Electrotechnics
ET

Course 6
Year I-ISA English

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= Course 6 =

1. Electromagnetic Induction Law

2. Energies and Forces in Magnetic Field

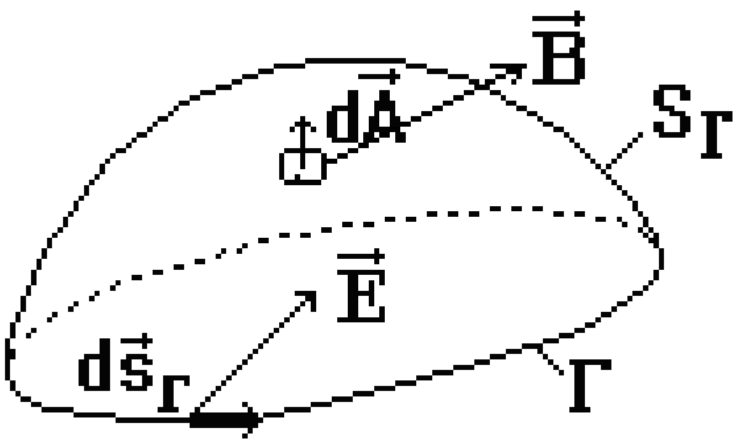
1. Electromagnetic Induction Law

(Faraday's Law)

❑ General (global) form of the law:

The electromotive voltage, e_Γ, produced through electromagnetic induction along the closed curve Γ is equal with the decrease speed of the magnetic flux through any surface supported on the curve Γ.

$$\boxed{e_\Gamma = -\frac{\mathrm{d}\Phi_{S_\Gamma}}{\mathrm{d}t}} \qquad \textbf{(1)}$$



$$\boxed{e_\Gamma = -\frac{d\Phi_{S_\Gamma}}{dt}} \quad \textbf{(1)}$$

- where we know:
  - o the electromotive voltage, e_Γ, respects the expression: $$e_\Gamma = \oint_\Gamma \bar{E}\, d\bar{s} \qquad (2)$$
  - o the magnetic flux can be expressed through the relation: $$\Phi_{S_\Gamma} = \int_{S_\Gamma} \bar{B}\, d\bar{A} \qquad (3)$$
- introducing (2) and (3) in (1) we obtain:

❑ Integral form of the law (IFL):

$$\boxed{\oint_\Gamma \bar{E}\, d\bar{s} = -\frac{d}{dt}\int_{S_\Gamma} \bar{B}\, d\bar{A}} \qquad (4)$$

Notes

- o if the considered medium is in motion the curve Γ is attached to the bodies in their movement;
- o the direction of integration on the curve Γ, and the direction $d\bar{A}$ of the surface for the flux calculation S_Γ are associated with the right-hand rule (regula mâinii drepte).

❖ The derivative of a flux integral of a vector $\bar{B}$:

$$\frac{d}{dt}\int_{S_\Gamma}\bar{B}\,d\bar{A}=\int_{S_\Gamma}\left[\frac{\partial\bar{B}}{\partial t}+\bar{v}\,\mathrm{div}\bar{B}+\mathrm{rot}\,\left(\bar{B}\times\bar{v}\right)\right]d\bar{A}$$

- **from the Magnetic Flux Law we know that:** $\mathrm{div}\bar{B}=0$

$$\Rightarrow\quad \frac{d}{dt}\int_{S_\Gamma}\bar{B}\,d\bar{A}=\int_{S_\Gamma}\left[\frac{\partial\bar{B}}{\partial t}+\mathrm{rot}\,\left(\bar{B}\times\bar{v}\right)\right]d\bar{A}\qquad(5)$$

- **introducing relation (5) in relation (4) the result is:**

$$\Rightarrow\quad \oint_{\Gamma}\bar{E}\,d\bar{s}=-\int_{S_\Gamma}\frac{\partial\bar{B}}{\partial t}d\bar{A}-\int_{S_\Gamma}\mathrm{rot}\,\left(\bar{B}\times\bar{v}\right)d\bar{A}\qquad(6)$$

● **Stokes theorem :** $\int_{S_\Gamma}\mathrm{rot}\left(\bar{B}\times\bar{v}\right)d\bar{A}=\oint_{\Gamma}\left(\bar{B}\times\bar{v}\right)d\bar{s}\ \Rightarrow$

- relation (6) can be put in the form:

$$\oint_{\Gamma} \overline{E} d\overline{s} = -\int_{S_{\Gamma}} \frac{\partial \overline{B}}{\partial t} d\overline{A} + \oint_{\Gamma} \left(\overline{v} \times \overline{B} \right) d\overline{s} \qquad (7)$$

where:

- The first term of the right member is called induced electromotive voltage through transformation or in romanian tensiune electromotoare indusă prin transformare and arises through the existence of a time variation of the magnetic induction vector $\overline{B}$;

- The second term of the right member is called induced electromotive voltage through movement and arises through the existence of a relative movement between the curve Γ and the magnetic induction vector $\overline{B}$.

❑ Local form of the law:

- is deduced from the integral form of the law, more specifically, starting from relation (6):

from IFL

$$\oint_{\Gamma} \bar{E}\, d\bar{s} = -\int_{S_\Gamma} \frac{\partial \bar{B}}{\partial t} d\bar{A} - \int_{S_\Gamma} \text{rot}\,(\bar{B} \times \bar{v}) d\bar{A} \qquad \textbf{(6)}$$

- we apply Stokes Theorem to the term from the left member in order to have here also a surface integral and the result is:

$$\oint_{\Gamma} \bar{E}\, d\bar{s} = \int_{S_\Gamma} rot \bar{E}\, d\bar{A} \qquad \textbf{(8)}$$

- we introduce relation (8) in relation (6): $\int_{S_\Gamma} \boldsymbol{rot\bar{E}\, \overline{dA}} = -\int_{S_\Gamma} \boldsymbol{\frac{\partial \bar{B}}{\partial t} \overline{dA}} - \int_{S_\Gamma} \boldsymbol{rot(\bar{B} \times \bar{v}) \cdot \overline{dA}}$

❑ Local form of the law:

$$\boxed{rot\,\bar{E} = -\frac{\partial \bar{B}}{\partial t} + rot\left(\bar{v} \times \bar{B}\right)} \qquad \textbf{(9)}$$

❑ Particular case – immobile mediums

❖ **in immobile mediums**, meaning if we don't have movement, $\bar{v} = 0$, the local from of the law becomes:

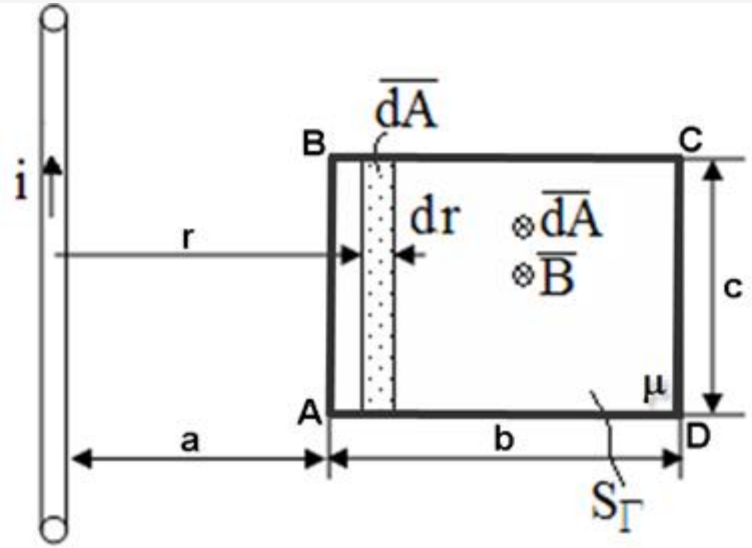
$$\boxed{rot\,\bar{E} = -\frac{\partial \bar{B}}{\partial t}} \qquad \textbf{(10)}$$

➞ Relation (10) is called **The 2nd Maxwell Equation**

Application

Electromotive Voltage Calculation

Find the electromotive voltage induced in the rectangular frame *ABCD* with the dimensions presented in the figure, due to the current i passing through a rectilinear, filiform, infinitely long conductor, present in the same plane as the frame, in the following situations:



1) The frame *ABCD* is fixed and the current $i = I_{max} \sin\omega t$;

2) The frame *ABCD* moves with the constant speed $\bar{v}$, at the moment $t=0$ being placed at the distance a from the conductor and the current i *being considered constant.*

3) The frame *ABCD* moves with the constant speed $\bar{v}$, at the moment $t=0$ being placed at the distance a from the conductor and the current $i = I_{max} \sin\omega t$.

