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Complex Numbers

$z = x + iy$ is called a complex number where $x, y \in \mathbb{R}$

$x = \operatorname{Re} z$, $y = \operatorname{Im} z$, i = imaginary unit

$i^2 = -1$

$\bar{z} = x - iy$ the conjugate of z

$|z| = \sqrt{x^2+y^2}$ absolute value of z

$z = x + iy$ the algebraic form

$z = r \cdot (\cos\varphi + i \sin\varphi)$ the polar form (the trygonometric form)

$r = |z| = \sqrt{x^2+y^2}$ polar radius

$\varphi = \arctan\frac{y}{x} + k\pi$, where $k = \begin{cases} 0, \text{ if } z \in C_{I} \\ 1, \text{ if } z \in C_{II}, C_{III} \\ 2, \text{ if } z \in C_{IV} \end{cases}$

the reduced argument of z

$$\begin{aligned} z_1 &= r_1(\cos\varphi_1 + i\sin\varphi_1) \\ z_2 &= r_2(\cos\varphi_2 + i\sin\varphi_2) \end{aligned} \Rightarrow z_1 z_2 = r_1 r_2[\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2)]$$

$$z_1^n = r_1^n(\cos n\varphi_1 + i\sin n\varphi_1)$$

$$\sqrt[n]{z_1} = \sqrt[n]{r}\left(\cos\frac{\varphi_1+2k\pi}{n} + i\sin\frac{\varphi_1+2k\pi}{n}\right), \quad k = \overline{0, n-1}$$

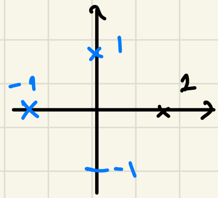
the roots of order n of z_1

The equation: $|z-z_0| = r$ is the equation of the circle centered in z_0 and of radius r.

① Represent the complex numbers in polar form.

a) $z_1 = 1$, $z_2 = -1$, $z_3 = i$, $z_4 = -i$

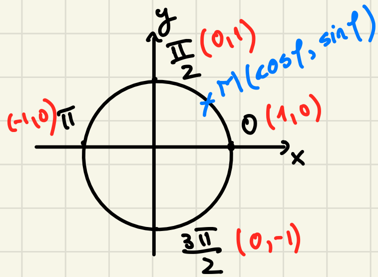
$r_1 = \sqrt{1} = 1 \Rightarrow z_1 = 1\cdot(\cos 0 + i\sin 0)$



$r_2 = 1 \Rightarrow z_2 = 1\cdot(\cos\pi + i\sin\pi)$

$r_3 = 1 \Rightarrow z_3 = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$r_4 = 1 \Rightarrow z_4 = 1\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$



b) $z_5 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$

$r = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$

$\varphi = \arctan\frac{y}{x} + k\pi = \arctan\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} + k\pi = \arctan\frac{\sqrt{3}}{3} = \frac{\pi}{6}$

$z_5 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

c) $z_6 = (-1+i)^{105} \quad \in Q_2$

$r = \sqrt{2}$

$\varphi = \arctan(-1) + k\pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

$z_6 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$(-1+i)^{105} = \sqrt{2}^{105}\left(\cos\frac{315\pi}{4} + i\sin\frac{315\pi}{4}\right)$

$= 2^{\frac{105}{2}}\left[\cos\left(78\pi + \frac{3\pi}{4}\right) + i\sin\left(78\pi + \frac{3\pi}{4}\right)\right]$

$$z_6 = 2^{\frac{105}{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

② . Solve the following equations

a) $z^4 = 1 - i \quad \in Q_4$

$r = \sqrt{2}$

$\varphi = \arctan\frac{y}{x} + k\pi = \arctan(-1) + 2\pi$

$= -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$

$z = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

$z_k = \sqrt[4]{\sqrt{2}}\left(\cos\frac{\frac{7\pi}{4} + 2k\pi}{4} + i\sin\frac{\frac{7\pi}{4} + 2k\pi}{4}\right) \qquad k = \overline{0,3}$

$z_0 = 2^{\frac{1}{8}}\left(\cos\frac{7\pi}{16} + i\sin\frac{7\pi}{16}\right)$

$z_1 = 2^{\frac{1}{8}}\left(\cos\frac{\frac{7\pi}{4} + 2\pi}{4} + i\sin\frac{\frac{7\pi}{4} + 2\pi}{4}\right)$

$= 2^{\frac{1}{8}}\left(\cos\frac{15\pi}{16} + i\sin\frac{15\pi}{16}\right)$

$z_2 = 2^{\frac{1}{8}}\left(\cos\frac{23\pi}{16} + i\sin\frac{23\pi}{16}\right)$

$z_3 = 2^{\frac{1}{8}}\left(\cos\frac{31\pi}{16} + i\sin\frac{31\pi}{16}\right)$

b) $z^6 = 1 \quad \in Q_1$

$r = 1$

$\varphi = 0$

$z = \cos 0 + i\sin 0$

$z_k = \cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6}, \quad k = \overline{0,5}$

$z_0 = \cos 0 + i\sin 0$

$z_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$\vdots$

c) $z^4+(1-i)z^2-i=0$

Subst: $z^2=t$

$$t^2+(1-i)t-i=0$$
$$t^2+t-ti-i=0$$
$$t(t+1)-i(t+1)=0$$
$$(t+1)(t-i)=0$$

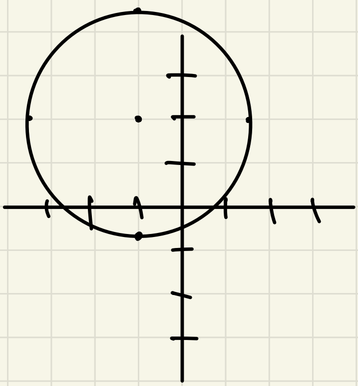
$$\Rightarrow \begin{cases} t_1=-1 \\ t_2=i \end{cases}$$

3 Specify and draw the curves represented by the following equations

a) $|z+1-2i|=3$

$$|z-(-1+2i)|=3$$

the circle centered at $(-1,2)$ and of radius 3



The gen. eq. of circle

$$(x-x_0)^2+(y-y_0)^2=r^2$$

b) $Re\, z^2 = 4$

$$z=x+iy$$
$$z^2=x^2+2ixy-y^2$$
$$Re z^2=x^2-y^2$$
$$x^2-y^2=4 \quad |:4$$
$$\frac{x^2-y^2}{4}=1 \text{ - hypherbola}$$

$a^2=4$

$a=\pm 2$

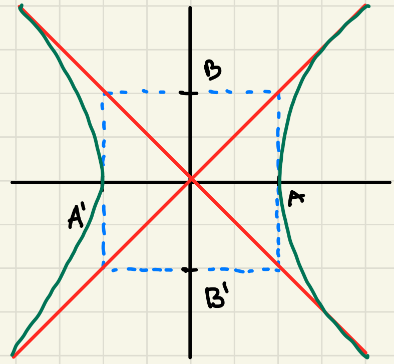
$A(2,0)$

$A'(-2,0)$

$b^2=4 \Rightarrow b=\pm 2$

$B(0,2)$

$B'(0,-2)$



c) $|z+2|+|z-2|=6$

$|x+iy+2|+|x+iy-2|=6$

$\sqrt{(x+2)^2+y^2}+\sqrt{(x-2)^2+y^2}=6$

$\sqrt{(x+2)^2+y^2}=6-\sqrt{(x-2)^2+y^2} \quad \Big|\,(\;)^2$

$(x+2)^2+y^2=36-12\sqrt{(x-2)^2+y^2}+(x-2)^2+y^2$

$\cancel{x^2}+4x+\cancel{4}=36-12\sqrt{(x-2)^2+y^2}+\cancel{x^2}-4x+\cancel{4}$

$12\sqrt{(x-2)^2+y^2}=36-8x$

$3\sqrt{(x-2)^2+y^2}=9-2x$

$9(x-2)^2+9y^2=81-36x+4x^2$

$9x^2-\cancel{36x}+36+9y^2=81-\cancel{36}x+4x^2$

$5x^2+9y^2=45$

$\frac{x^2}{9}+\frac{y^2}{5}=1$

elipse

$A^2 = 9 \Rightarrow A = \pm 3$

$B^2 = 5 \Rightarrow B = \pm\sqrt{5}$

Homework:

1) Represent in the complex plane the numbers:

a) $Re(z-3i+2) = 0$

b) $Re \frac{z-i}{z+i} = 1$

c) $|z-5+2i| = 1$

2) Find the domain given by the inequations

a) $\left|\frac{z}{z+3i}\right| < 1$; b) $\left|\frac{1-z}{1+z}\right| > 3$

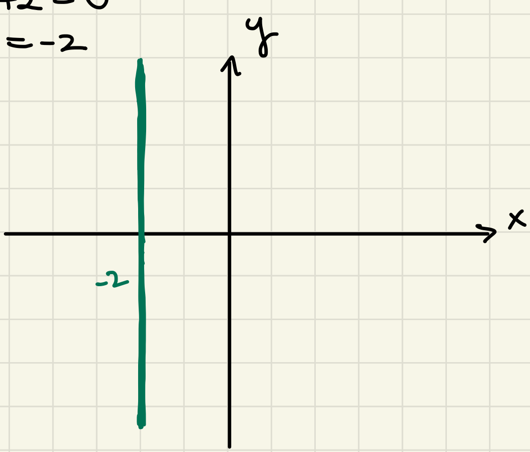
Homework

1) a) $Re(z-3i+2)=0$

$$z-3i+2 = x+iy-3i+2 = x+2+i(y-3)$$

$x+2=0$

$x=-2$

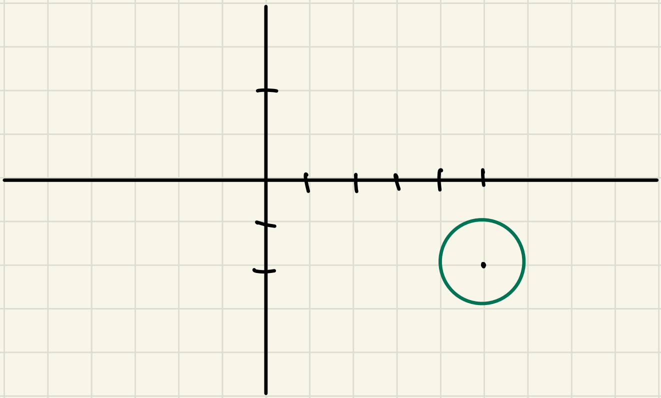


c) $|z-5+2i| = 1$

$|z - z_0| = 1$

$z_0 = 5 - 2i$

circle centered in $A(5,-2)$ and of radius 1



b) $\operatorname{Re} \frac{z-i}{z+i}$

$x - i(y+1)$

$$\frac{x+iy-i}{x+iy+i} = \frac{x+i(y-1)}{x+i(y+1)} = \frac{(x+i(y-1))(x-i(y+1))}{x^2+(y+1)^2}$$

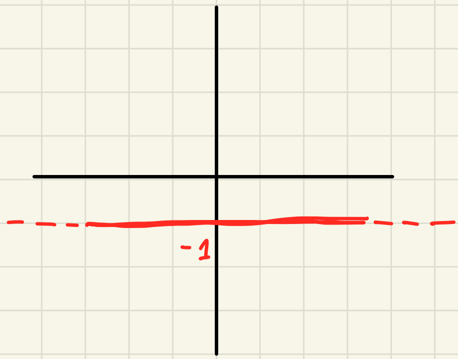
$$\frac{x^2 - xi(y+1) + xi(y-1) + (y-1)(y+1)}{x^2+y^2+2y+1}$$

$$\frac{x^2+y^2-1 - xi((y+1)-(y-1))}{x^2+y^2+2y+1}$$

$$\frac{x^2+y^2-1}{x^2+y^2+2y+1} = 1$$

$$\cancel{x^2}+\cancel{y^2}-1 = \cancel{x^2}+\cancel{y^2}+2y+1$$

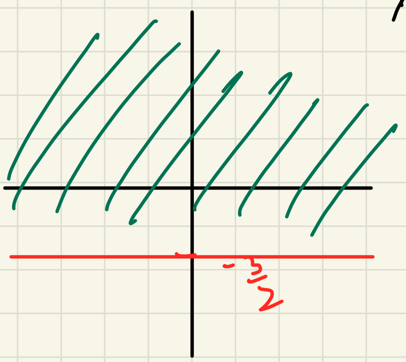
$$2y=-2$$
$$y=-1$$



2 a) $\left|\frac{z}{z+3i}\right| < 1$

$$\frac{|z|}{|z+3i|} < 1 \Rightarrow |z| < |z+3i|$$
$$\sqrt{x^2+y^2} < \sqrt{x^2+(y+3)^2}$$
$$\cancel{x^2}+\cancel{y^2} < \cancel{x^2}+\cancel{y^2}+6y+9$$

$$6y+9>0$$
$$6y>-9$$
$$2y>-3$$
$$y>-\frac{3}{2}$$



2) b) $\left|\frac{1-z}{1+z}\right| > 3$

$$\frac{|1-z|}{|1+z|} > 3$$

$$|1-z| > 3|1+z|$$

$$|1-x-iy| > 3|1+x+iy|$$

$$\sqrt{(1-x)^2+y^2} > 3\sqrt{(1+x)^2+y^2}$$

$$1-2x+x^2+y^2 > 9\left(1+2x+x^2+y^2\right)$$

$$1-2x+x^2+y^2 > 9+18x+9x^2+9y^2$$

$$8y^2+8x^2+20x+8<0 \quad |:8$$

$$y^2+x^2+\frac{20}{8}x+1<0$$

$$x^2+2\cdot\frac{10}{8}x+\left(\frac{10}{8}\right)^2+y^2+1-\left(\frac{10}{8}\right)^2<0$$

$$\left(x+\frac{10}{8}\right)^2+y^2<0$$

Probleme din curs:

1.4 Represent the complex numbers in trigonometric form:

i). $-1-i$

ii). $\frac{\sqrt{3}}{2}+i\frac{1}{2}$

iii). 1

iv). i

v). -1

vi). $-i$

vii). $\cos\frac{\pi}{8}-i\sin\frac{\pi}{8}$

viii). $-\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}$

ix). $(-1+i)^{105}$

i) $z=-1-i \quad \in Q_3$

$r=\sqrt{2}$

$\varphi = \arctan\frac{-1}{-1}+\pi = \arctan 1 + \pi = \frac{\pi}{4}+\pi = \frac{5\pi}{4}$

