Lecture 13: ASK, PSK, FSK and Coherent Detection

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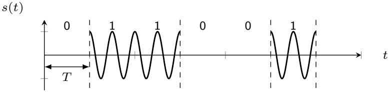
Outline

- Amplitude-shift keying (ASK)
- Phase-shift keying (PSK)
- Frequency-shift keying (FSK)
- Coherent detection
  - ✓ BER of ASK, PSK, and FSK
- Minimum-shift keying (MSK)
- References
  - ✓ [Haykin] Chapter 9

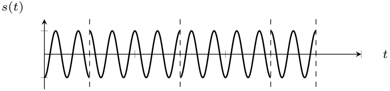
- Amplitude-shift keying (ASK)

$$s(t) = \begin{cases} A\cos(2\pi f_c t), & \text{if transmitting “1”} \\ 0, & \text{otherwise} \end{cases}$$



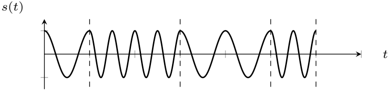
- Phase-shift keying (PSK)

$$s(t) = \begin{cases} A\cos(2\pi f_c t), & \text{if transmitting “1”} \\ A\cos(2\pi f_c t + \pi), & \text{otherwise} \end{cases}$$



- Frequency-shift keying (FSK)

$$s(t) = \begin{cases} A\cos(2\pi f_0 t), & \text{if transmitting “0”} \\ A\cos(2\pi f_1 t), & \text{if transmitting “1”} \end{cases}$$



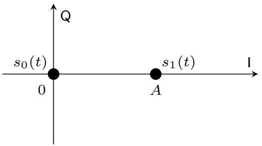
Coherent ASK Detection

- Amplitude shift keying (ASK) = on-off keying (OOK)

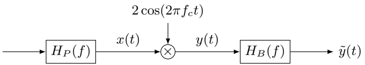
$$s_0(t) = 0, \quad s_1(t) = A\cos(2\pi f_c t)$$

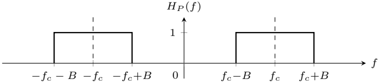
or

$$s(t) = A(t)\cos(2\pi f_c t), \quad A(t) \in \{0, A\}$$

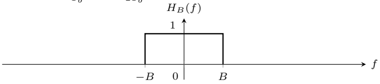


- Coherent detection





$2B = \frac{1}{T_b}$, $B = \frac{1}{2T_b}$



Coherent Demodulation

- Pre-detection signal:

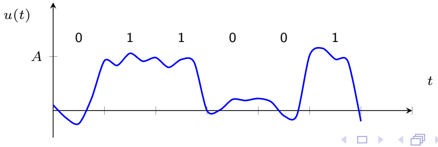
$$\begin{aligned} x(t) &= s(t) + n(t) \\ &= A(t)\cos(2\pi f_c t) + n_{\mathsf{I}}(t)\cos(2\pi f_c t) - n_{\mathsf{Q}}(t)\sin(2\pi f_c t) \\ &= \big(A(t) + n_{\mathsf{I}}(t)\big)\cos(2\pi f_c t) - n_{\mathsf{Q}}(t)\sin(2\pi f_c t) \end{aligned}$$

- After multiplication with $2\cos(2\pi f_c t)$:

$$\begin{aligned} y(t) &= \big(A(t) + n_{\mathsf{I}}(t)\big)2\cos^2(2\pi f_c t) - n_{\mathsf{Q}}(t)2\sin(2\pi f_c t)\cos(2\pi f_c t) \\ &= \big(A(t) + n_{\mathsf{I}}(t)\big)\big(1 + \cos(4\pi f_c t)\big) - n_{\mathsf{Q}}(t)\sin(4\pi f_c t) \end{aligned}$$

- After low-pass filtering:

$$\tilde{y}(t) = A(t) + n_{\mathsf{I}}(t)$$

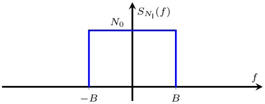


Bit-Error Rate (BER) of ASK

- PSD of $n_{\mathsf{I}}(t)$: N_0
- For equiprobable transmission of 0s and 1s: decision threshold $\lambda = A/2$
- Probability of error:

$$P_{e,\mathsf{ASK}} = Q\left(\frac{A}{2\sigma}\right) \quad \text{(from Lect. 11)}$$

- Transmission energy for a pulse: $E = A^2 T_b/2$
- Average energy per bit: $E_b = A^2 T_b/4$
- Noise variance: $\sigma^2 = N_0 \times 2B = N_0/T_b$



- Probability of error in terms of energy per bit and noise PSD

$$P_{e,\mathsf{ASK}} = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

ASK Modulation System with Coherent Demodulation

- Carrier Amplitude $A = 0.7$ V
- Standard Deviation of White Gaussian Noise $\sigma = 0.125$ V
- Symbols "0" and "1" with equal probability

What is BER?

