Lecture 11: Quadrature Amplitude Modulation (QAM)

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Outline

- Binary baseband communication (continued)
  - ✓ Optimal threshold
  - ✓ Average probability of error
- Digital baseband modulation
  - ✓ Waveform, bandwidth, symbol duration, and rate
- Digital passband modulation
  - ✓ QAM modulation and demodulation
- References
  - ✓ [Haykin] Chapter 8, 9

Optimal Threshold

- Total error probability:

$$P_e(\lambda) = P_{\mathsf{I}} + P_{\mathsf{II}} = p_0 P_{e0} + p_1 P_{e1}$$
$$= (1 - p_1) \int_{\lambda}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{y^2}{2\sigma^2}\Big) dy + p_1 \int_{-\infty}^{\lambda} \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{(y-A)^2}{2\sigma^2}\Big) dy$$

- Optimal threshold: setting $dP_e(\lambda)/d\lambda = 0$, then

$$-(1 - p_1)\frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{\lambda^2}{2\sigma^2}\Big) + p_1 \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{(\lambda - A)^2}{2\sigma^2}\Big) = 0 \Rightarrow$$
$$\lambda_{\text{opt}} = -\frac{\sigma^2}{A} \log \frac{p_1}{1 - p_1} + \frac{A}{2}$$

- Equal symbol probability ($p_0 = p_1 = 0.5$):

$$\lambda_{\text{opt}} = \frac{A}{2}, \quad P_{e0} = P_{e1}$$

- Unequal symbol probability: if $p_0 > p_1$, then

$$\lambda_{\text{opt}} > \frac{A}{2}, \quad P_{e0} < P_{e1}$$

- Define a new variable of integration

$$z \triangleq \frac{y}{\sigma} \Rightarrow dy = \sigma dz$$

  ✓ When $y = A/2$, $z = A/2\sigma$
  ✓ When $y = \infty$, $z = \infty$

- Then

$$P_{e0} = \frac{1}{\sigma\sqrt{2\pi}} \int_{A/2\sigma}^{\infty} e^{-z^2/2} \sigma dz = \frac{1}{\sqrt{2\pi}} \int_{A/2\sigma}^{\infty} e^{-z^2/2} dz = Q\left(\frac{A}{2\sigma}\right)$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt$$

$$P_e = P_{e_0} = P_{e_1} = Q\left(\frac{A}{2\sigma}\right)$$

Probability of Error

$$P_e = Q\left(\frac{A}{2\sigma}\right)$$

- Energy of a pulse is:

$$E = A^2 T_b$$

- We transmit a pulse only half of the time on average $\Rightarrow$ Average energy per bit (E_b):

$$E_b = \frac{A^2 T_b}{2}$$

- Noise variance:

$$\sigma^2 = \frac{N_0}{2T_b}$$

(from slide **??**)

- Probability of error in terms of energy per bit and noise PSD:

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Example

Example 1

$$A/\sigma = 7.4(17.4\ \text{dB}) \Rightarrow P_e = 10^{-4}$$

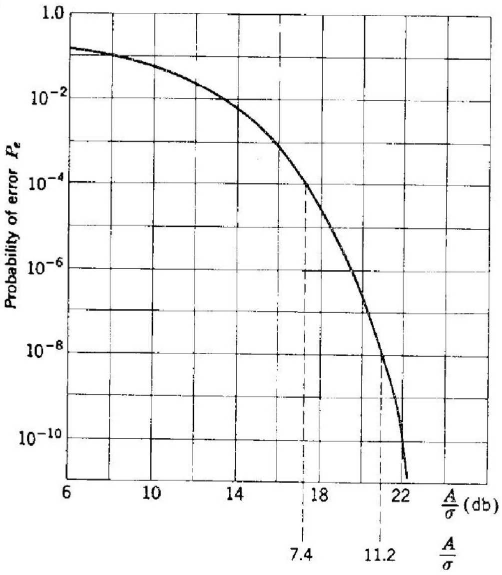
For a transmission rate of 10^5 bits/sec, there will be an error every 0.1 seconds on the average.

Example 2

$$A/\sigma = 11.2(21\ \text{dB}) \Rightarrow P_e = 10^{-8}$$

For a transmission rate of 10^5 bits/sec, there will be an error every 17 mins on the average.

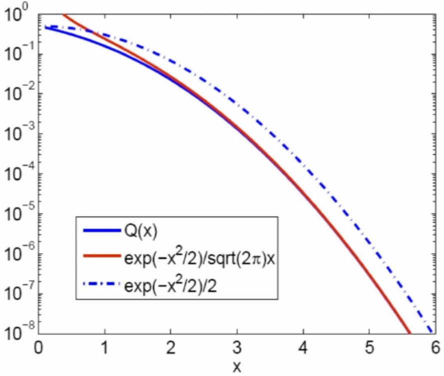
Enormous increase in reliability by a relatively small increase in SNR (if that is affordable).


