Lecture 9: Digital Representation of Signals

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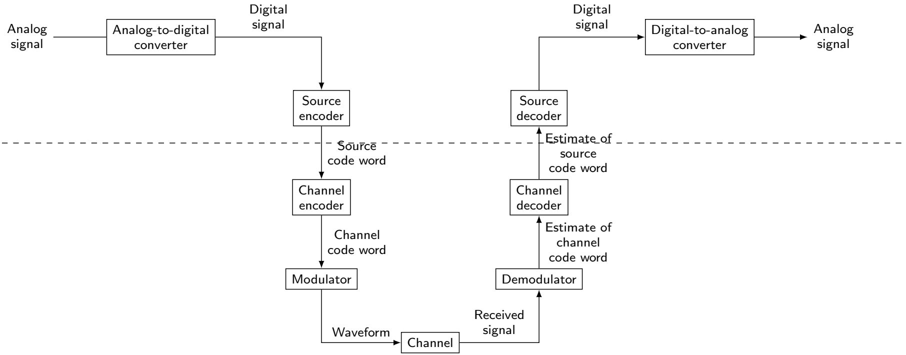
Outline

- Digital communication
- Quantization (A/D) and noise
- Pulse-Coded Modulation (PCM)
- Companding and expanding
- Reference
  - ✓ [Haykin] Chapter 7

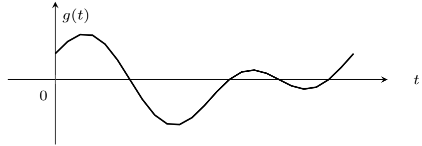
Digital Communication

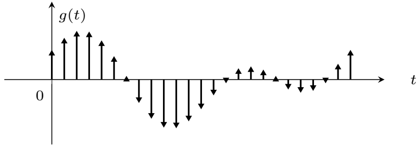


Why Digital?

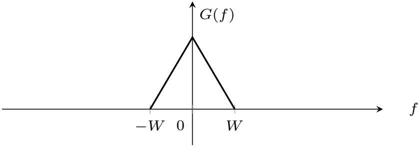
- More immune to channel noise by using channel coding
- Repeaters along the transmission path: error-correction and retransmission
- Representing different analog sources using digital signals, a uniform format
- Easily processed using microprocessors and VLSI (e.g., digital signal processors, FPGA)
- More and more things are digital . . .

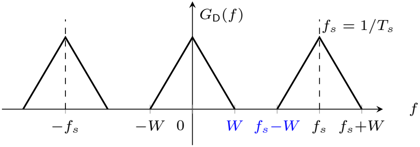
How densely should we sample?





$$g_{\mathsf{D}}(t) = g(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$





No overlapping in $G(f)$ if $W \leq f_s - W$, namely $f_s \geq 2W$

$$G_{\mathsf{D}}(f) = G(f) * \Big(f_s \sum_{k=-\infty}^{\infty} \delta(f - kf_s)\Big) = f_s \sum_{k=-\infty}^{\infty} G(f - kf_s)$$

Sampling Theorem

Sampling theorem

For distortionless recovery, sampling rate $f_s \geq 2W$ for a (real) signal with bandwidth W. The Nyquist frequency is

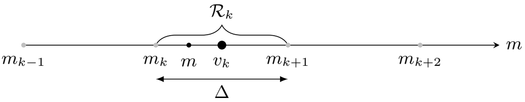
$$f_{\mathrm{N}} = 2W$$

Quantization: I

- Quantization: transforming the sample amplitude into a discrete amplitude taken from a finite set of possible amplitudes
- The more levels, the better approximation
- No need to transmit exact values
- Memoryless and instantaneous quantization: quantization at time t is independent of other samples
- Quantizers: uniform or nonuniform

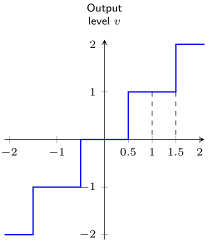
Continuous sample m $\longrightarrow$ $\boxed{\text{Quantizer } g(\cdot)}$ $\longrightarrow$ Discrete sample v_k

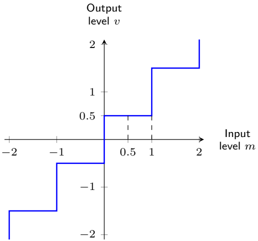
$$v_k = g(m)$$



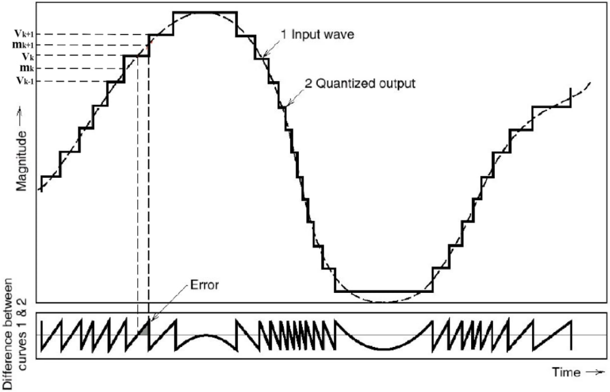
$$v_k = \frac{m_k + m_{k+1}}{2}, \quad \Delta = m_{k+1} - m_k$$

