Lecture 5: Noise

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Outline

- What is noise?
- White noise and Gaussian noise
- Lowpass noise
- Bandpass noise
  - ✓ In-phase/quadrature representation
  - ✓ Phasor representation
- Reference
  - ✓ [Haykin] Chapter 5

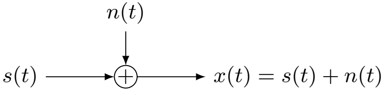
Noise

Noise: unwanted waves disturbing the transmission of signals.

- Where does noise come from?
  - ✓ External sources: e.g., atmospheric, galactic noise, interference.
  - ✓ Internal sources: generated by communication devices themselves.
    - A basic limitaion on communication systems
    - Shot noise: usually in vacuum tubes or transistors
    - Thermal noise: caused by rapid and random motion of electrons due to thermal agitation
- Stationary and zero-mean Gaussian distributions.

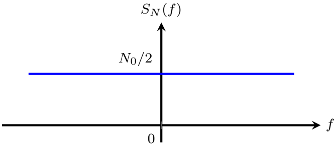
White Noise

- The additive noise channel
  - ✓ $n(t)$ models all types of noise
  - ✓ Zero mean



- White noise
  - ✓ Flat PSD over *all frequencies*

$$S_N(f) = \frac{N_0}{2}, \quad -\infty < f < \infty$$

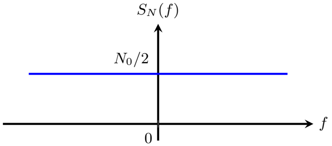


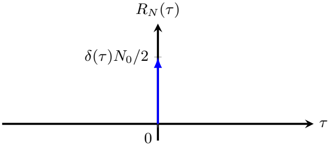
- Half the power $\frac{N_0}{2}$ associated with positive frequencies and half with negative
- The term white analogous to white light, indicating the shape of the PSD!
- Defined for stationary noise
- There are also non-stationary noises, but definitions are complicated

- *Infinite* bandwidth: a purely theoretic assumption, valid for flat PSD over the bandwidth of interest

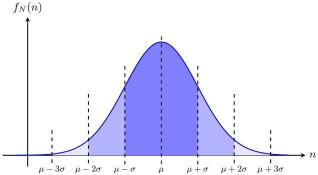
White and Gaussian Noise

- White noise: shape of PSD is flat!
  - ✓ Autocorrelation function of $n(t)$: $R_N(\tau) = \mathcal{F}^{-1}\{S_N(f)\} = \frac{N_0}{2}\delta(\tau)$
  - ✓ Uncorrelated samples at different time instants
  - ✓ Also colored noise
  - ✓ White noise: either Gaussian or non-Gaussian





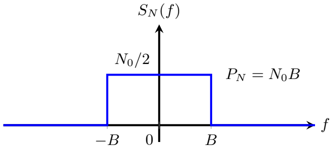
- Gaussian noise:
  - ✓ Gaussian distribution for a sample at any *time instant*
  - ✓ Colored or white Gaussian noise



- White noise (shape of PSD) and Gaussian noise (distribution) are different concepts
- In communications, we typically consider additive white Gaussian noise (AWGN)

- White noise passing through an ideal low-pass filter of bandwidth B (counting only positive part!)

$$S_N(f) = \begin{cases} \frac{N_0}{2}, & |f| \leq B, \\ 0, & \text{otherwise} \end{cases}$$

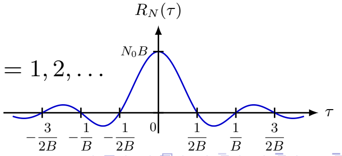


- By Einstein-Wiener-Khinchine relation, autocorrelation function

$$R_N(\tau) = \mathcal{F}^{-1}\{S_N(f)\} = N_0 B \text{sinc}(2B\tau), \quad \text{sinc}(x) = \frac{\sin \pi x}{\pi x} \quad \text{(Normalized definition)}$$

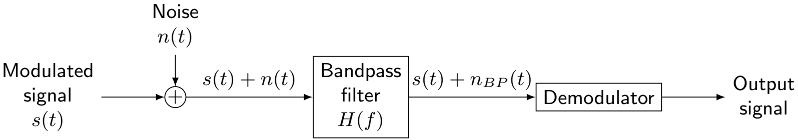
- Samples at Nyquist frequency $2B$ are uncorrelated, also independent if it is Gaussian

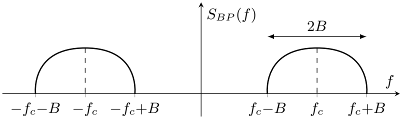
$$R_N(\tau) = 0, \quad \text{for } 2\pi B\tau = k\pi \text{ or } \tau = \frac{k}{2B},\ k = 1, 2, \ldots$$



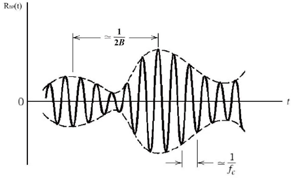
WSS Bandpass Noise

Noise in a communication system with a bandpass filter of bandwidth $2B$ (counting only positive frequency, but both sides of f_c)





PSD

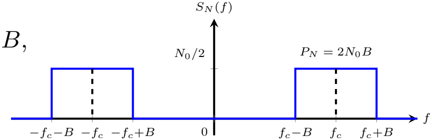


Autocorrelation

Example

White noise with PSD of $N_0/2$ passing through an ideal band-pass filter

$$S_N(f) = \begin{cases} \frac{N_0}{2}, & |f - f_c| \leq B \text{ or } |f + f_c| \leq B, \\ 0, & \text{otherwise} \end{cases}$$

