ELEMENTARY BOUNDS ON DIGITAL SUMS OF POWERS, FACTORIALS, AND LCMS

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Abstract. We prove lower bounds on digital sums of powers, multiples of powers, factorials, and the least common multiple of $\{1, \ldots, n\}$, using only elementary number theory.

1. Introduction

This expository article establishes lower bounds on digital sums of powers, multiples of powers, factorials, and the least common multiple of $\{1, \ldots, n\}$, using only elementary number theory.

We were inspired by the following problem, which was posed and solved by Wacław Sierpiński [9, Problem 209]:

> *Prove that the sum of digits of the number 2^n (in decimal system) increases to infinity with n.*

The reader is urged to attempt this problem on their own before proceeding. Note that it is not enough to prove that the sum of digits of 2^n is unbounded, since the sequence is not monotonic.

Consider the sequence of powers of 2 (sequence A000079 in the OEIS):

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots.$$

This sequence grows very rapidly. Now, define another sequence by adding the decimal digits of each term. For example, 16 becomes $1 + 6 = 7$, and 32 becomes $3 + 2 = 5$. The first few terms of this new sequence (A001370) are listed below.

$$1, 2, 4, 8, 7, 5, 10, 11, 13, 8, 7, \ldots.$$

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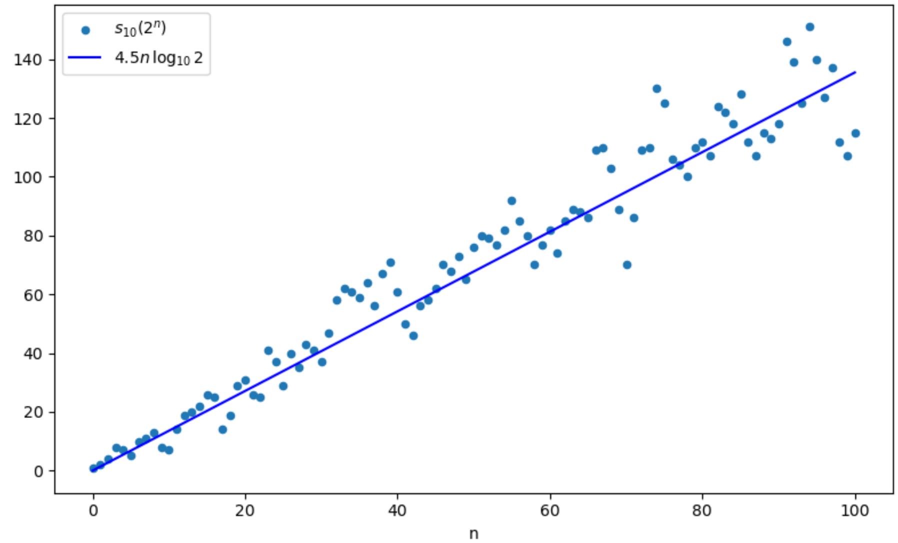

FIGURE 1. Scatter plot of the digital sum of 2^n for $n \leq 100$ together with the conjectured linear approximation.

This sequence of digital sums grows much more slowly, and it is not monotonic. Nevertheless, it is reasonable to conjecture that it tends to infinity. Indeed, one might guess that the sum of the decimal digits of 2^n is approximately equal to $4.5n\log_{10} 2$, since 2^n has $\lfloor n \log_{10} 2 \rfloor + 1$ decimal digits, and the digits seem to be approximately uniformly distributed among $0, 1, 2, \ldots, 9$. However, this stronger conjecture remains to be proved. See Figure 1.

We prove in Section 3 that the digital sum of 2^n is greater than $\log_4 n$ for all $n \geq 1$. But first, we review the relevant notation and terminology.

2. Notation and terminology

For integers $N \geq 0$ and $b \geq 2$, the *base-b expansion* of N is the unique representation of the form

$$N = \sum_{i=0}^{\infty} d_i b^i, \quad d_i \in \{0, 1, \ldots, b-1\}.$$

The integers d_i are the *base-b digits* of N; all but finitely many of these digits are zero.

