Lecture 2: Random Variables and Stochastic Processes

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Outline

- Random Variables
  - ✓ Joint distribution
  - ✓ Independence and uncorrelation
- Stochastic Processes
  - ✓ Mean and autocorrelation function
  - ✓ Gaussian process
  - ✓ Wide-sense stationary (WSS) process
- References
  - ✓ [Haykin] Chapter 5
  - ✓ [Lathi] Chapter 8

Joint Distribution

- Joint cdf for two random variables X and Y

$$F_{XY}(x,y) = P(X \leq x, Y \leq y)$$

- Joint pdf

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

- Properties
  1. $F_{XY}(\infty,\infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(u,v)dudv = 1$
  2. $f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)dy,\ f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)dx$
  3. Independent: $f_{XY}(x,y) = f_X(x)f_Y(y)$
  4. Uncorrelated: $\mathbb{E}\{XY\} = \mathbb{E}\{X\}\mathbb{E}\{Y\}$ or $\mathbb{E}\big\{(X-\mu_X)(Y-\mu_Y)\big\}$

Independent and Uncorrelated

- Independent $\Rightarrow$ uncorrelated
- Uncorrelated $\not\Rightarrow$ independent
- For jointly Gaussian random variables, uncorrelated $\iff$ independent

- Joint cdf

$$F_{X_1 X_2 \ldots X_n}(x_1 x_2 \ldots x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots X_n \leq x_n)$$

- Joint pdf

$$f_{X_1 X_2 \ldots X_n}(x_1 x_2 \ldots x_n) = \frac{\partial^n F_{X_1 X_2 \ldots X_n}(x_1 x_2 \ldots x_n)}{\partial x_1 \partial x_2 \ldots \partial x_n}$$

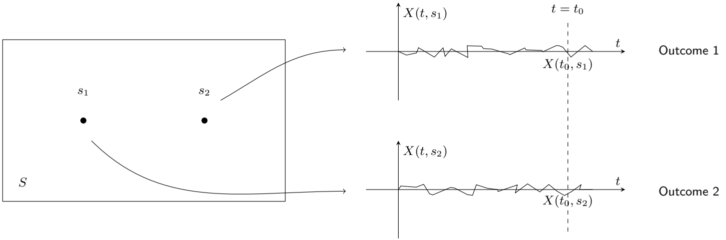
- Independent

$$F_{X_1 X_2 \ldots X_n}(x_1 x_2 \ldots x_n) = F_{X_1}(x_1) F_{X_2}(x_2) \ldots F_{X_n}(x_n)$$
$$f_{X_1 X_2 \ldots X_n}(x_1 x_2 \ldots x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \ldots f_{X_n}(x_n)$$

- i.i.d. (independent and identically distributed)
  - ✓ Independent random variables with the same distribution (e.g., flipping n coins)

Stochastic Process

- Stochastic process $X(t, s)$: a collection of random variables over time. It represents the *evolution* of a random system.
- At a given time t_0, $X(t_0, s)$ is a random variable.
- At a sample outcome s_j, $X(t, s_j)$ is a deterministic function over time.
- Stochastic process $X(t, s)$ is often denoted by $X(t)$ for simplicity.
- Noise is often modelled as a Gaussian stochastic process.



- Probability density function
  - ✓ 1st order: $f_X(x;t)$
  - ✓ 2nd order: $f_X(x_1, x_2; t_1, t_2)$
  - ✓ n-th order: $f_X(x_1, \ldots, x_n; t_1, \ldots, t_n)$
- Mean is usually a function of t:

$$\mu_X(t) = \mathbb{E}\{X(t)\} = \int_{-\infty}^{\infty} x f_X(x;t) dx$$

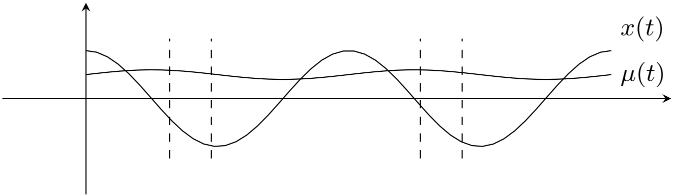
- Autocorrelation function is usually a function of t_1 and t_2. It measures the correlation between samples:

$$R_X(t_1, t_2) = \mathbb{E}\{X(t_1)X(t_2)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2) dx_1 dx_2$$

- A stochastic process is Gaussian if and only if the pdf $f_X(x;t)$ is Gaussian at any time t_n.

Wide-Sense Stationary (WSS) Stochastic Processes

- A stochastic process is wide-sense stationary (WSS) if and only if:
  1. The mean is not a function of time:
  $$\mu_X(t) = \mu_X, \quad \forall t$$
  2. The autocorrelation function only depends on time difference:
  $$R_X(t+\tau, t) = R_X(\tau), \quad \forall t, \tau$$



- Noise and message signals are often modelled as WSS processes.

Properties of Autocorrelation Function

For a real WSS process $X(t)$ with autocorrelation function $R_X(\tau)$:

1. $R_X(0) = \mathbb{E}\{X^2(t)\}$
2. $R_X(\tau)$ is an even function

$$R_X(\tau) = \mathbb{E}\big\{x(t+\tau)x(t)\big\} = \mathbb{E}\big\{x(t)x(t+\tau)\big\} = R_X(-\tau)$$

3. $R_X(\tau)$ takes maximum magnitude at $\tau = 0$ (Homework 1 Problem 5)

$$|R_X(\tau)| \leq R_X(0)$$

$R_X(\tau)$ can tell how predictable $X(t)$ is based on $X(t-\tau)$.

Exercise

WSS example

Show that a sinusoidal wave with a random uniformly distributed phase is WSS.

$$X(t) = A\cos(\omega_c t + \Theta), \quad f_\Theta(\theta) = \frac{1}{2\pi}, \quad \theta \in [0, 2\pi).$$

Note