P 30 $$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \qquad x \in (0, 2\pi).$$

We have (♦1)

$$\boxed{\frac{t}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt, \qquad t \in (-\pi, \pi).}$$

With $t :=$ $\boxed{\pi - x,}$ we obtain

$$\boxed{\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n(\pi - x), \qquad x \in (0, 2\pi).}$$

i.e., $$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx, \qquad x \in (0, 2\pi) \quad ✔$$

P 31 $$1 = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}, \qquad x \in (0,\pi).$$

Consider the odd function $f : \mathbb{R} \longrightarrow \mathbb{R}$,

$$f(x) = \begin{cases} 0, & x = -\pi, \\ -1, & x \in (-\pi, 0), \\ 0, & x = 0, \\ 1, & x \in (0,\pi), \\ 0, & x = \pi, \end{cases} \qquad \text{with period } 2\pi.$$

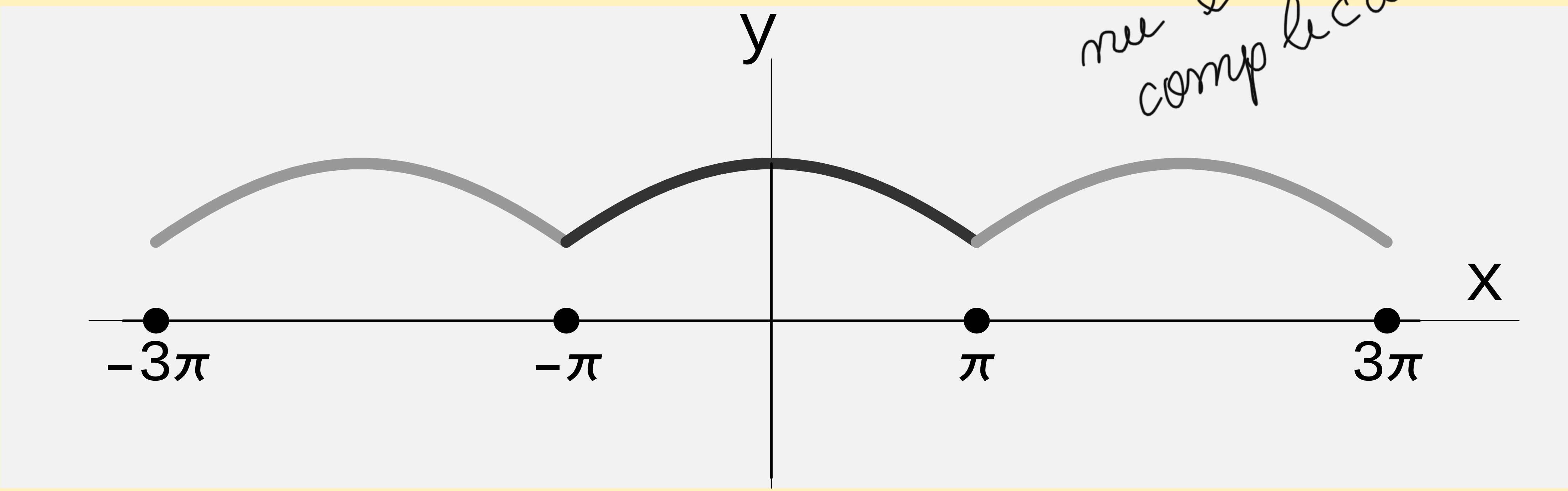
We obtain:

$$a_n = 0, \qquad n \in \mathbb{N};$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin nx \, \mathrm{d}x = \frac{2}{\pi}\frac{1 + (-1)^{n+1}}{n}, \qquad n \in \mathbb{N}^*;$$

P 32 $$\cos ax = \frac{2\sin a\pi}{\pi}\left(\frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^n a}{a^2 - n^2}\cos nx\right),$$
$$a \in \mathbb{C} \setminus \mathbb{Z},\ x \in [-\pi, \pi].$$

We expand the even function $\boxed{x \to \cos ax,}$ $x \in [-\pi, \pi]$, **with period** $\boxed{2\pi}$, **into a Fourier series.**



nu e complicat

We have:

$$b_n = 0, \quad n \in \mathbb{N}^*,$$

$$\boxed{a_n = \frac{2}{\pi}\int_0^\pi \cos ax \cos nx \, dx = \frac{1}{\pi}\int_0^\pi \left(\cos(a+n)x + \cos(a-n)x\right) dx}$$

$$= \boxed{(-1)^n \frac{2a \sin a\pi}{\pi(a^2-n^2)}, \quad n \in \mathbb{N}}$$

hence

$$\cos ax = \frac{2\sin a\pi}{\pi}\left(\frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^n a}{a^2-n^2}\cos nx\right),$$

$x \in [-\pi, \pi]$ ✔

The Infinite Product Expansion of the Sine Function

P 33

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right), \qquad x \in (-\pi, \pi).$$

Taking $x = \boxed{\pi}$ **in**

$$\cos ax = \frac{2\sin a\pi}{\pi}\left(\frac{1}{2a}+\sum_{n=1}^{\infty}\frac{(-1)^n a}{a^2-n^2}\cos nx\right)$$

we obtain

$$\boxed{\cot a\pi - \frac{1}{a\pi} = \sum_{n=1}^{\infty}\frac{2a\pi}{a^2\pi^2-n^2\pi^2}}$$

