ELEMENTARY BOUNDS ON DIGITAL SUMS OF POWERS, FACTORIALS, AND LCMS

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ABSTRACT. We prove that the sum of the base-b digits of a^n grows at least logarithmically in n if $\log(d)/\log(b)$ is irrational, where d is the smallest factor of a such that $\gcd(a/d, b) = 1$. Our approach uses only elementary number theory and applies to a wide class of sequences, including factorials and $\Lambda(n) = \mathrm{lcm}(1, 2, \ldots, n)$. We conclude with an expository proof of the previously known result that the sum of the base-b digits of a^n tends to infinity with n if and only if $\log(a)/\log(b)$ is irrational.

1. INTRODUCTION

This article was inspired by the following problem, which was posed and solved by Wacław Sierpiński[6, Problem 209]:

> *Prove that the sum of digits of the number 2^n (in decimal system) increases to infinity with n.*

The reader is urged to attempt this problem on their own before proceeding. Note that it is not enough to prove that the sum of digits of 2^n is unbounded, since the sequence is not monotonic.

Consider the sequence of powers of 2 (sequence A000079 in the OEIS):

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots.$$

This sequence grows very rapidly. Now, let us define another sequence, by adding the decimal digits of each power of 2. For example, 16 becomes $1 + 6 = 7$, and 32 becomes $3 + 2 = 5$. The first few terms of this new sequence (A001370) are listed below.

$$1, 2, 4, 8, 7, 5, 10, 11, 13, 8, 7, \ldots.$$

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Our new sequence grows much more slowly, and it is not monotonic. Nevertheless, it is reasonable to conjecture that it tends to infinity. Indeed, one might guess that the sum of the decimal digits of 2^n is asymptotic to $4.5n\log_{10} 2$, since 2^n has $\lfloor n\log_{10} 2\rfloor+1$ decimal digits, and the digits seem to be approximately uniformly distributed among $0, 1, 2, \ldots, 9$. However, this stronger conjecture remains to be proved.

For an integer $b \geq 2$, we write $s_b(n)$ for the sum of the base-b digits of n, and $c_b(n)$ for the number of nonzero digits in that expansion. These functions are asymptotic to each other, since $c_b(n) \leq s_b(n) \leq (b-1)c_b(n)$ for all n and b; so we will restrict our attention to $c_b(n)$. For a prime p, $\nu_p(n)$ denotes the exponent of p in the prime factorization of n. If p does not divide n then $\nu_p(n) = 0$.

2. Digital sums of powers of two

We will present an informal proof that $c_{10}(2^n)$ tends to infinity as $n \to \infty$. The key idea, which might seem too obvious to state, is that a positive multiple of a number cannot be smaller than that number.

Let n be a positive integer, and write the decimal expansion of 2^n as $2^n = \sum_{i=0}^{\infty} d_i 10^i$, where $d_i \in \{0, \ldots, 9\}$ and $d_i = 0$ for all but finitely many terms. Note that the final digit d_0 of 2^n cannot be zero, since 2^n is not divisible by 10.

If $2^n > 10$, then 2^n is divisible by 16; so $2^n \bmod 10^4$, the number formed by the last four digits of 2^n, is also divisible by 16. If the first three of these digits were zero, then $2^n \bmod 10^4$ would be less than 10, which is impossible. So at least one of the digits d_1, d_2, d_3 is nonzero.

If $2^n > 10^4$, then 2^n is divisible by 2^{14}; so $2^n \bmod 10^{14}$, the number formed by the last 14 digits of 2^n, is also divisible by 2^{14}. If the first 10 of these digits were zero, then $2^n \bmod 10^{14}$ would be less than 10^4, which is impossible. So at least one of the digits $d_4, d_5, \ldots, d_{13}$ is nonzero.

We can continue in this way, finding longer and longer non-overlapping blocks of digits, each containing at least one nonzero digit, as illustrated in Figure 1. This proves that $c_{10}(2^n)$ tends to infinity as $n \to \infty$.

$$2^{103} = \boxed{101412048018258352}\boxed{1197362564}\boxed{300}\boxed{8}$$

Figure 1. Each block contains at least one nonzero digit.

Let us formalize this argument.

