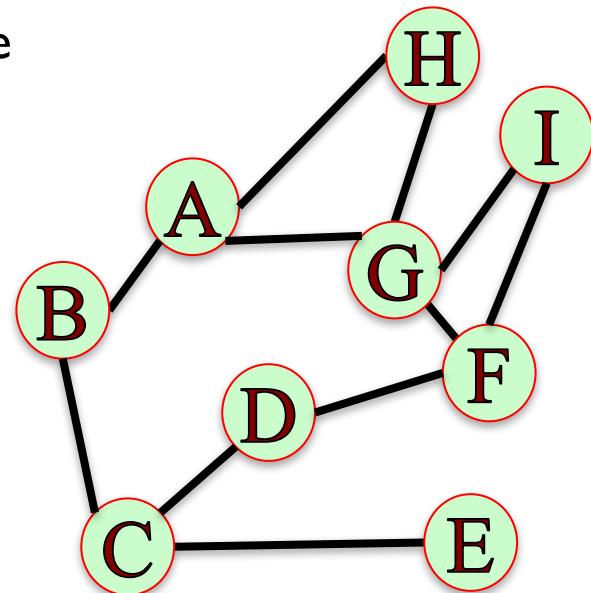


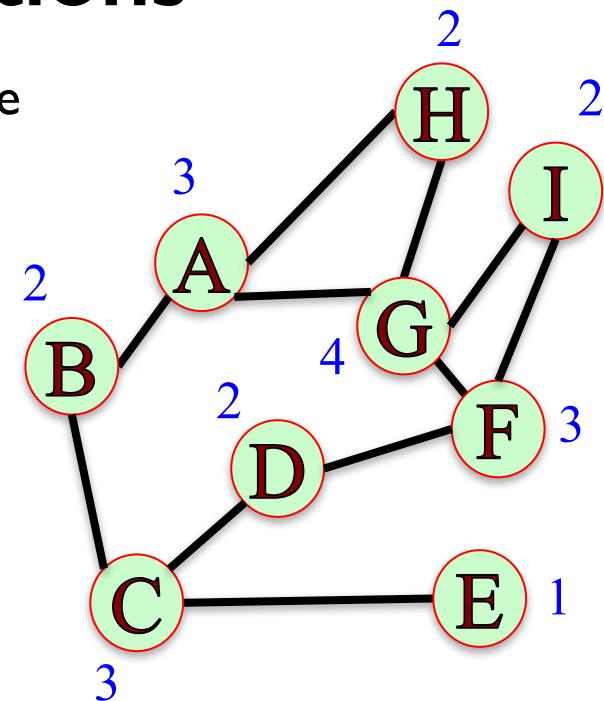
# Degree Distributions

The **degree** of a node in an undirected graph is the number of neighbors it has.



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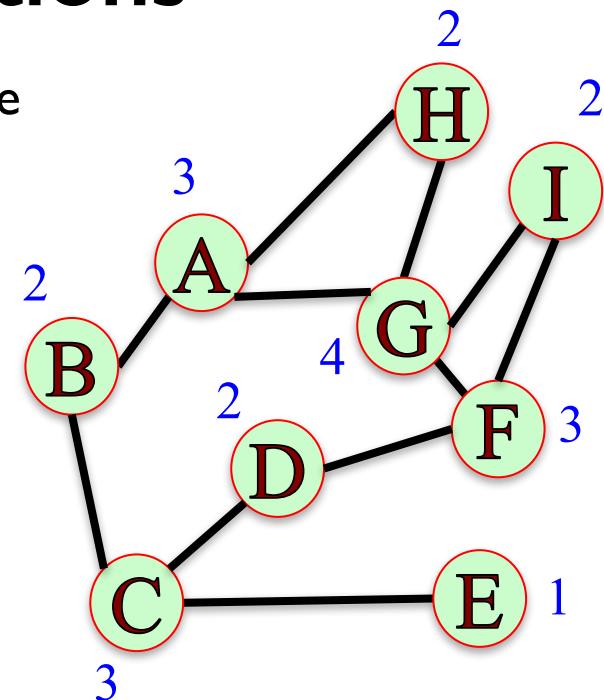
# Degree Distributions

The **degree** of a node in an undirected graph is the number of neighbors it has.

The **degree distribution** of a graph is the probability distribution of the degrees over the entire network.

The degree distribution,  $P(k)$ , of this network has the following values:

$$P(1) = \frac{1}{9}, P(2) = \frac{4}{9}, P(3) = \frac{1}{3}, P(4) = \frac{1}{9}$$

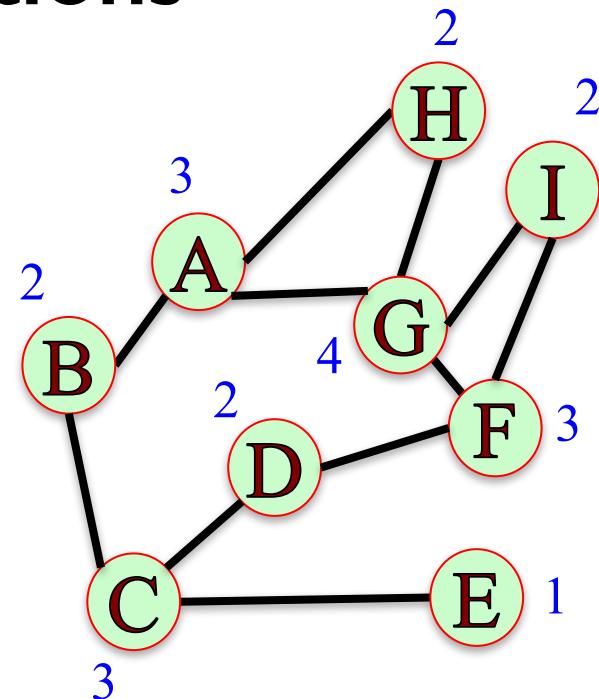


# Degree Distributions

Plot of the degree distribution of this network:

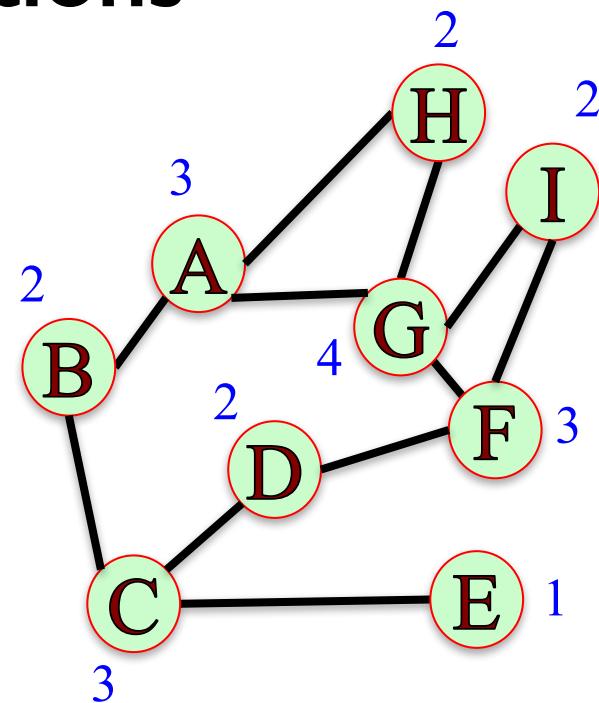
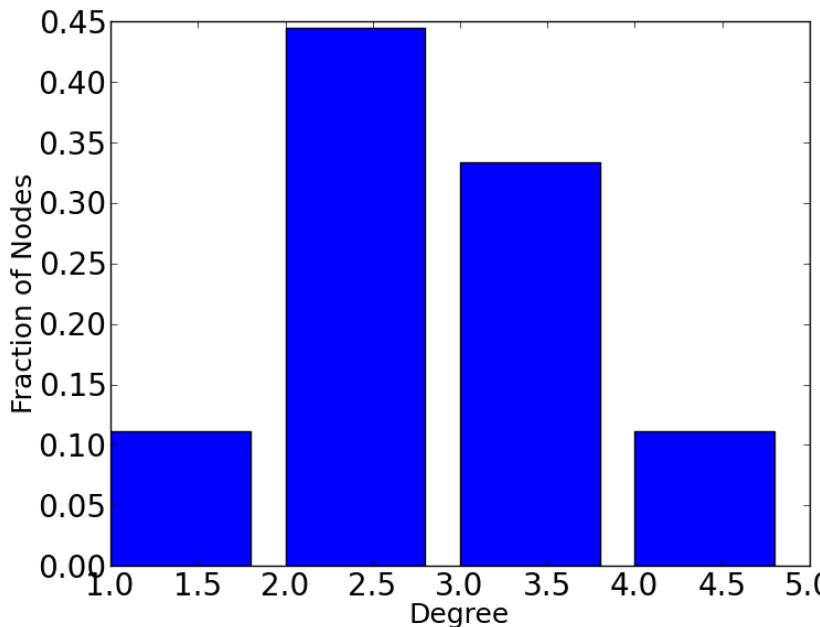
```
degrees = G.degree()  
degree_values = sorted(set(degrees.values()))  
histogram =  
[list(degrees.values()).count(i)/float(nx.number_of_nodes(  
G)) for i in degree_values]
```

```
import matplotlib.pyplot as plt  
plt.bar(degree_values, histogram)  
plt.xlabel('Degree')  
plt.ylabel('Fraction of Nodes')  
plt.show()
```



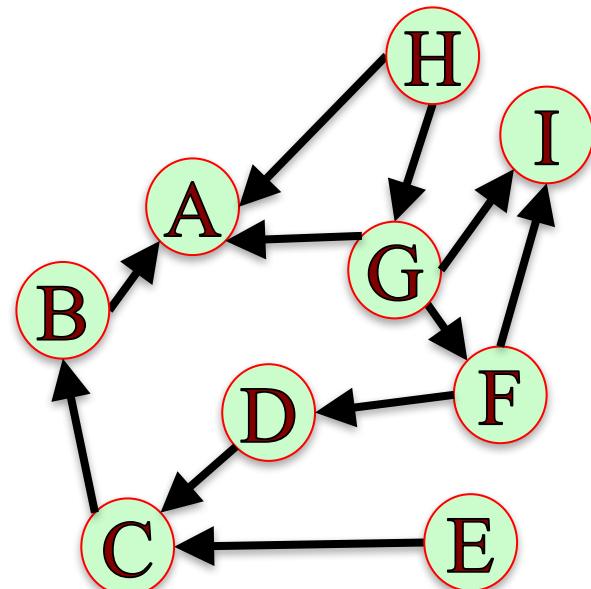
# Degree Distributions

Plot of the degree distribution of this network:



# In-Degree Distributions

The **in-degree** of a node in a directed graph is the number of in-links it has.

