MS. Suplimentar pentru sesiune

1.3 Elementary functions of a complex variable

$$e^z = e^x(\cos y + i\sin y)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

exponential form

$$z = \rho e^{i\varphi} \qquad \rho = |z| \qquad \varphi = \frac{y}{x}$$

$$e^{z_1+z_2} = e^{z_1}e^{z_2}, \quad z_1, z_2 \in \mathbb{C}$$

$$e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z(\cos 2k\pi + i\sin 2k\pi) = e^z$$

2. The polynomial function is defined by

$$w = a_0 z^n + a_1 z^{n-1} + \dots + a_n, \quad a_i, i = 1, \dots, n$$

3. The rational function

4. Trigonometric functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}$$

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}$$

5. hyperbolic functions

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}$$

6. Logarithmic function. Let $z \in \mathbb{C}$, $z \neq 0$

w logarithm of z if $e^w = z$

$$z = \rho(\cos\theta + i\sin\theta)$$

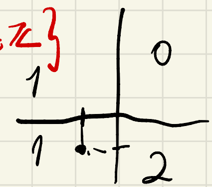
$$w = \operatorname{Log} z = \{\ln|z| + i(\arg z + 2k\pi) \mid k \in \mathbb{Z}\}$$

$$\log z = \ln|z| + i\arg z$$

$$\log(-1) = \ln|-1| + i\arg(-1) = i\pi$$

$$\operatorname{Log}(-1) = \{\ln|-1| + i(\arg(-1) + 2k\pi) \mid k \in \mathbb{Z}\}$$

$$= i(\pi + 2k\pi)$$



$A(-1,-1)$

$$\arg(\overset{x}{-1} - \overset{y}{i}) = \arctan 1 + \pi = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

7. general power function

$$w = z^a = e^{a \operatorname{Log} z}$$

principal value $z^a = e^{a \log z}$

Exerciții:

$$\operatorname{Log}(1-i) = \{\ln|1-i| + i(\arg(1-i)) + 2k\pi\}$$

$$= \{\ln\sqrt{2} + i(\frac{5\pi}{4} + 2k\pi)\}$$

$$z^a = i^{1-i} = e^{(1-i)\operatorname{Log} i}$$

$$\operatorname{Log} i = \{ \ln 1 + i\left(\frac{\pi}{2} + 2k\pi\right)$$

$$= e^{(1-i)i\left(\frac{\pi}{2}+2k\pi\right)} = e^{(i+1)\left(\frac{\pi}{2}+2k\pi\right)}$$

1.20 $\cos z = \frac{3+i}{4}$

$$\frac{e^{iz} + e^{-iz}}{2} = \frac{3+i}{4}$$

$$t = e^{iz} \Rightarrow iz = \operatorname{Log} t$$

$$\frac{t + \frac{1}{t}}{2} = \frac{3+i}{4}$$

$$2(3+i) = 4\left(t + \frac{1}{t}\right)$$

$$6 + 2i = 4t + \frac{4}{t}$$

$$t(6+2i) = 4t^2 + 4$$

$$4t^2 - 2(3t + it) + 4 = 0$$

$$2t^2 - t(3+i) + 2 = 0$$

$$\Delta = 6i - 8 = (1+3i)^2 \Rightarrow \quad t_1 = \frac{3+i-1+3i}{2} = 1+i$$

$$t_2 = \frac{3+i-1-3i}{2} = \frac{1}{2} - \frac{1}{2}i$$

$iz = \mathrm{Log}(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right) / i \qquad 1 = i^4$

$\frac{i^4}{i} = i^3 = -i$

$z = \frac{\ln\sqrt{2}}{i} + \frac{\pi}{4} + 2k\pi = -\frac{i}{2}\ln 2 \qquad \ln\sqrt{2} = \ln 2^{\frac{1}{2}}$

$iz = \mathrm{Log}\left(\frac{1}{2} - \frac{1}{2}i\right) = \ln\frac{1}{\sqrt{2}} + i\left(\frac{7\pi}{4} + 2k\pi\right)$

$\arctan\frac{-\frac{1}{2}}{\frac{1}{2}} + 2\pi$

$-\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$

$v = -e^{-x}\sin y$

$\frac{1}{2+3} = \frac{1}{2}$

1.24 $\quad w = u + iv$

① $w = e^{-z} = e^{-x-iy} = e^{-x}(\cos y - i\sin y)$

$= e^{-x}\cos y - ie^{-x}\sin y \qquad -i^2 = -1 \cdot -1 = 1$

② $w = \sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+yi)} - e^{-i(x+yi)}}{2i}$

$= \frac{e^{xi-y} - e^{-ix+y}}{2i} = \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i}$

$= \frac{e^{-y}\cos x + ie^{-y}\sin x - e^{y}\cos x + ie^{y}\sin x}{2i}$

$= \frac{\cos x(e^{-y} - e^{y}) + i\sin x(e^{-y} + e^{y})}{2i}$

$$-\frac{\cos x\,(e^{y}-e^{-y})}{2}$$

$$-\sinh y \cos x = -1^3 = -(-1) = 1$$

SEMINAR 9

Applications of the residue theorem to evaluate real integrals

I) $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$

$$z = e^{i\theta} \quad \begin{cases} d\theta = \dfrac{dz}{iz} \\ \cos\theta = \dfrac{z^2+1}{2z} \\ \sin\theta = \dfrac{z^2-1}{2iz} \end{cases}$$

II) $\int_{-\infty}^{+\infty} f(x)\,dx \quad , \quad f(x) = \dfrac{P(x)}{Q(x)}$

P, Q polynomials of degrees m, n

$n \geq m+2 \quad , \quad Q(x) \neq 0$

$$I = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}_{z=z_k} f(z)$$

① $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+1)(x^2+9)}\,dx$

$$f(z) = \frac{z^2}{(z^2+1)(z^2+9)}$$

$z^2+1 = 0 \Rightarrow z^2 = -1 \Rightarrow z = \pm i \Rightarrow z_0 = i$

$z^2+9 = 0 \Rightarrow z^2 = -9 \Rightarrow z = \pm 3i \Rightarrow z_1 = 3i$

$(\operatorname{Im} z_0 > 0, \operatorname{Im} z_1 > 0)$ poles of order 1

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=i} \frac{\frac{z^2}{z^2+9}}{2z}\Bigg|_{z=i} = \frac{-\frac{1}{8}}{2i} = \frac{-1}{16i}$$

$$\operatorname*{Res}_{z=z_1} f(z) = \operatorname*{Res}_{z=3i} \frac{\frac{z^2}{z^2+1}}{2z}\Bigg|_{z=3i} = \frac{\frac{-9}{-8}}{6i} = \frac{9}{48i} = \frac{3}{16i}$$

$$I = 2\pi i\left(-\frac{1}{16i} + \frac{3}{16i}\right) = 2\pi i + \frac{2}{16i} = \frac{\pi}{4}$$

② $$I = \int_{-\infty}^{+\infty} \frac{dx}{x^4+9x^2+20}$$

$$f(z) = \frac{1}{z^4+9z^2+20}$$

$$z^2 = t$$

$$t^2+9t+20 = 0$$

$$\Delta = 1$$

$$t_1 = -4 \;,\; t_2 = -5$$

$$z^2 = -4 \Rightarrow z_1 = \pm 2i \Rightarrow z_1 = 2i \text{ (int } C)$$

$$z^2 = -5 \Rightarrow z_2 = \pm\sqrt{5}i \Rightarrow z_2 = \sqrt{5}i \text{ (int } C)$$

poles of order 1

$$\operatorname*{Res}_{z=z_1} f(z) = \operatorname*{Res}_{z_1=2i} \frac{1}{4z^3+18z}\Bigg|_{z=2i} = \frac{1}{4\cdot 8\cdot i + 18\cdot 2\cdot i} = \frac{1}{4i}$$

$$\operatorname*{Res}_{z=z_2} f(z) = \operatorname*{Res}_{z_2=\sqrt{5}i} \left.\frac{1}{4z^3+18z}\right|_{z=\sqrt{5}i} = \frac{1}{4\cdot 5\sqrt{5}i+18\sqrt{5}i} = \frac{1}{-2\sqrt{5}i}$$

$$I = 2\pi i\left(\frac{1}{4i} - \frac{1}{2\sqrt{5}i}\right)$$

III $I = \int_0^{+\infty} f(x)\,dx$, $f(x) = \frac{P(x)}{Q(x)}$; P, Q polynomials of degrees m and n

$n \geq 2+m$, $Q(x) \neq 0$

f is even function

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} f(z)\,dz = \frac{1}{2}\cdot 2\pi i \sum_{\operatorname{Im} z_k>0} \operatorname*{Res}_{z=z_k} f(z)$$

③ $I = \int_0^{\infty} \frac{dx}{(x^2+4)(x^2+16)}$

$f(z) = \frac{1}{(z^2+4)(z^2+16)}$ even function

$z^2+4=0 \Rightarrow z^2=-4 \Rightarrow z=\pm 2i$, $z_1 = 2i$

$z^2+16=0 \Rightarrow z^2=-16 \Rightarrow z=\pm 4i$, $z_2 = 4i$

$\operatorname{Im} z_1, z_2 > 0$ poles of order 1

$$\operatorname*{Res}_{z=2i} f(z) = \left.\frac{\frac{1}{z^2+16}}{2z}\right|_{z=2i} = \frac{\frac{1}{-4+16}}{4i} = \frac{1}{-16i+64i} = \frac{1}{48i}$$

$$\operatorname*{Res}_{z=4i} f(z) = \left.\frac{\frac{1}{z^2+4}}{2z}\right|_{z=4i} = \frac{\frac{1}{-16+4}}{8i} = -\frac{1}{96i}$$

$$I = \frac{1}{2} 2\pi i \left(\frac{1}{48i} - \frac{1}{96i} \right) = \frac{\pi}{96}$$

