' \vee x \in B'$$
$$\iff x \in A' \cup B',$$

we deduce

$$(A \cup B)' = A' \cup B'.$$

(D_3) Suppose that there exists $x \in A''$ and $x \notin A \cup A'$. We deduce

$$x \in A'' \wedge x \notin A' \wedge x \notin A.$$

The relation $x \notin A'$ gives

$$(\exists G)(G \in (\mathcal{V}_x \cap \mathcal{T}) \wedge G \cap (A \setminus \{x\}) = \emptyset),$$

therefore, since $x \notin A$, we obtain

$$(\exists G)(G \in (\mathcal{V}_x \cap \mathcal{T}) \wedge (G \cap A = \emptyset)). \qquad (\diamond)$$

The relation $x \in A''$ implies

$$G \cap (A' \setminus \{x\}) \neq \emptyset, \qquad \text{therefore,} \qquad G \cap A' \neq \emptyset.$$

Let $y \in G \cap A'$. The relations $G \in \mathcal{V}_y$ and $y \in A'$ imply

$$G \cap (A \setminus \{y\}) \neq \emptyset, \qquad \text{moreover} \qquad G \cap A \neq \emptyset,$$

which contradicts $(\diamond)$. It follows that $x \in A \cup A'$ and finally

$$A'' \subseteq A \cup A'.$$

(D_4) From the relations

$$\emptyset' \subseteq \overline{\emptyset} = \emptyset$$

it follows

$$\emptyset' = \emptyset \ .$$

Properties D_1–D_4 are specific for the derivation operator as one can see from the following theorem.

3.6.9 THEOREM. *If, for all sets $A, B \subseteq X$, the function $f : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies the conditions:*

(d_1) $(\forall x)(x \in X \to x \notin f(\{x\}))$;
(d_2) $f(A \cup B) = f(A) \cup f(B)$;
(d_3) $f(f(A)) \subseteq A \cup f(A)$;
(d_4) $f(\emptyset) = \emptyset$,

then:

(1) *the family $\mathcal{T} = \{\complement(M) \mid (M \subseteq X) \wedge (f(M) \subseteq M)\}$ is a topology on X.*

(2) *in the topological space $(X, \mathcal{T})$ the function f is the derivation operator.*

Proof. (1) We note that any function $f : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the condition

$$f(A \cup B) = f(A) \cup f(B),$$

for all $A, B \subseteq X$, is increasing, i.e.,

$$A \subseteq B \to f(A) \subseteq f(B).$$

- Let $\{G_i\}_{i\in I}$ be a collection of sets in $\mathcal{T}$. Then, there exist $F_i \subseteq X$, $i \in I$ such that

$$(\forall i)(i \in I \rightarrow (G_i = \complement(F_i) \ \wedge \ f(F_i) \subseteq F_i)).$$

From the relations

$$(\forall i)(i \in I \rightarrow \bigcap_{i\in I} F_i \subseteq F_i),$$

taking into account that the function f is increasing, we deduce

$$(\forall i)(i \in I \rightarrow f(\bigcap_{i\in I} F_i) \subseteq f(F_i)),$$

hence

$$(\forall i)(i \in I \rightarrow f(\bigcap_{i\in I} F_i) \subseteq F_i),$$

and consequently

$$f(\bigcap_{i\in I} F_i) \subseteq \bigcap_{i\in I} F_i.$$

It follows that

$$\bigcup_{i\in I} G_i = \complement(\bigcap_{i\in I} F_i) \in \mathcal{T}.$$

- Let $G_1, G_2 \in \mathcal{T}$. Then, there exist $F_1, F_2 \subseteq X$ such that

$$G_1 = \complement(F_1), \quad G_2 = \complement(F_2), \quad f(F_1) \subseteq F_1, \quad f(F_2) \subseteq F_2.$$

We have:

$$G_1 \cap G_2 = \complement(F_1) \cap \complement(F_2) = \complement(F_1 \cup F_2)$$

and

$$f(F_1 \cup F_2) = f(F_1) \cup f(F_2) \subseteq F_1 \cup F_2,$$

therefore

$$G_1 \cap G_2 \in \mathcal{T}.$$

- From $f(\emptyset) = \emptyset$ we deduce

$$X \in \mathcal{T}.$$

- From the obvious inclusion

$$f(X) \subseteq X,$$

we deduce

$$\emptyset \in \mathcal{T}.$$

We have proved that $\mathcal{T}$ is a topology on X.

(2) From the fact that $\overline{A}$ is a closed set it follows

$$f(\overline{A}) \subseteq \overline{A}.$$

From:

$$f(A \cup f(A)) = f(A) \cup f(f(A)) \subseteq f(A) \cup (A \cup f(A)) = A \cup f(A)$$

we deduce that the $A \cup f(A)$ is closed and consequently

$$A \cup f(A) = \overline{A \cup f(A)}.$$

Since the function f is increasing,

$$A \cup f(A) \subseteq A \cup f(\overline{A}) \subseteq A \cup \overline{A} = \overline{A} \subseteq \overline{A \cup f(A)} = A \cup f(A),$$

and therefore

$$A \cup f(A) = \overline{A}.$$

We have:

$$x \in A' \iff x \in \overline{A \setminus \{x\}} = (A \setminus \{x\}) \cup f(A \setminus \{x\})$$

$$\iff x \in f(A \setminus \{x\}),$$

but $x \notin f(\{x\})$, and hence

$$x \in A' \iff x \in f(A \setminus \{x\}) \cup f(\{x\}) = f((A \setminus \{x\}) \cup \{x\}) = f(A).$$

Finally

$$f(A) = A',$$

i.e., f is derivation operator.

3.6.10 REMARK. *In general, there is no inclusion relation between the sets A' and A''.*

As an example, consider the topology $\mathcal{T} = \{\emptyset, \{1, 2\}\}$, where:

$$\{1\}' = \{2\}, \quad \{1\}'' = \{1\},$$

so

$$\{1\}' \cap \{1\}'' = \emptyset.$$

Using the result

$$\mathsf{int}(A) = \complement(\overline{\complement(A)}),$$

or

$$\mathsf{bd}(A) = \overline{A} \cap \overline{\complement(A)},$$

which will be proven in (3.7.3), we can show that the interior operator (int) and the boundary operator (bd) can be written in terms of the closure, the intersection and the complement operators.

We also tried to express the derivation operator using a formula that involves the other usual topological operators. A negative answer to this problem is presented in the following theorem.

3.6.11 THEOREM. **[16, M. Ivan, 1996]** *The derivation operator cannot be written in terms of a finite number of closure, intersection and complement operators.*

A relation between the derivation operator and the closure operator is given in the following theorem.

3.6.12 THEOREM. *In a topological space $(X, \mathcal{T})$, the following equality holds*

$$\overline{A} = A \cup A',$$

for all $A \subseteq X$.

Proof. We know that

$$A' \subseteq \overline{A} \quad \text{and} \quad A \subseteq \overline{A},$$

therefore

$$A \cup A' \subseteq \overline{A}. \tag{$\diamond$}$$

Let $x \in X$, $x \notin A \cup A'$. It follows that

$$x \notin A \quad \text{and} \quad x \notin A'.$$

Hence, there exists $V \in \mathcal{V}_x$ such that

$$V \cap (A \setminus \{x\}) = \emptyset.$$

Since $x \notin A$,

$$V \cap A = \emptyset,$$

i.e., $x \notin \overline{A}$ and consequently

$$A \cup A' \supseteq \overline{A}. \tag{$\diamond\diamond$}$$

From ($\diamond$), ($\diamond\diamond$) we obtain

$$\overline{A} = A \cup A' \ .$$

◆◆◆

3.6.13 THEOREM. *A set is closed iff it contains all its limit points.*

Proof. Using the equality

$$\overline{A} = A \cup A',$$

we deduce the relations:

$$A\text{–closed} \iff \overline{A} = A \iff A \cup A' = A \iff A' \subseteq A.$$

◆◆◆

3.7 Exterior and Boundary of a Set

Let $(X, \mathcal{T})$ be a topological space and A be a subset of X.

3.7.1 DEFINITION. *The set* $\mathsf{int}(\complement(A))$ *is called the* `exterior` *of the set* A *and is denoted by* $\mathsf{ext}(A)$,

$$\mathsf{ext}(A) := \mathsf{int}(\complement(A)).$$

We have:

$$\mathsf{ext}(A) = \mathsf{int}(\complement(A)) = \complement(\overline{\complement(\complement(A))}) = \complement(\overline{A}).$$

3.7.2 DEFINITION. *The set of all points which are neither in the interior nor in the exterior of* A *is called the* **boundary** *of* A *and is denoted by* $\mathsf{bd}(A)$, *or* ∂A, *i.e.,*

$$\mathsf{bd}(A) = \{x \in X \mid x \notin \mathsf{int}(A) \wedge x \notin \mathsf{ext}(A)\}.$$

We deduce that:

$$\begin{aligned}\mathsf{bd}(A) &= \complement(\mathsf{int}(A) \cup \mathsf{ext}(A)) = \complement(\mathsf{int}(A)) \cap \complement(\mathsf{ext}(A)) \\ &= \complement(\complement(\overline{\complement(A)})) \cap \complement(\complement(\overline{A})) \\ &= \overline{\complement(A)} \cap \overline{A},\end{aligned}$$

therefore, the boundary of any set is a closed set.

The relation between the boundary operator and the closure and the interior operators is given by the following theorem.

3.7.3 THEOREM. *The boundary operator has the following properties:*

(F_1) $A \cap B \cap \mathsf{bd}(A \cap B) = A \cap B \cap (\mathsf{bd}(A) \cup \mathsf{bd}(B))$;
(F_2) $\mathsf{bd}(A) = \mathsf{bd}(\complement(A))$;
(F_3) $\mathsf{bd}(\mathsf{bd}(A)) \subseteq \mathsf{bd}(A)$;
(F_4) $\mathsf{bd}(\emptyset) = \emptyset$,

for all $A, B \subseteq X$.

Proof. (F_1). We have:

$$\begin{aligned}A \cap B \cap \mathsf{bd}(A \cap B) &= A \cap B \cap \overline{A \cap B} \cap \overline{\complement(A \cap B)} \\ &= A \cap B \cap \overline{\complement(A \cap B)} = A \cap B \cap \overline{\complement(A) \cup \complement(B)} \\ &= A \cap B \cap (\overline{\complement(A)} \cup \overline{\complement(B)}).\end{aligned} \qquad (\diamond)$$

On the other hand, we have:

$$\begin{aligned}A \cap B \cap (\mathsf{bd}(A) \cup \mathsf{bd}(B)) &= A \cap B \cap ((\overline{A} \cap \overline{\complement(A)}) \cup (\overline{B} \cap \overline{\complement(B)})) \\ &= (A \cap B \cap \overline{A} \cap \overline{\complement(A)}) \cup (A \cap B \cap \overline{B} \cap \overline{\complement(B)}) \\ &= (A \cap B \cap \overline{\complement(A)}) \cup (A \cap B \cap \overline{\complement(B)}). \\ &= (A \cap B) \cap (\overline{\complement(A)} \cup \overline{\complement(B)})\end{aligned} \qquad (\diamond\diamond)$$

From ($\diamond$) and ($\diamond\diamond$) we deduce relation (F_1).
(F_2). We have:

$$\mathsf{bd}(A) = \overline{A} \cap \overline{\complement(A)} = \overline{\complement(A)} \cap \overline{A} = \overline{\complement(A)} \cap \overline{\complement(\complement(A))} = \mathsf{bd}(\complement(A)).$$

(F_3). Using the fact that $\mathsf{bd}(A)$ is closed we obtain:

$$\mathsf{bd}(\mathsf{bd}(A)) = \overline{\mathsf{bd}(A)} \cap \overline{\complement(\mathsf{bd}(A))} = \mathsf{bd}(A) \cap \overline{\complement(\mathsf{bd}(A))} \subseteq \mathsf{bd}(A).$$

(F_4). We have:

$$\mathsf{bd}(\emptyset) = \overline{\emptyset} \cap \overline{\complement(\emptyset)} = \emptyset \cap X = \emptyset \ .$$

We note that properties F_1–F_4 are specific for boundary operator.

3.7.4 THEOREM. *If the function* $f : \mathcal{P}(X) \to \mathcal{P}(X)$ *satisfies the conditions:*

(f_1) $A \cap B \cap f(A \cap B) = A \cap B \cap (f(A) \cup f(B))$;
(f_2) $f(A) = f(\complement(A))$;
(f_3) $f(f(A)) \subseteq f(A)$;
(f_4) $f(\emptyset) = \emptyset$,

for all sets $A, B \subseteq X$, *then:*

(1) *the family* $\mathcal{T} = \{\complement(M) \mid (M \subseteq X) \wedge (f(M) \subseteq M)\}$ *is a topology on* X;

(2) *in the topological space* $(X, \mathcal{T})$ *the function* f *is the boundary operator.*

Proof. (1) Define the auxiliary function $g : \mathcal{P}(X) \to \mathcal{P}(X)$, by

$$g(A) = A \cup f(A).$$

From the definition of g we deduce

$$A \subseteq g(A) \tag{$\diamond$}$$

From condition (f_1) we deduce

$$\complement(A) \cap \complement(B) \cap f(\complement(A) \cap \complement(B)) = \complement(A) \cap \complement(B) \cap (f(\complement(A)) \cup f(\complement(B))),$$

therefore, using (f_2), we obtain

$$\complement(A \cup B) \cap f(A \cup B) = \complement(A \cup B) \cap (f(A) \cup f(B)).$$

Adding $A \cup B$ to both sides of the previous equality we obtain

$$A \cup B \cup f(A \cup B) = A \cup B \cup f(A) \cup f(B),$$

that is

$$g(A \cup B) = g(A) \cup g(B) \qquad (\diamond\diamond)$$

For $B = f(A)$, the equality

$$A \cup B \cup f(A \cup B) = A \cup f(A) \cup B \cup f(B),$$

and relation (f_3) gives

$$A \cup f(A) \cup f(A \cup f(A)) = A \cup f(A),$$

that is

$$g(g(A)) = g(A) \qquad (\diamond\diamond\diamond)$$

From (f_4) we obtain

$$g(\emptyset) = \emptyset \qquad (\diamond\diamond\diamond\diamond)$$

Taking into account the relations

$$g(M) = M \iff f(M) \subseteq M,$$

from $(\diamond)$, $(\diamond\diamond)$, $(\diamond\diamond\diamond)$, $(\diamond\diamond\diamond\diamond)$ and by virtue of theorem (3.5.9), it follows that $\mathcal{T}$ is a topology on X and $g(A) = \overline{A}$ for all $A \subseteq X$.

(2) For all $A \subseteq X$, the following equalities hold:

$$\begin{aligned}\mathsf{bd}(A) &= \overline{A} \cap \overline{\complement(A)} = g(A) \cap g(\complement(A)) = (A \cup f(A)) \cap (\complement(A) \cup f(\complement(A))) \\ &= (A \cup f(A)) \cap (\complement(A) \cup f(A)) = f(A).\end{aligned}$$

More properties of the boundary operator are presented in the following theorem.

3.7.5 THEOREM. *The boundary operator satisfies the equalities:*

(1) $\mathsf{bd}(A) = \overline{A} \setminus \mathsf{int}(A)$;

(2) $\overline{A} = \mathsf{int}(A) \cup \mathsf{bd}(A)$.

Proof. (1). We have:

$$\mathsf{bd}(A) = \overline{A} \cap \overline{\complement(A)} = \overline{A} \setminus \complement(\overline{\complement(A)}) = \overline{A} \setminus \mathsf{int}(A).$$

(2)

$$\mathsf{int}(A) \cup \mathsf{bd}(A) = \mathsf{int}(A) \cup (\overline{A} \setminus \mathsf{int}(A)) = \overline{A} \ .$$

3.7.6 THEOREM. *A set A is closed iff* $\mathsf{bd}(A) \subseteq A$.

Proof. Adding A to both sides of the equality

$$\overline{A} = \mathsf{int}(A) \cup \mathsf{bd}(A),$$

we obtain

$$\overline{A} = A \cup \mathsf{bd}(A),$$

therefore

$$A \text{ closed} \iff \overline{A} = A \iff A = A \cup \mathsf{bd}(A) \iff \mathsf{bd}(A) \subseteq A \ .$$

3.7.7 EXAMPLE. In case of the natural topology of the real axis we have:

$$\mathsf{bd}([0,1]) = \{0,1\}; \quad \mathsf{bd}([0,1)) = \{0,1\}; \quad \mathsf{bd}(\{0\}) = \{0\};$$

$$\mathsf{bd}(\mathbb{N}) = \mathbb{N}; \quad \mathsf{bd}(\mathbb{Q}) = \mathbb{R}.$$

3.7.8 EXAMPLE. In case of the discrete topology we have:

$$\mathsf{bd}(A) = \overline{A} \setminus \mathsf{int}(A) = A \setminus A = \emptyset.$$

3.7.9 EXAMPLE. In case of the topology $\mathcal{T} = \{\emptyset, \{1\}, \{1,2\}\}$ we have:

$$\mathsf{bd}(\{1\}) = \overline{\{1\}} \setminus \mathsf{int}(\{1\}) = \{1,2\} \setminus \{1\} = \{2\}, \quad \mathsf{bd}(\{2\}) = \{2\}.$$

3.8 Density. Connectedness. Compactness

3.8.1 DEFINITION. *A set A in the topological space $(X, \mathcal{T})$ is said to be* **dense** *in X iff $\overline{A} = X$.*

3.8.2 EXAMPLE. In the natural topology of the real axis we have $\overline{\mathbb{Q}} = \mathbb{R}$.

3.8.3 DEFINITION. *A topological space is said to be* `connected` *if it cannot be represented as a union of two nonempty, disjoint and closed sets. Otherwise it is called* `disconnected`.

3.8.4 REMARK. *We note that a topological space is connected if it cannot be represented as a union of two nonempty, disjoint and open sets.*

Let A, B be two sets in the topological space $(X, \mathcal{T})$.

3.8.5 DEFINITION. *The set*

$$J(A,B) := (\overline{A} \cap B) \cup (A \cap \overline{B})$$

is called `junction` *of the sets A and B.*

3.8.6 DEFINITION. *The sets A and B are said to be* `separated` *if*

$$J(A,B) = \emptyset.$$

3.8.7 THEOREM. *A topological space $(X, \mathcal{T})$ is disconnected iff it can be represented as the union of two nonempty separated sets.*

Proof. Necessity. Suppose that X is disconnected. Then there exist two nonempty, disjoint and closed sets A and B such that

$$X = A \cup B.$$

We have:

$$J(A,B) = (\overline{A} \cap B) \cup (A \cap \overline{B}) = (A \cap B) \cup (A \cap B) = \emptyset \cup \emptyset = \emptyset.$$

Sufficiency. Suppose that there exist two sets $A, B \subseteq X$, such that

$$A \neq \emptyset, \quad B \neq \emptyset, \quad X = A \cup B, \quad J(A, B) = \emptyset.$$

From

$$\emptyset = J(A, B) = (\overline{A} \cap B) \cup (A \cap \overline{B}) = \emptyset.$$

we deduce

$$\overline{A} \cap B = \emptyset \qquad \text{and} \qquad A \cap \overline{B} = \emptyset, \qquad \text{hence} \qquad A \cap B = \emptyset.$$

We have:

$$\overline{A} = \overline{A} \cap X = \overline{A} \cap (B \cup A) = (\overline{A} \cap B) \cup (\overline{A} \cap A) = \emptyset \cup A = A,$$

therefore A is closed. Similarly, we can show that B is also closed. It follows that the space X can be represented as the union of two disjoint, nonempty and closed sets. Hence it is disconnected.

3.8.8 DEFINITION. *A set A in the topological space $(X, \mathcal{T})$ is said to be* `connected` *if the topological subspace $(A, \mathcal{T}_A)$ is connected.*

3.8.9 DEFINITION. *An open connected set is called a* `domain`.

3.8.10 REMARK. *We can prove that, in case of the natural topology on $\mathbb{R}^n$, an open set D is domain iff any two points in D can be united by a polygonal line included in D.*

3.8.11 DEFINITION. *The family $\mathcal{A} \subseteq \mathcal{P}(X)$ satisfying the equality*

$$\bigcup_{A \in \mathcal{A}} A = X$$

is said to be a `cover` *or* `covering` *of X. If the sets in the covering $\mathcal{A}$ are open then $\mathcal{A}$ is said to be an* `open covering`.

3.8.12 DEFINITION. *A topological space X is said to be* `compact` *if from any open covering of X a finite covering can be selected.*

3.8.13 DEFINITION. *In a topological space* $(X, \mathcal{T})$ *a set* A *is said to be* `compact` *if the topological subspace* $(A, \mathcal{T}_A)$ *is compact.*

3.8.14 DEFINITION. *In a topological space a connected compact set* A *is called a* `continuum`.

In the following theorem we shall present an important result concerning the compact sets.

3.8.15 THEOREM. *In case of the natural topology of* $\mathbb{R}^n$ *a set is compact iff it is closed and bounded.*

3.9 Exercises: Topological Spaces

P
3.9.1 Let X be an infinite set and $\mathcal{C}$ be a collection of sets consisting of the empty set and all subsets of X whose complement is finite.

1. Prove that $\mathcal{C}$ is a topology on X;
2. Find the family of the closed sets in the topological space $(X, \mathcal{C})$;
3. Prove that any neighborhood in the topological space $(X, \mathcal{C})$ is an open set.

P
3.9.2 Let X be an infinite set and $\mathcal{T}$ be a topology on X such that any infinite subset of X is an open set. Prove that $\mathcal{T}$ is the discrete topology.

P
3.9.3 Let A and B be two sets in a topological space $(X, \mathcal{T})$. Prove that:

$$A \cup B = X \Rightarrow \overline{A} \cup \mathrm{int}(B) = X; \qquad (1)$$

$$A \cap B = \emptyset \Rightarrow \overline{A} \cap \mathrm{int}(B) = \emptyset. \qquad (2)$$

P
3.9.4 Prove that in a topological space $(X, \mathcal{T})$ a set G is open iff

$$A \cap G = \emptyset \Rightarrow \overline{A} \cap G = \emptyset,$$

for all $A \subseteq X$.

P
3.9.5

Prove that in a topological space $(X, \mathcal{T})$ the following statements are equivalent:

$$G \in \mathcal{T} \Rightarrow \overline{G} \in \mathcal{T}; \tag{a}$$
$$(G_1, G_2 \in \mathcal{T}) \wedge (G_1 \cap G_2 = \emptyset) \Rightarrow \overline{G_1} \cap \overline{G_2} = \emptyset. \tag{b}$$

P
3.9.6

Prove that in a topological space $(X, \mathcal{T})$ the following relations are satisfied:

$$G \cap \overline{A} \subseteq \overline{G \cap A}; \tag{1}$$
$$\overline{G \cap \overline{A}} = \overline{G \cap A}, \qquad \text{for all } G \in \mathcal{T},\ A \subseteq X. \tag{2}$$

P
3.9.7

Let $(X, \mathcal{T})$ be a topological space such that for all $x \in X$, $\complement(\{x\}) \in \mathcal{T}$. Prove that $A'' \subseteq A'$ for all $A \subseteq X$.

P
3.9.8

Prove that, in a compact space, any closed subset is compact.

P
3.9.9

Let X be a nonempty set, $A \subseteq X$ and $x_0 \in X$. Consider the topology $\mathcal{T} = \mathcal{P}(X)$. Find:

$$\mathcal{F}, \ \mathcal{V}_{x_0}, \ \mathsf{int}(A), \ \overline{A}, \ \mathsf{bd}(A), \ A'.$$

P
3.9.10

Let $\mathcal{T} = \{\emptyset, \{1, 2\}\}$ be a topology. Find the following sets:

$$\mathcal{F}, \ \mathcal{V}_1, \ \mathcal{V}_2, \ \mathsf{int}\{1\}, \ \overline{\{1\}}, \ \mathsf{bd}\{1\}, \ \{1\}'.$$

P
3.9.11 Let $\mathcal{T} = \{\mathbb{R},\ G \mid (G \subseteq \mathbb{R}) \wedge (0 \notin G) \wedge (1 \notin G)\}$ be a topology. Find the following sets:

$$\mathcal{F},\ \mathcal{V}_0,\ \mathcal{V}_2,\ \overline{\{0\}},\ \mathsf{int}\{0\},\ \mathsf{bd}\{0\},\ \{0\}',$$

$$\overline{[0,1]},\ \mathsf{bd}([0,1]),\ \overline{\{5\}},\ \mathsf{int}\{5\},\ \mathsf{bd}\{5\},\ \{5\}'.$$

P
3.9.12 In the case of the natural topology of the real axis consider the set

$$A = ([0,1] \cap \mathbb{Q}) \cup (1,2) \cup \{3\}.$$

Find the following sets:

$$\overline{A},\quad \mathsf{int}(A),\quad A',\quad \mathsf{bd}(A),\quad \mathsf{bd}(\mathsf{bd}(A)).$$

4

Metric Spaces

Contents

4.1 Metric

Let X be a nonempty set.

4.1.1 DEFINITION. *A function $\rho : X \times X \to \mathbb{R}$ satisfying the following conditions:*

$(\mathcal{M}_1)$ $\rho(x, y) = 0 \Leftrightarrow x = y$;

$(\mathcal{M}_2)$ $\rho(x, y) \le \rho(x, z) + \rho(y, z)$ *(the triangle inequality);*

for all $x, y, z \in X$, is called a **metric** *on X.*

4.1.2 DEFINITION. *The set X along with a metric ρ is called a* **metric space** *and is denoted by (X, ρ).*

The elements of (X, ρ) are called **points** and the number $\rho(x, y)$ is called the **distance** between the points x and y.

4.1.3 THEOREM. *A metric $\rho : X \times X \to \mathbb{R}$ has the following properties:*

(M_1) $\rho(x, y) \ge 0$ *(non − negativity)* and
$\rho(x, y) = 0 \Leftrightarrow x = y$ *(identity of indiscernibles);*

(M_2) $\rho(x, y) = \rho(y, x)$ *(symmetry);*

(M_3) $\rho(x, y) \le \rho(x, z) + \rho(z, y)$ *(triangle inequality);*

for all $x, y, z \in X$.

Proof. By virtue of axioms $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$ it remains to show that ρ is non-negative and symmetric.

From axiom $(\mathcal{M}_2)$, for $y = x$, it follows that

$$\rho(x, x) \le \rho(x, z) + \rho(x, z),$$

therefore, by virtue of axiom $(\mathcal{M}_1)$, we obtain

$$0 \le 2\rho(x, z),$$

that is

$$\rho(x, z) \ge 0,$$

for all $x, z \in X$.

For $z = x$, from $(\mathcal{M}_2)$, we get

$$\rho(x, y) \le 0 + \rho(y, x).$$

Similarly, we obtain

$$\rho(y,x) \le \rho(x,y),$$

and consequently,

$$\rho(y,x) = \rho(x,y).$$

As an example, a metric space can be obtained if we define

$$\rho(x,y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

We refer to this type of space as a `discrete metric space`.

4.1.4 EXAMPLE. The function $\rho : \mathbb{R} \times \mathbb{R} \to [0,\infty)$,

$$\rho(x,y) = |x-y|,$$

is a metric on $\mathbb{R}$ called the `Euclidean`[1] `metric`.

4.1.5 EXAMPLE. The function $\rho : \mathbb{C} \times \mathbb{C} \to [0,\infty)$,

$$\rho(z_1,z_2) = |z_1 - z_2|,$$

is a metric on $\mathbb{C}$.

1 

Euclid of Alexandria
(about 325–265 BC)

4.1.6 EXAMPLE. The function $\rho_1 : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$,

$$\rho_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|,$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, is a metric called the `Minkowski`[2] `metric` on $\mathbb{R}^n$.

Also known as `Manhattan distance`, `taxi-cab metric`, or `city block distance`, the Minkowski metric is often used in integrated circuits where wires only run parallel to the X or Y axis.

4.1.7 EXAMPLE. The function $\rho_2 : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$,

$$\rho_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, is a metric called the `Euclidean metric` on $\mathbb{R}^n$.

[2] 

Hermann Minkowski (1864–1909), a German mathematician.

It is easy to verify axiom $(\mathcal{M}_1)$. Let us verify axiom $(\mathcal{M}_2)$.

$$\begin{aligned}
\rho^2(x,y) &= \sum_{i=1}^{n}(x_i-y_i)^2 = \sum_{i=1}^{n}(x_i-z_i+z_i-y_i)^2 \\
&= \sum_{i=1}^{n}(x_i-z_i)^2 + 2\sum_{i=1}^{n}(x_i-z_i)(z_i-y_i) + \sum_{i=1}^{n}(z_i-y_i)^2 \\
&\le \rho^2(x,z) + 2\sqrt{\sum_{i=1}^{n}(x_i-z_i)^2}\sqrt{\sum_{i=1}^{n}(z_i-y_i)^2} + \rho^2(z,y) \\
&= \rho^2(x,z) + 2\rho(x,z)\rho(z,y) + \rho^2(z,y) \\
&= (\rho(x,z)+\rho(z,y))^2,
\end{aligned}$$

therefore

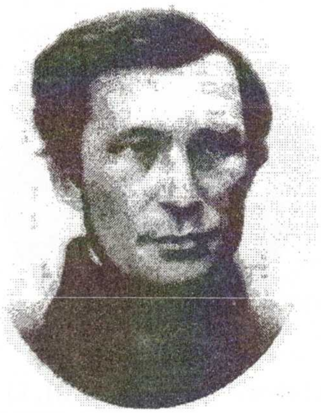
$$\rho(x,y) \le \rho(x,z) + \rho(y,z).$$

4.1.8 EXAMPLE. The function $\rho_\infty : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$,

$$\rho_\infty(x,y) = \max_{i=1,\ldots,n}\{|x_i-y_i|\},$$

where $x=(x_1,x_2,\ldots,x_n)$ and $y=(y_1,y_2,\ldots,y_n)$, is a metric called the **Chebyshev**[3] **metric** on $\mathbb{R}^n$.

Let (X,ρ) be a metric space, $\emptyset \ne A, B \subseteq X$ and $r>0$. The following notions are frequently used:

3 

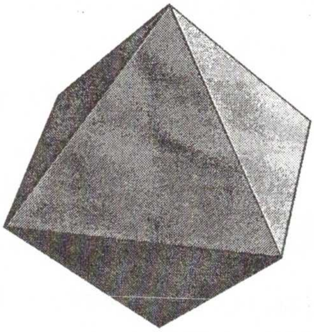
Pafnuty Lvovich Chebyshev (1821–1894), a Russian mathematician.

- the open ball centered at x of radius r,

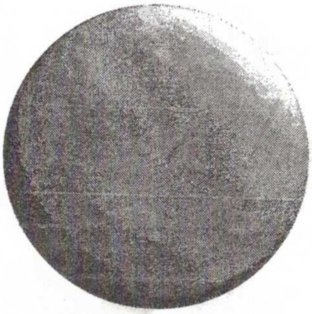
$$B(x, r) = \{y \mid (y \in X) \wedge (\rho(x, y) < r)\};$$

- the closed ball centered at x of radius r,

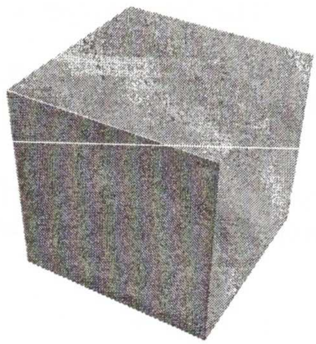
$$\overline{B}(x, r) = \{y \mid (y \in X) \wedge (\rho(x, y) \leq r)\};$$



The Minkowski ball $B((0,0,0),1)$
$= \left\{(x, y, z) \in R^3 \,\middle|\, |x| + |y| + |z| \leq 1\right\}$



The Euclidean ball $B((0,0,0),1)$
$= \left\{(x, y, z) \in R^3 \,\middle|\, \sqrt{x^2 + y^2 + z^2} \leq 1\right\},$



The Chebyshev ball $B((0,0,0),1)$
$= \Big\{(x,y,z) \in R^3 \Big| \max\{|x|,|y|,|z|\} \le 1\Big\}$,

- the distance between the point x and the set A,

$$\mathrm{d}(x,A) = \inf_{a\in A} \rho(x,a);$$

- the distance between the set A and the set B,

$$\mathrm{d}(A,B) = \inf_{a\in A,\, b\in B} \rho(a,b).$$

4.1.9 REMARK. *We note that the term "distance" is somewhat misleading because the set of non-empty subsets of the space X does not constitute a metric space with the distance* $\mathrm{d}(A,B)$. *We can define a distance between sets which has many more common features with the metric occurring in the definition of metric spaces* (see problem (4.5.9)).

4.1.10 EXAMPLE. In the metric space X along with the metric

$$\rho(x,y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases}$$

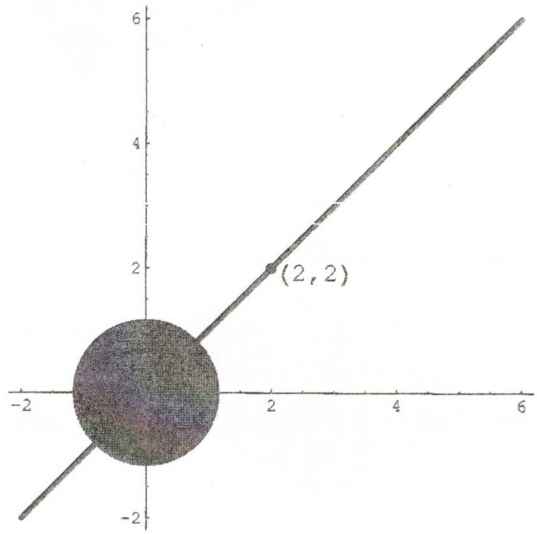
we have

$$B(x,1) = \{x\}, \qquad B(x,2) = X, \qquad \text{forall } x \in X.$$

4.1.11 EXAMPLE. The `French Metro Metric` $d\colon \mathbb{R}^2 \to [0,\infty)$,

$$d(x,y) = \begin{cases} |x-y|, & \text{if } x = cy \text{ for some } c \in \mathbb{R} \\ |x|+|y|, & \text{otherwise,} \end{cases}$$

where $|x| = \rho_2(x,0)$, is an example for disproving apparently intuitive but false properties of metric spaces.



The French Metro ball $\boldsymbol{B((2,2),4)}$

4.1.12 EXAMPLE. The `Hamming metric` $\rho\colon \mathbb{R}^n \to \mathbb{N}$,

$$\rho(x,y) = \mathsf{card}\{i \mid x_i \neq y_i,\ 1 \leq i \leq n\}.$$

is used in comparing codewords.

4.1.13 EXAMPLE. Let $P \in \mathbb{R}^n$. The `Post Office metric`, $\rho_P\colon \mathbb{R}^n \to [0,\infty)$,

$$\rho_P(x,y) = \begin{cases} \rho_2(x,P) + \rho_2(P,y), & \text{if } x \neq y \\ 0, & \text{if } x = y, \end{cases}$$

is an example of a hybrid Euclidean metric, since it involves a distance calculation using the Euclidean measures but taken via an intermediate point P. It applies to situations in which all traffic (physical, electronic, data etc) must be routed via a specified location.

4.2 Topology of a Metric Space

4.2.1 THEOREM. *In a metric space (X, ρ) we can define the following topology*

$$\mathcal{T} = \{\emptyset, G \mid (G \subseteq X) \wedge (\forall x)(\exists r)((x \in G) \rightarrow (r > 0 \wedge B(x,r) \subseteq G))\}.$$

Proof. $(\mathcal{T}_1)$ Let $\{G_i\}_{i \in I}$ be a family of sets in $\mathcal{T}$. We have:

$$x \in \bigcup_{i \in I} G_i \iff (\exists i)(i \in I, x \in G_i)$$

$$\rightarrow (\exists r_i)(r_i > 0, B(x, r_i) \subseteq G_i \subseteq \bigcup_{i \in I} G_i),$$

hence

$$\bigcup_{i \in I} G_i \in \mathcal{T}.$$

$(\mathcal{T}_2)$ Let $G_1, G_2 \in \mathcal{T}$. Let $x \in G_1 \cap G_2$. We have:

$$x \in G_1 \wedge x \in G_2$$

$$\rightarrow (\exists r_1)(\exists r_2)(r_1 > 0, r_2 > 0, B(x, r_1) \subseteq G_1, B(x, r_2) \subseteq G_2);$$

therefore, choosing $r = \min\{r_1, r_2\}$, we obtain

$$B(x, r) = B(x, r_1) \cap B(x, r_2),$$

hence

$$B(x, r) \subseteq G_1 \cap G_2,$$

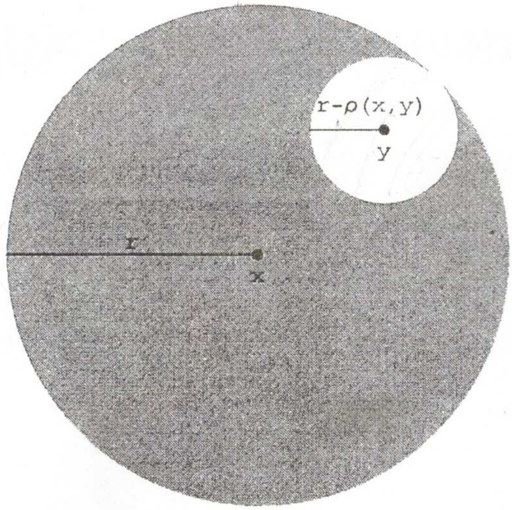
and consequently,

$$G_1 \cap G_2 \in \mathcal{T}.$$

$(\mathcal{T}_3)$ For all $x \in X$ we have $B(x, 1) \subseteq X$, hence $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.

The topology defined in theorem (4.2.1) is called the `topology induced` by the metric ρ.

4.2.2 REMARK. *In the topology induced by the metric ρ the open balls are open sets.*



Indeed, for all $y \in B(x,r)$, we have the inclusion

$$B(y, r - \rho(x,y)) \subseteq B(x,r),$$

that is

$$B(x,r) \in \mathcal{T}.$$

4.2.3 REMARK. *Unless differently specified, by the topology of a metric space we mean the topology induced by the metric of the space (cf. Theorem 4.2.1).*

4.3 Sequences in Metric Spaces

Let (X, ρ) be a metric space. We consider the set

$$X^{\mathbb{N}} = X \times X \times \cdots \times X \times \cdots = \{f \mid f : \mathbb{N} \to X\}.$$

4.3.1 DEFINITION. *An element f in the set $X^{\mathbb{N}}$ is called a* sequence.

Using the notation $f(n) = x_n$, we shall denote the sequence $(x_0, x_1, \ldots, x_n, \ldots)$ by the symbol

$$(x_n).$$

The element x_n is called the nth **term** of the sequence.

4.3.2 DEFINITION. [4, V. Câmpian] *A function $f \circ g$, where $g : \mathbb{N} \to \mathbb{N}$ is strictly increasing, is called a* **subsequence** *of the sequence f.*

In order to denote the subsequence $f \circ g$ we shall use the notation (x_{n_k}), where $x_{n_k} = f(g(k)) = f(n_k), \quad k \in \mathbb{N}$.

4.3.3 DEFINITION. *The sequence (x_n) is said to be* **convergent** *if there exists a point $x \in X$ such that, for all $\varepsilon > 0$, the number of terms x_n lying outside the ball $B(x, \varepsilon)$ is finite.*

The point x is called the **limit of the sequence** (x_n). The following notations will be used:

$$\lim_{n \to \infty} x_n = x \quad \text{or} \quad x_n \to x.$$

4.3.4 DEFINITION. *A sequence (x_n) in the metric space (X, ρ) is said to be a* **fundamental sequence** *or a* **Cauchy sequence** *iff, given any $\varepsilon > 0$ there exists an r_ε such that*

$$\rho(x_n, x_m) < \varepsilon,$$

provided $n, m \in \mathbb{N}, \ n, m > r_\varepsilon$.

4.3.5 DEFINITION. *A metric space (X, ρ) is said to be* **complete** *iff any Cauchy sequence in X converges to a point of X.*

4.3.6 EXAMPLE. The space $\mathbb{Q}$ along with the Euclidean metric

$$\rho(x, y) = |x - y|$$

is not a complete space. Although the sequence

$$\left(\left(1 + \frac{1}{n}\right)^n\right) \in \mathbb{Q}^{\mathbb{N}^*}$$

is a Cauchy sequence, however

$$\left(1+\frac{1}{n}\right)^n \to \mathrm{e} \notin \mathbb{Q}.$$

4.3.7 REMARK. *Every convergent sequence is a Cauchy sequence.*

4.3.8 REMARK. *In order to prove that a sequence (x_n) is a Cauchy sequence it is enough to find a sequence of positive real numbers (ε_n), which converges to zero, such that*

$$\rho(x_{n+p}, x_n) \leq \varepsilon_n,$$

provided $n, p \in \mathbb{N}$.

4.4 Bounded Sets in Metric Spaces

4.4.1 DEFINITION. *A set A in a metric space (X, ρ) is said to be* bounded *iff there exists a point $x \in X$ and a number $r > 0$ such that*

$$A \subseteq B(x, r).$$

4.4.2 THEOREM. *Any compact set in a metric space is bounded.*

Proof. Let (X, ρ) be a metric space, $A \subseteq X$ be a compact set and $x \in X$. Since the family

$$\{B(x, n)\}_{n \in \mathbb{N}}$$

is an open covering of X (implicitly an open cover of A) and A is compact, there exists a finite open subcover

$$\{B(x, n_i)\}_{i=\overline{1,m}}$$

of A, where $n_1 < n_2 < \cdots < n_m$. We have

$$A \subseteq \bigcup_{i=1}^{m} B(x, n_i) = B(x, n_m),$$

i.e., the set A is bounded.

4.5 Exercises: Metric Spaces

P 4.5.1 Prove that if ρ is a metric on X, then $\sigma = \dfrac{\rho}{1+\rho}$ is also a metric on X.

P 4.5.2 Consider the space $\mathbb{R}$ along with the metric $\rho(x,y) = \dfrac{|x-y|}{1+|x-y|}$. Find the balls $B(0,\frac{1}{2})$, $B(0,1)$.

P 4.5.3 Let (X,ρ) be a metric space, where $\rho(x,y) = 1$ for $x \neq y$. Find $B(x,r)$, $\overline{B}\,(x,r)$, $\overline{B(x,r)}$.

P 4.5.4 Consider the space $X = [-2,1] \cup \{2\}$ along with the metric $\rho(x,y) = |x-y|$. Find: $B(0,2)$, $\overline{B}\,(0,2)$, $\overline{B(0,2)}$.

P 4.5.5 Let X be the set of all bounded sequences of complex numbers with the metric $\rho((x_n),(y_n)) = \sum\limits_{k=0}^{\infty} \dfrac{|x_k - y_k|}{2^k}$. Find the distance between the sequences (x_n) and (y_n), $x_n = 1$, $y_n = \sin n$.

P 4.5.6 Let (X,ρ) be a metric space and $A \subseteq X$. Prove that

$$x \in \overline{A} \iff (\forall \varepsilon)(\exists a)(\varepsilon > 0,\ (a \in A,\ \rho(x,a) < \varepsilon)).$$

P
4.5.7 Let (X, ρ) be a metric space and $A \subseteq X$. Prove that

$$x \in \overline{A} \iff d(x, A) = 0.$$

P
4.5.8 Let A and B be two subsets of the metric space (X, ρ). Prove that

$$d(A, B) = d(\overline{A}, \overline{B}).$$

P
4.5.9 Let (X, ρ) be a metric space and $\mathcal{B}$ be the family of all nonempty, bounded and closed subsets of X. Define $D : \mathcal{B} \times \mathcal{B} \to [0, \infty)$,

$$D(A, B) = \max\{\sup_{a \in A}\{\inf_{b \in B}\{\rho(a, b)\}\}, \sup_{b \in B}\{\inf_{a \in A}\{\rho(a, b)\}\}\}.$$

Prove that $(\mathcal{B}, D)$ is a metric space.

P
4.5.10 Consider a nonempty compact set A in the Euclidian metric space $\mathbb{R}^2$. Prove that there exists a straight line which intersects the set A in only one point.

5

Sequences and Series of Numbers

Contents

5.1 Sequences of Numbers

A function $f : \mathbb{N} \to \mathbb{C}$ is called a sequence of complex numbers. We recall that the number $x_n = f(n)$ is called the nth term of the sequence. To denote the sequence f we use the notation (x_n).

5.1.1 DEFINITION. *An element $\ell \in \overline{\mathbb{C}}$ is called the* limit *of the sequence (x_n) if the number of points x_n lying outside any neighborhood of ℓ is finite.*

We write:

$$\lim_{n\to\infty} x_n = \ell \quad \text{or} \quad x_n \to \ell.$$

5.1.2 DEFINITION. *If $\ell \in \mathbb{C}$, then we say that the sequence (x_n) is* convergent.

5.1.3 DEFINITION. *The sequence (x_n) is called a* fundamental sequence *or a* Cauchy sequence[1] *if for any $\varepsilon > 0$ there is r_ε such that the condition $|x_n - x_m| < \varepsilon$ is satisfied for all $n, m \in \mathbb{N}$ and $n, m > r_\varepsilon$.*

5.1.4 THEOREM. *A sequence of complex numbers is convergent iff it is a fundamental sequence.*

1 

Augustin Louis Cauchy (1789–1857), a great French mathematician.

Let X be a nonempty set of real numbers.

5.1.5 DEFINITION. *The number M is called the* `greatest element` *(*`maximum`*) of the set X if $M \in X$ and, for any $x \in X$, the inequality $x \leq M$ is satisfied.*

The notion of the `least element` (`minimum`) of the set X is defined in a similar way.

The set X is said to be `bounded above` if there exists a real number a such that $x \leq a$ for all $x \in X$.

Any number a possessing this property is called the `upper bound` of the set X.

For a given bounded above set X, the set of all of its upper bounds has the least element (minimum), which is called the `least upper bound` or `supremum`, and is denoted by the symbol $\sup X$.

The notions of a set bounded below, `lower bound` and the `greatest lower bound` or `infimum` are defined in a similar way. The latter is denoted by the symbol $\inf X$. A set bounded above and below is said to be `bounded`.

We can prove that the number M is the supremum of the set X if and only if:

(1) $(\forall x)(x \in X \to x \leq M)$;

(2) $(\forall \varepsilon)(\exists x)((\varepsilon > 0) \to (x \in X) \wedge (x > M - \varepsilon))$,

and the number m is the infimum of the set X if and only if:

(1) $(\forall x)(x \in X \to x \geq m)$;

(2) $(\forall \varepsilon)(\exists x)((\varepsilon > 0) \to (x \in X) \wedge (x < m + \varepsilon))$.

The supremum and the infimum of a sequence (x_n) are respectively the supremum and the infimum of the set $\{x_0, x_1, \ldots\}$, i.e.,

$$\inf(x_n) = \inf(\{x_0, x_1, \ldots\}),$$

$$\sup(x_n) = \sup(\{x_0, x_1, \ldots\}).$$

5.1.6 REMARK. *If A is a closed bounded set then $\inf(A), \sup(A) \in A$.*

5.1.7 DEFINITION. *An element $x \in \mathbb{C} \cup \{\pm\infty\}$ is called a* `limit point` *of a sequence (x_n), if every neighborhood of x contains an infinite number of terms x_n.*

The set of all limit points of a sequence (x_n) is denoted by $\mathsf{LIM}(x_n)$.

5.1.8 DEFINITION. *The infimum of the set* $\mathsf{LIM}(x_n)$ *is called the* `limit inferior` *of the sequence* (x_n) *and denoted by* $\underline{\lim}(x_n)$. *Likewise, the* `limit superior` *is defined by*

$$\overline{\lim}\,(x_n) = \sup(\mathsf{LIM}(x_n)).$$

The following four theorems are well known.

5.1.9 THEOREM. *Any bounded sequence of complex numbers contains a convergent subsequence.*

5.1.10 THEOREM. *Any monotone sequence of real numbers has a limit.*

5.1.11 THEOREM. *For any sequence* (x_n) *of real numbers the following inequalities hold:*

$$\inf(x_n) \le \underline{\lim}(x_n) \le \overline{\lim}\,(x_n) \le \sup(x_n).$$

5.1.12 THEOREM. *A sequence* (x_n) *of real numbers has a limit iff*

$$\underline{\lim}(x_n) = \overline{\lim}\,(x_n).$$

Suppose that (α_n) and (β_n) are two sequences such that there exists a positive constant K with

$$|\alpha_n| \le K|\beta_n|, \quad \text{for large } n.$$

Then we say that (α_n) is a ``big oh'' of β_n and write

$$\alpha_n = O(\beta_n).$$

5.1.13 REMARK. *In what follows we shall denote by* 1_n *and* 0_n *the terms of a sequence convergent to* 1 *and* 0 *respectively.*

5.1.14 THEOREM. *The following equalities are true:*

1. $\sin 0_n = 0_n - \dfrac{0_n^3}{6}\, 1_n;$

2. $\cos 0_n = 1 - \dfrac{0_n^2}{2}\, 1_n;$

3. $\tan 0_n = 0_n + \dfrac{0_n^3}{3}\, 1_n;$

4. $\arcsin 0_n = 0_n + \dfrac{0_n^3}{6}\, 1_n;$

5. $\arctan 0_n = 0_n - \dfrac{0_n^3}{3}\, 1_n;$

6. $\ln(1 + 0_n) = 0_n - \dfrac{0_n^2}{2}\, 1_n;$

7. $a^{0_n} = 1 + 0_n \ln a + \dfrac{0_n^2 \ln^2 a}{2}\, 1_n, \quad a > 0;$

8. $(1 + 0_n)^a = 1 + a0_n + 0_n^2 \dfrac{a(a-1)}{2}\, 1_n, \quad a \in \mathbb{R}.$

Theorem (5.1.14) can be used to find the limit of several sequences, but we must pay attention to the relation

$$1_n - 1_n = 0_n.$$

5.1.15 EXAMPLE. Find the limit of the sequence

$$x_n = n^2 - \cot^2 \frac{1}{n}, \quad n \geq 1.$$

We have:

$$n^2 - \cot^2 \frac{1}{n} = n^2 - 1/\left(\frac{1}{n} + \frac{1_n}{3n^3}\right)^2 = \frac{\frac{2}{3}}{1 + \frac{2}{3}\frac{1_n}{n^2}} \to \frac{2}{3}.$$

Consider the `forward difference operator`, $\Delta : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$,

$$\Delta f(n) = f(n+1) - f(n).$$

We have

5.1.16
$$\boxed{\Delta^p f(n) = \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} f(n+k).}$$

The relationship between the derivative of a function and the forward finite difference is presented in the following theorem.

5.1.17 THEOREM. *If the function $f : \mathbb{R} \to \mathbb{R}$ has an n^{th} derivative, then for each $x \in \mathbb{R}$ there exists $\xi \in (x, x+p)$, such that*

$$\boxed{\Delta^p f(x) = f^{(p)}(\xi).}$$

5.2 The Stirling numbers

Let $x \in \mathbb{R}$. The `rising factorial` $x^{\overline{k}}$ and the `falling factorial` $x^{\underline{k}}$ are defined respectively by:

$$\begin{aligned} x^{\overline{0}} &= 1, \qquad x^{\overline{k}} = x\,(x+1)\ldots(x+k-1)\,, \\ x^{\underline{0}} &= 1, \qquad x^{\underline{k}} = x\,(x-1)\ldots(x-k+1)\,, \qquad k = 0, 1, \ldots \end{aligned}$$

Let n be a positive integer. If

$$x^{\underline{n}} = \sum_{k=0}^{n} s(n,k) x^k, \tag{5.1}$$

$$x^n = \sum_{k=1}^{n} S(n,k) x^{\underline{k}}, \qquad k = 0, 1, \ldots, n, \tag{5.2}$$

then the coefficients $s(n,k)$ and $S(n,k)$, $n, k \geq 0$, are called the

Stirling numbers of the first and second kind, respectively. [2]

We have:

$$\begin{aligned} s(0,0) &= 1, \quad s(0,k) = 0, \\ S(0,0) &= 1, \quad S(0,k) = 0, \qquad k = 0, 1, \ldots, n, \end{aligned}$$

and

$$\begin{aligned} S(n,k) &= \left.\frac{\Delta^k x^n}{k!}\right|_{x=0}, \\ s(n,k) &= \left.\frac{D^k x^n}{k!}\right|_{x=0}, \quad n, k = 0, 1, \ldots, \end{aligned}$$

where $D^k f(x) = f^{(k)}(x)$.

A great deal has been written about these numbers, and they have important combinatorial significance (see Riordan [29, 30]). Some Stirling numbers are listed in Table 5.1 and Table 5.2.

$n \setminus k$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	−1	1	0	0	0
3	0	2	−3	1	0	0
4	0	−6	11	−6	1	0
5	0	24	−50	35	−10	1

Table 5.1: Stirling numbers of the first kind $s(n,k)$.

Since

$$\sum_k S(n,k)s(k,m) = \delta_{n,m}, \qquad n, m \in \mathbb{N},$$

[2] James Stirling, Born: May 1692 in Garden (near Stirling), Scotland; Died: 5 Dec 1770 in Edinburgh, Scotland.
Stirling was able to show that for large n we get the approximation $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. This is usually called *Stirling's formula*, although in fact it have been known earlier to Abraham de Moivre.