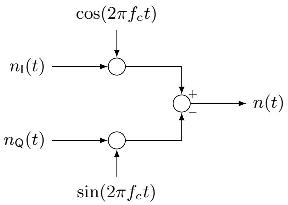


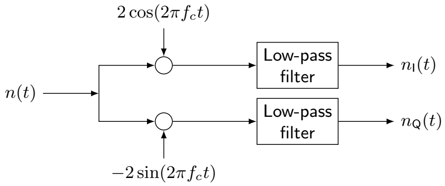
- $n(t)$ in canonical form:

$$n(t) = \underbrace{n_{\mathsf{I}}(t)}_{\text{real}} \cos(2\pi f_c t) - \underbrace{n_{\mathsf{Q}}(t)}_{\text{real}} \sin(2\pi f_c t)$$

where $\mathbb{E}\{n_{\mathsf{I}}(t)\} = \mathbb{E}\{n_{\mathsf{Q}}(t)\} = 0$, $R_{N_{\mathsf{I}}}(\tau) = R_{N_{\mathsf{Q}}}(\tau)$, $R_{N_{\mathsf{I}}N_{\mathsf{Q}}}(\tau) = -R_{N_{\mathsf{I}}N_{\mathsf{Q}}}(-\tau)$

- $n_{\mathsf{I}}(t)$ and $n_{\mathsf{Q}}(t)$: fully representative of the band-pass noise
  - ✓ Given band-pass noise, one may extract in-phase and quadrature components (using LPF of bandwidth B).
  - ✓ Given the two components, one may generate band-pass noise. This is useful in computer simulation.





Properties of Baseband Noise

- $R_N(\tau) = R_{N_\mathsf{I}}(\tau)\cos(2\pi f_c\tau) + R_{N_\mathsf{Q}}(\tau)\sin(2\pi f_c\tau)$
  - ✓ $R_N(\tau) = R_{N_\mathsf{I}}(\tau)\cos(2\pi f_c\tau)$ if $n_\mathsf{I}(t)$ and $n_\mathsf{Q}(t)$ are uncorrelated
- $n_\mathsf{I}(t)$ and $n_\mathsf{Q}(t)$ have the same variance (i.e., same power) as $n(t)$
- Both in-phase and quadrature components have the same PSD:

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f-f_c) + S_N(f+f_c), & |f| \leq B \\ 0, & \text{otherwise} \end{cases}$$

- If noise $n(t)$ is Gaussian, then so are $n_\mathsf{I}(t)$ and $n_\mathsf{Q}(t)$

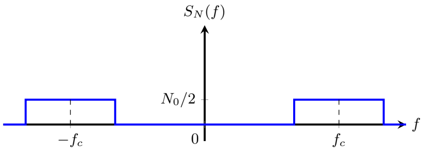
Noise Power: Narrowband White Noise

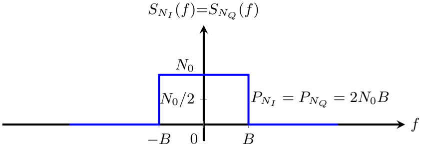
- For ideally filtered narrowband white noise, the PSDs of $n_{\mathsf{I}}(t)$ and $n_{\mathsf{Q}}(t)$:

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} N_0, & |f| \leq B, \\ 0, & \text{otherwise} \end{cases}$$

- The average power in *each* of the baseband waveforms $n_{\mathsf{I}}(t)$ and $n_{\mathsf{Q}}(t)$ is identical to the average power in the bandpass noise waveform $n(t)$.
- For ideally filtered narrowband noise, the variance of $n_{\mathsf{I}}(t)$ and $n_{\mathsf{Q}}(t)$:

$$P_{N_I} = P_{N_Q} = 2N_0B$$





Bandpass noise in alternative form:

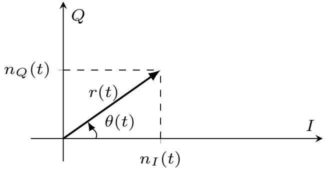
$$n(t) = n_{\mathsf{I}}(t)\cos(2\pi f_c t) - n_{\mathsf{Q}}(t)\sin(2\pi f_c t) = r(t)\cos\big(2\pi f_c t + \phi(t)\big)$$

- $r(t) = \sqrt{n_{\mathsf{I}}^2(t) + n_{\mathsf{Q}}^2(t)}$: the envelope of the noise following Rayleigh distribution

$$f(r) = \frac{r}{\sigma^2}\exp(-\frac{r^2}{2\sigma^2})$$

- $\phi(t) = \arg(n_{\mathsf{I}}(t) + jn_{\mathsf{Q}}(t))$: the phase of the noise following uniform distribution

$$f(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le x < 2\pi, \\ 0, & \text{otherwise} \end{cases}$$



Summary

- White noise: constant PSD over an infinite bandwidth
- Gaussian noise: Gaussian distribution for any sample
- Bandpass noise:
  - ✓ In-phase and quadrature components $n_{\mathsf{I}}(t)$ and $n_{\mathsf{Q}}(t)$ are low-pass random processes.
  - ✓ The same PSD for $n_{\mathsf{I}}(t)$ and $n_{\mathsf{Q}}(t)$
  - ✓ $\mathbb{E}\{n_{\mathsf{I}}^2(t)\} = \mathbb{E}\{n_{\mathsf{Q}}^2(t)\} = \mathbb{E}\{n^2(t)\}$

Note