④ $$I = \int_0^{\infty} \frac{x^2}{(x^2+1)^2} dx$$

$$f(z) = \frac{z^2}{(z^2+1)^2}$$ even function

$z^2+1=0 \Rightarrow z^2=-1 \Rightarrow z=\pm i$ poles of order 2

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to i} \cancel{(z-i)^2} \left(\frac{z^2}{(z+i)^2 \cancel{(z-i)^2}} \right) =$$

$$= \lim_{z\to i} \frac{2z(z+i)^2 - z^2 2(z+i)}{(z+i)^4} = \lim_{z\to i} \cancel{(z+i)} \frac{(2z(z+i) - 2z^2)}{(z+i)^{\cancel{4}}}$$

$$= \frac{1}{4i}$$

$$I = \frac{1}{2} 2\pi i \operatorname*{Res}_{z=z_0} f(z) = \pi i \cdot \frac{1}{4i} = \frac{\pi}{4}$$

IV $$\int_{-\infty}^{+\infty} f(x) e^{i\lambda x} dx, \quad f(x) = \frac{P(x)}{Q(x)},$$

P, Q are polynomials of degrees m, n $n \geq m+2$
$Q(x) \neq 0$, $Q(x)$ has no real roots

$$I = \int_{-\infty}^{+\infty} f(z) e^{i\lambda z}\, dz = \begin{cases} 2\pi i \sum\limits_{\operatorname{Im} z_k > 0} \operatorname{Res}\limits_{z=z_k} f(z), & \lambda > 0 \\ -2\pi i \sum\limits_{\operatorname{Im} z_k > 0} \operatorname{Res}\limits_{z=z_k} f(z), & \lambda < 0 \end{cases}$$

5. $$\int_{-\infty}^{+\infty} \frac{e^{-ix}}{x^2-2x+5}\, dx$$

$$f(z) = \frac{e^{-iz}}{z^2-2x+5}$$

$$z^2 - 2z + 5 = 0$$

$$\Delta = -16$$

$$z_{1,2} \begin{cases} 1+2i \\ 1-2i \end{cases}$$

$z_1 = 1-2i$ pole of order 1, $\operatorname{Im} z_1 < 0$

$\lambda = -1$

$$\operatorname{Res}_{z=1-2i} f(z) = \left. \frac{e^{-iz}}{2z-2} \right|_{z=1-2i} =$$

$$= \frac{e^{-i(1-2i)}}{2-4i-2} = \frac{e^{-i(1-2i)}}{-4i} = \frac{-1}{4i} e^{-2}(\cos 1 + i \sin 1)$$

$$I = -\cancel{2}\pi i \left(-\frac{1}{\cancel{4}i}\right) e^{-2}(\cos 1 + i\sin 1) = \frac{\pi}{2} \cdot e^{-2}(\cos 1 + i\sin 1)$$

⑥ $$\int_{-\infty}^{+\infty} \frac{xe^{ix}}{(x^2+4)^2}\,dx$$

$$f(z) = \frac{ze^{iz}}{(z^2+4)^2}$$

$z = 2i$ pole of order 2
$\lambda > 0$

$$\operatorname*{Res}_{z=2i} f(z) = \lim_{z\to 2i}\left[\cancel{(z-2i)^2}\,\frac{z\cdot e^{2iz}}{\cancel{(z-2i)^2}(z+2i)^2}\right]' =$$

$$= \lim_{z\to 2i} \frac{\left(e^{2iz} + z\cdot 2i\cdot e^{2iz}\right)\cdot(z+2i)^{\cancel{2}} - z\cdot e^{iz}\cdot 2\cancel{(z+2i)}}{(z+2i)^{\cancel{4}}}$$

$$= \frac{\left(e^{-4} - 4\cdot e^{-4}\right).4i - 4i\cdot e^{-4}}{(4i)^3} = +\frac{4e^{-4}}{16} = \frac{e^{-4}}{4}$$

$$I = 2\pi i\cdot\frac{e^{-4}}{4} = \frac{\pi\cdot i\cdot e^{-4}}{2}$$

Ⅴ $$\underbrace{\int_{-\infty}^{+\infty} f(x)\cos\alpha x\,dx}_{I} \quad \text{and} \quad \underbrace{\int_{-\infty}^{+\infty} f(x)\sin\alpha x\,dx}_{y}$$

$$K = I + iy = \int_{-\infty}^{+\infty} f(x)\left[\cos\alpha x + i\sin\alpha x\right]dx$$

$$= \int_{-\infty}^{+\infty} f(x) e^{i\alpha x}\, dx$$

(7) $$\underbrace{\int_{-\infty}^{+\infty} \frac{x \sin x}{x^2+9}\, dx}_{J} \quad ; \quad I = \int_{-\infty}^{+\infty} \frac{x \cos x}{x^2+9}\, dx$$

$$K = I + iJ = \int_{-\infty}^{+\infty} \frac{x \cos x}{x^2+9}\, dx + i \int_{-\infty}^{+\infty} \frac{x \sin x}{x^2+9}\, dx$$

$$= \int_{-\infty}^{+\infty} \frac{x e^{ix}}{x^2+9}\, dx$$

$$f(z) = \frac{z e^{iz}}{z^2+9}$$

$z^2 + 9 = 0 \Rightarrow z = \pm 3i$

$\lambda > 0 \Rightarrow z_0 = 3i$ pole of order 1

$$\operatorname*{Res}_{z=3i} f(z) = \left. \frac{z e^{iz}}{z^2+9} \right|_{z_0 = 3i} = \frac{e^{-3}}{2}$$

$$K = 2\pi i \operatorname*{Res}_{z=3i} f(z) = \cancel{2}\pi i \cdot \frac{e^{-3}}{\cancel{2}} = \pi i e^{-3}$$

$$\Rightarrow \begin{cases} I = 0 \\ y = \pi e^{-3} \end{cases}$$

A FOST LA EXAMEN

(8) $$\int_0^{\infty} \frac{1}{(x^2+4)(x^2+25)}\, dx$$

$$f(z) = \frac{1}{(z^2+4)(z^2+25)}$$

$$z^2+4=0 \Rightarrow z = \pm 2i \Rightarrow z_0 = 2i$$

$$z^2+25=0 \Rightarrow z = \pm 5i \Rightarrow z_1 = 5i$$

$$\operatorname*{Res}_{z=2i} f(z) = \left.\frac{\frac{1}{z^2+25}}{2z}\right|_{z=2i} = \frac{1}{\frac{21}{4i}} = \frac{1}{84i}$$

$$\operatorname*{Res}_{z=5i} f(z) = \left.\frac{\frac{1}{z^2+4}}{2z}\right|_{z=5i} = \frac{1}{\frac{-21}{10i}} = \frac{-1}{210i}$$

$$I = \frac{1}{2} 2\pi i \left(\frac{1}{84i} - \frac{1}{210i}\right) = \pi\left(\frac{1}{84i} - \frac{1}{210i}\right) = \frac{\pi}{140}$$

⑨ $$\int_0^{2\pi} \frac{\sin^2 x}{10+6\cos x}\,dx$$

$$z = e^{ix}$$

$$dx = \frac{dz}{iz}$$

$$\sin x = \frac{z^2-1}{2iz} \qquad \cos x = \frac{z^2+1}{2z}$$

$$I = \int_{C:|z|=1} \frac{\left(\frac{z^2-1}{2iz}\right)^2}{10+6\frac{(z^2+1)}{2z}}\,\frac{dz}{iz} = \int_{C:|z|=1} \frac{\frac{z^4+2z^2-1}{-4z^2}}{10z+6z^2+6}\cdot\frac{dz}{iz}$$

SEMINAR 10

Laplace transform

$f : \mathbb{R} \to \mathbb{C}$ the original

$$F(p) = \int_0^\infty f(t) e^{-pt}\, dt = \mathcal{L}[f(t)](p)$$

$$\underbrace{x(t)}_{\text{the time domain}} \underset{ILT}{\overset{LT}{\rightleftarrows}} \underbrace{X(p)}_{\text{Laplace domain}}$$

$$\mathcal{L}[a](p) = \frac{a}{p}$$

$$\mathcal{L}[e^{\lambda t}](p) = \frac{1}{p-\lambda}$$

$$\mathcal{L}[t^n](p) = \frac{n!}{p^{n+1}}, \quad n \in \mathbb{N}$$

$$\mathcal{L}[f(at)](p) = \frac{1}{a}\mathcal{L}[f(t)]\left(\frac{p}{a}\right)$$

$$\mathcal{L}[e^{at} \cdot f(t)](p) = \mathcal{L}[f(t)](p-a)$$

$$\mathcal{L}[t^1 \cdot f(t)](p) = (-1)^1 \left(\mathcal{L}[f(t)](p)\right)'$$

$$\mathcal{L}[t^n \cdot f(t)](p) = (-1)^n \left(\mathcal{L}[f(t)](p)\right)^{(n)}$$

$$\mathcal{L}[\sin(at)](p) = \frac{a}{p^2+a^2}$$

$$\mathcal{L}[\cos(at)](p) = \frac{p}{p^2+a^2}$$

$$\mathcal{L}[\sinh(at)](p) = \frac{a}{p^2-a^2}$$

$$\mathcal{L}[\cosh(at)](p) = \frac{p}{p^2-a^2}$$

$$\mathcal{L}\left[\int_0^t f(s)\,ds\right](p) = \frac{1}{p}\mathcal{L}[f(t)](p)$$

$$\mathcal{L}\left[\frac{f(t)}{t}\right](p) = \int_p^\infty \mathcal{L}[f(t)](q)\,dq$$

① Find the images by Laplace transform

a) $\mathcal{L}[e^{2t}\cos 3t + e^{3t}\sin 2t](p) =$

$= \mathcal{L}[e^{2t}\cos 3t](p) + \mathcal{L}[e^{3t}\sin 2t](p) =$

$= \mathcal{L}[\cos 3t](p-2) + \mathcal{L}[\sin 2t](p-3) =$

$= \left.\frac{p}{p^2+9}\right|_{p=p-2} + \left.\frac{2}{p^2+4}\right|_{p=p-3} =$

$= \frac{p-2}{(p-2)^2+9} + \frac{2}{(p-3)^2+4}$

b) $\mathcal{L}[t^3 \cdot e^{-t}](p) \overset{\text{I method}}{=\!=} \mathcal{L}[t^3](p+1)$