Solution:

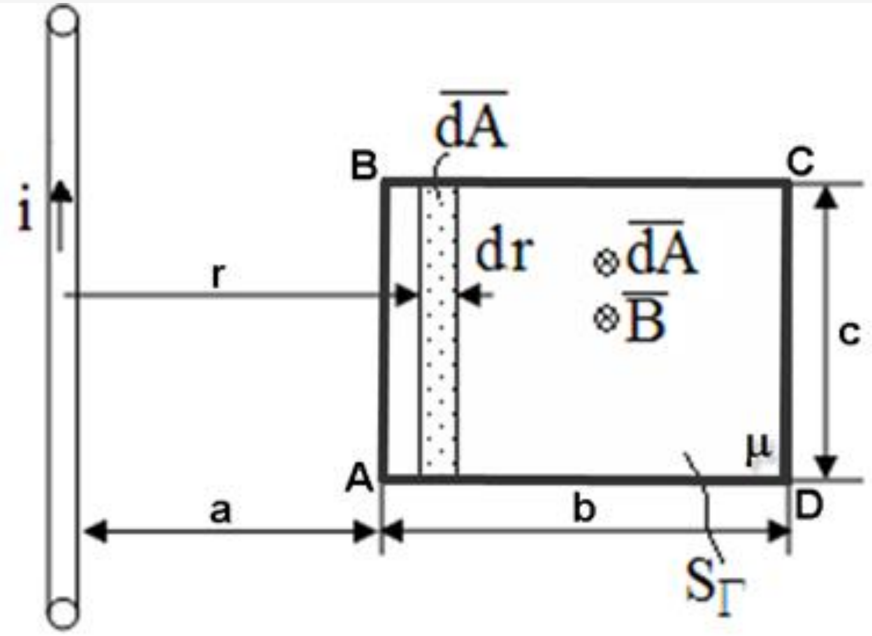
Solution:

✓ In order to find the induced electromotive voltage in the rectangular frame *ABCD* we will use *the Electromagnetic Induction Law* which says that the electromotive voltage, e_Γ, produced by electromotive induction along the closed curve Γ (which in our case is the contour of the rectangular frame *ABCD*) is equal to the decreasing speed of the magnetic flux through any surface supported by the curve Γ (which in our case is the surface of the frame ABCD, denoted S_Γ):

$$\boxed{e_\Gamma = -\frac{d\Phi_{S_\Gamma}}{dt}} \quad (1)$$

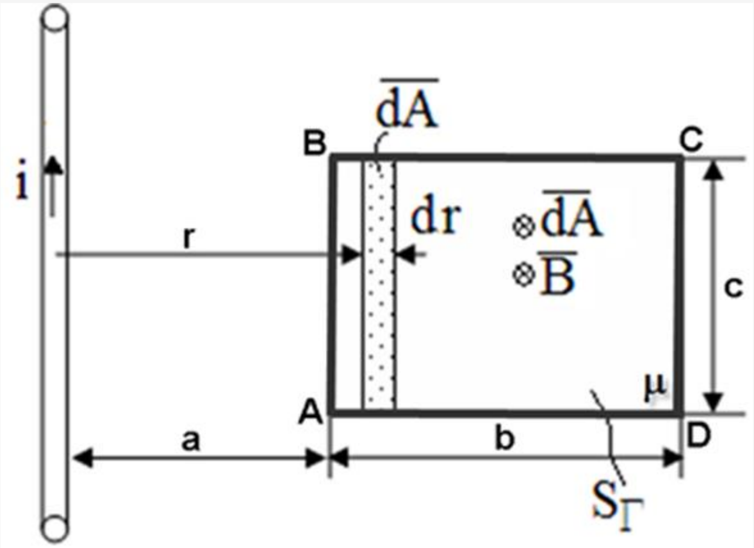
The magnetic flux is:

$$\Phi_{S_\Gamma} = \iint_{S_\Gamma} \bar{B} d\bar{A}$$



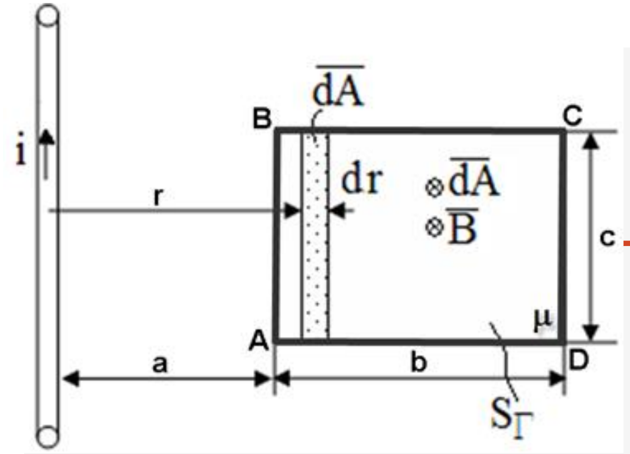
✓ From the data of the problem we observe that it is desired to determine the induced electromotive voltage in 3 different situations; thus considering the information regarding *the Electromagnetic Induction Law* presented in *Course 6* we see that, besides the relation (1), in this problem, the expression of the induced electromagnetic voltage decomposed in the induced electromagnetic voltage through transformation (the first term from the right member of relation (2)) and the induced electromagnetic voltage induced through movement (the second term from the right member of relation (2)) is very useful

$$e_\Gamma = -\iint_{S_\Gamma} \frac{\partial \vec{B}}{\partial t} d\vec{A} + \int_\Gamma \left(\vec{v} \times \vec{B}\right) d\vec{s} \quad (2)$$



a) the frame *ABCD* is fixed and the current $i_1 = I_{max}\, sin\omega t$

- First, we must analyze the problem's data in order to choose the simpler and most appropriate relation for the calculation of the induced electromotive voltage in the rectangular frame *ABCD:*
  - thus, in the data of the problem it is said that:
    - $\boldsymbol{i = I_{max} \cdot sin\omega t}$, from here it results that we have time variation, meaning that we have the transformation component of the induced electromotive voltage, $e_{\Gamma transformation} \neq 0$;
    - **the frame *ABCD* is immobile**, from here resulting that $\vec{v}=0$ meaning that we don't have movement, the movement component of the induced electromotive voltage, $e_{\Gamma movement} = 0$;
      - from these 2 observations it results that, for this study case the induced voltage is given only by the transformation component, meaning $e_\Gamma = e_{\Gamma transformation}$, thus the simplest and fastest way to solve is to use relation (2), because in this relation only the first term of the right member will be considered:

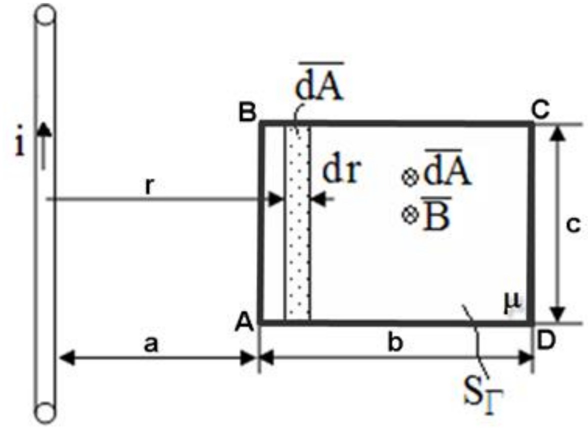


The frame ***ABCD*** **is immobile** $\Rightarrow \bar{v} = 0 \Rightarrow e_{\Gamma_{movement}} = 0$

$$\left.\begin{array}{r} i = \mathrm{I}_{\max} \sin \omega t \\ e_{\Gamma_{movement}} = 0 \end{array}\right\} \Rightarrow e_{\Gamma} = e_{\Gamma_{transformation}}$$

$$\Rightarrow e_{\Gamma_{transformation}} = -\iint_{S_\Gamma} \frac{\partial \vec{B}}{\partial t} d\vec{A}$$

- we observe that under the integral we have the scalar product of the vectors: magnetic induction $\bar{B}$ and area element $\overline{dA}$, thus in order to find the angle between these two vectors is necessary to represent them in the figure, thus:
    - we know that the vector of the magnetic field (weather we refer to $\bar{B}$ or $\bar{H}$) forms circles around the conductor in which an electrical current in passing through, thus in the case of this problem we can observe that these field lines, formed around the conductor, perpendicularly enter in the surface of the rectangular frame *ABCD;*

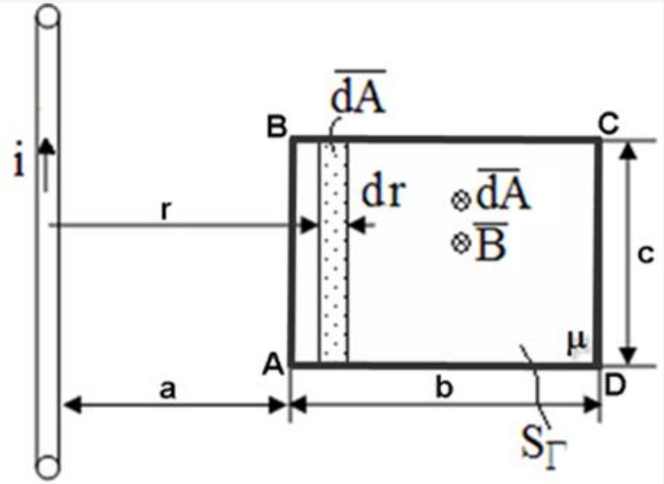


- the vector of the area element $\overline{dA}$ is perpendicular in any point from the S_Γ surface and we choose its direction towards the surface, meaning it enters in the integration surface S_Γ (the surface of the frame *ABCD):*
- from these observations it results that $\bar{B} \parallel \overline{dA}$, meaning between the two vectors we have an angle of *0* degrees:

$$\left.\begin{aligned} e_{\Gamma_{transformation}} &= -\iint_{S_\Gamma} \frac{\partial \vec{B}}{\partial t} d\vec{A} \\ \frac{\partial \vec{B}}{\partial t} d\vec{A} &= \left| \frac{\partial B}{\partial t} dA \right| \cos 0 = \frac{\partial B}{\partial t} dA \end{aligned}\right\} \Rightarrow e_{\Gamma_{transformation}} = -\int_{S_\Gamma} \frac{\partial B}{\partial t} dA \quad (3)$$

To find the magnetic induction we have first to know the magnetic field intensity:

The magnetic field intensity produced by a rectilinear infinitely long conductor in a point placed in the space of the frame at a distance r from the thread is:



$$H = \frac{i}{2\pi r} \quad (4)$$

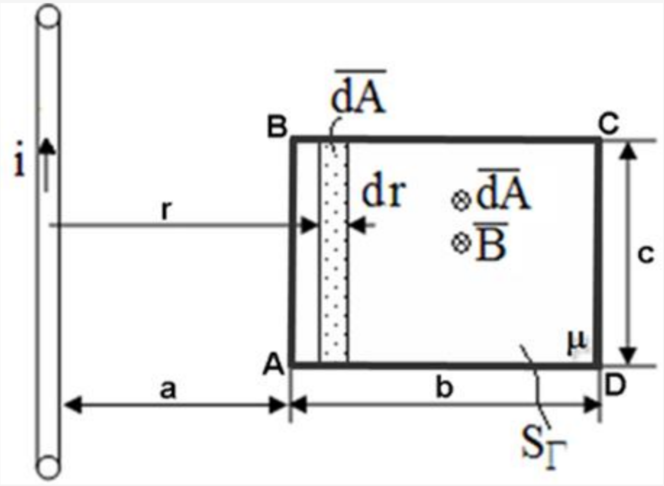
We know that the magnetic induction respects the relation:

$$\left.\begin{aligned} B &= \mu H \ (5) \\ H &= \frac{i}{2\pi r} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} B &= \frac{\mu i}{2\pi r} \\ i &= I_{max} \sin \omega t \end{aligned}\right\} \Rightarrow B = \frac{\mu i}{2\pi r} = \frac{\mu I_{max} \sin \omega t}{2\pi r}$$

To find the inducted electromotive voltage we have to magnetic induction time partial derivation:

$$\Rightarrow \frac{\partial B}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\mu I_{max} \sin \omega t}{2\pi r}\right) = \frac{\mu}{2\pi r}\frac{\partial}{\partial t}\left(I_{max} \sin \omega t\right) = \frac{\mu}{2\pi r}\omega I_{max} \cos \omega t \quad (6)$$

- ✓ we introduce relation (6) in relation (3) and we observe that in this problem, as in the problem from *Course 5*, it doesn't vary „all the area" of the surface S_Γ , only a side of it, meaning we randomly choose a point, placed at a distance *r*, somewhere inside the frame *ABCD* of the surface S_Γ, in which we determined the magnetic field intensity; thus, for us the variable is *dr*, so we must make the variable change and to express the area element *dA* depending on the length element *dr*:



$$\left.\begin{aligned} e_{\Gamma_{transformation}} &= -\int_{S_\Gamma} \frac{\mu I_{\max}\omega\cos\omega t}{2\pi r}\, dA \\ dA &= c\,dr \end{aligned}\right\} \Rightarrow$$

$$e_{\Gamma_{transformation}} = -\int_a^{a+b} \frac{\mu}{2\pi r}\,\omega I_{\max}\cos\omega t \cdot c \cdot dr$$

$$e_{\Gamma_{transformation}} = -\frac{\mu\omega I_{\max}\cos\omega t\cdot c}{2\pi}\int_a^{a+b}\frac{1}{r}dr$$

