THE GROWTH OF DIGITAL SUMS OF POWERS

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Abstract. We establish sufficient conditions for the sum of the base-b digits of a^n to diverge to infinity, and prove that under mild hypotheses this sum grows at least logarithmically. Our approach uses only elementary number-theoretic arguments and applies to a wide class of sequences, including factorials and least common multiples.

1. Motivation

This article was inspired by the following problem, which was posed and solved by Wacław Sierpiński[4, Problem 209]:

> *Prove that the sum of digits of the number 2^n (in decimal system) increases to infinity with n.*

We will prove a sufficient condition on positive integers a and b which implies that the sum of the base-b digits of a^n grows at least logarithmically in n. This condition includes Sierpiński's problem as a special case. The behavior of digital sums of powers has been studied by Senge and Straus (1973) and Stewart (1980), but their results rely on deeper tools from transcendence theory. In contrast, the arguments here are elementary.

Consider the sequence of powers of 2:

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots.$$

This sequence grows very rapidly. Now, let us define another sequence by adding the decimal digits of each power of 2. For example, 16 becomes $1 + 6 = 7$, and 32 becomes $3 + 2 = 5$. The first few terms of this new sequence are listed below.

$$1, 2, 4, 8, 7, 5, 10, 11, 13, 8, 7, \ldots.$$

It is apparent that this sequence grows much more slowly, and it is not monotone. Nevertheless, it is reasonable to conjecture that the sequence diverges to infinity. Indeed, we should expect that the sum of the decimal digits of 2^n is asymptotic to $4.5n \log_{10} 2$,

Date: November 5, 2025.

since 2^n has $\lfloor n \log_{10} 2 \rfloor + 1$ decimal digits and the digits seem to be approximately uniformly distributed among $0, 1, 2, \ldots, 9$. However, this stronger statement remains to be proved.

For an integer $b \geq 2$, we write $s_b(n)$ for the sum of the base-b digits of n, and $c_b(n)$ for the number of nonzero digits in that expansion. These functions are asymptotic to each other, since $c_b(n) \leq s_b(n) \leq (b-1)c_b(n)$ for all n and b, so we will restrict our attention to $c_b(n)$. For a prime p, $\nu_p(n)$ denotes the exponent of p in the prime factorization of n.

Let us prove that $\lim_{n\to\infty} c_{10}(2^n) = \infty$. Observe that the final digit of 2^n cannot be 0, since only multiples of 10 can end in 0.

If 2^n has four or more digits, then the last four digits cannot start with three consecutive zeros. This is because 2^n is divisible by 16, so $2^n \bmod 10^4$, the number formed by the last four digits of 2^n, is also divisible by 16. If the first three of these digits were zero, then $2^n \bmod 10^4$ would be less than 10, which is a contradiction.

Similarly, if 2^n has 14 or more digits, then $2^n \bmod 10^{14} \geq 2^{14} > 10^4$. So the last 14 digits of 2^n cannot start with 10 consecutive zeros.

We can continue in this way, finding longer and longer non-overlapping blocks of digits, each containing at least one nonzero digit. This shows that as n increases to infinity, the number of nonzero digits of 2^n also increases to infinity. Let us formalize this argument.

Theorem 1. *Let $\{e_k\}$ be a sequence of positive integers such that $e_1 = 1$ and $2^{e_{k+1}} > 10^{e_k}$ for all $k \geq 1$. If n is a positive integer that is divisible by 2^{e_k} but not divisible by 10, then $c_{10}(n) \geq k$.*

Proof. The proof is by induction on k. The case $k = 1$ is trivial, so let us assume that $k \geq 2$. By the division algorithm, there exist integers $q \geq 0$ and $0 \leq r < 10^{e_{k-1}}$ such that

$$n = 10^{e_{k-1}}q + r.$$

Since $n \geq 2^{e_k} > 10^{e_{k-1}}$, it follows that $q \geq 1$.