$= \left.\frac{3!}{p^4}\right|_{p=p+1} = \frac{3!}{(p+1)^4}$

II method $(-1)^3\left(\mathcal{L}[e^{-t}](p)\right)''' = -\left(\frac{1}{p+1}\right)''' =$

$$= -\left(\frac{-1}{(p+1)^2}\right)'' = \left((p+1)^{-2}\right)'' = -2\left((p+1)^{-3}\right)'$$

$$= 6(p+1)^{-4} = \frac{6}{(p+1)^4}$$

c) $\mathcal{L}[t\cdot e^{2t}\cdot \cos t](p) = \mathcal{L}[t\cdot\cos t](p-2)$

$$= (-1)^1\left(\mathcal{L}[\cos t](p)\right)'\Big|_{p=p-2} = -\left(\frac{p}{p^2+1}\right)'\Big|_{p=p-2}$$

$$= -\frac{p^2+1-2p^2}{(p^2+1)^2}\Big|_{p=p-2} = \frac{p^2-1}{(p^2+1)^2}\Big|_{p=p-2} = \frac{(p-2)^2-1}{((p-2)^2+1)^2}$$

d) $\mathcal{L}\left[\int_0^t \sin 3u\, du\right](p) = \frac{1}{p}\mathcal{L}[\sin 3t](p) = \frac{1}{p}\,\frac{3}{p^2+9} = \frac{3}{p^3+9p}$

e) $\mathcal{L}\left[\int_0^t u^2 e^{-3u}\, du\right](p) = \frac{1}{p}\mathcal{L}[t^2\cdot e^{-3t}](p) =$

$$= \frac{1}{p}\mathcal{L}[t^2](p+3) = \frac{1}{p}\left(\frac{2!}{p^3}\right)\Big|_{p=p+3} = \frac{1}{p}\cdot\frac{2}{(p+3)^3}$$

f) $\mathcal{L}\left[\int_0^t \frac{\sin u}{u}\, du\right](p) = \frac{1}{p}\mathcal{L}\left[\frac{\sin t}{t}\right](p)$

$$= \frac{1}{p}\int_p^\infty \mathcal{L}[\sin t](q)\, dq =$$

$$= \frac{1}{p}\int_p^{\infty} \frac{1}{q^2+1}\,dq = \frac{1}{p}\cdot \arctan q \Big|_p^{\infty} = \frac{1}{p}(\arctan p + \frac{\pi}{2})$$

$$= \frac{1}{p}\arctan\left(\frac{1}{p}\right)$$

g) $$\mathcal{L}[t^2\cdot \cos t](p) = (-1)^2\left(\mathcal{L}[\cos t](p)\right)''$$

$$= \left(\frac{p}{p^2+1}\right)'' = \left(\frac{p'(p^2+1)-(p^2+1)'p}{(p^2+1)^2}\right)' = \left(\frac{p^2+1-2p^2}{(p^2+1)^2}\right)' =$$

$$= \frac{1-p^2}{} = \frac{-2p(p^2+1)^2-(1-p^2)2\cdot 2p(p^2+1)}{(p^2+1)^4} = \frac{2p^3-6p}{(p^2+1)^3}$$

h) $$\mathcal{L}\left[\frac{e^{-2t}-e^{-3t}}{t}\right](q) = \mathcal{L}\left[\frac{e^{-2t}}{t}\right](p) - \mathcal{L}\left[\frac{e^{-3t}}{t}\right](p) =$$

$$= \int_0^{\infty}\mathcal{L}[e^{-2t}](y)\,dy - \int_0^{\infty}\mathcal{L}[e^{-3t}](y)\,dy =$$

$$= \int_0^{\infty}\frac{1}{y+2}\,dy - \int_0^{\infty}\frac{1}{y+3}\,dy = \ln\frac{y+2}{y+3}\Big|_p^{\infty} = -\ln\frac{p+2}{p+3} = \ln\frac{p+3}{p+2}$$

i*) $$\mathcal{L}\left[e^{-t}\cdot\int_0^t \frac{\sin 3s}{s}\,ds\right](p) =$$

$$\mathcal{L}\left[\int_0^{\infty}\frac{\sin 3s}{s}\,ds\right](p+1) =$$

$$= \left(\frac{1}{p} \mathcal{L}\left[\frac{\sin 3t}{3} \right](p) \right) \Big|_{p=p+1}$$

EXAMEN

$$= \left(\frac{1}{p} \int_p^\infty \mathcal{L}[\sin 3t](y)\,dy \right) \Big|_{p=p+1}$$

$$= \left(\frac{1}{p} \int_p^\infty \frac{3}{y^2+9}\,dy \right) \Big|_{p=p+1}$$

$$= \left(\frac{1}{p} \, 3 \, \frac{1}{3} \arctan \frac{y}{3} \Big|_p^\infty \right) \Big|_{p=p+1}$$

$$= \left(\frac{1}{p} \left(\frac{\pi}{2} - \arctan \frac{p}{3} \right) \right) \Big|_{p=p+1}$$

$$= \frac{1}{p+1} \cdot \arctan \frac{3}{p+1}$$

$$X(p) \xrightarrow{\text{ILT}} x(t)$$

② Find the originals $f(t)$ corresponding to the following images

a) $\mathcal{L}^{-1}\left[\frac{1}{p-3}\right] = e^{3t}$

b) $\mathcal{L}^{-1}\left[\frac{1}{2p-3}\right] = \mathcal{L}^{-1}\left[\frac{1}{2(p-\frac{3}{2})}\right] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{p-\frac{3}{2}}\right] = \frac{1}{2} e^{\frac{3}{2}t}$

c) $\mathcal{L}^{-1}\left[\frac{1}{p^2+9}\right] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{3}{p^2+9}\right] = \frac{1}{3}\sin 3t$

d) $\mathcal{L}^{-1}\left[\frac{1}{p^3}\right] = \frac{1}{2!}\mathcal{L}\left[\frac{2!}{p^3}\right] = \frac{1}{2}t^2$

$\mathcal{L}[t^n] = \frac{n!}{p^{n+1}}$

e) $\left[\frac{1}{(p-1)^4}\right] = \frac{1}{3!}\mathcal{L}^{-1}\left[\frac{3!}{(p-1)^4}\right] = \frac{1}{3!}\mathcal{L}^{-1}\left[\frac{3!}{p^4}\bigg|_{p=p-1}\right] = \frac{1}{3!}t^3e^t$

f) $\mathcal{L}^{-1}\left[\frac{1}{p(p-3)}\right] = \mathcal{L}^{-1}\left[\frac{1}{3}\,\frac{p-(p-3)}{p(p-3)}\right] = \mathcal{L}^{-1}\left[\frac{1}{3}\left(\frac{1}{p-3}-\frac{1}{p}\right)\right]$

$= \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{p-3}\right] - \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{p}\right] = \frac{1}{3}e^{3t} - \frac{1}{3}$

g) $\mathcal{L}^{-1}\left[\frac{1}{p^2-6p+25}\right] = \mathcal{L}^{-1}\left[\frac{1}{p^2-6p+9+16}\right] = \mathcal{L}^{-1}\left[\frac{1}{(p+3)^2+4^2}\right]$

$= \frac{1}{4}\mathcal{L}^{-1}\left[\frac{4}{(p-3)^2+4^2}\right] = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{4}{p^2+4^2}\bigg|_{p=p-3}\right]$

$\mathcal{L}[\sin at] = \frac{a}{a^2+p^2}$

EXAMEN

$= \frac{1}{4}e^{3t}\sin 4t$

h) $\mathcal{L}^{-1}\left[\frac{1}{p^2-5p+6}\right]$ EXAMEN

i) $\mathcal{L}^{-1}\left[\frac{p}{p^2-4p+11}\right] = \mathcal{L}^{-1}\left[\frac{p}{(p-2)^2+7}\right] = \mathcal{L}^{-1}\left[\frac{p}{(p-2)^2+(\sqrt{7})^2}\right]$

$\mathcal{L}^{-1}\left[\frac{p-2+2}{(p-2)^2+(\sqrt{7})^2}\right] = \mathcal{L}^{-1}\left[\frac{p-2}{(p-2)^2+(\sqrt{7})^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(p-2)^2+(\sqrt{7})^2}\right]$

$$\mathcal{L}^{-1}\left[\frac{p-2}{(p-2)^2+(\sqrt{7})^2}\right]+\frac{2}{\sqrt{7}}\mathcal{L}^{-1}\left[\frac{\sqrt{7}}{(p-2)^2+(\sqrt{7})^2}\right]$$

$$= e^{2t}\cos\sqrt{7}t + \frac{2}{\sqrt{7}}e^{2t}\sin\sqrt{7}t$$

K) $\mathcal{L}\left[\frac{\sin^2 2t}{t}\right](p) = \mathcal{L}\left[\frac{1-\cos 4t}{2t}\right](p) =$

$$\cos 2x = 1-2\sin^2 x$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\int_p^\infty \mathcal{L}\left[\frac{1-\cos t}{2}\right](q)\,dq = \frac{1}{2}\int_p^\infty \mathcal{L}[1-\cos 4t](q)\,dq$$

$$\frac{1}{2}\int_p^\infty \frac{1}{q} - \frac{q}{q^2+16}\,dq = \frac{1}{2}\left(\int_p^\infty \frac{1}{q}\,dq - \int_p^\infty \frac{q}{q^2+16}\,dq\right)$$

$$\frac{1}{2}\left(\ln q - \frac{1}{2}\ln(q^2+16)\right)\Big|_p^\infty = \frac{1}{2}\left(\ln q - \ln(q^2+16)^{\frac{1}{2}}\right)$$

$$\frac{1}{2}\ln\frac{q}{\sqrt{q^2+16}}\Big|_p^\infty = -\frac{1}{2}\ln\frac{p}{\sqrt{p^2+16}}$$

SEMINAR 11

Applications of the Laplace Transnsform

$f(t)$; $f'(t)$; $f''(t)$ originals ; $F(p)$ - the image

$$\mathcal{L}[f(t)](p) = F(p)$$

$$\mathcal{L}[f'(t)](p) = pF(p) - f(0)$$

$$\mathcal{L}[f''(t)](p) = p^2 F(p) - pf(0) - f'(0)$$

Algorithm: 1°) we convert the ODE to an algebraic equation
2°) We solve the algebraic eq. for the unknown $F(p) (X(p), Y(p))$
3°) We decompose $F(p)$ using partial fraction decomposition
4°) We apply the ILT to obtain the sol. of the initial form.