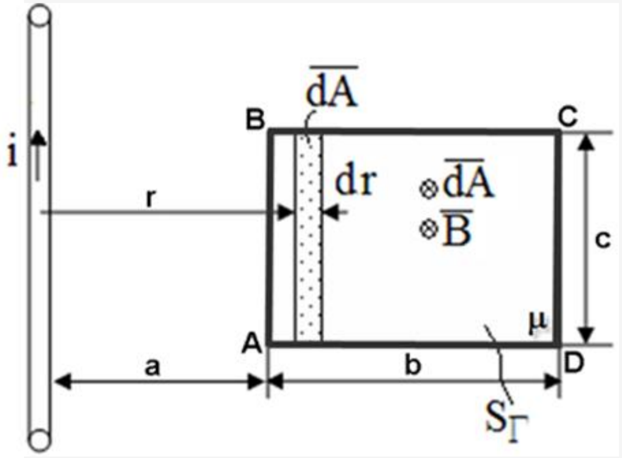
$$e_{\Gamma_{transformation}} = -\frac{\mu\omega I_{\max}\cos\omega t\cdot c}{2\pi}\ln r\Big|_{a}^{a+b}$$

$$\Rightarrow \boxed{e_{\Gamma_{transformation}} = -\frac{\mu\omega I_{\max}\cos\omega t\cdot c}{2\pi}\ln\frac{a+b}{a}} \quad (7)$$

b) the frame ABCD moves with constant speed $\overline{v}$, at the moment t = 0 being placed at the distance a from the conductor and the current i is constant

✓ Again, we start by analyzing the problem's data in order to choose the simpler and most appropriate relation for the calculation of the voltage induced in the rectangular frame *ABCD*:

- thus, this time, in the problem's data it is said that:
  - **the frame *ABCD* moves with constant speed** $\bar{v}$ meaning we have movement; thus, we have movement component for the induced electromotive voltage, $e_{\Gamma movement} \neq 0$;
  - $\boldsymbol{i = ct}$, from here we can deduce that there is no time variation, thus we don't have transformation component of the induced electromotive voltage, $e_{\Gamma transformation} = 0$;



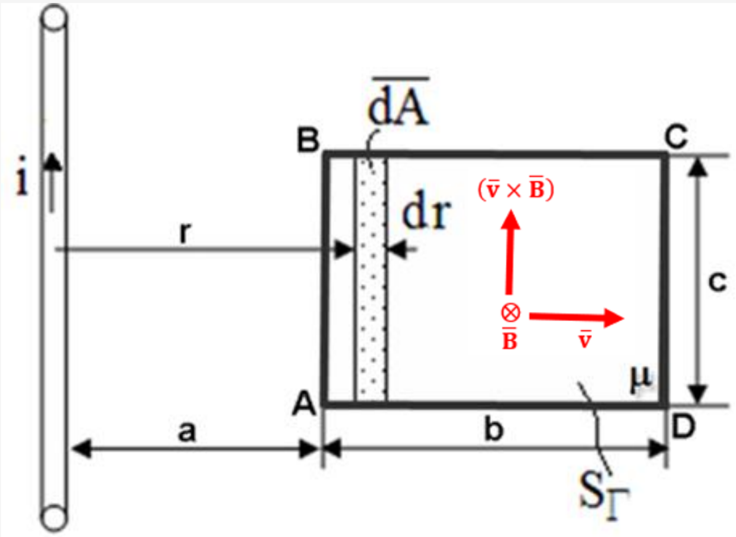
i=ct ➡ B=ct ➡ $e_{\Gamma_{transformation}} = 0$

➢ From the 2 observations it results that, for this study case the induced voltage is given only by the movement component, $e_{\Gamma} = e_{\Gamma movement}$, thus the simplest and fastest we solve with relation (2), because in this relation only the second term of the right member remains:

➡ $e_{\Gamma} = e_{\Gamma_{movement}}$

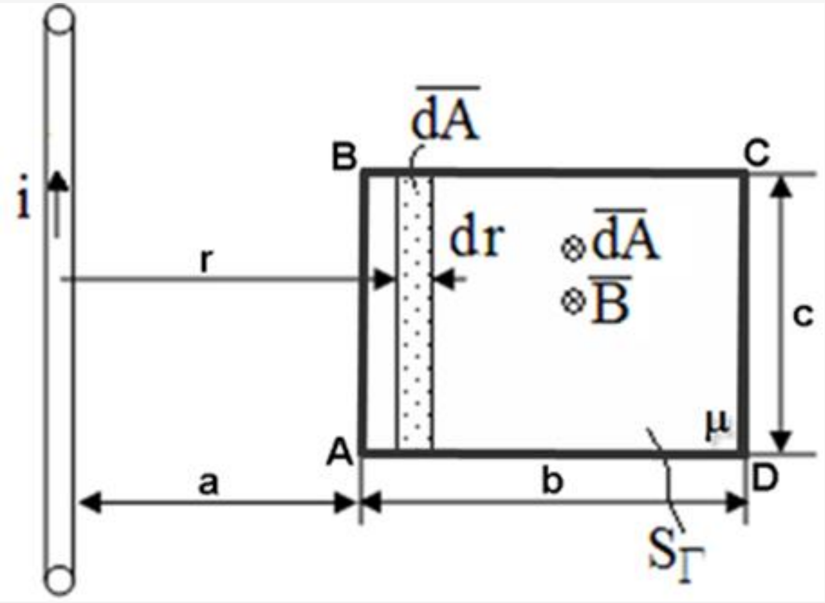
$$e_\Gamma = e_{\Gamma_{movement}} = \int_\Gamma \left(\vec{v} \times \vec{B}\right) d\vec{s} \qquad (8)$$

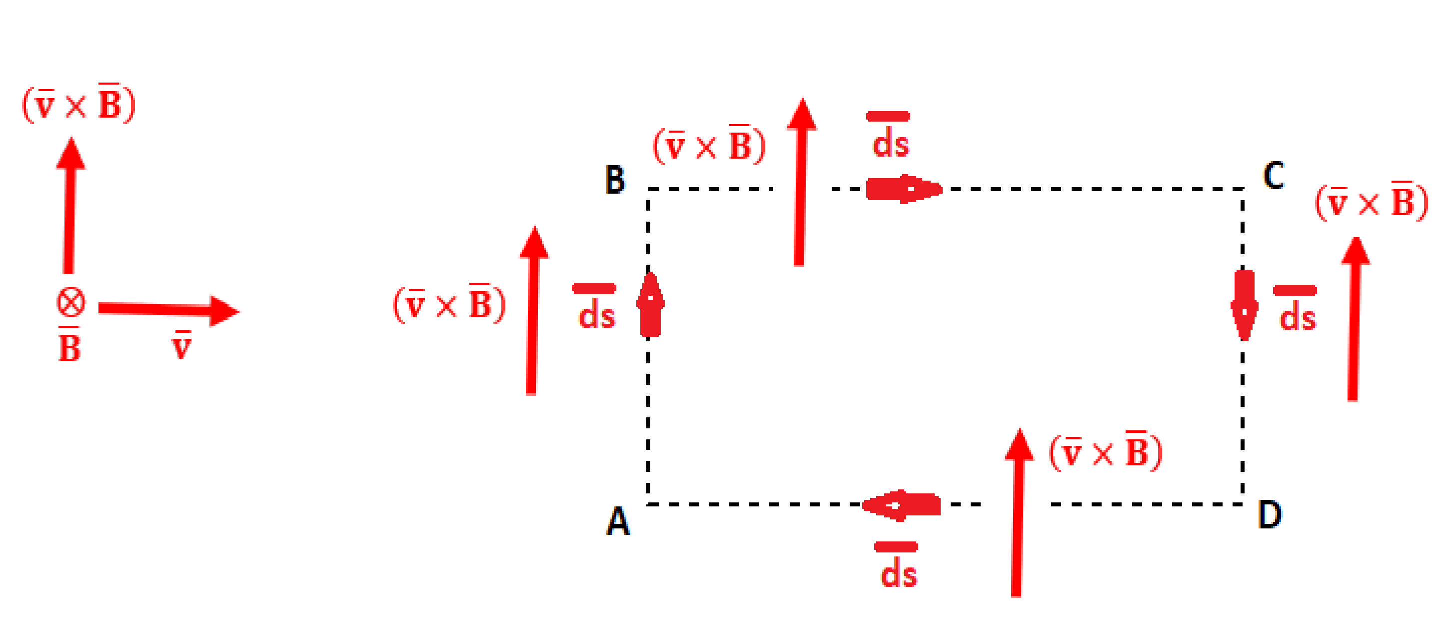
- We observe that under the integral we have a vectorial product of the vectors: speed $\bar{v}$ and magnetic induction $\bar{B}$, thus in order to find the angle between these two vectors is necessary to place them in the figure:
    - the magnetic field vector was already placed in order to solve *the 1st point of the problem;*
    - from the fact that in the problem's data it is said that the frame *ABCD* is moving with a constant speed $\bar{\boldsymbol{v}}$, at a moment t = 0 being at a distance *a* from the conductor, it results that this frame is moving towards right;
    - we observe that the vector $\bar{B}$ enters perpendicularly in the surface S_Γ, the vector $\bar{\boldsymbol{v}}$ is oriented towards right, thus it results that the vectorial product between the two vectors $(\bar{v} \times \bar{B})$ is in any point oriented perpendicularly and upwards;

