Advanced Set Representation Methods

AVL trees. 2-3(-4) Trees.
Union-Find Set ADT

Advanced Set Representation. AVL Trees

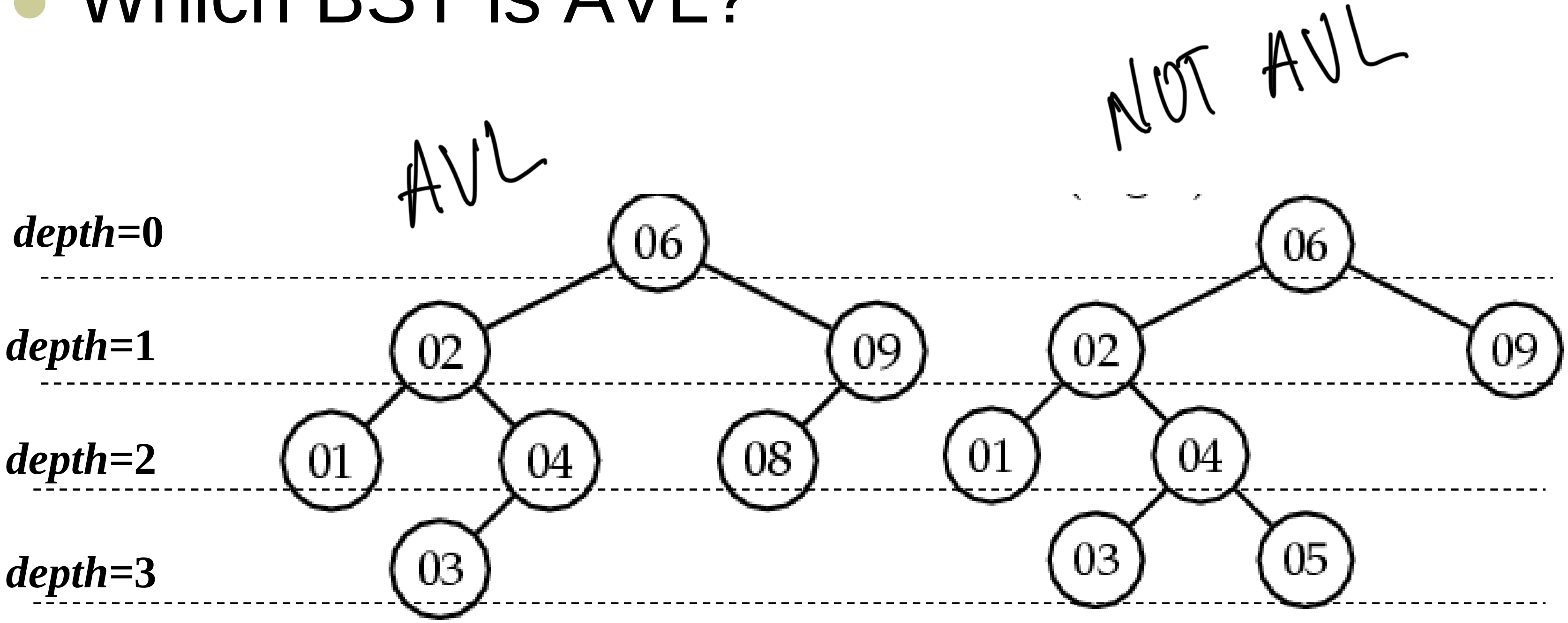
- Problem with BSTs: worst case operation may take $O(n)$ time. One solution:
- **AVL tree**: binary search tree with a *balance condition*:

  For every node in an AVL tree T, the height of the left (T_L) and right (T_R) subtrees can differ by at most 1: $|h_L - h_R| \leq 1$

  - Balance condition must be easy to maintain, and it ensures that the tree depth is $O(\log n)$
  - Requiring that the left and right subtrees have the same height does not suffice (tree may not be shallow)

Two BSTs

- Which BST is AVL?



For every node in an AVL tree T, the height of the left (T_L) and right (T_R) subtrees can differ by at most 1: $|h_L - h_R| \leq 1$

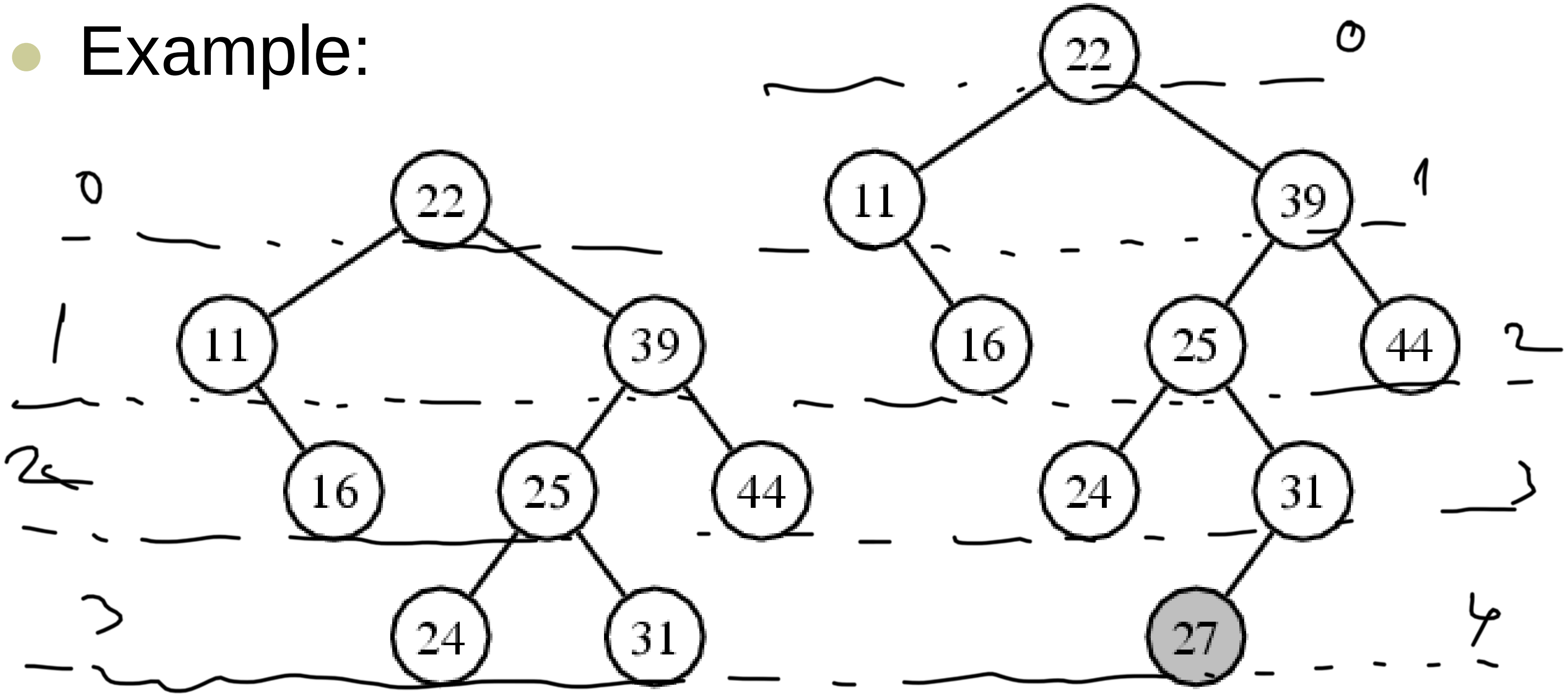
AVL Tree Height

- Claim: height of an AVL tree storing n keys is $\mathbf{O(\log n)}$
- Proof
  - **Let $n(h)$ denote the number of nodes of an AVL tree of height h.**
  - **It is easy to see that $n(1) = 1$ and $n(2) = 2$**
  - **For $n > 2$, an AVL tree of height h contains:**
    - **the root node,**
    - **one AVL subtree of height h-1 and**
    - **another of height h-2.**
  - **That is, $n(h) = 1 + n(h-1) + n(h-2)$**
  - **Knowing $n(h-1) > n(h-2)$, we get $n(h) > 2n(h-2)$. By induction**

  $n(h) > 2n(h-2)$, $n(h) > 4n(h-4)$, $n(h) > 8n(n-6)$, …, $n(h) > 2^i n(h-2i)$
  - **Solving the base case we get: $n(h) > 2^{h/2-1}$**
  - **Taking logarithms: $h < 2\log n(h) + 2$**
- Thus the height of an AVL tree is $\mathbf{O(\log n)}$

Insertion in an AVL Tree

- Insertion is as in a binary search tree
- Always done by attaching an external node.
- Example:



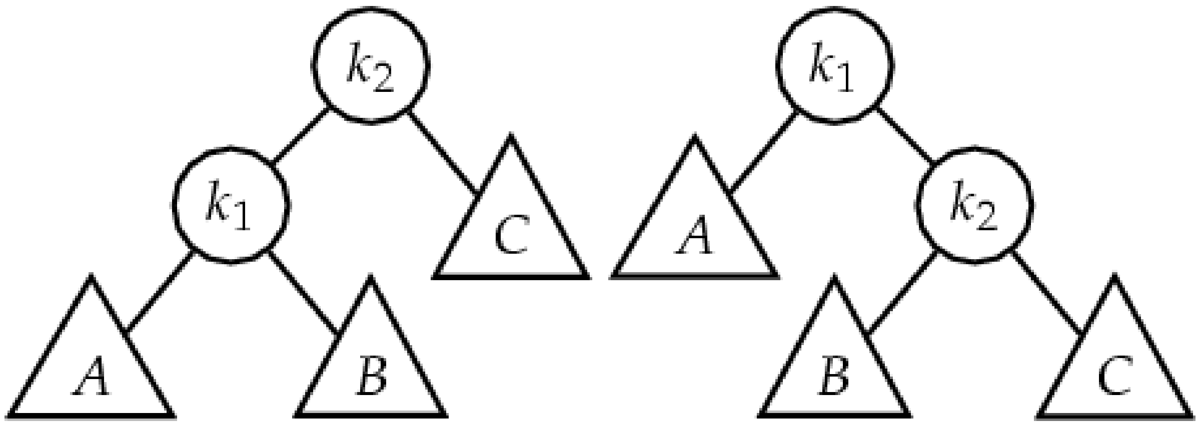
- After insertion the tree may need rebalancing