For an integer $b \geq 2$, we write $s_b(N)$ for the sum of the base-b digits of N, and $c_b(N)$ for the number of nonzero digits in that expansion. These functions are equivalent up to a constant factor, since $c_b(N) \leq s_b(N) \leq (b-1)c_b(N)$ for all N and b; so we focus on $c_b(N)$.

The function s_b is *subadditive*: $s_b(M+N) \leq s_b(M) + s_b(N)$ for all nonnegative integers M and N. Equality holds if no carries occur in the digitwise addition of M and N. Otherwise, each carry reduces the digital sum by $b-1$. The function c_b is likewise subadditive.

For a prime p, the *p-adic valuation* of N, denoted $\nu_p(N)$, is the exponent of p in the prime factorization of N. If p does not divide N then $\nu_p(N) = 0$. The function ν_p is *completely additive*: $\nu_p(MN) = \nu_p(M) + \nu_p(N)$ for all positive integers M and N.

The *floor* of a real number x, denoted $\lfloor x \rfloor$, is the greatest integer n such that $n \leq x$. The *ceiling* of x, denoted $\lceil x \rceil$, is the least integer n such that $n \geq x$.

We use Bachmann-Landau-Knuth notations [3] to describe the approximate size of functions. Let f and g be real-valued functions defined on a domain D, usually the set of positive integers.

One writes

$$f(n) = O(g(n))$$

if there exists a positive real number C such that

$$|f(n)| \leq Cg(n) \quad \text{for all } n \in D.$$

The notation

$$f(n) = o(g(n))$$

means that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

In particular, $O(1)$ denotes a bounded function, and $o(1)$ denotes a function that tends to 0 as $n \to \infty$.

Finally,

$$f(n) \asymp g(n)$$

means that there exist positive real numbers C and C' such that

$$Cg(n) < |f(n)| < C'g(n) \quad \text{for all } n \in D.$$

$$
\begin{array}{lcr}
2^0 & = & 1 \\
2^4 & = & 1\ 6 \\
2^{14} & = & 1\ 638\ 4 \\
2^{47} & = & 1\ 4073748835\ 532\ 8 \\
2^{157} & = & 1\ 826877046663628647754606040895353\ 7745699156\ 787\ 2
\end{array}
$$

FIGURE 2. Digits of 2^n subdivided into blocks. Each block contains at least one nonzero digit.

3. Digital sums of powers of two

We present an informal proof that $c_{10}(2^n)$ tends to infinity as $n \to \infty$. See [6] for an alternative approach.

Let n be a positive integer, and write the decimal expansion of 2^n as

$$2^n = \sum_{i=0}^{\infty} d_i 10^i,$$

where each $d_i \in \{0, \ldots, 9\}$ and all but finitely many d_i are zero. Since 2^n is not divisible by 10, its final digit d_0 is nonzero.

Assume first that $n \geq 4$. Then 2^n is divisible by $2^4 = 16$. Consider the last four digits of 2^n, that is,

$$2^n \bmod 10^4.$$

This number is divisible by 16. If the digits d_1, d_2, d_3 were all zero, then this remainder would be less than 10, and hence could not be divisible by 16. Therefore at least one of the digits d_1, d_2, d_3 is nonzero.

Now assume $n \geq 14$. Then 2^n is divisible by 2^{14}. Since $2^{14} > 10^4$, any number divisible by 2^{14} must be at least 10^4. Thus the last 14 digits of 2^n,

$$2^n \bmod 10^{14},$$

cannot be less than 10^4. If the digits $d_4, d_5, \ldots, d_{13}$ were all zero, this remainder would be less than 10^4, which is impossible. Hence at least one digit in this block is nonzero.

Continuing in this way, as n increases, we obtain new blocks of decimal digits, each containing at least one nonzero digit. These blocks are disjoint, and the number of such blocks grows without bound as $n \to \infty$.

Therefore the number of nonzero decimal digits of 2^n tends to infinity as $n \to \infty$. See Figure 3

Let us formalize this argument.