$z=\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$

ii) $z=\frac{\sqrt{3}}{2}+i\frac{1}{2} \quad \in Q_1$

$r=\sqrt{\frac{3}{4}+\frac{1}{4}}=1$

$\varphi = \arctan\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \arctan\frac{\sqrt{3}}{3}=\frac{\pi}{6}$

$z=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$

iii) $z=1$

$z=\cos 0+i\sin 0$

iv) $z=i$

$z=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

v) $z=-1$

$z=\cos\pi+i\sin\pi$

vi) $z=-i$

$z=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}$

vii) $z = \cos\frac{\pi}{8} - i\sin\frac{\pi}{8} = \cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right) =$

$$= \cos\left(-\frac{\pi}{8} + 2\pi\right) + i\sin\left(-\frac{\pi}{8} + 2\pi\right) = \cos\frac{15\pi}{8} + i\sin\frac{15\pi}{8}$$

viii) $z = -\sin\frac{\pi}{8} - i\cos\frac{\pi}{8}$

ix) $(-1 + i)^{105} \quad \in Q_3$

$$r = \sqrt{2}$$

$$\varphi = \arctan\frac{1}{-1} = \arctan(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$$

$$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$z^{105} = \sqrt{2}^{105}\left(\cos\frac{315\pi}{4} + i\sin\frac{315\pi}{4}\right)$$

$$= \sqrt{2}^{105}\left(\cos\left(78\pi + \frac{3\pi}{4}\right) + i\sin\left(78\pi + \frac{3\pi}{4}\right)\right)$$

$$= \sqrt{2}^{105}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

i). $z^4 = 1 - i$

ii). $z^6 = 1$

iii). $z^4 + (1-i)z^2 - i = 0$

iv). $z^4 - z^3 + z^2 - z + 1 = 0$

i) $z^4 = 1 - i \quad \in Q_4$

$r = \sqrt{2}$

$\varphi = \arctan(-1) + 2\pi = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$

$z_k = \sqrt[4]{\sqrt{2}}\left(\cos\frac{\frac{7\pi}{4} + 2k\pi}{4} + i\sin\frac{\frac{7\pi}{4}+2k\pi}{4}\right), \quad k = \overline{0,3}$

$z_0 = 2^{\frac{1}{8}}\left(\cos\frac{7\pi}{16} + i\sin\frac{7\pi}{16}\right)$

$z_1 = 2^{\frac{1}{8}}\left(\cos\frac{15\pi}{16} + i\sin\frac{15\pi}{16}\right)$

$\vdots$

ii) $z^6 = 1$

$r = 1$

$\varphi = 0$

$z_k = \cos\frac{0+2k\pi}{6} + i\sin\frac{0+2k\pi}{6}, \quad k \in \overline{0,5}$

$z_0 = \cos 0 + i\sin 0$

$z_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$z_3 = \cos\pi + i\sin\pi$

$z_4 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$\vdots$

iii) $z^4 + (1-i)z^2 - i = 0$

$z^2 = t$

$t^2 + (1-i)t - i = 0$

$t^2 + t - ti - i = 0$

$t(t+1) - i(t+1) = 0$

$(t+1)(t-i) = 0$

$\Rightarrow t_1 = -1$ & $t_2 = i$

$t_1 = i \Rightarrow \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$t_2 = -1 \Rightarrow \cos\pi + i\sin\pi$

Seminar 2

22.02.2022

Functions of a complex variable
The Cauchy-Riemann conditions

$f: D \to \mathbb{C}$, $D \subset \mathbb{C}$, $w, z \in \mathbb{C}$

$f(x) = y$

$f(z) = w$

Ex.1 $f(z) = z^2 + 2z - 3\bar{z} + 1$

$z = x + iy$

$w = u + iv$, $u(x,y)$
$v(x,y)$

Find $u = u(x,y)$, $v = v(x,y)$

$f(z) = u(x,y) + iv(x,y)$

$(x+iy)^2 + 2(x+iy) - 3(x-iy) + 1 = u(x,y) + iv(x,y)$

$x^2 + 2ixy - y^2 + 2x + 2iy - 3x + 3iy + 1 = u(x,y) + iv(x,y)$

$x^2 - y^2 - x + 1 + i(2xy + 5y) = u(x,y) + iv(x,y)$

$$\begin{cases} u(x,y) = x^2 - y^2 - x + 1 \\ v(x,y) = 2xy + 5y \end{cases}$$

(T1) Let $f = u + iv$. If f is monogenic at $z_0 = x_0 + iy_0$ then u and v have partial derivatives at (x_0, y_0) and the Cauchy-Rieman condition hold

C-R cond: $\begin{cases} \dfrac{\partial u}{\partial x}(x_0, y_0) = \dfrac{\partial v}{\partial y}(x_0, y_0) \\ \dfrac{\partial u}{\partial y}(x_0, y_0) = -\dfrac{\partial v}{\partial x}(x_0, y_0) \end{cases}$

f is monogenic at z_0 : $\exists \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}$
(f has a derivative at z_0)
f is holomorphic on D if f is monogenic at every point of D

Ex 2 Determine the points where the function f is monogenic

a) $f(z) = 3 \cdot z^2 - 2iz$

$f(z) = u + iv$

$f(z) = 3(x + iy)^2 - 2i(x + iy)$

$= 3x^2 + 6iyx - 3y^2 - 2ix + 2y$

$f(z) = \begin{cases} u(x,y) = 3x^2 - 3y^2 + 2y \\ v(x,y) = 6yx - 2x \end{cases}$

f monogenic $\Leftrightarrow$ $\dfrac{\partial u}{\partial x}(x,y) = \dfrac{\partial v}{\partial y}(x,y) \Rightarrow$

$\dfrac{\partial u}{\partial y}(x,y) = -\dfrac{\partial v}{\partial x}(x,y) \Rightarrow$

$\Rightarrow$ $6x = 6x$ "T"

$\Rightarrow$ $-6y+2 = -(6y-2)$ "T"

f monogenic at z, $\forall z \in \mathbb{C} \Rightarrow f$ holomorphic func.

b) $f(z) = 2 \cdot \operatorname{Im} z - i \operatorname{Re} z$

$f(z) = u(x,y) + iv(x,y)$

$f(z) = 2y - ix$

$$f(z) = \begin{cases} u(x,y) = 2y \\ v(x,y) = -x \end{cases}$$

C.R $\quad \dfrac{\partial u}{\partial x}(x,y) = \dfrac{\partial v}{\partial y}(x,y) \Rightarrow 0 = 0$ "T"

$\dfrac{\partial u}{\partial y}(x,y) = -\dfrac{\partial v}{\partial x}(x,y) \Rightarrow 2 = +1$ "F"

f is nowhere monogenic

c) $f(z) = (z^2 + (\bar{z})^2)(1-i) + 2z\bar{z}(1+i) - 4iz$

$f(z) = u(x,y) + iv(x,y)$

$f(z) = ((x+iy)^2 + (x-iy)^2)(1-i) + 2(x+iy)(x-iy)(1+i) - 4iz$

$f(z) = (x^2 + \cancel{2xiy} - y^2 + x^2 - \cancel{2ixy} - y^2)(1-i) + 2(x^2+y^2)(1+i) - 4iz$

$f(z) = (2x^2 - 2y^2)(1-i) + 2(x^2 + ix^2 + y^2 + iy^2) - 4iz$

$f(z) = 2x^2 - \cancel{2ix^2} - \cancel{2y^2} + 2iy^2 + 2x^2 + \cancel{2ix^2} + \cancel{2y^2} + 2iy^2 - 4iz$

$f(z) = 4x^2 + 4iy^2 - 4i(x+iy)$

$f(z) = 4x^2 + 4iy^2 - 4ix + 4y$

$$f(z) = \begin{cases} u(x,y) = 4x^2 + 4y \\ v(x,y) = 4y^2 - 4x \end{cases}$$

C.R $\quad \frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \Rightarrow 8x = 8y$

$\frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) \Rightarrow 4 = -(-4)$ "T"

f is monogenic $\Leftrightarrow x = y$

Ex. 3 Can the following function be the real part of a holomorphic function $f = u + iv$?

$u(x,y) = \arctan \frac{y}{x}$

If $\Delta u(x,y) = 0 \Rightarrow u$ is harmonic function

$\Delta u(x,y) = 0$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=0 \Rightarrow$$

$$\frac{\partial}{\partial x}\left(\frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}}\right)+\frac{\partial}{\partial y}\left(\frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}\right)=0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(-\frac{y}{x^2}\cdot\frac{x^2}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x}\cdot\frac{x^2}{x^2+y^2}\right)=0 \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)=0 \Rightarrow \frac{2yx}{(x^2+y^2)^2}-\frac{2yx}{(x^2+y^2)}=0$$

"T"

Ex 4 Find (Reconstruct) the holomorphic function $f=u+iv$ from its known real part $u(x,y)$ or imaginary part $v(x,y)$ and the value $f(z_0)$:

a) $u(x,y)=3xy$, $f(0)=i$

b) $v(x,y)=\frac{y}{x^2+y^2}$, $f(1)=1$

a) $u(x,y)=3xy$, f holomorphic C-R conditions hold

$$\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{cases} \Leftrightarrow \begin{cases}3y=\frac{\partial v}{\partial y} & \Big| \int dy\\ 3x=-\frac{\partial v}{\partial x} & \Big| \int dx\end{cases}$$

$\Rightarrow \dfrac{3y^2}{2} + \varphi(x) = v(x,y)$

$\dfrac{\partial v}{\partial x} = 0 + \varphi'(x)$

I now (2) we get $: 3x = -\varphi'(x) \Leftrightarrow \varphi'(x) = -3x \mid \int dx \Leftrightarrow$

$\varphi(x) = -\dfrac{3x^2}{2} + c$

$\Rightarrow v(x,y) = -\dfrac{3x^2}{2} + \dfrac{3y^2}{2} + c$

$f(x,y) = u(x,y) + i v(x,y)$

$f(x,y) = 3xy + i\left(\dfrac{3y^2}{2} - \dfrac{3x^2}{2} + c\right)$

$f(0,0) = i$

$f(0,0) = 0 + i(0+0+c) = i$

$ic = i \Rightarrow c = 1$

$f(x,y) = 3xy + i\left(\dfrac{3y^2}{2} - \dfrac{3x^2}{2} + 1\right)$

We need $f(z)$

I $f(z) = 3xy - \dfrac{3}{2} i (x^2 - y^2) + i$

$= \dfrac{3}{2} i (x^2 - y^2 + 2xyi) + i$

$= -\dfrac{3}{2} i (x + iy)^2 + i \Rightarrow$

$f(z) = -\frac{3}{2} i z^2 + i$

$$\text{II} \quad f(z) \underset{\substack{x=z \\ y=0}}{=\!=} 3\cdot 0 + i\left(-\frac{3}{2}z^2 + 0\right) + i = -\frac{3}{2} i z^2 + i$$

Remark : $f(z) = u(z;0) + i v(z;0)$

b) $v(x,y) = \dfrac{y}{x^2+y^2}$, $f(1) = 1$

$$\left(\frac{y}{x^2+y^2}\right)' = \frac{y'(x^2+y^2) - (x^2+y^2)' y}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

f - holomorphic $\Rightarrow$ CR conditions hold $\Rightarrow$

$$\left.\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= \frac{\partial v}{\partial y}(x,y) \\ \frac{\partial u}{\partial y}(x,y) &= -\frac{\partial v}{\partial x}(x,y) \end{aligned}\right\} \Rightarrow \frac{\partial u}{\partial x}(x,y) = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x}(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2} \Bigg| \int dy$$

$$\Leftrightarrow u(x,y) = \int \frac{2xy}{(x^2+y^2)}\, dy = \frac{-x}{x^2+y^2} + \varphi(x)$$

$$\frac{\partial u}{\partial x} = \frac{-x^2-y^2+2x^2}{(x^2+y^2)^2} + \varphi'(x) \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial u}{\partial x} = \frac{x^2-y^2}{(x^2+y^2)^2} + \varphi'(x)$$

From (1) we obtain $\Rightarrow$

$$\frac{x^2-y^2}{(x^2-y^2)^2} + \varphi'(x) = \frac{x^2-y^2}{(x^2+y^2)^2} \Leftrightarrow \varphi'(x) = 0 \;\Big|\int dx$$

$\varphi(x) = c$, $c \in \mathbb{R}$

$$u(x,y) = \frac{-x}{x^2+y^2} + c$$

$$f(x,y) = \frac{-x}{x^2+y^2} + i\,\frac{y}{x^2+y^2} + c$$

$$f(1,0) = \frac{-1}{1+0} + i\cdot\frac{0}{1+0} + c$$

$f(1,0) = -1+c$, $f(1,0) = 1$

$c = 2$

$$f(x,y) = -\frac{x}{x^2+y^2} + i\,\frac{y}{x^2+y^2} + 2$$

$$f(z) = \frac{-z}{z^2+0} + i\cdot\frac{0}{z^2+0} + 2, \quad f(z) = \frac{-z}{z^2} + 2$$

$$f(z) = -\frac{1}{z} + 2$$

Ex 5 Find a holomorphic function $f = u + iv$ such that:

$$u(x,y) - v(x,y) = x^2 - y^2 - 2xy - 2x - 2y \quad \text{and} \quad f(i) = 3$$

f holm $\Rightarrow$ the (C-R) cond hold.