AVL tree rebalancing

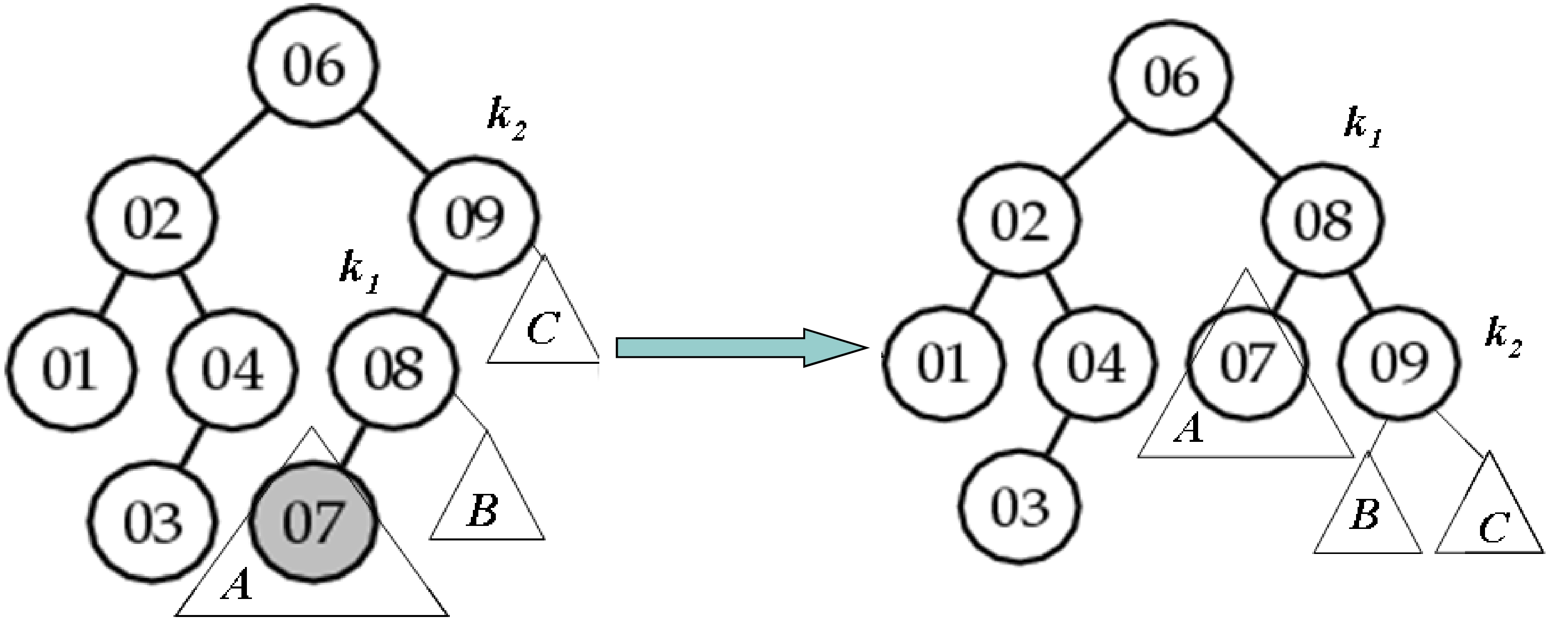
- Two kinds of rotation:
  - single rotation (left or right)
  - double rotation
    - **left-right:** *left rotation* around the *left child* of a node followed by a *right rotation* around the *node itself*
    - **right-left:** *right rotation* around the *left child* of a node followed by a *left rotation* around the *node itself*
- Rotation features:
  - Nodes not in the subtree of the node rotated are unaffected
  - A rotation takes constant time
  - Before and after the rotation tree is still BST
  - Code for left rotation is symmetric to code for a right rotation

AVL Single Rotations

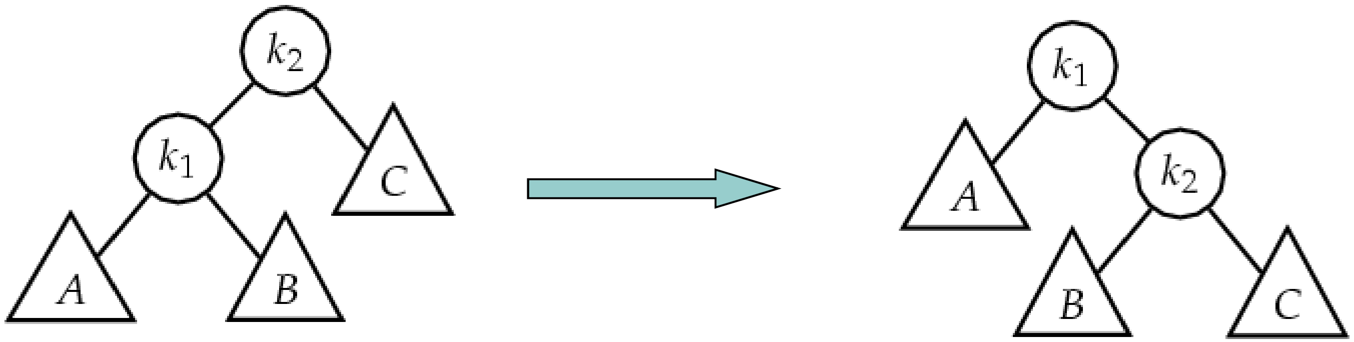
- Single Rotation
  - $k_1 < k_2$
  - all elements in subtree *A* are smaller than k_1
  - all elements in subtree *C* are larger than k_2
  - all elements in subtree *B* are in between k_1 and k_2



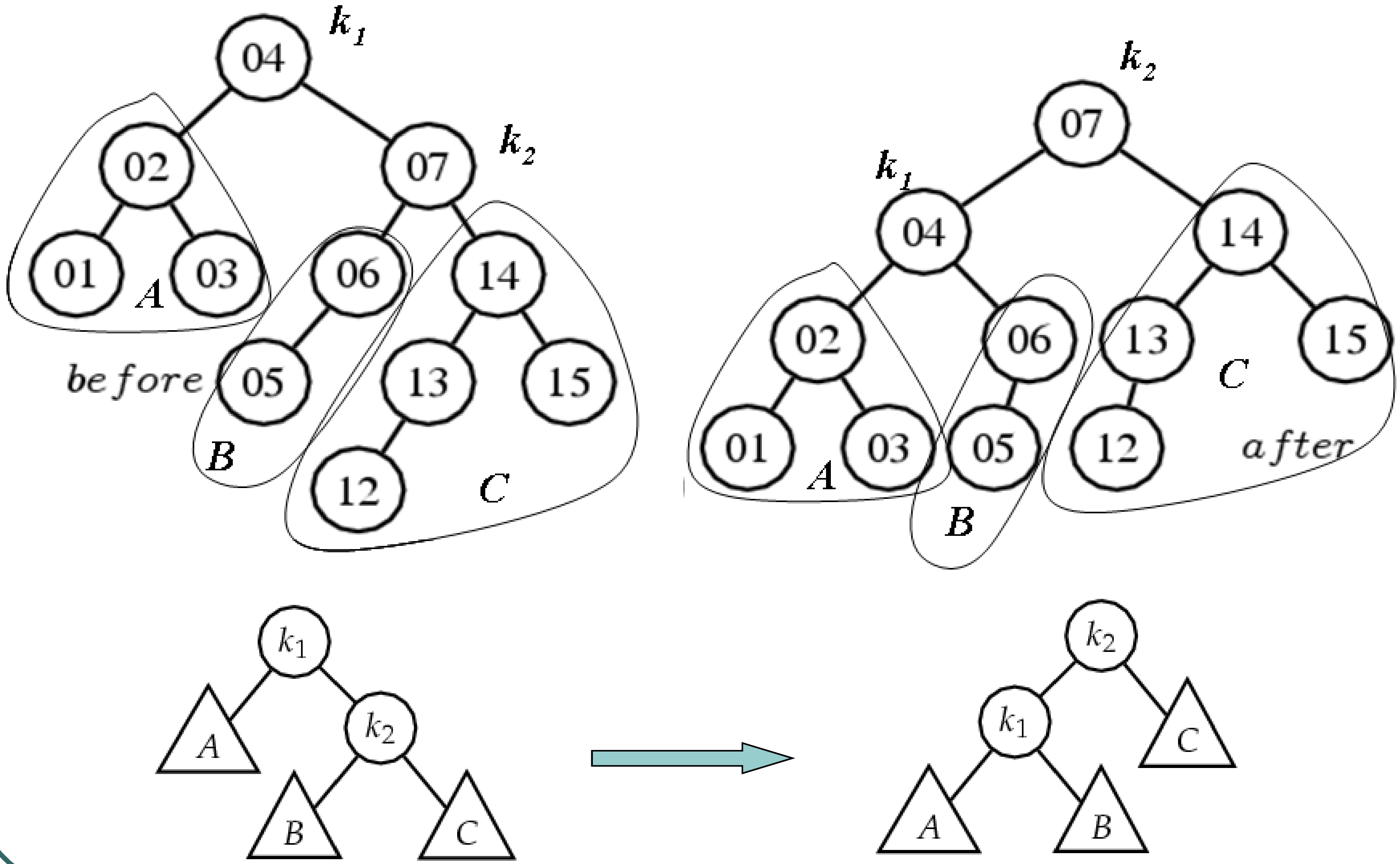
Single Rotation Right Example



- Note that subtrees B and C are empty



Single Rotation Left Example



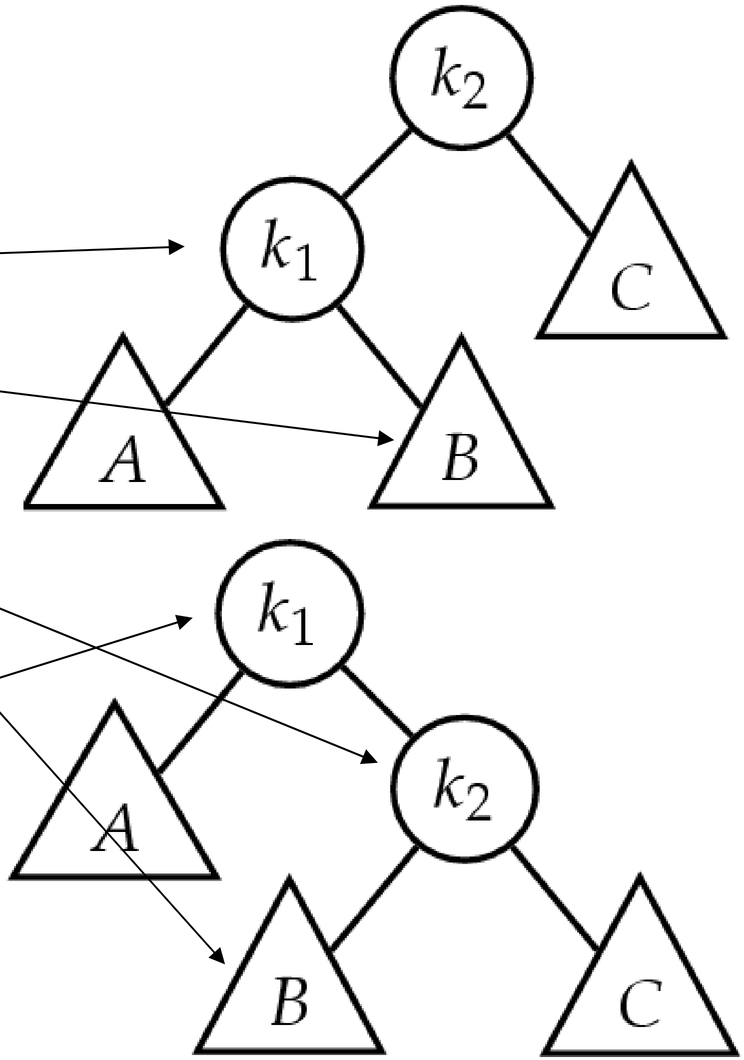
AVL Implementation Detail

```
typedef struct
{
   ElementT element;
   AVLPtr left;
   AVLPtr right;
   int height;
} AVLNode;
typedef AVLNode *AVLPtr;
```