$n \backslash k$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	1	0	0	0
3	0	1	3	1	0	0
4	0	1	7	6	1	0
5	0	1	15	25	10	1

Table 5.2: Stirling numbers of the second kind $S(n,k)$.

they satisfy the inverse relations

$$a_n = \sum_{k=1}^{n} S(n,k) b_k, \qquad b_n = \sum_{k=1}^{n} s(n,k) a_k.$$

We obtain:

$$\begin{aligned} S(n,k) &= \left.\frac{\Delta^k x^n}{k!}\right|_{x=0}, \\ s(n,k) &= \left.\frac{D^k x^n}{k!}\right|_{x=0}, \quad n,k = 0,1,\ldots. \end{aligned}$$

5.2.1 EXAMPLE. *Prove that the following equality is satisfied*

$$\sum_{k=1}^{n} k^p = \sum_{i=1}^{p} \left(\sum_{m=0}^{i} \frac{(-1)^{i-m}\, m^p}{m!(i-m)!(i+1)} \right) \prod_{j=0}^{i} (n+1-j)$$

Proof. The identity

5.2.2
$$\sum_{k=1}^{n} \prod_{j=0}^{i-1} (k-j) = \frac{1}{i+1} \prod_{j=0}^{i} (n+1-j),$$

is satisfied for $i = 1, 2, \ldots$ From

$$\sum_{k=1}^{n} k^p = \sum_{i=1}^{p} S(p,i) \prod_{j=0}^{i-1} (k-j),$$

by using (5.2.2), we obtain

$$
\begin{aligned}
\sum_{k=1}^{n} k^p &= \sum_{i=1}^{p} S(p,i)\,\frac{1}{i+1}\prod_{j=0}^{i}(n+1-j) \\
&= \sum_{i=1}^{p} \frac{1}{(i+1)!} \sum_{m=0}^{i} (-1)^{i-m}\binom{i}{m} m^p \prod_{j=0}^{i}(n+1-j) \\
&= \sum_{i=1}^{p} \left(\sum_{m=0}^{i} \frac{(-1)^{i-m}\, m^p}{m!(i-m)!(i+1)} \right) \prod_{j=0}^{i}(n+1-j).
\end{aligned}
$$

For the convenience of the reader we list the following sums.

$$
\begin{aligned}
\sum_{k=1}^{n} k \ &= \frac{1}{2}\, n\,(1+n), \\
\sum_{k=1}^{n} k^2 &= \frac{1}{6}\, n\,(1+n)\,(1+2\,n), \\
\sum_{k=1}^{n} k^3 &= \frac{1}{4}\, n^2\,(1+n)^2, \\
\sum_{k=1}^{n} k^4 &= \frac{1}{30}\, n\,(1+n)\,(1+2\,n)\,(-1+3\,n+3\,n^2), \\
\sum_{k=1}^{n} k^5 &= \frac{1}{12}\, n^2\,(1+n)^2\,(-1+2\,n+2\,n^2).
\end{aligned}
$$

```
In[1]:= (* Mathematica *)

In[2]:= (* Stirling numbers show up in many combinatorial enumeration problems. The Stirling
        numbers of the first kind StirlingS1[n,k], satisfy the generating function relation
        x (x-1) ... (x-n+1) = Σ_{k=0}^n StirlingS1[n,k] x^k.*)

In[3]:=
        (* The Stirling numbers of the second kind StirlingS2[n, m] give the
        number of ways of partitioning a set of n elements into m non-empty
        subsets. They satisfy the relation x^n = Σ_{m=0}^n S_n^(m) x (x-1) ... (x-m+1) . *)

In[4]:=
        (* The Pochhammer symbol or rising factorial
        Pochhammer[a, n] is (a)_n = a (a+1) ... (a+n-1) = Γ (a+n) / Γ (a) *)

In[5]:= Factor[Table[(-1)^n Pochhammer[-x, n], {n, 0, 4}]]

Out[5]= {1, x, (-1 + x) x, (-2 + x) (-1 + x) x, (-3 + x) (-2 + x) (-1 + x) x}

In[6]:= Simplify[Factor[Table[(-1)^n Pochhammer[-x, n] == Σ_{k=0}^n StirlingS1[n, k] x^k, {n, 0, 4}]]]

Out[6]= {True, True, True, True, True}

In[7]:= Simplify[Factor[Table[x^n == Σ_{k=0}^n StirlingS2[n, k] (-1)^k Pochhammer[-x, k] , {n, 0, 4}]]]

Out[7]= {True, True, True, True, True}
```

5.2.3 EXAMPLE. *Prove that*

$$\lim_{n\to\infty} n^p \sum_{k=0}^{p} (-1)^k \binom{p}{k} \sqrt{1+\frac{k}{n}} = \frac{-(2p-3)!!}{2^p}.$$

Hint: $f(x) = \sqrt{x}$.

5.2.4 EXAMPLE.

$$\lim_{n\to\infty} \left(\frac{\prod_{k=0}^{p} (n+2k)^{\binom{2p}{2k}}}{\prod_{k=0}^{p-1} (n+2k+1)^{\binom{2p}{2k+1}}} \right)^{n^{2p}} = e^{-(2p-1)!}, \quad p \in \mathbb{N}^*.$$

Hint: $f(x) = \ln x$.

5.2.5 THEOREM. (**Stolz**[3] – **Cesaro**[4]) *If the sequences (x_n) and (y_n) satisfy at least one of the conditions:*

(1) *(y_n) is strictly monotone, $x_n \to 0$, $y_n \to 0$;*

(2) *(y_n) is strictly monotone, $|y_n| \to \infty$,*

and the limit $\lim\limits_{n\to\infty} \dfrac{x_{n+1} - x_n}{y_{n+1} - y_n}$ exists, then

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}.$$

3 

Otto Stolz (1842–1905), a German mathematician.

4 

Ernesto Cesaro (1859–1906), an Italian mathematician.

Proof. Consider that the sequences (x_n), (y_n) satisfy condition (1), (y_n) is increasing and

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \ell \in \mathbb{R}.$$

Then, given any arbitrary positive ε, there exists r_ε such that for any $n > r_\varepsilon$,

$$\left| \frac{x_{n+1} - x_n}{y_{n+1} - y_n} - \ell \right| < \varepsilon.$$

Hence

$$-\varepsilon < \frac{x_{n+1} - x_n}{y_{n+1} - y_n} - \ell < \varepsilon,$$

that is

$$(y_{n+1} - y_n)(\ell - \varepsilon) < x_{n+1} - x_n < (y_{n+1} - y_n)(\ell + \varepsilon),$$

for all $n > r_\varepsilon$.
We obtain:

$$(y_{n+1} - y_n)(\ell - \varepsilon) < x_{n+1} - x_n < (y_{n+1} - y_n)(\ell + \varepsilon),$$

$$(y_{n+2} - y_{n+1})(\ell - \varepsilon) < x_{n+2} - x_{n+1} < (y_{n+2} - y_{n+1})(\ell + \varepsilon),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(y_{n+p} - y_{n+p-1})(\ell - \varepsilon) < x_{n+p} - x_{n+p-1} < (y_{n+p} - y_{n+p-1})(\ell + \varepsilon).$$

Consequently, we have

$$(y_{n+p} - y_n)(\ell - \varepsilon) < x_{n+p} - x_n < (y_{n+p} - y_n)(\ell + \varepsilon),$$

hence

$$-\varepsilon < \frac{x_{n+p} - x_n}{y_{n+p} - y_n} - \ell < \varepsilon,$$

for all $n > r_\varepsilon$. For $p \to \infty$, we get:

$$-\varepsilon \le \frac{x_n}{y_n} - \ell \le \varepsilon,$$

for all $n > r_\varepsilon$, that is

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \ell.$$

5.2.6 COROLLARY. *If the limit* $\lim\limits_{n\to\infty} a_n$ *exists, then*

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = \lim_{n\to\infty} a_n.$$

Proof. By theorem (5.2.5) we have

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = \lim_{n\to\infty} \frac{a_1 + \cdots + a_{n+1} - a_1 - \cdots - a_n}{n+1-n}$$
$$= \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n.$$

5.2.7 COROLLARY. *If* $a_n > 0$ *and the limit* $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ *exists, then the limit* $\lim\limits_{n\to\infty} \sqrt[n]{a_n}$ *also exists and*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Proof. By theorem (5.2.5) we have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} e^{\ln \sqrt[n]{a_n}} = \lim_{n\to\infty} e^{\frac{\ln a_n}{n}}$$
$$= \lim_{n\to\infty} e^{\frac{\ln a_{n+1} - \ln a_n}{n+1-n}} = \lim_{n\to\infty} e^{\ln \frac{a_{n+1}}{a_n}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

5.2.8 COROLLARY. *If* $a_n > 0$ *and the limit* $\lim\limits_{n\to\infty} a_n$ *exists, then the limit* $\lim\limits_{n\to\infty} \sqrt[n]{a_1 \ldots a_n}$ *also exists such that*

$$\lim_{n\to\infty} \sqrt[n]{a_1 \ldots a_n} = \lim_{n\to\infty} a_n.$$

Proof. By theorem (5.2.5) we have

$$\lim_{n\to\infty} \sqrt[n]{a_1 \ldots a_n} = \lim_{n\to\infty} e^{\ln \sqrt[n]{a_1 \ldots a_n}} = \lim_{n\to\infty} e^{\frac{\ln a_1 + \cdots + \ln a_n}{n}}$$
$$= \lim_{n\to\infty} e^{\frac{\ln a_{n+1}}{n+1-n}} = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n.$$

5.2.9 REMARK. *If the sequence*

$$\left(\frac{x_{n+1}-x_n}{y_{n+1}-y_n}\right)$$

has no limit, then we cannot draw any conclusion concerning the convergence of the sequence

$$\left(\frac{x_n}{y_n}\right).$$

Indeed, for $x_n = (-1)^n$ and $y_n = n$, the sequence

$$\left(\frac{x_{n+1}-x_n}{y_{n+1}-y_n}\right) = \left(2(-1)^{n+1}\right)$$

has no limit. However,

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{(-1)^n}{n} = 0.$$

Let $p \in \mathbb{N}^*$. One can easily prove the following corollary.

5.2.10 COROLLARY. *If the sequences (x_n) and (y_n) satisfy at least one of the conditions:*

(1) $(\Delta^k y_n)$ *is strictly monotone,* $\Delta^k x_n \to 0$, $\Delta^k y_n \to 0$;
(2) $(\Delta^k y_n)$ *is strictly monotone,* $|\Delta^k y_n| \to \infty$,

$k = 0, \ldots, p-1$, *and the limit* $\lim\limits_{n\to\infty}\dfrac{\Delta^p x_n}{\Delta^p y_n}$ *exists, then*

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{\Delta^p x_n}{\Delta^p y_n}.$$

One can also prove:

5.2.11 COROLLARY. *If* $a_n > 0$ *and the limit*

$$\lim_{n\to\infty}\frac{\prod\limits_{k=0}^{\lfloor p/2\rfloor} a_{n+p-2k}{}^{\binom{p}{2k}}}{\prod\limits_{k=0}^{\lfloor (p-1)/2\rfloor} a_{n+p-2k-1}{}^{\binom{p}{2k+1}}}$$

exists, then the limit $\lim\limits_{n\to\infty} \sqrt[n^p]{a_n}$ *also exists and*

$$\lim_{n\to\infty} \sqrt[n^p]{a_n} = \lim_{n\to\infty} \left(\frac{\prod\limits_{k=0}^{\lfloor p/2 \rfloor} a_{n+p-2k}{}^{\binom{p}{2k}}}{\prod\limits_{k=0}^{\lfloor (p-1)/2 \rfloor} a_{n+p-2k-1}{}^{\binom{p}{2k+1}}} \right)^{1/p!}.$$

5.2.12 Prove that

$$\lim_{n\to\infty} \left(\frac{\prod\limits_{k=0}^{p} (n+2k)^{\binom{2p}{2k}}}{\prod\limits_{k=0}^{p-1} (n+2k+1)^{\binom{2p}{2k+1}}} \right)^{n^{2p}} = e^{-(2p-1)!}, \qquad p \in \mathbb{N}^*.$$

5.2.13 THEOREM. **(M. Ivan's Test)** *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers different from zero,* $\alpha > 0$ *and* $\lambda \in \mathbb{R} \setminus \{0\}$. *If the limit*

$$\lim_{n\to\infty} n^\alpha \left(\frac{x_{n+1}}{x_n} - 1 \right) = \lambda$$

exists, then the limit $\lim\limits_{n\to\infty} x_n$ *also exists, and*

$$\lim_{n\to\infty} x_n = \begin{cases} 0, & \text{for } \alpha \le 1, \quad \lambda < 0; \\ \pm\infty, & \text{for } \alpha \le 1, \quad \lambda > 0; \\ \in \mathbb{R}\setminus\{0\}, & \text{for } \alpha > 1. \end{cases}$$

Proof. (Please read Section 5.5 first). We have

$$\frac{x_{n+1}}{x_n} = 1 + \frac{\lambda_n}{n^\alpha},$$

where $\lambda_n \to \lambda$. Let $p \in \mathbb{N}$ such that

$$1 + \frac{\lambda_n}{n^\alpha} > 0 \quad \text{and} \quad \operatorname{sgn} \lambda_n = \operatorname{sgn} \lambda_p = \operatorname{sgn} \lambda \qquad (n \ge p).$$

We obtain

$$x_n = x_p \prod_{k=p}^{n-1} \left(1 + \frac{\lambda_k}{k^\alpha}\right), \quad (n > p),$$

hence

$$\lim_{n\to\infty} x_n = x_p \exp\left(\sum_{k=p}^{\infty} \ln\left(1 + \frac{\lambda_k}{k^\alpha}\right)\right).$$

We deduce

$$\sum_{k=p}^{\infty} \ln\left(1 + \frac{\lambda_k}{k^\alpha}\right) \approx \sum_{k=p}^{\infty} \frac{\lambda_k}{k^\alpha} = \begin{cases} -\infty, & 0 < \alpha \le 1,\ \lambda < 0; \\ \infty, & 0 < \alpha \le 1,\ \lambda > 0; \\ \in \mathbb{R} \setminus \{0\}, & \alpha > 1. \end{cases}$$ ❖❖❖

A poorer criterion is given in [27].

5.2.14 REMARK. *In Th. 5.2.13 one can replace n^α by a certain positive sequence (a_n).*

Use Theorem 5.2.13 to test the following sequences for convergence:

5.2.15 $$a_n = \sqrt{n}\,\frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots(2n)}, \quad n\ge 1,$$

Hint: $\lim_{n\to\infty} n^2\left(\frac{x_{n+1}}{x_n}-1\right) = \lim_{n\to\infty}\frac{\sqrt{n}\,(1+2n-2\sqrt{n}\sqrt{1+n})}{2\sqrt{1+n}} = \frac{1}{8}$.
In fact, $x_n \to 1/\sqrt{\pi}$.

5.2.16 $$x_n = \frac{a(a+1)\ldots(a+n-1)\,e^n\sqrt{n}}{n^{n+a}}, \quad n\ge 1, \quad a>0.$$

Hint: $\lim_{n\to\infty} n^2\left(\frac{x_{n+1}}{x_n}-1\right) = -\frac{6a^2-6a+1}{12}$.

5.2.17 $$y_n = \frac{n!\,n^{a-1}}{a(a+1)\ldots(a+n-1)}, \quad n\ge 1, \quad a>0.$$

Hint: $\lim_{n\to\infty} n^2\left(\frac{y_{n+1}}{y_n}-1\right) = -\frac{1}{2}a+\frac{1}{2}a^2$.

5.2.18 $$z_n = \frac{n!\,e^n}{n^n\sqrt{n}}, \quad n\ge 1.$$

Hint: $\lim_{n\to\infty} n^2\left(\frac{z_{n+1}}{z_n}-1\right) = -\frac{1}{12}$.

5.2.19 $$t_n = \frac{\left(1-\left(\frac{a}{2}\right)^2\right)\left(1-\left(\frac{a}{4}\right)^2\right)\ldots\left(1-\left(\frac{a}{2n}\right)^2\right)}{\left(1-\left(\frac{a}{1}\right)^2\right)\left(1-\left(\frac{a}{3}\right)^2\right)\ldots\left(1-\left(\frac{a}{2n-1}\right)^2\right)}, \; n\ge 1, \; 0<a<1.$$

Hint: $\lim_{n\to\infty} n^3\left(\frac{t_{n+1}}{t_n}-1\right) = \frac{1}{4}a^2$.

5.3 Exercises: Sequences of Numbers (I)

Find the limit of each of the following sequences:

5.3.1 P $x_n = n - n^2 \ln \frac{n+1}{n}$;

5.3.2 P $x_n = \left(n\left(\arctan \frac{\sqrt{n}}{\sqrt{n}+\sqrt{2}} - \arctan \frac{\sqrt{n}-\sqrt{2}}{\sqrt{n}}\right)\right)^{n^2}$;

5.3.3 P $x_n = \frac{(n+1)^a - n^a - an^{a-1}}{n^{a-2}}, \quad a \in \mathbb{R}$;

5.3.4 P $x_n = \sqrt[n]{n}$;

5.3.5 P $x_n = \frac{\sqrt[n]{n!}}{n}$;

5.3.6 P $x_n = \left(\frac{\ln(n+1)^{n+1}}{\ln n^n}\right)^n$;

5.3.7 P $x_n = \frac{1}{n}\int_1^n \left(1+\frac{1}{x}\right)^x \mathrm{d}x$;

5.3.8 P $x_n = n \sin(2\pi \mathrm{e} n!)$;

5.3.9 P $x_n = n^2\left(\left(1+\frac{1}{n}\right)^{n+0.5} - \mathrm{e}\right)$;

5.3.10 P $x_n = \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right)$;

5.3.11 [P] $x_n = \sum_{k=1}^{n} \frac{\sin k}{n+k}$;

5.3.12 [P] $x_n = n^{\ln \frac{n+a}{n+b}}, \quad a, b \in \mathbb{R}$;

5.3.13 [P] $x_n = n \ln \frac{a\sqrt{n+1} + b\sqrt{n+2} + c\sqrt{n+3}}{a\sqrt{n+2} + b\sqrt{n+3} + c\sqrt{n+4}}, \quad a, b, c \in \mathbb{R}$;

5.3.14 [P] $x_n = n^2 \left(a^{\frac{1}{n+b}} - a^{\frac{1}{n+c}} \right), \quad a > 0,\ b, c \in \mathbb{R}$;

5.3.15 [P] $x_n = n^a \frac{\sqrt[n]{n} - 1}{\ln n}, \quad a \in \mathbb{R}$;

5.3.16 [P] $x_n = \sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n-1}}$;

5.3.17 [P] $x_n = \frac{a^{6n}}{(1+a^n)(1+2\,a^{2n})(1+3\,a^{3n})}, \quad a > 0$;

5.3.18 [P] $x_n = \frac{\ln n!}{n^a}, \quad a > 1$;

5.3.19 [P] $x_n = \frac{[n\,a]}{n}, \quad a \in \mathbb{R}$;

5.3.20 [P] $x_n = \frac{1}{\binom{n}{0}} + \frac{1}{\binom{n}{1}} + \cdots + \frac{1}{\binom{n}{n}}$;

5.3.21 [P] $x_n = \left(1 + \frac{a}{n}\right)\left(1 + \frac{2a}{n}\right)\cdots\left(1 + \frac{na}{n}\right), \quad a > 0$;

5.3.22 [P] $x_n = \frac{1^r + 3^r + \cdots + (2n-1)^r}{n^{r+1}}, \quad r > 0$;

5.3.23 P $x_n = \frac{1}{n} + \frac{1}{2n} + \cdots + \frac{1}{n^2}$;

5.3.24 P $x_n = \frac{1}{\ln 2^n} + \frac{1}{\ln 3^n} + \cdots + \frac{1}{\ln n^n}$;

5.3.25 P $x_n = \frac{1}{\ln n^2} + \frac{1}{\ln n^3} + \cdots + \frac{1}{\ln n^n}$;

5.3.26 P $x_n = \sum_{k=1}^{n} \left(\frac{1}{3k-2} + \frac{1}{3k-1} - \frac{2}{3k} \right)$;

5.3.27 P $x_n = \sum_{k=1}^{n} \frac{3k^2+3k+1}{k^3(k+1)^3}$;

5.3.28 P $x_n = \sum_{k=1}^{n} \frac{k+2}{k(k+1)2^k}$;

5.3.29 P $x_n = \sum_{k=1}^{n} \frac{k!}{(k+p+1)!}, \quad p \in \mathbb{N}^*$;

5.3.30 P $x_n = \sum_{k=1}^{n} \frac{k-2}{2^k}$;

5.3.31 P $x_n = \sum_{k=1}^{n} \frac{k\,2^k}{(k+2)!}$;

5.3.32 P $x_n = \sum_{k=1}^{n} \frac{a^{k-1}}{(1+a^k)(1+a^{k+1})}, \quad a \in \mathbb{R} \setminus \{-1, 0, 1\}$;

5.3.33 P $x_n = \sum_{k=1}^{n} \frac{1}{\sqrt[k]{n^k+1}+1}$;

5.3.34 P $x_n = \sum_{k=1}^{n} \frac{k^2}{k!}$;

5.3.35 (P) $x_n = \dfrac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n}$;

5.3.36 (P) $x_n = \dfrac{n^2+1}{n^2-1} \cdot \dfrac{n^2+2}{n^2-2} \cdots \dfrac{n^2+n}{n^2-n}$;

5.3.37 (P) $x_n = \dfrac{2a\, x_{n-1}}{a + x_{n-1}}, \quad a > 0, n \ge 1, x_0 > 0$;

5.3.38 (P) $x_n = x_{n-1}^2 - 2\, x_{n-1} + 2, \quad x_0 \in \mathbb{R},\ n \ge 1$;

5.3.39 (P) $x_n = \mathrm{e}^{-1+x_{n-1}}, \quad x_0 \in \mathbb{R},\ n \ge 1$;

5.3.40 (P) $x_n = 2^{\frac{x_{n-1}}{2}}, \quad x_0 \in \mathbb{R},\ n \ge 1$;

5.3.41 (P) $x_n = \mathrm{e}^{\frac{x_{n-1}}{\mathrm{e}}}, \quad x_0 \in \mathbb{R},\ n \ge 1$;

5.3.42 (P) $x_n = \dfrac{1}{2}\left(x_{n-1} + \dfrac{a}{x_{n-1}}\right), \quad x_0 > 0,\ a > 0,\ n \in \mathbb{N}$;

5.3.43 (P) $x_n = \sqrt{x_{n-1} x_{n-2}}, \quad x_0 = 1, x_1 = \mathrm{e}, \quad n \ge 2$;

5.3.44 (P) $x_n = a^{1+\ln x_{n-1}}, \quad x_0 = a > 0,\ n \ge 1$;

5.3.45 (P) $x_n = n\left(\left(1+\dfrac{a}{n}\right)^n - \left(1+\dfrac{1}{n}\right)^{a\,n}\right), \quad a \in \mathbb{R}$;

5.3.46 (P) $x_n = n\left(\dfrac{1}{\mathrm{e}-1} - \dfrac{1^n + 2^n + \cdots + (n-1)^n}{n^n}\right)$.

5.3.47 (P) Let $(x_n)_{n\geq 0}$ be a sequence recurrently defined by

$$x_n = \frac{x_{n-1}}{1+(n-1)x_{n-1}^2}, \quad n \geq 1,\ x_0 > 0.$$

Prove that the sequences (x_n) and $(n\,x_n)$ are convergent and compute their limits.

5.3.48 (P) Let $(x_n)_{n\geq 1}$ be a sequence. Prove that the sequence $(y_n)_{n\geq 1}$,

$$y_n = \frac{1}{n}\ln\left(e^{n\,x_1} + e^{n\,x_2} + \cdots + e^{n\,x_n}\right),$$

has limit.

5.3.49 (P) Let $(G,*)$, $G \subseteq \mathbb{R}$ be a group such that there exists a continuous isomorphism $f : (G,*) \to (\mathbb{R}_+^*,\cdot)$. Study the sequence $(x_n)_{n\geq 0}$,

$$x_{n+2} = x_n * x_{n+1}, \qquad n \in \mathbb{N}, \quad x_0,\ x_1 \in G,$$

for convergence.

5.3.50 (P) Let $f\colon \mathbb{R} \to \mathbb{R}$ be strictly increasing and strictly convex such that $f(a) = a$ and $f(b) = b$, for some $a < b$. Study the sequence $(x_n)_{n\geq 0}$, defined by

$$x_{n+1} = f(x_n), \qquad n \in \mathbb{N},\ x_0 \in \mathbb{R},$$

for convergence.

5.4 Exercises: Sequences of Numbers (II)

P
5.4.1 Consider the sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$, respectively given by:

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n+1),$$

$$b_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n,$$

$$c_n = \frac{a_n + b_n}{2} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\sqrt{n(n+1)}.$$

Prove that:
(1) The sequences (a_n) and (b_n) have the same limit γ ($\gamma = 0.577\ldots$ is known as the Euler constant) and they satisfy:

$$a_n < a_{n+1} < \gamma < b_{n+1} < b_n, \quad n \geq 1;$$

$$\lim_{n\to\infty} n(\gamma - a_n) = \lim_{n\to\infty} n(b_n - \gamma) = \frac{1}{2}; \tag{2}$$

$$\lim_{n\to\infty} n^2(c_n - \gamma) = \frac{1}{6}; \tag{3}$$

$$0.576 < \gamma < 0.578. \tag{4}$$

P
5.4.2 Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two increasing sequences of integers such that $0 < a_n \leq b_n$, $n = 1, 2, \ldots$. Prove that

$$\lim_{n\to\infty} \frac{a_n}{b_n} \prod_{k=a_n}^{b_n} \mathrm{e}^{1/k} = 1.$$

P
5.4.3 Find the following limits:

$$\lim_{n\to\infty} \frac{1}{n} \prod_{k=n!}^{(n+1)!} \sqrt[k]{e}; \tag{1}$$

$$\lim_{n\to\infty} \sum_{k=n^n}^{(n+1)^n} \frac{1}{k}; \tag{2}$$

$$\lim_{n\to\infty} \sum_{k=2^n}^{2^{(n+1)}} \frac{1}{k}. \tag{3}$$

P
5.4.4 Consider a periodic function $f : \mathbb{R} \to \mathbb{R}$ with period $T > 0$, continuous at a point $x \in \mathbb{R}$. Let (S_n) be a sequence satisfying the conditions:

$$\lim_{n\to\infty} S_n = \infty; \tag{1}$$

$$\lim_{n\to\infty} (S_{n+1} - S_n) = 0. \tag{2}$$

Prove that $f(x)$ is a limit point of the sequence $(f(S_n))$.

P
5.4.5 Find the sets of limit points of the following sequences:

$$\sin \ln n; \tag{1}$$

$$\cos \sqrt{n}; \tag{2}$$

$$\tan\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right); \tag{3}$$

$$\sqrt{n} - [\sqrt{n}]. \tag{4}$$

P
5.4.6 Find the following limits:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^a}{n^{a+1}}, \quad a > -1; \tag{1}$$

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} \frac{n^a}{k^{a+1}}, \quad a > 0; \tag{2}$$

$$\lim_{n\to\infty} n\left(\frac{1}{a+1} - \sum_{k=1}^{n} \frac{k^a}{n^{a+1}}\right), \quad a > 0; \tag{3}$$

$$\lim_{n\to\infty} n\left(\frac{1}{a} - \sum_{k=n+1}^{\infty} \frac{n^a}{k^{a+1}}\right), \quad a > 0. \tag{4}$$

P
5.4.7 Consider the sequence $f_0 : \mathbb{N}^* \longrightarrow \mathbb{R}$,

$$f_0(n) = \left(1 + \frac{1}{n}\right)^n,$$

and define

$$f_k(n) = \left(\frac{f_{k-1}(n)}{\lim\limits_{p\to\infty} f_{k-1}(p)}\right)^n, \quad n, k \in \mathbb{N}^*.$$

Find $\lim\limits_{n\to\infty} f_k(n)$.

P
5.4.8 Find the limit

$$\lim_{n\to\infty} \left(\frac{\left(\left(\frac{\left(1+\frac{1}{n}^{n}\right)}{\mathrm{e}}\right)^n \sqrt{\mathrm{e}}\right)^n}{\sqrt[3]{\mathrm{e}}}\right)^n.$$

5.4.9 (P)

Let p be a positive integer, $a \geq 0$ and $f : [0,1] \to \mathbb{R}$ be a function possessing a continuous derivative of order p. Prove that, if

$$f(0) = f'(0) = \cdots = f^{(p-1)}(0) = 0,$$

then,

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k^a}{n^{a+1/p}}\right) = \frac{f^{(p)}(0)}{p!(ap+1)}.$$

5.4.10 (P)

Calculate the following limits:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \sin\frac{k}{n^2}; \tag{1}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \sin^2\frac{\sqrt{k}}{n}; \tag{2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} (2^{k/n^2} - 1); \tag{3}$$

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{k}{n^2}\right). \tag{4}$$

5.4.11 (P) Let $f : [0, a) \to [0, a)$ be a function possessing a continuous derivative of order k, $k \geq 2$. Suppose that the following conditions are satisfied:

$$f(0) = 0, \quad f'(0) = 1, \quad f^{(k)}(0) \neq 0; \tag{1}$$

$$f(x) < x, \text{ for all } x \in (0, a); \tag{2}$$

$$\text{if } 2 \leq p \leq k-1, \text{ then } f^{(p)}(0) = 0. \tag{3}$$

Prove that

$$\lim_{n\to\infty} n^{\frac{1}{k-1}} \underbrace{f(f(\cdots f(x)\cdots))}_{n\text{times}} = \left(\frac{-k(k-2)!}{f^{(k)}(0)}\right)^{\frac{1}{k-1}}, \qquad \forall x \in (0, a).$$

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5.4.12 (P) Prove each of the following equalities:

$$\lim_{n\to\infty} \sqrt{n}\, \underbrace{\sin(\sin(\cdots \sin x \cdots))}_{n \text{ times}} = \sqrt{3}, \quad x \in (0, \pi); \tag{1}$$

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$$\lim_{n\to\infty} \sqrt{n}\, \underbrace{\arctan(\arctan(\cdots \arctan x \cdots))}_{n \text{ times}} = \sqrt{\frac{3}{2}}, \quad x \in (0, \infty); \tag{2}$$

$$\lim_{n\to\infty} n\, \underbrace{\ln(1 + \ln(1 + \cdots \ln(1 + x)\cdots))}_{n \text{ times}} = 2, \quad x \in (0, \infty). \tag{3}$$

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5.4.13 (P) Let (a_n) be a sequence such that

$$\lim_{n\to\infty} n(a_{n+1} - a_n) = K, \quad K > 0.$$

Prove that $\lim_{n\to\infty} a_n = \infty$.

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5.4.14 P

Let $f : [0,\infty) \to [0,\infty)$ be a function possessing a continuous derivative of order 2. Suppose that the following conditions are satisfied:

$$f(x) \le x, \quad \forall x \ge 0; \tag{1}$$

$$f'(0) = 1, \quad f''(0) \ne 0. \tag{2}$$

Define a sequence (a_n) by the recurrence formula

$$\frac{a_{n+1}}{n} = f\left(\frac{a_n}{n}\right), \quad n \ge 1, \quad a_1 \ge 0.$$

Prove that

$$\lim_{n\to\infty} a_n = 0.$$

5.4.15 P

Find the limit of each of the sequences defined by the following recurrence formulas:

$$\frac{a_{n+1}}{n} = \ln\left(1 + \frac{a_n}{n}\right), \quad n \ge 1,\ a_1 \ge 0; \tag{1}$$

$$\frac{b_{n+1}}{n+1} = \ln\left(1 + \frac{b_n}{n}\right), \quad n \ge 1,\ b_1 > 0. \tag{2}$$

5.4.16 [P] Let $0 < b \le \infty$, $f : [0, b) \to [0, b)$ such that:

$$f(0) = 0;$$

$$0 < f(x) \le x, \text{ for } x \text{ in some neighbourhood } (0, \delta);$$

$$f'(0) = 1;$$

$$\exists\, f''(0),$$

and a sequence $(x_n)_{n\ge1}$, $x_n \in [0, b)$ such that $\lim\limits_{n\to\infty} n\, x_n = L \in \mathbb{R}$. Calculate

$$\lim_{n\to\infty} n \underbrace{f \circ f \circ \cdots \circ f}_{n-times}(x_n).$$

5.4.17 [P] Prove that

$$\lim_{n\to\infty} \left\lfloor 100 \left(\tan 1 + \tan \frac{1}{2} + \cdots + \tan \frac{1}{n} - \ln n \right) \right\rfloor = 120.$$

5.4.18 [P] Prove that

$$\lim_{n\to\infty} \left\lfloor 1000 \sqrt{1 + \sqrt{2 + \cdots + \sqrt{n}}} \right\rfloor = 1757.$$

5.4.19 [P] Prove that there exists a sequence $(k_n)_{n\ge1}$ of positive integers such that $\lim\limits_{n\to\infty} \sin k_n = 0$.

5.4.20 [P] Find a sequence (a_n) of real numbers such that the sequence $(\sqrt[n]{a_n})$ is convergent but the sequence $\left(\dfrac{a_{n+1}}{a_n}\right)$ has no limit.

P
5.4.21 Let (a_n) be a sequence of positive real numbers such that

$$\limsup \frac{a_{n+1}}{a_n} < 1.$$

Prove that

$$\lim_{n\to\infty} a_n = 0.$$

P
5.4.22 Prove that, if $\sin x \neq 0$, then the sequence $(\sin nx)_{n\geq 0}$ has no limit.

P
5.4.23 Let (a_n) be a sequence of real numbers and $q \in (0,1)$ such that

$$\lim_{n\to\infty} (a_{n+1} - q\, a_n) = 0.$$

Prove that $\lim\limits_{n\to\infty} a_n = 0$.

P
5.4.24 Find the limit of the sequence

$$x_n = \left(1+\frac{1}{n}\right)^{1+\frac{1}{n}} \left(1+\frac{1}{n+1}\right)^{1+\frac{1}{n+1}} \cdots \left(1+\frac{1}{2\,n}\right)^{1+\frac{1}{2\,n}}.$$

P
5.4.25 Let $(a_n)_{n\geq 1}$ be defined by

$$a_{n+1} = a_n(1-a_n), \quad n \geq 1, \quad 0 < a_1 < 1.$$

Prove that

(a) $\lim\limits_{n\to\infty} n a_n = 1$;

(b) $\lim\limits_{n\to\infty} \dfrac{n(1-na_n)}{\ln n} = 1$.

5.4.26 (P) Let the sequence $(a_n)_{n\geq 1}$ be defined by the recurrence formula:

$$a_{n+1} = \ln(1 + a_n), \quad n \geq 1, \quad a_1 > 0.$$

Prove that:

$$\lim_{n\to\infty} \frac{n(na_n - 2)}{\ln n} = \frac{2}{3}.$$

5.4.27 (P) Let $a > 0$. Find $\lim\limits_{n\to\infty} n \log\left(1 + \log\left(1 + \left(\cdots \log\left(1 + \frac{a}{n}\right)\cdots\right)\right)\right)$ where the parentheses are nested to depth n.

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5.4.28 (P) Let $(a_n)_{n\geq 1}$ be a sequence of positive numbers such that:

$$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = l, \quad l \in \mathbb{R}.$$

Prove that

$$\lim_{n\to\infty} \frac{\ln \frac{1}{a_n}}{\ln n} = l.$$

5.5 Series of Numbers

Let (x_n) be a sequence of complex numbers. Consider the sequence (S_n),

$$S_n = x_0 + \cdots + x_n, \qquad (n \in \mathbb{N}).$$

The pair $((x_n), (S_n))$ is called a **series** and is denoted by

$$\sum x_n, \qquad \sum_{n \geq 0} x_n \quad \text{or} \quad \sum_{n=0}^{\infty} x_n.$$

The number S_n is called the **n-th partial sum** of the series and x_n is called the **general term** of the series.

If the sequence (S_n) has a finite limit as $n \to \infty$, then the series is said to be **convergent** and the limit is called the **sum** of series. In order to denote the sum of a series we shall use the same notations:

$$\sum_{n \geq 0} x_n \quad \text{or} \quad \sum_{n=0}^{\infty} x_n,$$

that is

$$\sum_{n=0}^{\infty} x_n \quad \stackrel{def.}{=} \quad \lim_{n \to \infty} S_n.$$

If the sequence (S_n) has no limit, or the limit is infinite, then the series is said to be **divergent**.

If two series, $\sum x_n$ and $\sum y_n$, are simultaneously convergent or divergent, we use the notation

$$\sum x_n \approx \sum y_n.$$

The series

$$\sum_{k=n+1}^{\infty} x_k$$

is called the **n-th remainder series**.

A series of positive terms is referred to as a **positive series**. The sequence of the partial sums of a positive series is increasing. Hence it has a limit and we deduce:

5.5.1 REMARK. *A positive series is convergent iff the sequence of the partial sums is bounded.*

5.5.2 DEFINITION. *The series $\sum x_n$ is said to be* `absolutely convergent` *if the series $\sum |x_n|$ is convergent.*

One can prove the following result:

5.5.3 THEOREM. *An absolutely convergent series is always convergent; any rearrangement of its terms does not violate its absolute convergence, and the sum of the new series remains the same.*

5.5.4 EXAMPLE.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \quad \text{is convergent;}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \quad \text{is divergent;}$$

$$\sum_{n\geq 0} (-1)^n \quad \text{is divergent;}$$

$$\sum_{n\geq 0} \frac{(-1)^n}{n} \quad \text{is convergent but not absolutely convergent.}$$

5.6 Convergence Tests for Series

From the relations:

$$\sum x_n \text{ is convergent}$$

$$\Longleftrightarrow \quad \text{the sequence } (S_n) \text{ is convergent}$$

$$\Longleftrightarrow \quad \text{the sequence } (S_n) \text{ is a Cauchy sequence,}$$

we deduce:

5.6.1 THEOREM. **(Cauchy's Criterion)** *A series $\sum x_n$ is convergent if and only if for any $\varepsilon > 0$ there exists r_ε such that the inequality*

$$|x_{n+1} + \cdots + x_{n+p}| < \varepsilon$$

is satisfied for all $n, p \in \mathbb{N}$ and $n > r_\varepsilon$.

Consequently, a series $\sum x_n$ is convergent if and only if there exists a sequence (ε_n) converging to zero, such that

$$|x_{n+1} + \cdots + x_{n+p}| < \varepsilon_n,$$

for all $n, p \in \mathbb{N}$.

For $p = 1$ we obtain

$$(\forall\varepsilon)(\exists r_\varepsilon)(\forall n)((\varepsilon > 0) \to ((r_\varepsilon \in \mathbb{N}) \wedge (n \in \mathbb{N}) \wedge (n > r_\varepsilon) \to |x_{n+1}| < \varepsilon))$$

and we deduce the following necessary condition for convergence.

5.6.2 THEOREM. *If a series $\sum x_n$ is convergent, then* $\lim_{n\to\infty} x_n = 0$.

5.6.3 THEOREM. *An absolutely convergent series is convergent.*

Proof. We use the inequality

$$|x_{n+p} + \cdots + x_{n+1}| \le |x_{n+p}| + \cdots + |x_{n+1}|$$

and Cauchy's criterion 5.6.1. ❖

Let (u_n) and (v_n) be two sequences of positive terms.

5.6.4 THEOREM. **(First Comparison Test)** *If* $u_n \le v_n$, *for* $n \in \mathbb{N}$, *then:*

(1) $\sum u_n$ *– divergent* $\Longrightarrow \sum v_n$ *– divergent;*

(2) $\sum v_n$ *– convergent* $\Longrightarrow \sum u_n$ *– convergent.*

Proof. Denote:

$$U_n = u_0 + \cdots + u_n,$$

$$V_n = v_0 + \cdots + v_n.$$

(1) $\sum u_n$ – divergent $\xrightarrow{(U_n)\text{ – increasing}} U_n \to \infty$ $\xrightarrow{U_n \le V_n} V_n \to \infty \iff \sum v_n$– divergent.

(2) $\sum v_n$ – convergent $\Longrightarrow (V_n)$ – bounded

$$\xrightarrow{U_n \le V_n} \qquad (U_n) \text{ – bounded} \iff \sum u_n \text{ – convergent.}$$

5.6.5 THEOREM. **(Second Comparison Test)** *If*

$$\frac{u_{n+1}}{u_n} \le \frac{v_{n+1}}{v_n},$$

for all $n \in \mathbb{N}$, then:

(1) $\sum u_n$ *– divergent* $\Longrightarrow \sum v_n$ *– divergent;*

(2) $\sum v_n$ *– convergent* $\Longrightarrow \sum u_n$ *– convergent.*

Proof. From the inequalities:

$$\frac{u_{n+1}}{u_n} \le \frac{v_{n+1}}{v_n}, \qquad (n \in \mathbb{N}),$$

we obtain:

$$\frac{u_1}{u_0}\frac{u_2}{u_1}\cdots\frac{u_n}{u_{n-1}} \le \frac{v_1}{v_0}\frac{v_2}{v_1}\cdots\frac{v_n}{v_{n-1}},$$

that is

$$\frac{u_n}{u_0} \le \frac{v_n}{v_0}, \qquad (n \in \mathbb{N}).$$

Using the first comparison test (5.6.4), the proof is completed.

5.6.6 THEOREM. **(Third Comparison Test)** *If*

$$\lim_{n\to\infty} \frac{u_n}{v_n} = L \in (0, \infty),$$

then $\sum u_n \approx \sum v_n$.

Proof. Let ε be a positive number such that $L - \varepsilon > 0$. Then there exists a number r_ε, such that the inequalities

$$L - \varepsilon < \frac{u_n}{v_n} < L + \varepsilon,$$

are satisfied for all $n \ge r_\varepsilon$. It follows that

$$(L - \varepsilon)v_n < u_n < (L + \varepsilon)v_n,$$

for $n \geq r_\varepsilon$.

Using the first comparison test (5.6.4), the proof of the theorem is completed.

5.6.7 THEOREM. **(Cauchy's Root Test)**

(1) *If* $\sqrt[n]{u_n} \leq q < 1$, *for a certain index* n_0 *onwards, then* $\sum u_n$ *is convergent;*

(2) *If* $\sqrt[n]{u_n} \geq 1$, *beyond a certain index* n_0, *then* $\sum u_n$ *is divergent.*

Proof.
(1) From the inequalities

$$\sqrt[n]{u_n} \leq q < 1, \qquad (n \geq n_0)$$

it follows that

$$u_n \leq q^n, \qquad (n \geq n_0).$$

Since the series $\sum q^n$ is convergent, by the first comparison test (5.6.4), the series $\sum u_n$ is convergent.

(2) $\sqrt[n]{u_n} \geq 1$, $\forall n \geq n_0$, $\Rightarrow u_n \geq 1 \;\Rightarrow u_n \not\to 0 \;\Rightarrow \sum u_n$ -divergent.

5.6.8 COROLLARY. *If* $\lim\limits_{n \to \infty} \sqrt[n]{u_n} = \ell$, *then:*

$\ell < 1 \quad \Rightarrow \quad \sum u_n$ – *convergent;*

$\ell > 1 \quad \Rightarrow \quad \sum u_n$ – *divergent;*

$\ell = 1 \quad \Rightarrow \quad$ *the test fails.*

Likewise one can prove the following more general test.

5.6.9 THEOREM.

(1) *If* $\overline{\lim}\, \sqrt[n]{u_n} < 1$, *then* $\sum u_n$ *is convergent;*

(2) *If* $\underline{\lim}\, \sqrt[n]{u_n} > 1$, *then* $\sum u_n$ *is divergent.*

5.6.10 THEOREM. **(D'Alembert[5] Ratio Test)**

(1) *If* $\dfrac{u_{n+1}}{u_n} \le q < 1$, *for a certain index* n_0 *onwards, then* $\sum u_n$ *is convergent;*

(2) *If* $\dfrac{u_{n+1}}{u_n} \ge 1$, *beyond a certain index* n_0, *then* $\sum u_n$ *is divergent.*

Proof.
(1) We have:

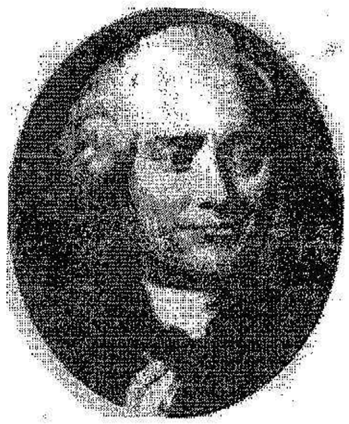
$$\frac{u_{n+1}}{u_n} \le q = \frac{q^{n+1}}{q^n}, \qquad (n \ge n_0)$$

and the series $\sum q^n$ is convergent. By the second comparison test (5.6.5) the series $\sum u_n$ is convergent.
(2) $\dfrac{u_{n+1}}{u_n} \ge 1, \ \Rightarrow u_{n+1} \ge u_n \ \Rightarrow (u_n)$ -increasing $\Rightarrow u_n \not\to 0 \ \Rightarrow$ $\sum u_n$ -divergent

5.6.11 COROLLARY. *If* $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \ell$, *then:*

$\ell < 1 \quad \Rightarrow \quad \sum u_n$ – *convergent;*

$\ell > 1 \quad \Rightarrow \quad \sum u_n$ – *divergent;*

5 

Jean le Rond d'Alembert (1717–1783), a French mathematician.

$\ell = 1 \quad \Rightarrow \quad$ *the test fails.*

In a similar way, a more general test can be established.

5.6.12 THEOREM.

(1) *If* $\overline{\lim}\,\frac{u_{n+1}}{u_n} < 1$, *then* $\sum u_n$ *is convergent;*

(2) *If* $\underline{\lim}\,\frac{u_{n+1}}{u_n} > 1$, *then* $\sum u_n$ *is divergent.*

5.6.13 REMARK. *If the limiting form of D'Alembert's test* (5.6.8) *fails, then the limiting form of Cauchy's test* (5.6.11) *also fails; this is a consequence of the relation*

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1 \quad \Rightarrow \quad \lim_{n\to\infty} \sqrt[n]{u_n} = 1$$

(cf. corollary (5.2.7)).

5.6.14 THEOREM. **(Kummer's Test)** [6] *Let* $\sum u_n$ *be a series with positive terms and* (c_n) *be a sequence of positive numbers. We have:*

[6] 

Ernst Eduard Kummer (1810–1893), a German mathematician.

(1) *if there exist* $r > 0$ *and* $p \in \mathbb{N}$ *such that*

$$c_n \frac{u_n}{u_{n+1}} - c_{n+1} \geq r,$$

for all $n \geq p$, *then* $\sum u_n$ *is convergent;*

(2) *if there exists* $p \in \mathbb{N}$ *such that*

$$c_n \frac{u_n}{u_{n+1}} - c_{n+1} \leq 0,$$

for all $n \geq p$, *and if* $\sum \frac{1}{c_n}$ *is divergent, then* $\sum u_n$ *is divergent.*

Proof. (1) We have:

$$r u_{n+1} \leq c_n u_n - c_{n+1} u_{n+1},$$

$n \geq p$, therefore

$$\begin{aligned} r u_{p+1} &\leq c_p u_p - c_{p+1} u_{p+1}, \\ r u_{p+2} &\leq c_{p+1} u_{p+1} - c_{p+2} u_{p+2}, \\ \ldots\ldots &\ \ldots \ \ldots\ldots\ldots\ldots\ldots\ldots \\ r u_n &\leq c_{n-1} u_{n-1} - c_n u_n. \end{aligned}$$

Consequently,

$$\begin{aligned} r(u_{p+1} + u_{p+2} + \cdots + u_n) &\leq c_p u_p - c_n u_n \leq c_p u_p, \\ u_{p+1} + u_{p+2} + \cdots + u_n &\leq c_p u_p / r, \end{aligned}$$

$n > p$.

The sequence of the partial sum of the series is bounded, hence the series is convergent.

(2) The inequalities

$$c_n \frac{u_n}{u_{n+1}} - c_{n+1} \leq 0,$$

$\forall n \geq p$, give

$$\frac{\frac{1}{c_{n+1}}}{\frac{1}{c_n}} \leq \frac{u_{n+1}}{u_n},$$

$\forall n \geq p$. Since the series $\sum \frac{1}{c_n}$ is divergent, using the second comparison test (5.6.5), we deduce that the series $\sum u_n$ is divergent.

If in Kummer's test 5.6.14 we take $c_n = n$ we obtain the following criterion.

5.6.15 THEOREM. **(Raabe[7] -Duhamel's[8] Test)**

7 

Joseph Ludwig Raabe (1801–1859) a Swiss mathematician.

8 

Jean Marie Constant Duhamel (1797–1872), a French mathematician.

(1) *if there exists $r > 1$ such that*

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) \geq r,$$

for all $n \in \mathbb{N}^*$, then the series $\sum u_n$ is convergent;

(2) *if for all $n \in \mathbb{N}^*$ we have*

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) \leq 1,$$

then the series $\sum u_n$ is divergent.

5.6.16 COROLLARY. *If* $\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \ell$, *then:*

$\ell > 1 \quad \Rightarrow \quad \sum u_n$ – *convergent;*

$\ell < 1 \quad \Rightarrow \quad \sum u_n$ – *divergent;*

$\ell = 1 \quad \Rightarrow \quad$ *the test fails.*

5.6.17 THEOREM. **(Gauss's Test)**[9] *If*

$$\frac{u_n}{u_{n+1}} = x + \frac{y}{n} + \frac{O(1)}{n^2},$$

9

Johann Carl Friedrich Gauss (1777–1855), a great German mathematician.

$n \geq 1$, *then*

$$x > 1 \quad \Rightarrow \quad \sum u_n \text{ – convergent;}$$
$$x < 1 \quad \Rightarrow \quad \sum u_n \text{ – divergent;}$$
$$x = 1 \quad \text{and} \quad y > 1 \quad \Rightarrow \quad \sum u_n \text{ -convergent;}$$
$$x = 1 \quad \text{and} \quad y \leq 1 \quad \Rightarrow \quad \sum u_n \text{ -divergent.}$$

5.6.18 THEOREM. **(Associativity Property)** *Let $\sum a_n$ be a series with positive terms and (k_n) be a strictly increasing sequence of positive integers with $k_0 = 0$. If*

$$b_0 = a_0, \quad b_n = a_{k_{n-1}+1} + a_{k_{n-1}+2} + \cdots + a_{k_n},$$

$n = 1, 2, \ldots$, *then*

$$\sum a_n = \sum b_n.$$

Proof. Denote:

$$A_n = \sum_{k=0}^{n} a_k, \qquad B_n = \sum_{k=0}^{n} b_k \qquad (n = 0, 1 \ldots).$$

We observe that:

$$B_n = A_{k_n} \quad \text{and} \quad k_n \geq n \qquad (n \in \mathbb{N}).$$

Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$, $m \geq k_n$. We have:

$$n \leq k_n \leq m \leq k_m,$$

therefore, using the fact that the sequence (A_n) is increasing, we deduce

$$A_n \leq A_{k_n} \leq A_m \leq A_{k_m},$$

i.e.,

$$A_n \leq B_n \leq A_m \leq B_m,$$

$\forall n \in \mathbb{N}$, $m \geq k_n$.

The previous inequalities, and the fact that each of the sequences (A_n) and (B_n) has a limit, imply $\sum a_n = \sum b_n$. ∎

5.6.19 THEOREM. **(Cauchy's Condensation Test)** *If (a_n) is a decreasing sequence of positive terms, and (k_n) is an increasing sequence of positive integers, such that*

$$\frac{k_{n+2}-k_{n+1}}{k_{n+1}-k_n}=O(1), \qquad n\in\mathbb{N},$$

then

$$\sum a_n \approx \sum (k_{n+1}-k_n)a_{k_n}.$$

Proof. Denoting

$$b_n = a_{k_n+1}+\cdots+a_{k_{n+1}},$$

we obtain

$$(k_{n+1}-k_n)a_{k_{n+1}} \le b_n \le (k_{n+1}-k_n)a_{k_n}, \qquad \forall\, n\ge 0.$$

There exists $M>0$ such that

$$0\le \frac{k_{n+2}-k_{n+1}}{k_{n+1}-k_n}\le M,$$

therefore

$$\frac{k_{n+2}-k_{n+1}}{M}a_{k_{n+1}} \le b_n \le (k_{n+1}-k_n)a_{k_n}, \qquad \forall n\ge 0,$$

and using the first comparison test (5.6.4), we deduce $\sum b_n \approx \sum(k_{n+1}-k_n)a_{k_n}$. By theorem (5.6.19), we get $\sum b_n \approx \sum a_n$, hence $\sum a_n \approx \sum(k_{n+1}-k_n)a_{k_n}$. ❖

For $k_n = 2^n$ we obtain:

5.6.20 COROLLARY. *If (a_n) is a decreasing sequence of positive terms, then*

$$\sum a_n \approx \sum 2^n a_{2^n}.$$

5.6.21 EXAMPLE. Let us examine the convergence of the `generalized harmonic series`

$$\sum_{n\ge 1}\frac{1}{n^\alpha}.$$

For $\alpha\le 0$ the sequence $\dfrac{1}{n^\alpha}$ does not converge to zero, therefore the series is divergent.

For $\alpha > 0$, using (5.6.20) we deduce that the series has the same behaviour as the infinite geometric progression

$$\sum_{n\geq 1} 2^n \left(\frac{1}{2^n}\right)^\alpha = \sum_{n\geq 1} \left(\frac{1}{2^{\alpha-1}}\right)^n,$$

which is convergent for $\frac{1}{2^{\alpha-1}} < 1$. Consequently,

$$\sum_{n\geq 1} \frac{1}{n^\alpha} \quad \text{is} \quad \begin{cases} \text{convergent,} & \text{for } \alpha > 1, \\ \text{divergent,} & \text{for } \alpha \leq 1. \end{cases}$$

5.6.22 EXAMPLE. Let us investigate the series $\sum_{n\geq 2} \frac{1}{n \ln n}$ for convergence.

Using Cauchy's Condensation Test (5.6.20) and the fact that the series

$$\sum_{n\geq 2} 2^n \frac{1}{2^n \ln 2^n} = \frac{1}{\ln 2} \sum_{n\geq 2} \frac{1}{n}$$

is divergent we deduce, that $\sum_{n\geq 2} \frac{1}{n \ln n}$ is divergent.

Mathematica is unable to take a decision:

```
In[1]:=  (* Mathematica *)

In[2]:=  Sum[1/(n Log[n]), {n, 2, ∞}]

Out[2]=  Sum[1/(n Log[n]), {n, 2, ∞}]
```

$$\text{Out[2]= } \sum_{n=2}^{\infty} \frac{1}{n\,\mathrm{Log}[n]}$$

5.6.23 THEOREM. **(Abel[10] - Dirichlet[11] Test)** *Let the sequence (a_n) be monotone decreasing to zero and the partial sum (S_n) of the series $\sum u_n$ be bounded. Then the series $\sum u_n a_n$ is convergent.*

Proof. Denote:

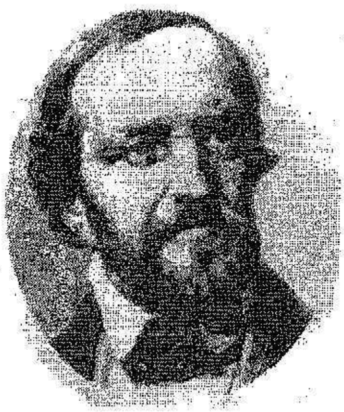
$$\sigma_n = a_0 u_0 + a_1 u_1 + \cdots + a_n u_n.$$

We have:

$$\sigma_{n+p} - \sigma_n = a_{n+1} u_{n+1} + \cdots + a_{n+p} u_{n+p}$$

10 

Niels Henrik Abel (1802–1829), a Norwegian mathematician.

11 

Peter Gustave Lejeune Dirichlet (1805-1859), a German mathematician.

$$= a_{n+1}(S_{n+1} - S_n) + a_{n+2}(S_{n+2} - S_{n+1}) + \cdots + a_{n+p}(S_{n+p} - S_{n+p-1})$$
$$= -a_{n+1}S_n + S_{n+1}(a_{n+1} - a_{n+2}) + \cdots + S_{n+p-1}(a_{n+p-1} - a_{n+p})$$
$$+ a_{n+p}S_{n+p},$$

therefore, we deduce

$$|\sigma_{n+p} - \sigma_n| \le M(a_{n+1} + a_{n+p}) + M(a_{n+1} - a_{n+2} + \cdots + a_{n+p-1} - a_{n+p})$$
$$= 2Ma_{n+1}.$$

The inequalities

$$|\sigma_{n+p} - \sigma_n| \le 2Ma_{n+1}, \qquad n, p \in \mathbb{N},$$

and the fact that the sequence (a_n) converges to zero show that the sequence (σ_n) is a Cauchy sequence, and consequently, it is convergent; so we deduce the convergence of the series $\sum u_n a_n$. ❖

5.6.24 THEOREM. **(Leibniz's[12] Test)** *If the sequence (a_n) decreases to zero, then the* **alternating series** *$\sum(-1)^n a_n$ is convergent.*

Proof. Denote:

$$S_n = 1 - 1 + 1 - \cdots + (-1)^n.$$

12

Gottfried Wilhelm Leibniz (1646–1716), a great German mathematician.

Observe that the sequence (S_n) is bounded. By virtue of theorem (5.6.23) the series $\sum(-1)^n a_n$ is convergent.