1.0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
Probability of error P_e
6
10
14
18
22
$\frac{A}{\sigma}$ (db)
7.4
11.2
$\frac{A}{\sigma}$

Q-Function

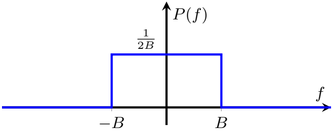
- Upper bounds and good approximations
- For $x \geq 0$, we have

$$Q(x) \leq \begin{cases} \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}, & \text{(often used for large } x\text{)} \\ \frac{1}{2} e^{-x^2/2}, & \text{(only good for small } x\text{)} \end{cases}$$



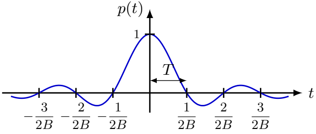
- A baseband waveform with bandwidth B

$$P(f) = \begin{cases} \frac{1}{2B}, & |f| \leq B \\ 0, & \text{otherwise} \end{cases}$$



- Corresponding time-domain waveform or modulation pulse

$$p(t) = \int_{-B}^{B} \frac{1}{2B} e^{j2\pi f t} df$$
$$= \frac{\sin(2\pi Bt)}{2\pi Bt} = \mathrm{sinc}(2Bt)$$



$T = 1/2B$ for ISI-free transmission

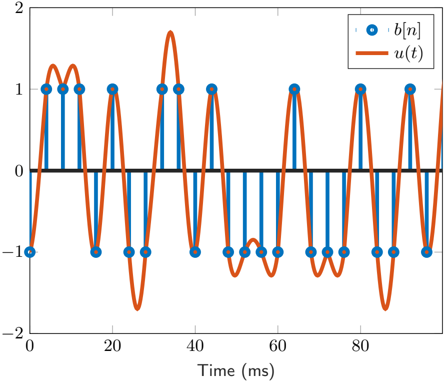
- Even if the *modulation pulse may be different*, we still regard the following to be true:
  - ✓ Symbol duration $T = \frac{1}{2B}$, or symbol rate $R = \frac{1}{T} = 2B$
  - ✓ For a modulated waveform with symbol duration T, bandwidth is $B = \frac{1}{2T} = \frac{R}{2}$

Baseband Modulation

- Baseband modulation, linearly modulated waveform

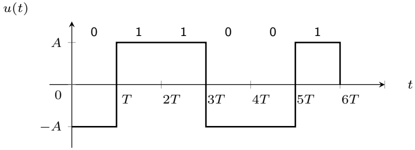
$$u(t) = \sum_n b[n]p(t - nT)$$

✓ $\{b[n]\}$: the sequence of symbols
✓ $p(t)$: the modulating pulse



Digital Passband Modulation → PSK

- Example: symbols $\{-1, +1\}$ and $p(t) = \begin{cases} 1, & 0 \le t \le T, \\ 0, & \text{otherwise} \end{cases}$



- Baseband → passband: $s(t) = u(t)\cos(2\pi f_c t)$

- For the nth symbol interval, $nT \le t \le (n+1)T$, we have

$$s(t) = \begin{cases} \cos(2\pi f_c t), & b[n] = +1, \\ \cos(2\pi f_c t + \pi), & b[n] = -1 \end{cases}$$

- Binary antipodal modulation switches the phase of the carrier between 0 and π, hence it is called binary phase-shift keying (BPSK)

PSK Signal → Constellations

- We can modulate both I and Q components (BPSK modulates only the I-component):

$$s(t) = u_{\mathsf{I}}(t)\cos(2\pi f_c t) - u_{\mathsf{Q}}(t)\sin(2\pi f_c t),$$

where

$$u_{\mathsf{I}}(t) = \sum_n b_{\mathsf{I}}[n]p(t-nT), \quad u_{\mathsf{Q}}(t) = \sum_n b_{\mathsf{Q}}[n]p(t-nT)$$