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial x} = 2x - 2y - 2 \\[2ex] \dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial y} = -2y - 2x - 2 \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2x - 2y - 2$$

$$-\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -2y - 2x - 2$$

$\oplus$

$$/ + 2\frac{\partial u}{\partial y} = -4y - 4$$

$$\frac{\partial u}{\partial y} = -2y - 2$$

From the system

$$\frac{\partial u}{\partial x} = 2x - \cancel{2y} - \cancel{2} + \cancel{2y} + \cancel{2}$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = 2x \quad \Big| \int dx \\ \dfrac{\partial u}{\partial y} = -2y-2 \end{cases} \implies \varphi'(y) = -2y-2 \,\Big|\int$$

$$\varphi(y) = -y^2 - 2y + c$$

$$\Rightarrow u(x,y) = 2\frac{x^2}{2} + \varphi(y)$$

$$\frac{\partial u}{\partial y} = \varphi'(y)$$

$$u(x,y) = x^2 - y^2 - 2y + c$$

From the initial condition we get $v(x,y)$ .... then $f(x,y)$

Homework: 1.12, 1.13, 1.14, 1.15 iii), iv), vi)

Chestii in Plus

f holomorphică $\Rightarrow$ Are derivate parțiale pe tot domeniu

1.2.1 Culegere

Let $w = f(z) = z^3 + i\bar{z} - 1$. Denoting $z = x+iy$ and $w = u+iv$ we find that

$$u + iv = (x+iy)^3 + i(x-iy) - 1 = x^3 - 3xy^2 + y - 1 + i(3x^2y - y^3 + x)$$

$$\Rightarrow u = x^3 - 3xy^2 + y - 1, \quad v = 3x^2y - y^3 + x$$

1.2.2. Culegere

Let $f(z) = z^2 - iz$, $z \in \mathbb{C}$. Then $f'(z) = 2z - i \Rightarrow f$ holomorphic in $\mathbb{C}$

$$f = u + iv = (x+iy)^2 - i(x+iy) = x^2 + 2xiy - y^2 - ix + y$$
$$= x^2 - y^2 + y + i(2xy - x)$$

$$u(x,y) = x^2 - y^2 + y$$
$$v(x,y) = 2xy - x$$

Let's verify the C-R cond.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2x = 2x \quad \text{"T"}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -2y+1 = -(2y-1) \quad \text{"T"}$$

1.2.3 Culegere

Let $f(z) = z\bar{z}$, Then $u+iv = (x+iy)(x-iy) = x^2+y^2$

$$u(x,y) = x^2+y^2$$
$$v(x,y) = 0$$

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \Rightarrow 2x = 0 \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \Rightarrow 2y = 0 \end{array}\right| \Rightarrow z=0 \text{ ca } f \text{ să fie monogenă}$$

1.2.4 Culegere

Let $f(z) = \bar{z}$. $u+iv = x-iy$

$$\Rightarrow u(x,y) = x$$
$$v(x,y) = -y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 1 = -1 \quad \text{"F"}$$ f is monogenic nowhere

Remark 1.5 $w = u(x,y) + iv(x,y)$ can be written in the form $w = f(z)$ noticing that $f(z) = u(z,0) + iv(z,0)$

1.8 $f = u + iv$, $u(x,y) = x^3 + 6x^2y - 3xy^2 - 2y^3$, $f(0) = 0$

$$\frac{\partial u}{\partial x} = 3x^2 + 12xy - 3y^2$$

hence, $\frac{\partial v}{\partial y} = 3x^2 + 12xy - 3y^2$

$$v(x,y) = 3x^2y + 6xy^2 - y^3 + p(x)$$

where $p(x)$ is yet to be found. From $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ we get

$$6x^2 - 6xy - 6y^2 = -6xy - 6y^2 - p'(x)$$

hence $p'(x) = -6x^2$, or $p(x) = -2x^3 + C$. Now

$$f = \underbrace{x^3 + 6x^2y - 3xy^2 - 2y^3}_{u(x,y)} + i(\underbrace{3x^2y + 6xy^2 - y^3 - 2x^3}_{v(x,y)}) + Ci$$

From $f(0) = 0$ we deduce that $C = 0$. Finally

$$f = (x^3 + 3x^2yi - 3xy^2 - y^3i)(1-2i) = (1-2i)(x+iy)^3$$
$$= (1-2i)z^3$$

Homework

1.12

a) $f(z) = ze^z$, $f(z) = u + iv$, $f(z) = (x+iy)e^{(x+iy)}$

$$f(z) = (x+iy)e^{x+iy} = (x+iy)(e^x \cdot e^{iy}) = xe^xe^{iy} + iye^xe^{iy}$$

b) $f(z) = z^2 + 3iz = (x+iy)^2 + 3i(x+iy) = x^2 + 2xiy - y^2 + 3ix - 3y$

$$u(x,y) = x^2 - y^2 - 3y$$
$$v(x,y) = 2xy + 3x$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2x = 2x \quad \text{"T"}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -2y-3 = -(2y+3) \quad \text{"T"}$$

f is monogenic in every point $\Rightarrow f$ holomorphic in $\mathbb{C}$

c) $f(z) = \bar{z}\,\mathrm{Im}\,z = (x-iy)y = xy - iy^2$

$$u(x,y) = xy$$
$$v(x,y) = -y^2$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow y = -2y$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow x = 0$$

f is monogenic for $y = 0$

d) $f(z) = |z|\,\mathrm{Re}\,z = \sqrt{x^2+y^2}\,x$

$$\Rightarrow u(x,y) = \sqrt{x^2+y^2}\,x$$
$$v(x,y) = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \sqrt{x^4+x^2y^2}' = \frac{1}{2\sqrt{x^4+x^2y^2}} \cdot (x^4+x^2y^2)'$$

$$= \frac{2x^3+xy^2}{\sqrt{x^4+x^2y^2}} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$$

f) $f(z) = \operatorname{Im} z + 2i \operatorname{Re} z$

$f(z) = y + 2ix$

$u(x,y) = y$
$v(x,y) = 2x$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 0 = 0$ "T"

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 1 = -2$ "F"

f is monogenic nowhere

g) $f(z) = (z^2 + \bar{z}^2)(1-i) + 2z\bar{z}(1+i) - 4iz$

$= \left((x+iy)^2 + (x-iy)^2\right) + 2(x+iy)(x-iy)(1+i) - 4i(x+iy)$

$= \left(x^2 + \cancel{2xyi} - y^2 + x^2 - \cancel{2xiy} - y^2\right) + 2(x^2+y^2)(1+i) - 4i(x+iy)$

$= (2x^2 - 2y^2) + 2(x^2+y^2)(1+i) - 4i(x+iy)$

$= ($

e) $f(z) = \frac{1}{z-1}$, $z \neq 1$

$$f(z) = \frac{1}{z-1} = \frac{1}{x+iy-1} = \frac{1}{x-1+iy} = \frac{x-1-iy}{(x-1)^2-(iy)^2} =$$

$$= \frac{x-1-iy}{x^2-2x+1+y^2} \quad , \quad u(x,y) = \frac{x-1}{x^2+2x+1+y^2}$$

$$v(x,y) = \frac{-y}{x^2-2x+1+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

SEMINAR 3

01.03.2022

Elementary functions of a complex variable

The exponential function

$$e^{z} = e^{x+iy} = e^{x}(\cos y + i\sin y)$$

$$e^{x-iy} = e^{x}(\cos y - i\sin y)$$

$$e^{iy} = \cos y + i\sin y$$

$$e^{-iy} = \cos y - i\sin y$$

The trigonometric functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}, \quad \sinh z = \frac{e^{z} - e^{-z}}{2}$$

1 a) $\sin a \cdot \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$

$$\sin a \cos b = \frac{e^{ia} - e^{-ia}}{2i} \cdot \frac{e^{ib} + e^{-ib}}{2} = \frac{1}{4i}\left[e^{i(a+b)} + e^{i(a-b)} - e^{-i(a-b)} - e^{-i(a+b)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{i(a+b)} - e^{-i(a+b)}}{2i} + \frac{e^{i(a-b)} - e^{-i(a-b)}}{2i}\right]$$

$$= \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

b) $(\cosh z)^2 - (\sinh z)^2 = 1$

$$\left(\frac{e^{z}+e^{-z}}{2}\right)^2 - \left(\frac{e^{z}-e^{-z}}{2}\right) = 1$$

$$= \frac{e^{2z}+2+e^{-2z}}{4} - \frac{e^{2z}-2+e^{-2z}}{4}$$

$$= \frac{4}{4} = 1 \quad \text{"T"}$$

The logarithmic function

$$z = e^{w}, \quad w, z \in \mathbb{C}, \quad z \neq 0$$

$$w = \log z = \{ \ln|z| + i(\arg z + 2k\pi), \ k \in \mathbb{Z} \}$$

The general power formula

$$z^{\alpha} = e^{\alpha \operatorname{Log} z}$$

$$z, \alpha \in \mathbb{C}, \quad z \neq 0$$

2 Write in algebraic form the following numbers:

a) $\cos(\pi + i)$, b) $\operatorname{Log}(1-i)$, c) i^{i}, d) $\operatorname{Log}(1+i\sqrt{3})$

a) $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$

$$\cos(\pi+i) = \frac{e^{i(\pi+i)} + e^{-i(\pi+i)}}{2} = \frac{e^{-1+i\pi} + e^{1-i\pi}}{2}$$

$$= \frac{e^{-1}\cdot(\cos\pi + i\sin\pi) + e^{1}(\cos\pi - i\sin\pi)}{2} = \frac{e^{-1}\cdot(-1) + e\cdot(-1)}{2}$$

$$= -\frac{e^{1}+e^{-1}}{2} = -\cosh 1$$

b) $\operatorname{Log}(1-i) = \{ \ln|1-i| + i(\arg(1-i) + 2k\pi), k \in \mathbb{Z} \}$

$\sqrt{1+1} = \sqrt{2}$

$\arg(1-i) \overset{Q_4}{=} \arctan\frac{-1}{1} + 2\pi = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$

$\operatorname{Log}(1-i) = \{ \ln\sqrt{2} + i(\frac{7\pi}{4} + 2k\pi), k \in \mathbb{Z} \}$

c) $i^{i} = e^{i \operatorname{Log} i}$

$\operatorname{Log} i = \{ \ln|i| + i(\arg i + 2k\pi), k \in \mathbb{Z} \}$

$\operatorname{Log} i = \{ \ln 1 + i(\frac{\pi}{2} + 2k\pi), k \in \mathbb{Z} \}$

$\operatorname{Log} i = \{ i(\frac{\pi}{2} + 2k\pi), k \in \mathbb{Z} \}$

$i^{i} = e^{i^{2}(\frac{\pi}{2} + 2k\pi)} = e^{-(\frac{\pi}{2} + 2k\pi)}$

d) $\mathrm{Log}(1+i\sqrt{3}) = \{\ln|1+i\sqrt{3}| + i(\arg(1+i\sqrt{3})+2k\pi), k\in\mathbb{Z}\}$

$|1+i\sqrt{3}| = \sqrt{1+3} = 2$

$1+i\sqrt{3}$

e) $1^i = e^{i\,\mathrm{Log}\,1} = e^{i i 2k\pi} = e^{-2k\pi}, k\in\mathbb{Z}$

$\mathrm{Log}\,1 = \{\ln 1 + i(0+2k\pi), k\in\mathbb{Z}\}$

f) $\mathrm{Log}(-1) = \{\ln 1 + i(\pi+2k\pi), k\in\mathbb{Z}\}$

$\mathrm{Log}(-1) = i(\pi+2k\pi), k\in\mathbb{Z}$

g) $\sin(1-i) = \dfrac{e^{iz}-e^{-iz}}{2i} = \dfrac{e^{i(1-i)}-e^{-i(1-i)}}{2i} = \dfrac{e^{i+1}-e^{-i-1}}{2i}$

$= \dfrac{e^{i+1}-e^{-(i+1)}}{2i} = \dfrac{e^{1}(\cos 1 + i\sin 1) - e^{-1}(\cos 1 - i\sin 1)}{2i}$

$= -i\dfrac{\cos 1(e^{1}-e^{-1}) + i\sin 1(e^{1}+e^{-1})}{2}$

$= -i(\cos 1 \sinh 1 + i\sin 1 \cosh 1)$

$= -i\cos 1 \sinh 1 + \sin 1 \cosh 1$

$= \sin 1 \cosh 1 - i\cos 1 \sinh 1$

EXAMEN

③ Solve the following equations

a) $\sin z = \frac{5}{3}$

$$\left.\begin{array}{l} \dfrac{e^{iz} - e^{-iz}}{2i} = \dfrac{5}{3} \\ e^{iz} = t \end{array}\right\} \Rightarrow \frac{t - \frac{1}{t}}{2i} = \frac{5}{3}$$

$$3t - \frac{3}{t} = 10i \quad \Big| \cdot t \neq 0$$

$$3t^2 - 10ti - 3 = 0$$

$$\Delta = 100i^2 + 36 = -64$$

$$t_{1,2} = \frac{10i \pm 8i}{6}$$

$$t_1 = 3i$$

$$t_2 = \frac{i}{3}$$

$$e^{iz} = 3i \,\Big|\, Log$$

$$iz = Log(3i)$$

$$iz = \ln|3i| + i(\arg(3i) + 2k\pi), \quad k \in \mathbb{Z}$$

$$iz = \ln 3 + i\left(\frac{\pi}{2} + 2k\pi\right) \Big| \cdot (-i)$$

$$z = \frac{\pi}{2} + 2k\pi - i\ln 3, \quad k \in \mathbb{Z}$$

$$e^{iz} = \frac{i}{3} \quad \Big| \text{Log}$$

$$iz = \text{Log}\left(\frac{i}{3}\right)$$

$$iz = \left\{ \ln\left|\frac{i}{3}\right| + i\left(\arg\left(\frac{i}{3}\right) + 2k\pi\right),\ k \in \mathbb{Z} \right\}$$

$$iz = \left\{ \ln\frac{1}{3} + i\left(\frac{\pi}{2} + 2k\pi\right),\ k \in \mathbb{Z} \right\} \quad \Big| \cdot(-i)$$

$$z_2 = \frac{\pi}{2} + 2k\pi + i\ln 3,\ k \in \mathbb{Z}$$

b) $\cosh z = \frac{1}{2}$

$$\left.\begin{aligned} \cosh z &= \frac{e^{z} + e^{-z}}{2} = \frac{1}{2} \\ e^{z} &= t \end{aligned}\right\} \Rightarrow \frac{t + \frac{1}{t}}{2} = \frac{1}{2}$$

$$2t + \frac{2}{t} = 2$$

$$2t^2 - 2t + 2 = 0$$

$$t^2 - t + 1 = 0$$

$$\Delta = -3$$

$$t_{1,2} = \frac{1 \pm i\sqrt{3}}{2},$$

I $\quad e^{z} = \frac{1}{2} + i\frac{\sqrt{3}}{2} \quad \Big| \text{Log}$

$$z = \mathrm{Log}\Big(\underbrace{\tfrac{1}{2} + \tfrac{i\sqrt{3}}{2}}_{w}\Big) = \Big\{ \ln\Big|\tfrac{1}{2} + \tfrac{i\sqrt{3}}{2}\Big| + i\Big(\arg\Big(\tfrac{1}{2} + \tfrac{i\sqrt{3}}{2}\Big) + 2k\pi\Big)\Big\}$$

$$z = \Big\{ \ln 1 + i\Big(\tfrac{\pi}{3} + 2k\pi\Big),\ k \in \mathbb{Z} \Big\}$$

$$\arg(w) = \arctan\sqrt{3} = \tfrac{\pi}{3}$$

II $e^{z} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

$$z = \mathrm{Log}\Big(\underbrace{\tfrac{1}{2} - \tfrac{i\sqrt{3}}{2}}_{w}\Big)$$

