# Lecture 13: Noncoherent Detection of Digital Modulation

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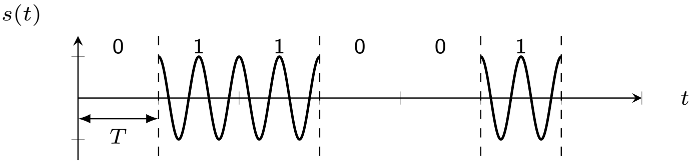
# Outline

- Noncoherent demodulation of ASK
- Noncoherent demodulation of FSK
- Differential demodulation of DPSK
- Reference
  - ✓ [Haykin] Chapter 9

# Coherent vs. Noncoherent Demodulation

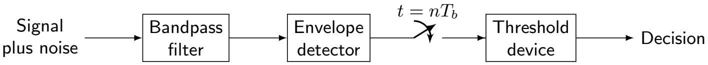
- Coherent demodulation assumes perfect synchronization
  - ✓ Needs a phase locked loop (very complicated)
- Accurate phase synchronization: difficult in a dynamic channel
  - ✓ Phase synchronization error is due to varying propagation delays, frequency drift, instability of the local oscillator, effects of strong noise ...
  - ✓ Performance of coherent detection will degrade severely
- For unknown phase, use non-coherent detection
  - ✓ No provision is made for carrier phase recovery
- Simpler circuitry/receiver but hard to analyze

# ASK Waveform




$$s(t) = \begin{cases} A\cos(2\pi f_c t), & \text{if transmitting "1"} \\ 0, & \text{otherwise} \end{cases}$$

# Noncoherent Demodulation of ASK




(Similar to noncoherent of AM)

- Output of the BPF

$$y(t) = \begin{cases} n(t), & \text{if 0 is sent,} \\ n(t) + A\cos(2\pi f_c t), & \text{if 1 is sent.} \end{cases}$$

- Recall

$$n(t) = n_{\mathsf{I}}(t)\cos(2\pi f_c t) - n_{\mathsf{Q}}(t)\sin(2\pi f_c t)$$

- Envelope

$$r(t) = \begin{cases} \sqrt{n_{\mathsf{I}}^2(t) + n_{\mathsf{Q}}^2(t)}, & \text{if 0 is sent,} \\ \sqrt{\left(A + n_{\mathsf{I}}(t)\right)^2 + n_{\mathsf{Q}}^2(t)}, & \text{if 1 is sent.} \end{cases}$$

$$r(t) = \begin{cases} \sqrt{n_{\mathsf{I}}^2(t) + n_{\mathsf{Q}}^2(t)}, & \text{if 0 is sent,} \\ \sqrt{\left(A + n_{\mathsf{I}}(t)\right)^2 + n_{\mathsf{Q}}^2(t)}, & \text{if 1 is sent.} \end{cases}$$

- Symbol 0 sent $\Rightarrow$ envelope $r(t) = \sqrt{n_{\mathsf{I}}^2(t) + n_{\mathsf{Q}}^2(t)}$: Rayleigh distribution

$$f(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}, \quad r \geq 0 \quad \text{(Lect 3)}$$

- Symbol 1 sent $\Rightarrow$ envelope $r(t) = \sqrt{\left(A + n_{\mathsf{I}}(t)\right)^2 + n_{\mathsf{Q}}^2(t)}$: Rician distribution

$$f(r) = \frac{r}{\sigma^2} e^{-(r^2+A^2)/2\sigma^2} I_0\left(\frac{Ar}{\sigma^2}\right), \quad r \geq 0, \quad \text{(HW 1)}$$

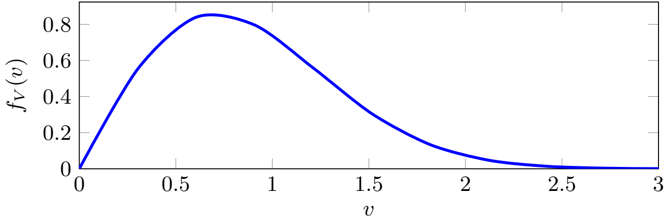
where $I_0(\cdot)$: the modified zero-order Bessel function of the first kind,

$$I_0(x) \triangleq \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos\theta} d\theta$$

- Define a random variable $R = \sqrt{X^2 + Y^2}$ where $X$ and $Y$ are i.i.d. Gaussian random variable with zero mean and variance $\sigma^2$