Since n and $10^{e_{k-1}}$ are divisible by $2^{e_{k-1}}$, r is also divisible by $2^{e_{k-1}}$. Moreover, r is not divisible by 10, so $c_{10}(r) \geq k-1$ by the induction hypothesis.

Note that $c_{10}(n) = c_{10}(q) + c_{10}(r)$, since the digit expansion of n is the concatenation of the digit expansions of q and r, possibly with leading zeros. Therefore,

$$c_{10}(n) = c_{10}(q) + c_{10}(r) \geq 1 + (k-1) = k.$$

□

Corollary 1.1. *Let a be a positive integer that is divisible by 2 but not divisible by 10. Then $\lim_{n\to\infty} c_{10}(a^n) = \infty$.*

Proof. Let k be a positive integer. If $n \geq e_k$ then a^n is divisible by 2^{e_k} but not divisible by 10, so $c_{10}(a^n) \geq k$ by the previous theorem. Therefore, $c_{10}(a^n) \geq k$ for all $n \geq e_k$. Since k is arbitrary, we conclude that $\lim_{n\to\infty} c_{10}(a^n) = \infty$. □

2. Generalizing to other bases

Our proofs rely only on divisibility properties and therefore extend naturally to arbitrary bases.

Theorem 2. *Let $b \geq 2$ be an integer that is not a power of a prime, and let p be a prime divisor of b. Let $\{e_k\}$ be a sequence of positive integers such that $e_1 = 1$ and $p^{e_{k+1}} > b^{e_k}$ for all $k \geq 1$. If n is a positive integer that is divisible by p^{e_k} but not divisible by b, then $c_b(n) \geq k$.*

Proof. The proof is by induction on k. The case $k = 1$ is trivial, so let us assume that $k \geq 2$. By the division algorithm, there exist integers $q \geq 0$ and $0 \leq r < b^{e_{k-1}}$ such that

$$n = b^{e_{k-1}}q + r.$$

Since $n \geq p^{e_k} > b^{e_{k-1}}$, it follows that $q \geq 1$.

Since n and $b^{e_{k-1}}$ are divisible by $p^{e_{k-1}}$, r is also divisible by $p^{e_{k-1}}$. Moreover, r is not divisible by b, so $c_b(r) \geq k - 1$ by the induction hypothesis.

Therefore,

$$c_b(n) = c_b(q) + c_b(r) \geq 1 + (k-1) = k.$$

□

Theorem 3. *Let $b \geq 2$ be an integer that is not a power of a prime, let p and q be distinct prime divisors of b, and let $\{e_k\}$ be defined as in Theorem 2. If*

$$\phi_{p,q}(n) := \nu_p(n) - \nu_q(n)\frac{\nu_p(b)}{\nu_q(b)} \geq e_k$$

then $c_p(n) \geq k$.

Proof. Write n as $b^r m$, where m is not divisible by b. The function $\phi_{p,q}$ satisfies $\phi_{p,q}(b) = 0$ and $\phi_{p,q}(uv) = \phi_{p,q}(u) + \phi_{p,q}(v)$ for all positive integers u and v, so $\phi_{p,q}(n) = \phi_{p,q}(m)$. Since $\nu_p(m) \geq \phi_{p,q}(m) \geq e_k$ and $b \nmid m$, Theorem 2 implies that $c_b(m) \geq k$. But $c_b(m) = c_b(n)$, since m and n have the same digits in base b, apart from trailing zeros. Therefore, $c_b(n) \geq k$. □

Corollary 3.1. *Let $a \geq 2$ and $b \geq 2$ be integers. Suppose that b has prime divisors p and q such that $\phi_{p,q}(a) > 0$. Then $\lim_{n\to\infty} c_b(a^n) = \infty$.*

Proof. Let $k > 0$ be given. There exists an integer N such that $\phi_{p,q}(a^n) = n\phi_{p,q}(a) \geq e_k$ for all $n \geq N$, so Theorem 3 implies that $c_b(a^n) \geq k$ for all $n \geq N$. Since k is arbitrary, we conclude that $\lim_{n\to\infty} c_b(a^n) = \infty$. □