$$z = \Big\{ \ln 1 + i\Big(\tfrac{5\pi}{3} + 2k\pi\Big),\ k \in \mathbb{Z} \Big\}$$

$$\arg(w) = -\tfrac{\pi}{3} + 2\pi = \tfrac{5\pi}{3}$$

$$z = z_1 \cup z_2$$

c) $\cot z = 2+i$

$$\frac{\cos z}{\sin z} = 2+i$$

$$\frac{e^{iz} + e^{-iz}}{\frac{e^{iz} - e^{-iz}}{i}} = 2+i$$

$$\frac{i(e^{iz} + e^{-iz})}{e^{iz} + e^{-iz}}$$

$$e^{iz} = t$$

$$\frac{i\left(t+\frac{1}{t}\right)}{t-\frac{1}{t}}=2+i$$

$$i\left(t+\frac{1}{t}\right)=(2+i)\left(t-\frac{1}{t}\right)$$

$$\cancel{t}i+\frac{i}{t}=2t-\frac{2}{t}+i\cancel{t}-\frac{i}{t}\quad\Big|\cdot t\neq 0$$

$$i=2t^2-2-i$$

$$\Leftrightarrow 2t^2=2i+2\quad\Leftrightarrow t^2=1+i$$

$$1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$t_k=\sqrt[4]{2}\cdot\left(\cos\frac{\frac{\pi}{4}+2k\pi}{2}+i\sin\frac{\frac{\pi}{4}+2k\pi}{2}\right)$$

$$t_0=\sqrt[4]{2}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right);\ t_1=\sqrt[4]{2}\left(\cos\frac{9\pi}{2}+i\sin\frac{9\pi}{2}\right)$$

$$e^{iz}=t_0\Rightarrow iz=\operatorname{Log}t_0$$

$$\Rightarrow iz_1=\left\{\ln(\sqrt[4]{2})+i\left(\frac{\pi}{8}+2k\pi\right),\ k\in\mathbb{Z}\right\}\quad\Big|\,(-i)$$

$$\Rightarrow z_1=\left\{\frac{\pi}{8}+2k\pi-\frac{i}{4}\ln 2,\ k\in\mathbb{Z}\right\}$$

$$\Rightarrow iz_2=\left\{\ln(\sqrt[4]{2})+i\left(\frac{9\pi}{2}+2k\pi\right),\ k\in\mathbb{Z}\right\}\quad\Big|\,(-i)$$

$$\Rightarrow z_2 = \left\{ \frac{9\pi}{2} + 2k\pi + \frac{i}{4}\ln 2 \right\}, k \in \mathbb{Z}$$

$$z = z_1 \cup z_2$$

4 Find the points from the complex plane where the function $\cos z$ takes real values

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} =$$

$$= \frac{e^{-y+ix} + e^{y-ix}}{2} = \frac{e^{-y}((\cos x + i\sin x) + e^{y}(\cos x + i\sin x))}{2}$$

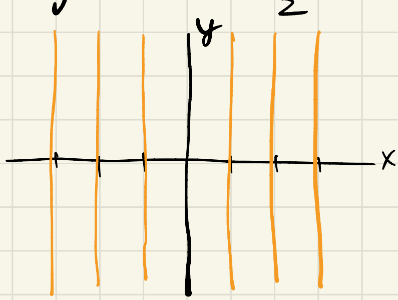
$$= \frac{\cos x}{2}\left(e^{-y} + e^{y}\right) + \frac{i\sin}{2}\left(e^{-y} + e^{y}\right)$$

$$= \cos x \cdot \cosh y - i\sin x \cdot \sinh y$$

$$\Rightarrow \sin x \sinh y = 0$$

1) $\sin x = 0 \Rightarrow x = k\pi, k \in \mathbb{Z}$

2) $\sinh y = 0 \Rightarrow \frac{e^{y} - e^{-y}}{2} = 0 \Rightarrow e^{y} = e^{-y} \Rightarrow y = 0$



SEMINAR 4

08.03.2022

Complex integrales

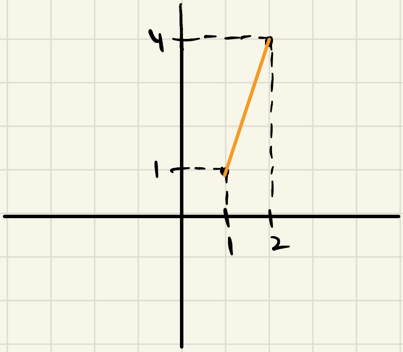
$$f: D \to \mathbb{C}, \quad C \subset D$$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy) \Longrightarrow$$

$$f(z) = u(x,y) + iv(x,y)$$
$$z = x+iy \Rightarrow dz = dx + idy$$

$$\Longleftarrow \int_{(x_1,y_1)}^{(x_2,y_2)} u\,dx - v\,dy + i\int_{(x_1,y_1)}^{(x_2,y_2)} v\,dx + u\,dy$$

1) $\displaystyle\int_{1+i}^{2+4i} z^2\,dz =$ — along the straight line joining the points $1+i$ and $2+4i$



$$= \int_{1+i}^{2+4i} (x+iy)^2(dx+idy) =$$

$$= \int_{(1,1)}^{(2,4)} (x^2-y^2+2xyi)(dx+idy) =$$

$$= \int_{(1,1)}^{(2,4)} (x^2-y^2)\,dx - 2xy\,dy + i\int_{(1,1)}^{(2,4)} 2xy\,dx + (x^2-y^2)\,dy$$

the eq. of the line : $\frac{x-1}{2-1} = \frac{y-1}{4-1} \Rightarrow x-1 = \frac{y-1}{3}$

$\Rightarrow y = 3x - 2$

$dy = 3dx$

$$I = \int_1^2 \left(x^2 - 9x^2 + 12x - 4 - 2x(3x-2)\cdot 3\right) dx + i \int_1^2 2x(3x-2) + 3x^2 - 27x^2 + 36x - 12 \,) dx$$

$$= \int_1^2 -8x^2 + 12x - 4 - 18x^2 + 12x \,) dx + i \int_1^2 6x^2 - 4x - 24x^2 + 36x - 12 \,) dx$$

$$= \int_1^2 \left(-26x^2 + 24x - 4\right) dx + i \int_1^2 \left(-18x^2 + 32x - 12\right) dx$$

$$= -\frac{26x^3}{3}\Big|_1^2 + \frac{24x^2}{2}\Big|_1^2 - 4x\Big|_1^2 + i\left(-\frac{18x^3}{3}\Big|_1^2 + \frac{32x^2}{2}\Big|_1^2 - 12x\Big|_1^2\right.$$

$$= -\frac{26}{3}(8-1) + 12(4-1) - 4(2-1) + i\left(-\frac{18}{3}(8-1) + 16(4-1) - 12(2-1)\right.$$

$\vdots$

$$= -\frac{86}{3} - 6i$$

1) $$\int_{1+i}^{2+4i} z^2 dz = \frac{z^3}{3}\Big|_{1+i}^{2+4i} = -\frac{86}{3} - 6i$$

The line integrals are independent of the path of integration

2) $$I = \int_C (z-a)^n dz$$

$C: \underbrace{|z-a| = r}$ the circle centered at a, of radius r

$$z = a + re^{i\theta} d\theta$$

$$a + r(\cos\theta + i\sin\theta)$$

$$dz = ire^{i\theta} d\theta$$

$$I = \int_0^{2\pi} r^n e^{in\theta} \cdot i \cdot r \cdot e^{i\cdot\theta} d\theta = r^{n+1} \cdot i \int_0^{2\pi} e^{i\theta(n+1)} d\theta$$

i) $n = -1 \Rightarrow I = i \cdot \theta \Big|_0^{2\pi} = i \cdot 2\pi$

ii) $n \neq -1 \Rightarrow I = r^{(n+1)} \cdot i \cdot \frac{1}{i(n+1)} e^{i\theta(n+1)} \Big|_0^{2\pi} = \frac{r^{n+1}}{n+1} \cdot (e^{i\cdot 2\pi(n+1)} - 1) = 0$

$$e^{i\cdot 2\pi(n+1)} = \underbrace{\cos 2\pi(n+1)}_{1} + i\cdot \underbrace{\sin 2\pi(n+1)}_{0}$$

$$I = \begin{cases} 2\pi i, & n = -1 \\ 0, & n \neq -1 \end{cases}$$

Remark: If the path of integration is a circle then we have to make the change of variables related to the parametric eq. of the circle

the result does not depend on the radius r or the center of the circle

Cauchy's integral formula

C curve, $C \subset D$
simple, closed, positively oriented w.r.t. it's interior
$G \subset C$, $G = \text{int } C$, $f: D \to \mathbb{C}$ holomorphic funct.
$z_0 \in G$

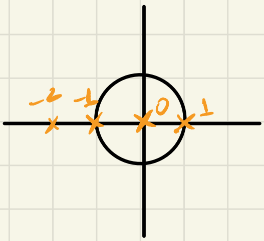
then $\int_C f(z)\,dz = 0$ $\quad \int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

1) $\int_C \frac{e^z}{z^2+2z}\,dz$ $\qquad C: |z| = 1$

$z^2 + 2z = 0$
$z(z+2) = 0$
$z_1 = 0 \in \text{int } C$
$z_2 = -2 \notin \text{int } C$

circle of center 0 of radius 1

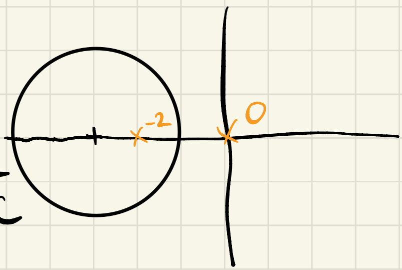


$$I = \int_C \frac{\frac{e^z}{z+2}}{z}\, dz = 2\pi i f(0) = 2\pi i \cdot \frac{e^0}{2} = \pi i$$

$f(z) = \frac{e^z}{z+2}$ holom on C

2) $\int_C \frac{e^z}{z^2+2z}\, dz$ $\quad C: |z+3| = 2$

$$I = \int \frac{e^z}{z(z+2)}\, dz \Rightarrow$$



$z(z+2) = 0 \Rightarrow z_1 = 0 \notin \text{int } C$

$z_2 = -2 \in \text{int } C$

$|0+3| < 2$ F

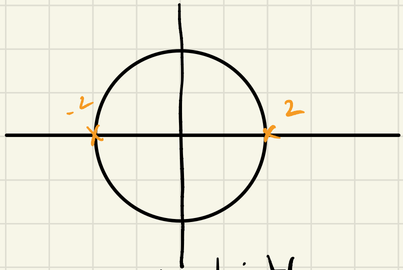
$|-2+3| < 2$ T

$$\Rightarrow \int \frac{\frac{e^z}{z}}{z+2}\, dz = 2\pi i f(-2)$$

$f(z) = \frac{e^z}{z}$ holom.

$$I = 2\pi i \cdot \frac{e^{-2}}{-2} = -\frac{\pi i}{e^2}$$

3) $I = \int_C \frac{\sin z}{z^2(z-4)}\, dz$ $\quad C: |z| = 2$



$z^2(z-4)=0 \rightarrow z_1 = 4 \notin \text{int}C$

$z_2 = 0 \in \text{int}C$

$$\int_C \frac{\frac{\sin z}{z-4}}{z^2} = \frac{2\pi i}{1!} \cdot f'(0)$$

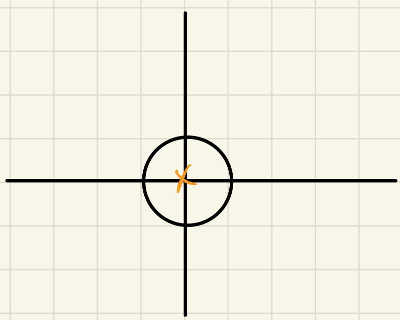
$f(z)$

$n+1=2 \Rightarrow n=1$

$$f(z) = \left(\frac{\sin z}{z-4}\right)' = \frac{\cos z \cdot (z-4) - \sin z}{(z-4)^2} =$$

$$f'(0) = \frac{-4}{16} = -\frac{1}{4}$$

4) $I = \int_C \frac{z+1}{z^4+2iz^3} dz$ $\qquad C: |z| = 1$



$z^3(z+2i)=0$

$z^3=0 \Rightarrow z_1=0,\ z_2=z_3 \in \text{int}C$

$z+2i=0$

$z_4=-2i \notin \text{int}C$

$$I = \int_C \frac{\frac{z+1}{z+2i}}{z^3}\, dz = \pi i \, f''(0)$$

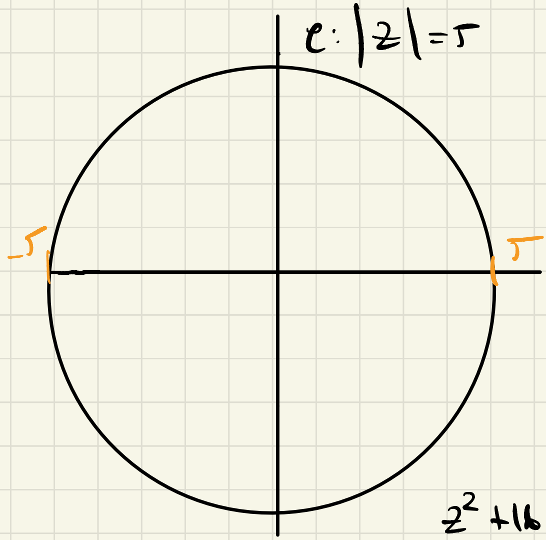
$$n+1 = 3 \Rightarrow n = 2$$

$$f(z) = \frac{z+1}{z+2i}$$

$$f'(z) = \frac{z+2i-z-1}{(z+2i)^2} = \frac{2i-1}{(z+2i)^2}$$

$$f''(z) = \frac{-2(z+2i)(2i-1)}{(z+2i)^4} = \frac{-4i(2i-1)}{(2i)^4} = \frac{8+4i}{16} = \frac{2+i}{4}$$

5) $$\int_C \frac{dz}{z^2+16}$$



$$z^2+16=0$$
$$z^2=-16$$
$$\Rightarrow z_1, z_2 = \pm 4i \in \operatorname{int} C$$