Theorem 1. *Let $(e_k)_{k \geq 1}$ be a sequence of integers such that $e_1 \geq 1$ and $2^{e_k} > 10^{e_{k-1}}$ for all $k \geq 2$. Suppose that N is divisible by 2^{e_k} but not 10. Then $c_{10}(N) \geq k$.*

Proof. We argue by induction on k. The case $k = 1$ is immediate, since any positive integer has at least one nonzero digit.

Assume now that $k \geq 2$, and that the statement holds for $k - 1$. Apply the division algorithm to write

$$N = 10^{e_{k-1}} q + r, \quad 0 \leq r < 10^{e_{k-1}},$$

for integers q, r.

Because $N \geq 2^{e_k} > 10^{e_{k-1}}$ by hypothesis, the quotient satisfies $q \geq 1$.

Next, both N and $10^{e_{k-1}} q$ are divisible by $2^{e_{k-1}}$, hence their difference

$$r = N - 10^{e_{k-1}} q$$

is also divisible by $2^{e_{k-1}}$.

Moreover, r is not divisible by 10, since N is not divisible by 10.

Therefore, $c_{10}(r) \geq k - 1$ by the induction hypothesis.

Finally, the decimal expansion of N is obtained by concatenating the decimal expansion of q with the (possibly zero-padded) expansion of r. Thus,

$$c_{10}(N) = c_{10}(q) + c_{10}(r) \geq 1 + (k - 1) = k.$$

This completes the proof.

□

Corollary 1. *Let a be a positive integer that is divisible by 2 but not divisible by 10. Then $c_{10}(a^n) \geq \log_4 n$ for all $n > 1$. In particular, $\lim_{n \to \infty} c_{10}(a^n) = \infty$.*

Proof. Let $e_k = 4^{k-1}$ for $k \geq 1$. This sequence satisfies $e_1 \geq 1$ and

$$2^{e_k} > 10^{e_{k-1}}$$

for all $k \geq 2$.

Let $n > 1$ be a positive integer, and let $k = \lceil \log_4 n \rceil$, so that $4^{k-1} < n \leq 4^k$. Then a^n is divisible by 2^n, so a^n is also divisible by 2^{e_k}. But a^n is not divisible by 10.

Therefore, $c_{10}(a^n) \geq k \geq \log_4 n$, by Theorem 1. □

A similar argument applies if a is divisible by 5 but not divisible by 10, or more generally, if the prime factorization of a contains unequal numbers of twos and fives. In the next section, we generalize this insight to non-decimal base expansions.

4. Digital sums of powers in other bases

Before extending the arguments of the previous section, we first situate our results in the broader context of earlier work on digital sums of powers. In 1973, Senge and Straus [8, Theorem 3] showed that for integers $a \geq 1$ and $b \geq 2$,

$$\lim_{n\to\infty} c_b(a^n) = \infty \quad \Longleftrightarrow \quad \frac{\log a}{\log b} \text{ is irrational.}$$

However, their result does not yield any explicit lower bound.

Subsequently, Stewart [10, Theorem 2] proved that if $\log(a)/\log(b)$ is irrational then

$$c_b(a^n) > \frac{\log n}{\log\log n + C} - 1$$

for all $n > 4$, where C depends only on a and b. This bound is extremely general but grows more slowly than logarithmic.

In this section we prove a logarithmic lower bound for $c_b(a^n)$ under conditions that are stronger than Stewart's. The first step (Theorem 2 below) generalizes the corresponding base-10 argument from the previous section.

Theorem 2. *Let a, b be integers such that $2 \leq a < b$ and a divides b. Let (e_k) be a sequence of integers such that $e_1 \geq 1$ and $a^{e_k} > b^{e_{k-1}}$ for all $k \geq 2$. Suppose that N is divisible by a^{e_k} but not b. Then $c_b(N) \geq k$.*

Proof. Follow the proof of Theorem 1, but replace 2 with a where needed, and replace 10 with b. □

The conclusion of Theorem 2 can be converted into an explicit logarithmic lower bound, as described below.

Theorem 3. *Let a, b be integers such that $2 \leq a < b$ and a divides b. Suppose that N is divisible by a^n but not b. Then there exists $C > 0$, depending only on a and b, such that*

$$c_b(N) > C \log n$$

for n sufficiently large. Indeed, we can choose any $0 < C < (\log(\log(b)/\log(a)))^{-1}$.