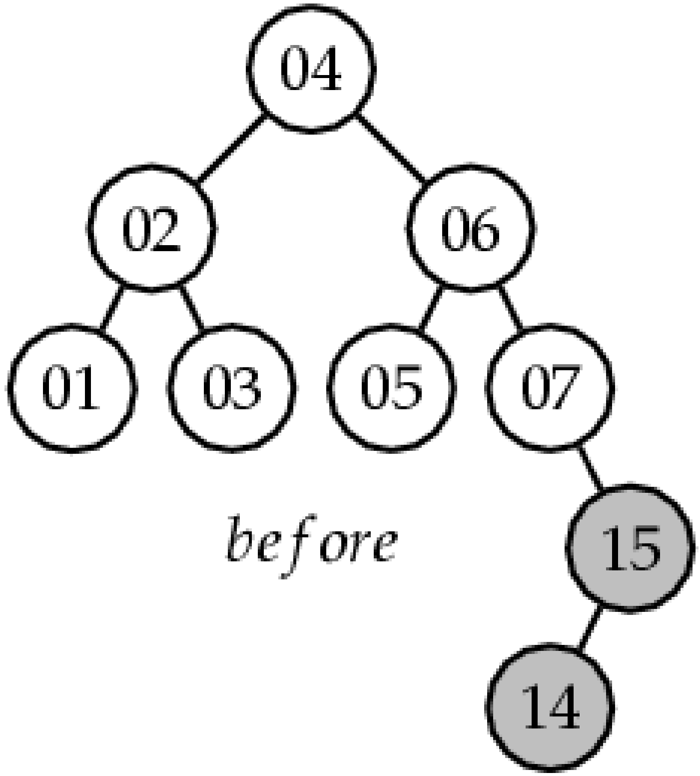
AVL Single Rotations

```
void snglRotRight( AVLPtr *k2 )
{
  AVLPtr k1 ;

  k1 = (*k2)->left ;
  (*k2)->left  = k1->right ;
  k1->right = *k2 ;
  (*k2)->height = max( height(
     (*k2)->left ),
                     height( (*k2)->right)
     ) +1 ;
  k1->height = max( height( k1-
     >left ),
                  (*k2)->height ) + 1 ;
  *k2 = k1 ; /* assign new  root */
}
/* snglRotLeft is symmetric */
```



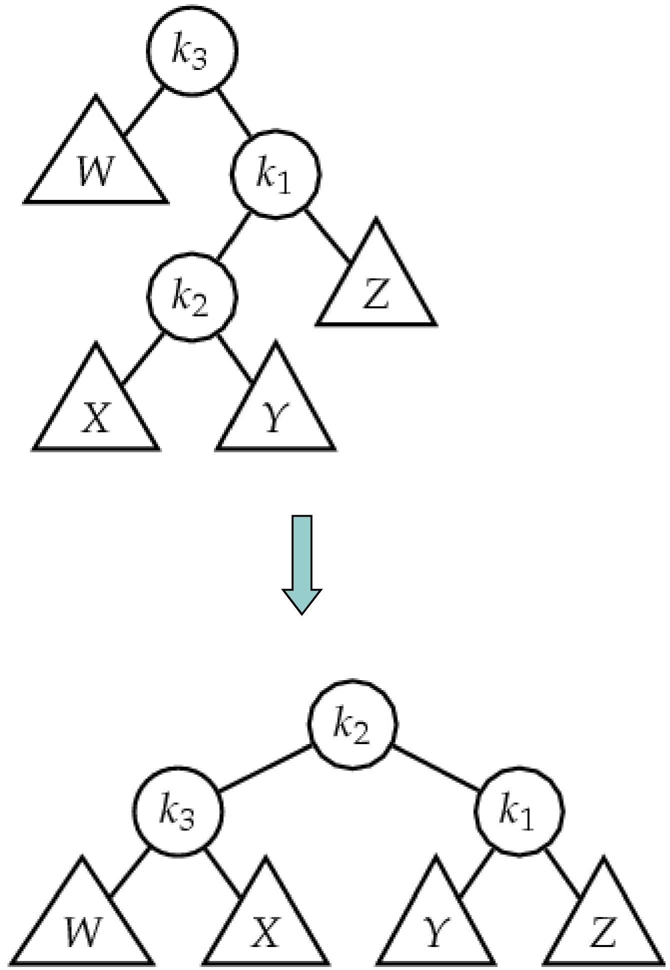
Single Rotation not enough



AVL Double Rotation

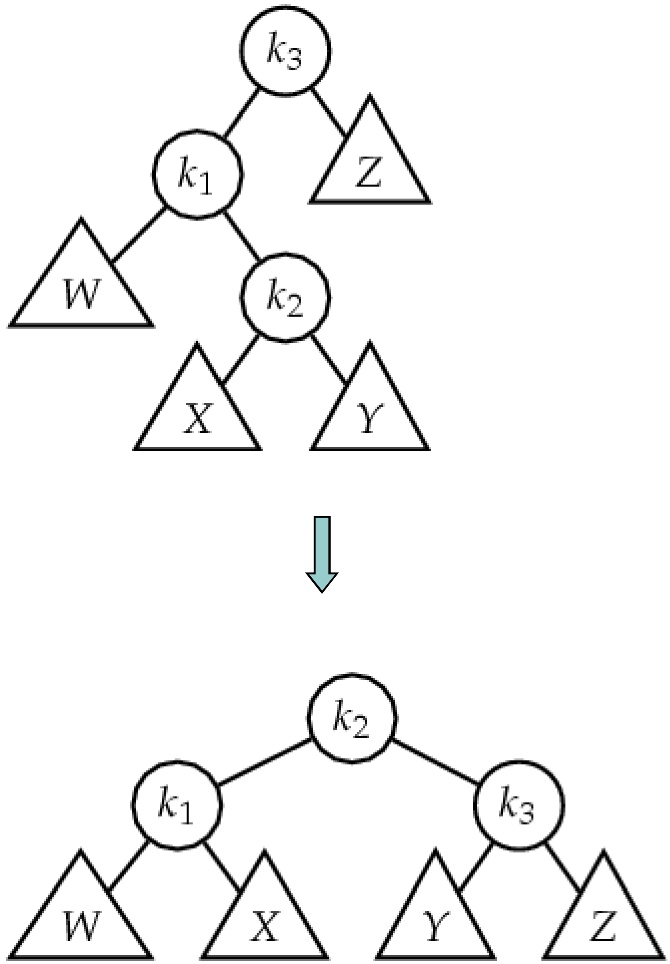
- Right-left
  - $k_2 < k_1, k_3 < k_1, k_3 < k_2$
  - right rotation around the left child of a node followed by
  - a left rotation around the node itself
  - Rotate to make k_2 topmost node

```
void  dblRotLeft( AVLPtr k3 )
{
  /* rotate between k1 and k2 */
  snglRotRight ( (*k3)->left );
  /* rotate between k3 and k2 */
  snglRotLeft ( k3 );
}
```



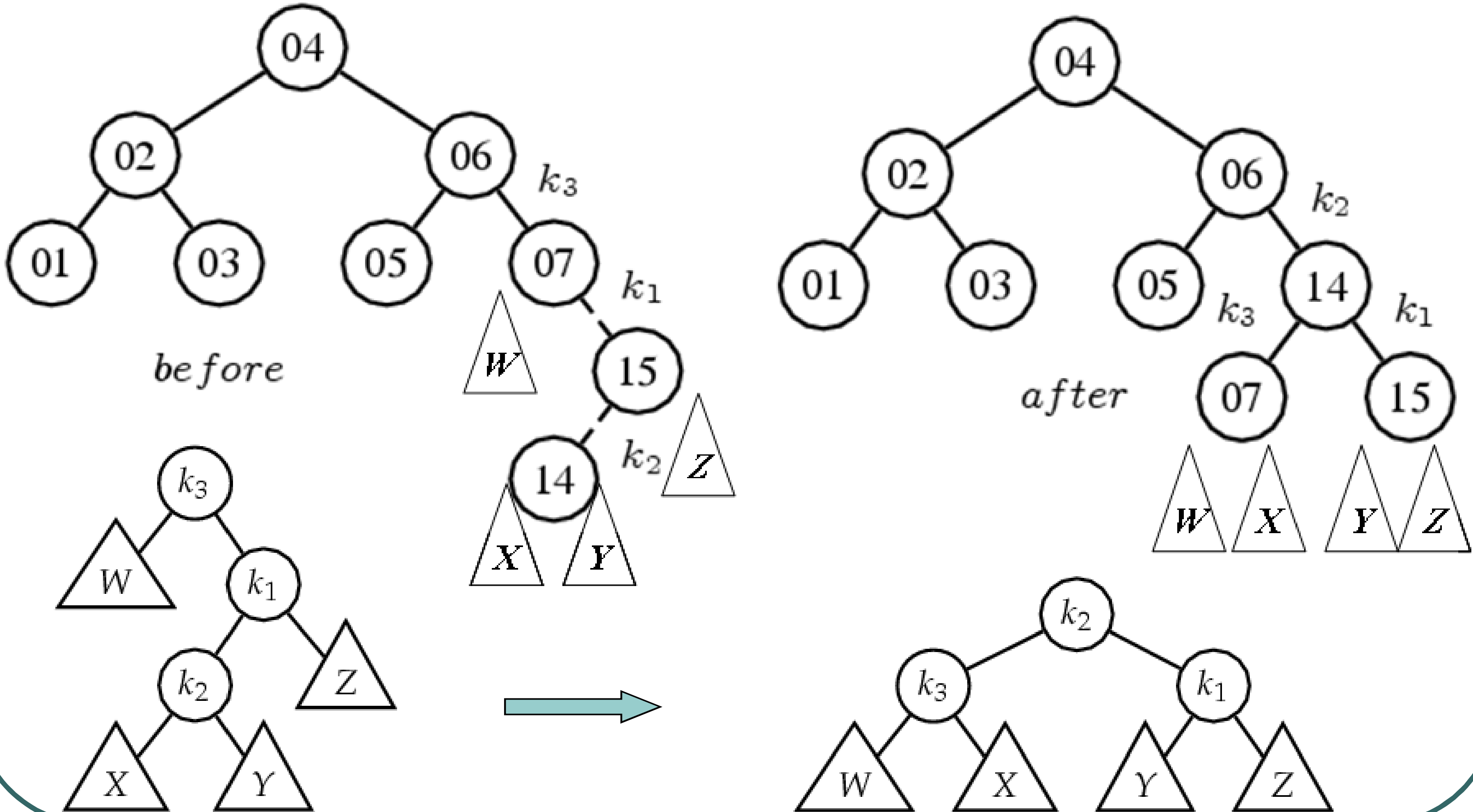
AVL Double Rotation

- Left-right:
  - $\boldsymbol{k_1 < k_2, k_1 < k_3, k_2 < k_3}$
  - left rotation around the left child of a node followed by a
  - right rotation around the node itself
  - Rotate to make $\boldsymbol{k_2}$ topmost node