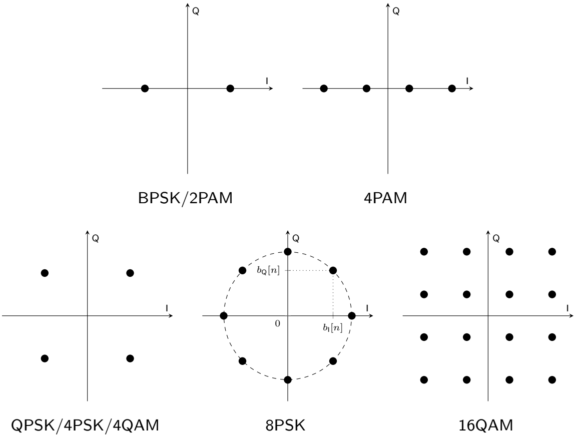
- If $b_{\mathsf{I}}[n]$ and $b_{\mathsf{Q}}[n]$ take values from $\{-1,+1\}$, for the nth symbol interval, $nT \leq t \leq (n+1)T$:

$$s(t) = \begin{cases} +\cos(2\pi f_c t) - \sin(2\pi f_c t) = \sqrt{2}\cos(2\pi f_c t + \pi/4), & \text{if } b_{\mathsf{I}}[n] = +1, b_{\mathsf{Q}}[n] = +1, \\ +\cos(2\pi f_c t) + \sin(2\pi f_c t) = \sqrt{2}\cos(2\pi f_c t - \pi/4), & \text{if } b_{\mathsf{I}}[n] = +1, b_{\mathsf{Q}}[n] = -1, \\ -\cos(2\pi f_c t) - \sin(2\pi f_c t) = \sqrt{2}\cos(2\pi f_c t + 3\pi/4), & \text{if } b_{\mathsf{I}}[n] = -1, b_{\mathsf{Q}}[n] = +1, \\ -\cos(2\pi f_c t) + \sin(2\pi f_c t) = \sqrt{2}\cos(2\pi f_c t + 3\pi/4), & \text{if } b_{\mathsf{I}}[n] = -1, b_{\mathsf{Q}}[n] = -1 \end{cases}$$

- Modulation switches the phase among $\pm\pi/4, \pm 3\pi/4$, called quadrature phase-shift keying (QPSK), 4-PSK, or 4-QAM

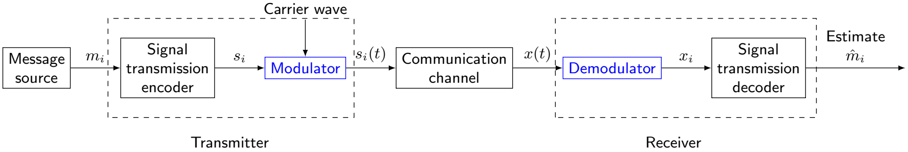
Quadrature Amplitude Modulation (QAM)

$$s(t) = u_{\mathsf{I}}(t)\cos(2\pi f_c t) - u_{\mathsf{Q}}(t)\sin(2\pi f_c t),$$

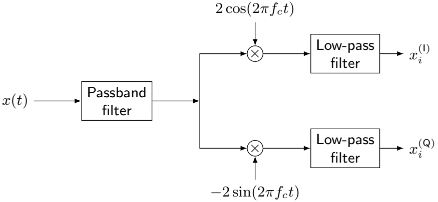


- ASK and PSK: special cases of QAM
- FSK: not a special case

QAM-Demodulation



- Coherent (synchronous) demodulation/detection
  - ✓ Use a band-pass filter (BPF) to reject out-of-band noise
  - ✓ Multiply the incoming waveform with a cosine and a sine of the carrier frequency
  - ✓ Use a low-pass filter (LPF)
  - ✓ Require carrier regeneration (both frequency and phase synchronization using a phase-locked loop)



Coherent Detection of QAM

QAM signal:

$$s(t) = u_{\mathsf{I}}(t)\cos(2\pi f_c t) - u_{\mathsf{Q}}(t)\sin(2\pi f_c t),$$

Received signal:

$$x(t) = s(t) + n(t) = u_{\mathsf{I}}(t)\cos(2\pi f_c t) - u_{\mathsf{Q}}(t)\sin(2\pi f_c t) + n(t)$$

After passband filter,

$$\begin{aligned}\hat{x}(t) &= u_{\mathsf{I}}(t)\cos(2\pi f_c t) - u_{\mathsf{Q}}(t)\sin(2\pi f_c t) + n_{\mathsf{I}}(t)\cos(2\pi f_c t) - n_{\mathsf{Q}}(t)\sin(2\pi f_c t)\\ &= \big(u_{\mathsf{I}}(t) + n_{\mathsf{I}}(t)\big)\cos(2\pi f_c t) - \big(u_{\mathsf{Q}}(t) + n_{\mathsf{Q}}(t)\big)\sin(2\pi f_c t)\end{aligned}$$

Outputs of coherent detector:

$$\begin{aligned}x_{\mathsf{I}}(t) &= u_{\mathsf{I}}(t) + n_{\mathsf{I}}(t)\\ x_{\mathsf{Q}}(t) &= u_{\mathsf{Q}}(t) + n_{\mathsf{Q}}(t)\end{aligned}$$

Note