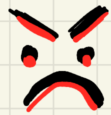
$$\frac{1}{z^2+16} = \frac{1}{(z+4i)(z-4i)} =$$

$$= \frac{z+4i-(z-4i)}{(z+4i)(z-4i)} \cdot \frac{1}{8i}$$

$$= \frac{1}{8i}\left(\frac{1}{z-4i} - \frac{1}{z+4i}\right)$$

$$= \int_C \frac{1}{8i}\left(\frac{1}{z-4i} - \frac{1}{z+4i}\right) dz = \frac{1}{8i}\left[\int_C \frac{\overset{=f(z)}{1}}{z-4i}\,dz - \int_C \frac{\overset{=f(z)}{1}}{z+4i}\,dz\right] \Rightarrow$$

$$= \frac{1}{8i}\left(2\pi i \cdot \underbrace{f(4i)}_{=1} - 2\pi i\, \underbrace{f(-4i)}_{=1}\right) = 0$$



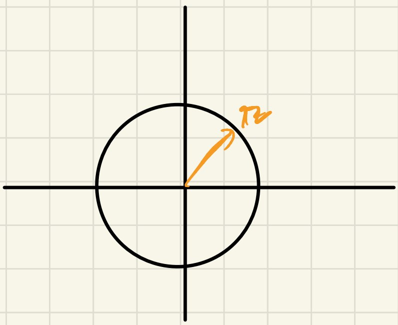
6) $\displaystyle\int_C \frac{\cosh^2(iz)}{z^3}\,dz$

$C: |z| = r \quad , r > 0$

$(\cosh z)' = \sinh z$

$(\sinh z)' = \cosh z$

$z^3 = 0 \quad , z_{1,2,3} = 0$



$$I = \frac{2\pi i}{2!} f^{(2)}(0)$$

$3 = n+1 \Rightarrow n = 2$

$f(z) = \cosh^2(iz)$

$f'(z) = 2i\cosh(iz)\sinh(iz)$

$$f^{(2)}(z) = 2i\left[i\sinh^2(iz) + i\cosh^2(iz)\right]$$

$$f^{(2)}(z) = 2i^2\left[\sinh^2(0) + i\cosh^2(0)\right]$$

$$= -2\left[\left(\frac{e^0 + e^{-0}}{2}\right)^2 + \left(\frac{e^0 + e^{-0}}{2}\right)\right]$$

$$= -2$$

$$I = \frac{2\pi i}{\cancel{2}}(-\cancel{2}) = -2\pi i$$

Temă: 1.35 Se dau la examen

SEMINAR 5

15.03.2022

Power Series

$c_i \in \mathbb{C}$, $i = 1, 2, \dots$

$z, z_0 \in \mathbb{C}$

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n = c_0 + c_1 (z+z_0)^1 + c_2 (z+z_0)^2 + \dots$$

power series centered at z_0

$$\sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2$$

power series centered at $z_0 = 0$

R = radius of convergence

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| \quad \text{or} \quad R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|c_n|}}$$

$\{ z \in \mathbb{C} \mid |z - z_0| < R \}$ the disk of convergence

① Find the radius and the disk of conv for the series:

a) $\sum_{n=0}^{\infty} (\cos(in)) z^n$; b) $\sum_{n=0}^{\infty} \left(\frac{z}{n+i} \right)^n$; c) $\sum_{n=0}^{\infty} \left(\cos\frac{i}{n} \right)^n (z+i)^n$

d) $\sum_{n=0}^{\infty} e^{in\pi} (z-1)^n$

a) $R = \lim_{n\to\infty} \left| \frac{(\cos in)}{\cos(i(n+1))} \right| = \lim_{n\to\infty} \left| \frac{e^{i^2 n} + e^{-i^2 n}}{2} \cdot \frac{2}{e^{i^2 n+1} + e^{-i^2 n+1}} \right|$

$$= \lim_{n\to\infty} \frac{e^{-n} + e^{n}}{e^{-n-1} + e^{n+1}} = \lim_{n\to\infty} \frac{e^{n}(1 + \overbrace{e^{-2n}}^{0})}{e^{n+1}(1+e^{-2(n+1)})} = \frac{1}{e}$$

$$\left\{ z \in \mathbb{C} \;\middle|\; |z| < \frac{1}{e} \right\}$$

b) $$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\left|\frac{1}{n+i}\right|^{n}}} = \left|\frac{1}{\frac{1}{n+i}}\right| = \lim_{n\to\infty} |n+i|$$

$\left\{ z \in \mathbb{C} \;\middle|\; |z| < \infty \right\}$ – the series is convergence in the whole plane

c) $$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|\cos\frac{i}{n}|^{n}}} = \lim_{n\to\infty} \frac{1}{|\cos\frac{i}{n}|} = \lim_{n\to\infty} \frac{1}{\frac{e^{\frac{i^2}{n}} + e^{-\frac{i^2}{n}}}{2}} = \lim_{n\to\infty} \frac{2}{e^{\frac{i}{n^2}} + e^{-\frac{i}{n^2}}} = 1$$

b) $R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\left|\frac{1}{n+i}\right|^n}} = \left|\frac{1}{\frac{1}{n+i}}\right| = \lim_{n\to\infty} |n+i|$

$\{z \in \mathbb{C} \mid |z| < \infty\}$ - the series is convergence in the whole plane

c) $R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\left|\cos\frac{i}{n}\right|^n}} = \lim_{n\to\infty} \frac{1}{\left|\cos\frac{i}{n}\right|} = \lim_{n\to\infty} \frac{1}{\left|\frac{e^{\frac{i^2}{n}} + e^{-\frac{i^2}{n}}}{2}\right|}$

$\frac{1}{\infty} = 0$

$e^0 = 1$

$= \lim_{n\to\infty} \frac{2}{e^{\frac{1}{n^2}} + e^{-\frac{1}{n^2}}} = 1$

$1 + 1 = 2$

$\{z \in \mathbb{C} \mid |z+i| < 1\}$

d) $R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|e^{in\pi}|^n}} = \lim_{n\to\infty} \frac{1}{e^{in\pi}} = \frac{1}{(\cos\pi + i\sin\pi)} = 1$

$\{z \in \mathbb{C} \mid |z-1| < 1\}$

Taylor Series

f analitic function inside $|z - z_0| < R$

$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots$

$z_0 = 0 \Rightarrow$ Mauclaurin series

Basic Mauclaurin series

$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}, z \in \mathbb{C}$ convergent

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, z \in \mathbb{C}$

$\sin z = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, z \in \mathbb{C}$

$$\frac{1}{1-z} = 1 + z + z^2 + \dots + z^n + \dots = \sum_{n=0}^{\infty} z^n, \ |z|<1$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots + = \sum_{n=0}^{\infty} (-1)^n z^n \quad |z|<1$$

(2) Expand the function using Taylor series in a neighborhood of z_0 (with powers $z - z_0$))

a) $f(z) = \frac{1}{3+6z}$, $z_0 = 0$, $z_0 = 1$ $(z-1)$

$$z_0 = 0 \quad f(z) = \frac{1}{3+6z} = \frac{1}{3(1+2z)} = \frac{1}{3}\cdot\frac{1}{1+2z} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n(2z)^n,$$

$$|2z|<1 \Rightarrow |z|<\frac{1}{2}$$

$$z_0 = 1 \quad f(z) = \frac{1}{3+6(z-1)+6} = \frac{1}{9+6(z-1)} = \frac{1}{9\left(1+\frac{6(z-1)}{9}\right)} =$$

$$= \frac{1}{9}\sum(-1)^n\left(\frac{2}{3}(z-1)\right)^n$$

$$\left|\left(\frac{2}{3}(z-1)\right)\right| < 1$$

$$|z-1| < \frac{3}{2}$$

$$f(z) = \frac{1}{3+6z} \quad z_0 = -1$$

$$f(z) = \frac{1}{3+6(z+1)-6} = \frac{1}{-3(1-2(z+1))} = \cancel{-\frac{1}{3}\sum(-1)^n\cdot(2(z+1))^n}$$

$$\cancel{|(2(z+1)|<1} \qquad \cancel{= -\frac{1}{3}\sum(-1)^n}$$

$$\cancel{|z+1|} \qquad = -\frac{1}{3}\sum\left[2(z+1)\right]^n$$

$$|2(z+1)|<1 \Rightarrow |z+1|<\frac{1}{2}$$

b) $f(z) = \frac{\ldots}{3-z}$

$$f(z) = -\frac{-z+1+3-3}{3-z} = -1 + \frac{2}{3-z} = -1 + \frac{2}{3-z+1-1}$$

$$= -1 + \frac{2}{2-(z-1)} = -1 + \frac{\cancel{2}}{\cancel{2}\left(1-\frac{1}{2}(z-1)\right)} = -1 + \sum_{n=0}^{\infty} (\ldots)^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{z-1}{2}\right)$$

$$\left|\frac{z-1}{2}\right| < 1 \Rightarrow |z-1| < 2$$

c) $f(z) = \frac{1}{z^2-4z+5}$ $z_0 = 0$

$$= \frac{1}{z^2-4z+5} = \frac{1}{(z-2+i)(z-2-i)} = \frac{z-2+i-(z-2-i)}{(z-2+i)(z-2-i)} \cdot \frac{1}{2i}$$

$z^2-4z+5=0$

$\Delta <$ $2+i$, $2-i$

$$= \frac{1}{2i}\left[\frac{1}{z-2-i} - \frac{1}{z-2+i}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{-2-i} \cdot \frac{1}{1+\frac{z}{-2-i}} - \frac{1}{-2+i} \cdot \frac{1}{1+\frac{z}{-2+i}}\right]$$

$$= \frac{1}{2i}\left[-\frac{1}{2+i} \cdot \frac{1}{1-\frac{z}{2+i}} + \frac{1}{2-i} \cdot \frac{1}{1-\frac{z}{2-i}}\right]$$

$$= \frac{1}{2i}\left[-\frac{1}{2+i}\sum\left(\frac{z}{2+i}\right)^n + \frac{1}{2-i}\sum\left(\frac{z}{2-i}\right)^n\right]$$

$$= \frac{1}{2i}\sum\left[\left(\frac{1}{2-i}\right)^{n+1} - \left(\frac{1}{2+i}\right)^{n+1}\right] z^n$$

d) $f(z) = \frac{1}{z^2+1}$, $z_0 = 1$ (exam)

$$f(z) = \frac{1}{(z-i)(z+i)} = \frac{z+i-(z-i)}{(z-i)(z+i)} \cdot \frac{1}{2i} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right) =$$

$$f(z) = \frac{1}{2i}\left(\frac{1}{z-1-i+1} - \frac{1}{z-1+i+1}\right)$$

$$= \frac{1}{2i}\left(\frac{1}{-i+1} \cdot \frac{1}{1+\frac{(z-1)}{-i+1}} - \frac{1}{i+1} \cdot \frac{1}{1-\frac{z-1}{i+1}} \right)$$

$$= \frac{1}{2i} \sum \frac{1}{2i}\left(\frac{1}{1-i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{1-i}\right)^n - \frac{1}{i+1} \sum (-1)^n \left(\frac{z-1}{i+1}\right)^n \right)$$

$$= \frac{1}{2i} \sum_{n=0}^{\infty} (-1)^n \left[\left(\frac{1}{1-i}\right)^{n+1} - \left(\frac{1}{1+i}\right)^{n+1} \right] \cdot (z-1)^n$$

$$\begin{cases} \left|\frac{z-1}{1-i}\right| < 1 \Rightarrow |z-1| < \underbrace{|1-i|}_{\sqrt{2}} \\ \left|\frac{z-1}{1+i}\right| < 1 \Rightarrow |z-1| < \underbrace{|1+i|}_{\sqrt{2}} \end{cases}$$

$$|z-1| < \sqrt{2}$$

② $f(z) = e^{2z} \cos z,\ z_0 = 0$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$f(z) = e^{2z} \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{z(2+i)} + e^{z(2-i)}}{2} = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(z(2+i))^n}{n!} + \sum_{n=0}^{\infty} \frac{(z(2-i))^n}{n!} \right] = \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{n!}\left((2+i)^n + (2-i)^n \right)$$

③ Evaluate $\int_C \frac{5z+7}{z^2+2z-3}$, $C: |z-2| = 2$

④ Evaluate $\int_C \frac{z}{z^2+9} dz$, $C: |z-2i| = 4$

SEMINAR 6

Laurent Series and singularities

f is holomorphic $R_1 < |z - z_0| < R_2$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$$

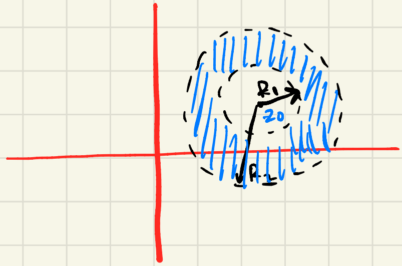
$a_{-1} = \operatorname{Res}_{z=z_0} f(z)$

on

$$f(z) = \underbrace{\dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)}}_{\substack{\text{the principal part} \\ \text{(conv. outside a circle centered at } z_0)}} + \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots}_{\substack{\text{the analytic part} \\ \text{the regular part} \\ \text{(conv. inside a circle centered at } z_0)}}$$

$R_1 < |z - z_0| < R_2$

the annulus domain



• $R_1 = 0$

• $R_2 = +\infty$

• $R_1 = 0$, $R_2 = \infty$

z_0 – isolated singulare point

$f(z)$ has a Laurent decomposition

z_0 – removable singular point: f has no princ. point

z_0 – essential singular point: the principle part of f has an infinite number of terms

z_0 - pole of order k : the princ part has a finite number of terms

$a_{-k} \neq 0$, $a_{-j} = 0$, $\forall j > k$

$$f(z) = \underbrace{\frac{a_{-k}}{(z-z_0)^k} + \ldots + \frac{a_{-1}}{z-z_0}}_{a_{-k} \neq 0} + a_0 + a_1(z-z_0) + \ldots$$

① $f(z) = \cos\frac{1}{z}$, $z_0 = 0$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$$

$$f(z) = 1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \frac{1}{6!z^6} + \ldots$$

z_0 - essential singularity

② $f(z) = \dfrac{\cos z - 1}{z^2}$, $z_0 = 0$

$$f(z) = \frac{1}{z^2}\left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots\right)$$

$$f(z) = -\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \frac{z^6}{8!} - \ldots$$

z_0 - removable singularity

3) $f(z) = \dfrac{\cos z}{z^4}$, $z_0 = 0$

$$f(z) = \frac{1}{z^4}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right)$$

$$= \frac{1}{z^4} - \frac{1}{z^2 2!} + \frac{1}{4!} - \frac{z^2}{6!} + \frac{z^4}{8!} - \dots$$

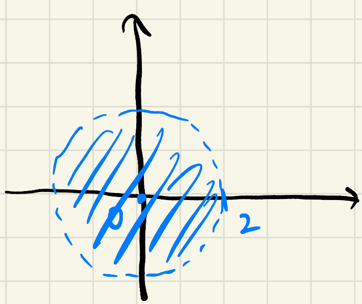
z_0 - pole of order 4

(4) Determine all possible Laurent expansions

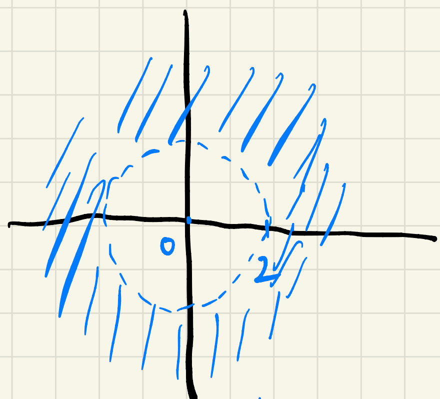
a) $f(z) = \dfrac{1}{z-2}$, $z_0 = 0$

b) $f(z) = \dfrac{1}{z-2}$, $z_0 = 1$ $\quad (z-1)$

a)



• $|z| < 2$



$|z| > 2 \quad | : |z|$

$\dfrac{2}{|z|} < 1$

$$\frac{1}{1-u} = 1 + u + u^2 + \dots + u^n + \dots \,, \quad |u| < 1$$

- $\dfrac{|z|}{2} < 1$

$$f(z) = \frac{1}{z-2} = \frac{-1}{2\left(1-\frac{z}{2}\right)} = -\frac{1}{2}\sum \left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} (-1)\frac{z^n}{2^{n+1}}$$

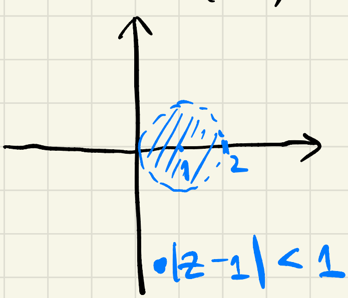
z_0 rem. sing.