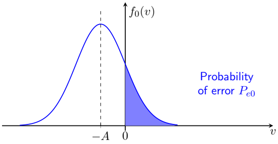
- PSK:

$$s(t) = A(t)\cos(2\pi f_c t), \quad A(t) \in \{-A, A\}$$

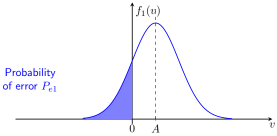
- Use coherent detection to eventually get detection signal:

$$\tilde{y}(t) = A(t) + n_{\mathsf{I}}(t)$$

- PDFs for PSK with equiprobable 0s and 1s in noise (use threshold 0 for detection):



Symbol 0 transmitted



Symbol 1 transmitted

BER of PSK

$$P_{e0} = P_{e1} = P_{e,\text{PSK}} = \int_A^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-y^2/2\sigma^2} dy$$

- Change variable of integration: $z \triangleq y/\sigma \Rightarrow dy = \sigma dz$

$$P_{e,\text{PSK}} = \frac{1}{\sqrt{2\pi}} \int_{A/\sigma}^\infty e^{-z^2/2} dz = Q\left(\frac{A}{\sigma}\right)$$

- Average energy per bit: $E_b = A^2 T_b/2$
- Noise variance: $\sigma^2 = N_0/T_b$

$$\frac{A^2}{\sigma^2} = \frac{2E_b}{N_0}$$

- Probability of error in terms of energy per bit and noise PSD:

$$P_{e,\text{PSK}} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

PSK Example

PSK Modulation System with Coherent Demodulation

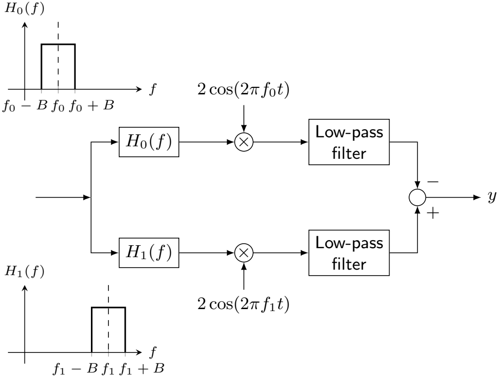
- Carrier Amplitude $A = 0.7$ V
- Standard Deviation of White Gaussian Noise $\sigma = 0.125$ V
- Symbols "0" and "1" with equal probability

What is BER?

$$s(t) = \begin{cases} s_0(t) = A\cos(2\pi f_0 t), & \text{if symbol 0 is transmitted} \\ s_1(t) = A\cos(2\pi f_1 t), & \text{if symbol 1 is transmitted} \end{cases}$$

Symbol recovery: two sets of coherent detectors at frequencies f_0 and f_1

$$\boxed{B = \frac{1}{2T_s}}$$



Coherent FSK demodulation. The two BPF's are non-overlapping in frequency domain.

FSK Output

- Each branch = an ASK detector

$$\text{LPF output on top branch} = \begin{cases} A + n_0(t), & \text{if symbol 0 is present} \\ n_0(t), & \text{if symbol 1 is present} \end{cases}$$

$$\text{LPF output on bottom branch} = \begin{cases} n_1(t), & \text{if symbol 0 is present} \\ A + n_1(t), & \text{if symbol 1 is present} \end{cases}$$

- $n_0(t), n_1(t)$: noise output of top and bottom branches, same statistics as $n_l(t)$!
- Output if transmitting "1"

$$y = y_1(t) = A + n_1(t) - n_0(t) \xlongequal{n(t)=n_1(t)-n_0(t)} A + n(t)$$

- Output if transmitting "0"

$$y = y_0(t) = -A + n(t)$$

- Average noise power

$$\mathbb{E}\{n^2(t)\} = \mathbb{E}\Big\{\big(n_1(t) - n_0(t)\big)^2\Big\} = \mathbb{E}\big\{n_0^2(t)\big\} + \mathbb{E}\big\{n_1^2(t)\big\} = 2\sigma^2$$