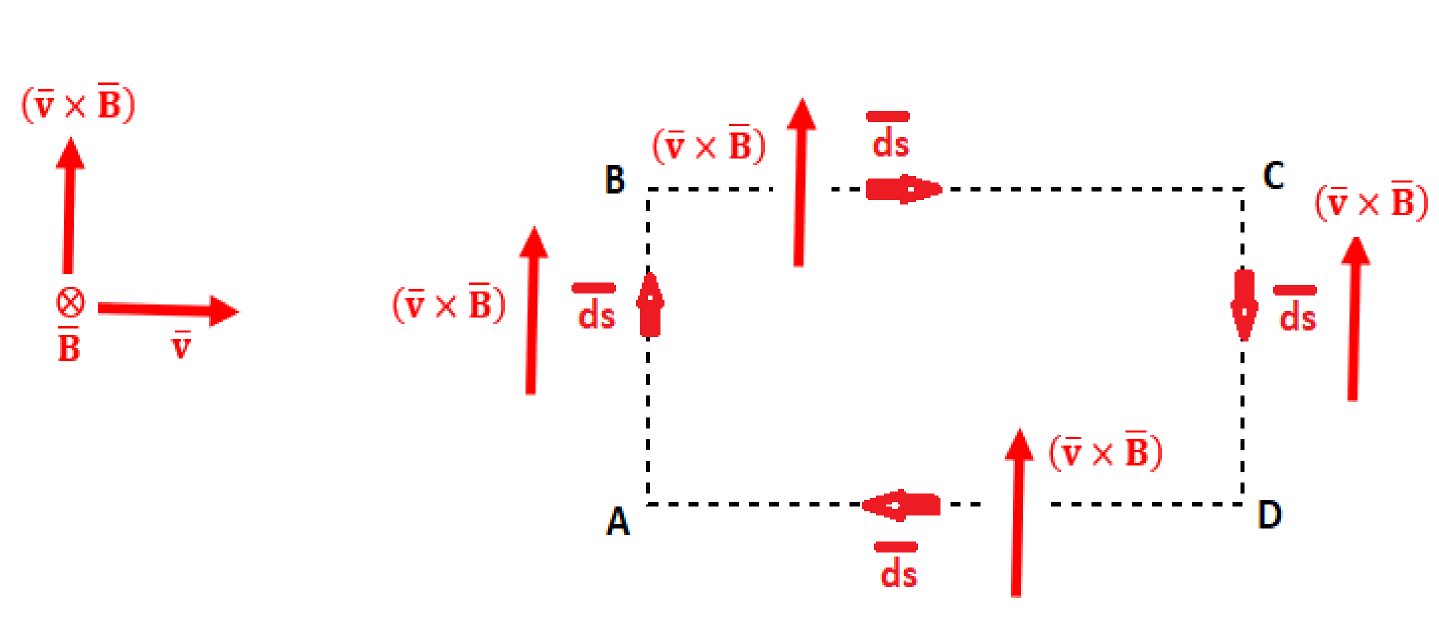


$$e_{\Gamma} = e_{\Gamma_{movement}} = \int_{\Gamma} \left(\vec{v} \times \vec{B}\right) d\vec{s} \quad (8)$$

- also, in relation (8) we observe that besides the vectorial product mentioned above we also have the scalar product between the vector $(\bar{v} \times \bar{B})$ and the vector of the length element $\overline{ds}$:
  - the length element $\overline{ds}$ represents a portion (a piece) of the contour *Γ*, thus, as it can be observed in Figure 2, this vector has on each side of the rectangle *ABCD* another orientation (direction), thus we must decompose the curve *Γ*, on each of the rectangle's sides:







$$e_{\Gamma_{miscare}}=\int_{\Gamma}\left(\vec{v}\times\vec{B}\right)d\vec{s}=\int_{AB}\left(\vec{v}\times\vec{B}\right)d\vec{s}+\int_{BC}\left(\vec{v}\times\vec{B}\right)d\vec{s}+\int_{CD}\left(\vec{v}\times\vec{B}\right)d\vec{s}+\int_{DA}\left(\vec{v}\times\vec{B}\right)d\vec{s}$$

- we know that the vectorial product of two vectors is determined with the relation:

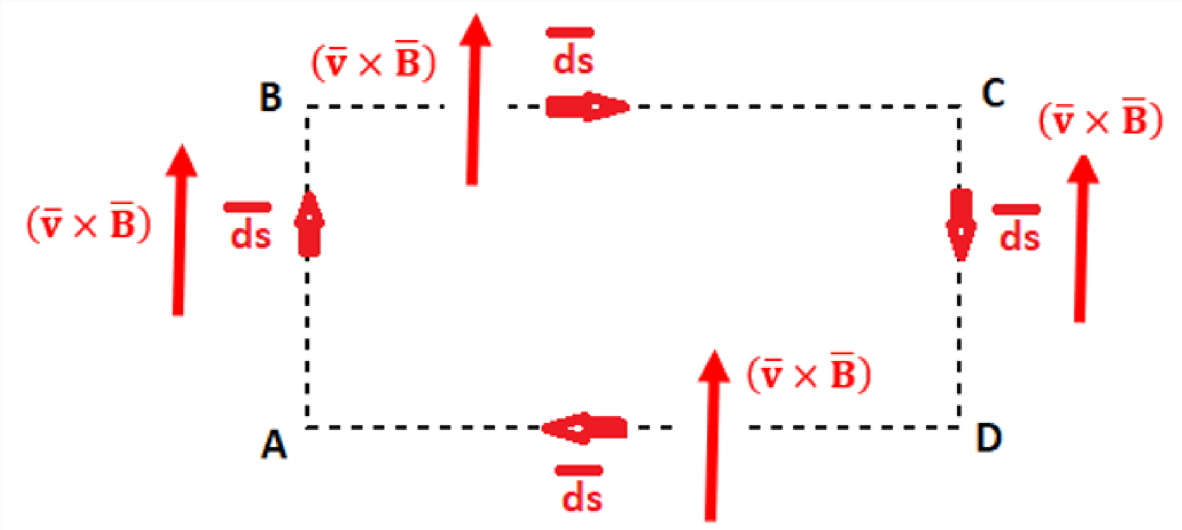
$$(\bar{v}\times\bar{B})=v\cdot B\cdot\sin(\widehat{\bar{v},\bar{B}})$$