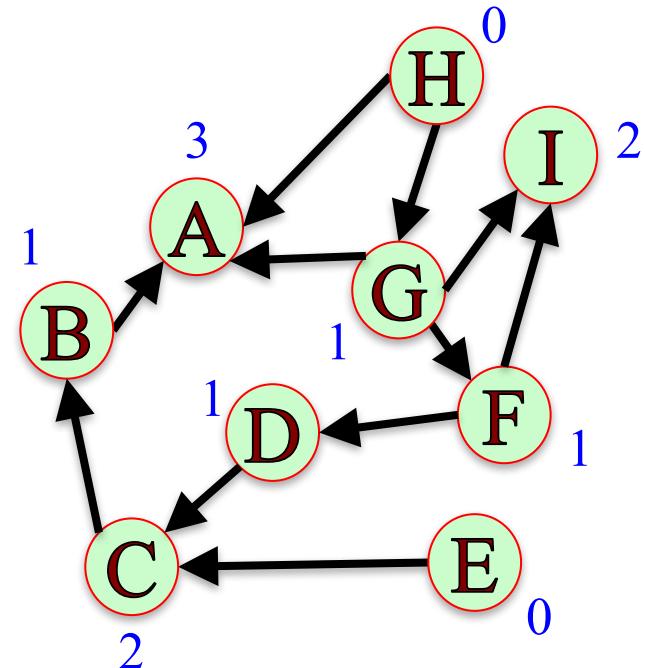


# In-Degree Distributions

The **in-degree** of a node in a directed graph is the number of in-links it has.

The in-degree distribution,  $P_{in}(k)$ , of this network has the following values:

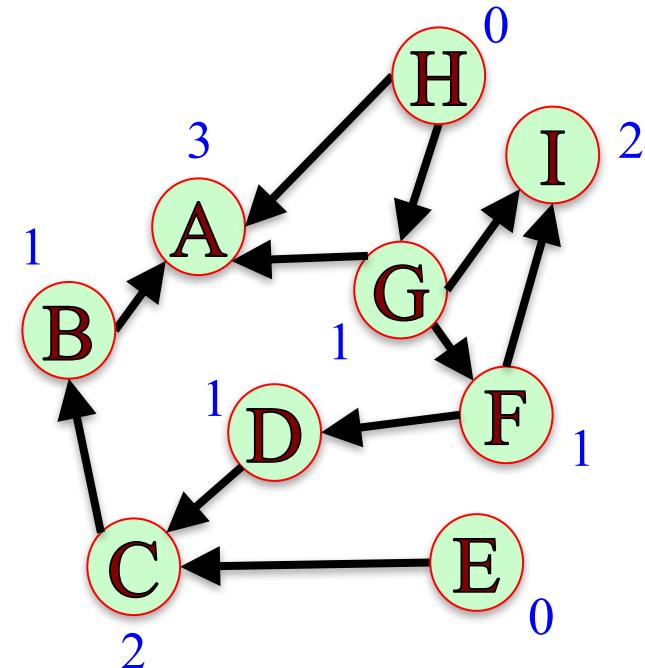
$$P_{in}(0) = \frac{2}{9}, P_{in}(1) = \frac{4}{9}, P_{in}(2) = \frac{2}{9}, P_{in}(3) = \frac{1}{9}$$



# In-Degree Distributions

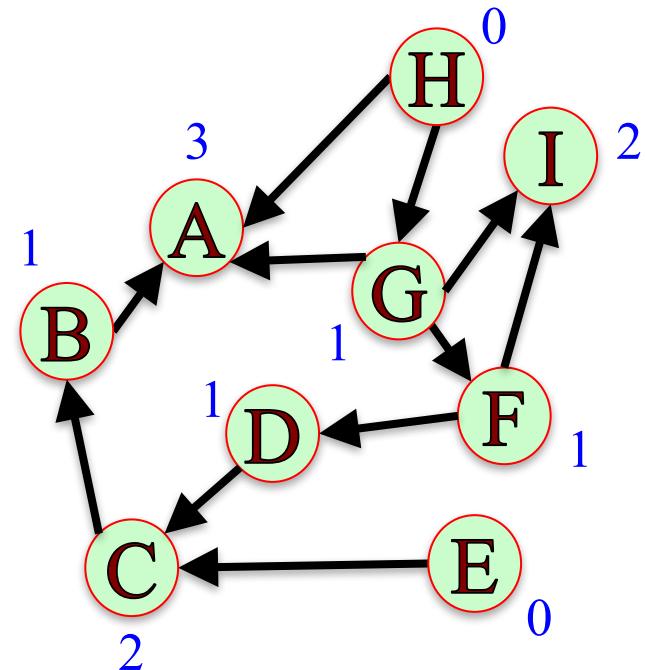
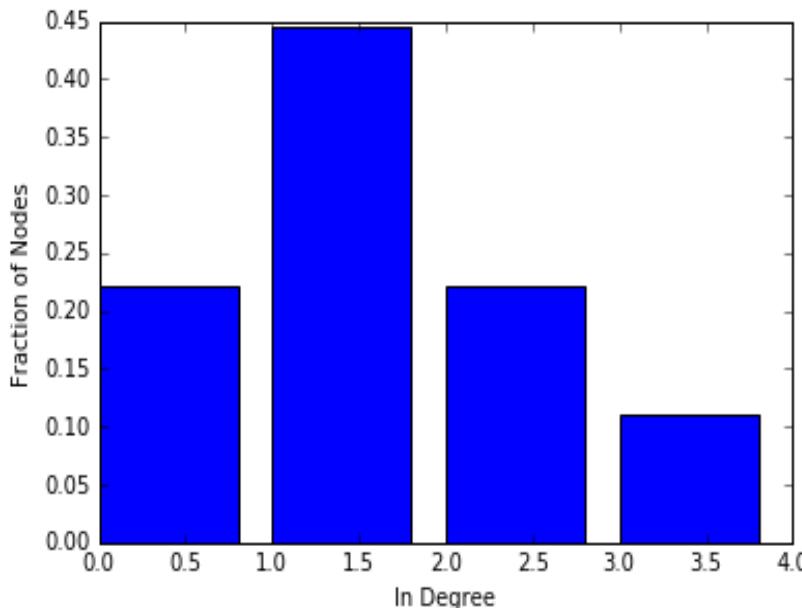
Plot of the degree distribution of this network:

```
in_degrees = G.in_degree()  
in_degree_values = sorted(set(in_degrees.values()))  
histogram =  
[list(in_degrees.values()).count(i)/float(nx.number_of_nodes(  
G)) for i in in_degree_values]  
  
plt.bar(in_degree_values, histogram)  
plt.xlabel('In Degree')  
plt.ylabel('Fraction of Nodes')  
plt.show()
```