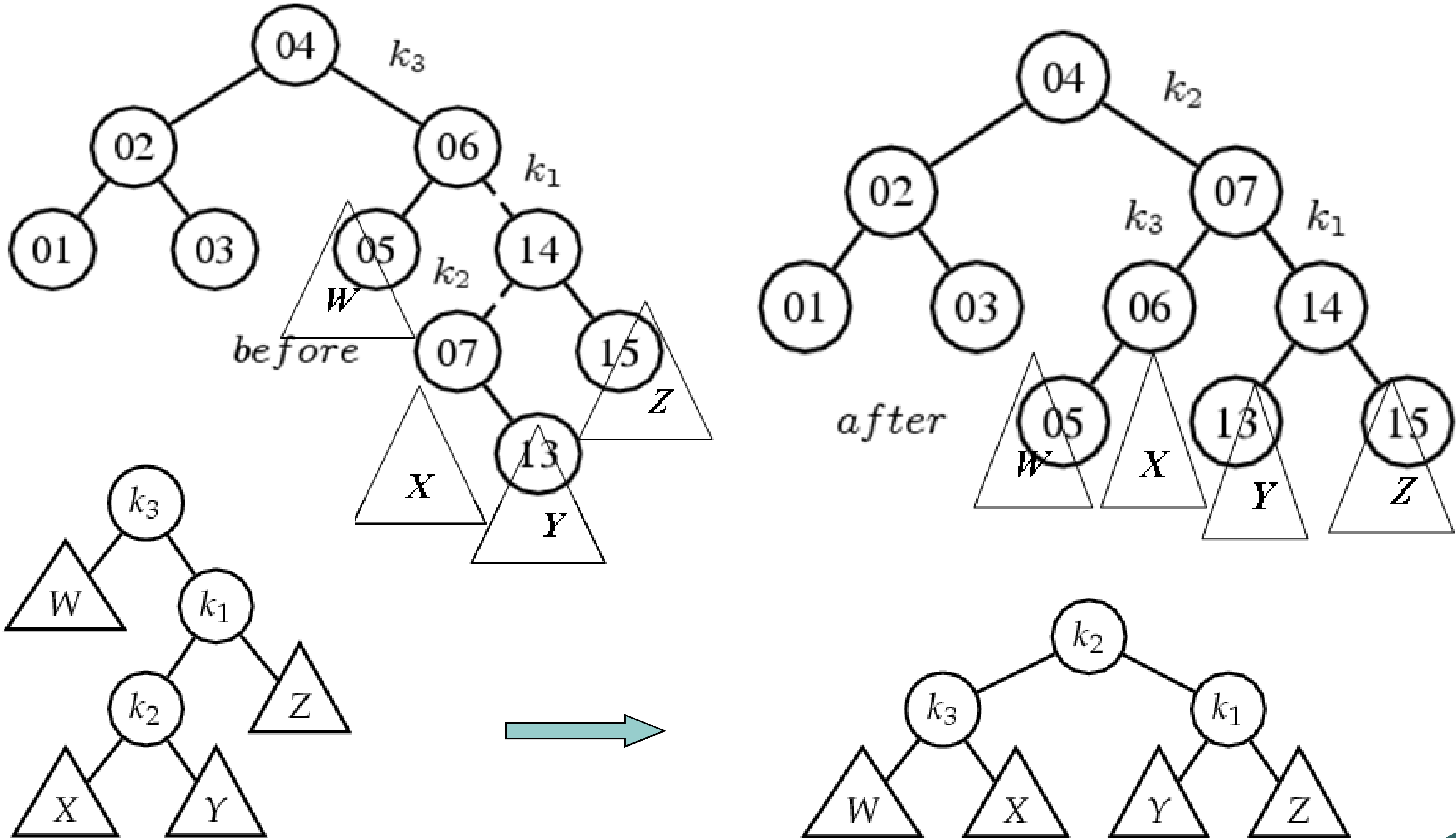
AVL Right-left Double Rotation Example 1

- Subtrees W, X, Y, Z are empty



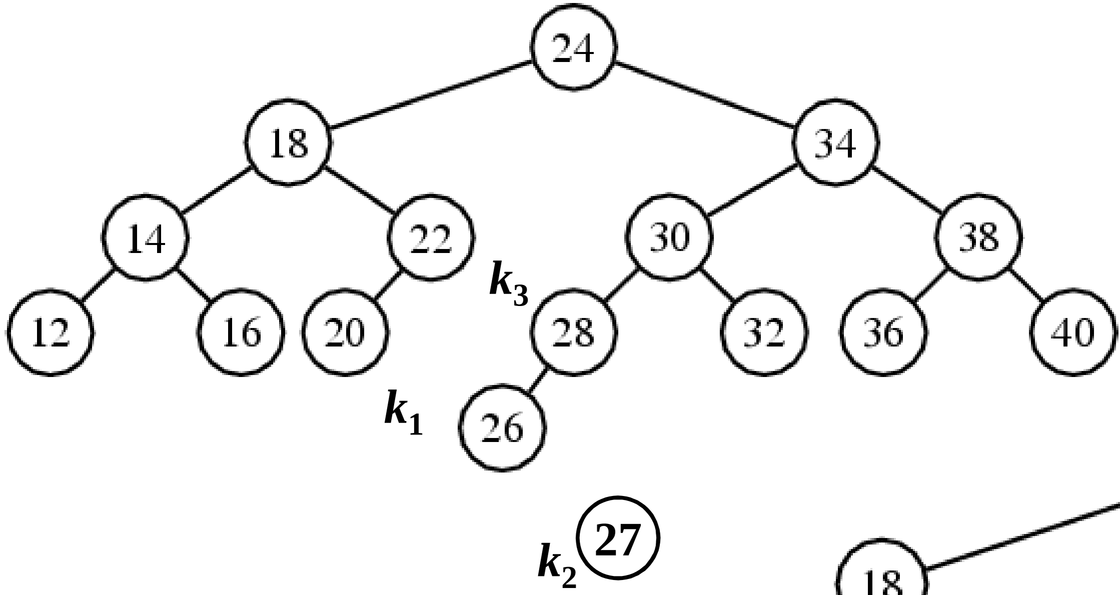
AVL Right-left Double Rotation Example 2

- Subtree X is empty



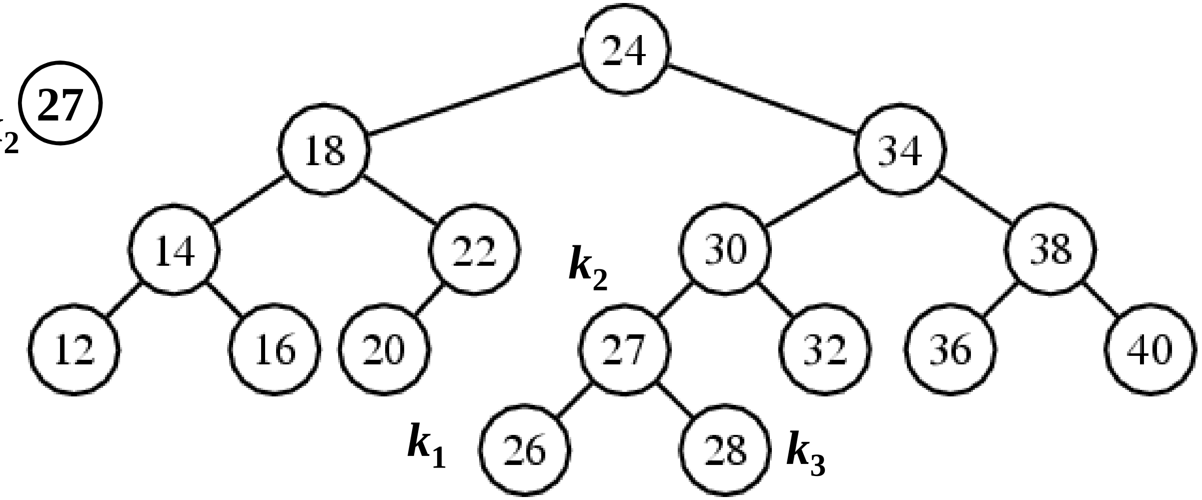
AVL Double Rotation example

- A symmetric case of double rotation



Tree before
insertion of 27

Tree after
insertion of 27
and re-balancing



AVL Trees. When to use rotations

- Balance factor: $height(T_L) - height(T_R)$
- Use
  - **Single rotation left**: when a node is inserted in the right subtree of the right child (B) of the nearest ancestor (A) with balance factor -2
  - **Single rotation right**: when a node is inserted in the left subtree of the left child (B) of the nearest ancestor (A) with balance factor +2.
  - **Right-left double rotation**: when a node is inserted in the left subtree of the right child (B) of the nearest ancestor (A) with balance factor -2.
  - **Left-right double rotation**: when a node is inserted in the right subtree of the left child (B) of the nearest ancestor (A) with balance factor +2.

AVL Trees. Deletions

- A node is deleted using the standard inorder successor (predecessor) logic for binary search trees
- Imbalance is fixed using rotations
- Identify the parent of the actual node that was deleted, then:
  - If the left child was deleted, the balance factor at the parent decreased by 1.
  - If the right child was deleted, the balance factor at the parent increased by 1.