Theorem 1. *Let $(\varepsilon(k))_{k\geq 1}$ be a sequence of integers such that $\varepsilon(1) \geq 1$ and $2^{\varepsilon(k)} > 10^{\varepsilon(k-1)}$ for all $k \geq 2$. If n is a positive integer that is divisible by $2^{\varepsilon(k)}$ but not divisible by 10, then $c_{10}(n) \geq k$.*

Proof. The proof is by induction on k. The case $k = 1$ is trivial, so let us assume that $k \geq 2$. By the division algorithm, there exist integers q and r such that

$$n = 10^{\varepsilon(k-1)}q + r,$$

where $q \geq 0$ and $0 \leq r < 10^{\varepsilon(k-1)}$.

Since $n \geq 2^{\varepsilon(k)} > 10^{\varepsilon(k-1)}$, it follows that $q \geq 1$.

Since n and $10^{\varepsilon(k-1)}$ are divisible by $2^{\varepsilon(k-1)}$, the integer r is also divisible by $2^{\varepsilon(k-1)}$. But r is not divisible by 10, so $c_{10}(r) \geq k - 1$, by the induction hypothesis.

Note that $c_{10}(n) = c_{10}(q) + c_{10}(r)$, since the digit expansion of n is the concatenation of the digit expansions of q and r, possibly with leading zeros.

Therefore, $c_{10}(n) \geq 1 + (k - 1) = k$. □

Corollary 1. *Let a be a positive integer that is divisible by 2 but not divisible by 10. Then $c_{10}(a^n) \geq \log_4(n)$ for all $n \geq 1$. In particular, $\lim_{n\to\infty} c_{10}(a^n) = \infty$.*

Proof. Let $\varepsilon(k) = 4^{k-1}$ for $k \geq 1$. This sequence satisfies $\varepsilon(1) \geq 1$ and

$$2^{\varepsilon(k)} > 10^{\varepsilon(k-1)}$$

for all $k \geq 2$.

Let $n \geq 4$ be a positive integer, and let $k = \lceil \log_4 n \rceil$, so that $4^{k-1} < n \leq 4^k$. Then a^n is divisible by 2^n, so a^n is also divisible by $2^{\varepsilon(k)}$. But a^n is not divisible by 10.

Therefore, $c_{10}(a^n) \geq k \geq \log_4(n)$, by Theorem 1. □

Exercise. Show that every power of 3 has a multiple m, not divisible by 10, such that $c_{10}(m) = 2$.

3. Generalizing to other bases

Our proofs rely only on divisibility properties and therefore extend naturally to other bases.

Theorem 2. *Let $b \geq 2$ be an integer that is not a power of a prime, and let p be a prime divisor of b. Let $(\varepsilon(k))_{k\geq 1}$ be a sequence of integers such that $\varepsilon(1) \geq 1$ and $p^{\varepsilon(k)} > b^{\varepsilon(k-1)}$ for all $k \geq 2$. If $\nu_p(n) \geq \varepsilon(k)$ and $b \nmid n$ then $c_b(n) \geq k$.*

Proof. The proof is by induction on k. The case $k = 1$ is trivial, so let us assume that $k \geq 2$. By the division algorithm, there exist integers q and r such that

$$n = b^{\varepsilon(k-1)}q + r,$$

where $q \geq 0$ and $0 \leq r < b^{\varepsilon(k-1)}$.

Since $n \geq p^{\varepsilon(k)} > b^{\varepsilon(k-1)}$, it follows that $q \geq 1$.

Since n and $b^{\varepsilon(k-1)}$ are divisible by $p^{\varepsilon(k-1)}$, the integer r is also divisible by $p^{\varepsilon(k-1)}$. But r is not divisible by b, so $c_b(r) \geq k - 1$, by the induction hypothesis.

Note that $c_b(n) = c_b(q) + c_b(r)$, since the digit expansion of n is the concatenation of the digit expansions of q and r, possibly with leading zeros.