Proof. Let $r = \log(b)/\log(a)$, and define (e_k) by $e_1 = 1$ and $e_k = \lceil re_{k-1} \rceil$ for $k \geq 2$. It is routine to verify that (e_k) satisfies the conditions of Theorem 2.

Then,

$$\begin{aligned} e_2 &< r+1, \\ e_3 &< r^2+r+1, \\ e_4 &< r^3+r^2+r+1, \end{aligned}$$

and for all $k \geq 1$,

$$e_k < \sum_{i=0}^{k-1} r^i < \frac{r^k}{r-1}. \tag{1}$$

Now suppose that n is an integer greater than $\dfrac{r}{r-1}$, and let

$$k = \left\lfloor \frac{\log((r-1)n)}{\log r} \right\rfloor.$$

Then

$$1 \leq k \leq \frac{\log((r-1)n)}{\log r},$$

which implies that

$$n \geq \frac{r^k}{r-1}.$$

Therefore $n > e_k$ by (1), hence $c_b(N) \geq k$ by Theorem 2.

Choose C so that $0 < C < (\log r)^{-1}$. Then

$$\left\lfloor \frac{\log((r-1)n)}{\log r} \right\rfloor > C \log n$$

for n sufficiently large, which implies that $c_b(N) > C \log n$. □

The following lemma allows us to relax the condition that b does not divide N.

Lemma 1. *Let $a, b \geq 2$ be integers such that $\log(a)/\log(b)$ is irrational. Suppose that $a^n = b^m t$, where $t \geq 1$ is an integer. Then there exists a prime factor p of a, and $C > 0$ depending only on a and b, such that $\nu_p(t) \geq Cn$.*

Proof. Since $\log(a)/\log(b)$ is irrational, there are no integers u, v with $a^v = b^u$, except $u = v = 0$. Equivalently, the vectors $(\nu_p(a))_p$ and $(\nu_p(b))_p$ are independent, so we can choose primes p and q such that

$$\nu_p(a)\nu_q(b) - \nu_q(a)\nu_p(b) > 0. \tag{2}$$

By comparing the valuations of a^n with respect to p and q, we obtain

$$n\nu_p(a) = m\nu_p(b) + \nu_p(t) \tag{3}$$

and

$$n\nu_q(a) \geq m\nu_q(b). \tag{4}$$

Combining (3) and (4) yields

$$\nu_p(t) \geq n\nu_p(a) - n\frac{\nu_q(a)}{\nu_q(b)}\nu_p(b) = Cn, \tag{5}$$

where

$$C = \frac{\nu_p(a)\nu_q(b) - \nu_q(a)\nu_p(b)}{\nu_q(b)}.$$

Finally, $C > 0$ by (2). □

We now come to the main result of this section.

Theorem 4. *Let $a, b \geq 2$ be integers. Let d be the smallest factor of a such that $\gcd(a/d, b) = 1$, and suppose that $\log(d)/\log(b)$ is irrational. Then $c_b(a^n) > C \log n$ for all sufficiently large n, where $C > 0$ depends only on a and b.*

Proof. Write $a^n = b^m s$ with $b \nmid s$, and set $g = a/d$. By the minimality of d, the prime divisors of d are exactly the primes of a that also divide b; in particular $\gcd(d, g) = 1$ and any prime $p \mid d$ satisfies $p \mid b$.

Since $g^n \mid a^n$ and $\gcd(g, b) = 1$, it follows that $g^n \mid s$. Define $t = s/g^n$; then $d^n = b^m t$.

By Lemma 1, applied with $a \leftarrow d$, there exists a prime divisor p of d such that

$$\nu_p(t) \geq C'n,$$

for some $C' > 0$ depending only on a and b.

Because $\gcd(d, g) = 1$, the prime p does not divide g, and hence $\nu_p(s) = \nu_p(t)$. Therefore, by Theorem 3, applied with $a \leftarrow p$ and $N \leftarrow s$,

$$c_b(s) > C \log n$$

for n sufficiently large.