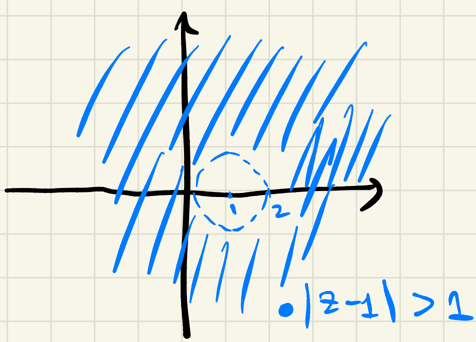
- $|z| > 2$

$$f(z) = \frac{1}{z-2} = \frac{1}{z\left(1-\frac{2}{|z|}\right)} = \frac{1}{z}\sum \left(\frac{2}{|z|}\right)^n = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$$

$z_0 =$ essential sing.

b) $f(z) = \dfrac{1}{(z-1)-1}$





$$f(z) = \frac{1}{(z-1)-1} = \frac{-1}{1-(z-1)} = -\sum_{n=0}^{\infty} (z-1)^n$$

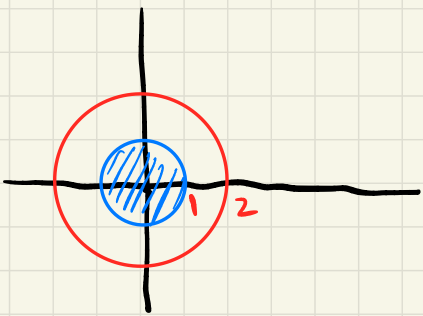
- $\frac{1}{|z-1|} < 1$

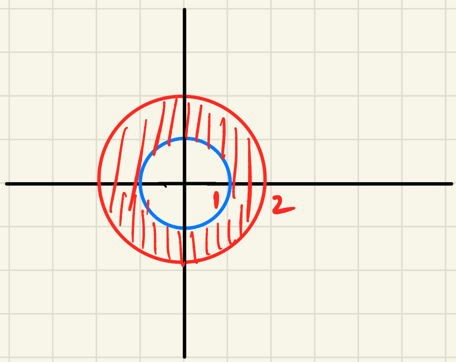
$$f(z) = \frac{1}{(z-1)-1} = \frac{1}{(z-1)\left(1-\frac{1}{z-1}\right)} = \frac{1}{z-1} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{z-1}\right)^n =$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{z-1}\right)^{n+1}$$

$z_0 = 1$ ess. sing.

c) $f(z) = \frac{1}{(z-1)(z-2)}$, $z_0 = 0$



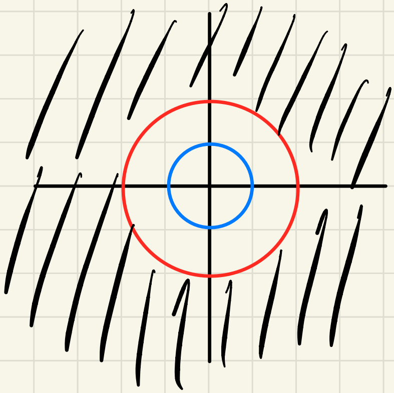


(i) $\cdot |z| < 1$

$(|z| < 2)$

$\frac{|z|}{2} > 1$

(ii) $1 < |z| < 2$



(iii) $\cdot\, |z| > 2$

$(|z| > 1)$

$$f(z) = \frac{z-1-(z-2)}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

i) $$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{-1}{2\left(1-\frac{z}{2}\right)} + \frac{1}{1-z} =$$

$$= -\frac{1}{2}\sum\left(\frac{z}{2}\right)^n + \sum z^n = \sum_{n=0}^{\infty}\left((-1)\frac{1}{2^{n+1}} + 1\right)z^n$$

ii) $1 < |z| \,\Big|\, \cdot |z| \Rightarrow \frac{1}{|z|} < 1$

$|z| < 2 \Rightarrow \frac{|z|}{2} < 1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} =$$

$$= -\frac{1}{2}\frac{1}{1-\frac{z}{2}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}} =$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

z_0 - essential sing.

iii) $1 < |z| : |z| \Rightarrow \frac{1}{|z|} < 1$ $\quad |z| > 2 : 2$

$\frac{|z|}{2} > 1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{-1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} =$$

$$= \frac{1}{z}\sum\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum\left(\frac{1}{z}\right)^n$$

z_0 - essential

Expand by Laurent series around z_0

a) $f(z) = z^3 e^{\frac{1}{z}}$, $z_0 = 0$, $0 < |z| < \infty$

b) $f(z) = \frac{2\sin^2 z}{z^5}$, $z_0 = 0$, $0 < |z| < \infty$

c) $f(z) = z e^{\frac{1}{z+i}}$, $z_0 = -i$, $0 < |z+i| < \infty$

d) $f(z) = \frac{1}{(z-1)(z-4)}$, $z_0 = 1$, $0 < |z-1| < 3$

a) $f(z) = z^3 e^{\frac{1}{z}} \iff f(z) = z^3 \cdot \left(1 + \frac{1}{z \cdot 1!} + \frac{1}{z^2 2!} \cdots \right)$

$$f(z) = z^3 + \frac{z^2}{1!} + \frac{z}{2!} + \frac{1}{3!} + \frac{1}{z4!} + \dots$$

z_0 - ess. sing.

b) $f(z) = \frac{2\sin^2 z}{z^5} = \frac{1}{z^5} \cdot (1 - \cos 2z) =$

$$= \frac{1}{z^5}\left[1 - \left(1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} - \frac{(2z)^6}{6!} + \dots\right)\right] =$$

$$= \frac{1}{z^5}\left(\frac{(2z)^2}{2!} - \frac{(2z)^4}{4!} + \frac{(2z)^6}{6!} - \dots\right) =$$

$$= \frac{1}{z^3} - \frac{2^4}{4 \cdot z} + \frac{2^6 z}{6!} - \dots$$

z_0 pole of order 3

c) $f(z) = (z+i-i)e^{\frac{1}{z+i}} = (z+i)e^{\frac{1}{z+i}} - ie^{\frac{1}{z+i}} =$

$$(z+i)\left[1 + \frac{1}{(z+i)1!} + \frac{1}{(z+i)^2 2!} + \dots\right] - i\left[1 + \frac{1}{(z+i)1!} + \frac{1}{(z+i)^2 2!} + \dots\right]$$

$$= z+i-i+1+1 + \frac{1}{z+i}\left(\frac{1}{2!} - i\right) + \frac{1}{(z+i)^2}\left(\frac{1}{3!} - \frac{i}{2!}\right) + \dots$$

$\Rightarrow z_0 = -i$ ess. sing.

d) $f(z) = \dfrac{1}{(z-1)(z-4)}$ $z_0 = 1$

$|z-1| < 3 \quad |:3$

$\dfrac{z-1}{3} < 1$

$$f(z) = \frac{1}{z-1} \cdot \frac{1}{(z-1)-3} = \frac{1}{(z-1)} \cdot \frac{1}{3} \cdot \frac{1}{\frac{z-1}{3}-1}$$

$$= -\frac{1}{z-1} \cdot \frac{1}{3} \cdot \frac{1}{1-\frac{z-1}{3}}$$

$$= -\frac{1}{(z-1)} \cdot \frac{1}{3} \cdot \sum \left(\frac{z-1}{3}\right)^n$$

$$= -\sum \frac{(z-1)^{n-1}}{3^{n+1}}$$

z_0 - pole of order 1

e) $f(z) = \dfrac{7-z}{2z^2+11z-6}$ all possible L. expansions

La examen vine pe o regiune

ex: $|z| > 6$

$2z^2 + 11z - 6 = 0$

$\Delta = 121 + 48$

$= 169 \Rightarrow \sqrt{\Delta} = 13$

$$z_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-11 \pm 13}{4} \quad \begin{cases} -\frac{24}{4} = -6 \\ \frac{2}{4} = \frac{1}{2} \end{cases}$$

$$f(z) = \frac{7-z}{(z-\frac{1}{2})(z+6)} \quad \dots$$

f) $$f(z) = \frac{1}{z^5} e^{3z} = \frac{1}{z^5}\left(1 + \frac{3z}{1!} + \frac{(3z)^2}{2!} + \dots\right) =$$

$$= \frac{1}{z^5} + \frac{3}{z^4} + \frac{3^2}{2! z^3} + \frac{3^3}{3! z^2} + \frac{3^4}{4! z} + \frac{3^5}{5!} + \dots$$

z_0 - pole of order 5

(viii)

$\sin z + \cos z = 1$

$\frac{e^{iz}-e^{-iz}}{2i} + \frac{e^{iz}+e^{-iz}}{2} = 1 \,/\cdot 2i$

$e^{iz} - e^{-iz} + ie^{iz} + ie^{-iz} = 2i$

$t - \frac{1}{t} + it + \frac{i}{t} - 2i = 0 \,/\cdot t$

$t^2 - 1 + it^2 + i - 2it = 0$

$t^2(1+i) - 2it + i - 1 = 0$

$\Delta = b^2 - 4ac = (2i)^2 - 4\cdot(i+1)(i-1)$
$= -4 - 4(i^2-1)$
$= -4 - 4i^2 + 4 = -4i^2 \Rightarrow -4\cdot(-1) = 4 \Rightarrow \sqrt{\Delta} = 2$

$t_{1;2} = \frac{-b \pm 2}{2a} = \frac{2i \pm 2}{2(1+i)} = \frac{i \pm 1}{i+1} \Rightarrow t_1 = \frac{i+1}{i+1} = 1$

$t_1 = \frac{i-1}{i+1} = \frac{(i-1)^2}{(i+1)(i-1)} = \frac{i^2+1-2i}{i^2-1} = \frac{-2i}{-2} = i$

$(1,0)$

$e^{iz} = t$

$\mathrm{arc}\,1 = \arctan 0 + 0 = 0$

$e^{iz} = 1 \Rightarrow e^{iz} = e^{\mathrm{Log}\,1}$

$\mathrm{Log}\,1 = \{ \ln 1 + i(\mathrm{arc}\,1 + 2k\pi) \}$

$\mathrm{Log}\,1 = \{ i(0 + 2k\pi) \} = i2k\pi$

$iz = i2k\pi$

$z = 2k\pi \quad (1)$

$e^{iz} = \mathrm{Log}\, i$

$\mathrm{Log}\, i = \{ \ln i + i(\mathrm{arc}\, i + 2k\pi)\ k\in\mathbb{Z} \}$

$\mathrm{arc}\, i = \arctan \frac{1}{0} = \frac{\pi}{2}$
$(0,1)$

$\mathrm{Log}\, i = \{ \underbrace{\ln 1}_{0} + i(\frac{\pi}{2} + 2k\pi), k\in\mathbb{Z} \}$

$iz = i(\frac{\pi}{2} + 2k\pi)$

$z = \frac{\pi}{2} + 2k\pi \quad (2)$

$z = (1) \cup (2)$

(ix) $\sin z - 2\cos z = 3$

$\frac{e^{iz}-e^{-iz}}{2i} - 2\frac{e^{iz}+e^{-iz}}{2} = 3 \,/\cdot 2i$

$e^{iz} - e^{-iz} - 2i(e^{iz}+e^{-iz}) = 6i$

$e^{iz} \overset{not}{=} t$

$t - \frac{1}{t} - 2i(t + \frac{1}{t}) = 6i$

$t - \frac{1}{t} - 2it - \frac{2i}{t} = 6i \,/\cdot t$

$t^2 - 1 - 2it^2 - 2i - 6it = 0$

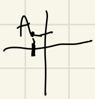
$a^2 - b^2 = (a+b)(a-b)$

$t^2(1-2i) - 6it - (1+2i) = 0$

$\Delta = (-6i)^2 - 4(1-2i)\cdot(-(1+2i)) =$
$= -36 + 4(1^2 - (2i)^2) =$
$= -36 + 4(1-(-4)) =$
$= -36 + 4(1+4) = -36 + 20 = -16$

$t_{1;2} = \frac{6i \pm \sqrt{16}\, i}{2(1-2i)} = \frac{6i \pm 4i}{2(1-2i)}$

$t_1 = \frac{6i + 4i}{2(1-2i)} = \frac{\overset{5}{\cancel{10}}i}{\cancel{2}(1-2i)} = \frac{5i(1+2i)}{1^2-(2i)^2} = \frac{5i+10i^2}{1-(-4)} = \frac{5i-10}{5} = i-2$



