Lecture 10: Matched Filter

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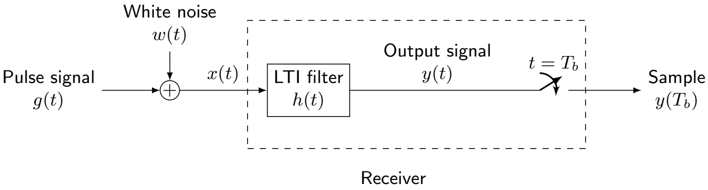
Outline

- Matched filter
  - ✓ Impulse response
  - ✓ Maximum peak SNR
- Binary baseband communication
  - ✓ Distribution of noise
  - ✓ Decision rule
  - ✓ Error cases and probabilities
- References
  - ✓ [Haykin] Chapter 8

Analog and Digital Communications

- Analog communication systems: reproducing transmitted waveform accurately
  - ✓ Signal-to-noise ratio (SNR) to assess the quality of the system
- Digital communication systems: recovering the transmitted symbol correctly
  - ✓ Probability of error or bit-error rate (BER) at the receiver to assess the quality of the system



$$x(t) = g(t) + w(t), \quad 0 \le t \le T_b$$
$$y(t) = x(t) * h(t)$$

- $w(t)$: white noise with zero mean and PSD $N_0/2$
- Goal:
  - ✓ Detect whether a pulse presents with known pulse shape $g(t)$
  - ✓ Design (and optimize) the receive filter $h(t)$ to minimize the noise effect

- Filter output:

$$y(t) = x(t) * h(t) = g(t) * h(t) + w(t) * h(t) = g_o(t) + n(t)$$

where

$$g_o(t) = g(t) * h(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau$$

$$n(t) = w(t) * h(t)$$

- We want instantaneous power of signal component $g_o(t)$ at time $t = T_b$ as large as possible compared to noise component $n(t)$

- Maximize peak signal-to-noise ratio

$$\eta = \frac{|g_o(T_b)|^2}{\mathbb{E}\{n^2(T_b)\}} = \frac{\text{instantaneous power}}{\text{average power}}$$

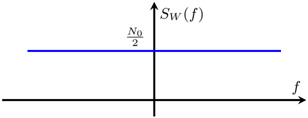
Matched Filter Derivation: I

- Noise

$$n(t) = w(t) * h(t)$$

$$S_N(f) = \underbrace{S_W(f)}_{\text{AWGN}} \underbrace{S_H(f)}_{\text{receive filter}} = \frac{N_0}{2}|H(f)|^2$$

$$\mathbb{E}\{n^2(t)\} = \int_{-\infty}^{\infty} S_N(f)df = \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2 df$$



- Signal

$$g_o(t) = g(t) * h(t) = \int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi ft}df$$

$$G_o(f) = H(f)G(f)$$

$$|g_o(T_b)|^2 = \left|\int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi fT_b}df\right|^2$$

Matched Filter Derivation: II

- Find $h(t)$ to maximize peak SNR

$$\eta = \frac{\left| \int_{-\infty}^{\infty} \overbrace{H(f)}^{\phi_1(x)} \overbrace{G(f) e^{j2\pi f T_b}}^{\phi_2^*(x)} df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df}$$

- Schwartz's inequality: Consider two energy signals $\phi_1(x)$ and $\phi_2(x)$,

$$\left| \int_{-\infty}^{\infty} \phi_1(x) \phi_2(x)^* dx \right|^2 \leq \int_{-\infty}^{\infty} |\phi_1(x)|^2 dx \int_{-\infty}^{\infty} |\phi_2(x)|^2 dx,$$

where equality holds if and only if $\phi_1(x) = k\phi_2(x)$ for an arbitrary constant k.

Matched Filter Derivation: III

Let $\phi_1(f) = H(f)$ and $\phi_2(f) = G^*(f)e^{-j2\pi f T_b}$,

$$\left|\int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi f T_b}df\right|^2 \leq \int_{-\infty}^{\infty}|H(f)|^2 df \int_{-\infty}^{\infty}|G(f)|^2 df$$

$$\eta = \frac{\left|\int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi f T_b}df\right|^2}{\frac{N_0}{2}|H(f)|^2} \leq \frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2 df,$$

where $\eta_{\max} = \frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2 df$ occurs if $H_{\text{opt}}(f) = kG^*(f)e^{-j2\pi f T_b}$.

$$G^*(f) \Leftrightarrow g^*(-t)$$

$$G^*(f)e^{-j2\pi f T_b} \Leftrightarrow g^*\big(-(t-T_b)\big) = g^*(T_b - t)$$

Hence,

$$h_{\text{opt}}(t) = kg^*(T_b - t) = kg(T_b - t)$$

Properties of Matched Filters

- Impulse response is

$$h_{\text{opt}}(t) = kg(T_b - t)$$

  - ✓ T_b : symbol period
  - ✓ $g(t)$: transmitter pulse shape
  - ✓ k : gain
  - ✓ scaled, time-reversed and shifted version of $g(t)$
  - ✓ duration and shape determined by pulse shape $g(t)$