5.6.25 EXAMPLE. Let investigate the series $\sum \frac{\sin n}{\ln(n+1)}$ for convergence. We have

$$S_n = \sin 1 + \cdots + \sin n = \frac{\sin \frac{n+1}{2} \sin \frac{n}{2}}{\sin \frac{1}{2}}.$$

We get

$$|S_n| \leq \frac{1}{\sin \frac{1}{2}}, \quad n \in \mathbb{N}^*,$$

and consequently, since $\frac{1}{\ln(n+1)} \to 0$, we deduce that the series $\sum \frac{\sin n}{\ln(n+1)}$ is convergent.

5.7 Infinite Products

Consider a sequence (a_n) of real numbers different from zero. Define the sequence (P_n),

$$P_n := a_0 \ldots a_n, \qquad n \in \mathbb{N}.$$

The pair $((a_n), (P_n))$ is called an `infinite product` and is denoted by

$$\prod a_n, \quad \prod_{n\geq 0} a_n \quad \text{or} \quad \prod_{n=0}^{\infty} a_n.$$

If the sequence (P_n) is convergent, then its limit is denoted by:

$$\prod_{n\geq 0} a_n \quad \text{or} \quad \prod_{n=0}^{\infty} a_n.$$

If $\lim\limits_{n\to\infty} P_n = 0$ we say that $\prod a_n$ diverges to zero.

If (P_n) is convergent to a number different from zero, then $\prod a_n$ is said to be convergent.

Let (a_n) be a sequence of positive numbers. Using the identity

$$\ln(a_0 \ldots a_n) = \ln a_0 + \cdots + \ln a_n,$$

we deduce that

$$\prod a_n \text{ — convergent} \iff \sum \ln a_n \text{ — convergent},$$

furthermore

$$\ln\left(\prod_{n\geq 0} a_n\right) = \sum_{n\geq 0} \ln a_n.$$

5.7.1 THEOREM. *If (a_n) is a sequence of positive numbers, then*

$$\prod(1+a_n) \approx \sum a_n.$$

Proof. The series $\sum a_n$ and $\sum \ln(1+a_n)$ are positive series. If $\sum a_n$ is convergent then $a_n \to 0$ and

$$\frac{\ln(1+a_n)}{a_n} \to 1,$$

therefore, according to the third comparison test (5.6.6), $\sum \ln(1+a_n)$ is convergent. Using $\sum \ln(1+a_n) \approx \prod(1+a_n)$, we deduce that the product $\prod(1+a_n)$ is convergent.

Similarly, we can show that if the product $\prod(1+a_n)$ is convergent, then the series $\sum a_n$ is also convergent.

```
In[1]:= (* Mathematica *)

In[2]:= (* Product[f, {i, imax}] evaluates the product ∏_{i=1}^{imax} f. *)

In[3]:= Product[k, {k, n}]

Out[3]= n!

In[4]:= FullSimplify[Product[(2 k - 1)/(2 k), {k, n}]
            == (2 n - 1)!!/(2 n)!!, n ∈ Integers]

Out[4]= True

In[5]:= Limit[√n Product[(2 k - 1)/(2 k), {k, n}], n → ∞]

Out[5]= 1/√π
```

5.8 Exercises: Series of Numbers

Determine whether each of the following series is convergent or not.

P 5.8.1 $$\sum_{n=1}^{\infty} \frac{(2n)!!}{n^n}.$$

P 5.8.2 $$\sum_{n=1}^{\infty} a^n \left(1+\frac{1}{n}\right)^{n^2+bn+c}, \qquad a,b,c \in \mathbb{R},\ a>0.$$

P 5.8.3 $$\sum_{n=1}^{\infty} a^{n^b}, \qquad a,b>0.$$

P 5.8.4 $$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}.$$

P 5.8.5 $$\sum_{n=1}^{\infty} \frac{n!}{(a+1)(a+2)\dots(a+n)}, \qquad a>-1.$$

P 5.8.6 $$\sum_{n\geq 1} n\left(\left(1+\frac{1}{n+1}\right)^{n+1}-\left(1+\frac{1}{n}\right)^{n}\right).$$

P 5.8.7 $$\sum_{n\geq 2} n^{-1-1/\sqrt{\log_2 n}}.$$

Find the sum of each of the following series:

P 5.8.8 $$\sum_{n=1}^{\infty} \frac{1}{(a+n)(a+n+1)\dots(a+n+p)}, \qquad a>-1,\ p\in\mathbb{N}^*.$$

5.8.9 [P] $$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}k(k+1)\dots(k+p)\right)^{-1}, \qquad p\in\mathbb{N}^*.$$

5.8.10 [P] $$\sum_{n=0}^{\infty}a^{3^n}\,\frac{a^{3^n}-1}{a^{3^{n+1}}+1}, \qquad a\in\mathbb{R},\ a\neq-1.$$

5.8.11 [P] $$\sum_{k=1}^{\infty}\frac{1}{k(k+1)(k+1)!}.$$

5.8.12 [P] $$\sum_{n=0}^{\infty}\arctan\frac{1}{n^2+n+1}.$$

5.8.13 [P] $$\sum_{n=1}^{\infty}\arctan\frac{1}{2\,n^2}.$$

5.8.14 [P] $$\sum_{n=1}^{\infty}\arctan\frac{2}{n^2}.$$

5.8.15 [P] $$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}.$$

5.8.16 [P] $$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}.$$

5.8.17 [P] $$\sum_{n=0}^{\infty}\left\lfloor\frac{x+2^n}{2^{n+1}}\right\rfloor, \qquad x\in\mathbb{R}.$$

5.8.18 [P] $$\sum_{n\geq0}\frac{(2n)!!}{(2n+1)!!}\,a^n, \qquad a\in[-1,1).$$

5.8.19 [P] $$\sum_{n\geq0}\frac{(2n-1)!!}{(2n)!!}\,a^n, \qquad a\in[-1,1).$$

P
5.8.20 $\displaystyle\sum_{n\geq 0} \frac{(n!)^2}{(2n+1)!}\, a^n,$ $\qquad a \in [-4, 4).$

P
5.8.21 $\displaystyle\sum_{n\geq 0} \frac{1}{\binom{2n}{n}}.$

Mircea Ivan

P
5.8.22

$$\sum_{n=0}^{\infty} \frac{2^n}{1+x^{2^n}}, \qquad |x| > 1.$$

P
5.8.23 $\displaystyle\sum_{n=0}^{\infty} \frac{P(n)}{n!},$ where P is a polynomial.

Mircea Ivan

P
5.8.24 Consider the sequence $(B_p)_{p\geq 1}$, $B_p = \dfrac{1}{\mathrm{e}} \displaystyle\sum_{n\geq 1} \frac{n^p}{n!}$, $p = 1, 2, \ldots$. Find a recurrence formula for B_p and calculate the terms B_1, B_2 and B_3.

P
5.8.25 Let $\sum u_n$ be a divergent series with positive terms and (S_n) be the sequence of the partial sums. Prove that:

(i) $\displaystyle\sum \frac{u_n}{S_n^{\alpha+1}}$ is convergent for all $\alpha > 0$;

(ii) $\displaystyle\sum \frac{u_n}{S_n}$ is divergent.

P
5.8.26 Let $(b_n)_{n\geq 1}$ be a sequence decreasing to zero. Prove that the series $\sum b_n$ is convergent if and only if there exists a constant $M > 0$ such that

$$b_1 + b_2 + \cdots + b_n < M + n\, b_n, \qquad \forall n \geq 1.$$

5.8.27 P Let $\sum u_n$ be a convergent series with positive terms such that the limit $\lim\limits_{n\to\infty} n\,u_n$ exists. Prove that $\lim\limits_{n\to\infty} n\,u_n = 0$.

5.8.28 P Let (b_n) be a decreasing sequence with positive terms such that the series $\sum b_n$ is convergent. Prove that $\lim\limits_{n\to\infty} n\,b_n = 0$.

5.8.29 P Find a convergent series with positive terms $\sum u_n$, such that $n\,u_n \not\to 0$.

5.8.30 P Use the notions presented in this course to prove the following equalities:

$$\sum_{n=1}^{\infty} \arctan\frac{1}{n^2} = \arctan\frac{\tan\frac{\pi}{\sqrt{2}} - \operatorname{th}\frac{\pi}{\sqrt{2}}}{\tan\frac{\pi}{\sqrt{2}} + \operatorname{th}\frac{\pi}{\sqrt{2}}};$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2-a^2} = \frac{1}{2a}\left(\frac{1}{a} - \pi\cot\pi a\right), \quad a \in (0,1).$$

6

Continuous Mappings

Contents

6.1 Continuous Mappings on Topological Spaces

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces.

6.1.1 DEFINITION. *A function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is said to be* `continuous at a point` *$x \in X$ if for each neighborhood $U \in \mathcal{V}_{f(x)}$ there exists a neighborhood $V \in \mathcal{V}_x$ such that*

$$f(V) \subseteq U.$$

6.1.2 THEOREM. *A function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is* `continuous on` *X iff the inverse image under f of every open (closed) set in $(Y, \mathcal{T}_Y)$ is an open (closed) set in $(X, \mathcal{T}_X)$.*

6.1.3 REMARK. *In general, the image of an open (closed) set under a continuous function is not an open (closed) set.*

6.1.4 THEOREM. *If $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a continuous mapping and $A \subseteq X$ is a compact set, then $f(A)$ is compact.*

Proof. Let $\{G_i\}_{i \in I}$ be an open cover of the set $f(A)$,

$$f(A) \subseteq \bigcup_{i \in I} G_i.$$

We have

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\bigcup_{i \in I} G_i) = \bigcup_{i \in I} f^{-1}(G_i).$$

Since f is continuous the family $\{f^{-1}(G_i)\}_{i \in I}$ is an open cover of A. We can select a finite sub-cover

$$\{f^{-1}(G_{i_k})\}_{k=\overline{1,n}}.$$

Using the relation

$$A \subseteq \bigcup_{k=1}^{n} f^{-1}(G_{i_k}),$$

we obtain

$$f(A) \subseteq f\Big(\bigcup_{k=1}^{n} f^{-1}(G_{i_k})\Big) = \bigcup_{k=1}^{n} f(f^{-1}(G_{i_k})) \subseteq \bigcup_{k=1}^{n} G_{i_k},$$

i.e., $\{G_{i_k}\}_{k=\overline{1,n}}$ is a finite sub-cover of $f(A)$. Hence, $f(A)$ is compact.

6.1.5 THEOREM. *If $f : (X, \mathcal{T}_X) \to \mathbb{R}$ is a continuous mapping and $A \subseteq X$ is a compact set, then f is bounded on A.*

Proof. Since A is compact $f(A)$ is also compact. But $f(A) \subseteq \mathbb{R}$ and hence $f(A)$ is bounded, i.e., f is bounded on A. ◊◊◊

6.1.6 REMARK. *The requirement of the last theorem that A be compact is essential.*

As for example, the function $f : (0,1) \to \mathbb{R}$, $f(x) = 1/x$ is continuous, but $f((0,1)) = (1, \infty)$. The conditions of theorem (6.1.5) are violated in this case since the domain $(0,1)$ is not compact.

6.1.7 THEOREM. **(Weierstrass)**[1] *If $f : (X, \mathcal{T}_X) \to \mathbb{R}$ is continuous and $A \subseteq X$ is a compact set, then f attains its greatest and least values on A.*

[1]

Karl Wilhelm Theodor Weierstrass (1815–1897), a great German mathematician.

Proof. By theorem (6.1.5) it follows that $f(A)$ is bounded and hence the supremum $M = \sup(f(A))$ is a real number. Suppose that $f(x) \neq M$ forall $x \in A$. The function $g : A \to \mathbb{R}$, defined by

$$g(x) = \frac{1}{M - f(x)},$$

is continuous on A. Consequently g is bounded on A, i.e., there exists $C > 0$ such that $g(x) \leq C$ $(\forall x \in A)$. We deduce

$$f(x) \leq M - \frac{1}{C} \qquad (\forall x \in A).$$

On the other hand, taking $\varepsilon = \frac{1}{C}$, by the definition of the supremum, there exists $x_\varepsilon \in A$ such that

$$f(x_\varepsilon) > M - \frac{1}{C}.$$

Therefore, there exists $\xi \in A$ such that $f(\xi) = M$.

The remaining part of the theorem concerning the infimum of the function can be proved in a similar way. ◆◆◆

6.1.8 REMARK. *The requirement of the last theorem that A be compact is essential.*

For example, the function $f : (0,1) \to \mathbb{R}$, $f(x) = x$, is continuous but $0 = m < f(x) < M = 1$ $(\forall x \in (0,1))$.

The conditions of the above theorem are violated in this case since the interval $(0,1)$ is not compact.

6.2 Continuous Mappings on Metric Spaces

Let (X, ρ) and (Y, σ) be two metric spaces. The following definition is deduced directly from the definition of the continuity on a topological space.

6.2.1 DEFINITION. *A mapping $f : (X, \rho) \to (Y, \sigma)$ is said to be* `continuous` *at a point $x \in X$, if for any $\varepsilon > 0$, there exists $\delta(x, \varepsilon) > 0$ such that $x' \in X$ and $\rho(x, x') < \delta(x, \varepsilon)$ imply $\sigma(f(x), f(x')) < \varepsilon$.*

6.2.2 DEFINITION. *A mapping $f:(X,\rho)\to(Y,\sigma)$ is said to be* `uniformly continuous` *if for every $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that $x,x'\in X$ and $\rho(x,x')<\delta(\varepsilon)$ imply $\sigma(f(x),f(x'))<\varepsilon$.*

6.2.3 DEFINITION. *A mapping $f:(X,\rho)\to(Y,\sigma)$ is said to be* `absolutely continuous` *if for any $\varepsilon>0$ there exists $\delta(\varepsilon)$ such that for all points $x_1,\dots,x_n, x'_1,\dots,x'_n\in X$ the relation*

$$\sum_{i=1}^{n}\rho(x_i,x'_i)<\delta(\varepsilon)$$

implies

$$\sum_{i=1}^{n}\sigma(f(x_i),f(x'_i))<\varepsilon.$$

6.2.4 REMARK. *In order to prove that a mapping $f:(X,\rho)\to(Y,\sigma)$ is not uniformly continuous it is sufficient to find two sequences $(x_n),(y_n)$ such that,*

$$\rho(x_n,y_n)\to 0,\quad\text{and}\quad \sigma(f(x_n),f(y_n))\not\to 0.$$

Note that a uniformly continuous mapping is a continuous mapping.

6.2.5 THEOREM. **(Heine–Cantor)**[2] *If a function $f:(X,\rho)\to(Y,\sigma)$ is continuous and X is compact, then f is uniformly continuous.*

[2] 

Heinrich Eduard Heine (1821–1881), a German mathematician.

Proof. Let $\boxed{\varepsilon > 0}$ and $x \in X$. Since f is continuous there exists $\delta(x, \varepsilon) > 0$ such that

$$\rho(x, x') < \delta(x, \varepsilon) \Rightarrow \sigma(f(x), f(x')) < \frac{\varepsilon}{2}.$$

The family

$$\left\{B\left(x, \frac{\delta(x, \varepsilon)}{2}\right)\right\}_{x \in X},$$

is an open cover of X. Since X is compact we can select a finite open sub-cover of X,

$$\left\{B\left(x_i, \frac{\delta(x_i, \varepsilon)}{2}\right)\right\}_{i=\overline{1,n}}.$$

Denote

$$\boxed{\delta = \min\left\{\frac{\delta(x_i, \varepsilon)}{2} \mid i = 1, \ldots, n\right\}.}$$

Let $\boxed{x, x' \in X}$ such that $\boxed{\rho(x, x') < \delta}$. There exists $i \in \{1, \ldots, n\}$ such that

$$x \in B\left(x_i, \frac{\delta(x_i, \varepsilon)}{2}\right).$$

We deduce:

$$\rho(x, x_i) < \frac{\delta(x_i, \varepsilon)}{2} < \delta(x_i, \varepsilon),$$

hence

$$\sigma(f(x), f(x_i)) < \frac{\varepsilon}{2}. \tag{$\diamond$}$$

From the inequalities:

$$\rho(x', x_i) \le \rho(x', x) + \rho(x, x_i)$$

$$\le \delta + \frac{\delta(x_i, \varepsilon)}{2} \le \frac{\delta(x_i, \varepsilon)}{2} + \frac{\delta(x_i, \varepsilon)}{2} = \delta(x_i, \varepsilon),$$

we get

$$\sigma(f(x'), f(x_i)) < \frac{\varepsilon}{2}. \tag{$\diamond\diamond$}$$

Taking into account the inequalities:

$$\sigma(f(x), f(x')) \le \sigma(f(x), f(x_i)) + \sigma(f(x_i), f(x')) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

by (∘) and (∘∘), we obtain

$$\boxed{\sigma(f(x), f(x')) < \varepsilon}.$$

The $\boxed{\text{boxed text}}$ in the previous proof represents the definition of uniform continuity. ◈◈◈

6.2.6 REMARK. *The requirement of the last theorem that A be compact is essential.*

For example, the function $f : (0,1) \to \mathbb{R}$, $f(x) = 1/x$, is continuous. Consider the sequences:

$$x_n = \frac{1}{n}, \quad y_n = \frac{1}{n+1}, \quad n \geq 1,$$

It can be seen that

$$x_n - y_n \to 0 \quad \text{and} \quad f(x_n) - f(y_n) = 1 \not\to 0.$$

The conditions of the above theorem are violated in this case since the interval $(0,1)$ is not compact.

6.2.7 THEOREM. *If $f : (X, \rho) \to \mathbb{R}$ is continuous and $f(x_0) \neq 0$, then there exists $\delta > 0$ such that $f(x) \neq 0$, for any x in the ball $B(x_0, \delta)$.*

Proof. Let $\varepsilon = |f(x_0)| > 0$. Then there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < |f(x_0)|,$$

for each $x \in B(x_0, \delta)$. Therefore

$$\begin{aligned} |f(x)| &= |f(x_0) + f(x) - f(x_0)| \\ &\geq |f(x_0)| - |f(x) - f(x_0)| > |f(x_0) - f(x_0)| = 0, \end{aligned}$$

i.e.,

$$f(x) \neq 0 \qquad (\forall x \in B(x_0, \delta)\,)$$ ◈◈◈

6.2.8 THEOREM. **(Heine)** *A mapping $f : (X, \rho) \to (Y, \sigma)$ is continuous at a point $x \in X$, iff for any sequence $(x_n) \in X^{\mathbb{N}}$ converging to x, the sequence $(f(x_n))$ tends to $f(x)$.*

Proof. Necessity. Let f be continuous at the point x and (x_n) be a sequence convergent to x. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $x' \in X$ and $\rho(x, x') < \delta$ imply $\sigma(f(x)), f(x')) < \varepsilon$.

Since (x_n) tends to x, there exists a number $r(\delta)$ such that $\rho(x, x_n) < \delta$, for all $n \in \mathbb{N}$, $n > r(\delta)$. Thus, for all $\varepsilon > 0$ there exists $r(\delta)$ such that $\sigma(f(x), f(x_n)) < \varepsilon$, for any $n > r(\delta)$, i.e., the sequence $(f(x_n))$ tends to $f(x)$.

Sufficiency. Suppose that f is discontinuous at the point x; therefore there exists $\varepsilon_0 > 0$ with the property: for all $\delta > 0$ there exists x_δ such that $\rho(x, x_\delta) < \delta$ and $\sigma(f(x), f(x_\delta)) \geq \varepsilon_0$. Taking $(\delta_n) = (1/n)$, there exists a sequence (x_n) such that $\rho(x, x_n) < \frac{1}{n}$ and $\sigma(f(x), f(x_n)) \geq \varepsilon_0$, i.e., (x_n) tends to x, and $(f(x_n))$ does not tend to $f(x)$. ♦♦♦

6.3 Contractions. Banach Fixed Point Theorem

6.3.1 DEFINITION. *A function $f : (X, \rho) \to (Y, \sigma)$ is called a* `contraction` *if there exists a number $\alpha \in [0, 1)$ satisfying the condition*

$$\sigma(f(u), f(v)) \leq \alpha\rho(u, v),$$

for all $u, v \in X$.

6.3.2 REMARK. *It is obvious from the above definition that any contraction is a continuous function.*

6.3.3 DEFINITION. *An element $x^* \in X$ is called a* `fixed point` *of a function $f : X \to X$ if*

$$f(x^*) = x^*.$$

6.3.4 THEOREM. **(Banach's Fixed Point Theorem)**[3] *If (X, ρ) is a complete metric space and $f : X \to X$ is a contraction then f has a unique fixed point.*

Proof. Let $x_0 \in X$. We define the sequence $(x_n) \in X^{\mathbb{N}}$ using the recurrence formula

$$x_{n+1} = f(x_n) \qquad (n \in \mathbb{N}).$$

We shall prove by induction that

$$\rho(x_{k+1}, x_k) \leq \alpha^k \rho(x_1, x_0) \qquad (k \in \mathbb{N}).$$

For $k = 0$ we obtain

$$\rho(x_1, x_0) = \alpha^0 \rho(x_1, x_0).$$

Suppose that

$$\rho(x_{k+1}, x_k) \leq \alpha^k \rho(x_1, x_0).$$

Then we get

$$\rho(x_{k+2}, x_{k+1}) = \rho(f(x_{k+1}), f(x_k)) \leq \alpha\rho(x_{k+1}, x_k) \leq \alpha\alpha^k \rho(x_1, x_0)$$
$$= \alpha^{k+1}\rho(x_1, x_0).$$

3 

Stephan Banach
(1892–1945),
a Polish mathematician.

We have:

$$\begin{aligned}
&\rho(x_{n+p}, x_n)\\
&\le \rho(x_{n+p}, x_{n+p-1}) + \rho(x_{n+p-1}, x_{n+p-2}) + \cdots + \rho(x_{n+1}, x_n)\\
&\le (\alpha^{n+p-1} + \alpha^{n+p-2} + \cdots + \alpha^n)\rho(x_1, x_0)\\
&\le \frac{\alpha^n}{1-\alpha}\rho(x_1, x_0).
\end{aligned}$$

From

$$\rho(x_{n+p}, x_n) \le \frac{\alpha^n}{1-\alpha}\rho(x_1, x_0),$$

and since

$$\alpha^n \to 0,$$

it follows that (x_n) is a Cauchy sequence (cf. remark (4.3.8)). Since (X, ρ) is a complete space the sequence (x_n) is convergent.

Let $x^* \in X$ be the limit of the sequence (x_n). We have

$$0 \le \rho(x^*, f(x^*)) \le \rho(x^*, x_n) + \rho(x_n, f(x^*))$$

$$= \rho(x^*, x_n) + \rho(f(x_{n-1}), f(x^*)) \le \rho(x^*, x_n) + \alpha\rho(x_{n-1}, x^*),$$

hence, for $n \to \infty$, we obtain

$$\rho(x^*, f(x^*)) = 0,$$

that is

$$f(x^*) = x^*,$$

i.e., x^* is a fixed point of f.

Let us prove the uniqueness of x^*. Suppose that x^* and x^{**} are two fixed points of f. Then:

$$f(x^*) = x^*, \quad f(x^{**}) = x^{**},$$

$$\rho(x^*, x^{**}) = \rho(f(x^*), f(x^{**})) \le \alpha\rho(x^*, x^{**}),$$

and, consequently,

$$(1-\alpha)\rho(x^*, x^{**}) \le 0,$$

therefore

$$\rho(x^*, x^{**}) = 0,$$

that is

$$x^* = x^{**}$$ ❖

6.3.5 REMARK. *The method used to determine the fixed point in the previous proof is called the* `fixed-point iteration` *or the* `functional iteration`.

The inequalities:

$$\rho(x_{n+p}, x_n) \le \frac{\alpha^n}{1-\alpha}\rho(x_1, x_0),$$

$n, p \in \mathbb{N}$, for $p \to \infty$, gives

$$\rho(x^*, x_n) \le \frac{\alpha^n}{1-\alpha}\rho(x_1, x_0),$$

$n \in \mathbb{N}$, which represents bounds for the error involved in the fixed-point iteration method.

6.3.6 THEOREM. *If I is a compact interval of the real axis and $f : I \to I$ is a function possessing derivative such that $|f'(x)| \le \alpha < 1$, for all $x \in I$, then the equation $f(x) = x$ has a unique root which can be approximated by the fixed-point iteration method.*

Proof. Let $x, y \in I$. By virtue of Lagrange mean-value theorem there exists $c \in (x, y)$ such that

$$f(x) - f(y) = f'(c)(x - y).$$

We deduce that

$$|f(x) - f(y)| \le \alpha|x - y|,$$

therefore f is a contraction and we can use Banach's theorem. ◆◆◆

To illustrate the technique of the fixed-point iteration method we consider the following examples.

6.3.7 EXAMPLE. Approximate the root of the equation

$$x^5 + 7x - 7 = 0$$

in the interval $[0, 1]$.

Write the equation in the form

$$1 - \frac{x^5}{7} = x.$$

Consider the function $f : [0,1] \to [0,1]$,

$$f(x) = 1 - \frac{x^5}{7}.$$

The equation can be written in the form $f(x) = x$, therefore the roots of the equation are the fixed-points of f. From the relations

$$|f'(x)| = \frac{5}{7}x^4 \leq \frac{5}{7} < 1,$$

by virtue of theorem (6.3.6), it follows that the sequence (x_n) defined by:

$$x_0 = 0, \quad x_{n+1} = 1 - \frac{x_n^5}{7}, \quad n = 0, 1, \ldots$$

converges to the root x^* of the equation $f(x) = x$ in the interval $(0,1)$. We obtain $x^* = 0.91057\ldots$.

6.3.8 EXAMPLE. Approximate the solutions of the system of equations

$$\begin{cases} \sin y &= 2x \\ \cos x &= 2y \end{cases} \qquad (x,y) \in [0,1]^2,$$

using the fixed-point iteration method.

To this end we shall find the fixed points of the function

$$f : ([0,1]^2, \rho) \to ([0,1]^2, \rho), \quad f(x,y) = \left(\frac{\sin y}{2}, \frac{\cos x}{2}\right).$$

We have:

$$\rho((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2},$$

$$\begin{aligned} \rho(f(x_1,y_1), f(x_2,y_2)) &= \frac{1}{2}\sqrt{(\sin y_1 - \sin y_2)^2 + (\cos x_1 - \cos x_2)^2} \\ &\leq \frac{1}{2}\sqrt{(y_1-y_2)^2 + (x_1-x_2)^2} = \frac{1}{2}\rho((x_1,y_1),(x_2,y_2)), \end{aligned}$$

hence f is a contraction. By virtue of Banach's theorem the solution of the system of equations is the limit of the sequence (x_n, y_n) defined by:

$$(x_0, y_0) = (0,0), \quad (x_{n+1}, y_{n+1}) = \left(\frac{\sin y_n}{2}, \frac{\cos x_n}{2}\right), \quad n = 0, 1, \ldots.$$

We obtain:

$$(x^*, y^*) = (0.233725\ldots, 0.486405\ldots).$$

6.4 Continuous Mappings on Euclidean Spaces

In this section we study the concept of continuity for the Euclidean space $\mathbb{R}^n$. We recall that, in the Euclidean space $\mathbb{R}^n$, the metric ρ is given by

$$\rho((x_1,\dots,x_n),(y_1,\dots,y_n)) = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}.$$

Consider the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ and a mapping

$$f:\mathbb{R}^n \to \mathbb{R}^m.$$

There exist the functions

$$f_i:\mathbb{R}^n \to \mathbb{R}, \quad i=1,\dots,m,$$

such that

$$f=(f_1,\dots,f_m),$$

i.e.,

$$f(x_1,\dots,x_n)=(f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n)).$$

We denote:

$$x=(x_1,\dots,x_n), \quad y=(y_1,\dots,y_n)\in\mathbb{R}^n.$$

From the definition of the continuity of a mapping on a metric space, we deduce:

A mapping $f:\mathbb{R}^n\to\mathbb{R}^m$ is continuous at the point $y\in\mathbb{R}^n$ if for any $\varepsilon>0$ there exists $\delta>0$ such that

$$\text{if} \quad \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}<\delta \quad \text{then} \quad \sqrt{\sum_{j=1}^{m}(f_j(x)-f_j(y))^2}<\varepsilon.$$

We deduce that the function f is continuous iff its component functions $f_1,\dots,f_m$ are continuous.

The limit of the mapping f at a point is defined as follows:

$$\lim_{x\to y} f(x)=\ell=(\ell_1,\dots,\ell_m)\in\mathbb{R}^m$$

if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if} \quad \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} < \delta \quad \text{then} \quad \sqrt{\sum_{j=1}^{m}(f_j(x) - \ell_j)^2} < \varepsilon.$$

6.4.1 EXAMPLE. Investigate the continuity of the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{|x| + |y|}, & \text{if} \quad (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\} \\ 0, & \text{if} \quad (x, y) = (0, 0) \end{cases}$$

at the point $(0, 0)$.

Let $\varepsilon > 0$ and take $\delta = \varepsilon/\sqrt{2}$. If

$$\rho((x, y), (0, 0)) = \sqrt{x^2 + y^2} \leq \delta = \varepsilon/\sqrt{2},$$

then

$$\sigma(f(x, y), f(0, 0)) = \frac{x^2 + y^2}{|x| + |y|} \leq \frac{x^2 + 2|xy| + y^2}{|x| + |y|} = |x| + |y|$$

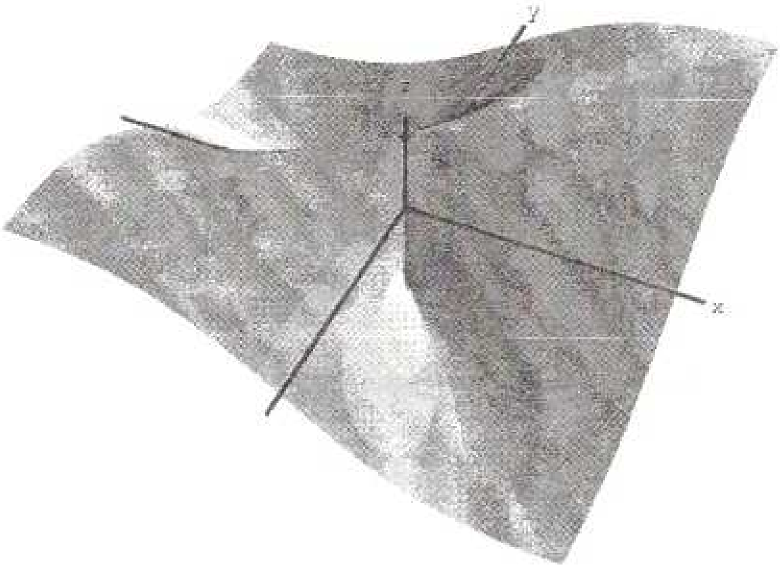
$$\leq \sqrt{2}\sqrt{x^2 + y^2} = \sqrt{2}\,\rho((x, y), (0, 0)) \leq \varepsilon,$$

therefore f is continuous at the point $(0, 0)$.

6.4.2 EXAMPLE. Examine the continuity of the function $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$,

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & \text{if} \quad (x, y) \in [-1, 1] \times [-1, 1] \setminus \{(0, 0)\} \\ 0, & \text{if} \quad (x, y) = (0, 0) \end{cases}$$

at the point $(0, 0)$.



Note that although $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$ however $f(\frac{1}{n}, \frac{1}{n}) = \frac{1}{2} \not\to 0 = f(0,0)$, therefore, by Heine's theorem, f is discontinuous at the point $(0,0)$.

6.4.3 EXAMPLE. Investigate the continuity of the function $f : \mathbb{R}^2 \to \mathbb{R}^2$,

$$f(x,y) = \begin{cases} (x, \sqrt{|xy|} \sin \dfrac{1}{\sqrt{|xy|}}), & \text{if} \quad xy \neq 0, \\ (x, 0), & \text{if} \quad xy = 0. \end{cases}$$

We have to study the continuity of the functions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$,

$$f_1(x,y) = x, \quad f_2(x,y) = \begin{cases} \sqrt{|xy|} \sin \dfrac{1}{\sqrt{|xy|}}, & \text{if} \quad xy \neq 0, \\ 0, & \text{if} \quad xy = 0. \end{cases}$$

Observe that f_1 is continuous on $\mathbb{R}^2$. Using the equality

$$\sqrt{|xy|} \leq \sqrt{\frac{x^2 + y^2}{2}}$$

we prove that f_2 is continuous on $\mathbb{R}^2$.

6.5 Exercises: Continuity of Functions

P
6.5.1 Let $A = \{a_1, a_2, \ldots\}$ be an infinite countable set of real numbers. Consider the function $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = a_n \in A, \\ 0, & \text{if } x \in \mathbb{R} \setminus A. \end{cases}$$

Prove that f is continuous on $\mathbb{R} \setminus A$ and discontinuous on A.

P
6.5.2 Consider the function $f : \mathbb{R}^m \to \mathbb{R}$, $m \geq 2$,

$$f(x_1, \ldots, x_m) = \begin{cases} \dfrac{x_1 \ldots x_m}{x_1^2 + \cdots + x_m^2}, & \text{if } (x_1, \ldots, x_m) \neq (0, \ldots, 0), \\ 0, & \text{if } (x_1, \ldots, x_m) = (0, \ldots, 0). \end{cases}$$

Determine whether the function f is continuous at the point $(0, \ldots, 0)$ or not.

P
6.5.3 Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = \begin{cases} -\frac{x^4 y^4}{x^8 + y^8}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0), \end{cases}$$

is discontinuous at $(0, 0)$.

P
6.5.4 Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0), \end{cases}$$

is discontinuous at $(0,0)$.

P
6.5.5 Find the points of discontinuity of the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} x \cdot \sin \frac{1}{xy}, & \text{if } xy \neq 0 \\ 0, & \text{if } xy = 0. \end{cases}$$

P
6.5.6 Let $f : (a,b) \to \mathbb{R}$ be a continuous injection. Prove that $f(a,b)$ is an open interval.

P
6.5.7 Let $f : [0,1] \to (0,1]$ be a bijection. Prove that f has an infinite number of points of discontinuity.

7

Differential Calculus for Functions of One Variable

Contents

7.1 Basic Differentiation Formulas

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $t \in A \cap A'$.

7.1.1 DEFINITION. *If the limit*

$$\lim_{\substack{x \to t \\ x \in A\setminus\{t\}}} \frac{f(x) - f(t)}{x - t},$$

exists, then it is called the **derivative of the function** f **at the point** t *and is denoted by* $\dfrac{df}{dx}(t)$ *or* $f'(t)$.

7.1.2 DEFINITION. *If, for all* $t \in A$, *the function* f *has a finite derivative, then the function* $t \mapsto f'(t)$ *is called the* **derivative** *of* f *and is denoted by* f'. *The process of finding the derivative is called* **differentiation**.

The **higher order derivatives** of the function f are defined by induction:

$$f^{(0)} = f, \quad f^{(n+1)} = \left(f^{(n)}\right)', \quad n = 0, 1, \ldots.$$

$f^{(n)}$ is called the **derivative of order** n, or the n**th derivative** of the function f.

For the derivative of order n of the function f we also use the notation $\dfrac{d^n f}{dx^n}$.

We define the set

$$C^k[a,b] = \{f \mid f : [a,b] \to \mathbb{R},\ \exists f^{(k)} - \text{continuous}\}.$$

When the following limits exist, we shall use the notations:

$$f_l'(t) = \lim_{\substack{x \to t \\ x<t}} \frac{f(x) - f(t)}{x - t}, \qquad f_r'(t) = \lim_{\substack{x \to t \\ x>t}} \frac{f(x) - f(t)}{x - t},$$

$$f'(t-0) = \lim_{\substack{x \to t \\ x<t}} f'(x), \qquad f'(t+0) = \lim_{\substack{x \to t \\ x>t}} f'(x).$$

7.1.3 EXAMPLE. Find $f_l'(0)$, $f_r'(0)$, $f'(0-0)$ and $f'(0+0)$ of the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \in \mathbb{R}^* \\ 0, & x = 0. \end{cases}$$

We have

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \in \mathbb{R}^* \\ 0, & x = 0, \end{cases}$$

$$f_l'(0) = f_r'(0) = 0;$$

$$f'(0-0) \quad \text{and} \quad f'(0+0) \quad \text{do not exist.}$$

Some of the basic differentiation rules are given below:

$$\begin{aligned}
(f+g)' &= f' + g'; \\
(fg)' &= f'g + fg'; \\
\left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2}; \\
(f \circ g)' &= f' \circ g \cdot g'; \\
(f^{-1})' &= \frac{1}{f' \circ f^{-1}}; \\
(u^v)' &= vu^{v-1}u' + u^v \ln u\, v'; \\
(f_1 f_2 \cdots f_n)' &= \sum_{k=1}^{n} f_1 f_2 \cdots f_{k-1} f_k' f_{k+1} \cdots f_n;
\end{aligned}$$

$$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}' = \sum_{k=1}^{n} \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{k-1,1} & f_{k-1,2} & \cdots & f_{k-1,n} \\ f_{k1}' & f_{k2}' & \cdots & f_{kn}' \\ f_{k+1,1} & f_{k+1,2} & \cdots & f_{k+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}.$$

In order to compute the higher order derivatives for the product of two functions we shall use the following result.

7.1.4 THEOREM. **(Leibniz's Formula)** *If the functions f and g possess a nth derivatives at a given point, the nth derivative of the product fg exists at that point and is expressed by*

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)},$$

where $\binom{n}{k}$ are the binomial coefficients.

Proof. The formula will be proved by induction. It is obvious that the formula is true when $n = 1$,

$$(fg)' = f'g + fg' = \binom{1}{0} f^{(1)} g^{(0)} + \binom{1}{1} f^{(0)} g^{(1)} = \sum_{k=0}^{1} \binom{1}{k} f^{(1-k)} g^{(k)}.$$

Suppose that it is true for n,

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}.$$

We have to show that it is valid for $n+1$. Consider

$$(fg)^{(n+1)} = \left((fg)^{(n)}\right)' = \left(\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}\right)'$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(n-k)} g^{(k)}\right)' = \sum_{k=0}^{n} \binom{n}{k} \left(f^{(n-k+1)} g^{(k)} + f^{(n-k)} g^{(k+1)}\right)$$

$$= \binom{n}{0} f^{(n+1)} g^{(0)}$$

$$+ \sum_{k=1}^{n} \binom{n}{k} f^{(n-k+1)} g^{(k)} + \sum_{k=0}^{n-1} \binom{n}{k} f^{(n-k)} g^{(k+1)}$$

$$+ \binom{n}{n} f^{(0)} g^{(n+1)}$$

$$= \binom{n+1}{0} f^{(n+1)} g^{(0)}$$

$$+ \sum_{i=1}^{n} \binom{n}{i} f^{(n+1-i)} g^{(i)} + \sum_{i=1}^{n} \binom{n}{i-1} f^{(n+1-i)} g^{(i)}$$

$$+ \binom{n+1}{n+1} f^{(0)} g^{(n+1)}$$

$$= \binom{n+1}{0} f^{(n+1)} g^{(0)}$$

$$+ \sum_{i=1}^{n} \left(\binom{n}{i} + \binom{n}{i-1} \right) f^{(n+1-i)} g^{(i)}$$

$$+ \binom{n+1}{n+1} f^{(0)} g^{(n+1)}$$

$$= \sum_{i=0}^{n+1} \binom{n+1}{i} f^{(n+1-i)} g^{(i)}.$$

7.1.5 EXAMPLE. Find the nth derivative of the function

$$f(x) = x^2 e^x, \quad x \in \mathbb{R}.$$

Using Leibniz's formula, we have:

$$(x^2 e^x)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} (e^x)^{(n-i)} (x^2)^{(i)}$$

$$= \sum_{i=0}^{n} \binom{n}{i} e^x (x^2)^{(i)} = e^x \left(\binom{n}{0} x^2 + 2\binom{n}{1} x + 2\binom{n}{2} \right)$$

$$= e^x (x^2 + 2nx + n(n-1)).$$

7.1.6 EXAMPLE. Find the derivative of order n of the function

$$f(x) = \frac{\arccos x}{\sqrt{1-x^2}}, \quad x \in (-1,1),$$

at zero.

We have:

$$f(x)\sqrt{1-x^2} - \arccos x = 0,$$

$$f'(x)\sqrt{1-x^2} - \frac{x}{\sqrt{1-x^2}}f(x) + \frac{1}{\sqrt{1-x^2}} = 0,$$

$$(1-x^2)f'(x) - xf(x) + 1 = 0.$$

Differentiating the previous identity $n-1$ times and using Leibniz's formula, we obtain:

$$f^{(n)}(x)(1-x^2) + \binom{n-1}{1}f^{(n-1)}(x)(-2x)$$

$$+\binom{n-1}{2}f^{(n-2)}(x)(-2) - f^{(n-1)}(x)x - \binom{n-1}{1}f^{(n-2)}(x) = 0$$

hence, we have derived the recurrence formula for $x = 0$:

$$f^{(n)}(0) - (n-1)^2 f^{(n-2)}(0) = 0, \quad n \geq 2,$$

or,

$$f^{(2k)}(0) = (2k-1)^2 f^{(2k-2)}(0), \quad k \geq 1,$$

$$f^{(2k+1)}(0) = (2k)^2 f^{(2k-1)}(0), \quad k \geq 0,$$

therefore,

$$f^{(2k)}(0) = ((2k-1)!!)^2 \frac{\pi}{2}, \quad k \geq 1,$$

$$f^{(2k+1)}(0) = -\left((2k)!!\right)^2, \quad k \geq 0,$$

where $(2k)!! = 2 \cdot 4 \cdots (2k), \quad (2k-1)!! = 1 \cdot 3 \cdots (2k-1)$.

7.2 Mean-value Theorems for Derivatives

7.2.1 DEFINITION. *A point $t \in A$ is called a* `point of local extremum` *(point of local maximum or point of local minimum) of a function $f : A \to \mathbb{R}$ if there exists $\delta > 0$ such that the difference $f(x) - f(t)$ does not change sign for $x \in A \cap (t-\delta, t+\delta)$.*

The value $f(t)$ is called a `local extremum` *(local maximum or local minimum).*

7.2.2 DEFINITION. *A point at which the derivative f' is zero is called a* `stationary point` *or* `turning point` *of the function f.*

7.2.3 THEOREM. **(Fermat)**[1] *For a function f possessing a derivative at an interior point x to attain a local extremum at this point it is necessary that the point x be a stationary point.*

Proof. Let $f : A \to \mathbb{R}$ be a function possessing a derivative at the point $t \in \operatorname{int}(A)$. We know that there exists $\delta > 0$ such that the difference $f(x) - f(t)$ does not change sign for $x \in A \cap (t - \delta, t + \delta)$. We deduce that

$$\frac{f(x) - f(t)}{x - t} \cdot \frac{f(x') - f(t)}{x' - t} \leq 0,$$

for all $x \in (t - \delta, t) \cap A$ and $x' \in (t, t + \delta) \cap A$. For $x \to t$, $x < t$ and $x' \to t$, $x' > t$, we obtain

$$f_l'(t) \cdot f_r'(t) \leq 0.$$

But the function f possesses a derivative at the point t, hence

$$\left(f'(t)\right)^2 \leq 0, \qquad \text{i.e.,} \qquad f'(t) = 0.$$

[1] 

Pierre Simon de Fermat (1601-1665), a French mathematician.

7.2.4 REMARK. *It is to be noted that a stationary point is not necessarily a point of local extremum and vice versa.*

As for example, the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$, has a local minimum at zero which is not a stationary point. On the other hand, the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, has no local extremum but a stationary point at $x = 0$.

7.2.5 THEOREM. **(Rolle)**[2] *If the function $f : [a, b] \to \mathbb{R}$ satisfies the conditions:*

- *is continuous on the closed interval $[a, b]$;*
- *possesses a derivative on the open interval (a, b);*
- $f(a) = f(b)$,

then there exists at least one point $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Proof. Let M and m be the supremum and respectively the infimum of f on the interval $[a, b]$. The existence of M and m is guaranteed because f is continuous on the compact interval $[a, b]$. If $M = m = f(a)$, then f is a constant and $f'(\xi) = 0$, for all $\xi \in (a, b)$. If f is not constant, then at least one of the two numbers

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Michel Rolle (1652–1719), a French mathematician.

M and m is distinct from the number $f(a) = f(b)$; for definiteness, let $M \neq f(a) = f(b)$. This means that f attains its supremum at an interior point ξ; Fermat's theorem implies that $f'(\xi) = 0$. The case $m \neq f(a)$ is treated in a similar way.

7.2.6 THEOREM. **(Cauchy)** *If the functions f and g satisfy the following conditions:*

- *are continuous on the closed interval $[a, b]$;*
- *possess derivatives on the open interval (a, b);*
- $g'(x) \neq 0, \forall x \in (a, b)$,

then there exists at least one point $\xi \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

Proof. Consider the auxiliary function $h : [a, b] \to \mathbb{R}$,

$$h(x) = \begin{vmatrix} f(x) & g(x) & 1 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix};$$

which is continuous on the closed interval $[a, b]$, possesses a derivative on the open interval (a, b) and satisfies the conditions $h(a) = h(b) = 0$. Therefore, by Rolle's theorem, there exists at least one point $\xi \in (a, b)$ such that

$$h'(\xi) = 0,$$

hence

$$\begin{vmatrix} f'(\xi) & g'(\xi) & 0 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix} = 0,$$

so,

$$\begin{vmatrix} g(a) & 1 \\ g(b) & 1 \end{vmatrix} f'(\xi) - \begin{vmatrix} f(a) & 1 \\ f(b) & 1 \end{vmatrix} g'(\xi) = 0,$$

therefore, using the fact that $g(a) \neq g(b)$ (see $g'(x) \neq 0$), we deduce

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

◇◇◇

As a consequence of the Cauchy theorem, when $g(x) = x$, one can obtain the following theorem.

7.2.7 THEOREM. **(Lagrange)**[3] *If the function $f : [a, b] \to \mathbb{R}$ satisfies the conditions:*

- *is continuous on the closed interval $[a, b]$;*
- *possesses a derivative on the interval (a, b),*

then there exists at least one point $\xi \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi).$$

The previous formula is called `Lagrange's formula of finite increments`.

7.2.8 THEOREM. **(Generalized Rolle's Theorem)** *If a function $h : [a, b] \to \mathbb{R}$ satisfies the conditions:*

- *is continuous on the closed interval $[a, b]$;*
- *possesses a nth derivative on the open interval (a, b),*

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Joseph Louis, Comte de Lagrange (1736–1813), an Italian–French mathematician.

- *has $n+1$ distinct roots in the interval $[a, b]$,*

then there exists at least one point $\xi \in (a, b)$ such that $h^{(n)}(\xi) = 0$.

Proof. Let $x_0 < x_1 < \cdots < x_n$ be the roots of the function h; then the conditions of Rolle's theorem are satisfied on the closed intervals

$$[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{n-1}, x_n].$$

Therefore, there exist the points:

$$\xi_0^1 \in (x_0, x_1),\ \xi_1^1 \in (x_1, x_2),\ \ldots,\ \xi_{n-1}^1 \in (x_{n-1}, x_n),$$

such that

$$h'(\xi_i^1) = 0 \qquad (i = 0, \ldots, n-1).$$

The conditions of Rolle's theorem are satisfied on the closed intervals

$$[\xi_0^1, \xi_1^1],\ [\xi_1^1, \xi_2^1],\ \ldots,\ [\xi_{n-2}^1, \xi_{n-1}^1],$$

for the function h'; then there exist points

$$\xi_i^2 \in (\xi_i^1, \xi_{i+1}^1) \qquad (i = 0, \ldots, n-2),$$

such that

$$h''(\xi_i^2) = 0 \qquad (i = 0, \ldots, n-2).$$

By induction, we prove the existence of the points

$$\xi_i^k \in (\xi_i^{k-1}, \xi_{i+1}^{k-1}) \qquad (i = 0, \ldots, n-k,\ \ k = 2, \ldots, n),$$

such that

$$h^{(k)}(\xi_i^k) = 0 \qquad (i = 0, \ldots, n-k,\ \ k = 2, \ldots, n).$$

Denoting $\xi = \xi_0^n$, the proof is completed. ∎

The Romanian mathematician Dimitrie Pompeiu[4] obtained the following variant of the Lagrange mean-value theorem.

7.2.9 THEOREM. [26, Dimitrie Pompeiu] *If the function* $f\colon [a,b] \to \mathbb{R}$ *satisfies the conditions:*

- *is continuous on* $[a,b]$;
- *if differentiable on* (a,b);
- $0 \notin [a,b]$,

then there exists a point $c \in (a,b)$ *such that*

$$\frac{a\,f(b) - b\,f(a)}{a-b} = f(c) - c\,f'(c).$$

◈◈◈

The geometric interpretation of Theorem 7.2.9 is given in Figure 7.1

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Dimitrie, Pompeiu
(1873–1954),
a Romanian
mathematician.

Another Pompeiu-type mean-value theorem is the following.

7.2.10 THEOREM. [11, Mircea Ivan, 1970] *Let $f\colon [a,b] \to \mathbb{R}$ satisfy the following conditions:*

- *is continuous on $[a,b]$;*
- *is differentiable on (a,b);*
- *has no roots in $[a,b]$;*
- $f(a) \neq f(b)$.

Then there exists a point $c \in (a,b)$ such that

$$\frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = c - \frac{f(c)}{f'(c)}.$$

We point out that no conditions are imposed to f'. The geometric interpretation of Theorem 7.2.10 is given in Figure 7.2

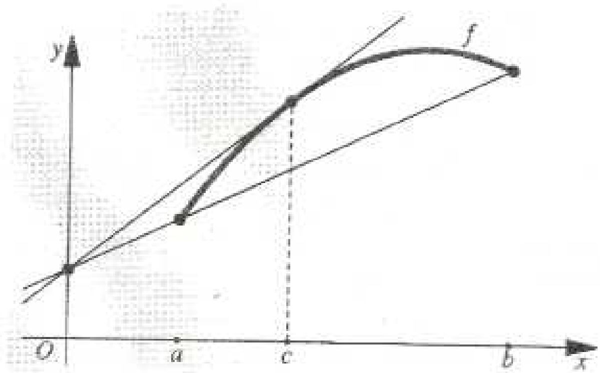

Figure 7.1: The geometric interpretation of Pompeiu's Theorem 7.2.9: the tangent to the graph of f at the point $(c, f(c))$ and the secant line joining the points $(a, f(a))$ and $(b, f(b))$ intersect the y-axis at the same point.

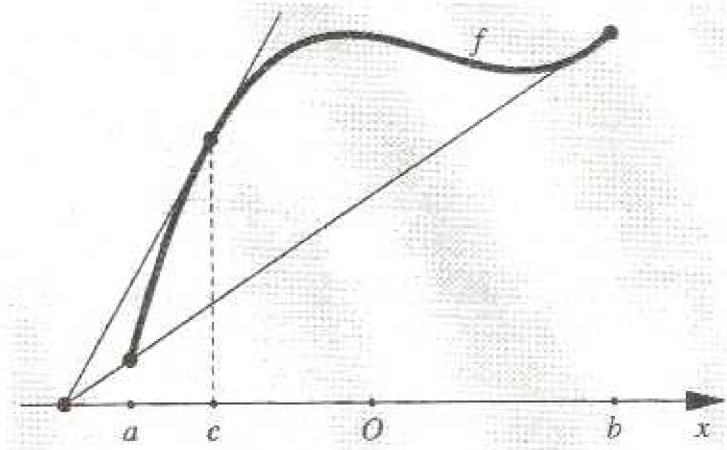

Figure 7.2: The geometric interpretation of Ivan's Theorem 7.2.10: the tangent to the graph of f at the point $(c, f(c))$ and the secant line joining the points $(a, f(a))$ and $(b, f(b))$ intersect the x-axis at the same point.

7.3 Applications of Mean-value Theorems

Let f be a function possessing derivative on an interval I. The monotonicity of the function f is determined by the sign of the ratio

$$\frac{f(x)-f(y)}{x-y}, \quad x, y \in I, \quad x \neq y.$$

Using the Lagrange mean-value theorem, we obtain:

$$\begin{aligned} f' > 0 \quad &\Rightarrow \quad f \text{ is increasing;} \\ f' \geq 0 \quad &\Longleftrightarrow \quad f \text{ is nondecreasing;} \\ f' \leq 0 \quad &\Longleftrightarrow \quad f \text{ is nonincreasing;} \\ f' < 0 \quad &\Rightarrow \quad f \text{ is decreasing.} \end{aligned}$$

7.3.1 REMARK. *If the derivative of a function f is zero on a union of disjoint intervals, then the function f is constant on each interval of the union.*

7.3.2 EXAMPLE. Consider the function

$$f : (-\infty, -1) \cup (-1, 1) \cup (1, \infty) \to \mathbb{R},$$

$$f(x) = 2 \arctan x - \arctan \frac{2x}{1-x^2}.$$

Find the set of all its values.

We have:

$$f'(x) = \frac{2}{1+x^2} - \frac{2\frac{1+x^2}{(1-x^2)^2}}{1+\frac{4x^2}{(1-x^2)^2}} = 0,$$

so,

$$f(x) = \begin{cases} C_1, & \text{if } x \in (-\infty, -1), \\ C_2, & \text{if } x \in (-1, 1), \\ C_3, & \text{if } x \in (1, \infty), \end{cases}$$

hence

$$C_1 = f(-\infty) = -\pi, \quad C_2 = f(0) = 0, \quad C_3 = f(\infty) = \pi,$$

therefore

$$f(x) = \begin{cases} -\pi, & \text{if } x \in (-\infty, -1), \\ 0, & \text{if } x \in (-1, 1), \\ \pi, & \text{if } x \in (1, \infty). \end{cases}$$

7.3.3 EXAMPLE. Find the elements of the set

$$A = \left\{ \arctan x + \arctan y - \arctan \frac{x+y}{1-xy} \mid x, y \in \mathbb{R},\ xy \neq 1 \right\}.$$

For $y = 0$, we have $\arctan x + \arctan y - \arctan \frac{x+y}{1-xy} = 0$, for all $x \in \mathbb{R}$. For $y \neq 0$, consider the function $g : \mathbb{R} \setminus \{\frac{1}{y}\} \to \mathbb{R}$, defined by

$$g(x) = \arctan x + \arctan y - \arctan \frac{x+y}{1-xy}.$$

Using the method of the previous example we obtain

$$A = \{-\pi, 0, \pi\}.$$

7.3.4 EXAMPLE. Find the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(2^{k/n^2} - 1\right).$$

Consider the function $f(x) = 2^x$. The Lagrange theorem on the intervals $[0, k/n^2]$, $k = 1, \ldots, n$ guarantees, the existence of the points $\xi_{nk} \in (0, k/n^2)$, $k = 1, \ldots, n$, such that

$$2^{k/n^2} - 1 = \frac{k}{n^2} 2^{\xi_{nk}} \ln 2,$$

therefore,

$$2^0 \frac{\ln 2}{n^2} \sum_{k=1}^{n} k < \sum_{k=1}^{n} \left(2^{k/n^2} - 1\right) < \sqrt[n]{2}\, \frac{\ln 2}{n^2} \sum_{k=1}^{n} k,$$

$$2^0 \frac{\ln 2}{n^2} \frac{n(n+1)}{2} < \sum_{k=1}^{n} \left(2^{k/n^2} - 1\right) < \sqrt[n]{2}\, \frac{\ln 2}{n^2} \frac{n(n+1)}{2},$$

hence,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(2^{k/n^2}-1\right)=\ln\sqrt{2}.$$

7.3.5 THEOREM. **(Darboux)**[5] *Each derivative has the Darboux property.*

Proof. Let $f:[a,b]\to\mathbb{R}$ be a function possessing derivative and $x_1,x_2\in[a,b]$, $x_1<x_2$. For definiteness, let $f'(x_1)<f'(x_2)$ and $\lambda\in(f'(x_1),f'(x_2))$. Consider the auxiliary function

$$F(x)=f(x)-\lambda x.$$

We obtain:

$$F'(x_1)=f'(x_1)-\lambda<0,\quad F'(x_2)=f'(x_2)-\lambda>0.$$

If F is injective on the interval $[x_1,x_2]$ then, being continuous, it is monotonous; therefore $F'(x_1)\cdot F'(x_2)>0$. We deduce that F is not injective on the interval $[x_1,x_2]$ hence, by Rolle's theorem there exists a point $\xi\in(x_1,x_2)$ such that $F'(\xi)=0$, i.e., $f'(\xi)=\lambda$. ◆◆◆

7.3.6 THEOREM. *If a function $f:(a,b)\to\mathbb{R}$ is continuous on the interval (a,b), possesses derivative on $(a,b)\setminus\{t\}$ and the limit $\lim_{x\to t}f'(x)$ exists, then f possesses a derivative at the point t and*

$$\lim_{x\to t}f'(x)=f'(t).$$

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Jean Gaston Darboux (1842–1917), a French mathematician.