1) $x''(t) - 5x'(t) + 6x(t) = 0 \quad / \mathcal{L}$, $x(0) = 1$, $x'(0) = -1$

$$\mathcal{L}[x''(t)](p) - 5\mathcal{L}[x'(t)](p) + 6\mathcal{L}[x(t)](p) = 0$$

$$\mathcal{L}[x(t)](p) = X(p)$$

$$p^2 X(p) - p\underbrace{x(0)}_{=1} - \underbrace{x'(0)}_{=-1} - 5\left[pX(p) - \underbrace{x(0)}_{=1}\right] + 6X(p) = 0$$

$$X(p) \cdot (p^2 - 5p + 6) = p - 1 - 5$$

$$X(p) = \frac{p-6}{(p-2)(p-3)} = \frac{A}{p-2} + \frac{B}{p-3} \Rightarrow p - 6 = A(p-3) + B(p-2)$$

$p = 3 \Rightarrow -3 = B$

$p = 2 \Rightarrow -4 = -A \Rightarrow A = 4$

$$X(p) = \frac{4}{p-2} - \frac{3}{p-3} \quad \Big| \mathcal{L}^{-1}$$

$$x(t) = \mathcal{L}^{-1}\left[\frac{4}{p-2}\right] - \mathcal{L}^{-1}\left[\frac{3}{p-3}\right]$$

$$x(t) = 4e^{2t} - 3e^{3t}$$

② $x''(t) + x(t) = 2\cos t$, $\quad x(0) = 1$, $x'(0) = -1$

$$\mathcal{L}[x''(t)](p) + \mathcal{L}[x(t)](p) = 2\frac{1}{p^2+1}$$

$$p^2 \cdot X(p) - p\cdot \underbrace{x(0)}_{=1} - \underbrace{x'(0)}_{=-1} + X(p) = 2\frac{p}{p^2+1}$$

$$p^2 \cdot X(p) + 1 + X(p) = 2\frac{p}{p^2+1}$$

$$X(p)(p^2+1) = 2\frac{p}{p^2+1} - 1 \quad \Big| :(p^2+1)$$

$$X(p) = \frac{2p}{(p^2+1)^2} - \frac{1}{p^2+1} \quad \Big| \mathcal{L}^{-1}$$

$$x(t) = \mathcal{L}^{-1}\left[\frac{2p}{(p^2+1)^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{p^2+1}\right]$$

$$x(t) = t\sin t - \sin t$$

$$\mathcal{L}^{-1}\left[\frac{2p}{(p^2+1)^2}\right] = \mathcal{L}^{-1}\left[\left(-\frac{1}{p^2+1}\right)'\right] =$$

$$= \mathcal{L}^{-1}\left[(-1)\left(\frac{1}{p^2+1}\right)'\right] = \mathcal{L}^{-1}\left[\left((-1)\mathcal{L}^{-1}[\sin(t)](p)\right)'\right] = t\sin t$$

③ $\begin{cases} x'(t) = 3x(t) - y(t) \\ y'(t) = -9x(t) + 3y(t) \end{cases} \Big|\ \mathcal{L} \qquad \begin{matrix} x(0) = 1 \\ y(0) = 0 \end{matrix}$

$$\begin{cases} \mathcal{L}[x'(t)](p) = 3\mathcal{L}[x(t)](p) - \mathcal{L}[z(t)](p) \\ \mathcal{L}[z'(t)](p) = -9\mathcal{L}[x(t)](p) + 3\mathcal{L}[z(t)](p) \end{cases}$$

$$\begin{cases} pX(p) - x(0) = 3X(p) - Y(p) \\ pY(p) - y(0) = -9X(p) + 3Y(p) \end{cases}$$

$$\begin{cases} X(p)(p-3) + Y(p) = 1 \quad \Big|\cdot (p-3) \\ 9X(p) + Y(p)(p-3) = 0 \end{cases}$$

$+$

$$X(p)\left[(p-3)^2 + 9\right] = -(p-3)$$

$$X(p) = \frac{-(p-3)}{(3+(p-3))(3-(p-3))}$$

$$X(p) = \frac{-(p-3)}{p(6-p)} = \frac{p-3}{p(p-6)} = \frac{A}{p} - \frac{B}{p-6}$$

$$p-3 = A(p-6) - pB$$

$$p = 0 \Rightarrow -3 = -6A \Rightarrow A = \frac{1}{2}$$

$$p = 6 \Rightarrow 6B \rightarrow \Rightarrow \frac{1}{2} = B$$

$$X(p) = \frac{1}{2p} + \frac{1}{2(p-6)} \quad \Big| \; \mathcal{L}^{-1}$$

$$x(t) = \frac{1}{2} + \frac{1}{2} \cdot e^{6t}$$

$$Y(p)\left(-9+(p-3)^2\right) = -9$$

$$Y(p) = \frac{-9}{(p-3-3)(p-3+3)} = \frac{p-(p-6)}{(p-6)p} \cdot -\frac{9}{6}$$

$$= -\frac{3}{2}\left(\frac{1}{p-6} - \frac{1}{p}\right) \Big| \; \mathcal{L}^{-1}$$

$$y(t) = -\frac{3}{2}\left(e^{6t} - 1\right)$$

Integral equation

The convultion product

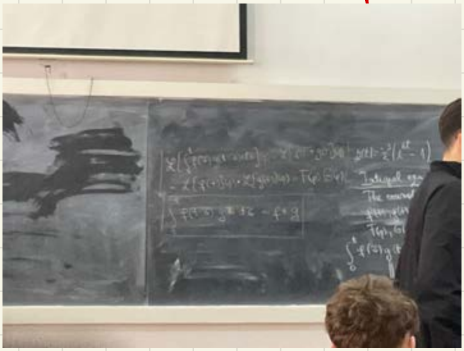
$f(t)$, $g(t)$ originals

$F(p)$, $G(p)$ images

$\int_0^t f(\tau)\, g(t-\tau)\, d\tau$ is the convolution product

$\Downarrow$

not $f * g$



$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)\,d\tau\right](p) = \mathcal{L}[f(t) * g(t)]$$

$$= \mathcal{L}[f(t)](p) \cdot \mathcal{L}[g(t)](p) = F(p)\,G(p)$$

① $\displaystyle\int_0^t \tau^3 \cos(t-\tau)\,d\tau = t^2 * \cos t$

② $\displaystyle\int_0^t e^{2(t-\tau)} \cdot \tau^3\,d\tau = e^{2t} * t^3$

③ $\displaystyle\int_0^t \tau^3\,d\tau = t^3 * 1$

④ $\displaystyle y'(t) + \int_0^t u \cdot y(t-u)\,du = t \;\Big/\; \mathcal{L}$, $y(0) = -1$

$$\mathcal{L}[y'(t)](p) + \mathcal{L}\left[\int_0^t u \cdot y(t-u)\,du\right](p) = \mathcal{L}[t](p)$$

$$p\,Y(p) - \underbrace{y(0)}_{-1} + \mathcal{L}[t * y(t)](p) = \frac{1}{p^2}$$

$\mathcal{L}[1](p) = \frac{1}{p}$

$\mathcal{L}[t](p) = \frac{1}{p^2}$

$$p\,Y(p) + 1 + \mathcal{L}[t](p) \cdot \mathcal{L}[y(t)](p) = \frac{1}{p^2}$$

$$p\,Y(p) + 1 + \frac{1}{p^2} \cdot Y(p) = \frac{1}{p^2}$$

$$Y(p)\left(p + \frac{1}{p^2}\right) = \frac{1}{p^2} + 1$$

$$Y(p) \cdot \left(\frac{p^3+1}{p^2}\right) = \frac{1-p^2}{p^2}$$

$$Y(p) = \frac{1-p^2}{\cancel{p^2}} \cdot \frac{\cancel{p^2}}{p^2+1}$$

$$Y(p) = \frac{(1-p)(1+p)}{(p+1)(p^2-p+1)} \Rightarrow Y(p) = \frac{1-p}{p^2-p+1} \Big| \mathcal{L}^{-1}$$

$$\mathcal{L}^{-1}[Y(p)] = \mathcal{L}^{-1}\left[\frac{1-p}{p^2-p+1}\right]$$

$$\mathcal{L}[\cos at] = \frac{p}{p^2+a^2}$$

$$\mathcal{L}[\sin at] = \frac{a}{p^2+a^2}$$

$$Y(p) = \frac{1-p}{p^2-2p\cdot\frac{1}{2}+\frac{1}{4}+1-\frac{1}{4}} = \frac{1-p}{(p-\frac{1}{2})^2+\frac{3}{4}} = \frac{1-(p-\frac{1}{2})-\frac{1}{2}}{(p-\frac{1}{2})^2+\frac{3}{4}}$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{\frac{1}{2}\cdot\frac{\sqrt{3}}{2}}{(p-\frac{1}{2})^2+\frac{3}{4}} - \frac{p-\frac{1}{2}}{(p-\frac{1}{2})^2+\frac{3}{4}}$$

$$= \frac{1}{\sqrt{3}} \cdot \frac{\frac{\sqrt{3}}{2}}{p^2+(\frac{\sqrt{3}}{2})^2}\Bigg|_{p=p-\frac{1}{2}} - \frac{p}{p^2+(\frac{\sqrt{3}}{2})^2}\Bigg|_{p=p-\frac{1}{2}} \Bigg| \mathcal{L}^{-1}$$

$$\Rightarrow y(t) = \frac{1}{\sqrt{3}} \cdot e^{\frac{1}{2}t} \sin\frac{\sqrt{3}}{2}t - e^{\frac{1}{2}t}\cdot\cos\frac{\sqrt{3}}{2}t$$