BER of FSK

- Detection threshold $\lambda = 0$
- Noise term: $n(t) = n_1(t) - n_0(t)$
- Independent noise in the two channels: $\hat{\sigma}^2 = 2\sigma^2 = \frac{2N_0}{T_b}, \hat{\sigma} = \sqrt{2}\sigma$

$$P_{e,\text{FSK}} = Q\left(\frac{A}{\sqrt{2}\sigma}\right)$$

- Average energy per bit: $E_b = A^2 T_b / 2$
- Probability of error in terms of energy per bit and noise PSD:

$$P_{e,\text{FSK}} = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

More on FSK:

- Symbol 0 $\rightarrow$ frequency f_0, symbol 1 $\rightarrow$ frequency f_1
- The unmodulated carrier frequency: $f_c = (f_0 + f_1)/2$
- Frequency separation: $\Delta f = |f_1 - f_0|$
- The symbol period: T
  - ✓ $f_c T \gg 1$ in practice, $\cos(2\pi f_1 t)$ and $\sin(2\pi f_0 t)$ orthogonal within the symbol period

Δf (or $1/T$) should be large enough to make $\cos(2\pi f_1 t)$ and $\sin(2\pi f_0 t)$ orthogonal!

Minimum-shift keying (MSK): using the minimum separation $\Delta f = \frac{1}{2T}$

Why $\Delta f = \frac{1}{2T}$ is the minimum separation?

$$\frac{1}{T}\int_0^T \cos(2\pi f_1 t)\cos(2\pi f_0 t)dt = \frac{1}{2T}\int_0^T \cos\big(2\pi(f_1 + f_0)t\big) + \cos\big(2\pi(f_1 - f_0)t\big)dt$$

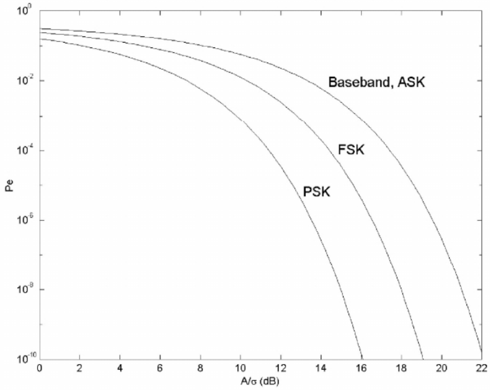
$$= \frac{1}{2T}\int_0^T \cos\big(4\pi f_c t\big) + \cos\big(2\pi \Delta f t\big)dt = \frac{1}{2}\left(\frac{\sin(4\pi f_c T)}{4\pi f_c T} + \frac{\sin(2\pi \Delta f T)}{2\pi \Delta f T}\right) \approx \frac{\sin(2\pi \Delta f T)}{4\pi \Delta f T}$$

$$2\pi \Delta f T = \pi \Rightarrow \Delta f = \frac{1}{2T} \text{ minimum separation!}$$

Whole bandwidth of MSK signal:

$$2B + \Delta f = 2 \times \frac{1}{2T} + \frac{1}{2T} = \frac{3}{2T}$$

Comparison of Three Schemes



ASK:

- $\frac{E_b}{N_0} = \frac{A^2}{4\sigma^2}$
- $P_{e,\text{ASK}} = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\frac{A}{2\sigma}\right)$

PSK:

- $\frac{E_b}{N_0} = \frac{A^2}{2\sigma^2}$
- $P_{e,\text{PSK}} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\frac{A}{\sigma}\right)$

FSK:

- $\frac{E_b}{N_0} = \frac{A^2}{2\sigma^2}$
- $P_{e,\text{FSK}} = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\frac{A}{\sqrt{2}\sigma}\right)$

Application: GMSK

Gaussian minimum shift keying (GMSK), a special form of FSK preceded by Gaussian filtering, is used in GSM (Global Systems for Mobile Communications), a leading cellular phone standard in the world.

- Also known as digital FM, it was used in (Advanced Mobile Phone System) AMPS, the first-generation analog system (30 KHz bandwidth).
- Binary data are passed through a Gaussian filter to satisfy stringent requirements of out-of-band radiation.
- Minimum Shift Keying: its spacing between the two frequencies of FSK is minimum in a certain sense.
- GMSK is allocated bandwidth of 200 kHz, shared among 32 users. This provides a $(30/200) \times 32 = 4.8$ times improvement over AMPS.

Note