- $R$ has Rayleigh distribution

$$f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}, \quad r \geq 0$$




Normalized Rayleigh distribution: $v = r/\sigma$, $f_V(v) = \sigma f_R(r)$

- Proving it requires change into polar coordinates

$$R = \sqrt{X^2 + Y^2}, \quad \Theta = \arctan \frac{Y}{X}$$

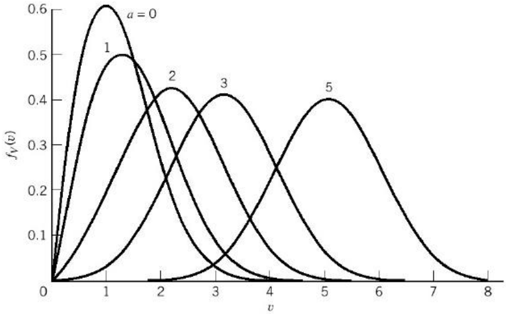
# Rician Distribution

- If $X$ has non-zero mean $A$, $R$ has Rician distribution:

$$f_R(r) = \frac{r}{\sigma^2} e^{-(r^2+A^2)/2\sigma^2} I_0\left(\frac{Ar}{\sigma^2}\right), \quad r \geq 0,$$

where $I_0(\cdot)$ is the modified zero-order Bessel function of the first kind

$$I_0(x) \triangleq \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos \theta} d\theta$$

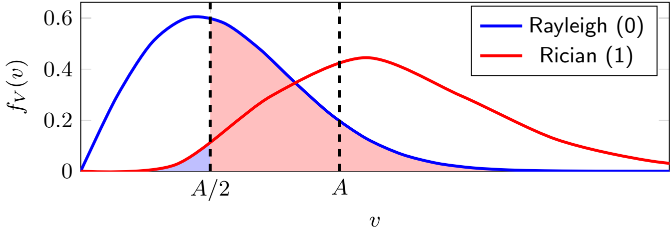



Normalized Rician distribution: $v = r/\sigma$, $a = A/\sigma$, $f_V(v) = \sigma f_R(r)$

# Error Probability

- Let threshold $\lambda = A/2$ for simplicity
- Error probability: *dominated by symbol* $0$ and given by

$$P_{e,\text{ASK,noncoherent}} \approx \frac{1}{2}\int_{A/2}^{\infty} \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} dr = \frac{1}{2} e^{-A^2/8\sigma^2}$$




- Coherent demodulation

$$P_{e,\text{ASK,coherent}} = Q\left(\frac{A}{2\sigma}\right) \leq \frac{1}{2} e^{-A^2/8\sigma^2}$$

- Noncoherent demodulation results in some performance degradation

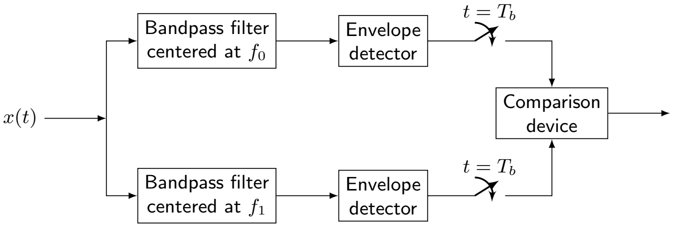
# ASK Example

## ASK Modulation System with Noncoherent Demodulation

- Carrier Amplitude $A = 0.7$ V
- Standard Deviation of White Gaussian Noise $\sigma = 0.125$ V
- Symbols "0" and "1" with equal probability

What is BER?

# Noncoherent Demodulation of FSK




$$\begin{cases} \text{choose } 0, & \text{if } R_0 > R_1 \\ \text{choose } 1, & \text{if } R_0 < R_1 \end{cases}$$

# Distribution of Envelope

- If a symbol 1 is sent, output of the BPFs

$$y_0(t) = n_0(t)$$
$$y_1(t) = n_1(t) + A\cos(2\pi f_1 t)$$

- First branch: Rayleigh distribution

$$f_{R_0}(r_0) = \frac{r_0}{\sigma^2} e^{-r_0^2/2\sigma^2}, \quad r_0 \geq 0$$

- Second: Rician distribution

$$f_{R_1}(r_1) = \frac{r_1}{\sigma^2} e^{-(r_1^2+A^2)/2\sigma^2} I_0\left(\frac{Ar_1}{\sigma^2}\right), \quad r_1 \geq 0,$$

- Envelopes $R_0$ and $R_1$ are independent

- Error probability (derivation omitted)

$$P_{e,\text{FSK,noncoherent}} = \Pr\{R_1 < R_0\} = \frac{1}{2} e^{-A^2/4\sigma^2}$$

- Error occurs if Rice < Rayleigh

$$
\begin{aligned}
P_e &= P(R_1 < R_0) \\
&= \int_0^\infty \int_{r_1}^\infty \frac{r_1}{\sigma^2} e^{-(r_1^2+A^2)/2\sigma^2} I_0\left(\frac{Ar_1}{\sigma^2}\right) \frac{r_0}{\sigma^2} e^{-r_0^2/2\sigma^2} dr_0 dr_1 \\
&= \int_0^\infty \frac{r_1}{\sigma^2} e^{-(r_1^2+A^2)/2\sigma^2} I_0\left(\frac{Ar_1}{\sigma^2}\right) \int_{r_1}^\infty \frac{r_0}{\sigma^2} e^{-r_0^2/2\sigma^2} dr_0 dr_1 \\
&= \int_0^\infty \frac{r_1}{\sigma^2} e^{-(2r_1^2+A^2)/2\sigma^2} I_0\left(\frac{Ar_1}{\sigma^2}\right) dr_1 \\
&\overset{x=\sqrt{2}r_1, \alpha=A/\sqrt{2}}{=\!=\!=\!=\!=\!=} \frac{1}{2} e^{-A^2/4\sigma^2} \int_0^\infty \frac{x}{\sigma^2} e^{(x^2+\alpha^2)/2\sigma^2} I_0\left(\frac{\alpha x}{\sigma^2}\right) dx
\end{aligned}
$$