Proof. Let (x_n) be a sequence convergent to t, where $x_n \in (a, b)$, $x_n \neq t$. The conditions of Lagrange's theorem are satisfied on the interval with end points x_n and t. Therefore, there exists a point ξ_n in the open interval with end points x_n and t, such that

$$\frac{f(x_n) - f(t)}{x_n - t} = f'(\xi_n).$$

This implies the existence of the limit

$$\lim_{n\to\infty} \frac{f(x_n) - f(t)}{x_n - t},$$

i.e., f possesses a derivative at the point t and

$$f'(t) = \lim_{x\to t} f'(x).$$

7.4 Taylor's Formula for Real Functions of One Variable

Let f be a real function of one variable possessing a nth derivative on a neighborhood of a point $a \in \mathbb{R}$. Taylor's[6] formula, which will be studied in this section, makes it possible to determine approximately, sometimes with a high accuracy, the values of a given function f, in a neighborhood of a point a, from the given values $f(a)$, $f'(a)$, ..., $f^{(n)}(a)$.

7.4.1 DEFINITION. *The polynomial*

$$T_n[f;a](x) := f(a) + \frac{x-a}{1!} f'(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a)$$

is called `the Taylor polynomial` *of degree n for the function f at the point a.*

The simplified notation T_n will be used for the polynomial $T_n[f;a]$. Note that it is closely related to f by:

$$T_n^{(k)}(a) = f^{(k)}(a), \quad (k = 0, \ldots, n).$$

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Brook Taylor (1685–1731), an English mathematician.

7.4.2 EXAMPLE.

$$\begin{aligned}
T_3[\sin, 0](x) &= x - \frac{1}{6}x^3, \\
T_7[\sin\circ\sin, 0](x) &= x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{8}{315}x^7, \\
T_7[\tan, 0](x) &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7, \\
T_7[\ln\cos, 0](x) &= -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6, \\
T_3[f, 0](x) &= \mathrm{e} - \frac{\mathrm{e}}{2}x + \frac{11\mathrm{e}}{24}x^2 - \frac{7\mathrm{e}}{16}x^3,
\end{aligned}$$

for

$$f(x) = \begin{cases} (1+x)^{1/x}, & \text{if } x > 0, \\ \mathrm{e}, & \text{if } x = 0. \end{cases}$$

If P is a polynomial of degree at most n, then $P = T_n[P, a]$, and consequently

$$P(x) = P(a) + \frac{x-a}{1!}P'(a) + \cdots + \frac{(x-a)^n}{n!}P^{(n)}(a),$$

for all $x, a \in \mathbb{R}$.

7.4.3 THEOREM. *If $f \in C^n[a, b]$ possesses a $(n+1)^{th}$ derivative on (a, b), then for all $x \in (a, b]$ and for all $p \in \mathbb{N}^*$ there exists $\xi \in (a, x)$ such that*

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a)$$
$$+ \frac{(x-a)^p(x-\xi)^{n+1-p}}{pn!}f^{(n+1)}(\xi).$$

Proof. Let $x \in (a, b]$. Consider the auxiliary function $F : [a, x] \to \mathbb{R}$,

$$F(t) = f(t) + \frac{x-t}{1!}f'(t) + \cdots + \frac{(x-t)^n}{n!}f^{(n)}(t) + K(x-t)^p,$$

where the constant K is determined by the condition $F(a) = F(x)$.

The function F is continuous on the closed interval $[a, x]$, possesses a derivative on (a, x) and $F(a) = F(x)$. By Rolle's theorem there exists at least one point $\xi \in (a, x)$ such that $F'(\xi) = 0$, hence, using the relations

$$\begin{aligned} F'(t) &= f'(t) - f'(t) + \frac{x-t}{1!} f''(t) + \cdots - n\frac{(x-t)^{n-1}}{n!} f^{(n)}(t) \\ &\quad + \frac{(x-t)^n}{n!} f^{(n+1)}(t) - pK(x-t)^{p-1} \\ &= \frac{(x-t)^n}{n!} f^{(n+1)}(t) - pK(x-t)^{p-1}, \end{aligned}$$

we deduce

$$K = \frac{(x-\xi)^{n+1-p}}{p\,n!} f^{(n+1)}(\xi)$$

The equality

$$F(x) = F(a),$$

i.e.,

$$\begin{aligned} f(x) = f(a) + \frac{x-a}{1!} f'(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) \\ + \frac{(x-a)^p (x-\xi)^{n+1-p}}{pn!} f^{(n+1)}(\xi) \end{aligned}$$

is known as `Taylor's formula with Schlömilch's`[7] `form of the remainder.`

Likewise we can show that:

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Oscar Xavier Schlömilch (1823–1901), a German mathematician.

7.4.4 THEOREM. *If $f : [a,b] \to \mathbb{R}$ possesses a $(n+1)^{th}$ derivative, then for all $x, t \in [a,b]$ there exists ξ such that $|t - \xi| \leq |t - x|$ for which*

$$f(x) = f(t) + \frac{x-t}{1!} f'(t) + \cdots + \frac{(x-t)^n}{n!} f^{(n)}(t)$$
$$+ \frac{(x-t)^p (x-\xi)^{n+1-p}}{pn!} f^{(n+1)}(\xi).$$

For $p = n+1$ and $x \in [a,b]$, we obtain

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi).$$

The above expression is known as `Taylor's formula with Lagrange's form of the remainder.`

In the case of $a = 0$, Taylor's formula is also called `Maclaurin's`[8] `formula` and given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\xi).$$

Let o be a function satisfying the condition

$$\lim_{y \to 0} \frac{o(y)}{y} = 0.$$

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Colin Maclaurin (1698–1746), a Scottish mathematician.

Suppose that $f^{(n+1)}$ is bounded on $[a, b]$; then Taylor's formula becomes

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + o((x-a)^n),$$

$x \in [a, b]$. This expression is known as `Taylor's formula with Peano's`[9] `form of the remainder`.

7.4.5 EXAMPLE. Find the Maclaurin formula with Peano's form of the remainder for the function $e^{\sin x}$.

We have

$$\sin x = x - \frac{x^3}{6} + o(x^4),$$

$$e^u = 1 + u + \frac{u^2}{2} + o(u^2),$$

and hence

$$e^{\sin x} = 1 + x + \frac{x^2}{2} + o(x^3).$$

7.4.6 THEOREM. **(Taylor's formula with integral form of the remainder)** *Let I be an interval and $f \in C^{n+1}(I)$. Then, for all $a, x \in I$, we have*

$$f(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt.$$

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Giuseppe Peano (1858–1932), an Italian mathematician.

Proof. We have:

$$\int_a^x \left(\frac{(x-t)^k}{k!} f^{(k)}(t)\right)' dt = \frac{(x-t)^k}{k!} f^{(k)}(t)\bigg|_{t=a}^{t=x} = -\frac{(x-a)^k}{k!} f^{(k)}(a),$$

for $k = 1, \ldots, n$, i.e.,

$$\int_a^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\, dt - \int_a^x \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t)\, dt = -\frac{(x-a)^k}{k!} f^{(k)}(a),$$

for $k = 1, \ldots, n$, hence we obtain

$$\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt - \int_a^x f'(t)\, dt = -\sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a),$$

so

$$f(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt.$$

7.4.7 Definition. *Let $f \in C^\infty[a,b]$ and $x_0 \in [a,b]$. The series*

$$\sum_{n=0}^{\infty} f^{(n)}(x_0)\, \frac{(x-x_0)^n}{n!}$$

is called `the Taylor series` *of f at the point x_0.*

In the case of $x_0 = 0$ the Taylor series is also called the `Maclaurin series`.

In general, the sum of the Taylor series of a function f is different from f. As an illustration, consider the function

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \in \mathbb{R}^*, \\ 0, & \text{if } x = 0. \end{cases}$$

We have

$$f^{(n)}(0) = 0, \qquad \forall n \in \mathbb{N},$$

i.e., the sum of the Taylor series of f is zero.

The case when Taylor's series of the function f is convergent to that very function is of particular importance:

7.4.8 THEOREM. *Let $f \in C^\infty[a,b]$ and $x_0 \in [a,b]$. If*

$$\lim_{n\to\infty} \sup_{x\in[a,b]} \frac{|f^{(n)}(x)|(b-a)^n}{n!} = 0,$$

then

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(x_0)\,\frac{(x-x_0)^n}{n!}$$

for all $x \in [a,b]$.

Proof. The Taylor formula with Lagrange's form of the remainder gives

$$|f(x) - \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0)| = \left|\frac{f^{(n+1)}(\xi)|(x-x_0)^{n+1}}{(n+1)!}\right|$$

$$\leq \sup_{x\in[a,b]} \frac{|f^{(n+1)}(x)|(b-a)^{n+1}}{(n+1)!} \to 0,$$

for $n \to \infty$, therefore

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(x_0)\,\frac{(x-x_0)^n}{n!}.$$ ◆◆◆

The above formula is also known as `Taylor's expansion` of f. When $x_0 = 0$, the Taylor expansion of f becomes

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\,\frac{x^n}{n!}.$$

It is also called `Maclaurin's expansion` of f.

Maclaurin's series for some elementary functions are presented below.

1. $\quad e^x \quad = \quad \sum_{n\geq 0} \frac{x^n}{n!}, \qquad x \in \mathbb{R};$

$$2. \qquad \sin x = \sum_{n\geq 0}(-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad x \in \mathbb{R};$$

$$3. \qquad \cos x = \sum_{n\geq 0}(-1)^n \frac{x^{2n}}{(2n)!}, \qquad x \in \mathbb{R};$$

$$4. \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots, \qquad x \in (-\pi/2, \pi/2);$$

$$5. \qquad \frac{1}{1-x} = \sum_{n\geq 0} x^n, \qquad x \in (-1,1);$$

$$6. \qquad \frac{1}{1+x} = \sum_{n\geq 0}(-1)^n x^n, \qquad x \in (-1,1);$$

$$7. \qquad \ln(1+x) = \sum_{n\geq 0}(-1)^n \frac{x^{n+1}}{n+1}, \qquad x \in (-1,1];$$

$$8. \qquad \ln\frac{1+x}{1-x} = \sum_{n\geq 0} 2\frac{x^{2n+1}}{2n+1}, \qquad x \in (-1,1);$$

$$9. \qquad \arctan x = \sum_{n\geq 0}(-1)^n \frac{x^{2n+1}}{2n+1}, \qquad x \in [-1,1];$$

$$10. \qquad (1+x)^\alpha = \sum_{n\geq 0}\binom{\alpha}{n} x^n, \qquad x \in (-1,1).$$

```
In[16]:= Sum[((-1)^n x^(2 n))/((2 n)!), {n, 0, ∞}] // FullSimplify
Out[16]= Cos[x]

In[17]:= Sum[((-1)^n x^(2 n+1))/((2 n+1)!), {n, 0, ∞}] // FullSimplify
Out[17]= Sin[x]

In[19]:= Sum[(-1)^n x^n, {n, 0, ∞}]
Out[19]= 1/(1 + x)

In[24]:= Sum[x^(2 n+1)/(2 n + 1), {n, 0, ∞}] == 1/2 Log[(1 + x)/(1 - x)] // FullSimplify
Out[24]= 2 ArcTanh[x] == Log[(1 + x)/(1 - x)]

In[21]:= TrigToExp[%]
Out[21]= -1/2 Log[1 - x] + 1/2 Log[1 + x]
```

7.5 Differential of Functions of One Variable

Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function possessing derivative at the point $x \in A \cap A'$. We have:

$$\lim_{h\to 0} \frac{|f(x+h) - f(x) - f'(x)\cdot h|}{|h|} = \lim_{h\to 0} \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| = 0,$$

i.e., the linear function $T : \mathbb{R} \to \mathbb{R}$, $T(h) = f'(x)\cdot h$, is the differential of the function f at the point x and therefore (cf. Definition 8.7.7)

$$df(x)(h) = f'(x)\cdot h.$$

Using the notations:

$$1_A : A \to A, \quad 1_A(x) = x, \quad \forall x \in A,$$

$$1_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}, \quad 1_{\mathbb{R}}(h) = h, \quad \forall h \in \mathbb{R},$$

we obtain:

$$d\,1_A(x)(h) = 1'_A(x)\cdot h = 1\cdot h = h, \quad \forall h \in \mathbb{R},$$

or

$$d\,1_A(x) = 1_{\mathbb{R}}, \quad \forall x \in A,$$

which can be written as

$$dx = 1_{\mathbb{R}},$$

where $1_A(x) = x$. We have:

$$df(x)(h) = f'(x) \cdot h = f'(x) \cdot 1_{\mathbb{R}}(h),$$

for all $h \in \mathbb{R}$, therefore

$$df(x) = f'(x)\,1_{\mathbb{R}} = f'(x)\,dx.$$

In conclusion, we obtain:

7.5.1 $$df(x) = f'(x)\,dx, \qquad \forall x \in A,$$

$$df = f'dx,$$

where

$$df \in \left(\mathbb{R}^{\mathbb{R}}\right)^A.$$

The second differential of f is defined by

7.5.2 $$d^2 f(x)(h) \quad \stackrel{def}{=} \quad \frac{d}{dt}(df(x+th)(h))\,\Big|_{t=0}.$$

We deduce

$$d^2 f = f''\,dx^2.$$

In general,

$$d^n f = f^{(n)}\,dx^n.$$

7.5.3 REMARK. *One should distinguish between the following notations*

$$\begin{aligned} dx^2 &= (dx)^2; \\ d(x^2) &= 2x\,dx; \\ d^2x &= x''\,(dx)^2 = 0. \end{aligned}$$

8

Differential Calculus for Functions of Several Variables

Contents

8.1 Partial Derivatives

Let $E \subseteq \mathbb{R}^n$, $f : E \to \mathbb{R}$ and $x = (x_1, \dots, x_n)$ be a point in $E \cap E'$.

8.1.1 DEFINITION. *The* `partial derivative` *of a function f at a point x, with respect to x_k, is defined as the limit*

$$\frac{\partial f}{\partial x_k}(x) := f'_{x_k}(x)$$

$$= \lim_{t \to x_k} \frac{f(x_1, \dots, x_{k-1}, t, x_{k+1} \dots, x_n) - f(x_1, \dots, x_n)}{t - x_k},$$

provided the limit exists.

The partial derivative $f'_{x_k}(x)$ is no more that the ordinary derivative of the function $f(x_1, \dots, x_{k-1}, \cdot, x_{k+1} \dots, x_n)$ of variable x_k alone, when $x_1, \dots, x_{k-1}, x_{k+1} \dots, x_n$ are fixed.

If a function f possesses a derivative with respect to x_k at each point in $E \cap E'$, then the function $x \mapsto f'_{x_k}(x)$ is denoted by

$$\frac{\partial f}{\partial x_k} \quad \text{or} \quad f'_{x_k},$$

and is called the `partial derivative of` f `with respect to` x_k.

8.1.2 EXAMPLE. Let $f : \mathbb{R}^2 \to \mathbb{R}$, such that

$$f(x, y) = x^2 y + e^{xy}.$$

Then

$$\frac{\partial f}{\partial x}(x, y) = 2xy + ye^{xy}, \qquad \frac{\partial f}{\partial y}(x, y) = x^2 + xe^{xy}.$$

8.1.3 REMARK. *The existence of partial derivatives of a function at a point does not imply the continuity of the function at that point.*

As an illustration, consider the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

We have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0}\frac{0-0}{x-0} = 0,$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0}\frac{0-0}{y-0} = 0,$$

despite the function f is discontinuous at the point $(0,0)$.

The **higher order partial derivatives** of a function f are defined by induction:

8.1.4
$$\frac{\partial^2 f}{\partial x \partial y} \stackrel{def}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

We write:

$$f''_{yx} = (f'_y)_x \stackrel{def}{=} \frac{\partial^2 f}{\partial x \partial y}.$$

In general, the mixed partial derivatives f''_{xy} and f''_{yx} are not equal to each other. As it will be shown later, in many important cases the operations of successive partial differentiation can be interchanged without affecting the final result.

Let $A \subseteq \mathbb{R}^2$. The following theorem presents a necessary condition for the independence of mixed partial derivatives of the order of differentiation.

8.1.5 THEOREM. **(Schwarz)**[1] *If $f : A \to \mathbb{R}$ possesses continuous mixed partial derivatives f''_{xy} and f''_{yx} in a neighborhood of a point $(x_0, y_0) \in A$, then these derivatives are equal to each other at that point:*

$$f''_{xy}(x_0, y_0) = f''_{yx}(x_0, y_0).$$

Proof. Let (x, y) be a point in an open disc centered at (x_0, y_0). Consider the auxiliary functions

$$h = f(\cdot, y) - f(\cdot, y_0), \qquad g = f(x, \cdot) - f(x_0, \cdot).$$

Then

$$h(x) - h(x_0) = g(y) - g(y_0).$$

By Lagrange's mean-value theorem, there exist two points ξ_0 and η_0 such that

$$h'(\xi_0)(x - x_0) = g'(\eta_0)(y - y_0),$$

where $|x_0 - \xi_0| < |x_0 - x|$, $|y_0 - \eta_0| < |y_0 - y,|$ or

$$(f'_x(\xi_0, y) - f'_x(\xi_0, y_0))(x - x_0) = (f'_y(x, \eta_0) - f'_y(x_0, \eta_0))(y - y_0).$$

Using, once again, the Lagrange's mean-value theorem, we get

$$f''_{xy}(\xi_0, \eta_1)(x - x_0)(y - y_0) = f''_{yx}(\xi_1, \eta_0)(x - x_0)(y - y_0),$$

1 

Karl Hermann Amandus Schwarz (1843–1921), a German mathematician.

i.e.,

$$f''_{xy}(\xi_0, \eta_1) = f''_{yx}(\xi_1, \eta_0).$$

If $(x, y) \to (x_0, y_0)$ then

$$(\xi_0, \eta_1) \to (x_0, y_0), \qquad (\xi_1, \eta_0) \to (x_0, y_0).$$

Using the continuity of f''_{yx} and f''_{xy} at the point (x_0, y_0), we can show that

$$f''_{xy}(x_0, y_0) = f''_{yx}(x_0, y_0).$$

It is clear from theorem (8.1.5) that if all partial derivatives of a function are continuous at a point, then it is possible to change the order of differentiation for any of these derivatives without affecting the final result.

8.1.6 EXAMPLE. If the partial derivatives f'''_{x^2y}, f'''_{xyx} and f'''_{yx^2} are continuous, then:

$$f'''_{x^2y} = f'''_{xyx} = f'''_{yx^2}.$$

8.2 Derivative of Composite Functions

Let $D \subseteq \mathbb{R}^2$ be a domain and $f : D \to \mathbb{R}$ be a continuous function. Consider two functions $u, v : (a, b) \to \mathbb{R}$ possessing derivatives such that $(u(x), v(x)) \in D$, for all $x \in (a, b)$.

8.2.1 THEOREM. *If at least one of the partial derivatives f'_u, f'_v is continuous, then the function $F : (a, b) \to \mathbb{R}$, defined by $F(x) = f(u(x), v(x))$ has derivative which can be expressed as*

$$F'(x) = f'_u(u(x), v(x))\, u'(x) + f'_v(u(x), v(x))\, v'(x).$$

Proof. Let $x_0 \in (a, b)$ be an arbitrary point. Consider, for definiteness, that f'_u is continuous. Define the function

$$G(x) = \begin{cases} \dfrac{f(u(x_0), v(x)) - f(u(x_0), v(x_0))}{v(x) - v(x_0)}, & \text{if } v(x) \neq v(x_0) \\ f'_v(u(x_0), v(x_0)), & \text{if } v(x) = v(x_0) \end{cases}.$$

One can see that the function G is continuous. We have:

$$\frac{F(x)-F(x_0)}{x-x_0} = \frac{f(u(x),v(x))-f(u(x_0),v(x))}{x-x_0} + G(x)\cdot\frac{v(x)-v(x_0)}{x-x_0}.$$

The Lagrange mean-value theorem, gives

$$\frac{F(x)-F(x_0)}{x-x_0} = f'_u(\xi,v(x))\cdot\frac{u(x)-u(x_0)}{x-x_0} + G(x)\cdot\frac{v(x)-v(x_0)}{x-x_0},$$

where $|u(x_0)-\xi| < |u(x_0)-u(x)|$. Taking the limit of both sides yields

$$\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0}$$
$$= f'_u(u(x_0),v(x_0))\cdot u'(x_0) + f'_v(u(x_0),v(x_0))\cdot v'(x_0).$$

The previous formula can be written in the form

$$\frac{d}{dx}(f(u,v)) = \frac{\partial f}{\partial u}\frac{du}{dx} + \frac{\partial f}{\partial v}\frac{dv}{dx}.$$

In general, we have

8.2.2
$$\frac{d}{dx}(f(u_1,\ldots,u_n)) = \sum_{i=1}^{n}\frac{\partial f}{\partial u_i}\frac{du_i}{dx}.$$

Let $u = u(x,y)$, and $v = v(x,y)$ be functions of two variables. We similarly find:

$$\frac{\partial}{\partial x}f(u,v) = \frac{\partial f}{\partial u}(u,v)\cdot\frac{\partial u}{\partial x}(x,y) + \frac{\partial f}{\partial v}(u,v)\cdot\frac{\partial v}{\partial x}(x,y),$$
$$\frac{\partial}{\partial y}f(u,v) = \frac{\partial f}{\partial u}(u,v)\cdot\frac{\partial u}{\partial y}(x,y) + \frac{\partial f}{\partial v}(u,v)\cdot\frac{\partial v}{\partial y}(x,y).$$

8.2.3 REMARK. *One should distinguish between the following notations:*

$$\frac{df}{dx}(u(x)) = f'(u(x))$$

and

$$\frac{d}{dx}f(u(x)) = (f\circ u)'(x) = (f'\circ u)(x)\cdot u'(x).$$

8.2.4 REMARK. *The continuity of the partial derivative is an essential requirement for theorem* (8.2.1).

As an example, consider

$$f(u,v) = \begin{cases} \dfrac{u^2 v}{u^2+v^2}, & \text{if } (u,v) \in \mathbb{R}^2 \setminus (0,0), \\ 0, & \text{if } (u,v) = (0,0), \end{cases}$$

$$u(x) = x, \quad v(x) = x.$$

We have:

$$\frac{\partial f}{\partial u}(0,0) = 0,\ \frac{\partial f}{\partial v}(0,0) = 0,\ F(x) = f(u(x), v(x)) = \frac{x}{2},\ F'(x) = \frac{1}{2},$$

for all $x \in \mathbb{R}$, but

$$\frac{\partial f}{\partial u}(0,0)u'(0) + \frac{\partial f}{\partial v}(0,0)v'(0) = 0.$$

The conditions of theorem (8.2.1) are violated in this case since none of the partial derivatives

$$\frac{\partial f}{\partial u}(u,v) = \begin{cases} \dfrac{2uv^3}{(u^2+v^2)^2}, & \text{if } (u,v) \in \mathbb{R}^2 \setminus (0,0), \\ 0, & \text{if } (u,v) = (0,0), \end{cases}$$

$$\frac{\partial f}{\partial v}(u,v) = \begin{cases} \dfrac{u^2(u^2-v^2)}{(u^2+v^2)^2}, & \text{if } (u,v) \in \mathbb{R}^2 \setminus (0,0), \\ 0, & \text{if } (u,v) = (0,0), \end{cases}$$

is continuous at the point (0,0).

8.3 Homogeneous Functions. Euler's Identity

8.3.1 DEFINITION. *A subset $D \subseteq \mathbb{R}^n$ is called a* cone *if the following condition holds*

$$(\forall x)(\forall t)((x \in D) \wedge (t \in (0,\infty)) \rightarrow tx \in D).$$

Let $D \subseteq \mathbb{R}^n$ be a cone and $\alpha \in \mathbb{R}$.

8.3.2 DEFINITION. *A function $f : D \to \mathbb{R}$ is said to be* **homogeneous** *of degree α if*

$$f(tx) = t^\alpha f(x),$$

for any $t > 0$, and for all $x \in D$.

8.3.3 THEOREM. **(Euler's Identity)** *Let $f : D \to \mathbb{R}$ be a function possessing continuous partial derivatives. The function f is homogeneous degree α if and only if the equality*

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) = \alpha f(x_1, \ldots, x_n)$$

is satisfied for all $x = (x_1, \ldots, x_n) \in D$.

Proof. Necessity. Suppose that f is homogeneous of degree α, i.e.,

$$f(tx_1, \ldots, tx_n) = t^\alpha f(x_1, \ldots, x_n),$$

for all $t > 0$. Differentiating with respect to t we obtain:

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n)\frac{d(tx_i)}{dt} = \alpha t^{\alpha-1} f(x_1, \ldots, x_n),$$

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n) = \alpha t^{\alpha-1} f(x_1, \ldots, x_n).$$

For $t = 1$ we have

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) = \alpha f(x_1, \ldots, x_n).$$

Sufficiency. Suppose that

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) = \alpha f(x_1, \ldots, x_n).$$

Consider the auxiliary function $F : (0, \infty) \to \mathbb{R}$,

$$F(t) = \frac{f(tx_1, \dots, tx_n)}{t^\alpha}.$$

We have:

$$F'(t) = \frac{1}{t^\alpha} \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n) - \alpha \frac{1}{t^{\alpha+1}} f(tx_1, \dots, tx_n)$$

$$= \frac{1}{t^{\alpha+1}} \left(\underbrace{\sum_{i=1}^{n} tx_i \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n) - \alpha f(tx_1, \dots, tx_n)}_{0} \right) = 0,$$

for all $t > 0$. This means that F is constant and therefore $F(t) = F(1)$, i.e.,

$$\frac{f(tx_1, \dots, tx_n)}{t^\alpha} = f(x_1, \dots, x_n),$$

and so

$$f(tx_1, \dots, tx_n) = t^\alpha f(x_1, \dots, x_n),$$

for all $t > 0$. ◆◆◆

8.4 Gradient

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function possessing partial derivatives.

8.4.1 DEFINITION. *The vector*

$$\boxed{\nabla f(x) := \operatorname{grad} f(x) := \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right).}$$

is called the **gradient of the function** f **at the point** x.

8.4.2 DEFINITION. *The mapping* $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$,

$$\boxed{\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).}$$

is called the **gradient of the function** f.

8.4.3 DEFINITION. *The operator*

$$\boxed{\nabla := \operatorname{grad} := \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right).}$$

is called **gradient**, *the* **nabla operator** *or the* **Hamiltonian** `operator`[2] .

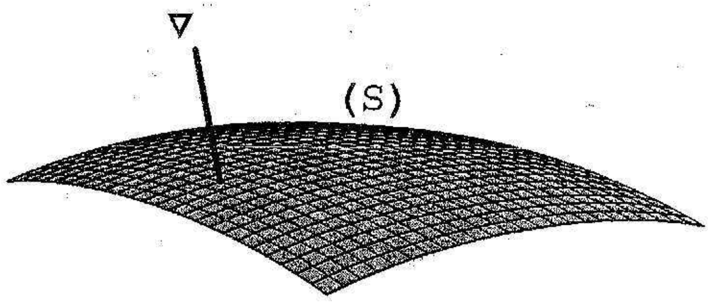
The term nabla originates from the Greek name "$\nu\alpha\beta\lambda\alpha$" of an ancient musical instrument of triangular form.

Consider a surface (S) specified by the equation

$$F(x, y, z) = 0,$$

where $F : \mathbb{R}^3 \to \mathbb{R}$ possesses continuous partial derivatives and let (x_0, y_0, z_0) be a point belonging to (S). The following theorem presents a remarkable property of the vector $\nabla F(x_0, y_0, z_0)$.

8.4.4 THEOREM. *The vector $\nabla F(x_0, y_0, z_0)$ is normal to the surface (S) at the point $(x_0, y_0, z_0) \in (S)$.*



2 

Sir William Rowan Hamilton (1805–1865), an Irish mathematician.

Proof. Let $(\Gamma) \subseteq (S)$ be an arbitrary differentiable curve lying on (S) and given by the parametric equations:

$$\begin{cases} x &= x(t), \\ y &= y(t), \qquad t \in (-1,1), \\ z &= z(t), \end{cases}$$

such that $(x_0, y_0, z_0) = (x(0), y(0), z(0))$. We must prove that the vector $\nabla F(x_0, y_0, z_0)$ is normal to the tangent line of the curve (Γ) at the point (x_0, y_0, z_0).

Since $(\Gamma) \subseteq (S)$

$$F(x(t), y(t), z(t)) = 0,$$

$t \in (-1,1)$, and

$$\frac{d}{dt}(F(x(t), y(t), z(t))) = 0,$$

so,

$$\frac{\partial F}{\partial x} \cdot x'(t) + \frac{\partial F}{\partial y} \cdot y'(t) + \frac{\partial F}{\partial z} \cdot z'(t) = 0,$$

i.e.,

$$\nabla F(x_0, y_0, z_0) \cdot (x'(0), y'(0), z'(0)) = 0$$

$$\iff \nabla F(x_0, y_0, z_0) \perp (x'(0), y'(0), z'(0)).$$

Therefore the vector $\nabla F(x_0, y_0, z_0)$ is normal to the surface (S) at the point $(x_0, y_0, z_0) \in (S)$. ❖

```
In[1]:= (* Mathematica *)

In[2]:= (* Gradient *)

In[3]:= D[f[x1, x2, x3], {{x1, x2, x3}}]

Out[3]= {f^(1,0,0)[x1, x2, x3], f^(0,1,0)[x1, x2, x3], f^(0,0,1)[x1, x2, x3]}

In[4]:= MatrixForm[%]
```

$$\text{Out[4]//MatrixForm=}\quad \begin{pmatrix} f^{(1,0,0)}[x_1,\ x_2,\ x_3] \\ f^{(0,1,0)}[x_1,\ x_2,\ x_3] \\ f^{(0,0,1)}[x_1,\ x_2,\ x_3] \end{pmatrix}$$

```
In[5]:= << Calculus`VectorAnalysis`

In[6]:= SetCoordinates[Cartesian[x, y, z]];

In[7]:= Grad[x y z]

Out[7]= {y z, x z, x y}

In[8]:= SetCoordinates[Spherical[r, theta, phi]];

In[9]:= Grad[r^2 Sin[theta] phi]

Out[9]= {2 phi r Sin[theta], phi r Cos[theta], r}
```

8.5 Directional Derivative

In this section we shall consider real functions f defined on an arbitrary open set $G \subseteq \mathbb{R}^n$. Let $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$, $s \neq 0$.

8.5.1 DEFINITION. *The* **directional derivative** *of a function f at a point $x \in G$ in the direction s (along s) is defined as the limit*

$$\lim_{t \to 0} \frac{f(x + ts) - f(x)}{t\|s\|},$$

provided that it exists, and is denoted by

$$\frac{df}{ds}(x) \qquad \text{or} \qquad \frac{\partial f}{\partial s}(x).$$

8.5.2 THEOREM. *If a function $f : G \to \mathbb{R}$ possesses continuous partial derivatives at a point $x \in G$, then it has a derivative along any unit vector $s \in \mathbb{R}^n$ given by*

8.5.3
$$\frac{\partial f}{\partial s}(x) = \nabla f(x) \cdot s\,.$$

Proof. We have:

$$\frac{\partial f}{\partial s}(x) = \lim_{t \to 0} \frac{f(x+ts) - f(x)}{t} = \lim_{t \to 0} \frac{\frac{d}{dt}\left(f(x+ts) - f(x)\right)}{1}$$

$$= \lim_{t \to 0} \frac{d}{dt} f(x_1 + ts_1, \ldots, x_n + ts_n)$$

$$= \lim_{t \to 0} \sum_{i=1}^{n} s_i \frac{\partial f}{\partial x_i}(x + ts) = \sum_{i=1}^{n} s_i \frac{\partial f}{\partial x_i}(x) = \nabla f(x) \cdot s.$$

8.5.4 REMARK. *Using the notations $\bar{\imath} = (1,0)$ and $\bar{\jmath} = (0,1)$, we have*

$$\frac{\partial f}{\partial \bar{\imath}} = \frac{\partial f}{\partial x}, \qquad \frac{\partial f}{\partial \bar{\jmath}} = \frac{\partial f}{\partial y}.$$

8.6 Lagrange's Mean-value Theorem

Let $D \subseteq \mathbb{R}^n$ be a convex set and $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ be distinct points in D.

8.6.1 THEOREM. *If $f : D \to \mathbb{R}$ possesses continuous partial derivatives on D, then there exists a point $\xi \in D$ such that*

$$f(b) - f(a) = \nabla f(\xi) \cdot (b - a).$$

Proof. Define the auxiliary function $F : [0, 1] \to \mathbb{R}$, such that

$$F(t) = f(a + t(b - a)).$$

By Lagrange's mean-value theorem for functions of one variable, there exists $c \in (0, 1)$ such that

$$F(1) - F(0) = F'(c).$$

But

$$F(0) = f(a), \quad F(1) = f(b)$$

and

$$F'(t) = \sum_{i=1}^{n} (b_i - a_i) \frac{\partial f}{\partial x_i}(a + t(b - a)) = \nabla f(a + t(b - a)) \cdot (b - a).$$

Choosing $\xi = a + c \cdot (b - a)$, we obtain

$$f(b) - f(a) = \nabla f(\xi) \cdot (b - a).$$

Let $D \subseteq \mathbb{R}^n$ be a domain.

8.6.2 THEOREM. *If $f : D \to \mathbb{R}$ is a continuous function and*

$$\frac{\partial f}{\partial x_i}(x) = 0,$$

for all $x \in D$, $i = 1, \ldots, n$, then f is constant on D.

Proof. Let $a, b \in D$. Since D is a domain there exist points $x_0, \ldots, x_m$, $x_0 = a$, $x_m = b$, such that the intervals $[x_{i-1}, x_i]$, $i = 1, \ldots, m$, are subsets of D.

By the Lagrange mean-value theorem (8.6.1) there exist $\xi_i \in (x_{i-1}, x_i)$, $i = 1, \ldots, m$, such that

$$f(x_i) - f(x_{i-1}) = \nabla f(\xi_i) \cdot (x_i - x_{i-1}) = 0,$$

$i = 1, \ldots, m$. Hence

$$f(x_0) = \cdots = f(x_m),$$

or

$$f(a) = f(b),$$

i.e., f is constant. ◆◆◆

8.6.3 EXAMPLE. Let $D = D_1 \cup D_2 \cup D_3$ and $f : D \to \mathbb{R}$,

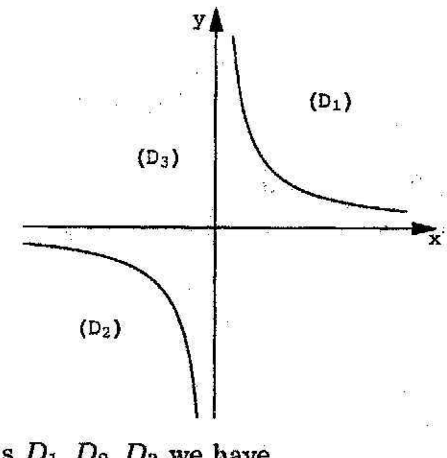
$$f(x,y) = \arctan x + \arctan y - \arctan \frac{x+y}{1-xy},$$

where:

$$D_1 = \{(x,y) \in \mathbb{R}^2 \mid xy > 1,\ y > 0\},$$

$$D_2 = \{(x,y) \in \mathbb{R}^2 \mid xy > 1,\ y < 0\},$$

$$D_3 = \{(x,y) \in \mathbb{R}^2 \mid xy < 1\}.$$



In the domains D_1, D_2, D_3 we have

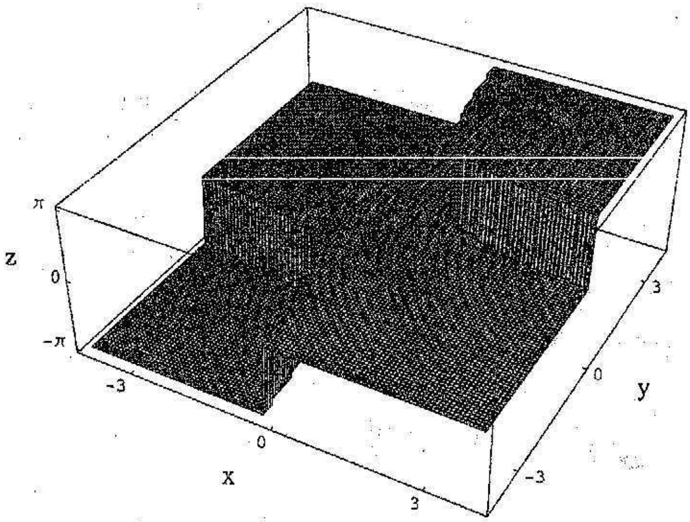
$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0,$$

for all $(x,y) \in D$. By virtue of theorem (8.6.2) it follows that f is a constant on each domain D_1, D_2 and D_3. Hence,

$$f(x,y) = f(0,0) = 0, \quad \text{on } D_3;$$

$$f(x,y) = \lim_{x\to\infty} f(x,1) = \pi, \quad \text{on } D_1;$$

$$f(x,y) = \lim_{x\to-\infty} f(x,-1) = -\pi, \quad \text{on } D_2.$$



8.7 Differential of a Function. Definitions. Properties

Let U and V be real linear spaces.

8.7.1 DEFINITION. *The mapping $T : U \to V$ is said to be a* `linear mapping` *if the following conditions*

- $T(x+y) = T(x) + T(y)$, *(additivity);*
- $T(\alpha x) = \alpha T(x)$, *(homogeneity),*

are satisfied for all $\alpha \in \mathbb{R}$, and for all $x, y \in U$.

8.7.2 DEFINITION. *A function $\|\cdot\| : U \to [0,\infty)$ satisfying the conditions:*

- $\|x\| = 0 \iff x = 0,$

- $\|\alpha x\| = |\alpha|\|x\|$,
- $\|x+y\| \leq \|x\| + \|y\|$,

*for all $\alpha \in \mathbb{R}$, and for all $x, y \in U$, is called a **norm** on U.*

8.7.3 EXAMPLE. The function $\|\cdot\| : \mathbb{R}^2 \to [0,\infty)$, $\|(x,y)\| = \sqrt{x^2+y^2}$, is a norm and called the **Euclidian norm**.

8.7.4 DEFINITION. *A linear space with a norm is called a **linear normed space** or a **normed space**.*

If the function $x \mapsto \|x\|$ is a norm, then $(x,y) \mapsto \|x-y\|$ is a metric. The topology of a normed space $(U, \|\cdot\|)$ is the topology induced by the metric $(x,y) \mapsto \|x-y\|$.

8.7.5 REMARK. *If U is a finite dimensional normed space, then any linear mapping defined on U is continuous.*

8.7.6 REMARK. *If $T : U \to V$ is a linear mapping, then T is continuous if and only if there exists a constant M such that*

$$\|T(x)\| \leq M\|x\|, \qquad \text{for all } x \in U.$$

8.7.7 DEFINITION. *A function $f : U \to V$ is said to be **Fréchet**[3]*

[3] 

Maurice Fréchet (1878–1973), a French mathematician.

differentiable at a point $x_0 \in U$ if there exists a continuous linear mapping $T : U \to V$ such that

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - T(x - x_0)\|}{\|x - x_0\|} = 0.$$

The function T is called the Fréchet differential of f at the point x_0 and is denoted by $df(x_0)$.

So, we have

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - df(x)(h)\|}{\|h\|} = 0.$$

8.7.8 THEOREM. *The differential of a function at a given point is unique.*

Proof. Let T_1 and T_2 be two differentials of a function f at a point $x \in U$ and $h \in U$, $h \neq 0$. We have

$$\frac{\|T_1(h) - T_2(h)\|}{\|h\|} = \frac{\|T_1(th) - T_2(th)\|}{\|th\|}$$

$$\leq \frac{\|T_1(th) - f(x+th) + f(x)\|}{\|th\|} + \frac{\|f(x+th) - f(x) - T_2(th)\|}{\|th\|} \to 0,$$

for $t \to 0$, hence $T_1(h) = T_2(h)$. Furthermore, any linear mapping is zero at the point $h = 0$, hence $T_1 = T_2$. ❖

8.7.9 REMARK. *If a function is differentiable at a point, then it is continuous at the same point.*

8.7.10 REMARK. *If $f : U \to V$ is differentiable on U, then for all $a \in U$ there exists a function $\varphi_a : U \to V$, continuous at the point a such that*

$$f(x) = f(a) + df(a)(x - a) + \varphi_a(x)\|x - a\|,$$

for all $x \in U$ and $\lim_{x \to a} \varphi_a(x) = 0$.

The function φ_a is given by

$$\varphi_a(x) = \begin{cases} \dfrac{f(x) - f(a) - df(a)(x-a)}{\|x - a\|}, & \text{if } x \in U \setminus \{a\}, \\ 0, & \text{if } x = a. \end{cases}$$

8.7.11 EXAMPLE. Let us prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = \sqrt{x^2 + y^2},$$

is not differentiable at the point $(0, 0)$.

Suppose that f is differentiable at the point $(0, 0)$. Then there exists a continuous linear mapping $T : \mathbb{R}^2 \to \mathbb{R}$, i.e., there exist the constants $a, b \in \mathbb{R}$ such that $T(x, y) = ax + by$ and

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2 + y^2} - ax - by}{\sqrt{x^2 + y^2}} = 0,$$

but the function

$$\frac{ax + by}{\sqrt{x^2 + y^2}}$$

has no limit at the point $(0, 0)$. It follows that f is not differentiable at $(0, 0)$.

Let U, V and W be normed spaces.

8.7.12 THEOREM. *If the functions $f : U \to V$, $g : V \to W$ are differentiable, then the function $g \circ f : U \to W$ is also differentiable and*

$$d(g \circ f)(a) = dg\big(f(a)\big) \circ df(a),$$

for all $a \in U$.

8.8 Differential of Functions of Several Variables

Let $D \subseteq \mathbb{R}^n$, $a \in \mathsf{int}(D)$ and $f : D \to \mathbb{R}$ be a continuous function.

8.8.1 THEOREM. *If a function f has continuous partial derivatives at a, then it is differentiable at that point such that*

$$df(a)(h) = \nabla f(a) \cdot h,$$

for all $h \in \mathbb{R}^n$.

Proof. Let $\varepsilon > 0$. Since f has continuous partial derivatives there exists a number $\delta > 0$, such that the inequality $\|\xi\| < \delta$ implies

$$\left|\frac{\partial f}{\partial x_i}(a+\xi) - \frac{\partial f}{\partial x_i}(a)\right| < \frac{\varepsilon}{\sqrt{n}},$$

for $i = 1, \ldots, n$. Let $h \in \mathbb{R}^n \setminus \{0\}$, $\|h\| < \delta$. Using the Lagrange mean-value theorem and the Cauchy-Buniakovski-Schwarz inequality, there exists a point $\xi \in D$, $\|\xi\| < \|h\|$, such that:

$$\begin{aligned}
\frac{|f(a+h) - f(a) - \nabla f(a) \cdot h|}{\|h\|} &= \frac{|(\nabla f(a+\xi) - \nabla f(a)) \cdot h|}{\|h\|} \\
&\leq \frac{\|\nabla f(a+\xi) - \nabla f(a)\| \cdot \|h\|}{\|h\|} \\
&= \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}(a+\xi) - \frac{\partial f}{\partial x_i}(a)\right)^2} < \sqrt{\sum_{i=1}^{n} \frac{\varepsilon^2}{n}} = \varepsilon,
\end{aligned}$$

for all h with $\|h\| < \delta$. Consequently

$$df(a)(h) = \nabla f(a) \cdot h = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) h_i,$$

for all $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$.

Using

$$df(a)(h) = \nabla f(a) \cdot h$$

and choosing

$$h = dx = (dx_1, \ldots, dx_n),$$

we obtain

$$df(a)(dx) = \nabla f(a) \cdot dx$$

or

$$df(a)(dx) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)\, dx_i,$$

which is usually written in the form

$$df(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x}(a)\, dx_i.$$

Using

8.8.2
$$d^k f(a)(h) \stackrel{def}{=} \frac{d}{dt}(d^{k-1}f(a+th)(h))\bigg|_{t=0},$$

we obtain

8.8.3
$$d^k f(a) = \left(dx_1\frac{\partial}{\partial x_1} + \cdots + dx_n\frac{\partial}{\partial x_n}\right)^k f(a)$$

8.8.4 EXAMPLE. If $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives of the second order, then

$$d^2 f = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^2 f = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dx\,dy + \frac{\partial^2 f}{\partial y^2}dy^2.$$

```
In[1]:= (* Mathematica *
         Dt[f] gives the differential "df"   *)

In[2]:= Dt[f[x, y]]

Out[2]= Dt[y] f^(0,1)[x, y] + Dt[x] f^(1,0)[x, y]

In[3]:= IDM := {Dt[Dt[x]] -> 0, Dt[Dt[y]] -> 0}

In[4]:= Dt[Dt[f[x, y]]] /. IDM // Simplify

Out[4]= Dt[y]^2 f^(0,2)[x, y] + 2 Dt[x] Dt[y] f^(1,1)[x, y] + Dt[x]^2 f^(2,0)[x, y]

In[5]:= Dt[Dt[Dt[f[x, y]]]] /. IDM // Simplify

Out[5]= Dt[y]^3 f^(0,3)[x, y] + 3 Dt[x] Dt[y]^2 f^(1,2)[x, y] +
        3 Dt[x]^2 Dt[y] f^(2,1)[x, y] + Dt[x]^3 f^(3,0)[x, y]
```

8.8.5 REMARK. *The Lagrange mean-value formula* (8.6.1) *can be written in the form*

8.8.6
$$f(b) - f(a) = df(\xi)\,(b-a)$$

8.9 Taylor's Formula for Functions of Several Variables

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function possessing $m+1$ order continuous partial derivatives in a convex neighborhood V of a point $a \in \mathbb{R}^n$.

8.9.1 THEOREM. *For an arbitrary point $x \in V$ there exists at least one point $\xi \in V$, $\|a-\xi\| < \|a-x\|$, such that*

$$\begin{aligned} f(x) &= f(a) + \frac{df(a)(x-a)}{1!} + \cdots + \frac{d^m f(a)(x-a)}{m!} \\ &\quad + \frac{d^{m+1} f(\xi)(x-a)}{(m+1)!}. \end{aligned}$$

Proof. Let $x \in V$ and define the function $F : [0,1] \to \mathbb{R}$, such that

$$F(t) = f(a + t(x-a)).$$

By virtue of Taylor's theorem for the functions of one variable, it follows that there exists at least one point $c \in (0,1)$ such that

$$F(1) = F(0) + \frac{F'(0)}{1!} + \cdots + \frac{F^{(m)}(0)}{m!} + \frac{F^{(m+1)}(c)}{(m+1)!}.$$

We have:

$$\begin{aligned} F(0) &= f(a), \quad F(1) = f(x), \\ F'(t) &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a + t(x-a))(x_i - a_i) = df(a + t(x-a))(x-a), \\ F'(0) &= df(a)(x-a), \\ F''(t) &= d^2 f(a + t(x-a))(x-a), \\ F''(0) &= d^2 f(a)(x-a), \end{aligned}$$

and, for higher order, we obtain

$$\begin{aligned} F^{(m)}(t) &= d^m f(a + t(x-a))(x-a), \\ F^{(m)}(0) &= d^m f(a)(x-a). \end{aligned}$$

Choosing $\xi = a + c(x-a)$, we have

$$F^{(m+1)}(c) = d^{m+1} f(\xi)(x-a).$$

8.9.2 REMARK. *Using the ∇ operator, Taylor's formula can be written in the form*

$$\begin{aligned} f(x) &= f(a) + \frac{((x-a)\nabla) f(a)}{1!} + \cdots + \frac{((x-a)\nabla)^m f(a)}{m!} \\ &\quad + \frac{((x-a)\nabla)^{m+1} f(\xi)}{(m+1)!}. \end{aligned}$$

8.9.3 EXAMPLE. For $n = 2$, $m = 1$, Taylor's formula becomes

$$f(x,y) = f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b)$$

$$+\frac{1}{2}\left((x-a)^2\frac{\partial^2 f}{\partial x^2}(\xi,\eta) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(\xi,\eta) + (y-b)^2\frac{\partial^2 f}{\partial y^2}(\xi,\eta)\right).$$

```
(* Mathematica *)

In[4]:=
    Series[f[x, y], {x, a, 1}, {y, b, 1}] // Normal

Out[4]= f[a, b] + (-a + x) f^(1,0)[a, b] + (-b + y) (f^(0,1)[a, b] + (-a + x) f^(1,1)[a, b])
```

8.10 Exercises: Derivatives

8.10.1 [P] Given $a_n \in \{-1, 1\}$, $n \in \mathbb{N}$, find $f\colon [0,1] \to \mathbb{R}$ such that

$$\operatorname{sgn}(f^{(n)}(x)) = a_n, \qquad \forall\, n \in \mathbb{N},\ \ \forall\, x \in [0,1].$$

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8.10.2 [P] Let $0 < a < b$. Calculate the derivative of the function

$$f(x) = \begin{cases} \displaystyle\int_{ax}^{bx} \sin\frac{1}{t}\,\mathrm{d}t, & \text{if } \ x \in \mathbb{R}\setminus\{0\}, \\ 0, & \text{if } \ x = 0, \end{cases}$$

at the point $x = 0$.

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8.10.3 [P] Determine whether the derivative of the function

$$f(x) = \begin{cases} \displaystyle\int_{\sin x}^{x} \sin\frac{1}{t}\,\mathrm{d}t, & \text{if } \ x \in (-1,1)\setminus\{0\}, \\ 0, & \text{if } \ x = 0, \end{cases}$$

is continuous at the point $x = 0$ or not.

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8.10.4 [P] Determine whether the derivative of the function

$$f(x) = \begin{cases} \displaystyle\int_{\ln(1+x)}^{x} \cos\frac{1}{t}\,\mathrm{d}t, & \text{if } \ x \in (-1,1)\setminus\{0\}, \\ 0, & \text{if } \ x = 0, \end{cases}$$

is continuous at the point $x = 0$ or not.

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P
8.10.5 Find the n-th derivative of the function

$$f(x) = \frac{1}{x^2 + 2x\cos\alpha + 1}, \qquad x \in \mathbb{R}, \quad \alpha \in (0, \pi).$$

P
8.10.6 Find the n-th derivative of the functions

$$e^{x\cos\alpha}\cos(x\sin\alpha), \qquad e^{x\cos\alpha}\sin(x\sin\alpha), \qquad x, \alpha \in \mathbb{R}.$$

P
8.10.7 Prove that:

$$(p+q)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} p^{\underline{k}} q^{\underline{n-k}},$$

$n, p, q \in \mathbb{N}$.

P
8.10.8 Find $\frac{dy}{dx}$ for the function defined parametrically

$$\begin{cases} x = R(t - \sin t), \\ y = R(1 - \cos t), \end{cases} \qquad t \in \mathbb{R} \quad (cycloid).$$

P
8.10.9 Find $\frac{dy}{dx}$ for the function defined parametrically

$$\begin{cases} x = \arccos \frac{1}{\sqrt{1+t^2}}, \\ y = \arcsin \frac{t}{\sqrt{1+t^2}}, \end{cases} \qquad t \in \mathbb{R}.$$

8.10.10 (P) Find $\frac{dy}{dx}$ for the function defined implicitly by the equation

$$\tan y = xy.$$

8.10.11 (P) Find $\frac{dy}{dx}$ for the function defined implicitly by the equation

$$\arctan \frac{y}{x} = \ln \sqrt{x^2 + y^2}.$$

8.10.12 (P) Find $\frac{d^2y}{dx^2}$ for the function defined parametrically

$$\begin{cases} x = \arctan t, \\ y = \ln(1 + t^2), \end{cases} \qquad t \in \mathbb{R}.$$

8.10.13 (P) Find $\frac{d^2y}{dx^2}$ for the function defined parametrically

$$\begin{cases} x = \ln t, \\ y = \frac{1}{1-t}, \end{cases} \qquad t \in (0,1).$$

8.10.14 (P) Find $\frac{d^ny}{dx^n}$ for the function defined parametrically

$$\begin{cases} x = \ln t, \\ y = t^k, \end{cases} \qquad t \in (0, \infty),$$

$k, n \in \mathbb{N}^*$.

P
8.10.15 Determine the equation of the tangent line to the curve

$$x^3 + y^3 - xy - 7 = 0$$

at the point $(1, 2)$.

P
8.10.16 Transform the equation

$$x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = 0,$$

by using the new independent variables u and v, such that

$$u = xy, \quad v = \frac{x}{y}.$$

P
8.10.17 Transform the equation

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

by taking

$$u = x + y, \quad v = \frac{y}{x}$$

as the new independent variables and

$$w = \frac{z}{x}$$

as the new function.

P
8.10.18 Transform the Laplace equation $z''_{x^2} + z''_{y^2} = 0$ by taking r and t,

$$\begin{cases} x = r\cos t, \\ y = r\sin t, \end{cases}$$ as new independent variables.

F

8.10.19 Find the nth differential of the function $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \mathrm{e}^{ax+by}$, $n \in \mathbb{N}^*$, $a, b \in \mathbb{R}$.

9

Functional Sequences and Series

Contents

9.1 Functional Sequences

Let $A \subseteq \mathbb{R}$. Consider a `functional sequence` (f_n) of real functions defined on A and $f : A \to \mathbb{R}$.

9.1.1 DEFINITION. *A functional sequence (f_n) is* **convergent** *to f on A, denoted by*

$$f_n \longrightarrow f,$$

if $\lim_{n\to\infty} f_n(x) = f(x)$, for all $x \in A$.

9.1.2 DEFINITION. *A functional sequence (f_n) is said to be* **uniformly convergent** *to f on A, denoted by*

$$f_n \xrightarrow{U} f,$$

if for all $\varepsilon > 0$ there exists r_ε such that the inequality

$$|f_n(x) - f(x)| < \varepsilon$$

holds for all $n \in \mathbb{N}$, $n > r_\varepsilon$ and for all $x \in A$.

