# Lecture 3: More on Stochastic Processes

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# Outline

- Stochastic Processes
  - ✓ Passing through linear time-invariant (LTI) system
  - ✓ Power spectral density (PSD)
- References
  - ✓ [Haykin] Chapter 5
  - ✓ [Lathi] Chapter 8

# Stochastic Process Through a Linear Time-Invariant (LTI) System: I

$$X(t) \longrightarrow \boxed{\begin{array}{c}\text{Impulse Response}\\ h(t)\end{array}} \longrightarrow Y(t) = X(t) * h(t) = \int h(\tau)X(t-\tau)d\tau$$

- If $\mathbb{E}\{X(t)\}$ is finite for all $t$ and the system is bounded-input bounded-output (BIBO) stable, we have

$$\mu_Y(t) = \mathbb{E}\{Y(t)\} = \mu_X(t) * h(t)$$

$$\mu_X(t) \longrightarrow \boxed{h(t)} \longrightarrow \mu_Y(t)$$

# Stochastic Process Through an LTI System: II

- If $\mathbb{E}\{X^2(t)\}$ is finite for all $t$ and the system is BIBO stable, we have

$$R_Y(t,u) = h(t) * h(u) * R_X(t,u)$$

$$R_{XX}(t,u) \longrightarrow \boxed{h(u)} \xrightarrow{\quad R_{XY}(t,u) \quad} \boxed{h(t)} \longrightarrow R_{YY}(t,u)$$

- If $X(t)$ is real WSS, then

$$\mu_Y(t) = \mu_X \underbrace{\int_{-\infty}^{\infty} h(\tau)d\tau}_{H(0)} = \mu_X \cdot \text{DC response} = \text{constant}$$

$$\mu_X \longrightarrow \boxed{h(t)} \longrightarrow \mu_Y$$

$$R_Y(\tau) = \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau - \tau_1 + \tau_2)d\tau_1 d\tau_2 = h(\tau) * h(-\tau) * R_X(\tau)$$

$$R_X(\tau) \longrightarrow \boxed{h(-\tau)} \longrightarrow \boxed{h(\tau)} \longrightarrow R_Y(\tau)$$

- For a complex WSS process $X(t)$, $R_X(\tau) = \mathbb{E}\{X(t+\tau)X^*(t)\}$

$$R_Y(\tau) = h(\tau) * h^*(-\tau) * R_X(\tau)$$

- If $X(t)$ is WSS, then $Y(t)$ is also WSS

$$\text{WSS Process} \longrightarrow \boxed{\text{LTI System}} \longrightarrow \text{WSS Process}$$

- If $X(t)$ is a Gaussian process, then $Y(t)$ is also a Gaussian process

$$\text{Gaussian Process} \longrightarrow \boxed{\text{LTI System}} \longrightarrow \text{Gaussian Process}$$

# Power Spectral Density (PSD): I

- PSD measures the distribution of power of a random process over its spectrum.
- PSD is defined only for WSS processes.

**Einstein-Wiener-Khintchine relation:**

The PSD of a wide sense stationary process is equal to the Fourier transform of its autocorrelation function:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} d\tau \geq 0$$

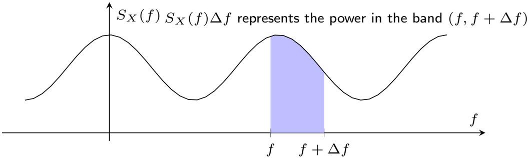
- The frequency content of a process depends on how rapidly the amplitude changes as a function of time (can be measured by the autocorrelation function).

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f \tau} df$$

$$R_X(0) = \int_{-\infty}^{\infty} S_X(f) df$$

- $S_X(f)$ is real and nonnegative.

# Power Spectral Density (PSD): II




The average power of a random process $X(t)$

$$P = \mathbb{E}\{X^2(t)\} = R_X(0) = \int_{-\infty}^{\infty} S_X(f)df$$

# Stochastic Process Through an LTI System: V

$$X(t) \longrightarrow \boxed{\begin{array}{c}\text{Impulse Response}\\ h(t)\end{array}} \longrightarrow Y(t) = X(t) * h(t) = \int h(\tau)X(t-\tau)d\tau$$

- For a real or complex WSS process $X(t)$ goes through an LTI system:

$$S_Y(f) = |H(f)|^2 S_X(f)$$

## WSS process and random variable

Let $X(t)$ be a WSS process with autocorrelation function $R_X(\tau)$, and $\Theta$ be a random variable independent of $X(t)$ and uniformly distributed in $[0, 2\pi)$.

1. Prove $Z(t) = X(t)\cos(2\pi f_c t)$ is not WSS;
2. Prove $Y(t) = X(t)\cos(2\pi f_c t + \Theta)$ is WSS, and find the PSD of $Y(t)$.

# Note