⑤ $$\int_0^t \sin(t-\tau) * (\tau)\, d\tau = \sin^2 t \Big| \mathcal{L},\ t \geq 0$$

$$\mathcal{L}[(\sin(t) * x(t)](p) = \mathcal{L}\left[\frac{1-\cos 2t}{2}\right](p)$$

$$\mathcal{L}[\sin(t)](p) \cdot \mathcal{L}[x(t)](p) = \mathcal{L}[\tfrac{1}{2}](p) - \tfrac{1}{2}\mathcal{L}[\cos 2t](p)$$

$$\frac{1}{p^2+1} X(p) = \frac{1}{2}\cdot\frac{1}{p} - \frac{1}{2}\frac{p}{p^2+4}$$

$$X(p) = (p^2+1)\frac{1}{2}\left(\frac{1}{p} - \frac{p}{p^2+4}\right)$$

$$X(p) = \frac{p^2+1}{2}\cdot\frac{p^2+4-p^2}{p(p^2+4)} = \frac{2(p^2+1)}{p(p^2+4)} = \frac{2(p^2+1)}{p(p^2+4)} =$$

$$= \frac{A}{p} + \frac{Bp+C}{p^2+4}$$

$$2p^2+2 = A(p^2+4)+(Bp+C)p$$

$$2p^2+2 = p^2(A+B)+Cp \quad +4A$$

$$\begin{cases} A+B = 2 \\ C = 0 \\ 4A = 2 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = \frac{3}{2} \\ C = 0 \end{cases}$$

$$X(p) = \frac{1}{2p} = \frac{1}{2p} + \frac{3p}{2(p^2+4)} \quad \Big| \mathcal{L}^{-1}$$

$$x(t) = \frac{1}{2} + \frac{3}{2}\cos 2t$$

$$⑥ \quad y(t) - 2\int_0^t y(t-u)\cdot \sin u \, du = \cos t$$

We denote by $\mathcal{L}[y(t)](p) = Y(p)$

$$\mathcal{L}[y(t)](p) - 2\mathcal{L}[y(t) * \sin(t)](p) = \mathcal{L}[\cos t](p)$$

$$Y(p) - 2Y(p)\cdot\frac{1}{p^2+1} = \frac{p}{p^2+1}$$

$$Y(p)\left(1-2\frac{1}{p^2+1}\right) = \frac{p}{p^2+1}$$

$$Y(p) = \frac{p}{p^2+1}\cdot\frac{p^2+1}{p^2-1} = \frac{p}{(p^2-1)} \quad \Big| \mathcal{L}^{-1}$$

$$y(t) = \cosh t$$

SEMINAR 12

① $t x''(t) - 2t x'(t) - 2x(t) = 0 \ / \ \mathcal{L}$

$x(0) = 0, \ x'(0) = 1$

$\mathcal{L}[t \cdot x''(t)](p) - \mathcal{L}[t \cdot x'(t)](p) + \mathcal{L}[x(t)](p) = 0$

$\mathcal{L}[x(t)](p) = X(p)$

$\mathcal{L}[t \cdot x''(t)](p) = (-1)^1 \cdot \left(\mathcal{L}[x''(t)](p)\right)' = -\left(p^2 X(p) - p x(0) - x'(0)\right)'$

$= -\left(p^2 X(p) - 1\right)' = -\left(2p X(p) + p^2 X'(p)\right)$

• $\mathcal{L}[t x'(t)](p) = (-1)^1 \left(\mathcal{L}[x'(t)](p)\right)' = -\left(p X(p) - p x(0)\right)' =$

$= -\left(p X(p)\right)' = -\left(X(p) + p X'(p)\right)$

$= -2p X(p) - p^2 X'(p) + 2 X(p) + 2p X(p) = 0 \Rightarrow$

$X'(p)(-p^2 + 2p) = 2p X(p)$

$\frac{X'(p)}{X(p)} = \frac{-2p}{p^2 - 2p}$

$\frac{X'(p)}{X(p)} = \frac{-2}{p-2} \ \Big| \int$

$\Rightarrow \ln|X(p)| = -2 \ln|p-2| + \ln C$

$\ln|X(p)| = \ln \frac{C}{(p-2)^2}$

$$\Rightarrow X(p) = \frac{C}{(p-2)^2} \Big/ \mathcal{L}^{-1}$$

$$\Rightarrow x(t) = C \cdot \mathcal{L}^{-1}\left[\frac{1}{(p-2)^2}\right]$$

$$x(t) = C\mathcal{L}^{-1}\left[\frac{1}{p^2}\Big|_{p=p-2}\right]$$

$$x(t) = C t \cdot e^{2t} \Rightarrow x(t) = te^{2t}$$

$$x'(t) = C \cdot 1 \cdot e^{2t} + 2Ct \cdot e^{2t}$$

$$x'(0) = 1 \Rightarrow \boxed{C = 1}$$

② $$x(t) = 2\sin 4t + \int_0^t \sin 4(t-u) \cdot x(u)\, du \Big/ \mathcal{L}$$

we denote $\mathcal{L}[x(t)](p) = X(p)$

$$\mathcal{L}[x(t)](p) = \mathcal{L}[2\sin 4t](p) + \mathcal{L}\left[\int_0^t \sin(4-u) \cdot x(u) \cdot du\right](p)$$

$$\Rightarrow x(p) = 2 \cdot \frac{4}{p^2+16} + \mathcal{L}[\sin 4t * x(t)](p)$$

$$\Rightarrow x(p) = \frac{8}{p^2+16} + \mathcal{L}[\sin 4t]\mathcal{L}[x(t)](p)$$

$$\Rightarrow x'(p) = \frac{8}{p^2+16} + \frac{4}{p^2+16} \cdot X(p)$$

$$\Rightarrow X(p) - X(p)\frac{4}{p^2+16} = \frac{8}{p^2+16} \Rightarrow X(p)\left[1 - \frac{4}{p^2+16}\right] = \frac{8}{p^2+16}$$

$$X(p) \cdot \frac{p^2+12}{p^2+16} = \frac{8}{p^2+16} \Rightarrow X(p) = \frac{8}{\cancel{p^2+16}} \cdot \frac{\cancel{p^2+16}}{p^2+12} = \frac{8}{p^2+12} \Big/ \mathcal{L}^{-1}$$

$$x(t) = \mathcal{L}^{-1}\left[\frac{8}{p^2+12}\right] = \mathcal{L}^{-1}\left[\frac{2\sqrt{3}}{p^2+(2\sqrt{3})^2}\right] \Rightarrow$$

$$x(t) = \frac{4}{\sqrt{3}} \sin 2\sqrt{3}(t)$$

③ $x''(t) + x(t) = \dfrac{1}{\cos t} \Big/ \mathcal{L}, \quad x(0) = 0,\ x'(0) = 2$ $\quad \mathcal{L}[x(t)](p) = X(p)$

$$\mathcal{L}[x''(t)](p) + \mathcal{L}[x(t)](p) = \mathcal{L}\left[\frac{1}{\cos t}\right]$$

$$p^2 X(p) - p\underbrace{x(0)}_{=0} - \underbrace{x'(0)}_{=2} + X(p) = \mathcal{L}\left[\frac{1}{\cos t}\right]$$

$$X(p)(p^2+1) = 2 + \mathcal{L}\left[\frac{1}{\cos t}\right] \Big| : (p^2+1)$$

$$X(p) = \frac{2}{p^2+1} + \frac{1}{p^2+1}\mathcal{L}\left[\frac{1}{\cos t}\right]$$

$$X(p) = \frac{2}{p^2+1} + \mathcal{L}[\sin t](p) \cdot \mathcal{L}\left[\frac{1}{\cos t}\right](p)$$

$$X(p) = 2 \cdot \mathcal{L}[\sin t](p) + \mathcal{L}\left[\sin t * \frac{1}{\cos t}\right](p)$$

$$\Rightarrow X(p) = 2\mathcal{L}[\sin t](p) + \mathcal{L}\left[\int_0^t \sin(t-u) \cdot \frac{1}{\cos u}\, du\right](p) \Big/ \mathcal{L}^{-1}$$

$$\Rightarrow x(t) = 2\sin t + \int_0^t \frac{\sin t \cdot \cos u - \cos t \sin u}{\cos u}\, du$$

$$\Rightarrow x(t) = 2\sin t + \int_0^t (\sin t - \cos t \cdot \tan u)\, du$$

$$\Rightarrow x(t) = 2\sin t + \sin t \cdot u \Big|_0^t + \cos t \cdot \ln(\cos u)\Big|_0^t$$

$\Rightarrow x(t) = 2\sin t + t\sin t + \cos t \cdot \ln|\cos t|$

④ $\sin t = \frac{t^3}{e^t} + \int_0^t x''(t)\cdot(t-u)^2\,du \;\Big/\; \mathcal{L},$ $\quad x(0)=0$, $x'(0)=2$

$$\mathcal{L}[\sin t](p) = \mathcal{L}\left[\frac{t^3}{e^t}\right](p) + \mathcal{L}\left[\int_0^t x''(t)\cdot(t-u)^2\,du\right](p)$$

$$\mathcal{L}[t^3e^{-t}] = \mathcal{L}[t^3](p+1)$$

$$= \frac{3!}{p^4}\Big|_{p \to p+1}$$

$$= \frac{6}{(p+1)^4}$$

$$\Rightarrow \frac{1}{p^2+1} = \frac{6}{(p+1)^4} + \mathcal{L}[x''(t) * t^2](p)$$

$$\Rightarrow \frac{1}{p^2+1} = \frac{6}{(p+1)^4} + \mathcal{L}[x''(t)]\cdot\mathcal{L}[t^2](p)$$

$$\Leftrightarrow \frac{1}{p^2+1} = \frac{6}{(p+1)^4} + \Big(\big(p^2X(p) - p\cdot \underset{0}{\underset{\parallel}{x(0)}} - \underset{2}{\underset{\parallel}{x'(0)}}\Big)\left(\frac{2}{p^3}\right)$$