Therefore, $c_b(n) \geq 1 + (k - 1) = k$. □

Notation. Given an integer $b \geq 2$ with distinct prime divisors p and q, define a function $\xi = \xi_{b,p,q}$ as follows:

$$\xi(n) = \nu_p(n) - \nu_q(n)\frac{\nu_p(b)}{\nu_q(b)}.$$

The reader can verify that $\xi(b^r u) = \xi(u)$ for all $r \geq 0$ and $u \geq 1$. Intuitively, ξ is a modified ν_p that ignores trailing zeros in the base-b representation of its argument.

Theorem 3. *Let $b \geq 2$ be an integer that is not a power of a prime, let p and q be distinct prime divisors of b, and let $(\varepsilon(k))_{k\geq 1}$ be defined as in Theorem 2. If $\xi(n) \geq \varepsilon(k)$, then $c_b(n) \geq k$. In particular, $\lim_{n\to\infty} c_b(a_n) = \infty$ for any sequence of positive integers that satisfies $\lim_{n\to\infty} \xi(a_n) = \infty$.*

Proof. Write n as $b^r u$, where u is not divisible by b. Note that $\xi(n) = \xi(u)$. Since $\nu_p(u) \geq \xi(u) \geq \varepsilon(k)$ and $b \nmid u$, Theorem 2 implies that $c_b(u) \geq k$. But $c_b(u) = c_b(n)$, since u and n have the same digits in base b, apart from trailing zeros. Therefore, $c_b(n) \geq k$. □

Theorem 4. *Let $a \geq 2$ and $b \geq 2$ be integers. Let d be the smallest factor of a such that $\gcd(a/d, b) = 1$, and suppose that $\log(d)/\log(b)$ is irrational. Then $c_b(a^n) \geq \log_r n$ for all $n \geq 1$, where $r \geq 2$ is an integer depending only on a and b. In particular, $\lim_{n\to\infty} c_b(a^n) = \infty$.*

Proof. Let $p_1^{e_1} \cdots p_t^{e_t}$ be the prime factorization of b. Then

$$d = p_1^{f_1} \cdots p_t^{f_t},$$

where $f_i = \nu_{p_i}(a)$. Note that some of the f_i may be zero. If

$$\frac{f_1}{e_1} = \cdots = \frac{f_t}{e_t}$$

then $\log(d)/\log(b) = f_1/e_1$, which is rational. Therefore, if $\log(d)/\log(b)$ is irrational, then the ratios f_i/e_i are not all equal, which implies that b has two prime factors $p = p_i$ and $q = p_j$ such that

$$\frac{f_i}{e_i} > \frac{f_j}{e_j},$$

and so $\xi(a) > 0$.

Let $r = \lceil \log(b)/\log(p) \rceil$, and let $\varepsilon(k) = r^{k-1}$ for $k \geq 1$. This sequence satisfies $\varepsilon(1) \geq 1$ and

$$p^{\varepsilon(k)} > b^{\varepsilon(k-1)}$$

for all $k \geq 2$.

Let n be a positive integer, and let $k = \lceil \log_r n \rceil$, so that $r^{k-1} < n \leq r^k$. Then a^n is divisible by p^n, so a^n is also divisible by $p^{\varepsilon(k)}$.

Therefore, $c_b(a^n) \geq k \geq \log_r(n)$, by Theorem 3. □

In 1973, Senge and Straus[5, Theorem 3] proved that if $a \geq 1$ and $b \geq 2$ are positive integers, then $\lim_{n\to\infty} c_b(a^n) = \infty$ if and only if $\log(a)/\log(b)$ is irrational. However, they did not demonstrate a lower bound. In 1980, Stewart[7, Theorem 2] proved that if $\log(a)/\log(b)$ is irrational, then

$$c_b(a^n) > \frac{\log n}{\log\log n + C} - 1$$

for $n > 4$, where C depends on a and b alone. Theorem 4 achieves a stronger bound, but with stricter conditions on a and b. We will replicate Stewart's result in Section 5.

4. Related sequences

Theorem 3 can be applied to many other sequences. We will give two examples here.

Theorem 5. *Let $b \geq 2$ be an integer. Suppose that b has prime divisors p and q such that*

$$(p-1)\nu_p(b) \neq (q-1)\nu_q(b).$$

Then $c_b(n!) > C \log n$ for some $C > 0$ depending only on b. In particular, $c_b(n!) \to \infty$ as $n \to \infty$.