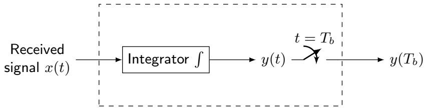
- Maximum peak SNR

$$\eta_{\max} = \frac{2}{N_0} \underbrace{\int_{-\infty}^{\infty} |G(f)|^2 df}_{E} = \frac{2}{N_0} \underbrace{\int_{-\infty}^{\infty} |g(t)|^2 dt}_{E} = \frac{2E}{N_0} = \text{SNR}$$

  - ✓ independent of pulse shape $g(t)$
  - ✓ proportional to signal energy (energy per bit) E
  - ✓ inversely proportional to noise power spectral density

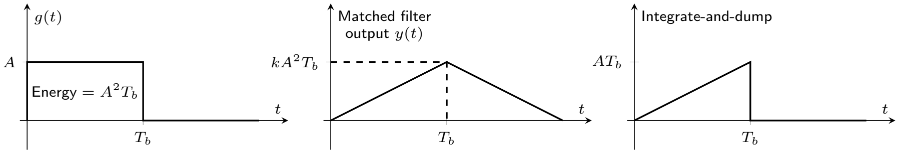
- Matched filter for rectangular pulse shape
  - ✓ matched filter: a rectangular pulse of same duration

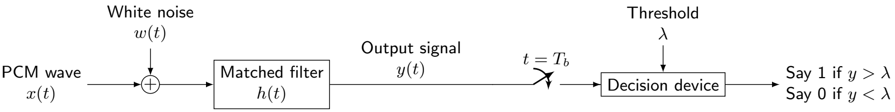
- Integrate and dump circuit



$$h(t) = kg(T_b - t), \quad y(t) = \int_0^{T_b} x(\tau)h(T_b - \tau)d\tau = kA\int_0^{T_b} x(\tau)d\tau$$

$$y(T_b) = kA^2T_b \text{ if no noise}$$





- For binary PCM with on-off signaling:
  - ✓ $0 \to 0$ and $1 \to A$ with bit duration T_b
- Assumptions:
  - ✓ AWGN channel with double-sided noise PSD of $N_0/2$
  - ✓ Rectangular matched filter (set $kT_b = 1$ for simplicity)
- Effect of additive noise: symbol 1 may be mistaken for 0, and vice versa $\Rightarrow$ bit errors
- The probability of a bit error?

Distribution of Noise

- After the matched filter, the pre-detection signal:

$$Y = y(T_b) = \frac{1}{T_b}\int_0^{T_b} x(t)dt = s + \underbrace{\frac{1}{T_b}\int_0^{T_b} w(t)dt}_{\text{noise } N}$$

  ✓ s : binary-valued function (either 0 or A volts)
  ✓ N : zero-mean additive white Gaussian noise with variance:

$$\sigma^2 = \frac{N_0}{2T_b}$$

- PDF of Gaussian random variable N:

$$p_N(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{n^2}{2\sigma^2}\right) = \mathcal{N}(0, \sigma^2)$$

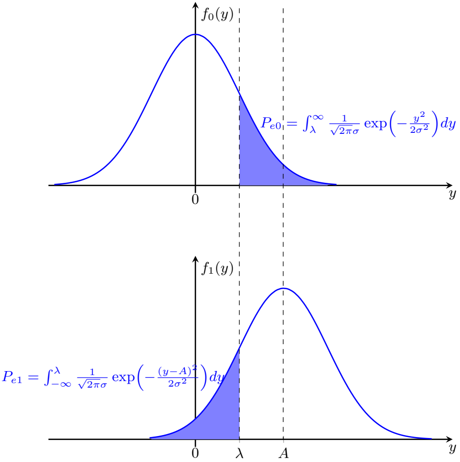
Decision

- If a symbol 0 was transmitted, $Y = N$
  - ✓ $Y \sim \mathcal{N}(0, \sigma^2)$, $f_0(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right)$
- If a symbol 1 was transmitted, $Y = A + N$
  - ✓ $Y \sim \mathcal{N}(A, \sigma^2)$, $f_1(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-A)^2}{2\sigma^2}\right)$
- Use λ as the decision threshold:
  - ✓ choose symbol 0 if $y < \lambda$
  - ✓ choose symbol 1 if $y > \lambda$

Errors

Two cases of decision error:

- Case I: Symbol 0 was transmitted, but symbol 1 is decided (with probability P_{e0})
- Case II: Symbol 1 was transmitted, but symbol 0 is decided (with probability P_{e1})



Case I:

$$\text{Prob(error|symbol 0 was transmitted)} \times \text{Prob(symbol 0 was transmitted)}$$
$$P_{\mathsf{I}} = P_{e0} \times p_0$$

- p_0 : a priori probability of transmitting a symbol 0
- P_{e0} : conditional probability of error if symbol 0 was transmitted

$$P_{e0} = \int_{\lambda}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{y^2}{2\sigma^2}\Big) dy$$

Case II:

$$\text{Prob(error|symbol 1 was transmitted)} \times \text{Prob(symbol 1 was transmitted)}$$
$$P_{\mathsf{II}} = P_{e1} \times p_1$$

- p_1 : a priori probability of transmitting a symbol 1
- P_{e1} : conditional probability of error if symbol 1 was transmitted

$$P_{e1} = \int_{-\infty}^{\lambda} \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{(y-A)^2}{2\sigma^2}\Big) dy$$

Note