$$\Leftrightarrow \frac{1}{p^2+1} = \frac{6}{(p+1)^4} + (p^2X(p) - 2)\cdot\frac{2}{p^3}$$

$$\Leftrightarrow \frac{1}{p^2+1} = \frac{6}{(p+1)^4} + \frac{2}{p}X(p) - \frac{4}{p^3}$$

$$\Leftrightarrow \frac{2}{p}\cdot X(p) = \frac{1}{p^2+1} - \frac{6}{(p+1)^4} + \frac{4}{p^3} \;\Big|\cdot\frac{p}{2}$$

$$\Leftrightarrow X(p) = \frac{p}{2(p^2+1)} - \frac{3p}{(p+1)^4} + \frac{2}{p^2} \Big/ \mathcal{L}^{-1}$$

$$\Rightarrow x(t) = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{p}{p^2+1}\right] - 3\mathcal{L}^{-1}\left[\frac{p}{(p+1)^4}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{p^2}\right]$$

$$x(t) = \frac{1}{2}\cos t + 2t - 3\mathcal{L}^{-1}\left[\frac{p+1}{(p+1)^4} - \frac{1}{(p+1)^4}\right]$$

$$x(t) = \frac{1}{2}\cos t + 2t - 3\mathcal{L}^{-1}\left[\frac{1}{(p+1)^3}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{(p+1)^4}\right]$$

$$x(t) = \frac{1}{2}\cos t + 2t - \frac{3\mathcal{L}^{-1}}{2!}\left[\frac{2!}{p^3}\right]\Bigg|_{p=p+1} + \frac{3}{3!}\mathcal{L}^{-1}\left[\frac{3!}{p^4}\right]\Bigg|_{p=p+1}$$

$$x(t) = \frac{1}{2}\cos t + 2t - \frac{3}{2}\cdot t^2 e^{-t} + \frac{1}{2}\cdot t^3\cdot e^{-t}$$

⑤ $x''(t) + 2x'(t) + x(t) = \left(e^t\cdot(t+1)\right)^{-1} \Big/ \mathcal{L}$, $x'(0) = x(0) = 0$

$\mathcal{L}[x(t)](p) = X(p)$

$$\mathcal{L}[x''(t)]^{(p)} + 2\mathcal{L}[x'(t)]^{(p)} + \mathcal{L}[x(t)]^{(p)} = \mathcal{L}\left[\left(e^t\cdot(t+1)\right)^{-1}\right](p)$$

$$p^2X(p) - px(0) - x'(0) + 2pX(p) - 2px(0) + X(p) = \mathcal{L}\left[\left(e^t\cdot(t+1)\right)^{-1}\right]$$

$$X(p)(p^2+2p+1) = \mathcal{L}\left[\left(e^t\cdot(t+1)\right)^{-1}\right](p) \Big| : (p+1)^2$$

$$X(p) = \frac{1}{(p+1)^2}\cdot\mathcal{L}\left[\left(e^t\cdot(t+1)\right)^{-1}\right](p)$$

$$X(p) = \mathcal{L}[t\cdot e^{-t}](p)\cdot\mathcal{L}\left[\left(e^t\cdot(t+1)\right)^{-1}\right](p)$$

$$\mathcal{L}[t\cdot e^{-t}] = \mathcal{L}[t](p+1)$$

$$= \frac{1}{p^2}\Big|_{p=p+1} = \frac{1}{(p+1)^2}$$

$$X(p) = \mathcal{L}\left[(t \cdot e^{-t}) * \left((e^{t} \cdot (t+1)\right)^{-1}\right](p) \quad \Big| \; \mathcal{L}^{-1}$$

$$x(t) = \int_0^t (t-u) e^{-t+u} \cdot e^{-u} \cdot (u+1)^{-1} \, du$$

$$x(t) = e^{-t} \int_0^t \frac{t-u}{u+1} \, du$$

$$x(t) = e^{-t} \left[t \ln|u+1| \Big|_0^t - \int_0^t \frac{u+1-1}{u+1} \, du \right]$$

$$x(t) = e^{-t} \left[t \ln|t+1| - u \Big|_0^t + \ln|u+1| \Big|_0^t \right]$$

$$x(t) = e^{-t} \left(t \cdot \ln|t+1| - t + \ln|t+1| \right)$$

⑥* $x(t) = 2t + 2\int_0^t \cos u \cdot x(t-u) \, du \quad \Big| \; \mathcal{L}$

$$\mathcal{L}[x(t)](p) = X(p)$$

$$X(p) = 2 \frac{1}{p^2} + 2 \mathcal{L}[\cos t](p) \cdot \mathcal{L}[x(t)](p)$$

$$X(p) = \frac{2}{p^2} + \frac{2p}{p^2+1} \cdot X(p)$$

$$X(p) - 2\frac{p}{p^2+1} \, X(p) = \frac{2}{p^2}$$

$$X(p)\left(1 - 2\frac{p}{p^2+1}\right) = \frac{2}{p^2}$$

$$X(p) = \frac{2}{p^2} \cdot \frac{1}{1-2\frac{p}{p^2+1}}$$

$$X(p) = \frac{2}{p^2} \cdot \frac{1}{1-2\frac{p}{p^2+1}}$$

$$X(p) = \frac{2}{p^2} \frac{p^2+1}{p^2+1-2p}$$

$$X(p) = \frac{2}{p^2} \frac{p^2+1}{(p-1)^2}$$

$$X(p) = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p-1} + \frac{D}{(p-1)^2} \Big| \mathcal{L}^{-1}$$

$$x(t) = A\mathcal{L}^{-1}\left[\frac{1}{p}\right] + B\mathcal{L}^{-1}\left[\frac{1}{p^2}\right] + C\mathcal{L}^{-1}\left[\frac{1}{p-1}\right] + D\mathcal{L}^{-1}\left[\frac{1}{(p-1)^2}\right]$$

$$x(t) = A + Bt + Ce^{t} + Dte^{t}$$

$$\frac{2p^2+2}{p^2(p-1)^2} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p-1} + \frac{D}{(p-1)^2}$$

$$2p^2+2 = p(p-1)^2A + (p-1)^2B + p(p-1)C + p^2D$$

$$p = 0 = B = 2$$

$$p = 1 \Rightarrow D = 4$$

$$p = 2 \Rightarrow 10 = 2A + 2 + 2C + 16$$

$$p = -1 \Leftrightarrow 4 = -4A + 8 - 4C + 4$$

(7) $x(t) = te^{t} + \int_0^t \tau \cdot x(t-\tau)\,d\tau \Big/ \mathcal{L}$

$$\mathcal{L}[x(t)](p) = X(p)$$

$$\mathcal{L}[x(t)](p) = \mathcal{L}[te^{t}](p) + \mathcal{L}\left[\int_0^t \tau \cdot x(t-\tau)\,d\tau\right](p)$$

După pași $\Rightarrow$ $X(p) = \dfrac{p^2}{(p-1)^3 \cdot (p+1)} = \dfrac{A}{p+1} + \dfrac{B}{p-1} + \dfrac{C}{(p-1)^2} + \dfrac{1}{2!}\dfrac{D \cdot 2!}{(p-1)^3} \Big/ \mathcal{L}^{-1}$

SESIUNE MS

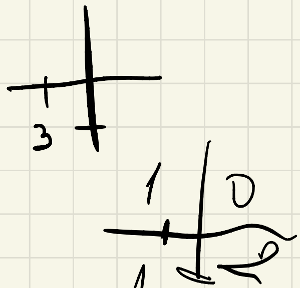
Seminar 1

1.4.

a) $-1-i$

$r=\sqrt{2}$, $\varphi=\arctan 1 = 1$

$z=\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{\pi}{4}\right)$

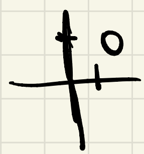


$\varphi=\arctan 1+\pi=\frac{\pi}{4}+\pi=\frac{5\pi}{4}$

b) $\frac{\sqrt{3}}{2}+i\frac{1}{2}$

$r=\sqrt{\frac{3}{4}+\frac{1}{4}}=1$ $\quad\varphi=\arctan\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\arctan\frac{\sqrt{3}}{3}=\frac{\pi}{6}$

$z=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+i\frac{1}{2}$



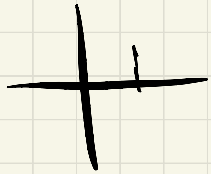
c) i

$r=1$, $\varphi=\arctan\frac{1}{0}$

$z=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

d) 1

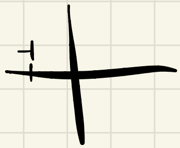
$r = 1$, $\varphi = \arctan \frac{0}{1} =$



$z = \cos 0 + i \sin 0$

e) -1

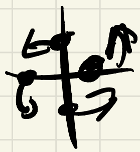
$r = 1$,



$z = \cos \pi + i \sin \pi$

f) $-i$

$r = 1$



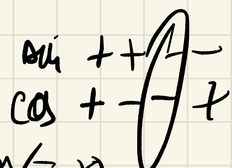
$z = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$

g) $\cos \frac{\pi}{8} - i \sin \frac{\pi}{8}$

$\cos\left(-\frac{\pi}{8}\right) + i \sin\left(-\frac{\pi}{8}\right)$

$\cos\left(-\frac{\pi}{8} + 2\pi\right) + i \sin\left(-\frac{\pi}{8} + 2\pi\right)$

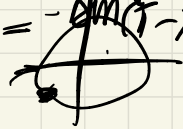
$\cos \frac{15\pi}{8} + i \sin \frac{15\pi}{8}$



h) $-\sin \frac{\pi}{8} - i \cos \frac{\pi}{8}$

$\cos x = -\cos(\pi - x)$

$\sin x = -\sin(\pi - x)$



$\sin \alpha = \cos\left(\frac{\pi}{2} - \alpha\right)$

$\cos \alpha = \sin\left(\frac{\pi}{2} - \alpha\right)$

$-\sin \frac{\pi}{8} = \sin\left(-\frac{\pi}{8}\right) = \cos\left(-\frac{\pi}{8} + \frac{\pi}{2}\right) = \cos \frac{3\pi}{4}$

$-\cos\frac{\pi}{8} = \cos\frac{\pi}{8} = \sin\left(\frac{\pi}{8}+\pi\right) = \sin\frac{9\pi}{8}$

180 : 8 = 2..