# In-Degree Distributions

Plot of the degree distribution of this network:

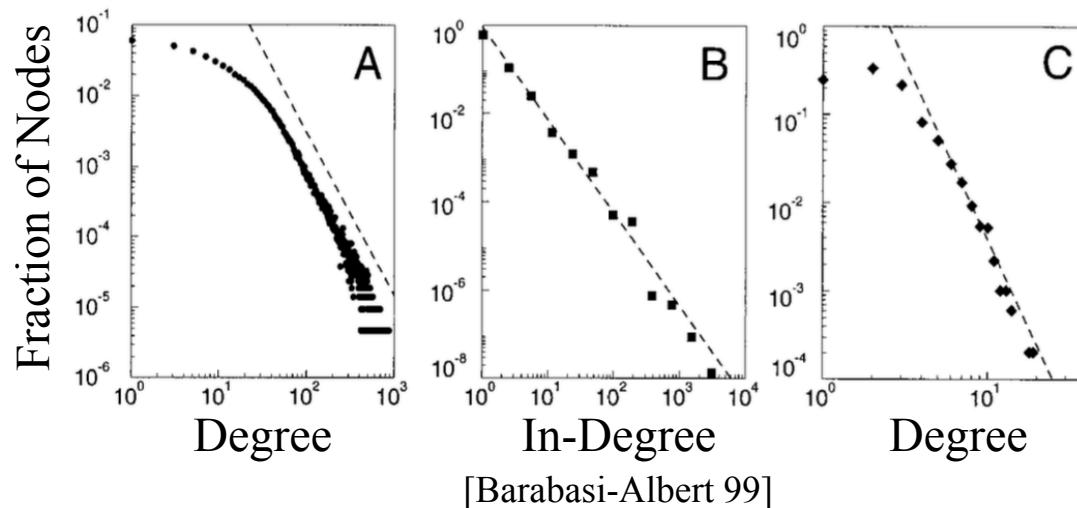


# Degree Distributions in Real Networks

A – **Actors**: network of 225,000 actors connected when they appear in a movie together.

B – **The Web**: network of 325,000 documents on the WWW connected by URLs.

C – **US Power Grid**: network of 4,941 generators connected by transmission lines.



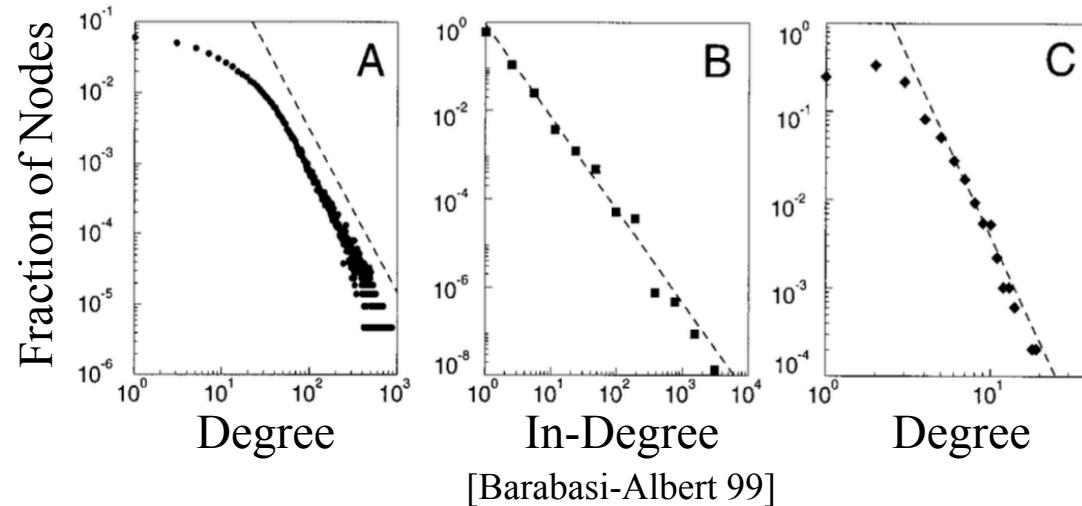
Degree distribution looks like a straight line when on a log-log scale. **Power law**:  $P(k) = Ck^{-\alpha}$ , where  $\alpha$  and C are constants.  $\alpha$  values: A: 2.3, B: 2.1, C: 4.

