II Oscillations

1. The harmonic oscillations

1.1. The elastic potential energy

$U = \frac{1}{2} k x^2$ – the elastic potential energy

Def $\vec{F} = -\nabla U$ – gradient

$U(x)$

$\vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(U(x))$

$\vec{i} = \left(\frac{\partial U(x)}{\partial x}\vec{i} + \frac{\partial U(x)}{\partial y}\vec{j} + \frac{\partial U(x)}{\partial z}\vec{k}\right) \Rightarrow \vec{F} = -\frac{\partial U(x)}{\partial x}\vec{i} \Rightarrow$

$F = -\frac{dU(x)}{dx} = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right) = -\frac{1}{2}k2x \Rightarrow$

$\boxed{F = -kx}$ – our state for force (a conservative force)

$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$

$\nabla \cdot f(x,y,z) = x^2 + xy + yz + z^2$

1.2. The law of motion

$F = ma$

$F_0 = -kx$

$I = m\ddot{a}$

We have the future $x = \frac{dx}{dt} = \dot{x}$

$a = \frac{dx}{dt} = \dot{v} = \ddot{x}$

$a = \frac{dx}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = \ddot{x}$

$m \cdot \ddot{x} = -kx \quad \left| \frac{1}{m} \right. \Rightarrow$

$\ddot{x} + \frac{k}{m}x = 0$ – linear order homogeneous differential equation

Note $\omega_0^2 = \frac{k}{m}$

$\ddot{x} + \omega_0^2 x = 0$

$\lambda^2 x + \omega_0^2 x = 0 \Rightarrow \quad \lambda^2 + \omega_0^2 = 0 \Rightarrow \quad \lambda^2 = -\omega_0^2 \Rightarrow$

$\lambda_{1,2} = \pm\sqrt{-\omega_0^2}$ Note $i^2 = -1$

$\lambda_{1,2} = \pm\sqrt{i^2\omega_0^2} \Rightarrow \lambda_{1,2} = \pm i\omega_0$

$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

$x(t) = C_1 e^{-i\omega_0 t} + C_2 e^{i\omega_0 t}$

$x(t) = A\sin(\omega_0 t + \varphi_0)$

Euler $e^{\pm i\alpha} = \cos(\alpha) \pm i\sin(\alpha)$

$x(t) = A\sin(\omega_0 t)\cos(\varphi_0) + A\sin(\varphi_0)\cos(\omega_0 t)$

$x(t) = C_1\left[\cos(\omega_0 t) - i\sin(\omega_0 t)\right] + C_2\left[\cos(\omega_0 t) + i\sin(\omega_0 t)\right]$

$\begin{cases} A\sin(\varphi_0) = C_1 + C_2 \\ A\cos(\varphi_0) = i(C_2 - C_1) \end{cases}$

I $-\operatorname{tg}(\varphi_0) = \frac{C_1 + C_2}{i(C_2 - C_1)} \Rightarrow \boxed{\operatorname{tg}(\varphi_0) = \frac{C_1 + C_2}{C_2 - C_1}}$

II $A^2 = C_1^2 + C_2^2 + 2C_1C_2 - C_1^2 + 2C_1C_2 \Rightarrow$

$A^2 = 4C_1C_2 \Rightarrow \boxed{A = 2\sqrt{C_1C_2}}$

$x(t) = A\sin(\omega_0 t + \varphi_0)$ – the harmonic motion for the harmonic oscillators

$x(t)$ – the elongation

A – the amplitude (maximum elongation)

$\omega_0 = \sqrt{\frac{k}{m}}$ – the self-pulsation

$\varphi(t) = \omega_0 t + \varphi_0$ – the phase

$\varphi(t=0) = \omega_0 0 + \varphi_0 = \varphi_0$ – the initial phase

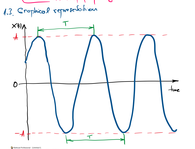
Note

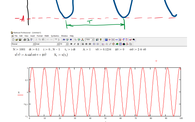
$\nu_0 = \frac{1}{T_0} \Rightarrow \omega_0 = 2\pi\nu_0 = \frac{2\pi}{T_0}$

the self period

the self frequency

1.3. Graphical representation





1.4. The energy

$K = \frac{mv^2}{2}$

$U = \frac{kx^2}{2}$

$E = K + U$

$x(t) = A\sin(\omega_0 t + \varphi_0)$

$\omega_0^2 = \frac{k}{m} \Rightarrow$

$k = m \cdot \omega_0^2$

$v = \frac{dx}{dt} = \frac{d}{dt}\left(A\sin(\omega_0 t + \varphi_0)\right) \Rightarrow$

$v = A\cos(\omega_0 t + \varphi_0) \cdot \omega_0$

$a = \frac{dv}{dt} = \frac{d}{dt}\left(\omega_0 A\cos(\omega_0 t + \varphi_0)\right) \Rightarrow$