Consequently, for $\boxed{a\pi} = t \in (0, \pi)$, we get

$$\boxed{\cot t - \frac{1}{t} = \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2\pi^2} \Big| \int_0^x}$$

Since

$$\int \frac{2t}{t^2 - n^2\pi^2}\,\mathrm{d}t = \log(n^2\pi^2 - t^2) \quad \boxed{-\log(n^2\pi^2)}$$

we obtain

$$\log \frac{\sin x}{x} = \log \sin x - \log x = \boxed{\int_0^x \left(\cot t - \frac{1}{t}\right) dt}$$

$$= \sum_{n=1}^{\infty} \log\left(1 - \frac{x^2}{n^2\pi^2}\right) = \log \boxed{\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)} \quad ✔$$

Resolution of $\boxed{1/\sin x}$ into Partial Fractions

P 34 $$\frac{1}{\sin x} = \boxed{\sum_{n=-\infty}^{\infty} \frac{A_n}{x-n\pi}}, \qquad x \in \mathbb{R} \setminus \pi\mathbb{Z}.$$

Taking $x = \boxed{0}$ in

$$\cos ax = \frac{2\sin a\pi}{\pi}\left(\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{(-1)^n a}{a^2-n^2}\cos nx\right),$$

we get

$$\boxed{\frac{1}{\sin a\pi} = \frac{1}{a\pi} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{a\pi-n\pi}+\frac{1}{a\pi+n\pi}\right);}$$

hence, for $\boxed{a\pi} = x$, we obtain

$$\boxed{\frac{1}{\sin x} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{x-n\pi}, \qquad x \in \mathbb{R}\setminus\pi\mathbb{Z}}$$ ✔

Resolution of 1/Sine into Partial Fractions

P 34
$$\frac{1}{\sin x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x-n\pi}, \qquad x \in \mathbb{R} \setminus \pi\mathbb{Z}.$$

Taking $x = 0$ **in**
$$\cos ax = \frac{2\sin a\pi}{\pi}\left(\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{(-1)^n a}{a^2-n^2}\cos nx\right),$$
we get

$$\frac{1}{\sin a\pi} = \frac{1}{a\pi} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{a\pi-n\pi} + \frac{1}{a\pi+n\pi}\right);$$

hence, for $a\pi = x$, **we obtain**

$$\frac{1}{\sin x} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{x-n\pi}, \qquad x \in \mathbb{R} \setminus \pi\mathbb{Z}$$

✔

The Infinite Product Expansion

of the Tangent Function

P 35 $\tan x = x\prod_{n=1}^{\infty} \dfrac{1-\dfrac{x^2}{n^2\pi^2}}{1-\dfrac{4x^2}{(2n-1)^2\pi^2}}$, $\qquad x \in (-\pi/2, \pi/2).$

Taking $x=0$ **in**

$$\cos ax = \frac{2\sin a\pi}{\pi}\left(\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{(-1)^n a}{a^2-n^2}\cos nx\right)$$

we obtain

$$\frac{1}{\sin a\pi} - \frac{1}{a\pi} = \sum (-1)^n \frac{2a\pi}{(a\pi)^2-(n\pi)^2}$$

For $\boxed{\alpha\pi} = 2t \in (0, \pi)$, we get

$$\boxed{\frac{1}{\sin 2t} - \frac{1}{2t} = \sum_{n=1}^{\infty} (-1)^n \frac{4t}{4t^2 - n^2\pi^2}}$$

$$\boxed{\int \frac{1}{\sin 2t}\,dt = \int \frac{1}{2\sin t\cos t}\,dt = \int \frac{1}{2\frac{\sin t}{\cos t}\cos^2 t} \qquad \frac{1}{\frac{\sin t}{\cos t}\cos^2 t}}$$

$$= \boxed{\frac{1}{2}\int \frac{(\tan t)'}{\tan t}\,dt = \frac{1}{2}\log|\tan t|}$$

$$\boxed{\frac{1}{2}\log\frac{\tan x}{x} = \int_0^x \left(\frac{1}{\sin 2t} - \frac{1}{2t}\right)dt = \sum_{n=1}^{\infty} (-1)^n \int_0^x \frac{4t}{4t^2 - n^2\pi^2}\,dt}$$

$$= \boxed{\frac{1}{2}\sum_{n=1}^{\infty} (-1)^n \log\left(1 - \frac{4x^2}{n^2\pi^2}\right) = \frac{1}{2}\log \prod_{n=1}^{\infty} \frac{1 - \frac{x^2}{n^2\pi^2}}{1 - \frac{4x^2}{(2n-1)^2\pi^2}} \quad ✔}$$

P 36 $$\sum_{n=0}^{\infty} \frac{\cos nx}{n!} = e^{\cos x} \cos(\sin x),$$
$$\sum_{n=0}^{\infty} \frac{\sin nx}{n!} = e^{\cos x} \sin(\sin x), \qquad x \in \mathbb{R}.$$

In $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{C}$, we take $z = \boxed{e^{ix}} = \cos x + i \sin x$,

we obtain:

$$e^{\cos x + i\sin x} = e^{e^{ix}} = \sum_{n=0}^{\infty} \frac{(e^{ix})^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{nix}}{n!} = \sum_{n=0}^{\infty} \frac{\cos nx + i \sin nx}{n!},$$

hence

$$e^{\cos x}\big(\cos(\sin x) + i \sin(\sin x)\big) = \sum_{n=0}^{\infty} \frac{\cos nx}{n!} + i \sum_{n=0}^{\infty} \frac{\sin nx}{n!}$$