# Modeling Networks

Networks with power law distribution have many nodes with small degree and a few nodes with very large degree.

What could explain power law degree distribution we observe in many networks?

Can we find a set of basic assumptions that explain this phenomenon?



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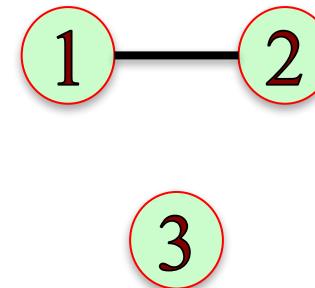
# Preferential Attachment Model

- Start with two nodes connected by an edge.
- At each time step, add a new node with an edge connecting it to an existing node.
- Choose the node to connect to at random with probability proportional to each node's degree.
- The probability of connecting to a node  $u$  of degree  $k_u$  is  $k_u / \sum_j k_j$ .



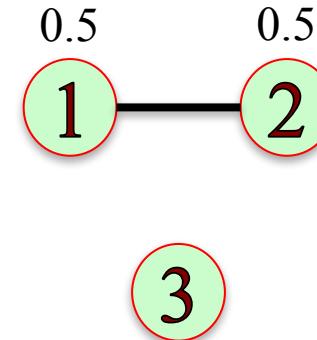
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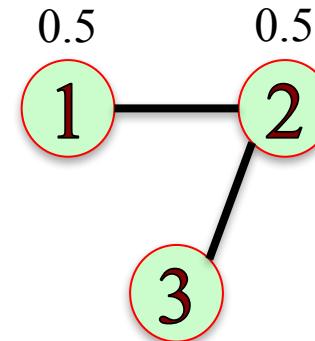