$t_1 = i-2$

$e^{iz} = i-2 \Rightarrow e^{iz} = e^{\mathrm{Log}(i-2)}$

$\mathrm{Log}(i-2) = \{ \ln|i-2| + i(\mathrm{arc}(i-2) + 2k\pi)\ k\in\mathbb{Z} \}$

$|i-2| = \sqrt{(-2)^2+1} = \sqrt{5}$

$\mathrm{arc}(i-2) = \arctan(\frac{1}{-2}) + \pi = -\arctan\frac{1}{2} + \pi =$

$A(-2,1)$

$\mathrm{Log}(i-2) = \{ \ln\sqrt{5} + i(-\arctan\frac{1}{2} + \pi + 2k\pi)\ k\in\mathbb{Z} \}$

$\mathrm{Log}(i-2) = \{ \ln 5^{\frac{1}{2}} + i(-\arctan\frac{1}{2} + \pi + 2k\pi)\ k\in\mathbb{Z} \}$

$iz = \frac{1}{2}\ln 5 + i(-\arctan\frac{1}{2} + \pi + 2k\pi)\ k\in\mathbb{Z} \,/:i$

$z = -\frac{i}{2}\ln 5 - \arctan\frac{1}{2} + \pi + 2k\pi \quad (A)$

$$t_2 = \frac{6i-4i}{2(1-2i)} = \frac{\cancel{2}i}{\cancel{2}(1-2i)} = \frac{i}{1-2i} \cdot \frac{1+2i}{} = \frac{i(1+2i)}{1^2-(2i)^2} = \frac{i-2}{1-(-4)} = \frac{i-2}{5} = \frac{i}{5} - \frac{2}{5}$$

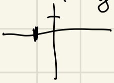
$$e^{iz} = \frac{i}{5} - \frac{2}{5} \Rightarrow iz = e^{\operatorname{Log}\frac{i}{5}-\frac{2}{5}}$$

$$\operatorname{Log}\left(\frac{i}{5}-\frac{2}{5}\right) = \left\{ \ln\left|\frac{i}{5}-\frac{2}{5}\right| + i\left(\operatorname{arc}\left(\frac{i}{5}-\frac{2}{5}\right) + 2k\pi\right)\right.$$

$$\sqrt{\left(\frac{1}{5}\right)^2+\left(-\frac{2}{5}\right)^2} = \sqrt{\frac{1+4}{25}} = \sqrt{\frac{5}{25}} = \frac{\sqrt{5}}{5} \Rightarrow \ln\frac{\sqrt{5}}{5} = \ln 5^{\frac{1}{2}-1} = \ln 5^{-\frac{1}{2}} = -\frac{1}{2}\ln 5$$

$$\operatorname{arc}\left(\frac{i}{5}-\frac{2}{5}\right) = \arctan\frac{\frac{1}{\cancel{5}}}{-\frac{2}{\cancel{5}}} + \pi = -\arctan\frac{1}{2} + \pi$$

$$\left(-\frac{2}{5}, \frac{1}{5}\right)$$
$x \qquad y$



$$\operatorname{Log}\left(\frac{i}{5}-\frac{2}{5}\right) = \left\{-\frac{1}{2}\ln 5 + i\left(-\arctan\frac{1}{2}+\pi+2k\pi\right)\right\} k\in\mathbb{Z}$$

$$iz = -\frac{1}{2}\ln 5 + i\left(-\arctan\frac{1}{2}+\pi+2k\pi\right\} \quad / :i$$

$$z = \frac{i}{2}\ln 5 - \arctan\frac{1}{2} + \pi(2k+1) \quad (B)$$

$$z = A \cup B$$

SEMINAR 7

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \dots \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0) + \dots$$

$$R_1 < |z-z_0| < R_2$$

$a_{-1} = \operatorname{Res}_{z=z_0} f(z)$ the Residue of $f(z)$

I Method - we expend $f(z)$ by Laurent series

$\Rightarrow a_{-1} =$ the coefficient of $\frac{1}{z-z_0}$

① Find the residue of $f(z) = \frac{\sin z}{z^2}$

$$f(z) = \frac{1}{z^2}\left(\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)$$

$$= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \dots$$

$$\operatorname{Res}_{z=0} f(z) = 1$$

Poles of z

$f(z) = \frac{\varphi(z)}{(z-z_0)^k}$, φ holom. function in the neighb. of z_0

$\varphi(z_0) \neq 0$

$\Rightarrow z_0$ pole of order K

② $f(z) = \dfrac{2z+5}{(z-1)(z+5)(z-2)^4}$

$z_1 = 1$, $z_2 = -5$, $z_3 = 2$

$z_1 = 1$ simple pole $\qquad \ell(1) \neq 0$

$z_2 = -5$ simple pole $\qquad \ell(-5) \neq 0$

$z_3 = 2$ pole of order 4 $\qquad \ell(2) \neq 0$

Method I : Evaluating the residue for poles

$$\operatorname*{Res}_{z=z_0} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=z_0}$$

z_0 pole of order I

$f(z) = \dfrac{g(z)}{h(z)}$ with the property $\quad g(z_0) \neq 0$

$h(z_0) = 0$

$h'(z_0) \neq 0$

Method II

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[(z-z_0)^n f(z) \right]^{(n-1)}$$

z_0 pole of order $n \geq 1$

③ Find the residues for the functions

$f(z) = \dfrac{z(\cos z)}{(z-\pi)^3}$

$z_0 = \pi$ pole of order 3

$$\operatorname{Res} f(z) = \frac{1}{2!} \lim_{z\to\pi} \left[\cancel{(z-\pi)}^3 \frac{z(\cos z)}{\cancel{(z-\pi)^3}} \right]^{(2)}$$

$$= \frac{1}{2} \lim_{z\to\pi} (z\cos z)''$$

$$= \frac{1}{2} \lim_{z\to\pi} (\cos z - z\sin z)'$$

$$= \frac{1}{2} \lim_{z-\pi} \left(-\sin z - \sin z - z\cos z\right)$$

$$= \frac{1}{2}\left(-2\underbrace{\sin\pi}_{0} - \pi\underbrace{\cos\pi}_{-1}\right) = \frac{\pi}{2}$$

b) $f(z) = \dfrac{\cos z}{z^4 - 1}$, $z_0 = i$

$z^4 = 1$ $\quad (z^2-1)(z^2+1) = 0$

$(z-1)(z+1)(z+i)(z-i)$

i - pole of order 1

$$\operatorname{Res} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=z_0}$$

$$f(z) = \frac{g(z)}{h(z)}$$

$$\operatorname*{Res}_{z=i} f(z) = \left.\frac{\cos z}{4z^3}\right|_{z=i} = \frac{\cos i}{4i^3} = \frac{\cos i}{-4i} = -\frac{1}{4i}\cdot\frac{e^{i^2}+e^{-i^2}}{2}$$

$$= -\frac{1}{8i}\cdot\left(e^{-1}+e^{1}\right)$$

④ Find the residues of the singular points of f

a) $f(z) = \frac{z^2}{(1+z)^3}$; b) $f(z) = \frac{1}{1+e^z}$ c) $f(z) = \frac{1}{\sin \pi z}$

d) $f(z) = \frac{\cos z}{(z-1)^2}$

a) $f(z) = \frac{z^2}{(1+z)^3}$ $(1+z)^3 = 0 \Rightarrow z_0 = -1$ pole of order 3

$$\operatorname{Res}_{z=-1} f(z) = \frac{1}{(3-1)!} \lim_{z\to -1} \left[\cancel{(z+1)^3} \frac{z^2}{\cancel{(1+z)^3}} \right]^{(2)}$$

$$= \frac{1}{2} \lim_{z\to -1} (z^2)'' = \frac{1}{2} \lim_{z\to -1} (2z)' = \frac{1}{2} \lim_{z\to -1} 2 = 1$$

b) $f(z) = \frac{1}{1+e^z}$

$1+e^z = 0 \Rightarrow e^z = -1 \mid \mathrm{Log}$

$z = \mathrm{Log}(-1) = \{ \ln|-1| + i(\arg(-1) + 2k\pi) \}$

$z = \{ i(\pi + 2k\pi) \mid k \in \mathbb{Z} \}$

$i(\pi+2k\pi)$ pole of order 1

$$\operatorname{Res}_{z=i(\pi+2k\pi)} f(z) = \frac{1}{(1+e^z)'} \Bigg|_{z=i(\pi+2k\pi)}$$

$$= \frac{1}{e^z}\Bigg|_{z=i(\pi+2k\pi)} = \frac{1}{e^{i(\pi+2k\pi)}} = \frac{1}{\cos(\pi+2k\pi)+i\sin(\pi+2k\pi)}$$

$$= \frac{1}{-1} = -1$$

c) $f(z) = \dfrac{1}{\sin \pi z}$

$\sin \pi z = 0 \Rightarrow \pi z = \pi k \;\Big|:\pi \Rightarrow z = k \;,\; k \in \mathbb{Z}$

$\Rightarrow z_0 = k$ pole of order 1

$$\operatorname*{Res}_{z=k} f(z) = \frac{1}{(\sin \pi z)'} = \frac{1}{\pi \cos \pi z}\Bigg|_{z=k} = \frac{1}{\pi \cos \pi k} = \frac{(-1)^k}{\pi}$$

d) $f(z) = \dfrac{\cos z}{(z-1)^2}$

$z-1=0 \Rightarrow z = 1 \Rightarrow z_0 = 1$ pole of order 2

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{1!}\lim_{z\to 1}\left[\cancel{(z-1)^2}\,\frac{\cos z}{\cancel{(z-1)^2}}\right]'$$

$$= \lim_{z\to 1}\left(\cos(z)\right)' = \lim_{z\to 1}(-\sin z) = -\sin 1$$

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} \operatorname*{Res}_{z=z_k} f(z)$$

$z_k \in \text{int } C$

① Evaluate the integrals : a) $\int_C \frac{dz}{z^4+1}$, $C: |z-1| = 1$,

b) $\int_C \frac{e^{\frac{\pi}{z-2i}}}{(z-1)(z-2i)} dz$, $C: |z| = 3$

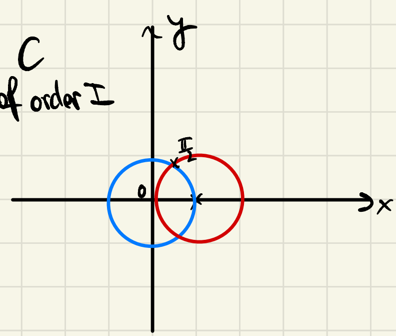
c) $\int_C \frac{e^{\frac{\pi}{z-1}}}{z^2-3z+2} dz$, $C: |z| = 3$

a) $\int_C \frac{dz}{z^4+1}$, $C: |z-1| = 1$,

$z^4 + 1 = 0 \Rightarrow z^4 = -1 \quad \Rightarrow -1 = \cos\pi + i\sin\pi$

$z_k = \cos\frac{\pi + 2k\pi}{4} + i\sin\frac{\pi+2k\pi}{4}$, $k = 0,1,2,3$

$z_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \in \text{int } C$
pole of order I



$z_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \notin \text{int } C$

$z_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \notin \text{int } C$

$z_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} \in \text{int } C$, pole of order 1

$I = 2\pi i \left(\operatorname{Res}_{z=z_0} f(z) + \operatorname{Res}_{z=z_3} f(z) \right)$

$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{4z^3}\Bigg|_{z=z_0} = \frac{1}{4\left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)^3} = \frac{1}{2\sqrt{2}(-1+i)}$

$$\operatorname*{Res}_{z=z_3} f(z) = \frac{1}{4z^3}\Big|_{z=z_3} = \frac{1}{4\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)^3} = \frac{1}{4\frac{2\sqrt{2}}{8}\cdot\left(1-3i+3i^2-i^3\right)} =$$

$$= \frac{1}{\sqrt{2}(-2-2i)} = \frac{1}{2\sqrt{2}(-1-i)}$$

$$I = 2\pi i\left(\frac{1}{2\sqrt{2}(-1+i)} + \frac{1}{2\sqrt{2}(-1-i)}\right)$$

z_k are the roots of $z^4+1=0$

Method II

$$I = 2\pi i \sum_{\substack{k=0\\k=3}} \frac{1}{4z_k^3} = 2\pi i \cdot \sum_{\substack{k=0\\k=3}} \frac{z_k}{4z_k^4} = 2\pi i \sum_{\substack{k=0\\k=3}} -\frac{1}{4} z_k = -\frac{2\pi i}{4}(z_0+z_3)$$

$$\int_C \frac{dz}{z^6+1} \qquad C: |z-1| = 1$$

b) $(z-1)(z-2i)=0$

$z_1 = 1 \in \operatorname{int} C$ pole of order 1

$z_2 = 2i \in \operatorname{int} C$ not a pole * pentru ca apare si sus *

is an essential singularity

$$I = 2\pi i\left(\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2i} f(z)\right)$$

$$\operatorname*{Res}_{z=1} f(z) = \frac{e^{\frac{\pi}{z-2i}}}{(z-1)'(z-2i)} = \frac{e^{\frac{\pi}{z-2i}}}{(z-2i)\cdot 1}\Bigg|_{z=1} = \frac{e^{\frac{\pi}{1-2i}}}{1-2i}$$

Calculam $\operatorname*{Res}_{z=2i} f(z) = a_{-1} \leftarrow$ coef lui $\frac{1}{z-2i}$

we expand the function using L. S.

$$f(z) = \frac{1}{z-2i} e^{\frac{\pi}{z-2i}} \cdot \frac{1}{z-2i+2i-1} = \frac{1}{z-2i} e^{\frac{\pi}{z-2i}} \frac{1}{2i-1}\left(\frac{1}{1-\frac{z-2i}{1-2i}}\right)$$

$$= \frac{1}{2i-1} \frac{1}{z-2i}\left(1+\frac{z-2i}{1-2i}+\left(\frac{z-2i}{1-2i}\right)^2+\dots\right) e^{\frac{\pi}{z-2i}}$$

$$= \frac{1}{2i-1}\left(\frac{1}{z-2i}+\frac{1}{1-2i}+\frac{z-2i}{(1-2i)^2}+\frac{z-2i}{(1-2i)^3}+\dots\right)\left(1+\frac{\pi}{1!(z-2i)}+\frac{\pi}{2!(z-2i)^2}+\dots\right)$$