AVL Trees. Deletions. Rebalancing (I)

- Let A be the node where balance must be restored.
- If the deleted node was in A's right subtree, then let B be the root of A's left subtree. Then:
  - **B has balance factor 0 or +1 after deletion** -- then perform a single right rotation
  - **B has balance factor -1 after deletion** -- then perform a left-right rotation
- If the deleted node was in A's left subtree, then let B be the root of A's right subtree. Then:
  - **B has balance factor 0 or -1 after deletion** -- then perform a single left rotation
  - **B has balance factor +1 after deletion** -- then perform a right-left rotation

AVL Trees. Deletions. Rebalancing (II)

- Unlike insertion, *one rotation may not be enough to restore balance to the tree*. If this is the case, then locate the next node where the balance factor is "bad" (call this A):
- If A's balance factor is positive, then let B be A's left child
  - If B's left subtree height is larger than B's right subtree height, then perform a **single right rotation**.
  - If B's left subtree height is smaller than B's right subtree height, then perform a **left-right rotation**.
  - If B's left subtree height is equal to B's right subtree height, then perform **either rotation**.

AVL Trees. Deletions. Rebalancing (III)

- If A's balance factor is negative, then let B be A's right child
  - If B's right subtree height is larger than B's left subtree height, then perform a **single left rotation**.
  - If B's right subtree height is smaller than B's left subtree height, then perform a **right-left rotation**.
  - If B's right subtree height is equal to B's left subtree height, then perform **either rotation**.

AVL trees demo

- Demos from:

  http://webpages.ull.es/users/jriera/Docencia/AVL/AVL%20tree%20applet.htm

  http://www.site.uottawa.ca/~stan/csi2514/applets/avl/BT.html

- Replacement for AVL trees: Red-Black trees (not discussed here)

Running times for AVL tree operations

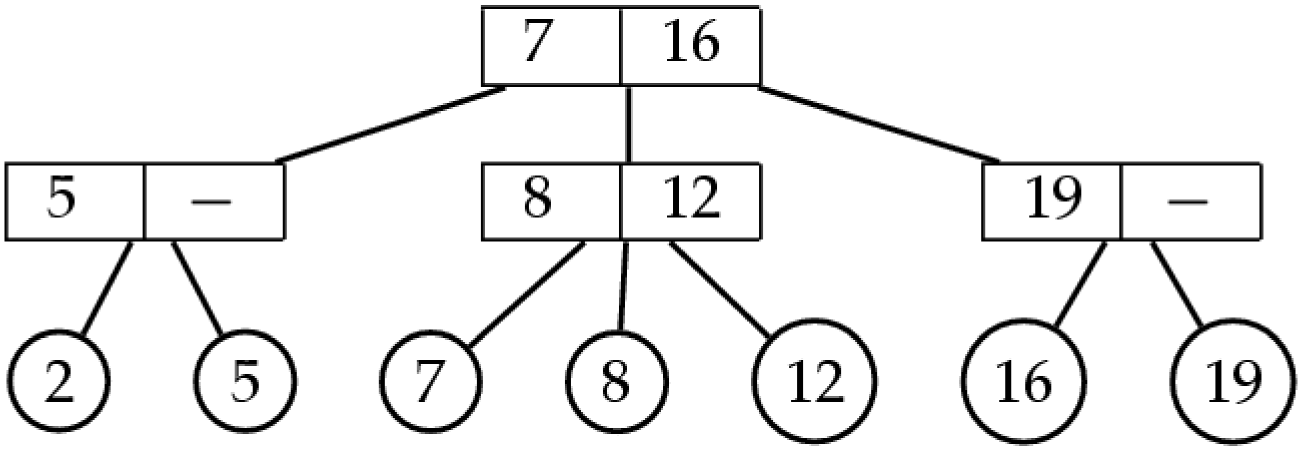
- a single restructure is $O(1)$
  - using a linked-structure binary tree
- find is $O(\log n)$
  - height of tree is $O(\log \boldsymbol{n})$, no restructures needed
- insert is $O(\log n)$
  - initial find is $O(\log \boldsymbol{n})$
  - restructuring up the tree, maintaining heights is $O(\log \boldsymbol{n})$
- remove is $O(\log n)$
  - initial find is $O(\log \boldsymbol{n})$
  - restructuring up the tree, maintaining heights is $O(\log \boldsymbol{n})$

2-3 Trees

- 2-3 tree properties:
  - Each interior node has two or three children.
  - Each path from the root to a leaf has the same length.
  - A tree with zero or one node(s) is a special case of a 2-3 tree.
- Representing sets with 2-3 trees:
  - Elements are placed at the leaves
  - If element a is to the left of element b, then $a < b$ must hold.
  - Ordering of elements based on one field of a record: a **key**.
  - At each **interior** node: key of the smallest descendant of the second child and, if there is a third child, key of the smallest descendant of third child.

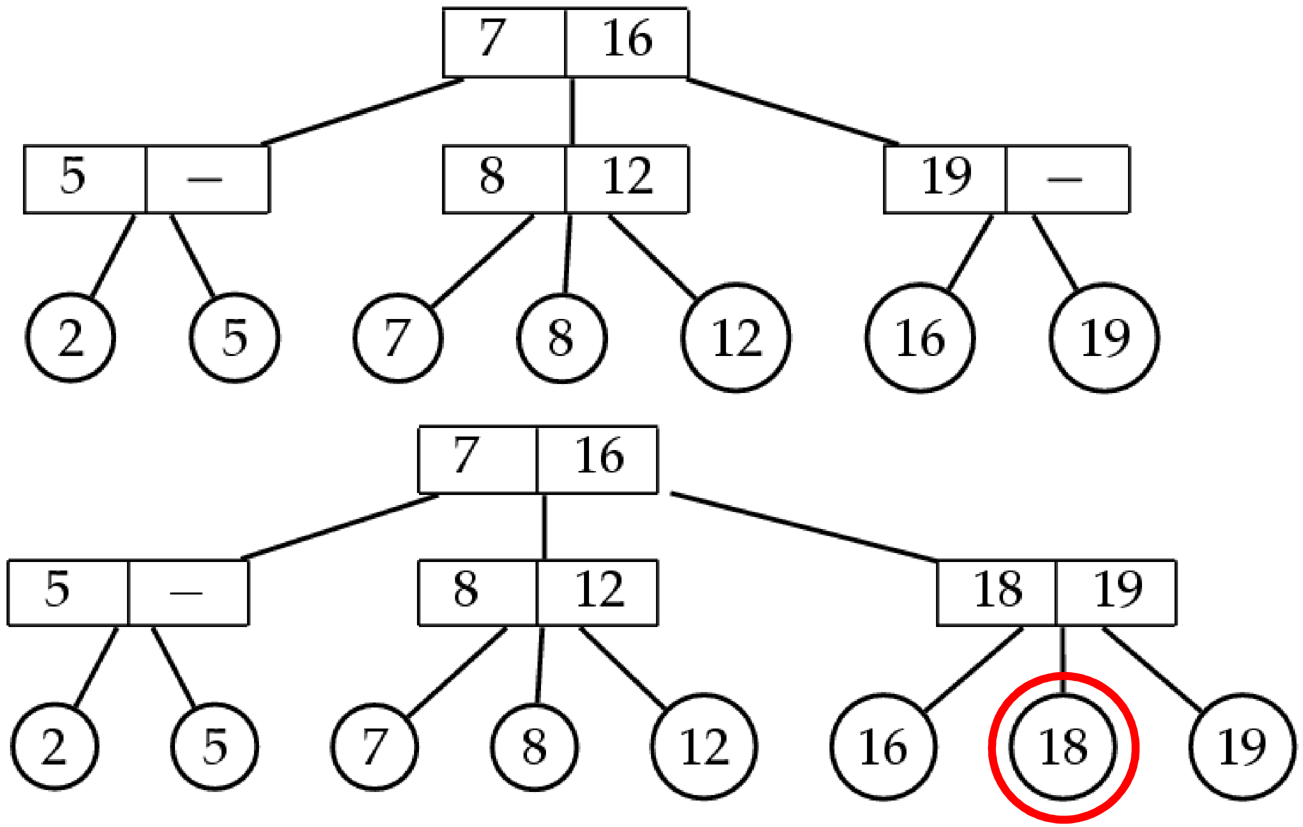
2-3 trees

- A 2-3 tree of *k* levels has between 2^{k-1} and 3^{k-1} leaves, i.e.
  - representing a set of n elements requires at least $1+\log_3 n$ levels and no more than $1+\log_2 n$ levels. Thus, path lengths in the tree are $O(\log n)$.



2-3 trees. Insertion – 2 children

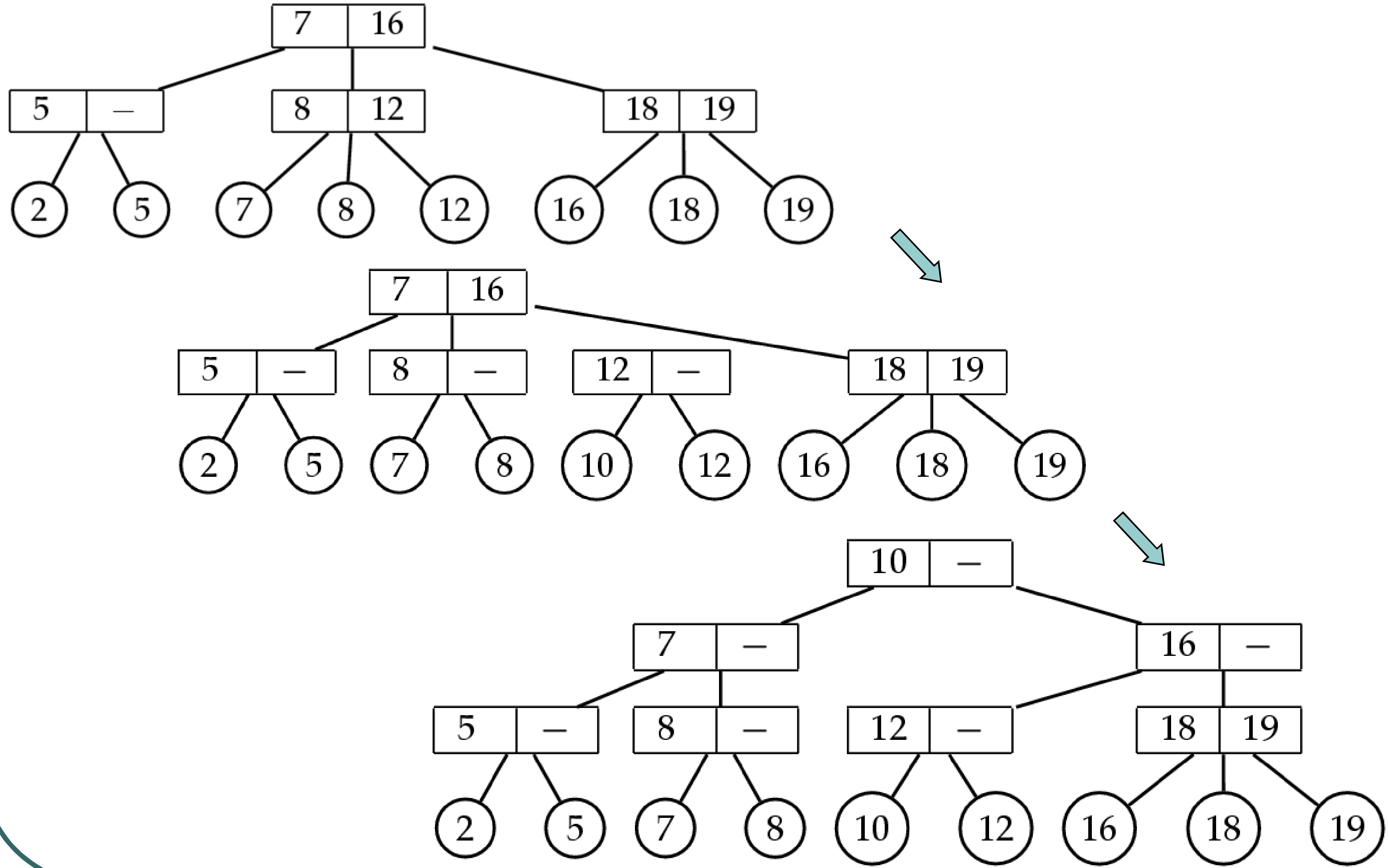
- If parent node has only two children:
  - insert in order, adjust keys