16

20

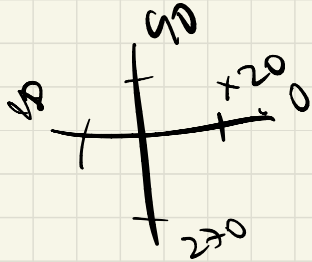
$\Rightarrow \cos\frac{3\pi}{4} + i\sin\frac{9\pi}{8}$

$-\sin\frac{\pi}{8} - i\cos\frac{\pi}{8} = \sin\left(\frac{\pi}{8}+\pi\right) + i\cos\left(\frac{\pi}{8}+\pi\right)$

$\cos x + i\sin x$

$\sin\ \frac{9\pi}{8} - \frac{\pi}{2}$

$-\sin\frac{\pi}{8} - i\cos\frac{\pi}{8} =$



$-\sin$

	I			
sin	+	+	−	−
cos	+	−	−	+

CI: $-\sin\frac{\pi}{8} \longrightarrow$ CIII: $-(-\sin(\pi+\frac{\pi}{8})) = \sin\frac{9\pi}{8}$

CI: $-\cos\frac{\pi}{8} \longrightarrow$ CIII: $-(-\cos(\frac{\pi}{8}+\pi)) = \cos\frac{9\pi}{8}$

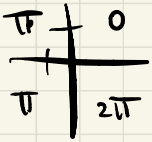
$\sin\frac{9\pi}{8} + i\cos\frac{9\pi}{8} = \cos\left(\frac{\pi}{2}-\frac{9\pi}{8}\right) + i\sin\left(\frac{\pi}{2}-\frac{9\pi}{8}\right)$

$= \cos\left(-\frac{5\pi}{8}\right) + i\sin\left(-\frac{5\pi}{8}\right)$

$= \cos\left(-\frac{5\pi}{8}+2\pi\right) + i\sin\left(-\frac{5\pi}{8}+2\pi\right)$

$= \cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}$

i) $(-1+i)^{105}$



$r=\sqrt{2}$

$\varphi = \arctan\frac{1}{-1} = \arctan(-1)+\pi = -\frac{\pi}{4}+\frac{4}{\pi} = \frac{3\pi}{4}$

$z^{105} = \sqrt{2}^{105}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)^{105}$

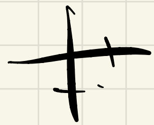
$z = \sqrt{2}^{105}\left(\cos\frac{315\pi}{4}+i\sin\frac{315\pi}{4}\right)$

$= \sqrt{2}^{105}\left(\cos\left(78\pi+\frac{3\pi}{4}\right)+i\sin\left(78\pi+\frac{3\pi}{4}\right)\right)$

$= 2^{\frac{105}{2}}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

1.5

a) $z^4 = 1-i$



$r=\sqrt{2}$, $\varphi=\arctan(-1) = -\frac{\pi}{4}+2\pi = \frac{7\pi}{4}$

$z = \sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$

$z_k = \sqrt[4]{\sqrt{2}}\left(\cos\frac{\frac{7\pi}{4}+2\pi k}{4}+i\sin\frac{\frac{7\pi}{4}+2\pi k}{4}\right)$, $k=\overline{0,3}$

$z_0 = 2^{\frac{1}{8}}\left(\cos\frac{7\pi}{16}+i\sin\frac{7\pi}{16}\right)$

$z_1 = 2^{\frac{1}{8}}\left(\cos\frac{15\pi}{16}+i\sin\frac{15\pi}{16}\right)$

$z_2 = 2^{\frac{1}{8}}\left(\cos\frac{23\pi}{16}+i\sin\frac{23\pi}{16}\right)$

$z_3 = 2^{\frac{1}{8}}\left(\cos\frac{31\pi}{16}+i\sin\frac{31\pi}{16}\right)$

b) $z^6 = 1$

$r = \sqrt{1} = 1$,

$z_k = \sqrt[6]{1}\left(\cos\frac{0+2k\pi}{6} + i\sin\frac{0+2k\pi}{6}\right)$, $k = \overline{0,5}$

$z_0 = \sqrt[6]{1}(\cos 0 + i\sin 0)$

$z_1 = \sqrt[6]{1}\left(\cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6}\right)$

$z_2 = \sqrt[6]{1}\left(\cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6}\right)$

c) $z^4 + (1-i)z^2 - i = 0$

not $z^2 = a$

$a^2 + (1-i)a - i = 0$

$\Delta = b^2 - 4ac$

$= (1-i)^2 - 4(-i)$

$= \cancel{1} - 2i - \cancel{1} + 4i$

$= 2i$

$a_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$ $\begin{cases} \frac{-1-i+\sqrt{2}i}{2} \\ \frac{-1-i-\sqrt{6}i}{2} \end{cases}$

$a_1 = \frac{-1}{2} - \frac{i - \sqrt{6}i}{2}$

$(a + (-b))^2$ with $a = 1$, $b = -i$

$= a^2 + b^2 + 2ab$

$= \cancel{1} - \cancel{1} + 2 \cdot 1 \cdot (-i)$

$= -2i$

$(a - b)^2$

$= a^2 - 2ab + b^2$

$1^2 - (2 \cdot 1 \cdot (-i)) - 1$

$1 + 2i - 1$

$(1-i)^2 =$

GREȘIT

c) $z^4+(1-i)z^2-i=0$

$a^2+(1-i)a-i=0$

$a^2+a-ai-i=0$

$a(a+1)-i(a+1)=0$

$(a+1)(a-i)=0$

$a_1=-1$, $a_2=i$

$a_1=\cos\pi+i\sin\pi$

$a_2=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

d) $z^4-z^3+z^2-z+1=0 \quad \Big|\cdot(z+1)$

$\Rightarrow z^5+1=0 \Rightarrow z^5=-1$

$z_k=\cos\frac{\pi+2k\pi}{5}+i\sin\frac{\pi+2\pi k}{5}$, $k=\overline{0,4}$

1.6

a) $\operatorname{Im}z^2=1$

$z=x+iy$

$z^2=x^2+2xyi+y^2$

$\operatorname{Im}z^2=2xy$

$2xy=1$

$xy=\frac{1}{2}$

b) $z^2+\overline{z}^2=2$

$a^2+\cancel{2abi}-b^2=(a+bi)^2$

$(a-bi)^2=a^2-\cancel{2abi}-b^2$ $\Rightarrow 2(a^2-b^2)=2$

$a^2-b^2=1$

c) $|z+c|+|z-c|=2a$, $a>c>0$

$|a+bi+c|+|a+bi-c|=2a$

$\sqrt{(a+c)^2+b^2}+\sqrt{(a-c)^2+b^2}=2a$

– elipsă

d) $\mathrm{Re}\, z=|z|$

$z=x+iy$

$|z|=\sqrt{x^2+y^2}$

$x=\sqrt{x^2+y^2}\quad ()^2$

$\cancel{x^2}=\cancel{x^2}+y^2$

$y^2=0 \xRightarrow{\Rightarrow y=0} Ox$

e) $\mathrm{Im}(z+i)=|z|$

$z+i=a+bi+i=a+i(b+1)$

$$|z| = \sqrt{a^2+b^2}$$

$$b+1 = \sqrt{a^2+b^2} \quad ()^2$$

$$\cancel{b^2} + 2b + 1 = a^2 \cancel{+b^2}$$

$$y = \frac{x^2}{2} - \frac{1}{2}$$

$$2b+1 = a^2$$

$$2y + 1 = x^2$$

$$2y = x^2 - 1$$

$$y = \frac{x^2-1}{2}$$

$f(x) = \ldots$

f) $\operatorname{Re}(\bar{z})^{-1} = 1$

$$(a-bi)^{-1} = \frac{\cancel{1}^{\,a+bi}}{a-bi} \Rightarrow \operatorname{Re}(\bar{z})^{-1}$$

SEMINAR 2

$f(z) = 3x + 4yi + 5y - 3xi$

i) $f(z) = ze^{z}$

$f(z) = (x+iy)e^{(x+iy)} = (x+iy)e^{x}(\cos y + i\sin y)$

$f(z) = (x+iy)(e^{x}\cos y + ie^{x}\sin y)$

$= xe^{x}\cos y + ixe^{x}\sin y + iye^{x}\cos y - ye^{x}\sin y$

$u(x,y) = e^{x}(x\cos y - y\sin y)$

$v(x,y) = e^{x}(x\sin y + y\cos y)$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \Rightarrow \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \Rightarrow \end{aligned}\right\} \Rightarrow$$

PRESPUNEM CĂ ȘTIU DERIVA

XD

b) $f(z) = z^2 + 3iz$

$f(z) = a^2 + 2abi - b^2 + 3ia - 3b$

$u(a,b) = a^2 - b^2 - 3b$

$v(a,b) = 2ab + 3a$

$$\frac{\partial u}{\partial a} = \frac{\partial v}{\partial b} \Rightarrow 2a = 2a \text{ "T"}$$

$$\frac{\partial u}{\partial b} = \frac{-\partial v}{\partial a} \Rightarrow -2b - 3 = -(2b+3)$$

"T"

c) $f(z) = \bar{z} \operatorname{Im} z$

$f(z) = (a - bi)b = ab - b^2 i$

$u(a,b) = ab$

$v(a,b) = -b^2$

$$\frac{\partial u}{\partial a} = \frac{\partial v}{\partial b} \Rightarrow b \neq -2b$$

$$\frac{\partial u}{\partial b} = -\frac{\partial v}{\partial a} \Rightarrow a = 0$$

f monogenic in origin

$a = b = 0$

d) $f(z) = |z| \operatorname{Re} z$

$f(z) = a\sqrt{a^2+b^2}$

$u(a,b) = a\sqrt{a^2+b^2}$

$v(a,b) = 0$

$$\frac{\partial u}{\partial a} = \frac{\partial v}{\partial b} \Rightarrow \left(a\sqrt{a^2+b^2}\right)'_a$$

$$a' \cdot \sqrt{a^2+b^2} + \left(\sqrt{a^2+b^2}\right)' a =$$

$$= \sqrt{a^2+b^2} + \frac{1}{\cancel{2}\sqrt{a^2+b^2}} \cancel{2}a\, a = \frac{\sqrt{a^2+b^2}^{\,2}}{\sqrt{a^2+b^2}} + \frac{a^2}{\sqrt{a^2+b^2}}$$

$$= \frac{2a^2+b^2}{\sqrt{a^2+b^2}} = 0$$

$$\frac{\partial u}{\partial b} = -\frac{\partial v}{\partial a} \Rightarrow$$ good