  - in our problem between the 2 vectors, we have an angle of 90 degrees:

$$(\bar{v}\times\bar{B})=v\cdot B\cdot\sin 90^{\circ}=v\cdot B$$

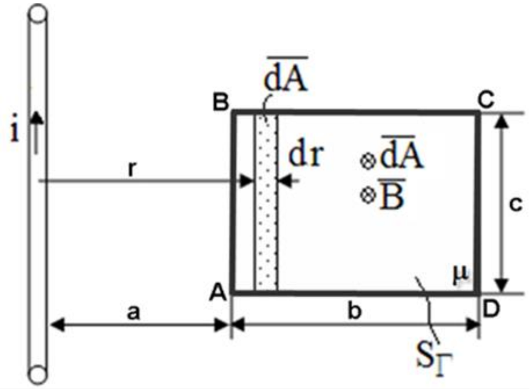
- we know that the scalar product of 2 vectors is determined with the relation:

$$\left(\vec{v}\times\vec{B}\right)d\vec{s} = \left|\vec{v}\times\vec{B}\right|\left|d\vec{s}\right|\cos\sphericalangle\left(\vec{v}\times\vec{B}, d\vec{s}\right)$$



$$e_{\Gamma_{miscare}} = \int_{AB}\left|\left(\vec{v}\times\vec{B}\right)\right|\left|d\vec{s}\right|\cos 0^\circ + \int_{BC}\left|\left(\vec{v}\times\vec{B}\right)\right|\left|d\vec{s}\right|\cos 90^\circ + \int_{CD}\left|\left(\vec{v}\times\vec{B}\right)\right|\left|d\vec{s}\right|\cos 180^\circ + \int_{DA}\left|\left(\vec{v}\times\vec{B}\right)\right|\left|d\vec{s}\right|\cos 90^\circ$$

$$\Rightarrow \quad e_{\Gamma_{miscare}} = \int_{AB} v\cdot B\cdot ds - \int_{CD} v\cdot B\cdot ds \qquad (9)$$



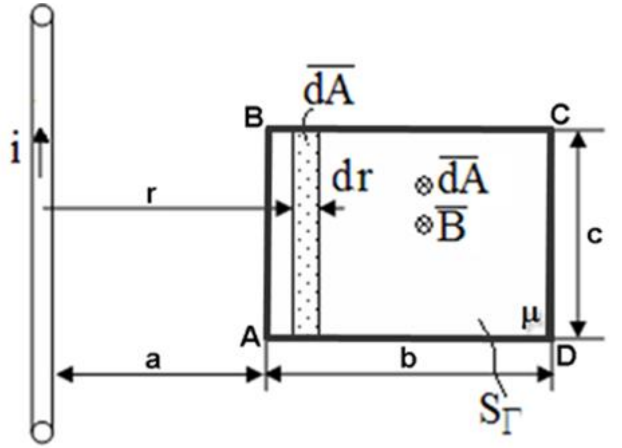
- ✓ from the problem's data we know that $\vec{v}$=***ct***;
- ✓ we must determine the magnetic induction at the distance between the conductor and the side *AB*, denoted B_{AB}, meaning the case in which we are on the *AB* side of the rectangle, at a distance $(a + v \cdot t)$ from the conductor (it is said that at the moment *t=0* the frame is at the distance *a* from the conductor, meaning that at a certain moment *t* we will be at a distance $(a + v \cdot t)$ and taking into account the relations (5) and (4) it results:

$$B_{AB} = \frac{\mu i}{2\pi\left(a + vt\right)} \qquad (10)$$

- ✓ similar we will do in order to determine the magnetic induction at a distance *CD* from the thread, B_{CD}, meaning the case in which we are on the *CD* side, thus at a distance $(a + b + v \cdot t)$ from the thread:

$$B_{CD} = \frac{\mu i}{2\pi\left(a + b + vt\right)} \qquad (11)$$

✓ we introduce the relation (10) and (11) in relation (9) and it results:



$$\left.\begin{aligned} B_{AB} &= \frac{\mu i}{2\pi(a+vt)} \\ B_{CD} &= \frac{\mu i}{2\pi(a+b+vt)} \\ \vec{v} &= ct \end{aligned}\right\} \Rightarrow e_{\Gamma_{movement}} = v\int_{AB}\frac{\mu i ds}{2\pi(a+vt)} - v\int_{CD}\frac{\mu i ds}{2\pi(a+b+vt)}$$

$$e_{\Gamma_{movement}} = v\left[\frac{\mu i}{2\pi(a+vt)}\int_{AB} ds - \frac{\mu i}{2\pi(a+b+vt)}\int_{CD} ds\right]$$

$$\Rightarrow e_{\Gamma_{movement}} = v\left[\frac{\mu i}{2\pi(a+vt)}c - \frac{\mu i}{2\pi(a+b+vt)}c\right] \Rightarrow e_{\Gamma_{movement}} = \frac{v\mu ic}{2\pi}\left(\frac{1}{a+vt} - \frac{1}{a+b+vt}\right)$$

$$\Rightarrow e_{\Gamma_{movement}} = \frac{v\mu ic}{2\pi}\left[\frac{a+b+vt-a-vt}{(a+vt)(a+b+vt)}\right] \Rightarrow \boxed{e_{\Gamma_{movement}} = \frac{v\mu icb}{2\pi(a+vt)(a+b+vt)}} \quad (12)$$

c) the frame ABCD is moving with the constant speed $\overline{v}$, at the moment *t* = *0* being at the distance *a* from the conductor and the current $i_1 = I_{max}\ sin\omega t$

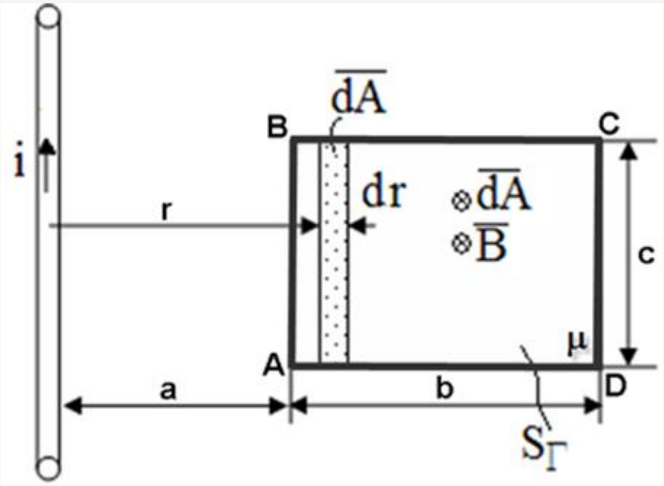
- ✓ in this situation we also analyze the problem's data in order to choose the simplest and the most appropriate relation for the induced electromotive voltage calculation in the rectangular frame *ABCD*:
  - this time in the problem's data it is said that:
    - **the frame *ABCD* is moving with the speed $\vec{v}$=*ct*** thus we have movement, and of course movement component of the induced electromotive voltage, $e_{\Gamma miscare} \neq 0$
    - $\boldsymbol{i = I_{max} \cdot sin\omega t}$, from here it results that we have time variation, thus we have transformation component of the induced electromotive voltage, $e_{\Gamma transformare} \neq 0$;
      - ➢ from the 2 observations it results that in this case the induced electromotive voltage is given by both movement component, $e_{\Gamma miscare}$, and transformation component, $e_{\Gamma transformare}$, of the induced electromotive voltage, thus the simplest and fastest way to solve is by using relation from point a) and b) being the two terms of the right member in relation (2), but it can be calculate also with relation (1):

$$i = I_{max} \sin \omega t \Rightarrow B \neq ct \Rightarrow \exists \; e_{\Gamma_{transformation}}$$

$$v \neq 0 \Rightarrow \exists \; e_{\Gamma_{movement}}$$

$$\Rightarrow \boxed{e_\Gamma = e_{\Gamma_{transformation}} + e_{\Gamma_{movement}}} \quad \text{or} \quad \boxed{e_\Gamma = -\frac{d\Phi_{S_\Gamma}}{dt}} \quad (13)$$