2-3 trees. Insertion – 3 children

- If parent node ***node*** already has three children:
  - We cannot have four nodes, and thus we have to split parent in two nodes ***node*** and ***node'*.**
  - The two smallest elements among the four children of ***node*** stay with ***node***,
  - The two larger will become the children of ***node'***
  - Continue process up the tree
  - Special case when splitting the root

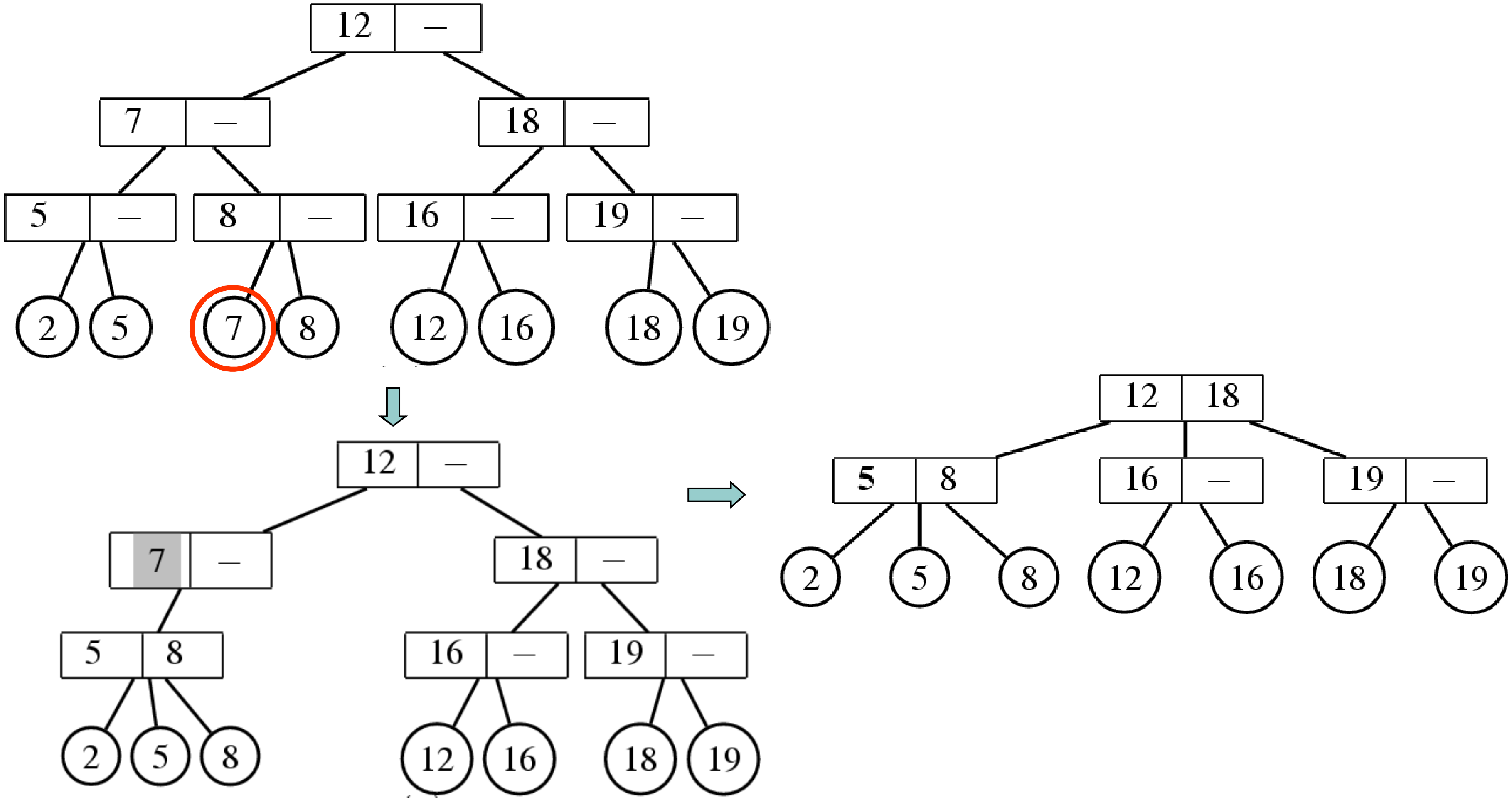
2-3 trees. Insertion



2-3 trees. Deletion

- Leaf deletion could leave parent with only 1 child
  - Parent=root: delete *node* and let its lone child be the new root
  - Otherwise, let *p* be the parent of *node*
    - if *p* has another child, and that child of *p* has three children, then we can transfer the proper one to *node;* done;
    - if the children of *p*, have only two children, transfer the lone child of *node* to an adjacent sibling of *node*, and delete *node*;
    - *p* has only one child, repeat all the above, recursively, with *p* in place of *node*.

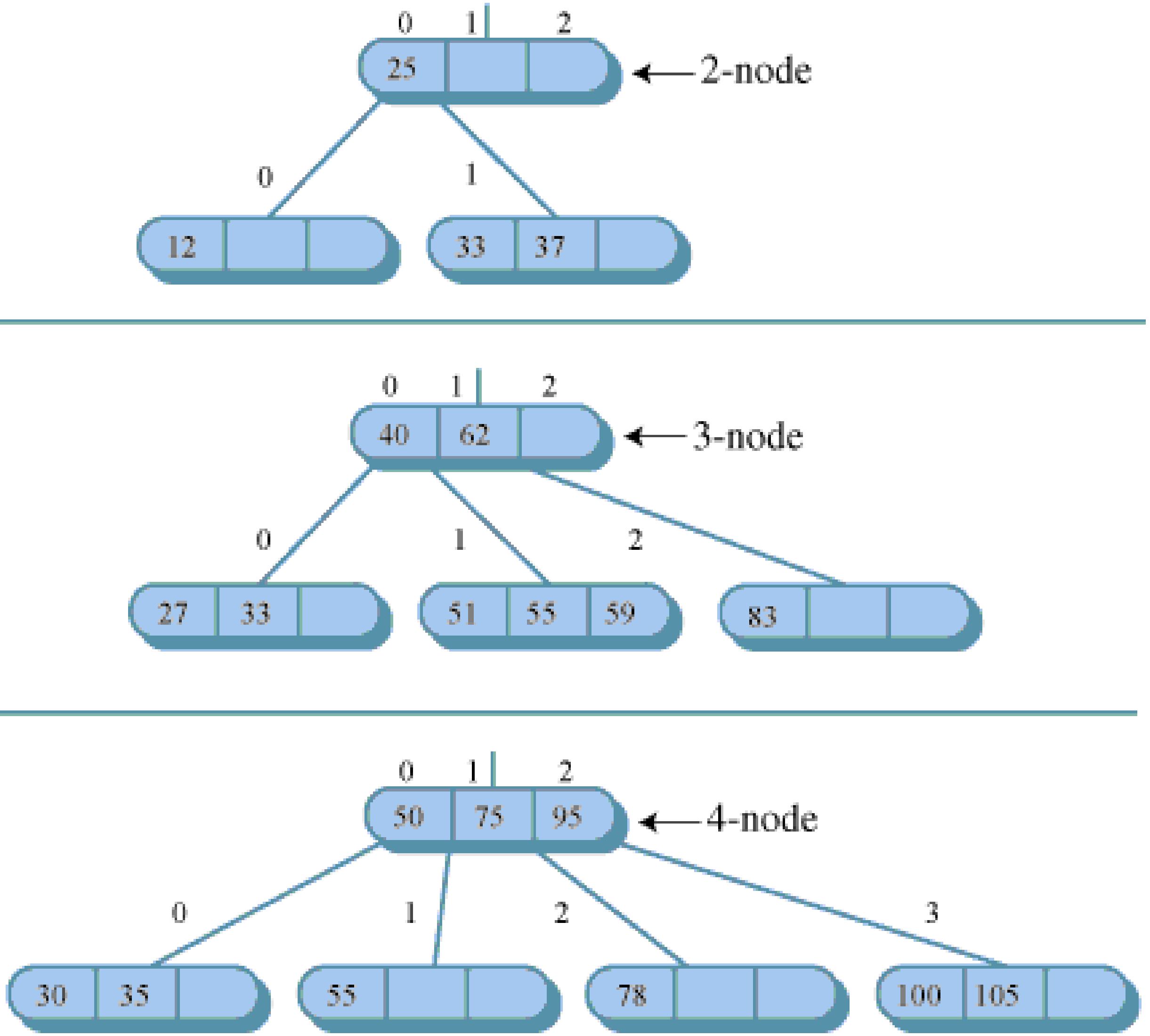
2-3 trees. Deletion



2-3-4 trees.