9.1.3 THEOREM. *If the functions $f_n : [a,b] \to \mathbb{R}$ are continuous and if $f_n \xrightarrow{U} f$, then f is continuous on the closed interval $[a,b]$.*

Proof. Let $x_0 \in [a,b]$ and $\varepsilon > 0$. The uniform convergence of the sequence (f_n) guarantees the existence of a number r_ε such that

$$|f_{r_\varepsilon}(x) - f(x)| < \frac{\varepsilon}{3}, \quad \forall x \in [a,b].$$

The function f_{r_ε} is continuous on the closed interval $[a,b]$. Then there exists $\delta > 0$ such that

$$|f_{r_\varepsilon}(x) - f_{r_\varepsilon}(x_0)| < \frac{\varepsilon}{3}, \quad \forall x \in (x_0 - \delta, x_0 + \delta) \cap [a,b].$$

We obtain:

$$\begin{aligned}
& |f(x) - f(x_0)| \\
\le\ & |f(x) - f_{r_\varepsilon}(x)| + |f_{r_\varepsilon}(x) - f_{r_\varepsilon}(x_0)| + |f_{r_\varepsilon}(x_0) - f(x_0)| \\
<\ & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

for all $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$, i.e., the function f is continuous at the point x_0.

9.1.4 THEOREM. *Let (f_n) be a uniformly convergent sequence of continuous functions on the interval $[a, b]$. Then the following equality holds*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx.$$

Proof. By virtue of the previous theorem it follows that the limit f of the sequence (f_n) is a continuous function. Therefore, f is integrable. Since the sequence (f_n) is uniformly convergent there exists $\varepsilon > 0$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a},$$

for all $n \geq r_\varepsilon$ and for all $x \in [a, b]$. We have:

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| \leq \int_a^b |f_n(x) - f(x)|\,dx < \int_a^b \frac{\varepsilon}{b-a}\,dx = \varepsilon,$$

for all $n \geq r_\varepsilon$.

By virtue of the previous theorem we deduce:

9.1.5 COROLLARY. *A uniformly convergent series of continuous functions on a closed interval $[a, b]$ can be integrated termwise.*

9.1.6 THEOREM. *Let (f_n) be a sequence of functions in $C^1[a, b]$. If (f_n) converges to a function f and the sequence of the derivatives (f_n') is uniformly convergent to a function g, then there exists the derivative f' and $f' = g$, i.e.,*

$$\lim_{n\to\infty} f_n' = \left(\lim_{n\to\infty} f_n\right)'.$$

Proof. The Newton[1] - Leibniz formula gives

$$\int_a^x f_n'(t)\,dt = f_n(x) - f_n(a),$$

for all $x \in [a, b]$ and for all $n \in \mathbb{N}$. Therefore, for $n \to \infty$, by making use of the previous theorem, we obtain

$$\int_a^x g(t)\,dt = f(x) - f(a),$$

for all $x \in [a, b]$. It follows that f is an antiderivative of g, and consequently f has derivative such that $f' = g$.

9.1.7 THEOREM. **(Weierstrass)** *If* $\sum a_n$ *is a convergent series of numbers, and the functional sequence of real functions* (f_n), *satisfies the inequalities*

$$|f_n(x)| \leq a_n, \quad (\forall x \in A, \quad \forall n \in \mathbb{N}),$$

then the functional series $\sum f_n$ *is uniformly and absolutely convergent on* A.

Proof. We use the inequality

$$|f_{n+1}(x) + \cdots + f_{n+p}(x)| \leq a_{n+1} + \cdots + a_{n+p},$$

for all $n, p \in \mathbb{N}$, $x \in A$, and the Cauchy's criterion 5.6.1.

1

Sir Isaac Newton
(1643–1727)
A famous English
mathematician and
physicist

9.2 Power Series

Let (a_n) be a sequence of complex numbers and $z_0 \in \mathbb{C}$.

9.2.1 DEFINITION. *The series*

$$\sum_{n\geq 0} a_n(z-z_0)^n$$

is called a **power series** *in powers of* $z-z_0$ *and coefficients* (a_n).

Making the substitution $x = z - z_0$ we obtain a power series in power of x. Some features of such series will be given.

9.2.2 THEOREM. **(Abel)** *If a power series* $\sum a_n x^n$ *converges at a point* $x_0 \neq 0$, *then it converges uniformly and absolutely in every closed disc*

$$\overline{D}\,(0,r) = \left\{x \in \mathbb{C} \;\middle|\; |x| \leq r\right\} \qquad (0 < r < |x_0|).$$

Proof. Let $x \in D$. Since the series $\sum a_n x_0^n$ is convergent, there exists a number M such that

$$|a_n x_0^n| \leq M, \quad \forall n \in \mathbb{N}.$$

We have:

$$\sum |a_n x^n| = \sum |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M \sum \left|\frac{r}{x_0}\right|^n,$$

for all $x \in D$. By Weierstrass's theorem, we deduce that the series $\sum a_n x^n$ is uniformly and absolutely convergent in the domain D. ❖

9.2.3 DEFINITION. *The element*

$$R = \frac{1}{\overline{\lim}\sqrt[n]{|a_n|}} \in [0,\infty],$$

on condition $\frac{1}{0} = \infty$, *and* $\frac{1}{\infty} = 0$, *is called the* **radius of convergence** *of the series* $\sum a_n x^n$. *The set*

$$D(0,R) = \left\{x \in \mathbb{C} \;\middle|\; |x| < R\right\}$$

is called the **disc of convergence** *of the series.*

9.2.4 THEOREM. *Let R be the radius of convergence of the series $\sum a_n x^n$. Then:*

(1) *if* $R = 0$, *then* $\sum a_n x^n$ *is convergent only at the point* $x = 0$;
(2) *if* $0 < R < \infty$, *then* $\sum a_n x^n$ *is convergent in the set* $D(0, R)$;
(3) *if* $R = \infty$, *then* $\sum a_n x^n$ *is convergent in* $\mathbb{C}$.

9.2.5 REMARK. *When the limit* $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ *exists, then the radius of convergence* R *of the series* $\sum a_n x^n$ *can be computed by the formula*

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$

9.2.6 REMARK. *Every power series possessing a radius of convergence* $R > 0$ *is uniformly and absolutely convergent in all closed discs* $\overline{D}(0, r)$, $r < R$.

One can show that any power series is uniformly and absolutely convergent in any compact set of the disc of convergence.

9.2.7 THEOREM. **(Abel-Tauber)**[2] *If* $0 < R < \infty$ *is the radius of convergence of a series* $\sum a_n x^n$ *and if the series of numbers* $\sum a_n R^n$ *is convergent, then*

$$\lim_{\substack{x\to R\\ x<R}} \sum a_n x^n = \sum a_n R^n.$$

9.2.8 REMARK. *Since the series*

$$\sum a_n x^n, \quad \sum n a_n x^{n-1}, \quad \text{and} \quad \sum \frac{a_n}{n+1} x^{n+1},$$

have the same radius of convergence, any power series can be formally differentiated and integrated term-by-term in the domain of convergence.

[2] Alfred Tauber (1866–1942), an Austrian mathematician.

Functions of complex variable which can be expanded into convergent power series are called `analytic functions`. They are studied as a branch of advanced mathematics called *the theory of analytic functions* or *the theory of functions of complex variables*.

Some of these functions will be presented. Each of the functions e^z, $\cos z$ and $\sin z$ of the complex variable z can be expanded in terms of a power series

$$
\begin{aligned}
e^z &:= 1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots \\
\cos z &:= 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots \\
\sin z &:= z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots
\end{aligned}
$$

It is obvious that for any complex number z we have

$$
\begin{aligned}
e^{iz} &= \cos z + i\sin z, \\
\cos z &= \frac{e^{iz}+e^{-iz}}{2}, \\
\sin z &= \frac{e^{iz}-e^{-iz}}{2i}
\end{aligned}
$$

(Euler's formulas). [3]

[3] 

Leonard Euler
(1707–1783)
a great Swiss
mathematician.

9.3 Exercises: Power Series

Determine the radius of convergence of each of the following power series:

9.3.1 (P) $\sum \sqrt[n]{n!}\, x^n$;

9.3.2 (P) $\sum \dfrac{(n!)^n \mathrm{e}^{n^2}}{n^{n^2+n/2}}\, x^n$;

9.3.3 (P) $\sum \left(1+\dfrac{1}{n^2}\right)^{n^4} x^{n^2}$;

9.3.4 (P) $\sum x^n \sin n$.

Expand each of the following functions into Maclaurin's series:

9.3.5 (P) $f(x) = \frac{1}{\sqrt{1-x^2}}$ $\quad x \in (-1,1)$.

9.3.6 (P) $f(x) = \arcsin x,$ $\quad x \in [-1,1]$;

9.3.7 (P) $f(x) = \arctan x,$ $\quad x \in [-1,1]$;

9.3.8 (P) $f(x) = \ln \frac{1+x}{1-x},$ $\quad x \in (-1,1)$;

9.3.9 (P) $f(x) = x \arctan x - \ln\sqrt{1+x^2},$ $\quad x \in [-1,1]$;

9.3.10 (P) $f(x) = \displaystyle\int_0^x \frac{\arctan t}{t}\,\mathrm{d}t, \qquad x \in [-1, 1];$

9.3.11 (P) Expand the function

$$f(x) = \ln(x^2 - 2x + 2)$$

into a Taylor's series in powers of $x - 1$.

Find the sum of each of the following series:

9.3.12 (P) $\displaystyle\sum_{n=0}^{\infty} \frac{\mathrm{e}^{nx}}{n!}, \qquad x \in \mathbb{R}.$

9.3.13 (P) $\displaystyle\sum_{n=1}^{\infty} n^2 x^n, \qquad x \in (-1, 1);$

9.3.14 (P) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{pn}}{(pn)!}, \qquad p \in \mathbb{N}^* \quad x \in \mathbb{R};$

9.3.15 (P) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}, \qquad x \in [-1, 1];$

9.3.16 (P) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n+k)n!}, \qquad k \in \mathbb{N}^*,\ x \in \mathbb{R}.$

9.3.17 (P) Evaluate the integral $\displaystyle\int_0^1 \frac{\ln(1+x)}{x}\,\mathrm{d}x.$

P
9.3.18 Prove that:

$$\sum_{n=0}^{\infty} r^n \cos nx = \frac{1 - r\cos x}{1 - 2r\cos x + r^2},$$

$$\sum_{n=0}^{\infty} r^n \sin nx = \frac{r\sin x}{1 - 2r\cos x + r^2},$$

$|r| < 1, \quad x \in \mathbb{R}.$

9.4 Trigonometric Series. Fourier Series

Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 1}$ be two sequences of real numbers.

9.4.1 DEFINITION. *The series*

$$\frac{a_0}{2} + \sum_{n\geq 1}(a_n \cos nX + b_n \sin nX)$$

is called a `trigonometric series` *of the coefficients* (a_n) *and* (b_n).

Let $S : \mathbb{R} \to \mathbb{R}$ be a periodic function with period 2π. In order to find two sequences of real numbers $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 1}$ such that

$$S(x) = \frac{a_0}{2} + \sum_{n\geq 1}(a_n \cos nx + b_n \sin nx),$$

holds for $x \in [-\pi, \pi]$, we need the following theorem.

9.4.2 THEOREM. *If the series*

$$\frac{a_0}{2} + \sum_{n\geq 1}(a_n \cos nX + b_n \sin nX)$$

is uniformly convergent on the interval $[-\pi, \pi]$, *then the function* $S : \mathbb{R} \to \mathbb{R}$, *such that*

$$S(x) = \frac{a_0}{2} + \sum_{n\geq 1}(a_n \cos nx + b_n \sin nx)$$

is periodic with period 2π *and*

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} S(x)\cos kx\, dx, \quad k \geq 0,$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} S(x)\sin kx\, dx, \quad k \geq 1.$$

Proof. Let

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx).$$

Then

$$S_n(x+2\pi) = S_n(x) \qquad (\forall x \in \mathbb{R},\ n \in \mathbb{N}),$$

and for $n \to \infty$, we obtain

$$S(x+2\pi) = S(x) \qquad (\forall x \in \mathbb{R}).$$

That is S has the period 2π. For $k \in \mathbb{N}$, the series

$$S(x)\cos kx = \frac{a_0}{2}\cos kx + \sum_{n\geq 1}(a_n \cos nx \cos kx + b_n \sin nx \cos kx)$$

is uniformly convergent on the interval $[-\pi, \pi]$. Therefore it can be integrated term-by-term. As a result, we get

$$\int_{-\pi}^{\pi} S(x)\cos kx\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos kx\,dx$$

$$+\sum_{n\geq 1}\left(a_n \int_{-\pi}^{\pi} \cos nx \cos kx\,dx + b_n \int_{-\pi}^{\pi} \sin nx \cos kx\,dx\right),$$

Using

$$\int_{-\pi}^{\pi} \cos nx \cos kx\,dx = \begin{cases} 0, & \text{if } n \neq k, \\ \pi, & \text{if } n = k \neq 0, \\ 2\pi, & \text{if } n = k = 0; \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \cos kx\,dx = 0,$$

we obtain

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} S(x)\cos kx\,dx, \quad k \geq 0.$$

Likewise, we obtain

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} S(x)\sin kx\,dx, \quad k \geq 1$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period 2π, Riemann integrable on the interval $[-\pi, \pi]$.

9.4.3 DEFINITION. *The trigonometric series*

$$\frac{a_0}{2} + \sum_{n\geq 1}(a_n \cos nX + b_n \sin nX),$$

where the coefficients a_n *and* b_n *are given by*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n \geq 0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n \geq 1,$$

is called the `Fourier`[4] `series` *of the function* f, *and we write*

$$f(x) \sim \frac{a_0}{2} + \sum_{n\geq 1}(a_n \cos nx + b_n \sin nx).$$

The set of the Fourier coefficients of a function is called the `spectrum` of that function. The terms of a Fourier series can be written in the form of harmonics

$$a_k \cos kx + b_k \sin kx = A_k \cos\left(\frac{k\pi}{\omega}x - \varphi_k\right)$$

having the amplitude A_k, the frequency $\frac{k\pi}{\omega}$ and the initial phase φ_k.

The Fourier series of a function f can be written also in complex form as

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

4

Jean Baptiste Joseph de Fourier (1768–1830), a French mathematician.

where the coefficients c_n are given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{e}^{-int}\,dt.$$

9.4.4 REMARK. *The Fourier series associated with a function f can be convergent or divergent. In some cases it converges to f.*

9.4.5 REMARK. *If f is an even function, then its Fourier coefficients are given by*

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \quad n \geq 0, \\ b_n &= 0, \quad n \geq 1; \end{aligned}$$

and if f is an odd function, then its Fourier coefficients are given by

$$\begin{aligned} a_n &= 0, \quad n \geq 0; \\ b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx, \quad n \geq 1. \end{aligned}$$

9.4.6 REMARK. *There exist trigonometric series which are not Fourier series.*

As for example, the series

$$\sum_{n\geq 1} \frac{\sin nx}{\sqrt{n}}$$

is not a Fourier series because it does not satisfy the Parseval equality (cf. 9.4.20).

9.4.7 DEFINITION. *A function possessing a finite number of points of discontinuity of the first kind and no points of discontinuity of the second kind is said to be* `piecewise continuous`.

9.4.8 DEFINITION. *A function f is said to be* `piecewise smooth` *if both f and f' are piecewise continuous.*

9.4.9 THEOREM. **(Dirichlet)** *The nth partial sum S_n of the Fourier series of f, satisfies the equality*

$$S_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{f(x+t)+f(x-t)}{2} \cdot \frac{\sin(2n+1)\frac{t}{2}}{\sin\frac{t}{2}}\,dt.$$

Proof. We have:

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos kx + b_k \sin kx) = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(t)}{2}\,dt$$

$$+\sum_{k=1}^{n}\left(\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt\right)\cos kx + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt\right)\sin kx\right)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{2}+\sum_{k=1}^{n}(\cos kt\cos kx + \sin kt \sin kx)\right)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{2}+\sum_{k=1}^{n}\cos k(t-x)\right)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(2n+1)\frac{t-x}{2}}{2\sin\frac{t-x}{2}}\,dt$$

$$\overset{t=x+u}{=} \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(x+u)\frac{\sin(2n+1)\frac{u}{2}}{2\sin\frac{u}{2}}\,du$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\frac{\sin(2n+1)\frac{u}{2}}{2\sin\frac{u}{2}}\,du$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x+u)\frac{\sin(2n+1)\frac{u}{2}}{2\sin\frac{u}{2}}\,du + \frac{1}{\pi}\int_{0}^{\pi} f(x+u)\frac{\sin(2n+1)\frac{u}{2}}{2\sin\frac{u}{2}}\,du$$

$$= \frac{1}{\pi}\int_{0}^{\pi} f(x-v)\frac{\sin(2n+1)\frac{v}{2}}{2\sin\frac{v}{2}}\,dv + \frac{1}{\pi}\int_{0}^{\pi} f(x+u)\frac{\sin(2n+1)\frac{u}{2}}{2\sin\frac{u}{2}}\,du$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{f(x+t)+f(x-t)}{2} \cdot \frac{\sin(2n+1)\frac{t}{2}}{\sin\frac{t}{2}}\,dt.$$

9.4.10 REMARK. *If $f=1$, then $S_n = 1$. By Dirichlet formula, we obtain the identity*

$$\frac{1}{\pi}\int_0^{\pi}\frac{\sin(2n+1)\frac{t}{2}}{\sin\frac{t}{2}}\,dt = 1.$$

9.4.11 THEOREM. *If the function $g : \mathbb{R} \to \mathbb{R}$, with period $T > 0$ and the function $f : [0, T] \to \mathbb{R}$ are integrable, then the following equality holds*

$$\lim_{n\to\infty} \int_0^T f(x)g(nx)\,dx = \frac{1}{T}\int_0^T f(x)\,dx \cdot \int_0^T g(x)\,dx.$$

Proof. Consider the case when $g \geq 0$. We denote:

$$m_k = \inf\{f(x) \mid x \in [k\frac{T}{n}, (k+1)\frac{T}{n}]\},$$

$$M_k = \sup\{f(x) \mid x \in [k\frac{T}{n}, (k+1)\frac{T}{n}]\},$$

$k = 1, \ldots, n-1$. There exist $f_k \in [m_k, M_k]$, $k = 0, \ldots, n-1$, such that;

$$\int_0^T f(x)g(nx)\,dx = \frac{1}{n}\int_0^{nT} f(\frac{t}{n})g(t)\,dt$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}\int_{kT}^{(k+1)T} f(\frac{t}{n})g(t)\,dt$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} f_k \int_{kT}^{(k+1)T} g(t)\,dt = \sum_{k=0}^{n-1} f_k \int_0^T g(t)\,dt$$

$$= \left(\frac{T}{n}\sum_{k=0}^{n-1} f_k\right)\cdot\left(\frac{1}{T}\int_0^T g(t)\,dt\right) \to \left(\int_0^T f(t)\,dt\right)\cdot\left(\frac{1}{T}\int_0^T g(t)\,dt\right).$$

Now, we consider the case when g does not satisfy condition $g \geq 0$; but g is integrable and hence it is bounded. Then there exists a constant C such that $g + C \geq 0$. As a consequence, we obtain

$$\lim_{n\to\infty}\int_0^T f(x)(g(nx) + C)\,dx = \frac{1}{T}\int_0^T f(x)\,dx \cdot \int_0^T (g(x) + C)\,dx,$$

i.e.,

$$\lim_{n\to\infty}\int_0^T f(x)g(nx)\,dx + C\int_0^T f(x)\,dx$$

$$= \frac{1}{T}\int_0^T f(x)\,dx \cdot \int_0^T g(x)\,dx + C\int_0^T f(x)\,dx,$$

hence

$$\lim_{n\to\infty}\int_0^T f(x)g(nx)\,dx = \frac{1}{T}\int_0^T f(x)\,dx \cdot \int_0^T g(x)\,dx.$$

The following result can be obtained by using theorem 9.4.11, when $g(x) = \sin x$.

9.4.12 LEMMA. **(Riemann)**[5] *If a function $h : [a, b] \to \mathbb{R}$ is piecewise continuous, then*

$$\lim_{n\to\infty}\int_a^b h(x)\sin nx\,dx = 0.$$

9.4.13 EXAMPLE. Let us calculate the limit

$$\lim_{n\to\infty}\int_0^1 \frac{\cos^2 n\pi x}{1+x^2}\,dx.$$

By theorem 9.4.11, we obtain

$$\lim_{n\to\infty}\int_0^1 \frac{\cos^2 n\pi x}{1+x^2}\,dx = \int_0^1 \cos^2 \pi x\,dx \cdot \int_0^1 \frac{1}{1+x^2}\,dx$$

5

Bernhard Riemann (1826–1866), a distinguished German mathematician.

$$= \frac{1}{2} \arctan x \Big|_0^1 = \frac{\pi}{8}.$$

9.4.14 THEOREM. **(Dirichlet)** *If a function f is piecewise smooth, then its Fourier series converges to*

$$\frac{f(x-0)+f(x+0)}{2},$$

for all $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$. By the Dirichlet formula we get

$$S_n(x) - \frac{f(x+0)+f(x-0)}{2}$$

$$= \frac{1}{\pi}\int_0^\pi \frac{f(x+t)+f(x-t)}{2} \cdot \frac{\sin(2n+1)\frac{t}{2}}{\sin\frac{t}{2}}\, dt - \frac{f(x+0)+f(x-0)}{2}$$

$$= \frac{1}{\pi}\int_0^\pi \left(\frac{f(x+t)-f(x+0)}{2} + \frac{f(x-t)-f(x-0)}{2}\right) \cdot \frac{\sin(2n+1)\frac{t}{2}}{\sin\frac{t}{2}}\, dt.$$

The function

$$g(t) = \begin{cases} \left(\dfrac{f(x+t)-f(x+0)}{2} + \dfrac{f(x-t)-f(x-0)}{2}\right) \cdot \dfrac{1}{\sin\frac{t}{2}}, & t \in (0,\pi], \\ f'(x+0)+f'(x-0), & t = 0, \end{cases}$$

satisfies the conditions of Riemann's lemma. Therefore

$$\lim_{n\to\infty}\left(S_n(x) - \frac{f(x+0)+f(x-0)}{2}\right) = 0.$$

9.4.15 EXAMPLE. Let expand the function $x \mapsto \frac{x}{2}$, $x \in (-\pi,\pi)$, as a Fourier series.

Consider the function $f : \mathbb{R} \to \mathbb{R}$, with period 2π

$$f(x) = \begin{cases} 0, & x = -\pi, \\ \dfrac{x}{2}, & x \in (-\pi,\pi), \\ 0, & x = \pi. \end{cases}$$

odd

Since the function f is ~~even~~ we get:

$$a_n = 0, \quad n \in \mathbb{N},$$

$$b_n = \frac{2}{\pi} \int_0^\pi \frac{x}{2} \sin nx \, dx = \frac{(-1)^{(n+1)}}{n}, \quad n \in \mathbb{N}^*.$$

From the equality

$$f(x) = \frac{f(x+0) + f(x-0)}{2}, \quad x \in \mathbb{R},$$

using Dirichlet's theorem, we obtain

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad x \in \mathbb{R},$$

so

$$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad x \in (-\pi, \pi).$$

9.4.16 REMARK. *If a function $f \in C^1(\mathbb{R})$ is periodic, then its Fourier series converges uniformly to f.*

9.4.17 REMARK. *If $f : \mathbb{R} \to \mathbb{R}$ has period $T > 0$, then using the substitution $x = \frac{Ty}{2\pi}$, we obtain the function $g(y) = f(\frac{Ty}{2\pi})$, with period 2π. Consequently, we have:*

$$g(y) \sim \frac{a_0}{2} + \sum_{n \geq 1} (a_n \cos ny + b_n \sin ny),$$

hence

$$f(x) \sim \frac{a_0}{2} + \sum_{n \geq 1} (a_n \cos n\frac{2\pi x}{T} + b_n \sin n\frac{2\pi x}{T}).$$

9.4.18 REMARK. *The Fourier series associated to any integrable function $f : [a, b] \to \mathbb{R}$ is*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi}{b-a} x + b_n \sin \frac{2n\pi}{b-a} x \right),$$

where the coefficients are given by

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi}{b-a}x\,dx, \quad n \in \mathbb{N},$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi}{b-a}x\,dx, \quad n \in \mathbb{N}^*.$$

9.4.19 REMARK. *In order to expand a function $f : (0,\pi) \to \mathbb{R}$ as a Fourier series of sines (or cosines) we define the function f_s (or f_c),*

$$f_s(x) = \begin{cases} 0, & x = -\pi, \\ -f(-x), & x \in (-\pi, 0), \\ 0, & x = 0, \\ f(x), & x \in (0,\pi), \\ 0, & x = \pi, \end{cases} \qquad f_c(x) = \begin{cases} 0, & x = -\pi, \\ f(-x), & x \in (-\pi, 0), \\ 0, & x = 0, \\ f(x), & x \in (0,\pi), \\ 0, & x = \pi. \end{cases}$$

Note that f_s is an odd function (f_c is an even function), hence it is expanded in Fourier series of sines (cosines respectively). The series converges to f on the interval $(0,\pi)$.

```
(* Mathematica *)

(* FourierTrigSeries[expr, t, k] gives the kth
    order Fourier trigonometric series approximation to the
    periodic function of t that is equal to  expr for -1/2 <=
  t <= 1/2, and has a period of 1.*)

(* FourierTrigSeries[expr, t, k, FourierParameters -> {a, b}] gives
    the kth order Fourier trigonometric series approximation to the
    periodic function of t that is equal to expr for -1/(2 Abs[b]) <=
  t <= 1/(2 Abs[b]), and has a period of 1/Abs[b].*)

(* FourierSeries[expr, t, k, FourierParameters -> {a, b}] gives the
    kth order Fourier exponential series approximation to the
    periodic function of t that is equal to expr for -1/(2 Abs[b]) <=
  t <= 1/(2 Abs[b]), and has a period of 1/Abs[b].*)

(* FourierSeries[expr, t, k] gives the kth order Fourier exponential
    series approximation to the periodic function of t that is
    equal to expr for -1/2 <= t <= 1/2, and has a period of 1. *)
```

In[1]:= `<< Calculus`FourierTransform``

In[2]:= `FourierTrigSeries[t / 2, t, 5, FourierParameters -> {0,` $\frac{1}{2\pi}$`}] // Expand`

Out[2]= $\mathrm{Sin}[t] - \frac{1}{2}\mathrm{Sin}[2\,t] + \frac{1}{3}\mathrm{Sin}[3\,t] - \frac{1}{4}\mathrm{Sin}[4\,t] + \frac{1}{5}\mathrm{Sin}[5\,t]$

In[3]:= `FourierTrigSeries[t - Floor[t],`
`t, 8, FourierParameters -> {0,` $\frac{1}{2}$`}] // Expand`

Out[3]= $\frac{1}{2} - \frac{\mathrm{Sin}[2\,\pi\,t]}{\pi} - \frac{\mathrm{Sin}[4\,\pi\,t]}{2\,\pi} - \frac{\mathrm{Sin}[6\,\pi\,t]}{3\,\pi} - \frac{\mathrm{Sin}[8\,\pi\,t]}{4\,\pi}$

In[4]:= `f[t_] := If[t < 0, -1, 1]`

In[5]:= `FourierTrigSeries[f[t], t, 8, FourierParameters -> {0,` $\frac{1}{2}$`}] // Expand`

Out[5]= $\frac{4\,\mathrm{Sin}[\pi\,t]}{\pi} + \frac{4\,\mathrm{Sin}[3\,\pi\,t]}{3\,\pi} + \frac{4\,\mathrm{Sin}[5\,\pi\,t]}{5\,\pi} + \frac{4\,\mathrm{Sin}[7\,\pi\,t]}{7\,\pi}$

9.4.20 THEOREM. **(Parseval)**[6] *If a function $f : [-\pi, \pi] \to \mathbb{R}$ and the square of its modulus are integrable, then its Fourier coefficients satisfy the equality*

$$\frac{a_0^2}{2} + \sum_{n\geq 1} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, dx.$$

9.4.21 EXAMPLE. Let us calculate the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Using the Fourier expansion

$$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad x \in (-\pi, \pi),$$

we obtain

$$\int_{-\pi}^{\pi} \frac{x^2}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{-\pi}^{\pi} \sin^2 nx\, dx,$$

i.e.,

$$\frac{\pi^3}{6} = \sum_{n=1}^{\infty} \frac{\pi}{n^2},$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

[6] Marc-Antoine Parseval des Chênes (1755–1836), a French mathematician.

9.5 Exercises: Fourier Series

Prove each of the following formulas:

9.5.1 [P] $$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \qquad x \in (-\pi, \pi).$$

9.5.2 [P] $$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \qquad x \in (0, 2\pi).$$

9.5.3 [P] $$1 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}, \qquad x \in (0, \pi).$$

9.5.4 [P] $$x - \lfloor x \rfloor = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n}, \qquad x \in \mathbb{R} \setminus \mathbb{Z},$$
($\lfloor x \rfloor$ is the "integer part" of x.)

9.5.5 [P] $$\arcsin(\cos x) = \frac{4}{\pi} \sum_{n \geq 0} \frac{\cos(2n+1)x}{(2n+1)^2}, \qquad x \in [-\pi, \pi].$$

9.5.6 [P] $$\sum_{n \geq 0} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

9.5.7 [P] $$\frac{\pi^2 - 3x^2}{12} = \sum_{n \geq 1} (-1)^{n+1} \frac{\cos nx}{n^2}, \qquad x \in [-\pi, \pi].$$

9.5.8 [P] $$\cos ax = \frac{2 \sin a\pi}{\pi} \left(\frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^n a}{a^2 - n^2} \cos nx \right), \; a \in \mathbb{C} \setminus \mathbb{Z}, \; x \in [-\pi, \pi].$$

9.5.9 P

$$\cosh ax = \frac{2\sinh a\pi}{\pi}\left(\frac{1}{2a} + \sum_{n\geq 1}(-1)^n \frac{a\cos nx}{a^2+n^2}\right), \qquad a \in \mathbb{R}\setminus\mathbb{Z},$$
$x \in [-\pi, \pi]$.

9.5.10 P

$$\frac{1}{\sin x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x - n\pi}, \qquad x \in \mathbb{R}\setminus\pi\mathbb{Z}.$$

9.5.11 P

$$\cot t = \sum_{n=-\infty}^{\infty} \frac{1}{t - n\pi}, \qquad t \in \mathbb{R}\setminus\pi\mathbb{Z}.$$

9.5.12 P

$$\sum_{n=0}^{\infty} \frac{\cos nx}{n!} = \mathrm{e}^{\cos x}\cos(\sin x), \quad \sum_{n=0}^{\infty} \frac{\sin nx}{n!} = \mathrm{e}^{\cos x}\sin(\sin x), \ x \in \mathbb{R}.$$

9.5.13 P

$$\sum_{n\geq 0} r^n \cos nx = \frac{1 - r\cos x}{1 - 2r\cos x + r^2},$$
$$\sum_{n\geq 1} r^n \sin nx = \frac{r\sin x}{1 - 2r\cos x + r^2}, \qquad -1 < r < 1,\ x \in \mathbb{R}.$$

9.5.14 P

$$\ln(1 - 2r\cos x + r^2) = -2\sum_{n\geq 1} \frac{r^n}{n}\cos nx, \ -1 < r < 1, \ x \in \mathbb{R}.$$

9.5.15 P

$$\ln\left(2\cos\frac{x}{2}\right) = \sum_{n\geq 1}(-1)^{n+1}\frac{\cos nx}{n}, \qquad x \in (-\pi, \pi).$$

9.5.16 P

$$\ln\left(2\sin\frac{x}{2}\right) = -\sum_{n\geq 1}\frac{\cos nx}{n}, \qquad x \in (0, 2\pi).$$

9.5.17 P

$$\ln\tan\frac{x}{2} = -2\sum_{n\geq 0}\frac{\cos(2n+1)x}{2n+1}, \qquad x \in (0, \pi).$$

9.5.18 P

$$\int_0^{\pi} \ln\sin x \,\mathrm{d}x = -\pi\ln 2.$$

9.5.19 [P] $$\int_0^{\pi/2} \ln \sin x \, \mathrm{d}x = -\frac{\pi}{2} \ln 2.$$

9.5.20 [P] $$\int_0^{\pi/2} \ln^2 \sin x \, \mathrm{d}x = \frac{\pi^3}{24} + \frac{\pi}{2} \ln^2 2.$$

9.5.21 [P] (Rayleigh-Ritz) Let $f \in C^1[0, \pi]$ such that $f(0) = f(\pi)$. Prove that

$$\int_0^{\pi} f^2(x) \, \mathrm{d}x \le \int_0^{\pi} \left(f'(x)\right)^2 \, \mathrm{d}x,$$

with equality iff $f(x) = C \sin x$.

10

Implicit Functions

Contents

10.1 Existence Theorems for Implicit Functions

Let $A, B \subseteq \mathbb{R}$ let $F : A \times B \to \mathbb{R}$. Consider the equation

10.1.1
$$F(x, y) = 0.$$

10.1.2 DEFINITION. *If, for all $x \in A$, equation (10.1.1) has a unique solution with respect to y, then a function $f : A \to B$ satisfying the equality $F(x, f(x)) = 0$, for all $x \in A$, is called an implicit function defined by equation (10.1.1) or, briefly,* `implicit function`.

The set of all points $(x, y) \in \mathbb{R}^2$ satisfying equation (10.1.1) represents the graph of the function f.

10.1.3 EXAMPLE. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$F(x, y) = y^3 - x + 1.$$

In this case, the implicit function is

$$f(x) = \sqrt[3]{x - 1}.$$

10.1.4 EXAMPLE. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$F(x, y) = y^5 + 2x^2y^3 + y + 7.$$

In this case, the implicit function cannot be represented explicitly.

Let $x_0, y_0 \in \mathbb{R}$, $U \in \mathcal{V}_{x_0}$, $V \in \mathcal{V}_{y_0}$ and $F : U \times V \to \mathbb{R}$ be a continuous function.

10.1.5 THEOREM. *If the function F satisfies the conditions:*

(1) $F(x_0, y_0) = 0$;

(2) *there exist* $\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}$ *continuous on* $U \times V$;

(3) $\dfrac{\partial F}{\partial y}(x_0, y_0) \neq 0$,

then there exist $U_0 \in \mathcal{V}_{x_0}$, $V_0 \in \mathcal{V}_{y_0}$, and a function $f : U_0 \to V_0$ such that:

(a) $f(x_0) = y_0$;

(b) $F(x, f(x)) = 0$, *for all* $x \in U_0$;

(c) *f possesses continuous derivative on U_0 and*

$$f'(x) = -\frac{\frac{\partial F}{\partial x}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))}.$$

Proof. Since the function $\frac{\partial F}{\partial y}$ is continuous, using the relation

$$\frac{\partial F}{\partial y}(x_0, y_0) \neq 0,$$

we deduce that there exist $U_1 \in \mathcal{V}_{x_0}$, $V_0 \in \mathcal{V}_{y_0}$ such that

$$\frac{\partial F}{\partial y}(x, y) \neq 0,$$

for all $x \in U_1$, and for all $y \in V_0$. This implies that, for all $x \in U_1$, the function $F(x, \cdot)$ is strictly monotone on V_0.

Let $\varepsilon > 0$ and choose $\varepsilon_1 \leq \varepsilon$ such that $[y_0 - \varepsilon_1, y_0 + \varepsilon_1] \subseteq V_0$. Using $F(x_0, y_0) = 0$ and the fact that the function $F(x_0, \cdot)$ is strictly monotone we obtain

$$F(x_0, y_0 - \varepsilon_1) \cdot F(x_0, y_0 + \varepsilon_1) < 0.$$

Since the function $F(\cdot, y_0 - \varepsilon_1)F(\cdot, y_0 + \varepsilon_1)$ is continuous there exists a neighborhood $U_\varepsilon \in \mathcal{V}_{x_0}$ such that

$$F(x, y_0 - \varepsilon_1) \cdot F(x, y_0 + \varepsilon_1) < 0,$$

for all $x \in U_\varepsilon$.

Let $U_0 = U_1 \cap U_\varepsilon$. Since the function $F(x, \cdot)$ is strictly monotone for all $x \in U_0$, we have defined a function $x \mapsto f(x) = y$ on U_0 such that

$$F(x, f(x)) = 0, \quad \forall x \in U_0.$$

But $F(x_0, y_0) = 0$, therefore

$$f(x_0) = y_0.$$

Moreover, for all $x \in U_\varepsilon$, we have

$$f(x) = y \in (f(x) - \varepsilon, f(x) + \varepsilon),$$

i.e., f is continuous at the point x_0.

Let $x' \in U_0$. The point $(x', f(x'))$ satisfies conditions (1), (2) and (3) from the statement of the theorem, so f is continuous at the point x', hence it is continuous on U_0.

Let us prove that f possesses a derivative in U_0. Let $x \in U_0$ and $h \neq 0$ such that $x + h \in U_0$. By virtue of Lagrange's mean-value theorem there exists $\theta \in (0,1)$ such that

$$\begin{aligned} 0 &= F(x+h, f(x+h)) - F(x, f(x)) \\ &= h\frac{\partial F}{\partial x}(x+\theta h, f(x) + \theta(f(x+h) - f(x))) \\ &+ (f(x+h) - f(x))\frac{\partial F}{\partial y}(x+\theta h, f(x) + \theta(f(x+h) - f(x))). \end{aligned}$$

Since the function f is continuous we obtain

$$\lim_{h\to 0}(f(x+h) - f(x)) = 0,$$

and

$$\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = -\frac{\frac{\partial F}{\partial x}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))},$$

i.e.,

$$f'(x) = -\frac{\frac{\partial F}{\partial x}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))}.$$

Since $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial x}$, are continuous in U_0, then f' is continuous on U_0.

Practically, by differentiating the equality $F(x, y) = 0$ with respect to x and taking into account that y is a function of x, we find

$$F'_x + y'F'_y = 0.$$

Differentiating again, we obtain

$$F''_{x^2} + y'F''_{xy} + y''F'_y + y'F''_{yx} + (y')^2F''_{y^2} = 0,$$

hence, using the equality

$$y' = -\frac{F'_x}{F'_y},$$

we obtain

$$y'' = -\frac{F''_{x^2}(F'_y)^2 - 2F'_xF'_yF''_{xy} + (F'_x)^2F''_{y^2}}{(F'_y)^3}.$$

The higher order derivatives are obtained similarly.

Let $x_0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n$, $z_0 \in \mathbb{R}$, $U \in \mathcal{V}_{x_0}$, $V \in \mathcal{V}_{z_0}$.

10.1.6 THEOREM. *If a function $F : U \times V \to \mathbb{R}$ is continuous and satisfies the conditions:*

(1) $F(x_0; z_0) = 0$;

(2) *there exist* $\dfrac{\partial F}{\partial x_i}, \dfrac{\partial F}{\partial z}$ *continuous on* $U \times V$, $i = 1, \ldots, n$;

(3) $\dfrac{\partial F}{\partial z}(x_0; z_0) \neq 0$, *then there exist* $U_0 \in \mathcal{V}_{x_0}$, $V_0 \in \mathcal{V}_{z_0}$, *and a function* $f : U_0 \to V_0$ *such that:*

(a) $f(x_0) = z_0$;

(b) $F(x; f(x)) = 0, \quad \forall x \in U_0$;

(c) *f has partial derivatives continuous on U_0 and*

$$\frac{\partial f}{\partial x_i}(x) = -\frac{\frac{\partial F}{\partial x_i}(x; f(x))}{\frac{\partial F}{\partial z}(x; f(x))},$$

$i = 1, \ldots, n$.

10.1.7 EXAMPLE. Show that the equation

$$F(x, y) = x + y + \tan y = 0,$$

$x \in \mathbb{R}$, $y \in (-\pi/2, \pi/2)$ defines a strictly decreasing function y on $\mathbb{R}$. From:

$$F(0,0) = 0,\ \frac{\partial F}{\partial y}(x, y) = 1 + \frac{1}{\cos^2 y} \neq 0,\ \forall x \in \mathbb{R},\ \forall y \in (-\pi/2, \pi/2),$$

we obtain

$$1 + y' + \frac{y'}{\cos^2 y} = 0, \quad y' = -\frac{\cos^2 y}{1 + \cos^2 y} < 0,$$

so y is a strictly decreasing function on $\mathbb{R}$.

10.2 Existence Theorems for Systems of Implicit Functions

Let $U \subset \mathbb{R}^n$, $a \in U$, $F : U \to \mathbb{R}^m$, $F = (f_1, \ldots, f_m)$. Suppose that the partial derivatives $\dfrac{\partial f_i}{\partial x_j}(a)$, $i = 1, \ldots, m$; $j = 1, \ldots, p$, $p \leq n$, exist.

10.2.1 DEFINITION. *The matrix*

$$J_F(a) := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \ldots & \frac{\partial f_1}{\partial x_p}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \ldots & \frac{\partial f_2}{\partial x_p}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \ldots & \frac{\partial f_m}{\partial x_p}(a) \end{bmatrix},$$

is called the `Jacobian matrix`[1] *of the function F at a point a with respect to the variables $x_1, \ldots, x_p$.*

Let $m = p$.

10.2.2 DEFINITION. *The determinant of the Jacobian matrix is called the* `Jacobian determinant` *or the* `functional determinant` *of the functions $f_1, \ldots, f_m$ with respect to the variables $x_1, \ldots, x_m$, at the point a, and is denoted by*

$$\frac{D(f_1, \ldots, f_m)}{D(x_1, \ldots, x_m)}(a).$$

[1] Karl Gustave Jacob Jacobi (1804–1851), a German mathematician.

```
In[1]:= (* Mathematica *)

In[2]:= (* The Jacobian matrix for a vector function {F1,F2} *)

In[3]:= D[{F1[x1, x2], F2[x1, x2]}, {{x1, x2}}]

Out[3]= {{F1^(1,0)[x1, x2], F1^(0,1)[x1, x2]}, {F2^(1,0)[x1, x2], F2^(0,1)[x1, x2]}}

In[4]:= MatrixForm[%]
```

Out[4]//MatrixForm= $\begin{pmatrix} F_1^{(1,0)}[x_1, x_2] & F_1^{(0,1)}[x_1, x_2] \\ F_2^{(1,0)}[x_1, x_2] & F_2^{(0,1)}[x_1, x_2] \end{pmatrix}$

```
In[5]:= (* The Hessian matrix for F *)

In[6]:= D[F[x1, x2], {{x1, x2}, 2}]

Out[6]= {{F^(2,0)[x1, x2], F^(1,1)[x1, x2]}, {F^(1,1)[x1, x2], F^(0,2)[x1, x2]}}

In[7]:= MatrixForm[%]
```

Out[7]//MatrixForm= $\begin{pmatrix} F^{(2,0)}[x_1, x_2] & F^{(1,1)}[x_1, x_2] \\ F^{(1,1)}[x_1, x_2] & F^{(0,2)}[x_1, x_2] \end{pmatrix}$

10.2.3 THEOREM. *If the functions F, G possess continuous partial derivatives in a neighborhood of a point $(x_0, y_0, u_0, v_0) \in \mathbb{R}^4$ and satisfy the conditions:*

(1) $F(x_0, y_0, u_0, v_0) = 0,$

(2) $G(x_0, y_0, u_0, v_0) = 0,$

(3) $\dfrac{D(F,G)}{D(u,v)}(x_0, y_0, u_0, v_0) \neq 0,$

then the system

$$\begin{cases} F(x,y,u,v) &= 0, \\ G(x,y,u,v) &= 0, \end{cases}$$

defines two implicit functions

$$u = u(x,y), \qquad v = v(x,y),$$

possessing continuous partial derivatives in a neighborhood V of the *point* (x_0, y_0), *and these functions satisfy the relations:*

$$\begin{cases} F(x, y, u(x, y), v(x, y)) &= 0, \\ G(x, y, u(x, y), v(x, y)) &= 0, \end{cases}$$

$$u(x_0, y_0) = u_0, \qquad v(x_0, y_0) = v_0,$$

for all $(x, y) \in V$.

The partial derivatives u'_x, u'_y, v'_x, v'_y can be found by differentiating the previous system. Hence,

$$\begin{cases} F'_x + u'_x F'_u + v'_x F'_v &= 0, \\ G'_x + u'_x G'_u + v'_x G'_v &= 0, \end{cases}$$

and

$$u'_x = -\frac{\frac{D(F,G)}{D(x,v)}}{\frac{D(F,G)}{D(u,v)}}, \qquad v'_x = -\frac{\frac{D(F,G)}{D(u,x)}}{\frac{D(F,G)}{D(u,v)}},$$

Likewise, we obtain:

$$u'_y = -\frac{\frac{D(F,G)}{D(y,v)}}{\frac{D(F,G)}{D(u,v)}}, \qquad v'_y = -\frac{\frac{D(F,G)}{D(u,y)}}{\frac{D(F,G)}{D(u,v)}}.$$

10.3 Change of Coordinates

Let $A, B \subseteq \mathbb{R}^n$ be open sets, $n \in \mathbb{N}^*$.

10.3.1 DEFINITION. *A continuous bijective function* $F : A \to B$ *with a continuous inverse is called a* `homeomorphism`.

10.3.2 DEFINITION. *A homeomorphism* $F : A \to B$ *possessing a continuous differentiable inverse is called a* `diffeomorphism`.

10.3.3 THEOREM. *If* $F : A \to B$ *is a diffeomorphism, then the Jacobian* $\det(J_F(a))$ *is different from zero and*

$$(J_F(a))^{-1} = J_{F^{-1}}(F(a)),$$

for all $a \in A$.

10.3.4 DEFINITION. *A mapping $F : A \to \mathbb{R}^n$, such that $F : A \to F(A)$ is a homeomorphism is called a* **change of coordinates** *in A.*

If $F = (f_1, \ldots, f_n)$, then $(f_1(x), \ldots, f_n(x))$ are the **coordinates** of the point x in the coordinate system $(f_1, \ldots, f_n)$.

10.3.5 EXAMPLE. For $A = \{(x, y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\}$ and $F : A \to \mathbb{R}^2$,

$$F(x, y) = (\rho(x, y), \theta(x, y)),$$

$$\rho = \rho(x, y) = \sqrt{x^2 + y^2},$$

$$\theta = \theta(x, y) = \arctan \frac{y}{x},$$

we have

$$\frac{D(\rho, \theta)}{D(x, y)} = \begin{vmatrix} \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ \frac{-y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2 + y^2}} > 0.$$

The function F changes the Cartesian coordinates into polar coordinates.

10.4 Change of Variables

The study of some sets of points (curves, surfaces, etc.) whose elements are determined by their coordinates with respect to a certain **coordinate system**, can be simplified by a convenient change of the coordinate system, which is called a **change of variables**.

Let $A, B \subset \mathbb{R}^2$ be two open sets and $F : A \to B$ be a diffeomorphism, $F \in C^k(A)$, $k \geq 2$,

$$F(u, v) = (h(u, v), g(u, v)) = (x, y).$$

Consider a curve $(\Gamma) \subseteq B$ which in the "old" coordinate system, (x, y), is specified by the equation

$$y = y(x).$$

In the "new" coordinate system, (u, v), the curve (Γ) is specified by the equation

$$g(u, v) = y(h(u, v)),$$

hence, by theorem (10.1.5), we obtain a local equality

$$v = v(u).$$

We shall find the relation between the derivatives of the functions

$$y = y(x), \qquad v = v(u).$$

We have:

$$y'(x) = \frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}} = \frac{\frac{\partial g}{\partial u} + \frac{\partial g}{\partial v}v'(u)}{\frac{\partial h}{\partial u} + \frac{\partial h}{\partial v}v'(u)},$$

$$y''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{\frac{dx}{du}} \cdot \frac{d}{du}\left(\frac{\frac{dy}{du}}{\frac{dx}{du}}\right) = \frac{\frac{d^2y}{du^2} \cdot \frac{dx}{du} - \frac{d^2x}{du^2} \cdot \frac{dy}{du}}{\left(\frac{dx}{du}\right)^3},$$

etc.

Some particular cases will be discussed.

(I). Interchange of Variables. Considering $F(u, v) = (v, u)$, i.e., $x = v$, $y = u$, we obtain:

$$y'(x) = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x'(y)},$$

$$y''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x'(y)}\right) = \frac{1}{\frac{dx}{dy}} \cdot \frac{d}{dy}\left(\frac{1}{x'(y)}\right) = -\frac{x''(y)}{(x'(y))^3}.$$

10.4.1 EXAMPLE. Transform the equation

$$y(y')^3 + y'' = 0,$$

where $y = y(x)$, by interchange of variables.

We have:

$$y' = \frac{1}{x'}, \qquad y'' = -\frac{x''}{(x')^3}.$$

The transformed equation is $x'' - y = 0$, where $x = x(y)$.

(II). Change of the Independent Variable. For $x = \varphi(u)$, choosing $Y = y \circ \varphi$, we obtain:

$$y'(x) = \frac{dy}{dx} = \frac{\frac{dY}{du}}{\frac{dx}{du}} = \frac{Y'(u)}{\varphi'(u)},$$

$$\begin{aligned} y''(x) &= \frac{d}{dx}\left(\frac{Y'(u)}{\varphi'(u)}\right) = \frac{1}{\varphi'(u)} \cdot \frac{d}{du}\left(\frac{Y'(u)}{\varphi'(u)}\right) \\ &= \frac{Y''(u)\varphi'(u) - Y'(u)\varphi''(u)}{(\varphi'(u))^3}. \end{aligned}$$

10.4.2 EXAMPLE. Transform the equation

$$x^2 y'' + xy' - y = 0,$$

where $y = y(x)$, by change of the independent variable

$$x = \mathrm{e}^u.$$

We have:

$$\begin{aligned} y'(x) &= \mathrm{e}^{-u} Y'(u), \\ y''(x) &= \mathrm{e}^{-2u}\left(Y''(u) - Y'(u)\right). \end{aligned}$$

The transformed equation is

$$Y'' - Y = 0,$$

where $Y = Y(u)$.

(III). Two Independent Variables. Let $A, B \subset \mathbb{R}^2$ be open sets and $T : A \to B$ be a diffeomorphism,

$$T(u,v) = (x(u,v), y(u,v)).$$

Consider the change of variables

$$\begin{cases} x = x(u,v), \\ y = y(u,v), \end{cases} \qquad (u,v) \in A.$$

We will find a relation between the operators $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}$, with respect to the "old" variables x, y and the operators $\dfrac{\partial}{\partial u}, \dfrac{\partial}{\partial v}$, with respect to the "new" variables u, v.

Let $z : B \to \mathbb{R}$ be an arbitrary function $z \in C^1(B)$. We define the function $Z : A \to \mathbb{R}$,

$$Z = z \circ T.$$

We have:

$$\begin{cases} \dfrac{\partial Z}{\partial u} = \dfrac{\partial x}{\partial u} \cdot \dfrac{\partial z}{\partial x} + \dfrac{\partial y}{\partial u} \cdot \dfrac{\partial z}{\partial y} \\ \dfrac{\partial Z}{\partial v} = \dfrac{\partial x}{\partial v} \cdot \dfrac{\partial z}{\partial x} + \dfrac{\partial y}{\partial v} \cdot \dfrac{\partial z}{\partial y} \end{cases},$$

hence:

10.4.3

$$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{1}{\frac{D(x,y)}{D(u,v)}} \left(\dfrac{\partial y}{\partial v} \cdot \dfrac{\partial Z}{\partial u} - \dfrac{\partial y}{\partial u} \cdot \dfrac{\partial Z}{\partial v} \right) \\ \dfrac{\partial z}{\partial y} = \dfrac{1}{\frac{D(x,y)}{D(u,v)}} \left(-\dfrac{\partial x}{\partial v} \cdot \dfrac{\partial Z}{\partial u} + \dfrac{\partial x}{\partial u} \cdot \dfrac{\partial Z}{\partial v} \right) \end{cases},$$

so:

$$\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{1}{\frac{D(x,y)}{D(u,v)}} \left(\dfrac{\partial y}{\partial v} \cdot \dfrac{\partial}{\partial u} - \dfrac{\partial y}{\partial u} \cdot \dfrac{\partial}{\partial v} \right) \\ \dfrac{\partial}{\partial y} = \dfrac{1}{\frac{D(x,y)}{D(u,v)}} \left(-\dfrac{\partial x}{\partial v} \cdot \dfrac{\partial}{\partial u} + \dfrac{\partial x}{\partial u} \cdot \dfrac{\partial}{\partial v} \right) \end{cases},$$

Since the domains of the the operators $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ are different from the domains of the operators $\dfrac{\partial}{\partial u}$ and $\dfrac{\partial}{\partial v}$, the previous equalities must be taken in the sense of formulas (10.4.3).

10.4.4 EXAMPLE. In the case of the change of variables

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases},$$

$\rho \in (0, \infty)$, $\theta \in [0, 2\pi)$, we obtain:

$$\begin{cases} \dfrac{\partial}{\partial x} = \cos\theta \cdot \dfrac{\partial}{\partial \rho} - \dfrac{\sin\theta}{\rho} \cdot \dfrac{\partial}{\partial \theta}, \\ \dfrac{\partial}{\partial y} = \sin\theta \cdot \dfrac{\partial}{\partial \rho} + \dfrac{\cos\theta}{\rho} \cdot \dfrac{\partial}{\partial \theta}. \end{cases}$$

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Extrema of Functions

Contents

11.1 Local Extremum of a Function

Let $D \subset \mathbb{R}^n$ and $f : D \to \mathbb{R}$.

11.1.1 DEFINITION. *The function f has a* `local maximum` *at a point $x_0 \in D$ if there exists a neighborhood $U \in \mathcal{V}_{x_0}$ such that*

$$f(x) \le f(x_0), \quad \forall x \in U.$$

On the other hand, f has a `local minimum` *at the point x_0 if*

$$f(x_0) \le f(x), \quad \forall x \in U.$$

Local minima and local maxima are referred to as `local extrema`.

A point at which a function has an extremum is called a `point of extremum`.

In what follows, assume that f possesses partial derivatives at the point $x_0 \in D$.

11.1.2 DEFINITION. *A point x_0 at which*

$$\nabla f(x_0) = 0$$

is called a `stationary` *or* `critical point` *of f.*

11.1.3 THEOREM. *Any interior point of local extremum of f is a critical point.*

Proof. Let $x_0 = (x_1^0, \ldots, x_n^0) \in \mathsf{int}(D)$ be a point of local extremum of the function f. Observe that the points x_k^0, $k = 1, \ldots, n$, are interior points of local extremum of the functions:

$$f(x_1^0, \ldots, x_{k-1}^0, \cdot, x_{k+1}^0, \ldots, x_n^0),$$

$k = 1, \ldots, n$, hence

$$\frac{d}{dx}\left(f(x_1^0, \ldots, x_{k-1}^0, x, x_{k+1}^0, \ldots, x_n^0)\right)\Big|_{x=x_k^0} = 0,$$

$k = 1, \ldots, n$, i.e.,

$$\frac{\partial f}{\partial x_k}(x_0) = 0,$$

$k = 1, \ldots, n$, and so

$$\nabla f(x_0) = 0.$$

In what follows assume that the function f possesses continuous derivatives of order $p \geq 2$ in D.