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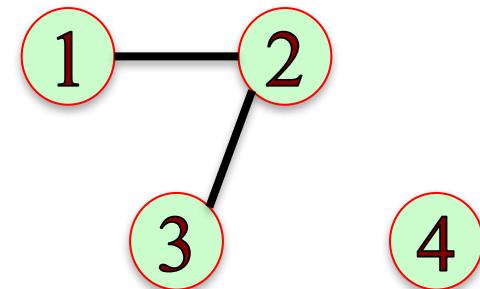
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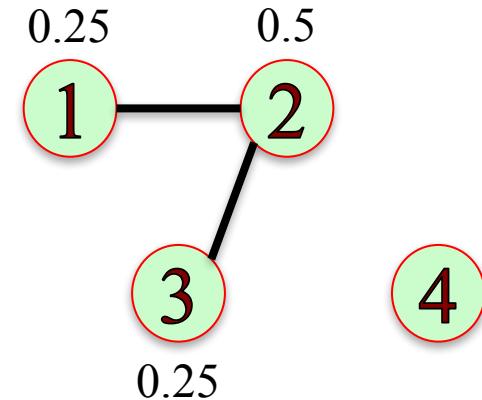
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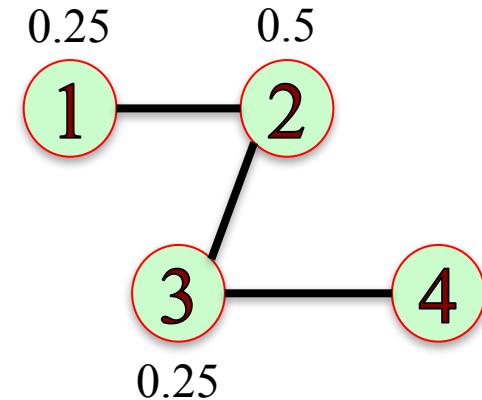
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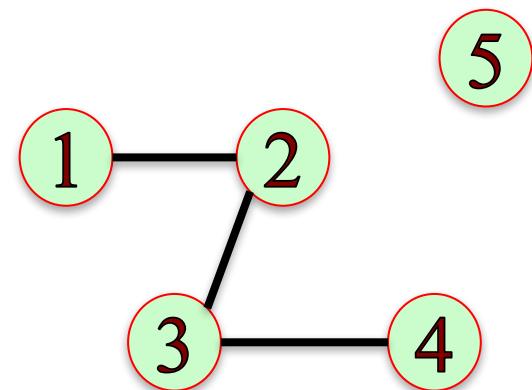
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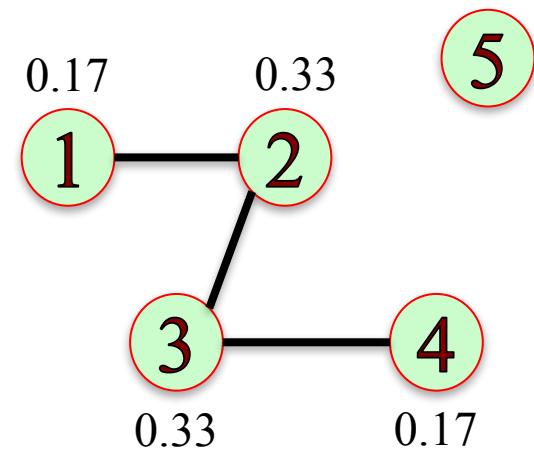
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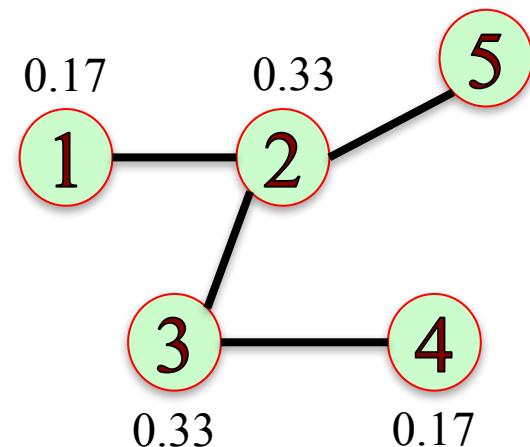
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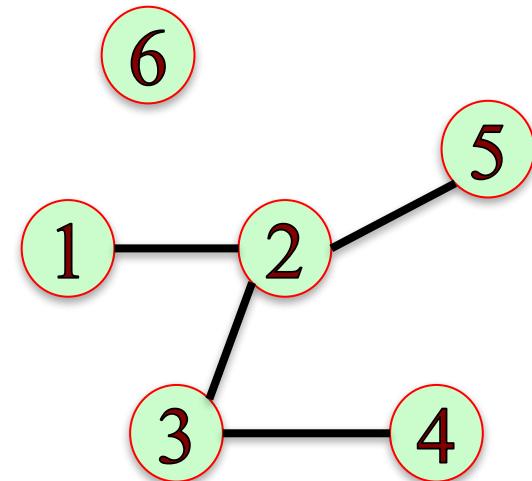
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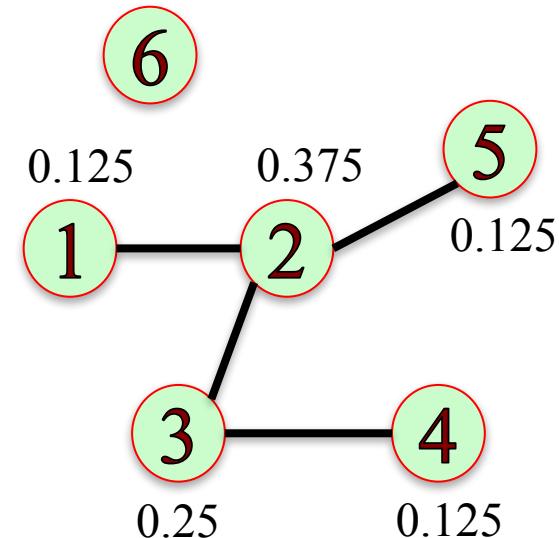