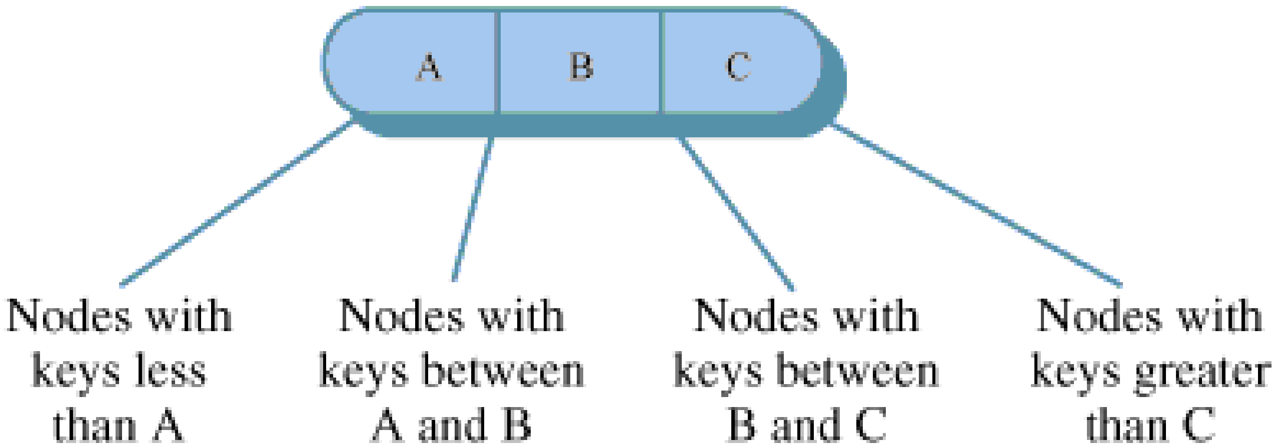
- *2-3-4 tree* refer to how many links to child nodes can potentially be contained in a given node. For non-leaf nodes, three arrangements :
  - A node with one data item always has two children
  - A node with two data items always has three children
  - A node with three data items always has four children
  - In short, a non-leaf node must always have one more child than it has data items.
- Symbolically, if the number of child links is L and the number of data items is D, then $L = D + 1$
- Empty nodes are not allowed.

2-3-4 trees. Node kinds.

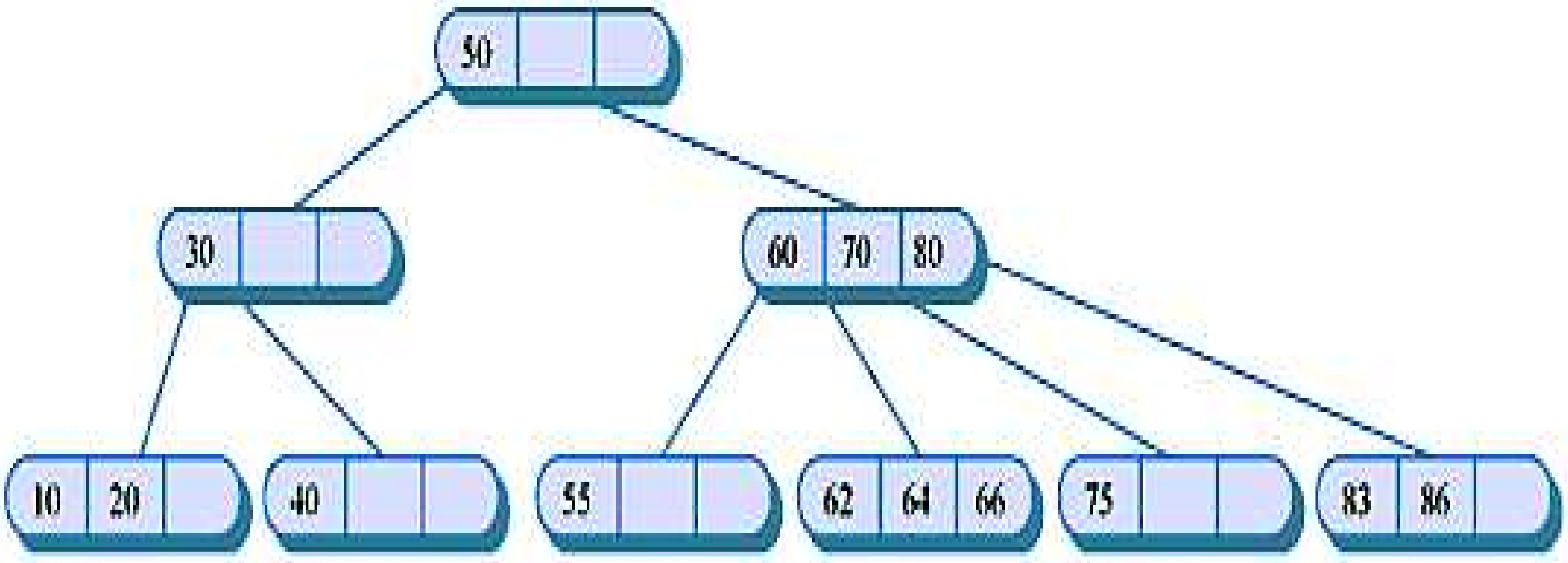


2-3-4 trees. Properties and example.

- All children in the subtree rooted at child 0 have key values less than key 0.
- All children in the subtree rooted at child 1 have key values greater than key 0 but less than key 1.
- All children in the subtree rooted at child 2 have key values greater than key 1 but less than key 2.
- All children in the subtree rooted at child 3 have key values greater than key 2.



2-3-4 trees. Example.



2-3-4 Trees. Insertion

- **Insertion procedure:**
  - similar to insertion in 2-3 trees
  - items are inserted at the leafs
  - since a 4-node cannot take another item,
    4-nodes are split up during insertion process

- **Strategy**
  - on the way from the root down to the leaf:
    split up all 4-nodes "on the way"
  - → insertion can be done in one pass
    (remember: in 2-3 trees, a reverse pass might be necessary)

2-3-4 Trees. Deletion

- **Deletion procedure:**
  - similar to deletion in 2-3 trees
  - items are deleted at the leafs
    → swap item of internal node with inorder successor
  - note: a 2-node leaf creates a problem
- **Strategy** (different strategies possible)
  - on the way from the root down to the leaf:
    turn 2-nodes (except root) into 3-nodes
  - → deletion can be done in one pass
    (remember: in 2-3 trees, a reverse pass might be necessary)

Demo: 2-3-4 trees

- Demo from:
  http://www.cs.unm.edu/~rlpm/499/ttft.html
- Local:
  Demos\2-3-4trees\TTFT.jar

Disjoint Sets with the UNION and FIND Operations

- Applicable to problems where:
  - start with a collection of objects, each in a set by itself;
  - combine sets in some order, and from time to time
  - ask which set a particular object is in
- Equivalence classes:
  - If set *S* has an equivalence relation (reflexive, symmetrical, transitive) defined on it, then the set *S* can be partitioned into disjoint subsets S_1, S_2,, ... S with $\bigcup_k S_k = S$
- Equivalence problem:
  - given a set *S* and a sequence of statements of the form $a \equiv b$
  - process the statements in order in such a way that at any time we are able to determine in which equivalence class a given element belongs

Union-find set ADT

- Example statements (see previous slide)

$$S = \{1, 2, \ldots, 7\}$$

$$1 \equiv 2,\ 5 \equiv 6,\ 3 \equiv 4,\ 1 \equiv 4$$

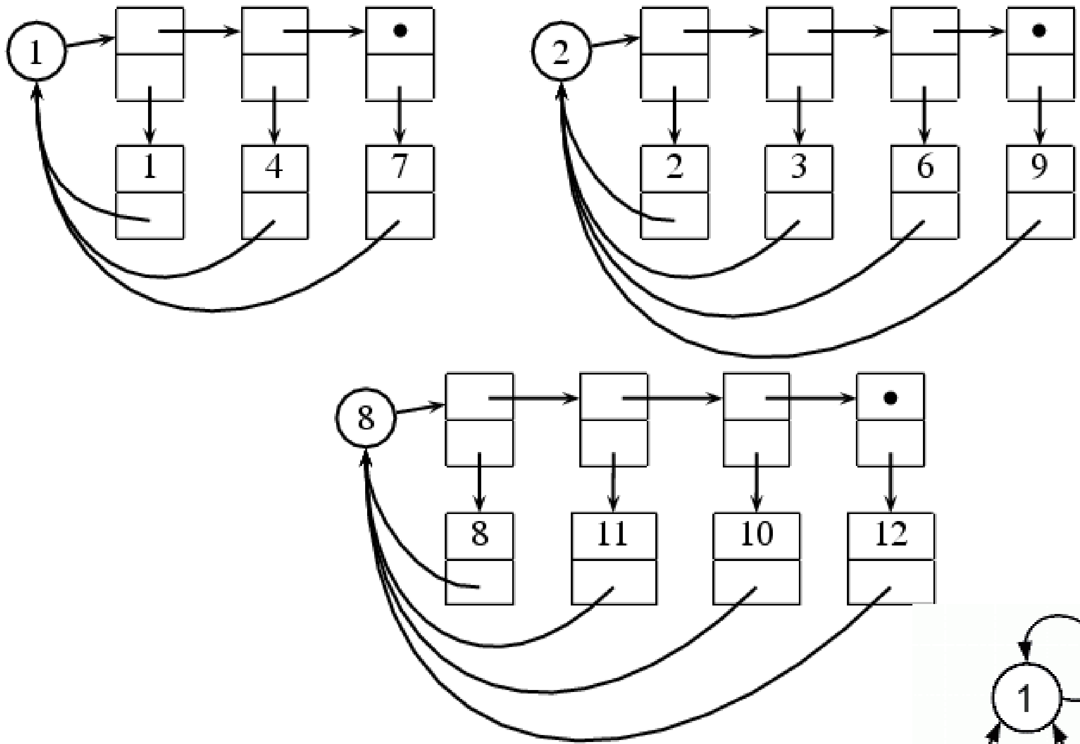
$$\begin{array}{ll}
1 \equiv 2 & \{1,2\}\{3\}\{4\}\{5\}\{6\}\{7\} \\
5 \equiv 6 & \{1,2\}\{3\}\{4\}\{5,6\}\{7\} \\
3 \equiv 4 & \{1,2\}\{3,4\}\{5,6\}\{7\} \\
1 \equiv 4 & \{1,2,3,4\}\{5,6\}\{7\}
\end{array}$$

- Operations:
  - union(*A*, *B*) takes the union of the components A and B and calls the result either A or B, arbitrarily.
  - find(*x*), a function that returns the name for the component of which *x* is a member.
  - initial(*A*, *x*) creates a component named *A* that contains only the element *x*.