- Observe that the integrand is a Rician density

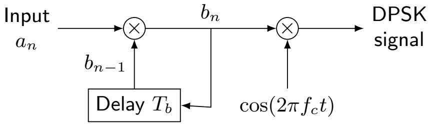
$$
P_{e,\text{FSK,noncoherent}} = \frac{1}{2} e^{-A^2/4\sigma^2}
$$

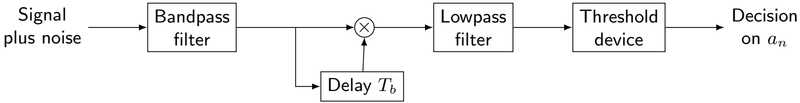
- Coherent modulation

$$
P_{e,\text{FSK,coherent}} = Q\left(\frac{A}{\sqrt{2}\sigma}\right) \leq \frac{1}{2} e^{-A^2/4\sigma^2}
$$

# DPSK: Differential PSK

- Impossible to demodulate PSK with an envelop detector since PSK signals have the same frequency and amplitude
- Demodulating PSK differentially, where phase reference is provided by a delayed version of the signal in the previous interval
- Essential to encode differentially: $b_n = b_{n-1} \times a_n$ where $a_n, b_n \in \pm 1$






- Probability of error ([Haykin] Chapter 9)

$$P_{e,\text{DPSK}} = \frac{1}{2} e^{-A^2/2\sigma^2}$$

- Coherent demodulation

$$P_{e,\text{PSK}} = Q\left(\frac{A}{\sigma}\right) \leq \frac{1}{2} e^{-A^2/2\sigma^2}$$

# Illustration of DPSK

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| Information symbols $\{a_n\}$ | | 1 | -1 | -1 | 1 | -1 | -1 | 1 | 1 |
| $\{b_{n-1}\}$ | | 1 | 1 | -1 | 1 | 1 | -1 | 1 | 1 |
| DPSK sequence $\{b_n = a_n \times b_{n-1}\}$ | 1 | 1 | -1 | 1 | 1 | -1 | 1 | 1 | 1 |
| Transmitted phase (radians) | 0 | 0 | $\pi$ | 0 | 0 | $\pi$ | 0 | 0 | 0 |
| Output of lowpass filter (polarity) | | $+$ | $-$ | $-$ | $+$ | $-$ | $-$ | $+$ | $+$ |
| Decision | | 1 | -1 | -1 | 1 | -1 | -1 | 1 | 1 |

Symbol 1 ($b_0 = 1$) is inserted at the beginning of the differentially encoded sequence.

# PSK Example

## DPSK Modulation System with Differential Demodulation

- Carrier Amplitude $A = 0.7$ V
- Standard Deviation of White Gaussian Noise $\sigma = 0.125$ V
- Symbols "0" and "1" with equal probability

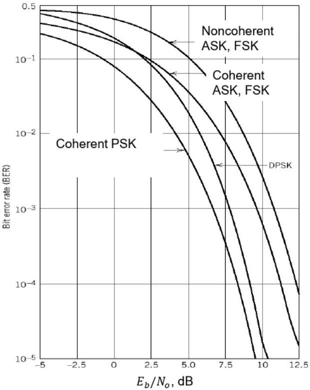
What is BER?

# Application: DPSK

- WLAN standard IEEE 802.11b
- Bluetooth2
- Digital audio broadcast (DAB): DPSK + OFDM (orthogonal frequency division multiplexing)
- Inmarsat (International Maritime Satellite Organization): now a London-based mobile satellite company

# Summary and Comparison

| Scheme | Bit-Error Rate (BER) |
|---|---|
| Coherent ASK | $Q\left(\frac{A}{2\sigma}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$ |
| Coherent FSK | $Q\left(\frac{A}{\sqrt{2}\sigma}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$ |
| Coherent PSK | $Q\left(\frac{A}{\sigma}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$ |
| Noncoherent ASK | $\frac{1}{2}\exp\left(-\frac{A^2}{8\sigma^2}\right) = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$ |
| Noncoherent FSK | $\frac{1}{2}\exp\left(-\frac{A^2}{4\sigma^2}\right) = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$ |
| Noncoherent PSK | $\frac{1}{2}\exp\left(-\frac{A^2}{2\sigma^2}\right) = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$ |




- ASK: $E_b/N_0 = A^2/4\sigma^2$
- FSK: $E_b/N_0 = A^2/2\sigma^2$
- PSK: $E_b/N_0 = A^2/2\sigma^2$

- Non-coherent demodulation retains the hierarchy of performance
- Non-coherent demodulation has error performance slightly worse than coherent demodulation, but approaches coherent performance at high SNR
- Non-coherent demodulators are considerably easier to build

# Note