11.1.4 DEFINITION. *The differential $d^2 f(x)$ is said to be:*
positive definite if $d^2 f(x)(h) > 0$;
negative definite if $d^2 f(x)(h) < 0$;
positive semidefinite if $d^2 f(x)(h) \geq 0$;
negative semidefinite if $d^2 f(x)(h) \leq 0$; *for all* $h \in \mathbb{R}^n$, $h \neq 0$.
Otherwise, it is indefinite.

11.1.5 THEOREM. **(Sylvester)**[1] *Using the notations*

$$a_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(x), \qquad i, j = 1, \ldots, n,$$

we have:

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \ \ldots, \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} > 0,$$

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James Joseph Sylvester (1814–1897), an English mathematician.

if and only if $d^2f(x)$ is positive definite;

$$a_{11} < 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \ldots, (-1)^n \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} > 0,$$

if and only if $d^2f(x)$ is negative definite.

The Sylvester theorem is proved in the theory of quadratic forms.

11.1.6 THEOREM. *Let $x_0 \in D$ be a critical point of the function f. If $d^2f(x_0)$ is positive definite, then x_0 is a point of local minimum. If $d^2f(x_0)$ is negative definite, then x_0 is a point of local maximum.*

Proof. Taylor's formula gives

$$f(x) = f(x_0) + df(x_0)(x - x_0) + \frac{1}{2}\, d^2f(x_0 + \theta(x - x_0))(x - x_0),$$

where $\theta \in (0, 1)$. But $df(x_0) = 0$, then the previous equality can be written as

$$f(x) - f(x_0) = \frac{1}{2}\, d^2f(x_0 + \theta(x - x_0))(x - x_0).$$

For x sufficiently close to x_0, the sign of the expression $f(x) - f(x_0)$ coincides with the sign of $d^2f(x_0)(x - x_0)$.

For definiteness, let V be a neighborhood of the point x_0 such that $d^2f(x_0)(x - x_0) > 0$, for all $x \in V \setminus \{x_0\}$. It follows that

$$f(x) - f(x_0) > 0, \quad \forall x \in V \setminus \{x_0\},$$

hence x_0 is a point of local minimum of f. Similarly we treat the case when $d^2f(x_0)$ is negative definite.

11.1.7 REMARK. *If $d^2f(x_0)$ is semidefinite then the question whether f has an extremum at x_0 remains open because there are examples where f has an extremum at x_0 or has no extremum.*

If $d^2f(x_0)$ is indefinite, then it is sure that f has no extremum at x_0.

In order to investigate a function f for an extremum, we can run through the following steps:

- find the interior critical points of f by solving the system of equations
$$\left\{\frac{\partial f}{\partial x_i}(x) = 0\,, \quad i = 1, \dots, n;\right.$$
- find the sign of the second differential of f at the critical points using, e.g., Sylvester's criterion;
- study the local extrema of f on the boundary of the domain.

11.1.8 EXAMPLE. Determine the local extrema of the function $f : \mathbb{R}^2 \to \mathbb{R}$, such that
$$f(x, y) = (x + y + 1)^3 - 27xy.$$

The critical points are the solutions of the system
$$\begin{cases} \dfrac{\partial f}{\partial x}(x, y) = 0, \\ \\ \dfrac{\partial f}{\partial y}(x, y) = 0, \end{cases} \iff \begin{cases} 3(x + y + 1)^2 - 27y = 0, \\ 3(x + y + 1)^2 - 27x = 0, \end{cases}$$

i.e.,
$$(1, 1) \quad \text{and} \quad \left(\frac{1}{4}, \frac{1}{4}\right).$$

We have:
$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2}(x, y) & = & 6(x + y + 1), \\ \\ \dfrac{\partial^2 f}{\partial x \partial y}(x, y) & = & 6(x + y + 1) - 27, \\ \\ \dfrac{\partial^2 f}{\partial y^2}(x, y) & = & 6(x + y + 1), \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2}(1,1) &=& 18, \\ \dfrac{\partial^2 f}{\partial x \partial y}(1,1) &=& -9, \\ \dfrac{\partial^2 f}{\partial y^2}(1,1) &=& 18, \end{cases} \qquad \begin{cases} \dfrac{\partial^2 f}{\partial x^2}\left(\dfrac{1}{4},\dfrac{1}{4}\right) &=& 9, \\ \dfrac{\partial^2 f}{\partial x \partial y}\left(\dfrac{1}{4},\dfrac{1}{4}\right) &=& -18, \\ \dfrac{\partial^2 f}{\partial y^2}\left(\dfrac{1}{4},\dfrac{1}{4}\right) &=& 9. \end{cases}$$

hence we deduce that the differential

$$d^2 f(1,1)(dx,dy) = 18\,dx^2 - 18\,dxdy + 18\,dy^2$$

is positive definite, so $(1,1)$ is a point of local minimum.

The differential

$$d^2 f\left(\frac{1}{4},\frac{1}{4}\right)(dx,dy) = 9\,dx^2 - 36\,dxdy + 9\,dy^2$$

is indefinite, therefore $\left(\dfrac{1}{4},\dfrac{1}{4}\right)$ is not a point of local extremum of f.

```
In[1]:= (* Mathematica *)

In[2]:= f = (x + y + 1)^3 - 27 x y;

In[3]:= FindMinimum[(x + y + 1)^3 - 27 x y, {{x, 0.5, -2, 2}, {y, 0.6, -2, 2}}]
Out[3]= {0., {x -> 1., y -> 1.}}
```

11.1.9 EXAMPLE. Let us investigate the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = x^2 + y^3,$$

for an extremum.

Solving the system

$$\begin{cases} 2x &=& 0, \\ 3y^2 &=& 0, \end{cases}$$

we find the critical points. The point $(0,0)$ is the only critical point. We have:

$$df(x,y) = 2x\,dx + 3y^2\,dy, \qquad d^2f(x,y) = 2\,dx^2 + 6y\,dy^2,$$

$$d^2f(0,0)(dx,dy) = 2dx^2 \geq 0,$$

so $d^2f(0,0)$ is positive semidefinite and therefore we can say nothing about the point $(0,0)$. But,

$$f(0,y) = y^3, \quad f(0,0) = 0,$$

hence $(0,0)$ is not a point of extremum.

11.2 Conditional Extrema

Let $U \subseteq \mathbb{R}^{n+m}$ be an open set and $f \in C^1(U)$.

Consider the functions $g_i \in C^1(U)$, $i = 1,\ldots,m$. Let $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, $y = (y_1,\ldots,y_m) \in \mathbb{R}^m$. Suppose that the variables x and y are related by

11.2.1
$$\begin{cases} g_1(x;y) = 0, \\ g_2(x;y) = 0, \\ \cdots\cdots\cdots \\ g_m(x;y) = 0, \end{cases}$$

which are called `constraints`, or `constraint equations`.

Let A be the set of points $(x;y) \in U$ for which the constraints (11.2.1), hold simultaneously, i.e.,

$$A = \{(x;y) \in U \mid g_i(x;y) = 0, \ i = 1,\ldots,m\}.$$

11.2.2 DEFINITION. *A point $(x_0;y_0) \in A$ is called a* `point of local extremum` *of a function f under the constraints (11.2.1) if there exists a neighborhood $W \in \mathcal{V}_{(x_0;y_0)}$ such that the difference $f(x;y) - f(x_0;y_0)$ does not change its sign on $A \cap W$.*

Notice that the point $(x_0;y_0)$ is a point of local extremum of the function f/A.

Let us present the **method of Lagrange multipliers** for determining the critical points.

11.2.3 THEOREM. **(Lagrange)** *If* $(x_0; y_0)$ *is a point of local extremum of* f *under the constraints* (11.2.1) *and*

$$\frac{D(g_1, \ldots, g_m)}{D(y_1, \ldots, y_m)}(x_0; y_0) \neq 0,$$

then there exists m *real numbers* $\lambda_1, \ldots, \lambda_m$, *called* **Lagrange multipliers**, *such that the auxiliary function*

$$\Phi = f + \sum_{i=1}^{m} \lambda_i g_i,$$

called **Lagrangian**, *satisfies the following conditions:*

$$\begin{cases} \dfrac{\partial \Phi}{\partial x_j}(x_0; y_0) = 0, \\ \dfrac{\partial \Phi}{\partial y_k}(x_0; y_0) = 0, \end{cases}$$

$j = 1, \ldots, n; \quad k = 1, \ldots, m.$

Proof. By theorem (10.2.3) there exists a neighborhood V of the point x_0 and there exist the functions $\varphi_1, \ldots, \varphi_m \in C^1(V)$ such that

$$(\varphi_1(x_0), \ldots, \varphi_m(x_0)) = y_0,$$

$$g_i(x; \varphi_1(x), \ldots, \varphi_m(x)) = 0,$$

for $i = 1, \ldots, m$, and for all $x \in V$.

It follows that

$$\frac{\partial g_i}{\partial x_j}(x_0; y_0) + \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k}(x_0; y_0) \frac{\partial \varphi_k}{\partial x_j}(x_0) = 0 \tag{1}$$

for $i = 1, \ldots, m; \quad j = 1, \ldots, n.$

Consider the function

$$h = f(\cdot; \varphi_1(\cdot), \ldots, \varphi_m(\cdot)).$$

Since $(x; \varphi_1(x), \ldots, \varphi_m(x)) \in A$, for all $x \in V$ and $(x_0; y_0)$ is a point of local extremum under constraints for f, x_0 is a point of local extremum of the function h, and hence

$$\frac{\partial h}{\partial x_j}(x_0) = 0,$$

$j = 1, \ldots, n$, i.e.,

$$\frac{\partial f}{\partial x_j}(x_0; y_0) + \sum_{k=1}^{m} \frac{\partial f}{\partial y_k}(x_0; y_0) \frac{\partial \varphi_k}{\partial x_j}(x_0) = 0 \tag{2}$$

$j = 1, \ldots, n$.

On the other hand, the determinant

$$\frac{D(g_1, \ldots, g_m)}{D(y_1, \ldots, y_m)}(x_0; y_0)$$

is different from zero and hence there exist real numbers $\lambda_1, \ldots, \lambda_m$, such that

$$\sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial y_k}(x_0; y_0) = -\frac{\partial f}{\partial y_k}(x_0; y_0) \tag{3}$$

$k = 1, \ldots, m$.

Let us prove that

$$\frac{\partial \Phi}{\partial x_j}(x_0; y_0) = 0, \qquad \frac{\partial \Phi}{\partial y_k}(x_0; y_0) = 0,$$

for $j = 1, \ldots, n$; $k = 1, \ldots, m$.

We have:

$$\begin{aligned}
\frac{\partial \Phi}{\partial x_j}(x_0; y_0) &= \frac{\partial f}{\partial x_j}(x_0; y_0) + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{x_j}(x_0; y_0) \\
&\overset{(1)}{=} \frac{\partial f}{\partial x_j}(x_0; y_0) - \sum_{i=1}^{m} \lambda_i \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k}(x_0; y_0) \frac{\partial \varphi_k}{\partial x_j}(x_0) \\
&= \frac{\partial f}{\partial x_j}(x_0; y_0) - \sum_{k=1}^{m} \frac{\partial \varphi_k}{\partial x_j}(x_0) \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial y_k}(x_0; y_0) \\
&\overset{(3)}{=} \frac{\partial f}{\partial x_j}(x_0; y_0) + \sum_{k=1}^{m} \frac{\partial \varphi_k}{\partial x_j}(x_0) \frac{\partial f}{\partial y_k}(x_0; y_0) \overset{(2)}{=} 0,
\end{aligned}$$

for $j = 1, \dots, n$.

$$\frac{\partial \Phi}{\partial y_k}(x_0; y_0) = \frac{\partial f}{\partial y_k}(x_0; y_0) + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial y_k}(x_0; y_0) \overset{(3)}{=} 0,$$

for $k = 1, \dots, m$.

11.2.4 REMARK. *From the constraints*

$$g_i(x; y) = 0, \qquad i = 1, \dots, m,$$

we deduce

$$dg_i(x_0; y_0) = 0, \qquad i = 1, \dots, m,$$

or

$$\sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j}(x_0; y_0)\, dx_j + \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k}(x_0; y_0)\, dy_k = 0,$$

$i = 1, \dots, m$, *hence, making use of the condition*

$$\frac{D(g_1, \dots, g_m)}{D(y_1, \dots, y_m)}(x_0; y_0) \neq 0,$$

we deduce that

$$dy_1, \dots, dy_m,$$

can be written in terms of

$$dx_1, \dots, dx_n.$$

In order to solve a problem of local extremum under constraints we can run through the following steps:

(1) Solve the system

$$\begin{cases} \dfrac{\partial \Phi}{\partial x_j}(x; y) & = & 0 \\ \dfrac{\partial \Phi}{\partial y_k}(x; y) & = & 0 \\ g_k(x; y) & = & 0, \end{cases}$$

$j = 1, \ldots, n$; $k = 1, \ldots, m$, with the unknowns:

$$x_1, \ldots, x_n, \quad y_1, \ldots, y_m, \quad \lambda_1, \ldots, \lambda_m;$$

(2) Let $(x_0; y_0; \lambda_0)$ be a solution of the previous system. Find the constraints between the differentials $dx_1, \ldots, dx_n$, and $dy_1, \ldots, dy_m$ at the point $(x_0; y_0)$.

(3) Find the sign of $d^2\Phi(x_0; y_0)$ taking into account the relations between the differentials $dx_1, \ldots, dx_n$.

11.2.5 EXAMPLE. Determine the local conditional extrema of the function

$$H(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \ln x_i,$$

$x_i > 0$, $i = 1, \ldots, n$, under the constraints

$$x_1 + \cdots + x_n = 1.$$

Consider the function

$$\Phi = \sum_{i=1}^{n} x_i \ln x_i + \lambda \left(\sum_{i=1}^{n} x_i - 1 \right).$$

The system

$$\begin{cases} \dfrac{\partial \Phi}{\partial x_1} & = & 0 \\ \dfrac{\partial \Phi}{\partial x_2} & = & 0 \\ \cdots\cdots\cdots & \cdots & \cdots \\ \dfrac{\partial \Phi}{\partial x_n} & = & 0 \\ x_1 + \cdots + x_n & = & 0 \end{cases}$$

can be written in the form

$$\begin{cases} \ln x_1 + 1 + \lambda & = & 0 \\ \ln x_2 + 1 + \lambda & = & 0 \\ \cdots\cdots\cdots & \cdots & \cdots \\ \ln x_n + 1 + \lambda & = & 0 \\ x_1 + \cdots + x_n & = & 0. \end{cases}$$

It has the solution $\left(\frac{1}{n}, \ldots, \frac{1}{n}, \ln n - 1\right)$. We have:

$$\Phi(x_1, \ldots, x_n) = x_1 \ln x_1 + \cdots + x_n \ln x_n + (\ln n - 1)(x_1 + \cdots + x_n - 1),$$

$$\begin{aligned}
d\Phi(x_1, \ldots, x_n) &= \sum_{i=1}^{n} (\ln x_i + \ln n)\, dx_i, \\
d^2\Phi(x_1, \ldots, x_n) &= \sum_{i=1}^{n} \frac{1}{x_i}\, dx_i^2, \\
d^2\Phi(\tfrac{1}{n}, \ldots, \tfrac{1}{n}) &= n \sum_{i=1}^{n} dx_i^2,
\end{aligned}$$

which is positive definite independent of the constraint

$$dx_1 + \cdots + dx_n = 0.$$

It follows that the point $\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ is a point of local minimum of the function H under the constraint

$$x_1 + \cdots + x_n = 1.$$

11.3 Exercises: Extrema of Functions

P
11.3.1 Prove that the function

$$f(x,y) = (1 + e^y) \cos x - y e^y, \qquad (x,y) \in \mathbb{R}^2,$$

has an infinite number of maxima and no minimum.

P
11.3.2 Find the distance from the point $M_0(x_0, y_0, z_0)$ to the plane

$$Ax + By + Cz + D = 0.$$

P
11.3.3 Find the distance between the straight lines

$$2(x-1) = y = 2z, \qquad x = y = z.$$

P
11.3.4 Among all rectangular parallelepipeds having the given volume 1, find the parallelepiped with the least area.

P
11.3.5 Prove that

$$\frac{x^n + y^n}{2} \geq \left(\frac{x+y}{2}\right)^n,$$

$x > 0, \quad y > 0, \quad n \geq 2.$

11.3.6 [P]

Find the greatest and the least values of the function

$$f(x, y) = x^2 + y^2$$

in the disc

$$D: \ (x-2)^2 + (y-2)^2 \leq 2.$$

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Answers to Exercises

Contents

12.1 Exercises: Sets. Functions (Solutions)

2.6.1 $\bigcap_{n=0}^{\infty} A_n = \bigcap_{n=0}^{\infty} [0, n] = \{0\};$

$\liminf A_n = \bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} [0, n] = \bigcup_{k=0}^{\infty} [0, k] = [0, \infty).$

2.6.2 $\liminf B_n = \bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} B_n = \bigcup_{k=0}^{\infty} \{0\} = \{0\};$

$\limsup B_n = \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} B_n = \bigcap_{k=0}^{\infty} (-\infty, \infty) = (-\infty, \infty).$

2.6.3 $\limsup C_n = \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} C_n = \bigcap_{k=0}^{\infty} \left[\frac{-1}{k+1}, \infty\right) = [0, \infty);$

$$\bigcup_{n=0}^{\infty} C_n = \bigcup_{n=0}^{\infty} \left[\frac{-1}{n+1}, n\right] = [-1, \infty).$$

2.6.4 The relations

$$\bigcap_{n=0}^{\infty} A_n \subseteq \bigcap_{n=k}^{\infty} A_n, \quad \bigcup_{n=k}^{\infty} A_n \subseteq \bigcup_{n=0}^{\infty} A_n, \quad (\forall k \in \mathbb{N}),$$

imply the inclusions

$$\bigcap_{n \in \mathbb{N}} A_n \subseteq \liminf A_n, \quad \limsup A_n \subseteq \bigcup_{n \in \mathbb{N}} A_n.$$

In order to prove the inclusion $\liminf A_n \subset \limsup A_n$ we consider an arbitrary element x such that

$$x \in \liminf A_n = \bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} A_n.$$

Hence, there exists an index k_0 such that

$$x \in \bigcap_{n=k_0}^{\infty} A_n;$$

therefore $x \in A_n$, for all $n \in \mathbb{N}$, $n \geq k_0$. It follows that $x \in \bigcup_{n=k}^{\infty} A_n$, $(\forall k \in \mathbb{N})$, so $x \in \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} A_n = \limsup A_n$.

2.6.5 (A) (a) From $B_{n+1} = B_n \cup A_{n+1}$, $n \in \mathbb{N}$, we deduce that $B_n \subseteq B_{n+1}$, $n \in \mathbb{N}$.
(b) From $A_n \subseteq B_n$, $n \in \mathbb{N}$, we deduce

$$\bigcup_{n=0}^{\infty} A_n \subseteq \bigcup_{n=0}^{\infty} B_n. \qquad (\diamond)$$

From

$$B_n = \bigcup_{k=0}^{n} A_k \subseteq \bigcup_{n=0}^{\infty} A_n$$

it follows that

$$\bigcup_{n=0}^{\infty} B_n \subseteq \bigcup_{n=0}^{\infty} A_n. \qquad (\diamond\diamond)$$

From $(\diamond)$ and $(\diamond\diamond)$ we deduce

$$\bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} B_n.$$

2.6.6 (A) (a) For definiteness, consider $i, j \in \mathbb{N}$, $i < j$. From

$$B_j = A_j \setminus \bigcup_{k=0}^{j-1} A_k$$

it follows that

$$B_j \cap A_k = \emptyset, \quad k = 0, \ldots, j-1,$$

hence

$$B_j \cap A_i = \emptyset.$$

Next, using the inclusion $B_i \subseteq A_i$, we deduce

$$B_i \cap B_j = \emptyset.$$

(b) From $B_n \subseteq A_n$, $n \in \mathbb{N}$, we deduce

$$\bigcup_{n=0}^{\infty} B_n \subseteq \bigcup_{n=0}^{\infty} A_n \,. \tag{$\diamond$}$$

Let

$$x \in \bigcup_{n=0}^{\infty} A_n.$$

It follows that there exists an index n_0 such that $x \in A_{n_0}$. Let m be the lowest index possessing the property that $x \in A_m$. If $m = 0$ then

$$x \in A_0 = B_0 \subset \bigcup_{n=0}^{\infty} B_n.$$

If $m \neq 0$, then

$$x \in A_m \setminus \bigcup_{k=0}^{m-1} A_k = B_m \subseteq \bigcup_{n=0}^{\infty} B_n,$$

hence

$$\bigcup_{n=0}^{\infty} A_n \subseteq \bigcup_{n=0}^{\infty} B_n \,. \tag{$\diamond\diamond$}$$

From ($\diamond$) and ($\diamond\diamond$) we deduce $\bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} B_n$.

2.6.7 $\neg(b) \Longrightarrow \neg(a)$ Suppose that there exists an x such that

$$x \in \bigcap_{k=1}^{n} A_k,$$

hence $x \in A_k$, $k = 1, \ldots, n$, therefore $x \notin A_k \setminus A_{k+1}$, $k = 1, \ldots, n$. It follows that

$$x \in \bigcup_{k=1}^{n} A_k \quad \text{and} \quad x \notin \bigcup_{k=1}^{n} (A_k \setminus A_{k+1}),$$

and consequently

$$\bigcup_{k=1}^{n} A_k \neq \bigcup_{k=1}^{n} (A_k \setminus A_{k+1}).$$

$\neg(a) \Longrightarrow \neg(b)$ Suppose that

$$\bigcup_{k=1}^{n} A_k \neq \bigcup_{k=1}^{n} (A_k \setminus A_{k+1}),$$

hence there exists an x such that

$$x \in \bigcup_{k=1}^{n} A_k \quad \text{and} \quad x \notin \bigcup_{k=1}^{n} (A_k \setminus A_{k+1}).$$

It follows that there exists an index m such that

$$x \in A_m \text{ and } x \notin A_k \setminus A_{k+1}, \quad k = 1, \dots, n.$$

From

$$x \in A_m \text{ and } x \notin A_m \setminus A_{m+1}$$

it follows that $x \in A_{m+1}$. Step by step we can show that

$$x \in A_{m+2}, \quad \dots, \quad A_1, \quad \dots, \quad A_{m-1},$$

i.e.,

$$\bigcap_{k=1}^{n} A_k \neq \emptyset.$$

$\overset{\text{A}}{2.6.8}$

$$A \in \bigcap_{i \in I} \mathcal{P}(A_i) \iff (\forall i)(i \in I \to A \in \mathcal{P}(A_i)) \qquad (a)$$

$$\iff (\forall i)(i \in I \to A \subseteq A_i)$$

$$\iff A \subseteq \bigcap_{i \in I} A_i$$

$$\iff A \in \mathcal{P}\left(\bigcap_{i \in I} A_i\right).$$

$$A \in \bigcup_{i \in I} \mathcal{P}(A_i) \iff (\exists i_0)(i_0 \in I,\ A \in \mathcal{P}(A_{i_0})) \qquad (b)$$

$$\iff (\exists i_0)(i_0 \in I,\ A \subseteq A_{i_0}) \Longrightarrow A \subseteq \bigcup_{i \in I} A_i$$

$$\iff A \in \mathcal{P}\left(\bigcup_{i\in I} A_i\right).$$

2.6.9 (A) $(a) \Longrightarrow (b)$ Suppose that

$$\mathcal{P}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \mathcal{P}(A_i).$$

We have:

$$\begin{aligned}
\bigcup_{i\in I} A_i \in \mathcal{P}\left(\bigcup_{i\in I} A_i\right) &= \bigcup_{i\in I} \mathcal{P}(A_i) \\
&\Longrightarrow (\exists i_0)(i_0 \in I,\ \bigcup_{i\in I} A_i \in \mathcal{P}(A_{i_0})) \\
&\iff (\exists i_0)(i_0 \in I,\ \bigcup_{i\in I} A_i \subseteq A_{i_0}) \\
&\iff (\exists i_0)(i_0 \in I,\ (\forall i)(i \in I \to A_i \subseteq A_{i_0})).
\end{aligned}$$

$(b) \Longrightarrow (a)$ Suppose that there exists $i_0 \in I$ such that

$$(\forall i)(i \in I \to A_i \subseteq A_{i_0}).$$

From

$$A_{i_0} \subseteq \bigcup_{i\in I} A_i \subseteq \bigcup_{i\in I} A_{i_0} = A_{i_0}$$

we deduce

$$\bigcup_{i\in I} A_i = A_{i_0}. \tag{$\diamond$}$$

From

$$\mathcal{P}(A_{i_0}) \subseteq \bigcup_{i\in I} \mathcal{P}(A_i) \subseteq \bigcup_{i\in I} \mathcal{P}(A_{i_0}) = \mathcal{P}(A_{i_0})$$

we obtain

$$\mathcal{P}(A_{i_0}) = \bigcup_{i\in I} \mathcal{P}(A_i). \tag{$\diamond\diamond$}$$

Finally, $(\diamond)$ and $(\diamond\diamond)$ yield

$$\mathcal{P}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \mathcal{P}(A_i).$$

2.6.10 (A) From $A_i \subseteq A_i \cup B_i$, $B_i \subseteq A_i \cup B_i$, $(\forall i \in I)$ we deduce

$$\bigcap_{i\in I} A_i \subseteq \bigcap_{i\in I}(A_i \cup B_i),$$

$$\bigcap_{i\in I} B_i \subseteq \bigcap_{i\in I}(A_i \cup B_i),$$

therefore

$$\left(\bigcap_{i\in I} A_i\right) \cup \left(\bigcap_{i\in I} B_i\right) \subseteq \bigcap_{i\in I}(A_i \cup B_i). \qquad (\diamond)$$

Consider an arbitrary element x such that

$$x \in \bigcap_{i\in I}(A_i \cup B_i).$$

It follows that

$$x \in A_i \cup B_i \quad (\forall i \in I).$$

For definiteness, suppose that there exists $i_0 \in I$ such that $x \in A_{i_0}$. Also suppose that there exists $j \in I$, $j \neq i_0$, such that $x \notin A_j$. Taking into account the relation $x \in A_j \cup B_j$, it follows that $x \in B_j$, and consequently $A_{i_0} \cap B_j \neq \emptyset$!!! Therefore $x \in A_i$, $(\forall i \in I)$, and

$$x \in \left(\bigcap_{i\in I} A_i\right) \cup \left(\bigcap_{i\in I} B_i\right),$$

hence

$$\bigcap_{i\in I}(A_i \cup B_i) \subseteq \bigcap_{i\in I} A_i \cup \bigcap_{i\in I} B_i. \qquad (\diamond\diamond)$$

From $(\diamond)$ and $(\diamond\diamond)$ we deduce

$$\bigcap_{i\in I}(A_i \cup B_i) = \left(\bigcap_{i\in I} A_i\right) \cup \left(\bigcap_{i\in I} B_i\right).$$

2.6.11 (A) $$\limsup(A_n \Delta A) = \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} (A \cap \complement(A_n)) \cup (A_n \cap \complement(A))$$
$$= \bigcap_{k=0}^{\infty} \left(\bigcup_{n=k}^{\infty} (A \cap \complement(A_n)) \cup \bigcup_{n=k}^{\infty} (A_n \cap \complement(A)) \right)$$

$$= \bigcap_{k=0}^{\infty}\left(\left(A \cap \bigcup_{n=k}^{\infty} \complement(A_n)\right) \cup \left(\bigcup_{n=k}^{\infty} A_n\right) \cap \complement(A)\right)$$

$$\overset{\text{P.2.6.10}}{=} \left(A \cap \bigcap_{k=0}^{\infty}\bigcup_{n=k}^{\infty} \complement(A_n)\right) \cup \left(\left(\bigcap_{k=0}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) \cap \complement(A)\right)$$

$$= \left(A \cap \complement\left(\bigcup_{k=0}^{\infty}\bigcap_{n=k}^{\infty} (A_n)\right)\right) \cup \left(\left(\bigcap_{k=0}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) \cap \complement(A)\right)$$

$$= (A \cap \complement(\liminf A_n)) \cup ((\limsup A_n) \cap \complement(A))$$

$$= (A \setminus \liminf A_n) \cup (\limsup A_n \setminus A).$$

2.6.12 [A] $(a) \Longrightarrow (b)$ From

$$\lim A_n = A \overset{def}{\Longleftrightarrow} \limsup A_n = \liminf A_n = A,$$

we deduce

$$\limsup A\Delta A_n = (A \setminus A) \cup (A \setminus A) = \emptyset.$$

$(b) \Longrightarrow (a)$ From $\limsup A\Delta A_n = \emptyset$, that is,

$$(A \cap \complement(\liminf A_n)) \cup ((\limsup A_n) \cap \complement(A)) = \emptyset,$$

we deduce

$$A \cap \complement(\liminf A_n) = (\limsup A_n) \cap \complement(A) = \emptyset,$$

therefore

$$A \subseteq \liminf A_n, \quad \limsup A_n \subseteq A,$$

and finally,

$$\liminf A_n \subseteq \limsup A_n = A, \quad \text{i.e., } \lim A_n = A.$$

2.6.13 (A) (*a*) We consider an element x such that

$$x \in \bigcap_{H \in A_k} \bigcup_{i \in H} X_i.$$

This is equivalent to

$$(\forall H)(H \in A_k \to (\exists i)(i \in H,\ x \in X_i)). \qquad (\diamond)$$

Let $X_{i_1}, \ldots, X_{i_p}$ be the sets of the family $\{X_i\}_{i\in I}$ containing the element x. If $p < k$, then $n - p > n - k \geq k - 1$, hence there exist at least k sets $X_{j_1}, \ldots, X_{j_k}$ not containing x in the family $\{X_i\}_{i\in I}$. Setting

$$H_0 := \{j_1, \ldots, j_k\},$$

we observe that $H_0 \in A_k$ and $(\forall i)(i \in H_0 \to x \notin X_i)$, which contradicts $(\diamond)$. It follows that $p \geq k$, and we can choose

$$H_1 = \{i_1, \ldots, i_k\} \in A_k.$$

We observe that

$$x \in \bigcap_{i \in H_1} X_i \subseteq \bigcup_{H \in A_k} \bigcap_{i \in H} X_i.$$

(*b*) We consider an element x such that

$$x \in \bigcup_{H \in A_k} \bigcap_{i \in H} X_i.$$

This is equivalent to

$$(\exists H_0)(H_0 \in A_k, \quad (\forall i)(i \in H_0 \to x \in X_i)). \qquad (\diamond\diamond)$$

We suppose that there exists $H \in A_k$ such that $H \cap H_0 = \emptyset$. We have

$$\begin{aligned} n &= \mathsf{card}(I) \geq \mathsf{card}(H \cup H_0) \\ &= \mathsf{card}(H) + \mathsf{card}(H_0) = 2k \geq n + 1 \text{ !!!} \end{aligned}$$

It follows that

$$(\forall H)(H \in A_k \to (\exists i)(i \in H \cap H_0)),$$

therefore, taking into account $(\diamond\diamond)$,

$$(\forall H)(H \in A_k \to (\exists i)(i \in H,\ x \in X_i)),$$

which is equivalent to

$$x \in \bigcap_{H \in A_k} \bigcup_{i \in H} X_i.$$

12.2 Exercises: Topological Spaces (Solutions)

3.9.1 (1) Let I be an index set and $\{G_i\}_{i\in I}$ be a collection of sets in $\mathcal{C}$. Let $j \in I$. Since $X \setminus G_j$ is finite, taking into account the inclusion

$$(X \setminus \bigcup_{i\in I} G_i) \subseteq (X \setminus G_j),$$

it follows that

$$X \setminus \bigcup_{i\in I} G_i \text{ is finite},$$

hence

$$\bigcup_{i\in I} G_i \in \mathcal{C}.$$

Let $G_1, G_2 \in \mathcal{C}$. The set

$$X \setminus (G_1 \cap G_2) = (X \setminus G_1) \cup (X \setminus G_2)$$

is a union of two finite sets, therefore it is finite, and consequently,

$$G_1 \cap G_2 \in \mathcal{C}.$$

The set $X \setminus X = \emptyset$ is finite, hence $X \in \mathcal{C}$. The empty set belongs to $\mathcal{C}$ by the hypothesis. We deduce that the family $\mathcal{C}$ is a topology on X.

(2) $\mathcal{F} = \{X,\ F \mid (F \subseteq X) \wedge (F\text{–finite})\}$.

(3) Let $x \in X$ and V be a neighborhood of the point x. It follows that there exists a set $G \in \mathcal{C}$ such that $G \subseteq V$. From the fact that $X \setminus G$ is finite, by using the inclusion $X \setminus V \subseteq X \setminus G$, we deduce that the set $X \setminus V$ is finite, hence $V \in \mathcal{C}$.

3.9.2 First, we prove that all unit sets in X are open. Let $x \in X$ and G_1, G_2 two infinite disjoint subsets in X. We deduce:

$$G_1 \cup \{x\} \in \mathcal{T}, \quad G_2 \cup \{x\} \in \mathcal{T},$$

$$\{x\} = (G_1 \cup \{x\}) \cap (G_2 \cup \{x\}) \in \mathcal{T}.$$

For all $A \subseteq X$ we write

$$A = \bigcup_{x \in A} \{x\},$$

therefore $A \in \mathcal{T}$ (as an union of open sets) and consequently, $\mathcal{T} = \mathcal{P}(X)$.

3.9.3 (1) Let $x \in X$. If $x \in \overline{A}$, then $x \in \overline{A} \cup \operatorname{int}(B)$. If $x \notin \overline{A}$ then there exists $V \in \mathcal{V}_x$ such that $V \cap A = \emptyset$.

We have:

$$\begin{aligned} V &= V \cap X = V \cap (A \cup B) = (V \cap A) \cup (V \cap B) \\ &= \emptyset \cup (V \cap B) = V \cap B, \end{aligned}$$

therefore $V \subseteq B$, and consequently $x \in \operatorname{int}(B)$.

We have proved that $X \subseteq \overline{A} \cup \operatorname{int}(B)$, therefore $X = \overline{A} \cup \operatorname{int}(B)$.

(2) Suppose that $\overline{A} \cap \operatorname{int}(B) \neq \emptyset$. Let $x \in \overline{A} \cap \operatorname{int}(B)$. From $x \in \operatorname{int}(B) \in \mathcal{T}$, it follows that $\operatorname{int}(B)$ is a neighborhood of x. Using the fact that $x \in \overline{A}$, we obtain $A \cap \operatorname{int}(B) \neq \emptyset$, and moreover, $A \cap B \neq \emptyset$.

3.9.4 *Necessity.* Suppose that G is open and $G \cap \overline{A} \neq \emptyset$. Then, there exists $x \in G \cap \overline{A}$. From $x \in G$ it follows that $G \in \mathcal{V}_x$. Using the fact that $x \in \overline{A}$, we get $G \cap A \neq \emptyset$.

Sufficiency. Suppose that, for all $A \subseteq X$,

$$A \cap G = \emptyset \Rightarrow \overline{A} \cap G = \emptyset.$$

Since

$$\complement(G) \cap G = \emptyset \Rightarrow \overline{\complement(G)} \cap G = \emptyset,$$

we get

$$G \subseteq \complement(\overline{\complement(G)}) = \operatorname{int}(G),$$

i.e., $G \in \mathcal{T}$.

3.9.5 (a) $\Rightarrow$ (b). Let $G_1, G_2 \in \mathcal{T}$, $G_1 \cap G_2 = \emptyset$. By virtue of Problem P.3.9.4 it follows that

$$G_1 \cap \overline{G_2} = \emptyset;$$

by virtue of Problem P.3.9.4 and statement (a), we obtain

$$\overline{G_1} \cap \overline{G_2} = \emptyset.$$

(b) $\Rightarrow$ (a) . Let $G \in \mathcal{T}$. Using

$$G \cap \complement(\overline{G}) = \emptyset,$$

and statement (b), we deduce

$$\overline{G} \cap \overline{\complement(\overline{G})} = \emptyset,$$

hence

$$\overline{G} \subseteq \complement(\overline{\complement(\overline{G})}) = \mathrm{int}(\overline{G}),$$

and hence $\overline{G} \in \mathcal{T}$.

3.9.6 (1). Let $x \in G \cap \overline{A}$. We deduce that G is a neighborhood of x. Consequently, for any neighborhood V of x it follows that $V \cap G$ is a neighborhood of x. Since $x \in \overline{A}$, we get

$$V \cap (G \cap A) = (V \cap G) \cap A \neq \emptyset,$$

therefore

$$x \in \overline{G \cap A},$$

and consequently

$$G \cap \overline{A} \subseteq \overline{G \cap A}.$$

(2). From Eq. (1) we deduce

$$\overline{G \cap \overline{A}} \subseteq \overline{\overline{G \cap A}} = \overline{G \cap A}.$$

On the other hand,

$$G \cap A \subseteq G \cap \overline{A} \Rightarrow \overline{G \cap A} \subseteq \overline{G \cap \overline{A}},$$

hence

$$\overline{G \cap \overline{A}} = \overline{G \cap A}.$$

3.9.7 Suppose that there exists a point $x \in X$ such that $x \in A''$ and $x \notin A'$.
From $x \notin A'$ it follows that there exists a set $G \in \mathcal{T}$, $x \in G$, such that

$$G \cap (A \setminus \{x\}) = \emptyset. \qquad (\diamond)$$

The relation $x \in A''$ and the fact that G is a neighborhood of x imply

$$G \cap (A' \setminus \{x\}) \neq \emptyset.$$

Then there exists

$$y \in G \cap (A' \setminus \{x\}) = G \cap \complement(\{x\}) \cap A',$$

hence,

$$y \in G \cap \complement(\{x\}) \text{ and } y \in A'.$$

Since $G \cap \complement(\{x\})$ is open, it is a neighborhood of y. We deduce the relation

$$(G \setminus \{x\}) \cap (A \setminus \{y\}) \neq \emptyset,$$

that is

$$(G \setminus \{y\}) \cap (A \setminus \{x\}) \neq \emptyset,$$

and moreover,

$$G \cap (A \setminus \{x\}) \neq \emptyset,$$

which contradicts ($\diamond$).

Briefly, $x \in A''$ implies $x \in A'$, i.e. $A'' \subseteq A'$.

3.9.8 (A) Let F be a closed set in the compact space X, and $\{G_i\}_{i \in I}$ be an open covering of F. From

$$\left(\bigcup_{i \in I} G_i\right) \cup \complement(F) = X,$$

it follows that

$$\bigcup_{i \in I} (G_i \cup \complement(F)) = X.$$

Since X is compact, there exists a finite subcovering of X,

$$\{G_{i_k} \cup \complement(F)\}_{k=\overline{1,n}}, \qquad \bigcup_{k=1}^{n} (G_{i_k} \cup \complement(F)) = X.$$

Consequently

$$F \subseteq \bigcup_{k=1}^{n} G_{i_k},$$

therefore the family

$$\{G_{i_k}\}_{k=\overline{1,n}}$$

is a finite subcover of F. Hence F is a compact set.

$\overset{\textbf{A}}{\textbf{3.9.9}}$ We have:

$$\mathcal{F} = \mathcal{T}, \quad \mathcal{V}_{x_0} = \{V \mid (V \subseteq X) \wedge (x_0 \in V)\},$$

$$A \in \mathcal{T} \Rightarrow \mathsf{int}(A) = A,$$

$$A \in \mathcal{F} \Rightarrow \overline{A} = A,$$

$$\mathsf{bd}(A) = \overline{A} \setminus \mathsf{int}(A) = A \setminus A = \emptyset.$$

Using

$$x \in A' \iff x \in \overline{A \setminus \{x\}} = A \setminus \{x\},$$

we deduce $A' = \emptyset$.

$\overset{\textbf{A}}{\textbf{3.9.10}}$ We have:

$$\mathcal{T} = \mathcal{F},$$

$$\mathcal{V}_1 = \mathcal{V}_2 = \{\{1,2\}\},$$

$$\mathsf{int}\{1\} = \emptyset,$$

$$\overline{\{1\}} = \{1,2\},$$

$$\mathsf{bd}\{1\} = \overline{\{1\}} \setminus \mathsf{int}\{1\} = \{1,2\}.$$

From

$$x \in \{1\}' \iff x \in \overline{\{1\} \setminus \{x\}},$$

we deduce:

$$1 \in \{1\}' \iff 1 \in \overline{\{1\} \setminus \{1\}} = \overline{\emptyset} = \emptyset,$$

$$2 \in \{1\}' \iff 2 \in \overline{\{1\} \setminus \{2\}} = \overline{\{1\}} = \{1,2\},$$

therefore

$$\{1\}' = \{2\}.$$

$\overset{\textbf{A}}{\textbf{3.9.11}}$ We can write

$$\mathcal{T} = \{\mathbb{R},\ G \mid (G \subseteq \mathbb{R}) \wedge (G \cap \{0,1\}) = \emptyset)\},$$

therefore

$$\mathcal{F} = \{\emptyset,\ F \mid (F \subseteq \mathbb{R}) \wedge (\{0,1\} \subseteq F)\} = \{\emptyset, \{0,1\} \cup A \mid A \subseteq \mathbb{R}\}.$$

We deduce:

$$
\begin{aligned}
\mathcal{V}_0 &= \{\mathbb{R}\}, \\
\mathcal{V}_2 &= \{V \mid (V \subseteq \mathbb{R}) \wedge (2 \in V)\}, \\
\overline{\{0\}} &= \{0,1\}, \\
\mathsf{int}\{0\} &= \emptyset, \\
\mathsf{bd}\{0\} &= \{0,1\}, \\
\{0\}' &= \{1\}, \\
\overline{[0,1]} &= [0,1], \\
\mathsf{bd}([0,1]) &= \{0,1\}, \\
\overline{\{5\}} &= \{0,1,5\}, \\
\mathsf{int}\{5\} &= \{5\}, \\
\mathsf{bd}\{5\} &= \{0,1\}, \\
\{5\}' &= \{0,1\}.
\end{aligned}
$$

3.9.12 (A) We have:

$$
\begin{aligned}
\overline{A} &= [0,2] \cup \{3\}, \\
\mathsf{int}(A) &= (1,2), \\
A' &= [0,2], \\
\mathsf{bd}(A) &= [0,1] \cup \{2,3\}, \\
\mathsf{bd}(\mathsf{bd}(A)) &= \{0,1,2,3\}.
\end{aligned}
$$

12.3 Exercises: Metric Spaces (Solutions)

4.5.1 (A) That condition $\mathcal{M}_1$ holds for σ is trivial. To verify $\mathcal{M}_2$ we proceed as follows:

$$\begin{aligned}
\sigma(x,y)+\sigma(z,y) &= \frac{\rho(x,y)}{1+\rho(x,y)}+\frac{\rho(z,y)}{1+\rho(z,y)} \\
&\geq \frac{\rho(x,y)}{1+\rho(x,y)+\rho(z,y)}+\frac{\rho(z,y)}{1+\rho(x,y)+\rho(z,y)} \\
&= \frac{\rho(x,y)+\rho(z,y)}{1+\rho(x,y)+\rho(z,y)}=\frac{1}{1+\frac{1}{\rho(x,y)+\rho(z,y)}} \\
&\geq \frac{1}{1+\frac{1}{\rho(x,z)}}=\frac{\rho(x,z)}{1+\rho(x,z)} \\
&= \sigma(x,z).
\end{aligned}$$

4.5.2 (A) We have:

$$\begin{aligned}
B\left(0,\frac{1}{2}\right) &= \left\{y \mid y\in\mathbb{R},\ \frac{|0-y|}{1+|0-y|}<\frac{1}{2}\right\} \\
&= \{y \mid y\in\mathbb{R},\ |y|<1\} \\
&= (-1,1).
\end{aligned}$$

Likewise, $B(0,1)=\mathbb{R}$.

4.5.3 (A) We have:

$$B(x,r)=\begin{cases}\{x\}, & 0<r\leq 1;\\ X, & r>1\end{cases},$$

$$\overline{B}(x,r)=\begin{cases}\{x\}, & 0<r<1;\\ X, & r\geq 1.\end{cases}$$

Note that the topology induced in X by the metric ρ is the discrete topology, therefore, taking into account that every element in $\mathcal{P}(X)$ is a closed set, it follows that $\overline{B(x,r)}=B(x,r)$.

4.5.4 (A) We have:

$$B(0,2)=(-2,1],$$

$$\overline{B}(0,2) = [-2,1] \cup \{2\},$$
$$\overline{B(0,2)} = [-2,1].$$

4.5.5 We have:

$$\rho((x_n),(y_n)) = \sum_{k=0}^{\infty} \frac{|1-\sin k|}{2^k} = \sum_{k=0}^{\infty} \frac{1-\sin k}{2^k}$$
$$= 2 + \Im\left(\sum_{k=0}^{\infty} \frac{\cos k + i \sin k}{2^k}\right) = 2 + \Im\left(\sum_{k=0}^{\infty}\left(\frac{e^i}{2}\right)^k\right)$$
$$= 2 + \Im\left(\frac{1}{1-\frac{e^i}{2}}\right) = 2 - \frac{2\sin 1}{5-4\cos 1}.$$

4.5.6 *Necessity.* Let $x \in \overline{A}$ and $\varepsilon > 0$. Since $B(x,\varepsilon)$ is a neighborhood of x then $B(x,\varepsilon) \cap A \neq \emptyset$, hence there exists $a \in B(x,\varepsilon) \cap A$, such that $\rho(x,a) < \varepsilon$.

Sufficiency. Let V an arbitrary neighborhood of the point x. Then there exists a $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq V$. From the hypothesis we deduce that there exists $a \in A$ such that $\rho(x,a) < \varepsilon$. Consequently, $a \in B(x,\varepsilon) \subseteq V$, therefore $V \cap A \neq \emptyset$, and finally, $x \in \overline{A}$.

4.5.7 By virtue of P.4.5.6 we have

$$x \in \overline{A} \iff (\forall\varepsilon)(\exists a)(\varepsilon > 0 \to a \in A,\ (\rho(x,a) < \varepsilon)),$$

hence

$$x \in \overline{A} \iff (\forall\varepsilon)(\varepsilon > 0 \to \inf_{a\in A} \rho(x,a) < \varepsilon),$$

that is,

$$x \in \overline{A} \iff (\forall\varepsilon)(\varepsilon > 0 \to d(x,A) < \varepsilon),$$

i.e.,

$$x \in \overline{A} \iff d(x,A) = 0.$$

4.5.8 From $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$ it follows that

$$d(\overline{A},\overline{B}) \leq d(A,B). \qquad (\diamond)$$

Let $\varepsilon > 0$. Then there exist the points $x \in \overline{A}, y \in \overline{B}$ such that

$$\rho(x,y) < d(\overline{A},\overline{B}) + \varepsilon,$$

and by virtue of Problem P.4.5.6 there exist points $a \in A, b \in B$ such that

$$\rho(x,a) < \varepsilon, \quad \rho(y,b) < \varepsilon,$$

therefore

$$d(A,B) \leq \rho(a,b) \leq \rho(a,x) + \rho(x,y) + \rho(y,b) < d(\overline{A},\overline{B}) + 3\varepsilon$$

for all $\varepsilon > 0$, and consequently

$$d(\overline{A},\overline{B}) \geq d(A,B). \tag{$\diamond\diamond$}$$

From ($\diamond$) and ($\diamond\diamond$) it follows that

$$d(\overline{A},\overline{B}) = d(A,B).$$

$\overset{\text{A}}{4.5.9}$ We can write

$$D(A,B) = \max\{\sup_{a\in A}\{d(a,B)\}, \sup_{b\in B}\{d(b,A)\}\}.$$

If $A = B$, then

$$d(a,B) = 0, \quad d(b,A) = 0,$$

hence

$$D(A,B) = 0.$$

If $D(A,B) = 0$, then

$$d(a,B) = 0, \quad d(b,A) = 0, \quad \forall\, a \in A,\ b \in B,$$

therefore, using Problem P.4.5.7, we obtain

$$A \subset \overline{B} = B, \quad B \subset \overline{A} = A,$$

i.e. $A = B$.

The symmetry of the mapping D is obvious.

From

$$(\forall a)(\forall b)(\forall c)(a \in A,\ b \in B,\ c \in C \rightarrow \rho(a,b) \leq \rho(a,c) + \rho(c,b)),$$

we deduce:

$$\begin{aligned}(\forall a)(\forall c)(a \in A,\ c \in C \to d(a,B) &\leq \rho(a,c) + d(c,B)\\ &\leq \rho(a,c) + D(C,B)),\end{aligned}$$

$$\begin{aligned}(\forall a)(a \in A \to d(a,B) &\leq d(a,C) + D(C,B)\\ &\leq D(A,C) + D(C,B)),\end{aligned}$$

hence

$$\sup_{a\in A}\{d(a,B)\} \leq D(A,C) + D(C,B)). \qquad (\diamond)$$

Similarly, we obtain

$$\sup_{b\in B}\{d(b,A)\} \leq D(A,C) + D(C,B)). \qquad (\diamond\diamond)$$

Relations $(\diamond)$ and $(\diamond\diamond)$ yield

$$D(A,B) \leq D(A,C) + D(C,B).$$

4.5.10 (A) Let $y_0 \in \mathbb{R}^2 \setminus A$. Consider the continuous function

$$\rho(\cdot, y_0) : \mathbb{R}^2 \to [0, \infty),$$

where $\rho(x, y_0)$ is the Euclidian distance between the points x and y_0. It attains its supremum on the compact set A. So, there exists a point $x_0 \in A$ such that

$$\rho(x_0, y_0) = \sup_{x\in A}\{\rho(x, y_0)\}.$$

The straight line perpendicular to the line segment $[x_0, y_0]$ is the solution of the problem.

12.4 Exercises: Sequences of Numbers (I) (Solutions)

5.$\overset{\text{A}}{3}$.1 $n - n^2 \ln \dfrac{n+1}{n} \to \dfrac{1}{2}$.

5.$\overset{\text{A}}{3}$.2 $x_n = \left(n \arctan \dfrac{1}{n}\right)^{n^2} \to 1/\sqrt[3]{\mathrm{e}}$.

5.$\overset{\text{A}}{3}$.3 $\dfrac{(n+1)^a - n^a - an^{a-1}}{n^{a-2}} \to \dfrac{a(a-1)}{2}$.

5.$\overset{\text{A}}{3}$.4 $\sqrt[n]{n} \to 1$.

5.$\overset{\text{A}}{3}$.5 $\dfrac{\sqrt[n]{n!}}{n} \to \dfrac{1}{\mathrm{e}}$.

5.$\overset{\text{A}}{3}$.6 $\left(\dfrac{\ln(n+1)^{n+1}}{\ln n^n}\right)^n \to \mathrm{e}$.

5.$\overset{\text{A}}{3}$.7 $\dfrac{1}{n}\displaystyle\int_1^n \left(1+\frac{1}{x}\right)^x \mathrm{d}x \to \mathrm{e}$.

5.$\overset{\text{A}}{3}$.8 $n \sin(2\pi \mathrm{e} n!) \to 2\pi$.

5.$\overset{\text{A}}{3}$.9 $n^2 \left(\left(1+\dfrac{1}{n}\right)^{n+0.5} - \mathrm{e}\right) = \dfrac{\mathrm{e}}{12}$.

5.$\overset{\text{A}}{3}$.10 $\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) \to \dfrac{1}{\mathrm{e}}$.

5.$\overset{\text{A}}{3}$.11 $\displaystyle\sum_{k=1}^{n} \frac{\sin k}{n+k} \to 0$.

5.$\overset{\text{A}}{3}$.12 $n^{\ln \frac{n+a}{n+b}} \to 1$.

5.$\overset{\text{A}}{3}$.13 $n \ln \dfrac{a\sqrt{n+1} + b\sqrt{n+2} + c\sqrt{n+3}}{a\sqrt{n+2} + b\sqrt{n+3} + c\sqrt{n+4}} \to -\dfrac{1}{2}$.

5.$\overset{\text{A}}{3}$.14 $n^2 \left(a^{\frac{1}{n+b}} - a^{\frac{1}{n+c}}\right) \to (c-b)\ln a$.

5.3.15 $$n^a \frac{\sqrt[n]{n}-1}{\ln n} \to \begin{cases} 0, & \text{if} \quad a<1, \\ 1, & \text{if} \quad a=1, \\ \infty, & \text{if} \quad a>1. \end{cases}$$

5.3.16 $$\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n-1}} \to \infty.$$

5.3.17 $$\frac{a^{6n}}{(1+a^n)(1+2\,a^{2n})(1+3\,a^{3n})} \to \begin{cases} 0, & \text{if} \quad |a|<1, \\ 1/24, & \text{if} \quad a=1, \\ 1/6, & \text{if} \quad a>1. \end{cases}$$

5.3.18 $$\frac{\ln n!}{n^a} \to 0.$$

5.3.19 $$\frac{[n\,a]}{n} \to a.$$

5.3.20 $$\frac{1}{\binom{n}{0}} + \frac{1}{\binom{n}{1}} + \cdots + \frac{1}{\binom{n}{n}} \to 2.$$

5.3.21 $$\left(1+\frac{a}{n}\right)\left(1+\frac{2a}{n}\right)\cdots\left(1+\frac{na}{n}\right) \to \infty.$$

5.3.22 $$\frac{1^r+3^r+\cdots+(2n-1)^r}{n^{r+1}} \to \frac{2^r}{r+1}.$$

5.3.23 $$\frac{1}{n}+\frac{1}{2n}+\cdots+\frac{1}{n^2} \to 0.$$

5.3.24 $$\frac{1}{\ln 2^n}+\frac{1}{\ln 3^n}+\cdots+\frac{1}{\ln n^n} \to 0.$$

5.3.25 $$\frac{1}{\ln n^2}+\frac{1}{\ln n^3}+\cdots+\frac{1}{\ln n^n} \to 1.$$

5.3.26 $$\sum_{k=1}^{n}\left(\frac{1}{3k-2}+\frac{1}{3k-1}-\frac{2}{3k}\right) \to \ln 3.$$

5.3.27 (A) $\displaystyle\sum_{k=1}^{n} \frac{3k^2+3k+1}{k^3(k+1)^3} = 1 - \frac{1}{(n+1)^3} \to 1.$

5.3.28 (A) $\displaystyle\sum_{k=1}^{n} \frac{k+2}{k(k+1)2^k} = 1 - \frac{1}{(n+1)2^n} \to 1.$

5.3.29 (A) $\displaystyle\sum_{k=1}^{n} \frac{k!}{(k+p+1)!} = \frac{1}{p}\left(\frac{1}{p!} - \frac{1}{(n+2)\cdots(n+p+1)}\right) \to \frac{1}{p\cdot p!}.$

5.3.30 (A) $\displaystyle\sum_{k=1}^{n} \frac{k-2}{2^k} = -\frac{n}{2^n} \to 0.$

5.3.31 (A) $\displaystyle\sum_{k=1}^{n} \frac{k\,2^k}{(k+2)!} = 1 - \frac{2^{n+1}}{(n+2)!} \to 1.$

5.3.32 (A) $\displaystyle\sum_{k=1}^{n} \frac{a^{k-1}}{(1+a^k)(1+a^{k+1})} = \frac{1}{a(a-1)}\left(\frac{1}{a+1} - \frac{1}{a^{n+1}+1}\right) a \in \mathbb{R} \setminus \{-1,0,1\}.$

5.3.33 (A) $\displaystyle\sum_{k=1}^{n} \frac{1}{\sqrt[k]{n^k+1}+1} \to 1.$

5.3.34 (A) $\displaystyle\sum_{k=1}^{n} \frac{k^2}{k!} \to 2\,\mathrm{e}.$

5.3.35 (A) $\displaystyle\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n} \to 0.$

5.3.36 (A) $\displaystyle\frac{n^2+1}{n^2-1}\cdot\frac{n^2+2}{n^2-2}\cdots\frac{n^2+n}{n^2-n} \to \mathrm{e}.$

5.3.37 (A) $\displaystyle x_n = \frac{2a\,x_{n-1}}{a+x_{n-1}} = \frac{2^n a x_0}{a+(2^n-1)x_0} \to a.$

5.3.38 (A) $x_n = x_{n-1}^2 - 2\,x_{n-1} + 2 = (x_0-1)^{2^n} + 1.$

5.3.39 (A) $x_n = \mathrm{e}^{-1+x_{n-1}} \to \begin{cases} 1, & \text{if} \quad x_0 \le 1, \\ \infty, & \text{if} \quad x_0 > 1. \end{cases}$

5.3.40 (A) $x_n = 2^{\frac{x_{n-1}}{2}} \to \begin{cases} 2, & \text{if} \quad x_0 < 4, \\ 4, & \text{if} \quad x_0 = 4, \\ \infty, & \text{if} \quad x_0 > 4; \end{cases}$

5.3.41 (A) $x_n = \mathrm{e}^{\frac{x_{n-1}}{\mathrm{e}}} \to \begin{cases} \mathrm{e}, & \text{if} \quad x_0 \le \mathrm{e}, \\ \infty, & \text{if} \quad x_0 > \mathrm{e}. \end{cases}$

5.3.42 (A) $x_n = \dfrac{1}{2}\left(x_{n-1} + \dfrac{a}{x_{n-1}}\right) \to \sqrt{a}.$

5.3.43 (A) $x_n = \exp\left(\dfrac{2}{3}\left(1 + \dfrac{(-1)^{n+1}}{2^n}\right)\right) \to \mathrm{e}^{2/3}.$

5.3.44 (A) $x_n = a^{1+\ln x_{n-1}}; \quad x_n = \exp(\ln a + \cdots + \ln^{n+1} a).$

5.3.45 (A) $x_n = n\left(\left(1 + \dfrac{a}{n}\right)^n - \left(1 + \dfrac{1}{n}\right)^{an}\right) \to \dfrac{a(1-a)}{2}\,\mathrm{e}^a.$

5.3.46 (A) $n\left(\dfrac{1}{\mathrm{e}-1} - \dfrac{1^n + 2^n + \cdots + (n-1)^n}{n^n}\right) \to \dfrac{\mathrm{e}(\mathrm{e}+1)}{2(\mathrm{e}-1)^3}$ (see [1]).