$a = -\omega_0^2 A\sin(\omega_0 t + \varphi_0) \Rightarrow \boxed{a = -\omega_0^2 x}$

$K = \frac{m\cdot\omega_0^2 A^2}{2}\cos^2(\omega_0 t + \varphi_0)$

$U = \frac{m\cdot\omega_0^2}{2}A^2\sin^2(\omega_0 t + \varphi_0) = \frac{m\cdot\omega_0^2 A^2}{2}\sin^2(\omega_0 t + \varphi_0)$

$E = k + U = \frac{1}{2}m\omega_0^2 A^2 \Rightarrow \boxed{E = \frac{1}{2}kA^2}$ – constant (is conserved)

2. The damped oscillations

1) $F = F_0 + F_d$

$F = m \cdot a = m \cdot \ddot{x}$

$F_0 = -kx$

$F_d = -b\dot{v} = -b\dot{x}$

the resistance coefficient

the resistance force

$\Rightarrow m \cdot \ddot{x} = -kx - b\dot{x} \quad \left| \frac{1}{m} \right.$

$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$

the second order homogeneous differential equation

Note $\omega_0^2 = \frac{k}{m}$, $2\delta = \frac{b}{m}$

$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$

the self pulsation

$\lambda^2 x + 2\delta\lambda x + \omega_0^2 x = 0 \Rightarrow \lambda^2 + 2\delta\lambda + \omega_0^2 = 0$

$\lambda_{1,2} = \delta \pm \sqrt{\delta^2 - \omega_0^2}$

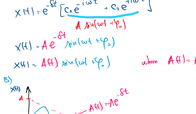
$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

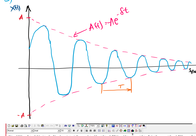
$x(t) = C_1 e^{\left(\delta - \sqrt{\delta^2 - \omega_0^2}\right)t} + C_2 e^{\left(\delta + \sqrt{\delta^2 - \omega_0^2}\right)t}$

$x(t) = e^{\delta t}\left[C_1 e^{\sqrt{\delta^2 - \omega_0^2}\,t} + C_2 e^{-\sqrt{\delta^2 - \omega_0^2}\,t}\right]$

I. Large resistance; λ – large

$\delta > \omega_0$ & $C_2 = 0 \Rightarrow x(t) = C_1 e^{\left(\delta - \sqrt{\delta^2 - \omega_0^2}\right)t}$





II. Small resistance; λ – small

$\delta < \omega_0$

$\sqrt{\delta^2 - \omega_0^2} = \sqrt{(-1)(\omega_0^2 - \delta^2)}$

Note $\omega_1^2 = \omega_0^2 - \delta^2$

Note $i^2 = -1$

$\Rightarrow \sqrt{\delta^2 - \omega_0^2} = \sqrt{i^2\omega_1^2} = i\omega_1$

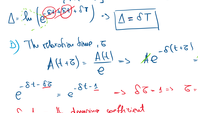
$x(t) = e^{-\delta t}\left[C_1 e^{-i\omega_1 t} + C_2 e^{i\omega_1 t}\right]$

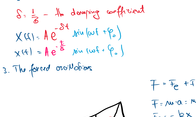
$A\sin(\omega_1 t + \varphi_0)$

$x(t) = Ae^{-\delta t}\sin(\omega_1 t + \varphi_0)$

$x(t) = A(t)\sin(\omega_1 t + \varphi_0)$ where $A(t) = Ae^{-\delta t}$

b)





$\omega = 2\pi\nu = \frac{2\pi}{T}$ – the pulsation

the period

the frequency

c) The logarithmic decrement

$\Delta = \ln\left[\frac{A(t)}{A(t+T)}\right] \Rightarrow \Delta = \ln\left[\frac{Ae^{-\delta t}}{Ae^{-\delta(t+T)}}\right] \Rightarrow$

$\Delta = \ln\left(e^{\delta T}\right) \Rightarrow \Delta = \delta T$

D) The relaxation time τ

$A(t+\tau) = \frac{A(t)}{e} \Rightarrow Ae^{-\delta(t+\tau)} = \frac{Ae^{-\delta t}}{e} \Rightarrow$

$e^{-\delta t - \delta\tau} = e^{-\delta t - 1} \Rightarrow \delta\tau - 1 \Rightarrow \tau = \frac{1}{\delta}$

$\delta = \frac{1}{\tau}$ – the damping coefficient

$x(t) = Ae^{-\delta t}\sin(\omega_1 t + \varphi_0)$

$x(t) = Ae^{-\frac{t}{\tau}}\sin(\omega_1 t + \varphi_0)$

3. The forced oscillations

$F = F_0 + F_d + F_{ext}$

$F = m \cdot a = m \cdot \ddot{x}$

$F_0 = -kx$

$F_d = -b \cdot v = -b \cdot \dot{x}$

$F_{ext} = F_0\sin(\omega_{ext} t)$