- ✓ the magnetic flux is determined from the *Magnetic Flux Law*, thus:



$$\Phi = \iint_{S_\Gamma} \vec{B} d\vec{A}$$

- we have shown above that $\bar{B} \parallel \overline{dA}$;
- the magnetic induction B in a point placed at a distance r from the conductor is: $B = \dfrac{\mu i}{2\pi r}$
- we observe that in this case we must make a variable change, and we express the element of the area dA depending on the length element dr: $dA = cdr$

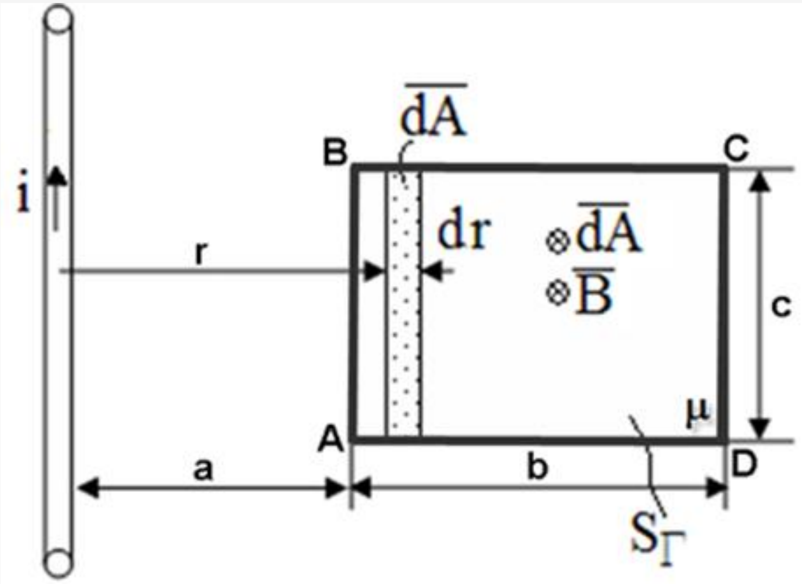
$$\Phi = \iint_{S_\Gamma} \vec{B} d\vec{A}$$

$$\vec{B} \parallel d\vec{A}$$

$$B = \frac{\mu i}{2\pi r}$$

$$dA = cdr$$

$$\Rightarrow \Phi = \iint_{S_\Gamma} BdA$$



$$\Phi = \int_{a+vt}^{a+b+vt} \frac{\mu i}{2\pi r} cdr = \frac{\mu i}{2\pi} c \int_{a+vt}^{a+b+vt} \frac{1}{r} dr = \frac{\mu i}{2\pi} c \ln \frac{a+b+vt}{a+vt}$$

$$i = \mathrm{I}_{\max} \sin \omega t$$

$$\Rightarrow$$

$$\Phi = \frac{\mu I_{max} \sin \omega t}{2\pi} c \ln \frac{a+b+vt}{a+vt}$$

✓ we come back to relation (13) in order to determine the induced electromotive voltage in the rectangular frame *ABCD*:

$$e_\Gamma = -\frac{d}{dt}\left(\frac{\mu I_{max} \sin \omega t}{2\pi} c \ln \frac{a+b+vt}{a+vt}\right)$$

- from mathematics we know that:

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$(\sin u)' = u' \cdot \cos u$$

$$(\ln u)' = \frac{u'}{u}$$

$$e_\Gamma = -\left\{\frac{\mu \omega I_{max} \cos \omega t \cdot c}{2\pi} \ln \frac{a+b+v \cdot t}{a+v \cdot t} + \frac{\mu I_{max} \sin \omega t \cdot c}{2\pi}\left[\ln\left(a+b+v \cdot t\right) - \ln\left(a+v \cdot t\right)\right]'\right\}$$

$$e_\Gamma = -\left[\frac{\mu \omega I_{max} \cos \omega t \cdot c}{2\pi} \ln \frac{a+b+v \cdot t}{a+v \cdot t} + \frac{\mu I_{max} \sin \omega t \cdot c}{2\pi}\left(\frac{v}{a+b+v \cdot t} - \frac{v}{a+v \cdot t}\right)\right]$$

$$\Rightarrow e_\Gamma = -\frac{\mu\omega I_{\max}\cos\omega t\cdot c}{2\pi}\ln\frac{a+b+v\cdot t}{a+v\cdot t}+\frac{\mu I_{\max}\sin\omega t\cdot c\cdot v\cdot b}{2\pi(a+v\cdot t)(a+b+v\cdot t)}$$

or

$$\boxed{e_\Gamma = e_{\Gamma_{transformation}} + e_{\Gamma_{movement}}}$$

$$e_{\Gamma_{transformation}} = -\frac{\mu\omega I_{\max}\cos\omega t\cdot c}{2\pi}\ln\frac{a+b}{a} \qquad e_{\Gamma_{movement}} = \frac{v\mu icb}{2\pi(a+vt)(a+b+vt)}$$

We must consider $a = a + vt$ and $i = I_{\max}\sin\omega t$

$$\boxed{e_\Gamma = -\frac{\mu\omega I_{\max}\cos\omega t\cdot c}{2\pi}\ln\frac{a+b+v\cdot t}{a+v\cdot t}+\frac{\mu I_{\max}\sin\omega t\cdot c\cdot v\cdot b}{2\pi(a+v\cdot t)(a+b+v\cdot t)}}$$

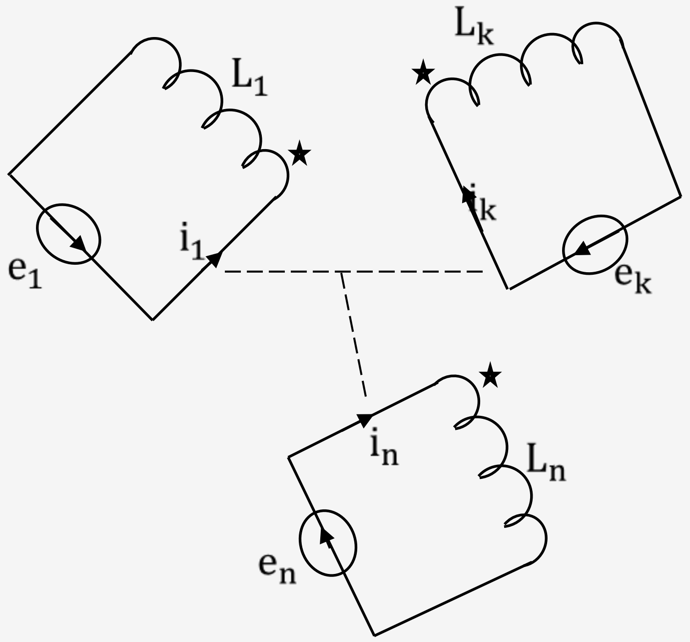
2. Energies and Forces in Magnetic Field

2.1 Magnetic Field Energy

It is considered a circuit system, where the currents $i_1, i_2, \ldots i_n$ are passing through and connected to the electromotive voltage sources $e_1, e_2, \ldots e_n$.

The circuits can be mobile, and the electromotive voltages can be variable in time.

