Mathematical Analysis
An 1
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Curs 12

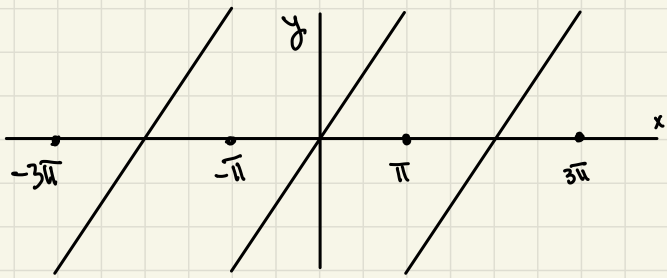
T_{23} (Dirichlet) If a function f is piecewise smooth, then it's Fourier Series converges to:

$$\frac{f(x-0)+f(x+0)}{2},$$

(left hand limit) (right hand limit) for all $x \in \mathbb{R}$

Ex 24 Let expand the function $x \longmapsto \frac{x}{2}$, $x \in (-\pi, \pi)$, as a Fourier series:
Consider the function $f: \mathbb{R}$, with period 2π

$$f(x) = \begin{cases} 0, & x = -\pi, \\ \frac{x}{2}, & x \in (-\pi, \pi), \\ 0, & x = \pi, \end{cases}$$



Since the function f is odd we have:

$$a_n = 0, \quad n \in \mathbb{N}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \frac{x}{2} \sin nx \, dx = \frac{(-1)^{n+1}}{n}, \quad n \in \mathbb{N}^*$$

From the equality

$$f(x) = \frac{f(x+0)+f(x-0)}{2}, \quad x \in \mathbb{R}$$

using Dirichlet's Theorem 23 we obtain:

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad x \in \mathbb{R}$$

Hence:

$$\frac{x}{2}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx \ , x\in(-\pi,\pi) \qquad (1)$$

R_{25} From Eq. (1), for $x=\frac{\pi}{2}$, we deduce:

$$\frac{\pi}{4}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2}=\sum_{k=0}^{\infty}\frac{(-1)^{k}}{2k+1},$$

and we re-obtain the Gregory - Leibniz Formula.

R_{26} If a function $f\in C^1(\mathbb{R})$ is periodic, then its Fourier series converges uniformly to f.

R_{27} The Fourier series associated to any integrable function $f:[a,b]\to\mathbb{R}$

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi}{b-a}x+b_n\sin\frac{2n\pi}{b-a}x\right),$$

where the coefficients are given by

$$a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi}{b-a}x\,dx$$

$$b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi}{b-a}x\,dx$$

T_{28} (Parseval) If a function $f:[-\pi,\pi]\to\mathbb{R}$ and the square of its modulus are integrable, then its Fourier coefficients satisfy the equality

$$\frac{a_0^2}{2}+\sum_{n\geq 1}(a_n^2+b_n^2)=\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)\,dx$$

E_{29} Prove that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

cf 1 $\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \, , x \in (-\pi, \pi)$

For odd functions, Parseval's equality becomes

$$\sum_{n=1}^{\infty} (b_n)^2 = \frac{2}{\pi} \int_0^{\pi} (f(x))^2 dx ,$$

hence

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2}{\pi} \int_0^{\pi} \frac{x^2}{4} dx = \frac{\pi^2}{6}$$

P_{30} $\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \, , x \in (0, 2\pi)$

We have (1)

$$\frac{t}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sin nt} \, , t \in (-\pi, \pi)$$

With $t := \pi - x$, we obtain:

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n(\pi - x) \, , x \in (0, 2\pi)$$

i.e.

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \, , x \in (0, 2\pi)$$