The coef of $\frac{1}{z-2i}$: $a_{-1} = \frac{1}{2i-1}\left(1+\frac{\pi}{(1-2i)1!}+\frac{\pi^2}{2!(1-2i)^2}+\dots\right)$

$$= \frac{1}{2i-1} e^{\frac{\pi}{1-2i}} \Rightarrow \operatorname*{Res}_{z=2i} f(z) = \frac{1}{2i-1} e^{\frac{\pi}{1-2i}}$$

$$I = 2\pi i\left(\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2i} f(z)\right) = 2\pi i\left(\frac{1}{1-2i} e^{\frac{\pi}{1-2i}} + \frac{1}{2i-1} e^{\frac{\pi}{1-2i}}\right)$$

$$I = 0$$

De EXAMEN

Compute the residues in the singular points

$$f(z) = \frac{1}{1-z} \sin\frac{1}{z}$$

$1-z=0 \Rightarrow z=1$ pole of order 1

$$\operatorname*{Res}_{z=1} f(z) = \left.\frac{\sin\frac{1}{z}}{-1}\right|_{z=1} = -\sin 1$$

$z=0$ essential singularity

We expand by Laurant Series

$$f(z) = \left(1 + z + z^2 + z^3 + \dots\right)\left(\frac{1}{1!\,z} - \frac{1}{3!\,z^3} + \frac{1}{5!\,z^5} - \dots\right)$$

$$\sin z = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} \dots$$

$$\operatorname*{Res}_{z=0} f(z) = a_{-1} = \frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots = \sin 1$$

coef $\frac{1}{z}$

$$I = \left(\operatorname*{Res}_{z=-1} f(z) + \operatorname*{Res}_{z=0} f(z)\right) = \left(-\sin 1 + \sin 1\right) = 0$$

SEMINAR 8

5.04.2022

1) $I = \int_C \frac{dz}{z\cos z^2}$, $C: x^2 + \frac{y^2}{4} = 1$

$z \cdot \cos z^2 = 0$

$z = 0 \quad \in \text{int } C$

pole of order 1

$\cos z^2 = 0 \Rightarrow z^2 = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$, $k = 0, \pm 1, \pm 2, \dots$

$\Rightarrow z = \pm\sqrt{\frac{\pi}{2} + k\pi}$

$k = 0 \Rightarrow z_0 = \underbrace{\pm\sqrt{\frac{\pi}{2}}}_{x}$ $\qquad \frac{\pi}{2} < 1 \Rightarrow \pi < 2$ F

$k = 1 \Rightarrow z = \pm\sqrt{\frac{\pi}{2} + \pi} \Rightarrow \pm\sqrt{\frac{3\pi}{2}}$; $\frac{3\pi}{2} < 1$ F

$k = -1 \Rightarrow z = \pm\sqrt{\frac{\pi}{2} - \pi} \Rightarrow \pm i\sqrt{\frac{\pi}{2}}$

$\Rightarrow \frac{\frac{\pi}{2}}{4} < 1$ T $= \pm i\sqrt{\frac{\pi}{2}} \in \text{int } C$

poles of order 1

$k = -2 \ \dots$ F

$I = 2\pi i \left(\operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=i\sqrt{\frac{\pi}{2}}} f(z) + \operatorname{Res}_{z=-i\sqrt{\frac{\pi}{2}}} f(z) \right)$

$f(z) = \frac{g(z)}{h(z)} = \frac{\frac{1}{\cos z^2}}{z}$

$\operatorname{Res}_{z=0} f(z) = \frac{g(z)}{h'(z)} \Big|_{z=0} = \frac{\frac{1}{\cos z^2}}{1} \Big|_{z=0} = 1$

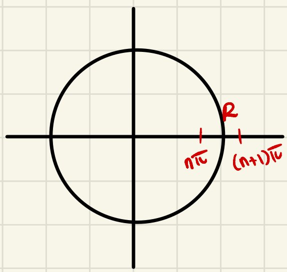
$$\operatorname*{Res}_{z=i\sqrt{\frac{\pi}{2}}} f(z) = \frac{\frac{1}{z}}{(\cos z^2)'}\Bigg|_{z=i\sqrt{\frac{\pi}{2}}} = \frac{\frac{1}{z}}{-\sin z^2 \cdot 2z}\Bigg|_{z=i\sqrt{\frac{\pi}{2}}} = \frac{-1}{\sin z^2 \cdot 2z}\Bigg|_{z=i\sqrt{\frac{\pi}{2}}}$$

$$= \frac{1}{\sin(-\frac{\pi}{2})\cdot\pi} = -\frac{1}{\pi}$$

$$\operatorname*{Res}_{z=-i\sqrt{\frac{\pi}{2}}} f(z) = \frac{\overbrace{\frac{1}{z}}^{g(z)}}{\underbrace{\cos z^2}_{h'(z)}} = \frac{\frac{1}{z}}{-(\sin z^2)2z}\Bigg|_{z=-i\sqrt{\frac{\pi}{2}}} = \frac{-1}{2z^2\sin z^2}\Bigg|_{z=-i\sqrt{\frac{\pi}{2}}}$$

$$= \frac{-1}{2\left(-i\sqrt{\frac{\pi}{2}}\right)^2\cdot\sin\left(-i\sqrt{\frac{\pi}{2}}\right)} = \frac{-1}{2\cdot\frac{\pi}{2}\cdot\sin\frac{\pi}{2}} = -\frac{1}{\pi\cdot\sin\frac{\pi}{2}} = -\frac{1}{\pi}$$

(2) $\displaystyle\int_{C:|z|=R} \frac{dz}{z^2\cdot\sin z}$ $\qquad n\pi < R < (n+1)\pi$



$z^2\sin z = 0 \Rightarrow z^2 = 0 \Rightarrow z = 0$

$\sin z = 0 \Rightarrow z = k\pi$, $k = 0, \pm1, \pm2, \dots$

dacă $k = 0 \Rightarrow z = 0$ pole of order 3

$z = k\pi$, $k = \pm1, \pm2, \dots$ pole of order 1

$$\operatorname*{Res}_{z=k\pi} f(z) = \frac{\frac{1}{z^2}}{(\sin z)'}\Bigg|_{z=k\pi} = \frac{1}{z^2\cdot\cos z}\Bigg|_{z=k\pi} = \frac{1}{(k\pi)^2(-1)^k} =$$

$$= \frac{(-1)^k}{k^2\pi^2}$$

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{(3-1)!} \cdot \lim_{z\to 0}\left(z^3 \cdot \frac{1}{z^2 \sin z}\right) = \frac{1}{2} \cdot \lim_{z\to 0}\left(\frac{z}{\sin z}\right)^{(2)} = \dots$$

Derivam de 2 ori (aplicam L'Hospital de 2 ori) Rezultat $= \frac{1}{6}$

$$\operatorname*{Res}_{z=0} f(z) = a_{-1} = \text{coef } \frac{1}{z}$$

$$f(z) = \frac{1}{z^2 \sin z} = \frac{1}{z^2 \cdot \left(\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)} =$$

$$= \frac{1}{z^3 \left(\frac{1}{1!} - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right)} = \frac{1}{z^3} \cdot \frac{1}{1 - \underbrace{\left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right)}_{u}} \qquad \frac{1}{1-z}$$

$$= \frac{1}{z^3} \cdot \left[1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right)^2 + (\dots)^3 + \dots\right]$$

$$I = 2\pi i \left(\operatorname*{Res}_{z=0} f(z) + 2\sum_{k=1}^{n} \operatorname*{Res}_{z=k\pi} f(z)\right) = 2\pi i \left(\frac{1}{6} + 2\sum_{k=1}^{n} \frac{(-1)^k}{k^2\pi^2}\right)$$

(3) $$\int_{C:\,|z|=R,\, R>0} \frac{e^{\frac{1}{z}}}{z-1}\, dz$$

$z=0$ essential singularity

$z-1=0 \Rightarrow z=1$ pole of order 1 $\in \operatorname{int} C$ if $R>1$

$$\operatorname*{Res}_{z=1} f(z) = \left.\frac{e^{\frac{1}{z}}}{(z-1)'}\right|_{z=1} = e$$

$z=0$ essential singularity

We expand by L. S.

$$f(z) = \frac{1}{z-1}\cdot\left(1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\dots\right)$$

$$= \frac{-1}{1-z}\left(1+\frac{1}{z}+\frac{1}{2!z^2}+\dots\right)$$

$$= -1\left(1+z+z^2+z^3+\dots\right)\left(1+\frac{1}{z}+\frac{1}{z^2 2!}+\frac{1}{z^3 3!}+\dots\right)$$

$$\operatorname*{Res}_{z=0} f(z) = d_{-1} = -\left(\underbrace{1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\dots}_{e-1}\right) = 1-e$$

$$e^1 = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots$$

I) if $R<1 \Rightarrow I = 2\pi i(1-e)$

II) if $R>1 \Rightarrow I = 2\pi i(e+1-e) = 2\pi i$

Applications of residues to real integrals

SE DAU LA EXAMEN

$$I \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta \Rightarrow I = \int_{C:|z|=1} f\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\cdot\frac{dz}{iz}$$

f is rational function $\Rightarrow 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{\substack{z=z_k \\ z_k\in \operatorname{int} C}} f(z)$

$$z(\theta) = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta$$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2} = \frac{z^2+1}{2z}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$$

① $$I = \int_0^{2\pi} \frac{dx}{\frac{5}{4}+\sin x}$$

$$z = e^{ix},\quad dz = ie^{ix}\,dx$$

$$\sin x = \frac{z^2-1}{2iz}$$

$$I = \int_{C:|z|=1} \frac{dz}{iz}\cdot\frac{1}{\frac{5}{4}+\frac{z^2-1}{2iz}} = \int_C \frac{8\cancel{iz}}{\cancel{iz}\left(10iz+4(z^2-1)\right)}\,dz =$$

$$= \int_C \frac{4}{2z^2+5iz-2}\,dz$$

$$2z^2+5iz-2=0,\quad \Delta=-9$$

$$z_{1,2} = \frac{-5i\pm 3i}{4} \Rightarrow z_1 = -\frac{i}{2},\quad z_2 = -2i$$

$\in \operatorname{int} C$ $\qquad$ $\notin \operatorname{int} C$

pole of order 1

$$\operatorname*{Res}_{z=-\frac{i}{2}} f(z) = \frac{4}{4z+5i}\Bigg|_{z=-\frac{i}{2}} = \frac{4}{-2i+5i} = \frac{4}{3i}$$

$$I = 2\pi i \cdot \frac{4}{3i} = \frac{8\pi}{3}$$

EXAMEN

② $$I = \int_0^{2\pi} \frac{dx}{\sqrt{2}-\cos x}$$

$$z = e^{ix} \quad , \quad dz = ie^{ix}dx$$

$$\cos x = \frac{z^2+1}{2z}$$

$$I = \int_{C:|z|=1} \frac{dz}{iz} \cdot \frac{1}{\sqrt{2}-\frac{z^2+1}{2z}} = \int_{C:|z|=1} \frac{2z}{2\sqrt{2}z-z^2+1} \cdot \frac{1}{zi}\,dz$$

$$= -\frac{2}{i}\int_{C:|z|=1} \frac{1}{z^2-2\sqrt{2}z+1}\,dz$$

$$z^2 - 2\sqrt{2}z + 1 = 0$$

$$\Delta = 4 \quad , \quad z_{1,2} \begin{cases} \sqrt{2}-1 \in \operatorname{int} C \\ \sqrt{2}+1 \notin \operatorname{int} C \end{cases}$$

$$\operatorname*{Res}_{z=\sqrt{2}-1} f(z) = \frac{1}{2z-2\sqrt{2}}\Bigg|_{z=\sqrt{2}-1} = \frac{1}{2\sqrt{2}-2-2\sqrt{2}} = -\frac{1}{2}$$

$$I = -\frac{2}{i} \cdot 2\pi i \cdot \left(-\frac{1}{i}\right) = 2\pi$$

TIP EXAMEN

① $\displaystyle\int_C z^2 \cdot e^{\frac{2z}{z+1}}\, dz$, $C: x^2+y^2+2x=0$

② $\displaystyle\int_0^{2\pi} \frac{1}{(\cos\theta+2)^2}\, d\theta$

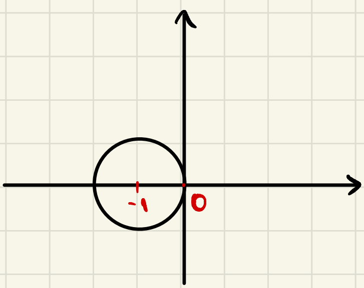
Rezolvare

① $\displaystyle\int_C z^2 \cdot e^{\frac{2z}{z+1}}\, dz$, $C: x^2+y^2+2x=0$

$(x+1)^2+y^2=1$

$\mathcal{C}((-1,0), 1)$

$(x-x_0)^2+(y-y_0)^2=R^2$



$z=-1 \in \operatorname{int} C$

essential singularity

$$f(z) = (z+1-1)^2 e^{2\frac{z+1-1}{z+1}} =$$

$$= \left[(z+1)^2 - 2(z+1) + 1\right] e^2 \cdot e^{-\frac{2}{z+1}} =$$

$$= e^2 \left[(z+1)^2 - 2(z+1) + 1\right] \cdot \left[1 + \left(\frac{-2}{z+1}\right) \cdot \frac{1}{1!} + \left(\frac{-2}{z+1}\right) \frac{1}{2!} + \dots\right.$$

$$\operatorname*{Res}_{z=-1} f(z) = a_{-1} = e^2\left(-\overset{6}{2} - \frac{8}{6} - \overset{6}{4}\right) = e^2\left(-\frac{22}{3}\right)$$

$$I = 2\pi i \, e^2 \left(-\frac{22}{3}\right)$$

② $\displaystyle\int_0^{2\pi} \frac{1}{(\cos\theta + 2)^2} d\theta$

$$z = e^{ix} \Rightarrow dz = ie^{ix} dx$$

$$\cos\theta = \frac{z^2+1}{2z}$$

$$\int_{C:|z|=1} \frac{dz}{iz} \cdot \frac{1}{\left(\frac{z^2+1}{2z} + 2\right)^2} = \int_{C:|z|=1} \frac{dz}{iz} \frac{1}{\left(\frac{z^2+1+4z}{2z}\right)^2}$$

$$\int_{C:|z|=1} \frac{dz}{iz} \cdot \frac{4z^2}{(z^2+4z+1)^2}$$

$z^2+4z+1=0$, $\Delta=12$

$$z_{1,2} \begin{cases} -2-\sqrt{3} \notin \operatorname{int} C \\ -2+\sqrt{3} \in \operatorname{int} C \end{cases}$$

$$\operatorname*{Res}_{z=-2+\sqrt{3}} f(z) =$$