From the time t to the time $t+dt$, the total energy ceded to the sources is $\sum_{k=1}^{n} e_k i_k dt$ and it must cover:



- the energy dissipated in the circuit's conductors due to Joule-Lenz effect:

$$\sum_{k=1}^{n} r_k i_k^2 dt$$

- the elementary mechanical work carried out by the generalized magnetic forces X, at the variation of the generalized coordinate x (the external mechanical work):

$$\delta L = X \cdot dx$$

- the magnetic energy variation dW_m located in the magnetic field of the n circuits (the internal mechanical work).

The energy balance for the n circuit system, in the elementary time *dt* is given by the equality:

$$\sum_{k=1}^{n} e_k i_k dt = \sum_{k=1}^{n} r_k i_k^2 dt + \delta L + dW_m \qquad (1)$$

Notes

Besides the electromotive voltage of the sources, in the circuits are induced electromotive voltages given by *electromagnetic induction law:*

$$e_{ki} = -\frac{d\Psi_k}{dt} \qquad (2)$$

We suppose that the system is imobile: $\Rightarrow$ $\delta L = 0$ (3)

Ohm's Law is applied on one side of the circuit : $e_k + e_{ki} = r_k\, i_k$ (4)

- We introduce relation (2) in relation (4): $\Rightarrow$ $e_k - \frac{d\Psi_k}{dt} = r_k\, i_K$

- we extend relation (5) from a branch to the entire circuit, meaning we will make the sum from relation (5) and multiply with $i_k dt$:

$$\Rightarrow \quad e_k = r_k i_k + \frac{d\Psi_k}{dt} \quad \Big| \cdot i_k dt \qquad \textbf{(5)}$$

$$\Rightarrow \quad \sum_{k=1}^{n} e_k i_k dt = \sum_{k=1}^{n} r_k i_k^2 dt + \sum_{k=1}^{n} d\Psi_k i_k \qquad \textbf{(6)}$$

- from relations (1) and (6):

$$\left.\begin{aligned} &\Rightarrow \quad \sum_{k=1}^{n} d\Psi_k i_k = \delta L + dW_m \quad \textbf{(7)} \\ &\delta L = 0 \end{aligned}\right\} \Rightarrow$$

- from relation (7):

$$\Rightarrow \quad dW_m = \sum_{k=1}^{n} d\Psi_k i_k \quad \textbf{(8)}$$

- we know that the flux through inductor *k is:*

$$\Psi_k = L_{kk}\, i_k + \sum_{\substack{j=1 \\ j\neq k}}^{n} L_{kj}\, i_j = \sum_{j=1}^{n} L_{kj}\, i_j \qquad (9)$$

$$\Rightarrow \quad d\Psi_k = \sum_{j=1}^{n} L_{kj} \cdot di_j \qquad (10)$$

- we introduce relation (10) in relation (8): $\Rightarrow \quad dW_m = \sum_{k=1}^{n}\sum_{j=1}^{n}\left(L_{kj} \cdot di_j\right) \cdot i_k$

- or by integration: $\Rightarrow \quad \boxed{W_m = \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} L_{kj} \cdot i_k \cdot i_j} \qquad (11)$

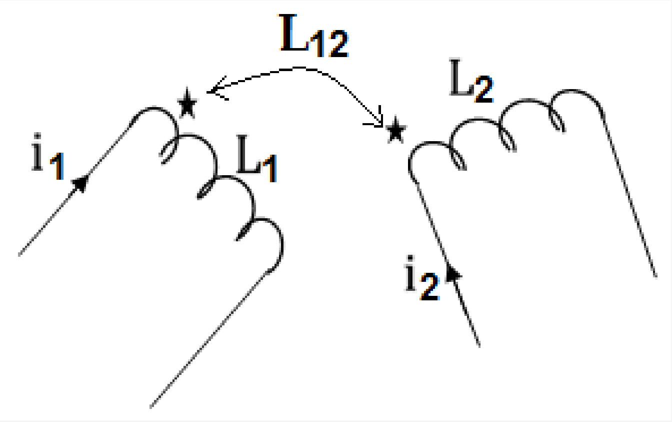
- using relation (9) (the relation of the magnetic flux), the magnetic energy can be written as:

$$\Rightarrow \quad \boxed{W_m = \sum_{k=1}^{n} \frac{\Psi_k \cdot i_k}{2}} \qquad (12)$$

Application

Magnetic Energy Calculation

Calculate the magnetic energy of two inductors magnetically coupled as seen in the figure:



Solution:

- in order to determine the magnetic energy, we apply the following relation to the circuit presented in the figure:

$$\boxed{W_m = \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} L_{kj}\cdot i_k \cdot i_j}$$

$$\left.\begin{aligned} W_m &= \frac{1}{2}\left(L_1\cdot i_1^2 + L_{12}\cdot i_1\cdot i_2 + L_{21}\cdot i_2\cdot i_1 + L_2\cdot i_2^2\right) \\ L_{12} &= L_{21} \end{aligned}\right\}$$

$$W_m = \frac{1}{2}\left(L_1\cdot i_1^2 + L_2\cdot i_2^2\right) + L_{12}\cdot i_1\cdot i_2$$

$$W_m = \frac{1}{2}\left(L_1 \cdot i_1^2 + L_2 \cdot i_2^2\right) + L_{12} \cdot i_1 \cdot i_2$$

where:

- $W_{m1} = \dfrac{L_1 \cdot i_1^2}{2}$ **represents the self-magnetic energy of *inductor 1***
- $W_{m2} = \dfrac{L_2 \cdot i_2^2}{2}$ **represents the self-magnetic energy of *inductor 2***
- $W_{m12} = L_{12} \cdot i_1 \cdot i_2$ **represents the interaction energy between the *two coupled inductors***

- ***total magnetic energy:***

$$W_m = W_{m1} + W_{m2} + W_{m12}$$

2.2 Magnetic Forces

The elementary mechanical work, which is carried at a movement *dx* of a body in the magnetic field, under the action of the generalized force *X*, can be calculated from relation (3):

$$X \cdot dx = \sum_{k=1}^{n} i_k d\Psi_k - dW_m \qquad (13)$$

<u>Two calculation situations</u> are considered:

a) If the fluxes are maintained constant ($d\psi_k = 0$),

$$\Rightarrow \quad X \cdot dx = -\left(dW_m\right)_{\Psi_k = const.}$$

$$\Rightarrow \quad \boxed{X = -\left(\frac{\partial W_m}{\partial x}\right)_{\Psi_k = const.}}$$

The generalized force *X*, corresponding to the generalized coordinate *x*, is equal to the partial derivative of the magnetic energy in relation to the generalized coordinate, with opposite sign, at constant magnetic fluxes through circuits.

b) If in the circuits the curents are supposed to be constant:

- relation (13) can be put in the form:

$$X \cdot dx = d\left[\sum_{k=1}^{n} i_k \cdot \Psi_k\right] - dW_m$$

- we know that the magnetic energy is equal with:

$$W_m = \sum_{k=1}^{n} \frac{\Psi_k \cdot i_k}{2}$$

$$\Rightarrow \sum_{k=1}^{n} i_k \cdot \Psi_k = 2W_m$$

$$\Rightarrow X \cdot dx = 2 \cdot dW_m - dW_m$$

$$X \cdot dx = dW_m$$ for i_k = const.

the generalized force:

$$\boxed{X = +\left(\frac{\partial W_m}{\partial x}\right)_{i_k = const.}}$$

The generalized force X, corresponding to the generalized coordinate x, is equal to the partial derivative of the magnetic energy (expressed in relation to the currents and the generalized coordinate), the corresponding generalized coordinate in relation to the constant currents from the circuits.

Thank you for your attention!!!

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Questions???