5.3.47 (A) $x_n = \dfrac{x_{n-1}}{1 + (n-1)x_{n-1}^2}; \quad x_n \to 0; \quad n \cdot x_n \to 1.$

5.3.48 (A) $y_n = \dfrac{1}{n}\ln\left(\mathrm{e}^{n x_1} + \mathrm{e}^{n x_2} + \cdots + \mathrm{e}^{n x_n}\right) \approx \max\{x_1, \ldots, x_n\}.$

5.3.49 (A) $x_n = f^{-1}\left(\exp\left(a\left(\dfrac{1-\sqrt{5}}{2}\right)^n + b\left(\dfrac{1+\sqrt{5}}{2}\right)^n\right)\right)$, etc.

5.3.50 (A) $\lim\limits_{n\to\infty} x_n = \begin{cases} a, & \text{if} \quad x_0 < b; \\ b, & \text{if} \quad x_0 = b; \\ \infty, & \text{if} \quad x_0 > b. \end{cases}$

12.5 Exercises: Sequences of Numbers (II) (Solutions)

5.4.1 (1) From

$$\left(1+\frac{1}{n+1}\right)^{n+1} < \mathrm{e} < \left(1+\frac{1}{n}\right)^{n+1},$$

we deduce

$$(n+1)\ \ln\left(1+\frac{1}{n+1}\right) < 1 < (n+1)\ \ln\left(1+\frac{1}{n}\right),$$

and consequently:

$$a_{n+1}-a_n = \frac{1}{n+1} - \ln\left(1+\frac{1}{n+1}\right) > 0,$$

$$b_{n+1}-b_n = \frac{1}{n+1} - \ln\left(1+\frac{1}{n}\right) < 0,$$

$$0 < b_n - a_n = \ln\left(1+\frac{1}{n}\right),$$

hence

$$a_n < a_{n+1} < b_{n+1} < b_n, \quad n \geq 1, \quad \text{and} \quad \lim_{n\to\infty}(b_n - a_n) = 0.$$

It follows that the sequences (a_n) and (b_n) are convergent and have the same limit (usually denoted by γ).

(2) We have:

$$\begin{aligned} &\lim_{n\to\infty} n(\gamma - a_n) \\ &= \lim_{n\to\infty} \frac{a_n - a_{n+1}}{\frac{1}{n+1} - \frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{\frac{1}{n+1} - \ln\left(1+\frac{1}{n+1}\right)}{\frac{1}{n(n+1)}} \\ &= \lim_{x\to 0} \frac{x - \ln(1+x)}{x^2} = \frac{1}{2}. \end{aligned}$$

(3) We have:

$$\lim_{n\to\infty} n^2(c_n - \gamma)$$

$$= \lim_{n\to\infty} \frac{-\frac{1}{n+1} + \frac{1}{2}\ln\frac{n+2}{n}}{\frac{1}{n^2} - \frac{1}{(n+1)^2}}$$

$$= \lim_{x\to 0} \frac{-\frac{x}{x+1} + \frac{1}{2}\ln(1+2x)}{2x^3} = \frac{1}{6}.$$

(4) $\quad 0.576 < a_{1000} < \gamma < b_{1000} < 0.578.$

5.4.2 We have:

$$\ln\left(\frac{a_n}{b_n}\prod_{k=a_n}^{b_n}\sqrt[k]{\mathrm{e}}\right) = \sum_{k=a_n}^{b_n}\frac{1}{k} + \ln a_n - \ln b_n$$

$$= \left(\sum_{k=1}^{b_n}\frac{1}{k} - \ln b_n\right) - \left(\sum_{k=1}^{a_n}\frac{1}{k} - \ln a_n\right) + \frac{1}{a_n}$$

$$\to \gamma - \gamma + 0 = 0,$$

hence

$$\lim_{n\to\infty}\frac{a_n}{b_n}\prod_{k=a_n}^{b_n}\sqrt[k]{\mathrm{e}} = 1.$$

5.4.3 Using Problem P.5.4.2, we obtain: (1) 1; (2) 1; (3) $\ln 2$.

5.4.4 Let n_0 be an arbitrary positive integer and $\varepsilon > 0$.

We must prove that there exists a positive integer p, $p \geq n_0$, such that $|f(x) - f(S_p)| < \varepsilon$.

From the continuity of the function f at the point x it follows that there exists a number $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \quad \forall y \in (x-\delta, x+\delta).$$

Condition (2) implies the existence of an integer m_0 such that

$$|S_{n+1} - S_n| < \delta, \quad \forall n > m_0.$$

We set $p_0 := \max\{n_0, m_0\}$. For sufficiently large integer k, we have

$$S_{p_0} < x + kT.$$

Condition (1) implies the existence of an integer p_1, $p_1 > p_0$, such that

$$x + k\,T < S_{p_1},$$

hence there exists an integer p, $p_0 \leq p < p_1$, such that

$$S_p \leq x + k\,T \leq S_{p+1}.$$

We have

$$|x + k\,T - S_p| \leq S_{p+1} - S_p < \delta,$$

hence

$$|f(x) - f(S_p - k\,T)| \leq \varepsilon,$$

i.e.,

$$|f(x) - f(S_p)| \leq \varepsilon.$$

5.4.5 Using Problem P.5.4.4 we obtain:

$$\mathrm{LIM}(\sin \ln n) = [-1, 1]; \tag{1}$$

$$\mathrm{LIM}(\cos \sqrt{n}) = [-1, 1]; \tag{2}$$

$$\mathrm{LIM}\left(\tan\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\right) = \mathbb{R}; \tag{3}$$

$$\mathrm{LIM}(\sqrt{n} - [\sqrt{n}]) = [0, 1]. \tag{4}$$

5.4.6 (1) Using the Stolz-Cesaro theorem, we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^a}{n^{a+1}} = \lim_{n\to\infty} \frac{(n+1)^a}{(n+1)^{a+1} - n^{a+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^a}{(a+1)\,n^a}\,1_n = \frac{1}{a+1}.$$

(2) Using the Stolz-Cesaro theorem, we have

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} \frac{n^a}{k^{a+1}} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^{a+1}}}{\frac{1}{n^a} - \frac{1}{(n+1)^a}}$$

$$= \lim_{n\to\infty} \frac{n^a}{(n+1)((n+1)^a - n^a)} = \lim_{n\to\infty} \frac{n^a}{a\,n^{a-1}(n+1)\,1_n} = \frac{1}{a}.$$

(3) The Stolz-Cesaro theorem yields

$$\lim_{n\to\infty} n\left(\frac{1}{a+1}-\sum_{k=1}^{n}\frac{k^a}{n^{a+1}}\right)=\lim_{n\to\infty}\frac{n^{a+1}-(a+1)\sum_{k=1}^{n}k^a}{(a+1)\,n^a}$$

$$=\lim_{n\to\infty}\frac{(n+1)^{a+1}-n^{a+1}-(a+1)(n+1)^a}{(a+1)((n+1)^a-n^a)}$$

$$=\lim_{n\to\infty}\frac{n^a+\frac{a}{2}\,n^{a-1}\,1_n-(n+1)^a}{(n+1)^a-n^a}=-1+\frac{\frac{a}{2}\,1_n}{a\,n^{a-1}\,1_n}=-\frac{1}{2}.$$

(4) By virtue of the Stolz-Cesaro theorem, we deduce

$$\lim_{n\to\infty} n\left(\frac{1}{a}-\sum_{k=n+1}^{\infty}\frac{n^a}{k^{a+1}}\right)=\lim_{n\to\infty}\frac{\dfrac{1}{n^a}-a\displaystyle\sum_{k=n+1}^{\infty}\frac{1}{k^{a+1}}}{\dfrac{a}{n^{a+1}}}$$

$$=\lim_{n\to\infty}\frac{\dfrac{1}{n^a}-\dfrac{1}{(n+1)^a}-\dfrac{a}{(n+1)^{a+1}}}{a\left(\dfrac{1}{n^{a+1}}-\dfrac{1}{(n+1)^{a+1}}\right)}$$

$$=\lim_{n\to\infty}\frac{(n+1)^{a+1}-(n+1)n^a-an^a}{a((n+1)^{a+1}-n^{a+1})}\,n$$

$$=\lim_{n\to\infty}\frac{n^{a+1}+(a+1)n^a+\frac{a(a+1)}{2}\,n^{a-1}\,1_n-(n+1)n^a-an^a}{a(a+1)\,n^{a-1}}\,1_n$$

$$=\frac{1}{2}.$$

5.4.7 We shall prove by induction that

$$f_k(n)=\frac{\left(1+\frac{1}{n}\right)^{n^{k+1}}}{\exp\left(n^k-\dfrac{n^{k-1}}{2}+\cdots+\dfrac{(-1)^{k-1}}{k}\,n\right)},\quad n,k\in\mathbb{N}^*.$$

Using

$$\ln\left(1+\frac{1}{n}\right)=\frac{1}{n}-\frac{1}{2n^2}+\cdots+\frac{(-1)^{k-1}}{kn^k}+\frac{(-1)^k}{(k+1)n^{k+1}}+R_k\left(\frac{1}{n}\right),$$

where

$$\lim_{n\to\infty} n^{k+1} R_k\left(\frac{1}{n}\right) = 0,$$

we obtain

$$\begin{aligned}
&\lim_{n\to\infty} f_k(n)\\
&= \lim_{n\to\infty} \exp\left(n^{k+1}\ln\left(1+\frac{1}{n}\right) - n^k + \frac{n^{k-1}}{2} - \dots + \frac{(-1)^{k+1}}{k} n\right)\\
&= \lim_{n\to\infty} \exp\left(\frac{(-1)^k}{k+1}\cdot n^{k+1} R_k\left(\frac{1}{n}\right)\right) = \exp\left(\frac{(-1)^k}{k+1}\right).
\end{aligned}$$

5.4.8 (A) Using Problem P.5.4.7 we obtain $\lim\limits_{n\to\infty} f_3(n) = \dfrac{1}{\sqrt[4]{e}}$.

5.4.9 (A) Using Maclaurin's formula then there exist the numbers $c_{n,k} \in (0, k^a/n^{a+1/p})$, $n \in \mathbb{N}^*$, $k = 1, 2, \dots, n$, such that

$$f\left(\frac{k^a}{n^{a+1/p}}\right) = \frac{k^{ap}}{n^{ap+1}} \frac{f^{(p)}(c_{n,k})}{p!}.$$

We can choose the numbers $x_n, y_n \in \{c_{n,1}, c_{n,2}, \dots, c_{n,n}\} \subset (0, 1/\sqrt[p]{n})$ such that

$$\frac{f^{(p)}(x_n)}{p!} \frac{k^{ap}}{n^{ap+1}} \le f\left(\frac{k^a}{n^{a+1/p}}\right) \le \frac{f^{(p)}(y_n)}{p!} \frac{k^{ap}}{n^{ap+1}},$$

$n \in \mathbb{N}^*$, $k = 1, 2, \dots, n$. Using

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0,$$

and

$$\frac{f^{(p)}(x_n)}{p!} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{ap} \le \sum_{k=1}^{n} f\left(\frac{k^a}{n^{a+1/p}}\right) \le \frac{f^{(p)}(y_n)}{p!} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{ap},$$

we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k^a}{n^{a+1/p}}\right) = \frac{f^{(p)}(0)}{p!(ap+1)}.$$

5.4.10 (A) We use Problem P.5.4.9.

(1) For $a = p = 1$, $f(x) = \sin x$, we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} \sin \frac{k}{n^2} = \frac{1}{2}.$$

(2) For $a = \frac{1}{2}, p = 2$, $f(x) = \sin^2 x$, we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} \sin^2 \frac{\sqrt{k}}{n} = \frac{1}{2}.$$

(3) For $a = p = 1$, $f(x) = 2^x - 1$, we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} (2^{k/n^2} - 1) = \ln \sqrt{2}.$$

(4) We have

$$\ln \prod_{k=1}^{n} \left(1 + \frac{k}{n^2}\right) = \sum_{k=1}^{n} \ln \left(1 + \frac{k}{n^2}\right).$$

For $a = p = 1$, $f(x) = \ln(1 + x)$, we get

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{k}{n^2}\right) = \sqrt{e}.$$

5.4.11 Let $x \in (0, a)$. Denote $x_1 = f(x)$, $x_{n+1} = f(x_n)$, $n = 1, 2, \ldots$

The sequence (x_n) is decreasing and lower bounded by 0. Taking into account the equality

$$f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} x_n,$$

from (2), we deduce

$$\lim_{n\to\infty} x_n = 0.$$

We have:

$$\lim_{n\to\infty} \frac{1}{n\, x_n^{k-1}} = \lim_{n\to\infty} \left(\frac{1}{x_{n+1}^{k-1}} - \frac{1}{x_n^{k-1}} \right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{(f(x_n))^{k-1}} - \frac{1}{x_n^{k-1}} \right) = \lim_{t\to 0} \left(\frac{1}{(f(t))^{k-1}} - \frac{1}{t^{k-1}} \right).$$

Using the Taylor formula, we have

$$f(t) = t + \frac{t^k}{k!} f^{(k)}(c), \qquad 0 < c < t,$$

and consequently,

$$\lim_{n\to\infty} \frac{1}{n\,x_n^{k-1}} = \lim_{t\to 0} \frac{t^{k-1} - (f(t))^{k-1}}{t^{k-1}(f(t))^{k-1}} = \lim_{t\to 0} \frac{1 - \left(1 + \frac{t^{k-1}}{k!} f^{(k)}(c)\right)^{k-1}}{(f(t))^{k-1}}$$

$$= \lim_{t\to 0} \frac{f^{(k)}(c)}{k!} \left(\frac{t}{f(t)}\right)^{k-1} \frac{1 - \left(1 + \frac{t^{k-1}}{k!} f^{(k)}(c)\right)^{k-1}}{\frac{t^{k-1}}{k!} f^{(k)}(c)}$$

$$= \frac{f^{(k)}(0)}{k!} \cdot 1 \cdot \lim_{u\to 0} \frac{1 - (1+u)^{k-1}}{u} = -\frac{f^{(k)}(0)}{k(k-2)!},$$

hence

$$\lim_{n\to\infty} n^{\frac{1}{k-1}} x_n = \left(\frac{-k(k-2)!}{f^{(k)}(0)}\right)^{\frac{1}{k-1}}.$$

5.4.12 (A) (1) Use Problem P.5.4.11 for $f(x) = \sin x$, ($f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$), $k = 3$.

(2) Use Problem P.5.4.11 for $f(x) = \arctan x$, ($f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -2$), $k = 3$.

(3) Use Problem P.5.4.11 for $f(x) = \ln(1+x)$, ($f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$), $k = 2$.

5.4.13 (A) Using the Stolz-Cesaro theorem, we have

$$\lim_{n\to\infty} \frac{a_n}{1 + \frac{1}{2} + \cdots + \frac{1}{n-1}} = \lim_{n\to\infty} n(a_{n+1} - a_n) = K,$$

hence $\lim_{n\to\infty} a_n = \infty$.

5.4.14 (A) Using condition (1) we deduce that the sequence (a_n) is monotonically decreasing and it is lower bounded by 0. It follows that it is convergent. Let a be its limit. Suppose that $a \neq 0$. We have:

$$n(a_{n+1} - a_n) = n^2 \left(f\left(\frac{a_n}{n}\right) - \frac{a_n}{n}\right) = \frac{a_n^2}{2} f''(c_n),$$

$0 \le c_n \le \frac{a_n}{n}$, $n \ge 1$, hence

$$\lim_{n\to\infty} n(a_{n+1} - a_n) = \frac{a^2}{2} f''(0) \ne 0.$$

Using Problem P.5.4.13, it follows that (a_n) is unbounded, which contradicts the fact that it is convergent. It follows that $a = 0$.

5.4.15 (A) (1) Problem P.5.4.14, for $f(x) = \ln(1+x)$, yields $\lim\limits_{n\to\infty} a_n = 0$.

(2) Problem P.5.4.11, for $f(x) = \ln(1+x)$, yields

$$b_n = \underbrace{f \circ f \circ \cdots \circ f}_{(n-1)-times}(b_1),$$

hence

$$\lim_{n\to\infty} b_n = 2.$$

5.4.16 (A) (The particular case $f(x) = 1 - e^{-x}$ was proposed by Ioan Raşa and solved by Mircea Rus).

$$\lim_{n\to\infty} n \underbrace{f \circ f \circ \cdots \circ f}_{n-times}(x_n) = \frac{2L}{2 - L f''(0)}.$$

5.4.17 (A) The sequence (a_n) is monotonically decreasing and

$$\gamma < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n < a_n,$$

hence it is convergent. We have

$$\gamma - 1 - \frac{1}{2} - \cdots - \frac{1}{5} + \tan 1 + \tan\frac{1}{2} + \cdots + \tan\frac{1}{5}$$

$$< 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n - 1 - \frac{1}{2} - \cdots - \frac{1}{5} + \tan 1 + \tan\frac{1}{2} + \cdots + \tan\frac{1}{5}$$

$$< \tan 1 + \tan\frac{1}{2} + \cdots + \tan\frac{1}{n} - \ln n = a_n,$$

hence

$$\lim_{n\to\infty} a_n$$

$$> \gamma - 1 - \frac{1}{2} - \cdots - \frac{1}{5} + \tan 1 + \tan \frac{1}{2} + \cdots + \tan \frac{1}{5}$$
$$= 1.20189\ldots$$

On the other hand, we have:

$$a_n = \tan 1 + \tan \frac{1}{2} + \cdots + \tan \frac{1}{5} - \ln n + \sum_{k=6}^{n} \int_0^{1/k} \frac{dx}{\cos^2 x}$$
$$< \tan 1 + \tan \frac{1}{2} + \cdots + \tan \frac{1}{5} - \ln n + \sum_{k=6}^{n} \int_0^{1/k} \frac{dx}{1 - x^2}$$
$$= \tan 1 + \tan \frac{1}{2} + \cdots + \tan \frac{1}{5} - \ln n + \frac{1}{2} \sum_{k=6}^{n} \ln \frac{k+1}{k-1}$$
$$= \tan 1 + \tan \frac{1}{2} + \tan \frac{1}{3} + \tan \frac{1}{4} + \tan \frac{1}{5} - \frac{1}{2} \ln \frac{30n}{n+1},$$

hence

$$\lim_{n\to\infty} a_n$$
$$< \tan 1 + \tan \frac{1}{2} + \tan \frac{1}{3} + \tan \frac{1}{4} + \tan \frac{1}{5} - \frac{1}{2} \ln 30$$
$$= 1.207417\ldots,$$

finally,

$$\lim_{n\to\infty} \lfloor 100\, a_n \rfloor = 120.$$

5.4.18 (A) Consider the sequences $(a_n)_{n\geq 5}$, $(b_n)_{n\geq 5}$,

$$a_n = \sqrt{1 + \sqrt{2 + \cdots + \sqrt{n}}},$$

$$b_n = \sqrt{1 + \sqrt{2 + \cdots + \sqrt{n + \sqrt{2n}}}}.$$

One can easily prove that (a_n) is monotonically increasing, (b_n) is monotonically decreasing, and

$$a_n < b_n, \qquad n \geq 5,$$

hence

$$1.7575 < a_6 < \lim_{n\to\infty} a_n < b_6 < 1.7579,$$

consequently,

$$\lim_{n\to\infty}\left[1000\sqrt{1+\sqrt{2+\cdots+\sqrt{n}}}\right] = 1757.$$

5.4.19 (A) It is known that if $w \in \mathbb{R}\setminus\mathbb{Q}$, then there exist k and m integers such that

$$|w - \frac{k}{m}| < \frac{1}{mn} \quad [28, \text{p. } 146].$$

It follows that there exist $k_n, m_n \in \mathbb{N}^*$ such that

$$|2\, m_n\, \pi - k_n| < \frac{1}{n},$$

consequently

$$|\sin k_n| < \sin\frac{1}{n} \to 0.$$

5.4.20 (A) An example is given by the sequence $a_n = \begin{cases} 1, & \text{for odd } n; \\ \frac{1}{n}, & \text{for even } n. \end{cases}$

5.4.21 (A) The relation $\limsup \frac{a_{n+1}}{a_n} < 1$ implies the existence of a positive integer n_0 and the existence of a number q, $0 < q < 1$ such that

$$\frac{a_{n+1}}{a_n} < q, \quad \forall n \geq n_0.$$

Consequently

$$\frac{a_{n_0+1}}{a_{n_0}} \cdot \frac{a_{n_0+2}}{a_{n_0+1}} \cdots \frac{a_n}{a_{n-1}} < q^{n-n_0},$$

$$a_n < a_{n_0} q^{n-n_0}, \quad n \geq n_0.$$

We obtain

$$0 \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} \frac{a_{n_0}}{q^{n_0}}\, q^n,$$

hence

$$0 \leq \lim_{n\to\infty} a_n \leq 0.$$

5.4.22 Suppose that the sequence $(\sin nx)$ is convergent. From

$$\cos nx = \frac{\sin(n+1)x - \sin(n-1)x}{2\,\sin x}$$

it follows that

$$\lim_{n\to\infty} \cos nx = 0.$$

Making use of

$$\sin nx = \frac{\cos(n+1)x - \cos(n-1)x}{2\,\sin x}$$

we deduce that

$$\lim_{n\to\infty} \sin nx = 0,$$

hence

$$\lim_{n\to\infty} (\sin^2 nx + \cos^2 nx) = 0,$$

which contradicts the identity

$$\sin^2 nx + \cos^2 nx = 1.$$

It follows that the sequence $(\sin nx)_{n\geq 0}$ is divergent.

5.4.23 Consider an arbitrary number $\varepsilon > 0$. There exists an integer n_0 such that

$$-\varepsilon < a_{n_0+n} - q\,a_{n_0+n-1} < \varepsilon, \quad n \in \mathbb{N}.$$

Using

$$\sum_{k=0}^{n-1} (a_{n_0+n-k} - q\,a_{n_0+p-k-1})\,q^k = a_{n_0+n} - q^n a_{n_0}, \quad n \geq 1,$$

we deduce

$$-\varepsilon(1+q+\cdots+q^{n-1}) < a_{n_0+n} - q^n a_{n_0} < \varepsilon(1+q+\cdots+q^{n-1})$$

Consequently, for $n \to \infty$, we obtain

$$-\frac{\varepsilon}{1-q} \leq \lim_{n\to\infty} a_{n_0+n} \leq \frac{\varepsilon}{1-q},$$

i.e.,

$$\lim_{n\to\infty} a_n = 0.$$

5.4.24 [A] From:

$$\left(\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\cdots\left(1+\frac{1}{2n}\right)\right)^{1+\frac{1}{2n}} < x_n,$$

$$\left(\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\cdots\left(1+\frac{1}{2n}\right)\right)^{1+\frac{1}{n}} > x_n,$$

i.e.,

$$\left(2+\frac{1}{n}\right)^{1+\frac{1}{2n}} < x_n < \left(2+\frac{1}{n}\right)^{1+\frac{1}{n}}, n \geq 1$$

we deduce

$$\lim_{n\to\infty} x_n = 2.$$

5.4.25 [A] (a) From $a_{n+1} - a_n = -a_n^2 < 0$ and $0 < a_n < 1$, we deduce that the sequence (a_n) is convergent. It follows that $\lim_{n\to\infty} a_n = 0$. Using the Stolz-Cesaro theorem we obtain

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{n}{\frac{1}{a_n}} = \lim_{n\to\infty} \frac{n+1-n}{\frac{1}{a_{n+1}} - \frac{1}{a_n}} = \lim_{n\to\infty} 1 - a_n = 1;$$

(b) $$\lim_{n\to\infty} \frac{n(1-na_n)}{\ln n} = \lim_{n\to\infty} \frac{\frac{1}{a_n} - n}{\ln n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{a_{n+1}} - \frac{1}{a_n} - 1}{\ln \frac{n+1}{n}} = \lim_{n\to\infty} \frac{a_n}{(1-a_n)\ln\frac{n+1}{n}}$$

$$= \lim_{n\to\infty} \frac{n\,a_n}{(1-a_n)\ln\left(1+\frac{1}{n}\right)^n} = 1.$$

5.4.26 [A] $$\lim_{n\to\infty} n\,a_n = \lim_{n\to\infty} \frac{n}{\frac{1}{a_n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{a_{n+1}} - \frac{1}{a_n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\ln a_n} - \frac{1}{a_n}}$$

$$= \lim_{x\to 0} \frac{1}{\frac{1}{\ln(1+x)} - \frac{1}{x}} = 2.$$

$$\lim_{n\to\infty} n\,a_n \frac{n - \frac{2}{a_n}}{\ln n} = 2 \lim_{n\to\infty} \frac{n - \frac{2}{a_n}}{\ln n} = 2 \lim_{n\to\infty} \frac{1 - \frac{2}{a_{n+1}} + \frac{2}{a_n}}{\ln\left(1+\frac{1}{n}\right)}$$

$$= 2\lim_{n\to\infty} \frac{n}{\frac{1}{1-\frac{2}{a_{n+1}}+\frac{2}{a_n}}} = 2\lim_{n\to\infty} \frac{1}{\frac{1}{1-\frac{2}{a_{n+2}}+\frac{2}{a_{n+1}}} - \frac{1}{1-\frac{2}{a_{n+1}}+\frac{2}{a_n}}}$$

$$= 2\lim_{n\to\infty} \frac{1}{\frac{1}{1-\frac{2}{\ln(1+\ln(1+x))}+\frac{2}{\ln(1+x)}} - \frac{1}{1-\frac{2}{\ln(1+x)}+\frac{2}{x}}} = \frac{2}{3}.$$

5.4.27 (A) $\dfrac{2a}{2+a}$.

5.4.28 (A) Using the Stolz-Cesaro theorem, we obtain:

$$\lim_{n\to\infty} \frac{\ln\frac{1}{a_n}}{\ln n} = \lim_{n\to\infty} \frac{-\ln\frac{a_{n+1}}{a_n}}{\ln\frac{n+1}{n}} = \lim_{n\to\infty} \frac{\ln\frac{a_{n+1}}{a_n}}{\frac{a_{n+1}}{a_n}-1} \frac{n\left(1-\frac{a_{n+1}}{a_n}\right)}{n\ln\frac{n+1}{n}} = 1\,\frac{l}{1}.$$

12.6 Exercises: Series of Numbers (Solutions)

5.8.1 (A) Using the d'Alembert ratio test, we have:

$$\frac{u_{n+1}}{u_n} = \frac{(2n+2)!!}{(n+1)^{n+1}} \frac{n^n}{(2n)!!} = \frac{2}{\left(1+\frac{1}{n}\right)^n} \to \frac{2}{\mathrm{e}} < 1,$$

hence the series is convergent.

5.8.2 (A) Using the Cauchy root test we have:

$$\sqrt[n]{u_n} = a\left(1+\frac{1}{n}\right)^{n+b+c/n} \to a\,\mathrm{e},$$

hence, for $a < 1/\mathrm{e}$, the series is convergent, and for $a > 1/\mathrm{e}$ the series is divergent.

When $a = 1/\mathrm{e}$, using the inequality

$$\left(1+\frac{1}{n}\right)^{n+1} > \mathrm{e},$$

we have

$$u_n = \frac{1}{\mathrm{e}^n}\left(1+\frac{1}{n}\right)^{n(n+1)+(b-1)n+c} > \left(1+\frac{1}{n}\right)^{n(b-1)+c} \to \mathrm{e}^{b-1} > 0,$$

hence $u_n \not\to 0$, and consequently the series is divergent.

5.8.3 (A) If $a \geq 1$, then $a^{n^b} \not\to 0$, hence the series is divergent. If $0 < a < 1$, taking into account that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we use the third comparison test: $\lim_{n\to\infty} a^{n^b} \cdot n^2 = \lim_{x\to\infty} a^x \cdot x^{2/b} = 0$, hence the series is convergent.

5.8.4 (A) Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, then using the third comparison test, we obtain $\lim_{n\to\infty} \frac{n}{n^{1+1/n}} = \lim_{n\to\infty} \sqrt[n]{n} = 1$, hence the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ is divergent.

5.8.5 (A) Using the Raabe-Duhamel test we obtain:

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{a\,n}{n+1} = a,$$

hence for $a < 1$ the series is divergent, and for $a > 1$ the series is convergent. For $a = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n+1}$, is divergent.

5.8.6 (A) We use the third comparison test and the fact that the harmonic series $\sum \frac{1}{n}$ is divergent. We have:

$$\lim_{n\to\infty} \frac{u_n}{\frac{1}{n}} = \lim_{n\to\infty} n^2 \left(\left(1+\frac{1}{n+1}\right)^{n+1} - \left(1+\frac{1}{n}\right)^n\right)$$

$$= \lim_{n\to\infty} n^2 \left(\mathrm{e}^{(n+1)\ln(1+1/(n+1))} - \mathrm{e}^{n\ln(1+1/n)}\right)$$

$$= \lim_{n\to\infty} n^2\, \mathrm{e}^{n\ln(1+1/n)} \left(\mathrm{e}^{(n+1)\ln(1+1/(n+1)) - n\ln(1+1/n)} - 1\right)$$

$$= \mathrm{e} \lim_{n\to\infty} n^2 \left((n+1)\ln\left(1+\frac{1}{n+1}\right) - n\ln\left(1+\frac{1}{n}\right)\right) = \frac{\mathrm{e}}{2}.$$

Therefore the series is divergent.

5.8.7 (A) Using the equivalent relations:

$$(n+1)^{1+1/\sqrt{\log_2(n+1)}} > n^{1+1/\sqrt{\log_2 n}}$$

$$\iff (1+1/\sqrt{\log_2(n+1)})\log_2(n+1) > (1+1/\sqrt{\log_2 n})\log_2 n$$

$$\iff \log_2(n+1) + \sqrt{\log_2(n+1)} > \log_2 n + \sqrt{\log_2 n},$$

we deduce that the general term of the series is decreasing. By virtue of the Cauchy condensation test, using the fact that the series

$$\sum 2^n \frac{1}{2^{n(1+1/\sqrt{n})}} = \sum \frac{1}{2^{\sqrt{n}}}$$

is convergent, we deduce that $\sum_{n\geq 2} n^{-1-1/\sqrt{\log_2 n}}$ is convergent.

5.$\overset{\mathrm{A}}{8}$.8 We have

$$\sum_{n=1}^{\infty} \frac{1}{(a+n)\dots(a+n+p)} = \frac{1}{p}\sum_{n=1}^{\infty} \frac{(a+p+n)-(a+n)}{(a+n)\dots(a+n+p)}$$

$$= \frac{1}{p}\sum_{k=1}^{\infty}\left(\frac{1}{(a+n)\dots(a+n+p-1)} - \frac{1}{(a+n+1)\dots(a+n+p)}\right)$$

$$= \lim_{n\to\infty}\frac{1}{p}\left(\frac{1}{(a+1)\dots(a+p)} - \frac{1}{(a+n+1)\dots(a+n+p)}\right)$$

$$= \frac{1}{p}\,\frac{1}{(a+1)(a+2)\dots(a+p)}.$$

5.$\overset{\mathrm{A}}{8}$.9 We have:

$$\sum_{k=1}^{n} k(k+1)\dots(k+p)$$

$$= \frac{1}{p+2}\sum_{k=1}^{n}((k+p+1)-(k-1))\cdot k(k+1)\dots(k+p)$$

$$= \frac{1}{p+2}\sum_{k=1}^{n}(k(k+1)\dots(k+p+1)-(k-1)k\dots(k+p))$$

$$= \frac{n(n+1)\dots(n+p+1)}{p+2},$$

hence

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} k(k+1)\dots(k+p)\right)^{-1} = (p+2)\sum_{n=1}^{\infty}\frac{1}{n(n+1)\dots(n+p+1)}.$$

Using P.5.8.8 with $a := 0$, $p := p+1$, we find that the sum of the series is $\dfrac{p+2}{(p+1)(p+1)!}$.

5.$\overset{\mathrm{A}}{8}$.10 We have

$$a^{3^n}\,\frac{a^{3^n}-1}{a^{3^{n+1}}+1} = \frac{1}{a^{3^n}+1} - \frac{1}{a^{3^{n+1}}+1}.$$

Consequently,

$$\sum_{n=0}^{\infty} a^{3^n} \frac{a^{3^n}-1}{a^{3^{n+1}}+1} = \lim_{n\to\infty}\left(\frac{1}{a+1} - \frac{1}{a^{3^{n+1}}+1}\right)$$

$$= \begin{cases} \frac{-a}{a+1} & \text{if } |a|<1, \\ \frac{1}{a+1} & \text{if } |a|>1, \\ 0 & \text{if } a=1. \end{cases}$$

5.$\overset{\text{A}}{8}$.11 We have:

$$\frac{1}{k(k+1)(k+1)!} = \frac{1}{k(k+1)!} - \frac{1}{(k+1)(k+1)!}$$

$$= \frac{1}{k(k+1)!} - \frac{k+2}{(k+1)(k+2)!} = \frac{1}{k(k+1)!} - \frac{1}{(k+1)(k+2)!} - \frac{1}{(k+2)!}$$

$$= \frac{1}{k(k+1)!} - \frac{1}{(k+1)k+2)!} - \frac{1}{(k+2)!}.$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+1)!} = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)!} + \sum_{k=1}^{n} \frac{1}{(k+2)!}\right)$$

$$= \frac{1}{2} - \frac{5}{2} + e = 3 - e.$$

5.$\overset{\text{A}}{8}$.12 Using the identity

$$\arctan x - \arctan y = \arctan \frac{x-y}{1+xy}, \quad xy > -1,$$

we obtain

$$\sum_{n=0}^{\infty} \arctan \frac{1}{n^2+n+1} = \sum_{n=0}^{\infty} (\arctan(n+1) - \arctan n)$$

$$= \lim_{n\to\infty} (\arctan(n+1) - \arctan 0) = \frac{\pi}{2}.$$

5.8.13 (marked A) Using the identity

$$\arctan x - \arctan y = \arctan \frac{x-y}{1+xy}, \quad xy > -1,$$

we obtain

$$\sum_{n=1}^{\infty} \arctan \frac{1}{2\,n^2} = \sum_{n=1}^{\infty} (\arctan(2n+1) - \arctan(2n-1))$$

$$= \lim_{n\to\infty} (\arctan(2n+1) - \arctan 1) = \frac{\pi}{4}.$$

5.8.14 (marked A) Using the identity

$$\arctan x - \arctan y = \arctan \frac{x-y}{1+xy}, \quad xy > -1,$$

we obtain

$$\sum_{n=1}^{\infty} \arctan \frac{2}{n^2} = \sum_{n=1}^{\infty} (\arctan(n+1) - \arctan(n-1))$$

$$= \lim_{n\to\infty} (\arctan(n+1) + \arctan n - \arctan 1) = \frac{3\,\pi}{4}.$$

5.8.15 (marked A) Using the Leibniz test it follows that the series is convergent. In order to find the sum of the series we can calculate the limit of the partial sum

$$S_{2n} = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$$

$$= \left(1 + \frac{1}{2} + \cdots + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)$$

$$= \left(1 + \frac{1}{2} + \cdots + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

$$= \left(1 + \frac{1}{2} + \cdots + \frac{1}{2n} - \ln 2n\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right) + \ln 2$$

$$\to C - C + \ln 2 = \ln 2.$$

5.8.16 We have:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} - 1$$
$$= 2 \ln 2 - 1 \quad \text{(see P.5.8.15).}$$

5.8.17 Using the identity

$$\left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2x \rfloor - \lfloor x \rfloor, \qquad x \in \mathbb{R},$$

we deduce

$$\sum_{n=0}^{\infty} \left\lfloor \frac{x+2^n}{2^{n+1}} \right\rfloor = \sum_{n=0}^{\infty} \left(\left\lfloor \frac{x}{2^n} \right\rfloor - \left\lfloor \frac{x}{2^{n+1}} \right\rfloor \right)$$
$$= \lim_{n\to\infty} \left(\lfloor x \rfloor - \left\lfloor \frac{x}{2^{n+1}} \right\rfloor \right)$$
$$= \begin{cases} \lfloor x \rfloor, & \text{if } x \geq 0, \\ \lfloor x \rfloor + 1, & \text{if } x < 0. \end{cases}$$

5.8.18 We start from the equality

$$\int_0^1 (1-x^2)^n \, dx = \frac{(2n)!!}{(2n+1)!!}, \qquad n \geq 0.$$

For $a \neq 0$, we obtain:

$$S(a) = \sum_{n\geq 0} \frac{(2n)!!}{(2n+1)!!} a^n = \int_0^1 \frac{dx}{1 - a(1-x^2)} = \frac{1}{a} \int_0^1 \frac{dx}{x^2 + \frac{1-a}{a}}.$$

If $a \in [-1, 0)$, then

$$S(a) = -\frac{1}{2\sqrt{a^2-a}} \ln \left| \frac{x - \sqrt{\frac{a}{a-1}}}{x + \sqrt{\frac{a}{a-1}}} \right| \Bigg|_0^1$$

$$= \frac{1}{2\sqrt{a^2 - a}} \ln \frac{\sqrt{1-a} + \sqrt{-a}}{\sqrt{1-a} - \sqrt{-a}}.$$

If $a \in (0, 1)$, then

$$S(a) = \frac{1}{\sqrt{a - a^2}} \arctan x \sqrt{\frac{a}{1-a}} \Bigg|_0^1 = \frac{1}{\sqrt{a-a^2}} \arctan \sqrt{\frac{a}{1-a}}.$$

Finally, if $a = 0$ then $S(a) = 1$.

5.8.19 (A) We have

$$\sum_{n \geq 0} \frac{(2n-1)!!}{(2n)!!} a^n = \frac{2}{\pi} \sum_{n \geq 0} a^n \int_0^{\pi/2} \sin^2 nx \, \mathrm{d}x$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \frac{\mathrm{d}x}{1 - a \sin^2 x} = \frac{1}{\sqrt{1-a}}.$$

5.8.20 (A) We obtain:

$$\sum_{n \geq 0} \frac{(n!)^2}{(2n+1)!} a^n = \sum_{n \geq 0} B(n+1, n+1)\, a^n = \sum_{n \geq 0} \int_0^1 x^n (1-x)^n \, \mathrm{d}x \, a^n$$

$$= \int_0^1 \frac{\mathrm{d}x}{1 - ax(1-x)}.$$

5.8.21 (A) Let $f(a) = \displaystyle\int_0^1 \frac{\mathrm{d}x}{1 - ax + ax^2}$, $\quad a \in [-4, 4)$. We have:

$$a\, f(a^2) = \sum_{n \geq 0} \frac{a^{2n+1}}{(2n+1)!} (n!)^2,$$

$$f(a^2) + 2a^2 f'(a^2) = \sum_{n \geq 0} \frac{a^{2n}}{(2n)!} (n!)^2,$$

hence

$$\sum_{n \geq 0} \frac{1}{\binom{2n}{n}} = \sum_{n \geq 0} \frac{(n!)^2}{(2n)!} = f(1) + 2f'(1) = \frac{2\left(18 + \sqrt{3}\,\pi\right)}{27}.$$

5.8.22 [A] For $|t| < 1$, we have,

$$\sum_{n=0}^{\infty} \frac{2^n t^{2^n}}{1+t^{2^n}} = t \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} \log\left(1+t^{2^n}\right) = t \frac{\mathrm{d}}{\mathrm{d}t} \sum_{n=0}^{\infty} \log\left(1+t^{2^n}\right)$$

$$= t \frac{\mathrm{d}}{\mathrm{d}t} \log \prod_{n=0}^{\infty} \left(1+t^{2^n}\right) = t \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{1}{1-t} = \frac{t}{1-t}$$

For $t = \frac{1}{x}$, we obtain

$$\sum_{n=0}^{\infty} \frac{2^n}{1+x^{2^n}} = \frac{1}{x-1}, \qquad |x| > 1.$$

5.8.23 [A] If m is the degree of P, Newton's expansion gives

$$P(x) = \sum_{i=0}^{m} \binom{x}{i} \Delta^i P(0)$$

We have

$$\sum_{n=0}^{\infty} \frac{P(n)}{n!} = \sum_{i=0}^{m} \Delta^i P(0) \sum_{n=0}^{\infty} \frac{\binom{n}{i}}{n!} = \sum_{i=0}^{m} \frac{\Delta^i P(0)}{i!} \sum_{n=i}^{\infty} \frac{1}{(n-i)!}$$

$$= \mathrm{e} \sum_{i=0}^{m} \frac{\Delta^i P(0)}{i!}.$$

5.8.24 [A] B_n is the number of ways a set of n elements can be partitioned into nonempty subsets and is called a Bell number [30].

We have:

$$\begin{aligned}
\mathsf{e}\, B_{p+1} &= \sum_{n\geq 1} \frac{n^{p+1}}{n!} = \sum_{n\geq 1} \frac{1+n^p-1}{(n-1)!} \\
&= \sum_{n\geq 1} \frac{1}{(n-1)!} + \sum_{n\geq 1} \frac{n^p-1}{(n-1)!} \\
&= \mathsf{e} + \sum_{n\geq 2} \frac{n^p-1}{(n-1)!} = \mathsf{e} + \sum_{n\geq 1} \frac{(n+1)^p-1}{n!} \\
&= \mathsf{e} + \sum_{n\geq 1} \frac{1}{n!} \sum_{k=1}^{p} \binom{p}{k} n^k \\
&= \mathsf{e} + \sum_{k=1}^{p} \binom{p}{k} \sum_{n\geq 1} \frac{n^k}{n!} \\
&= \mathsf{e} + \sum_{k=1}^{p} \binom{p}{k} \mathsf{e}\, B_k.
\end{aligned}$$

We obtain: $B_1 = 1, \quad B_2 = 2, \quad B_3 = 5.$

5.8.25 (A) Consider the function

$$f(x) = \frac{\ln x}{x^\alpha}, \quad \alpha > 0.$$

We have:

$$f'(x) = \frac{1 - \alpha \ln x}{x^{\alpha+1}},$$

$$f''(x) = \frac{-2\alpha - 1 + \alpha(\alpha+1)\ln x}{x^{\alpha+2}}.$$

For large x we have $f''(x) > 0$. Using Lagrange's mean value theorem, and the fact that $-f'$ is decreasing, we obtain:

$$\begin{aligned}
&\frac{\ln S_{n-1}}{S_{n-1}^\alpha} - \frac{\ln S_n}{S_n^\alpha} \\
&> (S_n - S_{n-1}) f'(S_n) \\
&= (S_n - S_{n-1}) \frac{\alpha \ln S_{n-1} - 1}{S_n^{\alpha+1}} > \frac{S_n - S_{n-1}}{S_n^{\alpha+1}}
\end{aligned}$$

$$= \frac{u_n}{S_n^{\alpha+1}},$$

from a certain index m onwards. Consequently,

$$\sum_{n>m} \frac{u_n}{S_n^{\alpha+1}}$$

$$< \sum_{n>m} \left(\frac{\ln S_{n-1}}{S_{n-1}^{\alpha}} - \frac{\ln S_n}{S_n^{\alpha}} \right)$$

$$= \frac{\ln S_m}{S_m^{\alpha}},$$

i.e., the series $\sum \frac{u_n}{S_n^{\alpha+1}}$ is convergent.

(ii) We have

$$\sum \frac{u_n}{S_n} = \sum \left(1 - \frac{S_{n-1}}{S_n}\right).$$

If $\frac{S_{n-1}}{S_n} \not\to 1$, then $\frac{u_n}{S_n} \not\to 0$, hence the series $\sum \frac{u_n}{S_n}$ is divergent.

If $\frac{S_{n-1}}{S_n} \to 1$, then $\lim\limits_{n\to\infty} \frac{1 - S_{n-1}/S_n}{\ln(S_{n-1}/S_n)} = \lim\limits_{x\to 1} \frac{x-1}{\ln x} = -1$. Therefore the series

$$\sum \left(1 - \frac{S_{n-1}}{S_n}\right)$$

is divergent (like the series $\sum(\ln S_n - \ln S_{n-1})$).

5.8.26 (A) Suppose that the series $\sum b_n$ is convergent and let S be its sum. Choosing $M = S$, we have

$$b_1 + b_2 + \cdots + b_n < S < S + n\,b_n, \quad \forall n \geq 1.$$

(2). Suppose that

$$b_1 + b_2 + \cdots + b_n < M + n\,b_n, \quad \forall n \geq 1.$$

We have:

$$b_1 - b_n + \cdots + b_k - b_n \leq b_1 - b_n + \cdots + b_{n-1} - b_n < M,$$

$\forall n, k \in \mathbb{N}$, $n > k$.

For $n \to \infty$, we obtain

$$b_1 + b_2 + \cdots + b_k \le M, \quad \forall k \in \mathbb{N},$$

hence the series $\sum b_n$ is convergent.

5.8.27 [A] Let $\ell = \lim_{n\to\infty} n\, u_n$. It is obviously that $\ell \ge 0$. If $\ell > 0$, then using the result

$$\lim_{n\to\infty} \frac{u_n}{\frac{1}{n}} = \ell,$$

and the fact that the harmonic series is divergent, we deduce that the series $\sum u_n$ is divergent. Therefore $\ell = 0$.

5.8.28 [A] Since the series $\sum b_n$ is convergent, then using P.5.8.26, we deduce that the sequence

$$a_n = b_1 + b_2 + \cdots + b_n - n\, b_n, \quad n \ge 1,$$

is bounded. Furthermore, from

$$a_{n+1} - a_n = n(b_n - b_{n+1}) > 0, \quad n \in \mathbb{N}^*,$$

we deduce that the sequence (a_n) is monotone and hence is convergent. This implies that the sequence $(n\, b_n)$ is convergent, and using P.5.8.27, it follows that $\lim_{n\to\infty} n\, b_n = 0$.

5.8.29 [A] We set

$$u_n = \begin{cases} \frac{1}{2^n}, & \text{if } \forall k \in \mathbb{N},\ n \ne 2^k; \\ \frac{1}{n}, & \text{if } \exists k \in \mathbb{N},\ n = 2^k. \end{cases}$$

We observe that:

$$\sum u_n < \sum_{n\ge 0} \frac{1}{2^n} + \sum_{n \ge 1} \frac{1}{2^n} = 3,$$

$$2^n\, u_{2^n} = 1 \not\to 0.$$

5.8.30 [A] See [14].

12.7 Exercises: Continuity of Functions (Solutions)

6.5.1 (A) Let $x = a_n$ be an arbitrary point of A. Since the set A is countable, each interval $(x - \frac{1}{k}, x + \frac{1}{k})$, $k \in \mathbb{N}^*$, contains a point $y_k \in \mathbb{R} \setminus A$. Hence, we obtain a sequence $(y_k)_{k \geq 1}$ convergent to x. We have

$$\lim_{k \to \infty} f(y_k) = 0 \neq \frac{1}{n} = f(x),$$

i.e., f is discontinuous at the point x.

Let $x \in \mathbb{R} \setminus A$ and $\varepsilon > 0$. Consider a point $p \in \mathbb{N}^*$ such that $\frac{1}{p} < \varepsilon$. By denoting $\delta = \min\limits_{k=\overline{1,p}} |x - a_k| > 0$, we deduce that $a_n \notin (x - \delta, x + \delta)$, $\forall n \leq p$. Consider an arbitrary $y \in (x - \delta, x + \delta)$. If $y \in A$, it follows that there exists $n \in \mathbb{N}$, $n > p$, such that $y = a_n$, hence $|f(y)| = \frac{1}{n} < \varepsilon$. If $y \in \mathbb{R} \setminus A$, then $f(y) = 0 < \varepsilon$; that is $|f(y)| < \varepsilon$ for all $y \in (x - \delta, x + \delta)$ and consequently, f is continuous at x.

6.5.2 (A) Case $m = 2$. Consider the sequence $\left(\frac{1}{n}, \frac{1}{n}\right) \to (0, 0)$;

$$f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{2} \not\to 0 = f(0, 0),$$

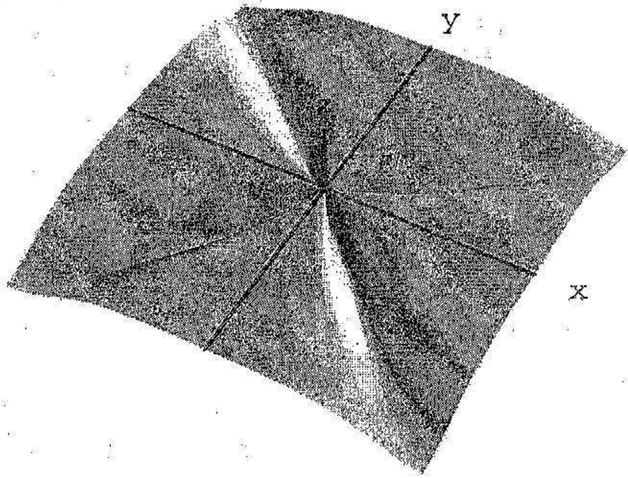
hence f is discontinuous at the point $(0, \ldots, 0)$.

Case $m > 2$. We have:

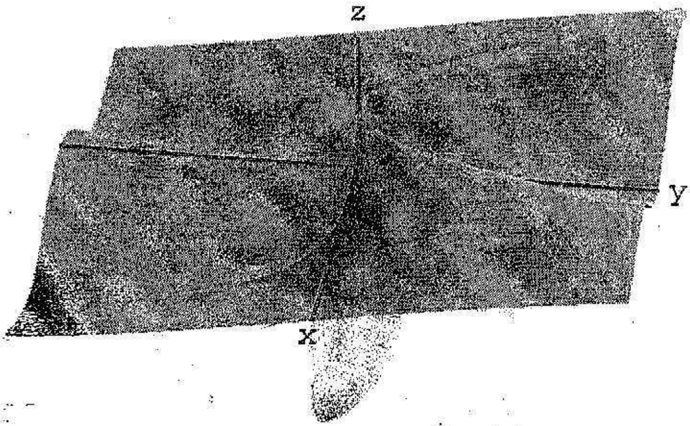
$$\begin{aligned} |x_1 \ldots x_m| &= |x_1 \ldots x_m|^{1-2/m} \sqrt[m]{x_1^2 \ldots x_m^2} \\ &\leq |x_1 \ldots x_m|^{1-2/m} \frac{x_1^2 + \cdots + x_m^2}{m}. \end{aligned}$$

Consequently, $|f(x_1, \ldots, x_m)| \leq \dfrac{|x_1 \ldots x_m|^{1-2/m}}{m} \to 0$, when $(x_1, \ldots, x_m) \to (0, \ldots, 0)$, hence f is continuous at the point $(0, \ldots, 0)$.

6.5.3 (A) Consider the sequence $\left(\frac{1}{n}, \frac{1}{n}\right) \to (0, 0)$; $f\left(\frac{1}{n}, \frac{1}{n}\right) = -\frac{1}{2} \not\to 0 = f(0, 0)$, hence, f is discontinuous at $(0, 0)$.



6.5.4 (A) Consider the sequence $\left(\frac{1}{n}, \frac{1}{n^2}\right) \to (0,0)$; $f\left(\frac{1}{n}, \frac{1}{n^2}\right) = \frac{1}{2} \not\to 0 = f(0,0)$, hence, f is discontinuous at $(0,0)$.



6.5.5 (A) Consider a point $(x, 0)$, $x \neq 0$. The sequence $\left(\left(x, \frac{1}{x(\pi/2+2n\pi)}\right)\right)_{n\in\mathbb{N}}$ tends to $(x, 0)$. However,

$$\left(f\left(x, \frac{1}{x(\pi/2+2n\pi)}\right)\right)_{n\in\mathbb{N}} = x \not\to 0 = f(x, 0).$$

It follows that f is discontinuous at the point $(x, 0)$.

Consider a point $(0, y)$. Let $(u, v) \to (0, y)$. We have $|f(u, v)| \leq |u| \to 0$, hence f is continuous at the point $(0, y)$. Consequently, the set of the points of discontinuity of the function f is

$$D = \{(x, 0) \mid x \in \mathbb{R} \setminus \{0\}\}.$$

6.5.6 (A) Since f is injective and continuous then the mapping $f^* : (a, b) \to f((a, b))$, $f^*(x) = f(x)$, is a continuous bijection. Since f is continuous, then $f((a, b))$ is a conex set (on the real axis, an interval). On the other hand, $f((a, b))$ is the counter image of the open set (a, b) under the continuous mapping $(f^*)^{-1}$, hence it is an open set.

6.5.7 (A) Suppose that f has a finite number of points of discontinuity, $x_1 < \ldots, x_n$. We add, if necessary, the points $x_0 = 0$, $x_{n+1} = 1$. We have

$$\bigcup_{i=1}^{n+1} f((x_{i-1}, x_i)) = (0, 1] \setminus \bigcup_{i=0}^{n+1} \{f(x_i)\} = (0, 1) \setminus \bigcup_{i=0}^{n} \{f(x_i)\}.$$

We remark the fact that the set in the right hand side of the equality is the union of $n + 2$ open intervals, at least. This contradicts the fact that the left hand side of the previous equality is the union of $n + 1$ open intervals (cf. P.6.5.6). Therefore, f has an infinite number of points of discontinuity.