Note that this argument also shows that $c_b(a^n)$ grows at least as fast as $\Omega(\log n)$ when the conditions of the corollary are satisfied. In 1973, Senge and Straus[3, Theorem 3] proved that if $a \geq 1$ and $b \geq 2$ are positive integers then $\lim_{n\to\infty} c_b(a^n) = \infty$ if and only if $\log(a)/\log(b)$ is irrational. However, they did not prove a lower bound on the rate of growth. In 1980, Stewart[5, Theorem 2] proved that if $\log(a)/\log(b)$ is irrational then

$$c_b(a^n) > \frac{\log n}{\log\log n + C} - 1$$

for $n > 4$, where C depends on a and b alone.

3. Related sequences

Theorem 3 can be applied to many other sequences. We will give two examples here. As before, let $b \geq 2$ be an integer that is not a prime power, let p and q be distinct prime divisors of b, and let

$$\phi_{p,q}(n) = \nu_p(n) - \nu_q(n)\frac{\nu_p(b)}{\nu_q(b)}.$$

Factorials. By Legendre's formula[1, p. 263], $\nu_p(n!) = (n - s_p(n))/(p-1)$. Thus,

$$\phi_{p,q}(n!) \approx n\left(\frac{1}{p-1} - \frac{\nu_p(b)}{(q-1)\nu_q(b)}\right)$$

whenever $(p-1)\nu_p(b) \neq (q-1)\nu_q(b)$, since $s_p(n) + s_q(n) \ll n$. Therefore, $\lim_{n\to\infty} c_b(n!) = \infty$ for any b with distinct factors p and q satisfying

$$(p-1)\nu_p(b) > (q-1)\nu_q(b).$$

In particular, $\lim_{n\to\infty} c_{10}(n!) = \infty$.

Cumulative LCMs. Let $\Lambda_n = \mathrm{lcm}(1, 2, \ldots, n)$. It is easy to see that $\nu_p(\Lambda_n) = \lfloor \log_p(n) \rfloor$. Thus,

$$\phi_{p,q}(\Lambda_n) = \lfloor \log_p n \rfloor - \lfloor \log_q n \rfloor \cdot \frac{\nu_p(b)}{\nu_q(b)} \approx \log n \cdot \left(\frac{1}{\log p} - \frac{1}{\log q} \cdot \frac{\nu_p(b)}{\nu_q(b)}\right).$$

The quantity in parentheses is nonzero, since $\log(p)/\log(q)$ is irrational, and we may assume that it is positive, else we can switch p and q. Therefore, $\lim_{n\to\infty} \phi_{p,q}(\Lambda_n) = \infty$, hence $\lim_{n\to\infty} c_b(\Lambda_n) = \infty$ for every base b that is not a prime power.

Remark. Sanna[2] proved that $s_b(n!)$ and $s_b(\Lambda_n)$ are greater than $C_b \log n \log\log\log n$ for each integer $n > e^e$ and every $b \geq 2$, where C_b is a constant depending only on b.

References

[1] Leonard Eugene Dickson. *History of the Theory of Numbers*. Vol. 1. Washington, Carnegie Institution of Washington, 1919. URL: https://archive.org/details/historyoftheoryo01dick.

[2] Carlo Sanna. "On the sum of digits of the factorial". In: *Journal of Number Theory* 147 (2015), pp. 836–841. DOI: 10.1016/j.jnt.2014.09.003.

[3] H. G. Senge and E. G. Straus. "PV-numbers and sets of multiplicity". In: *Periodica Mathematica Hungarica* 3.1 (1973), pp. 93–100. DOI: 10.1007/BF02018464.

[4] Wacław Sierpiński. *250 Problems in Elementary Number Theory*. American Elsevier Publishing Company, 1970. URL: https://archive.org/details/250problemsinele0000sier.

[5] C.L. Stewart. "On the representation of an integer in two different bases." In: *Journal für die reine und angewandte Mathematik* 319 (1980), pp. 63–72. DOI: 10.1515/crll.1980.319.63.