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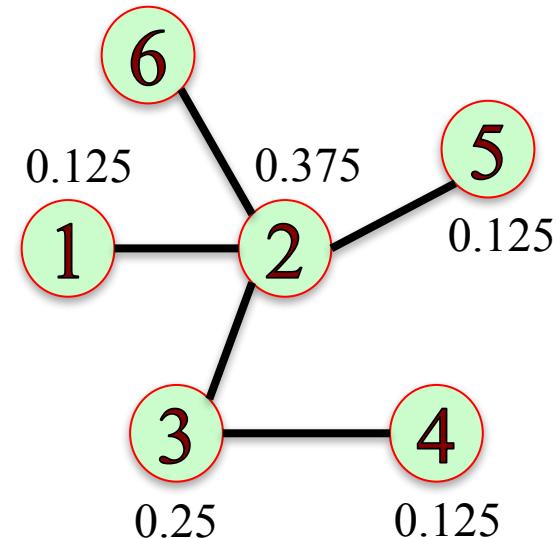
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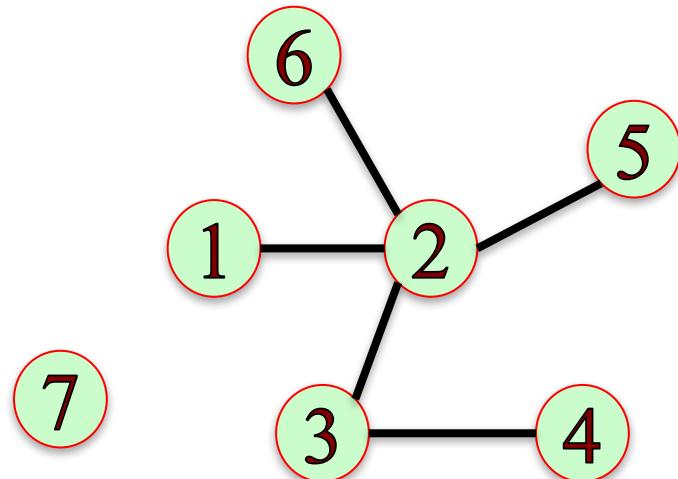
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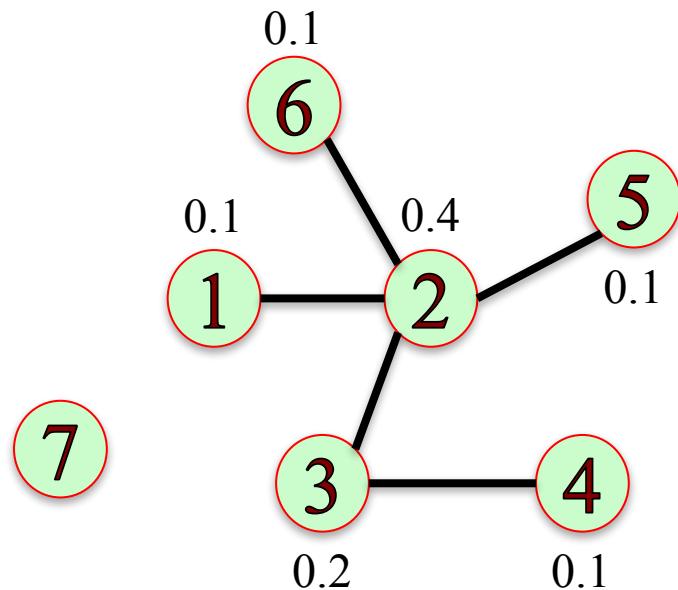
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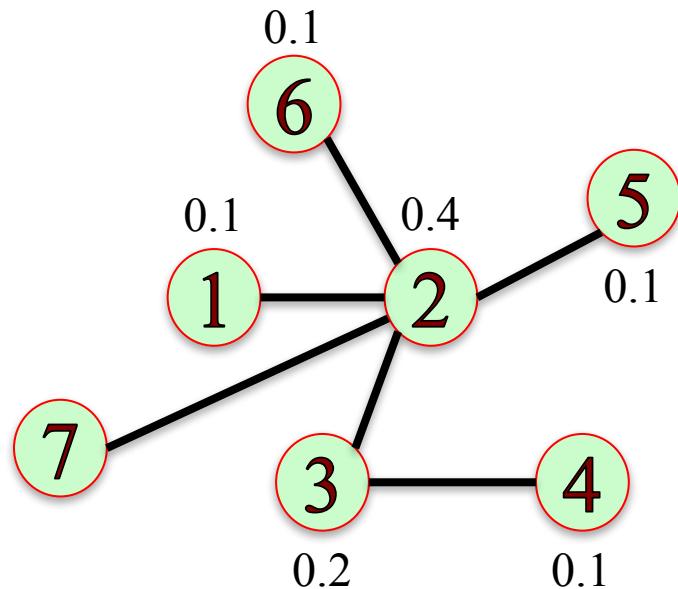
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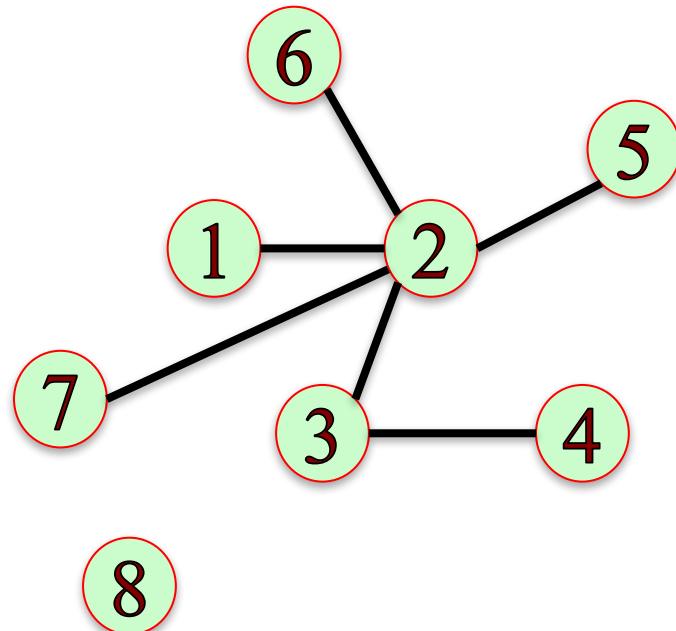
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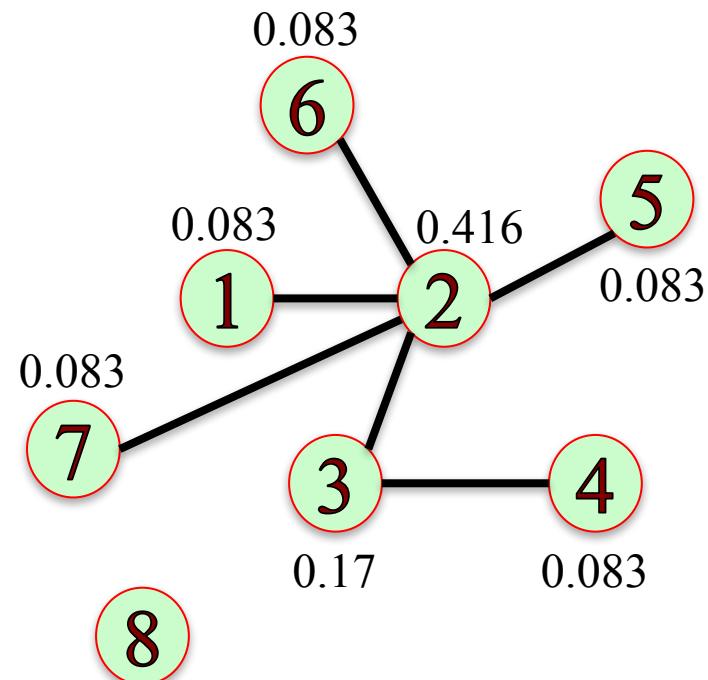
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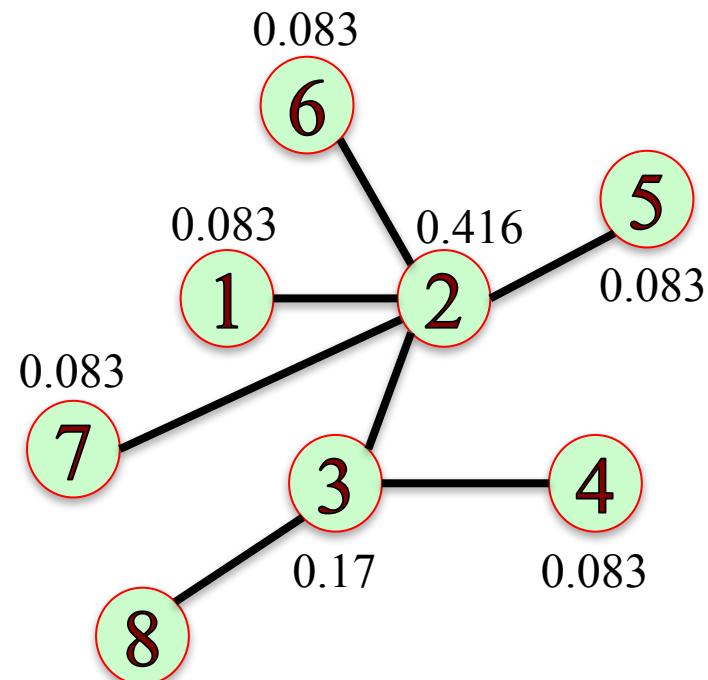
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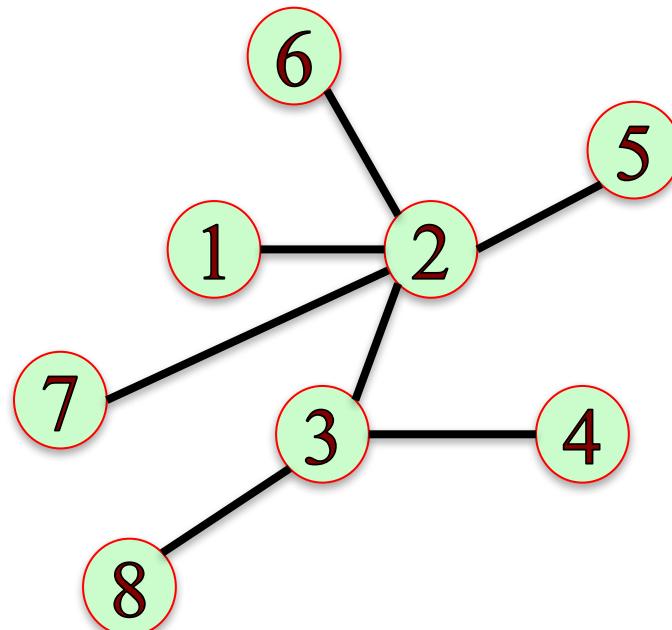
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# Preferential Attachment Model