Union-find set ADT. Implementations

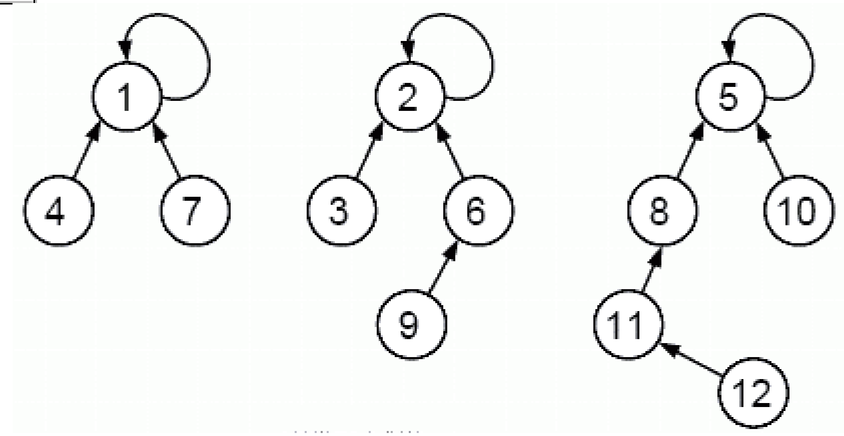
- List-based
  - Each set is stored in a sequence represented with a linked-list
  - Each node should store an object containing the element and a reference to the set name
- Tree based
  - Each element is stored in a node, which contains a pointer to a set name
  - A node *v* whose set pointer points back to *v* is also a set name
  - Each set is a tree, rooted at a node with a self-referencing set pointer

Examples



List based

Tree based



Union-find set. List based implementation

```
const int n = /* appropriate value */
typedef int NameT;
typedef int ElementT;
typedef struct ufset
{
  struct /* headers for set lists */
  {
    int count:;
    int firstElement ;
  } setHeaders[ n ] ;
  struct /* table giving set containing
      each member */
  {
    NameT setName ;
    int nextElement ;
  } setNames[ n ] ;
} UFSetT;
```

setHeaders

	count	*firstElement*
1	3	1
2	1	2
3	0	0
4	0	0
5	2	5
6	0	0

setNames

	setName	*nextElement*
1	1	3
2	2	0
3	1	4
4	1	0
5	5	6
6	5	0

Union-find sets. Tree representation

- Maintain *S* as a collection of trees (a forest), one per partition
- Initially, there are *n* trees, each containing a single element
- find(*i*) returns the root of the tree containing *i*
- union(*i*, *j*) merges the trees containing *i* and *j*
- Typical traversals not needed, so no need for pointers to children, but we need pointers to parent
- Parent pointers can be stored in an array: parent[*i*] (=-1 for root)

Union-find sets. Tree implementation

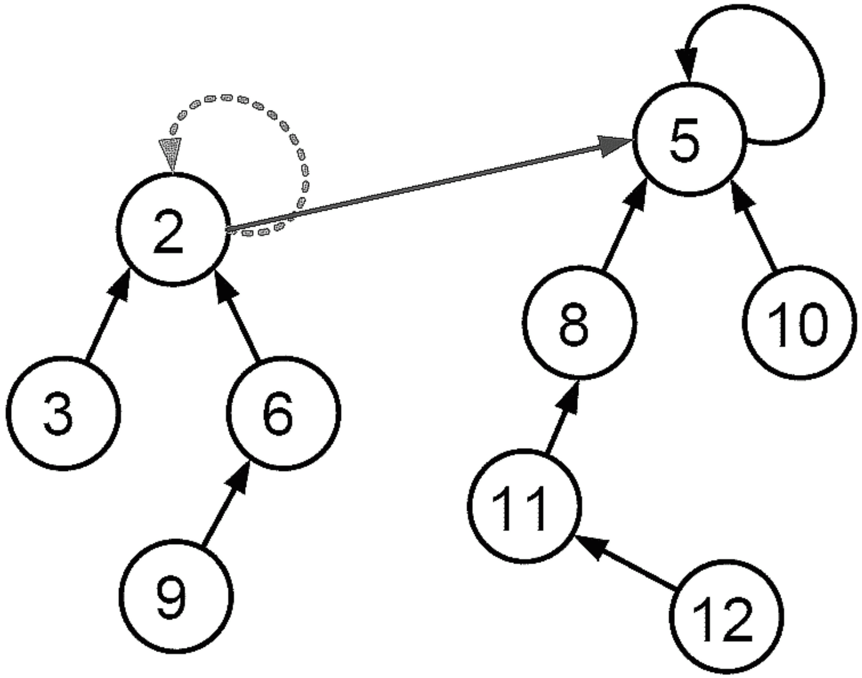
```
INIT()
    for i ← 1 to n
        do parent[i] ← 0
```

```
FIND(i)
    j ← i
    while parent[j] > 0
        do j ← parent[j]
    return j
```

```
UNION(i, j)
    root1 ← FIND(i)
    root2 ← FIND(j)
    if root1 ≠ root2
        then parent[root2] ← root1
```

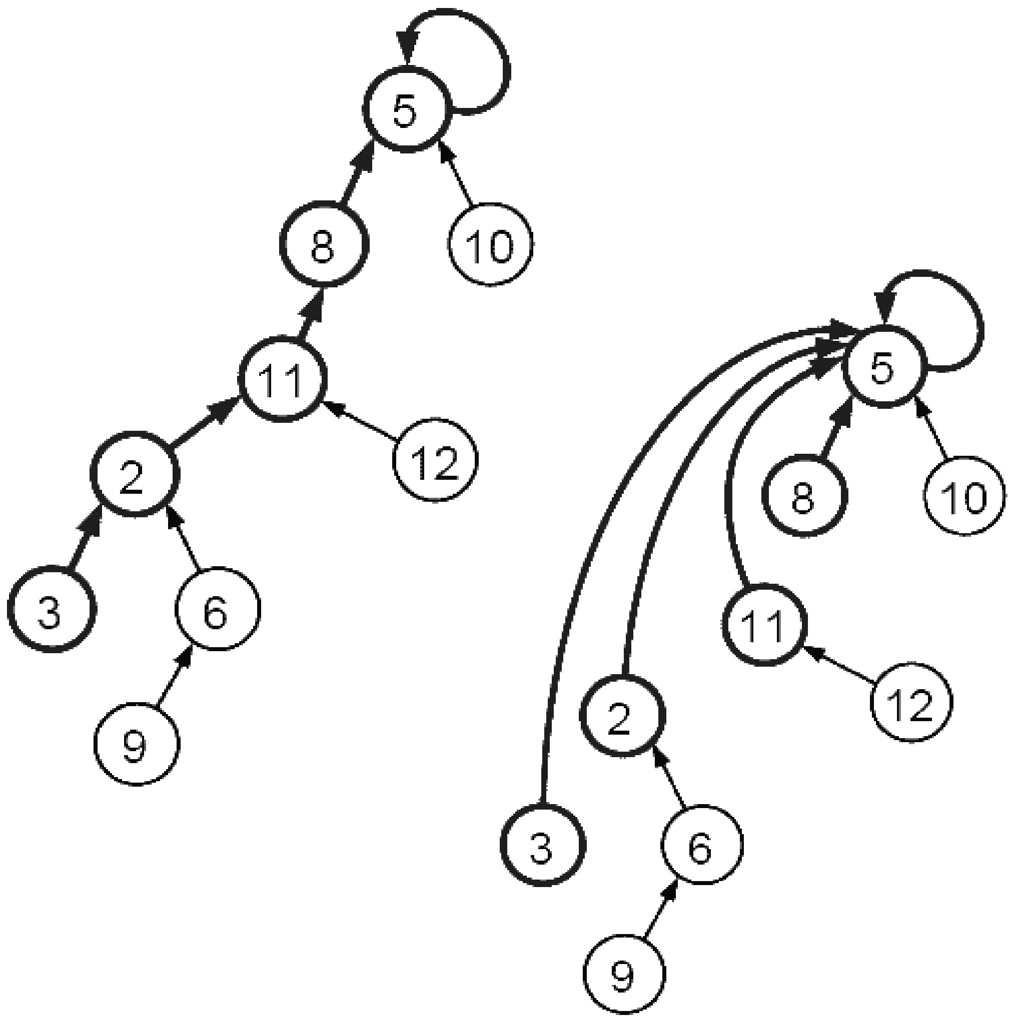
Union-find sets. Speeding up

- Union by size (rank):
  - When performing a union, make the root of smaller tree point to the root of the larger
- Implies $O(n \log n)$ time for performing n unionfind operations:
  - Each time we follow a pointer, we are going to a subtree of size at least double the size of the previous subtree
  - Thus, we will follow at most $\boldsymbol{O(\log n)}$ pointers for any find.



Union-find sets. Speeding up

- Path compression:
  - After performing a find, compress all the pointers on the path just traversed so that they all point to the root
- Implies $O(n \log^* n)$ time for performing n union- find



Using union by rank and path compression

MAKESET(x)

```
p[x] ← x
r[x] ← 0
```

FINDSET(x)

```
if x ≠ p[x]
    then p[x] ← FINDSET(p[x])
return p[x]
```

LINK(x, y)

```
if r[x] > r[y]
    then p[y] ← x
    else p[x] ← y
        if r[x] = r[y]
            then r[y] ← r[y] + 1
```

UNION(x, y)

```
LINK(FINDSET(x), FINDSET(y))
```

- Time bound of $O(m\alpha(n))$ where $\alpha(n)$ is a very slowly growing function

Union-find ADT demo

- Demo from:

  http://www.cs.unm.edu/~rlpm/499/uf.html
- Local:

  Demos\UnionFind\UnionFind.jar

A very quickly growing function and its inverse

- For integers $k \geq 0$ and $j \geq 1$, define $A_k(j)$:

$$A_k(j) = \begin{cases} j+1, & \text{if } k = 0 \\ A_{k-1}^{(j+1)}(j), & \text{if } k \geq 1 \end{cases}$$

  - where $A_{k-1}^{0}(j)=j$, $A_{k-1}^{(i)}(j)= A_{k-1}(A_{k-1}^{(i-1)}(j))$ for $i \geq 1$.
  - *k* is called the <u>level</u> of the function and
  - *i* in the above is called <u>iterations</u>.
- $A_k(j)$ strictly increase with both *j* and *k*.
- Let us see how quick the increase is

A very quickly growing function and its inverse

- Let us see $A_k(1)$: for k=0,1,2,3,4.
  - $A_0(1)=1+1=2$
  - $A_1(1)=2\times 1+1=3$
  - $A_2(1)=2^{1+1}\times(1+1)-1=7$
  - $A_3(1)=A_2^{(1+1)}(1)=A_2^{(2)}(1)=A_2(A_2(1))=A_2(7)=2^{7+1}(7+1)-1=2^8\times 8-1=2047$
  - $A_4(1)=A_3^{2}(1)=A_3(A_3(1)) =A_3(2047)=A_2^{(2048)}(2047)$
  - **>>** $A_2(2047) =2^{2048}\times 2048-1 >2^{2048} =(2^4)^{512} =(16)^{512}$
  - **$>>10^{80}$**. (estimated number of atoms in universe)

Inverse of $A_k(n)$: $\alpha(n)$

- $\alpha(n)=\min\{k: A_k(1) \geq n\}$
- $\alpha(n)=$ 0 for $0\leq n \leq 2$

  1 $n=3$

  2 for $4\leq n \leq 7$

  3 for $8\leq n \leq 2047$

  4 for $2048\leq n \leq A_4(1)$.
- Extremely slow increasing function.
- $\alpha(n) \leq 4$ for all practical purpose.

Reading

- AHU, chapter 5, sections 5.4, 5.5
- Preiss, chapter: Search Trees – section AVL Search Trees
- CLR, chapter 22, sections 1-4
- CLRS chapter 21
- Notes