1.13

i) $u(x,y) = x^2 - y^2 + 2xy$

$$f(z) = u(x,y) + iv(x,y)$$

If $\Delta u(x,y) = 0 \Rightarrow u$ is harmonic function

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u(x,y) = x^2 - y^2 + 2xy$

$$\Delta u(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = 0$$

$$\frac{\partial u}{\partial x} = 2x + 2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y + 2x$$

$$\frac{\partial^2 u}{\partial y^2} = -2$$

$$\Delta u(x,y) = 2-2=0$$ 4T [illegible]

b) $v(x,y) = x^2 + y^2$

$$\Delta v(x,y) = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial v}{\partial x} = 2x, \quad \frac{\partial^2 v}{\partial x^2} = 2$$

$$\frac{\partial v}{\partial y} = 2y, \quad \frac{\partial^2 v}{\partial y^2} = 2$$

$\Delta v(x,y)=2+2=4 \neq 0$ No

c) ... Mai sus in seminar 2

1.14 $f=u+iv$

a) $u(x,y)=3xy$

REFUZ

ESTEMAISUS

SEMINAR 3

$e^{z}=e^{x+iy}=e^{x}(\cos y+i\sin y)$

$e^{\bar{z}}=e^{x-iy}=e^{x}(\cos y-i\sin y)$

$e^{iy}=\cos y+i\sin y$

$e^{-iy}=\cos y-i\sin y$

Trigonometric Functions

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\cosh z = \frac{e^{z}+e^{-z}}{2}$$

$$\sinh z = \frac{e^{z}-e^{-z}}{2}$$

$z = e^{w}$, $w, z \in \mathbb{C}$, $z \neq 0$

$w = \mathcal{L}og\, z = \{ \ln|z| + i(\arg z + 2k\pi), k \in \mathbb{Z} \}$

$z^{\alpha} = e^{\alpha \mathcal{L}og z}$

$z, \alpha \in \mathbb{C}$, $z \neq 0$

1.24

a) $w = e^{-z} = e^{-(a+bi)} = e^{-a-bi} = e^{-a}(\cos b - i\sin b)$

$= e^{-a}\cos b - ie^{-a}\sin b$

$u(a,b) = e^{-a}\cos b$

$v(a,b) = e^{-a}\sin b$

b) $w = \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$$= \frac{(\cos z + i\sin z) - (\cos z - i\sin z)}{2i}$$

$$= \frac{\cos z + i\sin z - \cos z + i\sin z}{2i}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+yi)} - e^{-i(x+yi)}}{2i}$$

$$= \frac{e^{ix-y} - e^{-ix+y}}{2i}$$

$$= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i}$$

$$= \frac{\cos x(e^{-y} - e^{y}) + i\sin x(e^{-y} + e^{y})}{2i}$$

$$= \frac{\cos x \cdot (-1)(e^{y} - e^{-y}) + i\sin x(e^{y} + e^{-y})}{2i}$$

$$= \frac{-\cos x \sinh y + i\sin x \cosh y}{i}$$

$$= i\cos x \sinh y + \sin x \cosh y$$

$\frac{-i}{i} = 1$

c) $w = \cosh(z-i)$

$$w = \cosh(a+bi-i) = \cosh(a+i(b-1))$$

$$w = \frac{e^{(a+i(b-1))} + e^{-(a+i(b-1))}}{2}$$

$$= \frac{e^{x}(\cos(y-1)+i\sin(y-1)) + e^{-x}(\cos(y-1)-i\sin(y-1))}{2}$$

$$= \frac{\cos(y-1)(e^{x}+e^{-x}) + i\sin(y-1)(e^{x}-e^{-x})}{2}$$

$$= \cos(y-1)\cosh x + i\sin(y-1)\sinh x$$

d) $w = \sinh z$

$$\sinh z = \frac{e^{z}-e^{-z}}{2} = \frac{e^{x+yi}-e^{-x-yi}}{2}$$

$$= \frac{e^{x}(\cos y + i\sin y) - e^{-x}(\cos y - i\sin y)}{2}$$

$$= \frac{\cos y\,(e^{x} - e^{-x}) + i \sin y\,(e^{x} + e^{-x})}{2}$$

$$\cos y \sinh x + i \sin y \cosh x$$

e) $$w = \frac{z + \bar{z}}{z} = \frac{x + \cancel{iy} + x - \cancel{iy}}{x + iy} = \frac{\cancel{x-iy}\quad 2x}{x + iy}$$

$$= \frac{2x(x - iy)}{x^2 + y^2} = \frac{x^2 - xyi}{x^2 + y^2}$$

$$u(x,y) = \frac{2x^2}{x^2 + y^2}$$

$$v(x,y) = \frac{2xy}{x^2 + y^2}$$

1.25

i) $\sin(1 - i)$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(1-i)} - e^{-i(1-i)}}{2i}$$

$$= \frac{e^{i+1} - e^{-i-1}}{2i} = \frac{e(\cos 1 + i\sin 1) - \frac{1}{e}(\cos 1 - i\sin 1)}{2i}$$

$$= \frac{\cos 1\,(e - e^{-1}) + i\sin 1\,(e + e^{-1})}{2i}$$

$$= -i\cos 1 \sinh e + \sin 1 \cosh 1$$

b) $\cos(\pi + i)$

$$= \frac{e^{i(\pi+i)} + e^{-i(\pi+i)}}{2} = \frac{e^{\pi i - 1} + e^{-\pi i + 1}}{2}$$

$$= \frac{e^{-1}(\cos\pi + i\sin\pi) + e(\cos\pi - i\sin\pi)}{2}$$

$$= \frac{\cos\pi\,(e + e^{-1}) - i\sin\pi\,(e - e^{-1})}{2}$$

$$= \cos\pi \cosh 1 - i\sin\pi \sinh 1$$

$= -\cosh 1$

c) $\cosh(2-i) = \dfrac{e^{2-i} + e^{-(2-i)}}{2}$

$= \dfrac{e^{2}(\cos 1 - i\sin 1) + e^{-2}(\cos 1 + i\sin 1)}{2}$

$= \dfrac{\cos 1(e^{2} + e^{-2}) - i\sin 1(e^{2} - e^{-2})}{2}$

$= \cos 1 \cosh 2 - i \sin 1 \sinh 2$

d) $\sinh i = \dfrac{e^{i} - e^{-i}}{2} = \dfrac{\cancel{\cos 1} + i\sin 1 - \cancel{\cos 1} + i\sin 1}{2}$

$= i \sin 1$

e) $\tan(i\pi) = \dfrac{\sin(i\pi)}{\cos(i\pi)} = \dfrac{\dfrac{e^{i(\pi i)} - e^{-i(i\pi)}}{\cancel{2}i}}{\dfrac{e^{i(i\pi)} + e^{-i(i\pi)}}{\cancel{2}}}$

$$= \frac{e^{-\pi} - e^{\pi}}{i(e^{-\pi} + e^{\pi})} = ??$$

f) $\operatorname{Log}(1+i\sqrt{3}) = \{\ln 2 + i(\frac{\pi}{3} + 2k\pi), k \in \mathbb{Z}\}$

$= \{\ln 2 + i\pi(\frac{1}{3} + 2k), k \in \mathbb{Z}\}$

g) $\operatorname{Log}(-1) = \{\ln 1 + i(\pi + 2k\pi), k \in \mathbb{Z}\}$

h) $\operatorname{Log} e = \{\ln e + i(0 + 2k\pi), k \in \mathbb{Z}\}$

$= 1 + i2k\pi, k \in \mathbb{Z}$

i) $\operatorname{Log}(-i) = \{\ln 1 + i(\frac{3\pi}{2} + 2k\pi), k \in \mathbb{Z}\}$

$= i\pi(\frac{3}{2} + 2k), k \in \mathbb{Z}$

$$\int_{(1,1)}^{(2,4)} z^2\,dz = \int_{(1,1)}^{(2,4)} (x^2-y^2+2xyi)(dx+idy) = \int_{(1,1)}^{(2,4)} (x^2-y^2)\,dx - 2xy\,dy$$

$$+\,i\int_{(1,1)}^{(2,4)} (x^2-y^2)\,dy + 2xyi\,dx$$

$dz = dx + idy$

$z^2 = (x+yi)^2 = x^2 - y^2 + 2xyi$

$$I = \int_{(1,1)}^{(2,4)} (x^2-y^2)\,dx - 2xy\,dy + i\int_{(1,1)}^{(2,4)} (x^2-y^2)\,dy + 2xy\,dx$$

$(1,1) \quad (2,4)$

$$\frac{x-1}{2-1} = \frac{y-1}{4-1}$$

$$\frac{x-1}{1} = \frac{y-1}{3} \iff 3(x-1) = y-1$$

$$y = 3x-2$$

$$dy = 3\,dx$$

$$I = \int_1^2 (x^2-(3x-2)^2\,dx - \int_1^2 2x(3x-2)3\,dx + i\int_1^2 (x^2-(3x-2)^2$$

$$\cdot 3\,dx + i\int_1^2 2x(3x-2)\,dx$$

$$= \int_1^2 \overset{-8x^2}{x^2 - (9x^2} + 4 - 12x) \overset{-26x^2}{-18x^2} + 12x + i\int_1^2 (\overset{-8x^2}{x^2-(9x^2}+4-12x))\,dx$$

$$+\,i\int_1^2 \underset{-2x^2}{6x^2-4x}\,dx =$$