- Decision levels: $m_1, \ldots, m_L$
- Decision region: $\mathcal{R}_k = (m_k, m_{k+1}]$, $k = 1, \ldots, L$
- Reconstruction levels: v_k, $k = 1, \ldots, L$
- Quantizer output v_k represents decision region $\mathcal{R}_k$
- Mapping $v = g(m)$ is the quantizer characteristic





- Error between the input and the output signals



Variance of Quantization Noise

- Δ : gap between quantizing levels (of a uniform quantizer)
- q : quantization error = random variable within the range

$$-\frac{\Delta}{2} \leq q \leq \frac{\Delta}{2}$$

- For small Δ, q is uniform:

$$f_Q(q) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} \leq q \leq \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases}$$

- Quantization noise variance

$$P_N = \mathbb{E}\{q^2\} = \int_{-\infty}^{\infty} q^2 f_Q(q) dq = \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} q^2 dq = \frac{1}{\Delta}\left(\frac{\Delta^3}{24} - \frac{(-\Delta)^3}{24}\right) = \frac{\Delta^2}{12}$$

Quantization Gap Δ

For n-bit quantization:

- Maximum number of quantizing levels: $L = 2^n$
- Maximum peak-to-peak dynamic range: $2^n \Delta$
- Power of the message signal:

$$P = \mathbb{E}\{m^2(t)\} \xlongequal{\text{periodic}} \frac{1}{T} \int_{-T/2}^{T/2} |m(t)|^2 dt$$

- Maximum magnitude: $m_p = \max|m(t)|$
- Full load quantizer:

$$2m_p = 2^n \Delta \quad \text{or} \quad \Delta = 2^{-(n-1)} m_p$$

SNR

- SNR at the quantizer output:

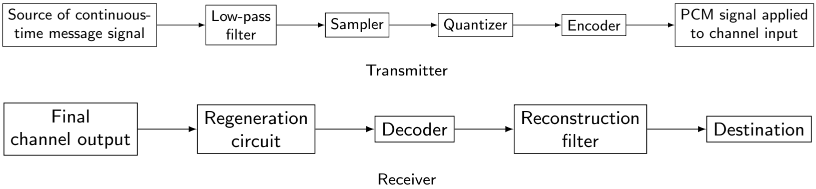
$$\text{SNR}_o = \underbrace{\frac{P_S}{P_N} = \frac{P}{\Delta^2/12}}_{\text{by quantizer principle}} = \frac{3P}{m_p^2} 2^{2n}$$

$$\text{SNR}_o(\text{dB}) = 10\log_{10}(2^{2n}) + 10\log_{10}\left(\frac{3P}{m_p^2}\right) = 6n + 10\log_{10}\left(\frac{3P}{m_p^2}\right)$$

- An extra bit in the encoder $\Leftrightarrow$ 6 dB more to the output SNR
- Recognize the tradeoff between SNR and n (i.e., rate, or bandwidth)

Example

Pulse-Coded Modulation (PCM)

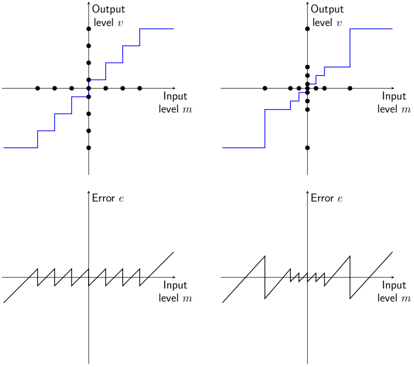


- Low-pass filter is applied to prevent aliasing
- Sample the message signal above Nyquist rate
- Quantize each sample
- Encode discrete amplitudes into a binary codeword
- PCM: not modulation in usual sense; type of Analog-to-Digital Converter

Problem With Uniform Quantization

- SNR: adversely affected by peak-to-average power ratio
- More often with small signals than large signals
- More quantization levels for smaller amplitudes

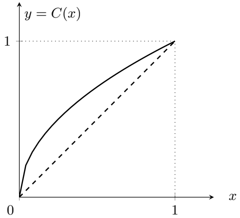
Nonuniform quantization: quantization levels of variable spacing, denser at small signal amplitudes, broader at large amplitudes.