Proof. Assume without loss of generality that $(p-1)\nu_p(b) < (q-1)\nu_q(b)$. This condition may be rewritten as

$$\frac{1}{p-1} - \frac{\nu_p(b)}{(q-1)\nu_q(b)} > 0.$$

By Legendre's formula[3, p. 263],

$$\nu_p(n!) = \frac{n - s_p(n)}{p-1}.$$

Since $s_p(n)/n \to 0$ and $s_q(n)/n \to 0$ as $n \to \infty$,

$$\xi(n!) = n\left(\frac{1}{p-1} - \frac{\nu_p(b)}{(q-1)\nu_q(b)} + o(1)\right) = \Theta(n).$$

Therefore, by Theorem 3,

$$c_b(n!) > C \log n$$

for some $C > 0$ depending only on b. $\square$

Theorem 6. *Let $\Lambda_n = \operatorname{lcm}(1, 2, \ldots, n)$, and let $b \geq 2$ be an integer that is not a power of a prime. Then $c_b(\Lambda_n) > C \log\log n$ for some $C > 0$ depending only on b. In particular, $c_b(\Lambda_n) \to \infty$ as $n \to \infty$.*

Proof. Let p and q be two distinct divisors of b. It is well known [1, p. 62] that $\nu_p(\Lambda_n) = \lfloor \log_p(n) \rfloor$. Thus,

$$\xi(\Lambda_n) = \lfloor \log_p n \rfloor - \lfloor \log_q n \rfloor \cdot \frac{\nu_p(b)}{\nu_q(b)} = \log n \cdot \left(\frac{1}{\log p} - \frac{1}{\log q} \cdot \frac{\nu_p(b)}{\nu_q(b)} + o(1)\right).$$

Since $\log(p)/\log(q)$ is irrational,

$$\frac{1}{\log p} - \frac{1}{\log q} \cdot \frac{\nu_p(b)}{\nu_q(b)} \neq 0,$$

and we may assume without loss of generality that it is positive.

Therefore,

$$\xi(\Lambda_n) = \Theta(\log n),$$

so by Theorem 3,

$$c_b(\Lambda_n) > C \log\log n$$

for some $C > 0$ depending only on b.

$\square$

Stronger bounds are known. In 2015, Sanna[4] proved that

$$s_b(n!) > C \log n \log\log\log n$$

for every integer $n > e^e$ and every $b \geq 2$, where C is a constant depending only on b, and they established the same inequality for $s_b(\Lambda_n)$.

5. Digital sums of powers: the general case

In this final section, we will prove that the number of nonzero digits in the base-b expansion of a^n tends to infinity as $n \to \infty$, provided that $\log(b)/\log(a)$ is irrational. This result appears in earlier work of Senge–Straus [5] and Stewart [7], but we present an argument that we hope is more accessible.

The irrationality condition is necessary. Indeed, if $\log(a)/\log(b) = r/s \in \mathbb{Q}$, then

$$a^{ns} = b^{nr}$$

for every integer n, so a^{ns} has only one nonzero digit in base b.

Recall that a nonzero algebraic number $\alpha \in \mathbb{C}$ is a root of a unique irreducible integer polynomial P with positive leading coefficient and coprime coefficients. The *degree* of α is equal to the degree of P, and the *height* of α is the maximum of the absolute values of the cofficients of P. A rational integer has degree 1 and height equal to its absolute value.

Our proof relies on a version of Baker's theorem on linear forms in logarithms, which we use as a black box.

Theorem 7. *[2, p. 23] Let $\alpha_1, \ldots, \alpha_n$ be nonzero algebraic numbers with degrees at most d. Suppose that $\alpha_1, \ldots, \alpha_{n-1}$ and α_n have heights at most $A' \geq 4$ and $A \geq 4$ respectively. Let $\beta_1, \ldots, \beta_n$ be integers of absolute value at most $B \geq 2$. If*

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \alpha_n \neq 0,$$

there exists $C > 1$, depending only on n, d, and A', such that

$$|\Lambda| > C^{-\log A \log B}.$$

Baker's theorem gives effective lower bounds on linear forms in logarithms of algebraic numbers. In our context, it guarantees that the expression in Equation 2 (below) cannot be too small, which limits how well powers of a can be approximated by powers of b.

