THE GROWTH OF DIGITAL SUMS OF POWERS OF TWO

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In this note, we prove that

$$\lim_{n\to\infty} s(2^n) = \infty$$

where $s(n)$ denotes the sum of the digits of n written in base 10.

Lemma 1. *Every positive integer N can be expressed in the form*

$$N = \sum_{i=1}^{m} d[i] \cdot 10^{e[i]}$$

where $d[i] \in \{1, 2, \ldots, 9\}$ and $e[i]$ are integers so that

$$0 \le e[1] < e[2] < \cdots < e[m] \ .$$

Furthermore,

$$s(N) = \sum_{i=1}^{m} d[i] \ge m$$

Proof. This is simply the decimal expansion of N with zeros omitted. □

Lemma 2. *Let $2^n = A + B \cdot 10^k$ where A, B, k, n are positive integers and $A < 10^k$. Then $A \ge 2^k$.*

Proof. Since $2^n > 10^k > 2^k$, it follows that $n > k$, so 2^k divides 2^n. But 2^k also divides 10^k, therefore 2^k divides A. But $A > 0$, so $A \ge 2^k$. □

We use these lemmas to establish a lower bound on $s(2^n)$. Write

$$2^n = \sum_{i=1}^{m} d[i] \cdot 10^{e[i]}$$

so the conditions of Lemma 1 hold, and let k be an integer between 2 and m. Then $2^n = A + B \cdot 10^{e[k]}$ where

$$A = \sum_{i=1}^{k-1} d[i] \cdot 10^{e[i]}$$

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and

$$B = \sum_{i=k}^{m} d[i] \cdot 10^{e[i]-e[k]} \ .$$

Since $A < 10^{e[k]}$, Lemma 2 implies that $A \geq 2^{e[k]}$. Therefore,

$$2^{e[k]} \leq A < 10^{e[k-1]+1}$$

which implies that

$$e[k] < (\log_2 10)(e[k-1]+1) \ .$$

We prove that $e[k] < 4^{k-1}$ for all k. It is clear that $e[1] = 0$, else 2^n would be divisible by 10. From the inequality above, we have $e[1] \leq 3$ and $e[2] \leq 13$. If $k \geq 3$ then $e[k-1] > 5$, so

$$\begin{aligned} e[k] &< (\log_2 10)e[k-1] + (\log_2 10) \\ &< \frac{10}{3}e[k-1] + \frac{10}{3} \\ &< \frac{10}{3}e[k-1] + \frac{2}{3}e[k-1] \\ &= 4e[k-1]. \end{aligned}$$

Therefore, $e[k] < 4^{k-1}$ for all k, by induction.

We are now able to prove the main result. Note that

$$2^n < 10^{e[m]+1} \leq 10^{4^{m-1}}$$

since $10^{e[m]}$ is the leading power of 10 in the decimal expansion of 2^n.

Taking logarithms gives

$$\begin{aligned} 4^{m-1} &> n \log_{10} 2 > n \\ m-1 &> \log_4 n \\ s(2^n) &> \log_4 n \end{aligned}$$

hence

$$\lim_{n\to\infty} s(2^n) = \infty \ .$$