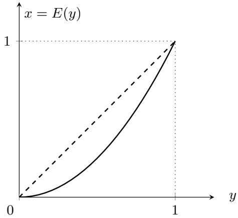


Uniform Nonuniform

Companding = Compressing + Expanding

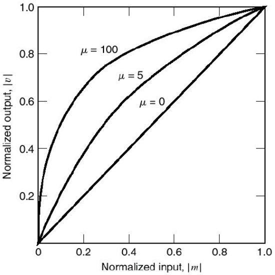
- A practical (and equivalent) solution to nonuniform quantization:
  - ✓ compress the signal
  - ✓ quantize it (using a uniform quantizer)
  - ✓ transmit it
  - ✓ expand it





- Companding/Expanding: pre-emphasising/de-emphasising as in FM
- Ideal compression and expansion: exactly inverse of each other
- Exact SNR gain: depending on the exact form of the compression

Compander Standards: μ-Law

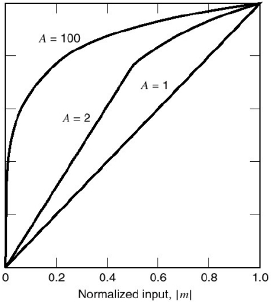


- μ-Law (North America and Japan, typical $\mu = 255$)

$$y = F(x) = \mathrm{sgn}(x) \frac{\log\big(1 + \mu|x|\big)}{\log(1+\mu)}, \quad |x| < 1$$

$$x = F^{-1}(y) = \mathrm{sgn}(y) \frac{(1+\mu)^{|y|} - 1}{\mu}$$

Compander Standards: A-Law



- A-Law (in most countries of the world; typical A = 87.6)

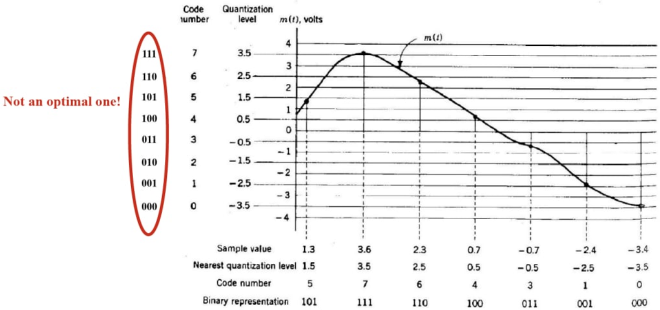
$$y = F(x) = \begin{cases} \text{sgn}(x)\frac{A|x|}{1+\log(A)}, & |x| < \frac{1}{A} \\ \text{sgn}(x)\frac{1+\log(A|x|)}{1+\log(A)}, & \frac{1}{A} \leq |x| \leq 1 \end{cases}$$

$$x = F^{-1}(y) = \begin{cases} \text{sgn}(y)\frac{|y|(1+\log(A))}{A}, & |y| < \frac{1}{1+\log(A)} \\ \text{sgn}(y)\frac{\exp(|y|(1+\log(A))-1)}{A}, & \frac{1}{1+\log(A)} \leq |y| \leq 1 \end{cases}$$

Applications of PCM & Variants

- Speech:
  - ✓ PCM: Voice signal is sampled at 8 kHz, quantized into 256 levels (8 bits). Thus, a telephone PCM signal requires 64 kbps (need to reduce bandwidth requirements).
  - ✓ DPCM (differential PCM): quantize the difference between consecutive samples; can save 8 to 16 kbps. ADPCM (Adaptive DPCM) can go further down to 32 kbps.
  - ✓ Delta modulation: 1-bit DPCM with oversampling; has even lower symbol rate (e.g., 24 kbps).
- Audio CD: 16-bit PCM at 44.1 kHz sampling rate.
- MPEG audio coding: 16-bit PCM at 48 kHz sampling rate compressed to a rate as low as 16 kbps.

PCM Process



Not an optimal one!
Code number
Quantization level
m(t), volts
m(t)
111
110
101
100
011
010
001
000
7
6
5
4
3
2
1
0
3.5
2.5
1.5
0.5
−0.5
−1.5
−2.5
−3.5
4
3
2
1
0
−1
−2
−3
−4
Sample value 1.3 3.6 2.3 0.7 −0.7 −2.4 −3.4
Nearest quantization level 1.5 3.5 2.5 0.5 −0.5 −2.5 −3.5
Code number 5 7 6 4 3 1 0
Binary representation 101 111 110 100 011 001 000

Summary

Digitization of signals requires

- Sampling: sampled at the Nyquist frequency $2W$
- Quantization: link between analog waveforms and digital representation
  - ✓ SNR (under high-resolution assumption)

$$\mathrm{SNR_o(dB)} = 6n + 10\log_{10}\left(\frac{3P}{m_p^2}\right)$$

  - ✓ Companding to improve SNR

PCM: a common method of representing audio signals

- "Pulse coded modulation": a simple source coding technique (i.e, method of digitally representing analog information)
- More advanced source coding (compression) techniques in information theory

Note