The next lemma deals with the possibility that some power of a might be divisible by b.

Lemma 1. *Let $a, b \geq 2$ be integers with $\log(a)/\log(b)$ irrational. Then there exists C and C', depending only on a and b, such that whenever*

$$a^n = b^r u \qquad \text{with } b \nmid u,$$

we have

$$Cn \leq \log u \leq C'n.$$

Proof. Since $\log(a)/\log(b)$ is irrational, we can choose primes p, q such that

$$\nu_p(a)\nu_q(b) > \nu_q(a)\nu_p(b).$$

Hence $\xi(a) > 0$ for the function $\xi = \xi_{b,p,q}$ defined in Section 3.

Because $\nu_p(u) \geq \xi(u) = \xi(a^n) = n\xi(a)$, the integer u is divisible by $p^{\lceil n\xi(a) \rceil}$. Therefore

$$\log u \geq Cn, \qquad C = \xi(a) \log p > 0.$$

On the other hand, $\log u \leq C'n$ for $C' = \log a$, since $u \leq a^n$. □

We now prove the main theorem.

Theorem 8. *Let $a, b \geq 2$ be integers, and suppose that $\log(a)/\log(b)$ is irrational. Then for all sufficiently large n,*

$$c_b(a^n) > \frac{\log n}{\log \log n + C}$$

for some $C > 0$ depending only on a and b. In particular, $c_b(a^n) \to \infty$.

Proof. Let

$$a^n = b^m \left(d_i b^{-m(1)} + \cdots + d_k b^{-m(k)} \right),$$

where $m = \lfloor \log_b a \rfloor$, $k = c_b(a^n)$, $d_i \in \{1, \ldots, b-1\}$, and

$$1 = m(1) < \cdots < m(k) \leq m.$$

That is, $d_1, \ldots, d_k$ are the nonzero digits of a^n in base b, and $m(1), \ldots, m(k)$ are their positions, counting from the left. Assume that $k \geq 2$.

Fix $i \in \{1, \ldots, k-1\}$. We estimate the ratio $m(i+1)/m(i)$, which will ultimately bound n.

Define

$$q = b^{m(i)}(d_1 b^{-m(1)} + \cdots + d_i b^{-m(i)}),$$
$$r = b^m(d_{i+1} b^{-m(i+1)} + \cdots + d_k b^{-m(k)}),$$

so that

$$a^n = b^{m-m(i)}q + r.$$

From the base-b expansion we obtain the bounds

$$b^{m-1} < a^n < b^m,$$
$$b^{m(i)-1} < q < b^{m(i)},$$
$$b^{m-m(i+1)} < r < b^{m-m(i+1)+1}.$$

These imply the approximation

$$\frac{-\log(a^{-n}r)}{\log(q)} = \frac{m(i+1)}{m(i)} + O(1). \tag{1}$$

Now set

$$\Lambda = \log(a^{-n}b^{m-m(i)}q) = -n\log a + (m - m(i))\log b + \log q. \tag{2}$$

Then

$$|\Lambda| = -\log(1 - a^{-n}r) < 2a^{-n}r.$$

By Theorem 7,

$$|\Lambda| > C^{-\log q \log n}$$

for some $C > 1$ depending only on a and b. Thus,

$$a^{-n}r > C^{-\log q \log n},$$

and hence, for n sufficiently large,

$$\frac{-\log(a^{-n}r)}{\log(q)} < C\log n \tag{3}$$

for some $C > 0$ depending only on a and b.

Combining 1 and 3 gives

$$\frac{m(i+1)}{m(i)} < C\log n. \tag{4}$$

Summing the logarithms of these ratios,

$$\log m(k) = \sum_{i=1}^{k-1} \log\left(\frac{m(i+1)}{m(i)}\right)$$

which yields

$$\log m(k) < (k-1)(\log\log n + C). \tag{5}$$

By Lemma 1, $\log m(k) = \log n + O(1)$, so (5) implies

$$k > \frac{\log n}{\log \log n + C}$$

for all sufficiently large n, as required. $\square$

Exercise. Prove that the number of nonzero decimal digits in the nth Fibonacci number tends to infinity as $n \to \infty$.

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