As the number of nodes increases, the degree distribution of the network under the preferential attachment model approaches the power law  $P(k) = Ck^{-3}$  with constant  $C$ .

The preferential attachment model produces networks with degree distributions similar to real networks.

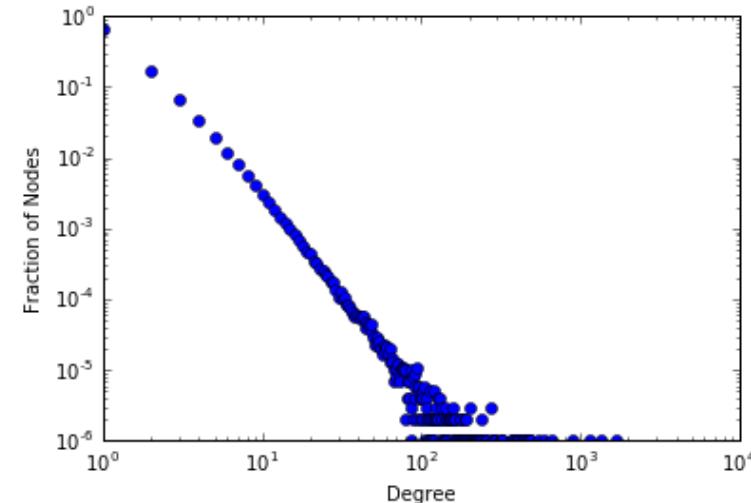


# Preferential Attachment in NetworkX

`barabasi_albert_graph( $n, m$ )` returns a network with  $n$  nodes. Each new node attaches to  $m$  existing nodes according to the Preferential Attachment model.

```
G = nx.barabasi_albert_graph(1000000,1)
degrees = G.degree()
degree_values = sorted(set(degrees.values()))
histogram =
[list(degrees.values().count(i))/float(nx.number_of_nodes(G)) for i in degree_values]

plt.plot(degree_values, histogram, 'o')
plt.xlabel('Degree')
plt.ylabel('Fraction of Nodes')
plt.xscale('log')
plt.yscale('log')
plt.show()
```



# Summary

- The degree distribution of a graph is the probability distribution of the degrees over the entire network.
- Many real networks have degree distributions that look like power laws ( $P(k) = Ck^{-\alpha}$  ).
- Models of network generation allow us to identify mechanisms that give rise to observed patterns in real data.
- The Preferential Attachment Model produces networks with a power law degree distribution.
- You can use `barabasi_albert_graph(n, m)` to construct a  $n$ -node preferential attachment network, where each new node attaches to  $m$  existing nodes.