12.8 Exercises: Derivatives (Solutions)

8.10.1 $f(x) = \sum_{n=0}^{\infty} \frac{a_n}{e^n} \frac{x^n}{n!}$.

8.10.2 We have:

$$\left| \int_{ax}^{bx} \sin \frac{1}{t} \, \mathrm{d}t \right| \le b^2 x^2 + a^2 x^2 + 2 \left| \int_{ax}^{bx} t \cos \frac{1}{t} \, \mathrm{d}t \right| < 2b^2 x^2,$$

hence

$$\left| \frac{f(x)}{x} \right| \le 2 \, b^2 \, x;$$

consequently, $f'(0) = 0$.

8.10.3 We have:

$$f'(x) = \sin \frac{1}{x} - \cos x \sin \frac{1}{\sin x}, \qquad x \in \mathbb{R} \setminus \{0\}.$$

The function f is continuous at the point 0 and

$$\begin{aligned} \lim_{x \to 0} f'(x) &= \lim_{x \to 0} \left(\sin \frac{1}{x} - \sin \frac{1}{\sin x} + (1 - \cos x) \sin \frac{1}{\sin x} \right) \\ &= \lim_{x \to 0} \left(\sin \frac{1}{x} - \sin \frac{1}{\sin x} \right) + 0 \\ &= 2 \lim_{x \to 0} \sin \frac{\frac{1}{x} - \frac{1}{\sin x}}{2} \cos \frac{\frac{1}{x} + \frac{1}{\sin x}}{2} = 0. \end{aligned}$$

It follows that there exists $f'(0) = 0$ and the derivative f' is continuous at the point 0.

8.10.4 We have:

$$\left| \int_{\ln(1+x)}^{x} \cos \frac{1}{t} \, \mathrm{d}t \right| \le x - \ln(1 + x), \quad x \in (-1, 1) \setminus \{0\}.$$

It follows that

$$\lim_{x \to 0} \left| \frac{f(x)}{x} \right| \le \lim_{x \to 0} \frac{x - \ln(1 + x)}{x} = 0,$$

hence there exists $f'(0) = 0$. We obtain:

$$f'(x) = \cos\frac{1}{x} - \frac{1}{x+1}\cos\frac{1}{\ln(x+1)}$$

$$= \cos\frac{1}{x} - \cos\frac{1}{\ln(x+1)} + \frac{x}{x+1}\cos\frac{1}{\ln(x+1)}.$$

We have

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(\cos\frac{1}{x} - \cos\frac{1}{\ln(x+1)}\right)$$

$$= 2\lim_{x\to 0}\sin\frac{\frac{1}{x}+\frac{1}{\ln(1+x)}}{2}\sin\frac{\frac{1}{\ln(x+1)}-\frac{1}{x}}{2}.$$

Let $x_n > 0$, $n \in \mathbb{N}$, be the root of the equation

$$\frac{\frac{1}{x}+\frac{1}{\ln(1+x)}}{2} = 2n\pi + \frac{\pi}{2}.$$

We remark that $x_n \to 0$. It follows that

$$\lim_{n\to\infty} f'(x_n) = \lim_{n\to\infty} 2\sin\frac{\frac{1}{x_n}+\frac{1}{\ln(1+x_n)}}{2}\sin\frac{\frac{1}{\ln(x_n+1)}-\frac{1}{x_n}}{2}$$

$$= \lim_{n\to\infty} 2\cdot 1\cdot\sin\frac{\frac{1}{\ln(x_n+1)}-\frac{1}{x_n}}{2} = 2\sin\frac{1}{4}.$$

We deduce that the derivative f' is discontinuous at the point 0.

8.10.5 (A) We have:

$$x^2 + 2x\cos\alpha + 1 = (x+\cos\alpha + i\sin\alpha)(x+\cos\alpha - i\sin\alpha).$$

Let $r > 0$, $t \in (0,\pi)$ be such that

$$x + \cos\alpha + i\sin\alpha = r(\cos t + i\sin t) = z$$

(cf. $\Im(z) = \sin\alpha > 0$). It follows that:

$$t = \operatorname{arccot}\frac{x+\cos\alpha}{\sin\alpha},$$

$$f(x) = \frac{1}{2i\sin\alpha}\left(\frac{1}{x+\cos\alpha - i\sin\alpha} - \frac{1}{x+\cos\alpha+i\sin\alpha}\right),$$

$$f^{(n)}(x)$$

$$= \frac{(-1)^n n!}{2i \sin\alpha}\left(\frac{1}{(x+\cos\alpha - i\sin\alpha)^{n+1}} - \frac{1}{(x+\cos\alpha + i\sin\alpha)^{n+1}}\right)$$

$$= \frac{(-1)^n n! \sin(n+1)t}{r^{n+1}\sin\alpha},$$

i.e.,

$$f^{(n)}(x) = \frac{(-1)^n n! \sin(n+1)\operatorname{arccot}\frac{x+\cos\alpha}{\sin\alpha}}{(\sqrt{x^2+2x\cos\alpha+1})^{n+1}\sin\alpha}.$$

8.10.6 (A) We have:

$$e^{x(\cos\alpha + i\sin\alpha)} = e^{x\cos\alpha}\cdot\cos(x\sin\alpha) + i\, e^{x\cos\alpha}\cdot\sin(x\sin\alpha).$$

By using the equalities:

$$(e^{xc})^{(n)} = c^n e^{xc},$$

$$(e^{x\cos\alpha}\cdot\cos(x\sin\alpha))^{(n)} + i\,(e^{x\cos\alpha}\cdot\sin(x\sin\alpha))^{(n)}$$
$$= (\cos\alpha + i\sin\alpha)^n e^{x(\cos\alpha+i\sin\alpha)}$$
$$= e^{x\cos\alpha}\left(\cos(x\sin\alpha + n\alpha) + i\sin(x\sin\alpha + n\alpha)\right),$$

it follows that

$$\left(e^{x\cos\alpha}\cos(x\sin\alpha)\right)^{(n)} = e^{x\cos\alpha}\cos(x\sin\alpha + n\alpha),$$

$$\left(e^{x\cos\alpha}\sin(x\sin\alpha)\right)^{(n)} = e^{x\cos\alpha}\sin(x\sin\alpha + n\alpha).$$

8.10.7 (A) We use

$$(x^n)^{(k)} = n^{\underline{k}}x^{n-k}, \qquad k=0,\ldots,n.$$

By virtue of Leibniz's formula, taking the n-th derivative of both sides of the identity

$$x^p\cdot x^q = x^{p+q},$$

we obtain

$$\sum_{k=0}^{n}\binom{n}{k}p^{\underline{k}}q^{\underline{n-k}}x^{p+q-n} = (p+q)^{\underline{n}}x^{p+q-n}.$$

Consequently,

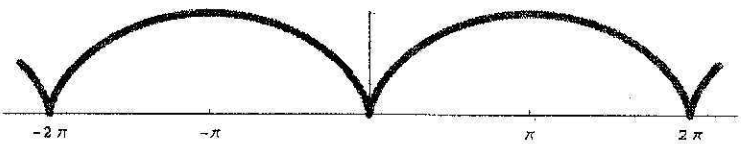
$$\sum_{k=0}^{n} \binom{n}{k} p^{\underline{k}} q^{\underline{n-k}} = (p+q)^{\underline{n}}.$$

8.10.8 We have:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = R \sin t;$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = R\,(1-\cos t);$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{R \sin t}{R\,(1-\cos t)} = \cot\frac{t}{2}, \qquad t \in \mathbb{R} \setminus 2\pi\mathbb{Z}.$$



8.10.9 We have:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{1+t^2};$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t}{|t|}\,\frac{1}{1+t^2};$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \operatorname{sgn} t, \quad t \in \mathbb{R} \setminus \{0\}.$$

8.10.10 We have:

$$y'(1+\tan^2 y) = y + xy';$$

$$y'(1+x^2y^2-x) = y;$$

$$y' = \frac{y}{x^2y^2-x+1}.$$

8.10.11 We have:

$$\frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{y'x-y}{x^2} = \frac{1}{2}\,\frac{1}{x^2+y^2}\,(2x+2yy');$$

$$y' = \frac{x+y}{x-y}.$$

8.1Â0.12 We have:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\frac{2t}{1+t^2}}{\frac{1}{1+t^2}} = 2t;$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(2t) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}(2t)}{\frac{\mathrm{d}x}{\mathrm{d}t}} = 2(1+t^2).$$

8.1Â0.13 We have:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\frac{1}{(1-t)^2}}{\frac{1}{t}} = \frac{t}{(1-t)^2};$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{d}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{t}{(1-t)^2}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{(1-t)^2}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}} = t\,\frac{1+t}{(1-t)^3}.$$

8.1Â0.14 We have:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{k\,t^{k-1}}{\frac{1}{t}} = k\,t^k;$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}(k\,t^k) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}(k\,t^k)}{\frac{\mathrm{d}x}{\mathrm{d}t}} = k^2t^k.$$

Similarly,

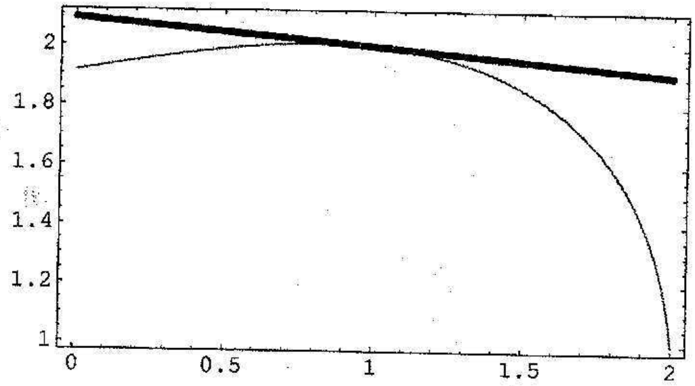
$$\frac{\mathrm{d}^ny}{\mathrm{d}x^n} = k^nt^k.$$

8.1Â0.15 We have:

$$3\,x^2 + 3\,y^2y' - y - xy' = 0,$$

therefore, with $x = 1$, $y = 2$, we obtain $y' = -1/11$. The equation of the tangent line at the point $(1, 2)$ is

$$x + 11\,y - 23 = 0.$$



8.10.16 From $u = xy$, $v = \dfrac{x}{y}$ we obtain:

$$uv = x^2, \quad \frac{u}{v} = y^2.$$

Let $z(x,y) = w(u(x,y), v(x,y))$. We have:

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x} = y\cdot\frac{\partial w}{\partial u} + \frac{1}{y}\cdot\frac{\partial w}{\partial v};\\
\frac{\partial^2 z}{\partial x^2} &= y\left(y\frac{\partial^2 w}{\partial u^2} + \frac{1}{y}\frac{\partial^2 w}{\partial u \partial v}\right) + \frac{1}{y}\left(y\frac{\partial^2 w}{\partial u \partial v} + \frac{1}{y}\frac{\partial^2 w}{\partial v^2}\right)\\
&= y^2\frac{\partial^2 w}{\partial u^2} + 2\frac{\partial^2 w}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 w}{\partial v^2} = \frac{u}{v}\frac{\partial^2 w}{\partial u^2} + 2\frac{\partial^2 w}{\partial u \partial v} + \frac{v}{u}\frac{\partial^2 w}{\partial v^2}.
\end{aligned}$$

$$\frac{\partial z}{\partial y} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y} = x\cdot\frac{\partial w}{\partial u} - \frac{x}{y^2}\cdot\frac{\partial w}{\partial v};$$

Similarly, we obtain

$$\frac{\partial^2 z}{\partial y^2} = x^2\frac{\partial^2 w}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 w}{\partial u \partial v} + \frac{x^2}{y^4}\frac{\partial^2 w}{\partial v^2} + \frac{2x}{y^3}\frac{\partial w}{\partial v}, \text{ etc.}$$

Substituting these expressions into the given equation we obtain:

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{1}{2u}\frac{\partial w}{\partial v}.$$

8.10.17 We have

$$z = w \cdot x.$$

Let $w(x,y) = W(u(x,y), v(x,y))$.

$$\frac{\partial z}{\partial x} = W + x\frac{\partial W}{\partial x}, \qquad \frac{\partial^2 z}{\partial x^2} = 2\frac{\partial W}{\partial x} + x\frac{\partial^2 W}{\partial x^2},$$

$$\frac{\partial z}{\partial y} = x\frac{\partial W}{\partial y}, \qquad \frac{\partial^2 z}{\partial y^2} = x\frac{\partial^2 W}{\partial y^2}, \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial W}{\partial y} + x\frac{\partial^2 W}{\partial x \partial y}.$$

Substituting into the given equation, we obtain:

$$x\frac{\partial^2 W}{\partial x^2} - 2x\frac{\partial^2 W}{\partial x \partial y} + x\frac{\partial^2 W}{\partial y^2} + 2\frac{\partial W}{\partial x} - 2\frac{\partial W}{\partial y} = 0.$$

We have:

$$\frac{\partial w}{\partial x} = \frac{\partial W}{\partial u} - \frac{y}{x^2}\frac{\partial W}{\partial v}, \qquad \frac{\partial w}{\partial y} = \frac{\partial W}{\partial u} + \frac{1}{x}\frac{\partial W}{\partial v},$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 W}{\partial u^2} - \frac{y}{x^2}\frac{\partial^2 W}{\partial u \partial v} + \frac{2y}{x^3}\frac{\partial W}{\partial v} - \frac{y}{x^2}\frac{\partial^2 W}{\partial u \partial v} + \frac{y^2}{x^4}\frac{\partial^2 W}{\partial v^2},$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 W}{\partial u^2} + \frac{1}{x}\frac{\partial^2 W}{\partial u \partial v} - \frac{1}{x^2}\frac{\partial W}{\partial v} - \frac{y}{x^2}\frac{\partial^2 W}{\partial u \partial v} - \frac{y}{x^3}\frac{\partial^2 W}{\partial v^2},$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 W}{\partial u^2} + \frac{2}{x}\frac{\partial^2 W}{\partial u \partial v} + \frac{1}{x^2}\frac{\partial^2 W}{\partial v^2}.$$

Substituting in the previous equation, we obtain

$$\frac{\partial^2 W}{\partial v^2} = 0.$$

8.10.18 [A] We have $\begin{cases} r = \sqrt{x^2+y^2}, \\ t = \arctan\dfrac{y}{x}, \quad x, y > 0. \end{cases}$

With $z(x,y) = w(r(x,y), t(x,y))$, we obtain:

$$z'_x = w'_r\, r'_x + w'_t\, t'_x = w'_r \frac{x}{\sqrt{x^2+y^2}} - w'_t \frac{y}{x^2+y^2},$$

$$z'_y = w'_r\, r'_y + w'_t\, t'_y = w'_r \frac{y}{\sqrt{x^2+y^2}} + w'_t \frac{x}{x^2+y^2},$$

$$
\begin{aligned}
z''_{x^2} &= w''_{r^2}\frac{x^2}{x^2+y^2} - w''_{rt}\frac{2xy}{(x^2+y^2)^{3/2}} + w''_{t^2}\frac{y^2}{(x^2+y^2)^2} \\
&\quad + w'_r\frac{y^2}{(x^2+y^2)^{3/2}} + w'_t\frac{2xy}{(x^2+y^2)^2},
\end{aligned}
$$

$$
\begin{aligned}
z''_{y^2} &= w''_{r^2}\frac{y^2}{x^2+y^2} + w''_{rt}\frac{2xy}{(x^2+y^2)^{3/2}} + w''_{t^2}\frac{x^2}{(x^2+y^2)^2} \\
&\quad + w'_r\frac{x^2}{(x^2+y^2)^{3/2}} - w'_t\frac{2xy}{(x^2+y^2)^2},
\end{aligned}
$$

and the new equation is

$$r^2\, w''_{r^2} + w''_{t^2} + r\, w'_r = 0.$$

8.10.19 (A) We have:

$$
\begin{aligned}
& \mathrm{d}^n f(x,y) \\
&= \left(\mathrm{d}x\,\frac{\partial}{\partial y} + \mathrm{d}y\,\frac{\partial}{\partial x}\right)^n f(x,y) \\
&= \left(\sum_{k=0}^{n}\binom{n}{k}\,\mathrm{d}x^{n-k}\,\mathrm{d}y^k\,\frac{\partial^n}{\partial x^{n-k}\partial y^k}\right) f(x,y) \\
&= \sum_{k=0}^{n}\binom{n}{k}\,\mathrm{d}x^{n-k}\,\mathrm{d}y^k\,\frac{\partial^n}{\partial x^{n-k}\partial y^k}\mathrm{e}^{a\,x+b\,y} \\
&= \sum_{k=0}^{n}\binom{n}{k}\,\mathrm{d}x^{n-k}\,\mathrm{d}y^k\,a^{n-k}\,b^k\,\mathrm{e}^{a\,x+b\,y} \\
&= \mathrm{e}^{a\,x+b\,y}\,(a\,\mathrm{d}x + b\,\mathrm{d}y)^n.
\end{aligned}
$$

12.9 Exercises: Power Series (Solutions)

9.3.1 (A) $R = \dfrac{1}{\overline{\lim}\sqrt[n]{|a_n|}} = \dfrac{1}{\lim\limits_{n\to\infty}\sqrt[n^2]{n!}} = 1.$

9.3.2 The Stirling asymptotic expansion,

$$n! = \frac{\sqrt{2\pi}\, n^n \sqrt{n}}{\mathrm{e}^n}\left(1 + \frac{1}{12n} + O(n^{-2})\right),$$

implies

$$R = \frac{1}{\overline{\lim}\sqrt[n]{|a_n|}} = \lim_{n\to\infty} \frac{n^{n+1/2}}{n!\,\mathrm{e}^n} = \frac{1}{\sqrt{2\pi}}.$$

9.3.3 We have:

$$a_k = 0, \quad \text{for} \quad k \neq n^2;$$

$$a_k = \left(1 + \frac{1}{k}\right)^{k^2}, \quad \text{for} \quad k = n^2.$$

We obtain:

$$R = \frac{1}{\overline{\lim}\sqrt[k]{|a_k|}} = \lim_{k\to\infty}\left(1 + \frac{1}{k}\right)^{-k} = \frac{1}{\mathrm{e}}.$$

9.3.4 For $|x| < 1$ the series is convergent. For $x = 1$ the series $\sum \sin n$ is divergent. Consequently, $R = 1$.

9.3.5 We have:

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$$

$$= \sum_{k=0}^{\infty} \frac{(-1/2)(-1/2-1)\cdots(-1/2-k+1)}{k!}(-x^2)^k, \quad |x| < 1,$$

i.e.,

$$\frac{1}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!}\, x^{2k}, \qquad |x| < 1,$$

where $(-1)!! = 1$.

9.3.6 We use the equality

$$\frac{1}{\sqrt{1-t^2}} = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!}\, t^{2k}, \quad |t| < 1.$$

We have:

$$\arcsin x = \int_0^x \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \frac{x^{2k+1}}{2k+1}, \quad |x| < 1.$$

Taking $x = 1$ in the previous series, we obtain the series of numbers

$$\sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \frac{1}{2k+1}.$$

By using Raabe-Duhamel test,

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{3}{2} > 1,$$

it follows that it is convergent. Similarly for $x = -1$. By virtue of Tauber's theorem [17, Th. 9.2.7, pag. 171] we obtain

$$\arcsin x = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \frac{x^{2k+1}}{2k+1}, \quad |x| \leq 1.$$

$\overset{\mathrm{A}}{9.3.7}$ We have:

$$\begin{aligned}\arctan x &= \int_0^x \frac{\mathrm{d}t}{1+t^2} \\ &= \int_0^x \left(\sum_{k=0}^{\infty} (-1)^k t^{2k}\right) \mathrm{d}t \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}, \quad |x| < 1.\end{aligned}$$

For $|x| = 1$, the previous series of numbers is convergent (cf. Leibniz test). Using the Tauber test it follows that

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}, \qquad |x| \leq 1.$$

$\overset{\mathrm{A}}{9.3.8}$ We have:

$$\ln(1-x) = -\int_0^x \frac{\mathrm{d}t}{1-t} = -\int_0^x \left(\sum_{k=0}^{\infty} t^k\right) \mathrm{d}t = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1};$$

$$\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x) = -\sum_{k=0}^{\infty}\frac{(-x)^{k+1}}{k+1} + \sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1};$$

consequently,

$$\ln\frac{1+x}{1-x} = 2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}, \quad |x| < 1.$$

9.3.9 (A) We have:

$$f''(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} \quad |x| < 1;$$

$$f'(x) = \int_0^x \left(\sum_{n=0}^{\infty}(-1)^n t^{2n}\right) dt = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \quad |x| \le 1;$$

$$f(x) = \int_0^x \left(\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{2n+1}\right) dt, \quad |x| \le 1;$$

consequently,

$$x \arctan x - \ln\sqrt{1+x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)} \quad |x| \le 1.$$

9.3.10 (A) We have:

$$\int_0^x \frac{\arctan t}{t}\, dt = \int_0^x \left(\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{2n+1}\right) dt$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)^2}, \quad |x| \le 1;$$

The number

$$\int_0^1 \frac{\arctan t}{t}\, dt$$

is referred to as `Catalan's constant` and is denoted by G. Therefore,

$$G = \int_0^1 \frac{\arctan t}{t}\, dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = 0.91\ldots$$

9.$\overset{\mathrm{A}}{3}$.11 $\ln(x^2-2x+2) = \ln(1+(x-1)^2) = \sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{(x-1)^{2n}}{n}, \quad x \in [0,2].$

9.$\overset{\mathrm{A}}{3}$.12 $\sum_{n=0}^{\infty}\dfrac{\mathrm{e}^{nx}}{n!} = \sum_{n=0}^{\infty}\dfrac{(\mathrm{e}^{x})^n}{n!} = \mathrm{e}^{\mathrm{e}^x}, \quad x \in \mathbb{R};$

9.$\overset{\mathrm{A}}{3}$.13 For $|x| < 1$, we have:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n;$$

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \sum_{n=1}^{\infty} n\,x^{n-1};$$

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n\,x^n;$$

$$\frac{1+x}{(1-x)^3} = \left(\frac{x}{(1-x)^2}\right)' = \sum_{n=1}^{\infty} n^2 x^{n-1};$$

finally,

$$x\,\frac{1+x}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^n;$$

9.$\overset{\mathrm{A}}{3}$.14 Let $\varepsilon_k \in \mathbb{C}$, $k = 1, \dots, p$, be the roots of the equation $z^n = 1$. It is known that

$$\sum_{k=1}^{p} \varepsilon_k^q = \begin{cases} p, & \text{if } q \text{ is a multiple of } p, \\ 0, & \text{if } q \text{ is not a multiple of } p. \end{cases}$$

We obtain:

$$\sum_{n=0}^{\infty}\sum_{k=1}^{p}\frac{(\varepsilon_k x)^n}{n!} = p\sum_{n=0}^{\infty}\frac{x^{np}}{(np)!};$$

$$\sum_{n=0}^{\infty}\frac{x^{np}}{(np)!} = \frac{1}{p}\sum_{k=1}^{p}\mathrm{e}^{\varepsilon_k x} = \frac{1}{p}\sum_{k=1}^{p}\cos(\varepsilon_k x)$$

$$= \frac{1}{p}\sum_{k=1}^{p}\cos\frac{(4k+1)\,\pi\,x}{2p}.$$

9.3.15 We have:

$$\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}=\sum_{n=1}^{\infty}x^n\left(\frac{1}{n}-\frac{1}{n+1}\right)=\sum_{n=1}^{\infty}\frac{x^n}{n}-\sum_{n=1}^{\infty}\frac{x^n}{n+1}$$

$$=-\ln(1-x)-\frac{1}{x}\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}=-\ln(1-x)-\frac{1}{x}\left(-x-\ln(1-x)\right)$$

$$=\frac{x+(1-x)\ln(1-x)}{x},\quad x\in(-1,1)\setminus\{0\}.$$

Using the Tauber Theorem [17, Th. 9.2.7, pag. 171] we obtain

$$\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}=\begin{cases}\dfrac{x+(1-x)\ln(1-x)}{x}, & \text{if } x\in(-1,1)\setminus\{0\},\\ \lim\limits_{t\to x}\dfrac{t+(1-t)\ln(1-t)}{t}, & \text{if } x\in\{-1,0,1\}.\end{cases}$$

$$=\begin{cases}\dfrac{x+(1-x)\ln(1-x)}{x}, & \text{if } x\in(-1,1)\setminus\{0\},\\ 1-2\ln 2, & \text{if } x=-1,\\ 0, & \text{if } x=0,\\ 1, & \text{if } x=1.\end{cases}$$

9.3.16 Let $f(x)=\sum_{n=0}^{\infty}\frac{x^{n+k}}{(n+k)n!}$. We have:

$$f'(x)=\sum_{n=0}^{\infty}\frac{x^{n+k-1}}{n!}=e^x x^{k-1}.$$

Since $f(0)=0$, we obtain

$$f(x)=\int_0^x e^t t^{k-1}\mathrm{d}t=(-1)^k(k-1)!+e^x\sum_{i=0}^{k-1}\frac{(-1)^i(k-1)!}{(k-i-1)!}x^{k-i-1},$$

hence

$$\sum_{n=0}^{\infty}\frac{x^n}{(n+k)n!}=\frac{1}{x^k}\left((-1)^k(k-1)!+e^x\sum_{i=0}^{k-1}\frac{(-1)^i(k-1)!}{(k-i-1)!}x^{k-i-1}\right),$$

$x \in \mathbb{R}^*$.

9.3.17 We have:

$$\int_0^x \frac{\ln(1+t)}{t}\,\mathrm{d}t = \sum_{k=1}^{\infty} \int_0^x (-1)^{k+1} \frac{t^{k-1}}{k}\,\mathrm{d}t = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k^2},$$

$x \in [0,1]$. We obtain:

$$\int_0^1 \frac{\ln(1+t)}{t}\,\mathrm{d}t = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - 2\sum_{k=1}^{\infty} \frac{1}{(2k)^2}$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{12} \quad \text{(cf. [17, 9.3.21, pag. 183])}.$$

9.3.18 Substituting z by $r\,\mathrm{e}^{ix}$ ($r \in (0,1)$, $x \in \mathbb{R}$) and identifying the real and imaginary part in the equality

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,$$

we obtain

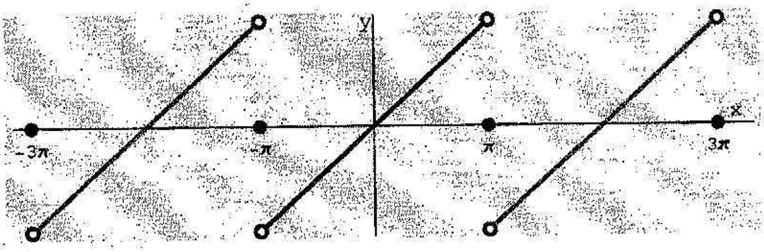
$$\sum_{n=0}^{\infty} r^n \cos nx = \frac{1 - r\cos x}{1 - 2r\cos x + r^2},$$

$$\sum_{n=0}^{\infty} r^n \sin nx = \frac{r\sin x}{1 - 2r\cos x + r^2}.$$

12.10 Exercises: Fourier Series (Solutions)

9.5.1 Consider the function $f : \mathbb{R} \to \mathbb{R}$, with period 2π,

$$f(x) = \begin{cases} 0, & x = -\pi, \\ \dfrac{x}{2}, & x \in (-\pi, \pi), \\ 0, & x = \pi. \end{cases}$$



The graph of the function f.

Since the function f is even we get:

$$a_n = 0, \quad n \in \mathbb{N},$$

$$b_n = \frac{2}{\pi} \int_0^\pi \frac{x}{2} \sin nx \, \mathrm{d}x = \frac{(-1)^{(n+1)}}{n}, \quad n \in \mathbb{N}^*.$$

Using the equality

$$f(x) = \frac{f(x+0) + f(x-0)}{2}, \quad x \in \mathbb{R},$$

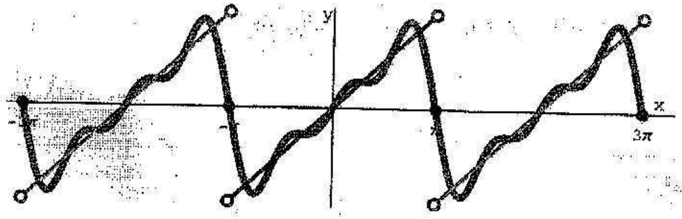
we obtain

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad x \in \mathbb{R},$$

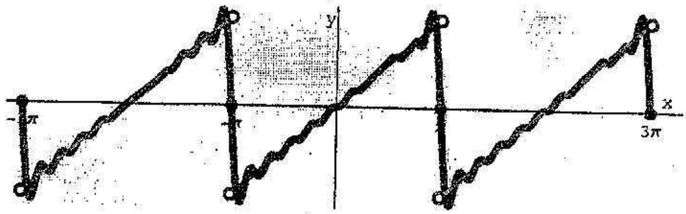
hence

$$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad x \in (-\pi, \pi).$$

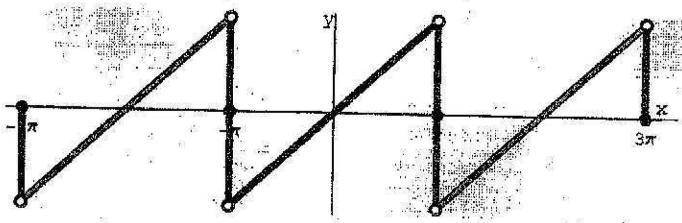
We present below the graphs of the partial sums S_n of the Fourier series attached to f, for $n = 3,\ 10,\ 100$.



S_3



S_{10}



S_{100}

9.5.2 Problem p09.0 yields

$$\frac{t}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt, \qquad t \in (-\pi, \pi).$$

With $t = \pi - x$, we obtain

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n(\pi - x), \qquad x \in (0, 2\pi),$$

that is,

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx, \qquad x \in (0, 2\pi).$$

9.5.3 (A) Consider the odd function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0, & x = -\pi, \\ -1, & x \in (-\pi, 0), \\ 0, & x = 0, \\ 1, & x \in (0, \pi), \\ 0, & x = \pi, \end{cases}$$

with period 2π. We obtain:

$$a_n = 0, \qquad n \in \mathbb{N};$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin nx \, dx = \frac{2}{\pi} \frac{1 + (-1)^{n+1}}{n}, \qquad n \in \mathbb{N}^*;$$

hence,

$$a_n = 0, \quad n \in \mathbb{N};$$

$$b_{2n} = 0, \quad n \in \mathbb{N}^*;$$

$$b_{2n+1} = \frac{4}{\pi(2n+1)}, \quad n \in \mathbb{N}.$$

It follows that

$$1 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}, \qquad x \in (0, \pi).$$

9.5.4 (A) The fractional part function $x \mapsto x - \lfloor x \rfloor$, $x \in \mathbb{R}$, has period 1. Consequently, we have:

$$a_n = 2 \int_0^1 f(x) \cos 2n\pi x \, dx, \qquad n \in \mathbb{N},$$

$$b_n = 2 \int_0^1 f(x) \sin 2n\pi x \, dx, \qquad n \in \mathbb{N}^*,$$

hence:

$$a_0 = 1, \quad a_n = 0, \quad n \in \mathbb{N}^*,$$

$$b_n = -\frac{1}{n\pi}, \quad n \in \mathbb{N}^*.$$

Therefore, we obtain

$$x - \lfloor x \rfloor \sim \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n},$$

that is,

$$x - \lfloor x \rfloor = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n}, \quad x \in \mathbb{R} \setminus \mathbb{Z}.$$

9.5.5 The function $f(x) = \arcsin(\cos x)$ is continuous on $\mathbb{R}$ and has period 2π. We can write

$$\arcsin(\cos x) = \frac{\pi}{2} - |x|, \quad x \in [-\pi, \pi].$$

We obtain:

$$a_{2n} = 0, \quad n \in \mathbb{N}; \qquad b_n = 0, \quad n \in \mathbb{N}^*,$$

hence

$$\arcsin(\cos x) = \frac{4}{\pi} \sum_{n \geq 0} \frac{\cos(2n+1)x}{(2n+1)^2}, \quad x \in [-\pi, \pi].$$

9.5.6 Taking $x = 0$ in P.9.5.6, we obtain $\displaystyle\sum_{n \geq 0} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

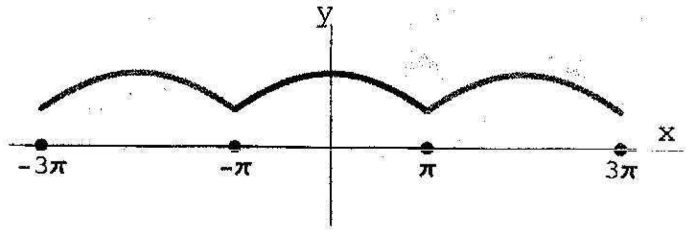
9.5.7 We have:

$$a_n = 0, \quad n \in \mathbb{N}; \qquad b_n = (-1)^{n+1}/n^2, \quad n \in \mathbb{N}^*.$$

Therefore,

$$\frac{\pi^2 - 3x^2}{12} = \sum_{n \geq 1} (-1)^{n+1} \frac{\cos nx}{n^2}, \qquad x \in [-\pi, \pi].$$

9.5.8 We expand the odd function $x \mapsto \cos ax$, $x \in [-\pi, \pi]$, with period 2π, into a Fourier series.



We have:

$$b_n = 0, \quad n \in \mathbb{N}^*,$$

$$a_n = \frac{2}{\pi}\int_0^\pi \cos ax \cos nx \,\mathrm{d}x = \frac{1}{\pi}\int_0^\pi (\cos(a+n)x + \cos(a-n)x)\,\mathrm{d}x$$

$$= (-1)^n \frac{2a\pi \sin a\pi}{a^2 - n^2}, \quad n \in \mathbb{N},$$

hence

$$\cos ax = \frac{2\sin a\pi}{\pi}\left(\frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^n a}{a^2 - n^2}\cos nx\right), \quad x \in [-\pi, \pi].$$

$\overset{\text{A}}{9.5.9}$ We know that:

$$\cosh ax = \cos iax, \qquad \sinh ax = \frac{\sin iax}{i}.$$

Substituting a by ia in P.9.5.8 we obtain

$$\cosh ax = \frac{2\sinh a\pi}{\pi}\left(\frac{1}{2a} + \sum_{n\geq 1}(-1)^n \frac{a\cos nx}{a^2 + n^2}\right),$$

$a \in \mathbb{R} \setminus \mathbb{Z};\ x \in [-\pi, \pi]$.

$\overset{\text{A}}{9.5.10}$ Taking $x = 0$ in P.9.5.8 we obtain

$$\frac{1}{\sin a\pi} = \frac{1}{a\pi} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{a\pi - n\pi} + \frac{1}{a\pi + n\pi}\right);$$

consequently, for $a\pi = x$, we get

$$\frac{1}{\sin x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x - n\pi}, \qquad x \in \mathbb{R} \setminus \pi\mathbb{Z}.$$

9.5.11 (A) We expand the function $t \mapsto \cot t$, $t \in \mathbb{R} \setminus \pi\mathbb{Z}$, into partial fractions. Taking $x = \pi$, in (P.9.5.8)

$$\cos ax = \frac{2\sin a\pi}{\pi}\left(\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{(-1)^n a}{a^2 - n^2}\cos nx\right), \qquad x \in [-\pi, \pi],$$

yields

$$\cot a\pi = \frac{1}{a\pi} + \sum_{n=1}^{\infty}\left(\frac{1}{a\pi - n\pi} + \frac{1}{a\pi + n\pi}\right),$$

hence, with $a\pi = t$, we obtain

$$\cot t = \sum_{n=-\infty}^{\infty}\frac{1}{t - n\pi}, \quad t \in \mathbb{R} \setminus \pi\mathbb{Z}.$$

9.5.12 (A) From $\mathrm{e}^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$, $z \in \mathbb{C}$, for $z = \mathrm{e}^{ix} = \cos x + i\sin x$, we obtain

$$\mathrm{e}^{\cos x + i\sin x} = \sum_{n=0}^{\infty}\frac{(\mathrm{e}^{ix})^n}{n!} = \sum_{n=0}^{\infty}\frac{(\mathrm{e}^{nix})}{n!} = \sum_{n=0}^{\infty}\frac{(\cos nx + i\sin nx)}{n!},$$

hence

$$\mathrm{e}^{\cos x}(\cos(\sin x) + i\sin(\sin x)) = \sum_{n=0}^{\infty}\frac{\cos nx}{n!} + i\sum_{n=0}^{\infty}\frac{\sin nx}{n!},$$

and consequently,

$$\sum_{n=0}^{\infty}\frac{\cos nx}{n!} = \mathrm{e}^{\cos x}\cos(\sin x), \qquad \sum_{n=0}^{\infty}\frac{\sin nx}{n!} = \mathrm{e}^{\cos x}\sin(\sin x),$$

$x \in \mathbb{R}$.

13 From the equality

$$\frac{1}{1-z} = \sum_{n\geq 0} z^n, \quad z \in \mathbb{C}, \quad |z| < 1,$$

for $z = r\mathrm{e}^{ix}$, $-1 < r < 1$, $x \in \mathbb{R}$, we obtain

$$\frac{1 - r\cos x + ir\sin x}{1 - 2r\cos x + r^2} = \sum_{n\geq 0} r^n(\cos nx + i\sin nx),$$

hence

$$\sum_{n\geq 0} r^n \cos nx = \frac{1 - r\cos x}{1 - 2r\cos x + r^2},$$

$$\sum_{n\geq 1} r^n \sin nx = \frac{r\sin x}{1 - 2r\cos x + r^2},$$

$-1 < r < 1$, $x \in \mathbb{R}$.

9.5.14 (A) Using P.9.5.13, we have

$$\sum_{n\geq 1} r^n \sin nx = \frac{r\sin x}{1 - 2r\cos x + r^2}, \qquad -1 < r < 1, \quad x \in \mathbb{R}.$$

Since the left hand side series converges uniformly for $x \in \mathbb{R}$, then we can integrate it term-by-term:

$$-\sum_{n\geq 1} r^n \frac{\cos nx - 1}{n} = \int_0^x \frac{r\sin t}{1 - 2r\cos t + r^2}\,\mathrm{d}t,$$

$$= \frac{1}{2}\ln(1 - 2r\cos t + r^2)\Big|_{t=0}^{t=x},$$

$$-\sum_{n\geq 1} \frac{r^n}{n}\cos nx + \sum_{n\geq 1} \frac{r^n}{n}$$

$$= \frac{1}{2}\ln(1 - 2r\cos x + r^2) - \frac{1}{2}\ln(1 - r)^2.$$

Therefore, using the equality

$$\sum_{n\geq 1} \frac{r^n}{n} = -\ln(1 - r),$$

we obtain

$$\ln(1 - 2r\cos x + r^2) = -2\sum_{n\geq 1} \frac{r^n}{n}\cos nx, \quad -1 < r < 1, \quad x \in \mathbb{R}.$$

9.5.15 (A) Problem P.9.5.14 yields

$$\ln(1 - 2r\cos x + r^2) = -2\sum_{n\geq 1} \frac{r^n}{n}\cos nx, \quad -1 < r < 1, \quad x \in \mathbb{R}.$$

By virtue of [17, Th. 9.2.7, pag. 171], and using the fact that, for $r = -1$ and $x \in (-\pi, \pi)$, the series

$$-2\sum_{n\geq 1}\frac{r^n}{n}\cos nx,$$

is convergent (see Abel-Dirichlet test [17, Th. 5.4.22, pag. 95]), we deduce

$$\ln\left(2\cos\frac{x}{2}\right) = \sum_{n\geq 1}(-1)^{n+1}\frac{\cos nx}{n}, \qquad x \in (-\pi, \pi).$$

9.5.16 (A) Substituting x by $\pi - t$ in P.9.5.15, we obtain

$$\ln\left(2\cos\frac{\pi - t}{2}\right) = \sum_{n\geq 1}(-1)^{n+1}\frac{\cos n(\pi - t)}{n}, \qquad t \in (0, 2\pi),$$

hence

$$\ln\left(2\sin\frac{x}{2}\right) = -\sum_{n\geq 1}\frac{\cos nx}{n}, \quad x \in (0, 2\pi).$$

9.5.17 (A) We have:

$$\begin{aligned}\ln\tan\frac{x}{2} &= \ln 2\sin\frac{x}{x} - \ln\cos\frac{x}{x}\\ &= -\sum_{n\geq 1}\frac{\cos nx}{n} - \sum_{n\geq 1}(-1)^{n+1}\frac{\cos nx}{n}\\ &= -2\sum_{n\geq 0}\frac{\cos(2n+1)x}{2n+1}, \qquad x \in (0, \pi).\end{aligned}$$

9.5.18 (A) From P.9.5.15 and P.9.5.16 we obtain:

$$\ln 2\sin x = \ln\left(2\cos\frac{x}{2}\right) + \ln\left(2\sin\frac{x}{2}\right)$$

$$= \sum_{n\geq 1}(-1)^{n+1}\frac{\cos nx}{n} - \sum_{n\geq 1}\frac{\cos nx}{n} = -\sum_{n\geq 1}\frac{\cos 2nx}{n}, \qquad x \in (0, \pi).$$

It follows that

$$\int_0^{\pi}\ln 2\sin x\, dx = 0,$$

hence

$$\int_0^{\pi} \ln \sin x \, dx = -\pi \ln 2.$$

9.5.19 From P.9.5.15 and P.9.5.16 we obtain:

$$\ln 2 \sin x = \ln\left(2 \cos \frac{x}{2}\right) + \ln\left(2 \sin \frac{x}{2}\right)$$

$$= \sum_{n \geq 1} (-1)^{n+1} \frac{\cos nx}{n} - \sum_{n \geq 1} \frac{\cos nx}{n} = -\sum_{n \geq 1} \frac{\cos 2nx}{n}, \qquad x \in (0, \pi).$$

It follows that

$$\int_0^{\pi/2} \ln 2 \sin x \, dx = 0,$$

hence

$$\int_0^{\pi/2} \ln \sin x \, dx = -\frac{\pi}{2} \ln 2.$$

9.5.20 We have:

$$\int_0^{\pi/2} \ln^2 2 \sin x \, dx = \sum_{m,n=1}^{\infty} \int_0^{\pi/2} \frac{\cos 2nx \, \cos 2mx}{nm} \, dx$$

$$= \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{24},$$

i.e.,

$$\int_0^{\pi/2} (\ln 2 + \ln \sin x)^2 \, dx = \frac{\pi^3}{24}.$$

By using P.9.5.19, we obtain

$$\int_0^{\pi/2} \ln^2 \sin x \, dx = \frac{\pi^3}{24} + \frac{\pi}{2} \ln^2 2.$$

9.5.21 We expand the function f' into a Fourier series of cosines on $[0, \pi]$. We have:

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f'(x) \, dx = 2 \, \frac{f(\pi) - f(0)}{\pi} = 0,$$

hence

$$f'(x) = \sum_{n=1}^{\infty} a_n \cos nx.$$

Integrating term-by-term, we obtain

$$f(x) - f(0) = \sum_{n=1}^{\infty} \frac{a_n}{n} \sin nx.$$

By virtue of the Parseval equality, we obtain

$$\frac{2}{\pi} \int_0^{\pi} f^2(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \frac{a_n^2}{n^2} \leq \sum_{n=1}^{\infty} a_n^2 = \frac{2}{\pi} \int_0^{\pi} (f'(x))^2 \mathrm{d}x.$$

The equality is valid iff for all $n \geq 2$, $a_n = 0$, i.e., $f(x) = C \sin x$.

12.11 Exercises: Extrema of a Function (Solutions)

11.3.1 We have:

$$\begin{cases} \frac{\partial f}{\partial x} = 0, \\ \frac{\partial f}{\partial y} = 0, \end{cases} \iff \begin{cases} -(1+\mathrm{e}^y)\sin x &= 0, \\ \mathrm{e}^y(\cos x - 1 - y) &= 0. \end{cases}$$

Solving the system, we find the stationary points:

$$(k\pi, (-1)^k - 1), \quad k \in \mathbb{Z},$$

i.e.,

$$(2n\pi, 0), \quad ((2n+1)\pi, -2), \qquad n \in \mathbb{Z}.$$

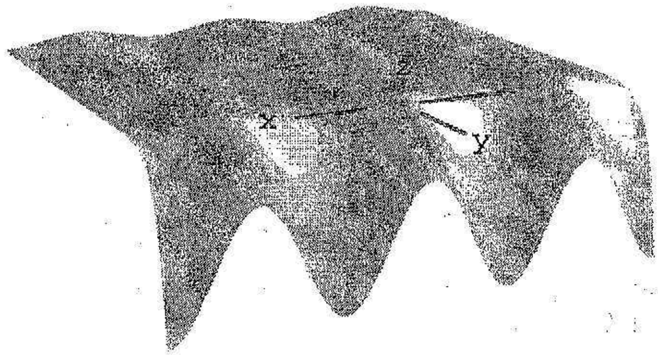
Further, we find the partial derivatives of second order:

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= -(1+\mathrm{e}^y)\cos x, \\ \frac{\partial^2 f}{\partial x \partial y} &= -\mathrm{e}^y \sin x, \\ \frac{\partial^2 f}{\partial y^2} &= \mathrm{e}^y(\cos x - 2 - y). \end{aligned}$$

We calculate the second differential at the stationary points:

$$\begin{aligned} d^2 f(2n\pi, 0) &= -2\,\mathrm{d}x^2 - \mathrm{d}y^2, \quad \text{negative definite}, \\ d^2 f((2n+1)\pi, -2) &= (1+\mathrm{e}^{-2})\,\mathrm{d}x^2 - \mathrm{e}^{-2}\,\mathrm{d}y^2, \quad \text{indefinite}. \end{aligned}$$

Consequently, $(2n\pi, 0)$ are points of maximum and $((2n+1)\pi, -2)$ are not points of extremum.



11.3.2 We find the minimum of the function

$$f(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

with the constraint

$$Ax + By + Cz + D = 0.$$

The problem is reduced to examine the Lagrange function

$$L(x, y, z) = f(x, y, z) + \lambda(Ax + By + Cz + D)$$

for an ordinary extremum. Consider the system

$$\begin{cases} \frac{\partial L}{\partial x} & = 0, \\ \frac{\partial L}{\partial y} & = 0, \\ \frac{\partial L}{\partial z} & = 0, \\ F(x, y, z) & = 0, \end{cases} \iff \begin{cases} 2(x - x_0) + \lambda A & = 0, \\ 2(y - y_0) + \lambda B & = 0, \\ 2(z - z_0) + \lambda C & = 0, \\ Ax + By + Cz + D & = 0. \end{cases}$$

We obtain

$$\lambda = 2\frac{Ax_0 + By_0 + Cz_0 + D}{A^2 + B^2 + C^2}.$$

The stationary point is

$$\left(x_0 - \frac{\lambda A}{2}, y_0 - \frac{\lambda B}{2}, z_0 - \frac{\lambda C}{2}\right).$$

The second differential

$$d^2L = 2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

is positive definite, hence the stationary point is a point of minimum.

$$f_{min} = \left(\frac{Ax_0 + By_0 + Cz_0 + D}{A^2 + B^2 + C^2}\right)^2,$$

hence, the distance from the point $M_0(x_0, y_0, z_0)$ to the plane $Ax + By + Cz + D = 0$ is

$$\frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

11.3.3 The parametric equations of the straight lines are:

$$\begin{cases} x &= u + 1, \\ y &= 2u, \\ z &= u, \end{cases} \quad \text{and} \quad \begin{cases} x &= v, \\ y &= v, \\ z &= v. \end{cases}$$

We examine the function $f(u, v) = (u + 1 - v)^2 + (2u - v)^2(u - v)^2$ for a minimum. Consider the system

$$\begin{cases} \frac{\partial f}{\partial u} &= 0, \\ \frac{\partial f}{\partial v} &= 0, \end{cases} \iff \begin{cases} 12u - 8v + 2 &= 0, \\ -8u + 6v - 2 &= 0. \end{cases}$$

The stationary point is $(\frac{1}{2}, 1)$. Further, we find the partial derivatives of second order:

$$\begin{cases} \frac{\partial^2 f}{\partial x^2} &= 12, \\ \frac{\partial^2 f}{\partial x \partial y} &= -8, \\ \frac{\partial^2 f}{\partial y^2} &= 6. \end{cases}$$

We calculate the second differential at the stationary point:

$$d^2 f\left(\frac{1}{2}, 1\right) = 12\,du^2 - 16\,du\,dv + 6\,dv^2.$$

We have

$$\begin{vmatrix} 12 & -8 \\ -8 & 6 \end{vmatrix} = 8 > 0,$$

hence, by Sylvester criterion, the second differential is positive definite; therefore $(\frac{1}{2}, 1)$ is a point of minimum. We obtain

$$f_{min} = f\left(\frac{1}{2}, 1\right) = \frac{1}{2}.$$

The distance is $\frac{\sqrt{2}}{2}$.

11.3.4 (A) We find the minimum of the function

$$f(x,y,z) = xy + xz + yz$$

with the constraint

$$xyz - 1 = 0, \quad x > 0, \quad y > 0, \quad z > 0.$$

The problem is reduced to examine the Lagrange function

$$L(x,y,z) = f(x,y,z) + \lambda(xyz - 1)$$

for an ordinary extremum. Consider the system

$$\begin{cases} \frac{\partial L}{\partial x} & = 0, \\ \frac{\partial L}{\partial y} & = 0, \\ \frac{\partial L}{\partial z} & = 0, \\ xyz - 1 & = 0, \end{cases} \iff \begin{cases} y + z + \lambda yz & = 0, \\ x + z + \lambda xz & = 0, \\ x + y + \lambda xy & = 0, \\ xyz - 1 & = 0. \end{cases}$$

We obtain $\lambda = -2$. The stationary point is $(1,1,1)$. From $xyz = 1$ we deduce

$$yz\mathrm{d}x + xz\mathrm{d}y + xy\mathrm{d}z = 0.$$

At the point $(1,1,1)$ we obtain

$$\mathrm{d}x + \mathrm{d}y + \mathrm{d}z = 0,$$

hence,

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + 2(\mathrm{d}x\mathrm{d}y + \mathrm{d}x\mathrm{d}z + \mathrm{d}y\mathrm{d}z) = 0. \qquad (\diamond)$$

The second differential at the stationary point is

$$d^2L(1,1,1) = -2(\mathrm{d}x\mathrm{d}y + \mathrm{d}x\mathrm{d}z + \mathrm{d}y\mathrm{d}z).$$

Using $(\diamond)$, we obtain

$$d^2L(1,1,1) = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

which is positive definite; consequently $(1,1,1)$ is a point of minimum and the smallest area is equal to 3.

11.3.5 Let a be an arbitrary positive number. We examine the function

$$f(x,y,z) = \frac{x^n + y^n}{2},$$

with the constraint

$$x + y - a = 0, \quad x > 0, \quad y > 0,$$

for extremum. The problem is reduced to examine the Lagrange function

$$L(x,y,z) = f(x,y,z) + \lambda(x + y - a)$$

for an ordinary extremum. Consider the system

$$\begin{cases} \dfrac{\partial L}{\partial x} & = 0, \\ \dfrac{\partial L}{\partial y} & = 0, \\ x + y - a & = 0, \end{cases} \iff \begin{cases} \dfrac{nx^{n-1}}{2} + \lambda & = 0, \\ \dfrac{ny^{n-1}}{2} + \lambda & = 0, \\ x + y - a & = 0. \end{cases}$$

We obtain $\lambda = -\frac{na^{n-1}}{2^n}$. The stationary point is $(\frac{a}{2}, \frac{a}{2})$. The second differential at the stationary point is

$$d^2L((\frac{a}{2}, \frac{a}{2})) = -\frac{n(n-1)}{2}\left(\frac{a}{2}\right)^{n-2}(\mathrm{d}x^2 + \mathrm{d}y^2),$$

which is positive definite; consequently $(\frac{a}{2}, \frac{a}{2})$ is a point of minimum, i.e.,

$$f(x,y) \geq f\left(\frac{a}{2}, \frac{a}{2}\right),$$

that is

$$\frac{x^n + y^n}{2} \geq \left(\frac{a}{2}\right)^n = \left(\frac{x+y}{2}\right)^n.$$

11.3.6 First we find the ordinary extrema of the function f on the interior of the disc. Consider the system

$$\begin{cases} \frac{\partial f}{\partial x} &= 0, \\ \frac{\partial f}{\partial y} &= 0, \end{cases} \iff \begin{cases} 2x &= 0, \\ 2y &= 0. \end{cases}$$

The stationary point $(0,0)$ does not belong to D. Furthermore, we find the extrema of the function f with the constraint

$$(x-2)^2 + (y-2)^2 - 2 = 0.$$

We examine the Lagrange function

$$L_\lambda(x,y) = f(x,y) + \lambda((x-2)^2 + (y-2)^2 - 2)$$

for an ordinary extremum. Consider the system

$$\begin{cases} \frac{\partial L}{\partial x} &= 0, \\ \frac{\partial L}{\partial y} &= 0, \\ (x-2)^2 + (y-2)^2 &= 2, \end{cases} \iff \begin{cases} 2(x + \lambda(x-2)) &= 0, \\ 2(y + \lambda(y-2)) &= 0, \\ (x-2)^2 + (y-2)^2 &= 2. \end{cases}$$

We obtain:

$\lambda_1 = -3$; the first stationary point is $(3,3)$;

$\lambda_2 = 1$; the second stationary point is $(1,1)$.

We have:

$$\begin{aligned} \mathrm{d}^2 L(x,y) &= (2+2\lambda)\,\mathrm{d}x^2 + (2+2\lambda)\,\mathrm{d}y^2, \\ \mathrm{d}^2 L(3,3) &= -4(\mathrm{d}x^2 + \mathrm{d}y^2), \quad \text{positive definite,} \\ \mathrm{d}^2 L(1,1) &= 4(\mathrm{d}x^2 + \mathrm{d}y^2), \quad \text{negative definite.} \end{aligned}$$

We obtain:

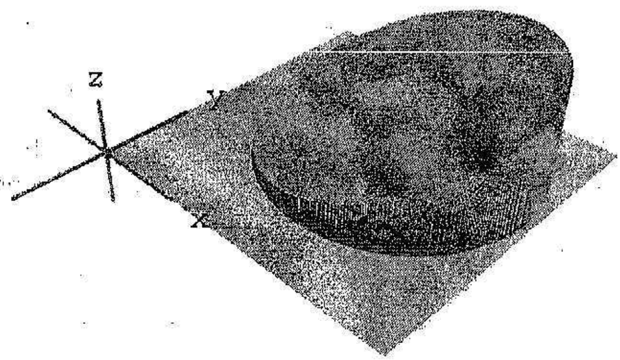
$$f(3,3) = 18,$$

$$f(1,1) = 2.$$

Comparing all the obtained values of the given function, we conclude that

$$f_{greatest} = 18 \quad \text{at} \quad (3,3),$$

$$f_{least} = 2 \quad \text{at} \quad (1,1).$$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APPENDIX A

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APPENDIX B

Name Index

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