Finally, $c_b(a^n) = c_b(s)$, since s and a^n differ only by a power of b and thus have the same base-b expansion up to trailing zeros.

Consequently, when n is sufficiently large, we have

$$c_b(a^n) > C \log n.$$

□

5. Digital sums of factorials and LCMs

In this section, we prove logarithmic lower bounds for the base-b digital sums of $n!$ and $\Lambda_n = \mathrm{lcm}(1, \dots, n)$. In contrast with the situation for a^n, where prime-power divisibility played a central role, the key feature for factorials and LCMs is that both $n!$ and Λ_n are divisible by large integers of the form $b^r - 1$.

The key insight is provided by the following lemma, which was originally proved by Stolarsky [11] for base 2, and extended to general bases by Balog and Dartyge [1].

Lemma 2. *Let $m, r \geq 1$ and $b \geq 2$ be integers. If m is divisible by $b^r - 1$, then $s_b(m) \geq (b-1)r$.*

Proof. Write the base-b expansion of m as a concatenation of r-digit blocks, so that

$$m = \sum_{i=0}^{k-1} B_i b^{ri}, \quad 0 \leq B_i < b^r, \quad B_{k-1} \geq 1.$$

Define the *block-sum* operator G by

$$G(m) = \sum_{i=0}^{k-1} B_i.$$

Observe that $G(m) \equiv m \pmod{b^r - 1}$, since $b^r \equiv 1 \pmod{b^r - 1}$. Also, $G(m) < m$ for $m \geq b^r$, and $G(m) = m$ for $0 \leq m < b^r$.

Iterate G on m: define $m_0 = m$ and $m_{t+1} = G(m_t)$. By the observations above, (m_t) is a sequence of positive multiples of $b^r - 1$, which is strictly decreasing while its terms exceed $b^r - 1$. Therefore, the sequence must eventually reach $b^r - 1$, which is the unique positive multiple of $b^r - 1$ that is less than b^r.

Since s_b is subadditive,

$$s_b(G(m)) \leq \sum_{i=0}^{k-1} s_b(B_i) = s_b(m).$$

Therefore,

$$s_b(m) \geq s_b(b^r - 1) = (b-1)r.$$

□

This lemma has an immediate consequence for factorials and least common multiples. If $n \geq b^r - 1$, then both $n!$ and Λ_n are divisible by $b^r - 1$, and hence

$$s_b(n!) \geq (b-1)r, \quad s_b(\Lambda_n) \geq (b-1)r.$$

Since one may choose $r = \lfloor \log_b(n+1) \rfloor$, this yields lower bounds of the form

$$s_b(n!) > C \log n, \quad s_b(\Lambda_n) > C \log n$$

for some $C > 0$ depending only on b.

Luca [4] proved the same results using similar methods. In 2015, Sanna [7] used the lemma above, together with more advanced methods, to prove that

$$s_b(n!) > C \log n \log\log\log n$$

for all integers $n > e^e$ and all $b \geq 2$, where C depends only on b. The same estimate holds for $s_b(\Lambda_n)$. Our interest here is not to compete with the best known results, but rather to show that simple divisibility arguments already imply logarithmic growth.

We conjecture that $s_b(n!) \asymp n \log n$ and $s_b(\Lambda_n) \asymp n$, but this remains to be proved. See Figure 3.

6. Stewart's Theorem

In this final section, we prove that the number of nonzero digits in the base-b expansion of a^n tends to infinity as $n \to \infty$, provided that $\log(a)/\log(b)$ is irrational. This result appears in earlier work of Senge and Straus [8] and Stewart [10], but we present an argument that we hope is more accessible.

The irrationality condition is necessary. Indeed, if $\log(a)/\log(b) = r/s \in \mathbb{Q}$, then

$$a^{ns} = b^{nr}$$

for every integer n, so a^{ns} has only one nonzero digit in base b.

We may assume without loss of generality that $a < b$. If not, replace b with b^r, where $b^r > a$. Since $c_{b^r}(n) \geq c_b(n)/r$, this replacement does not affect the order of growth.

The following lemma will permit us to disregard trailing zeros in the base-b expansion of a^n.

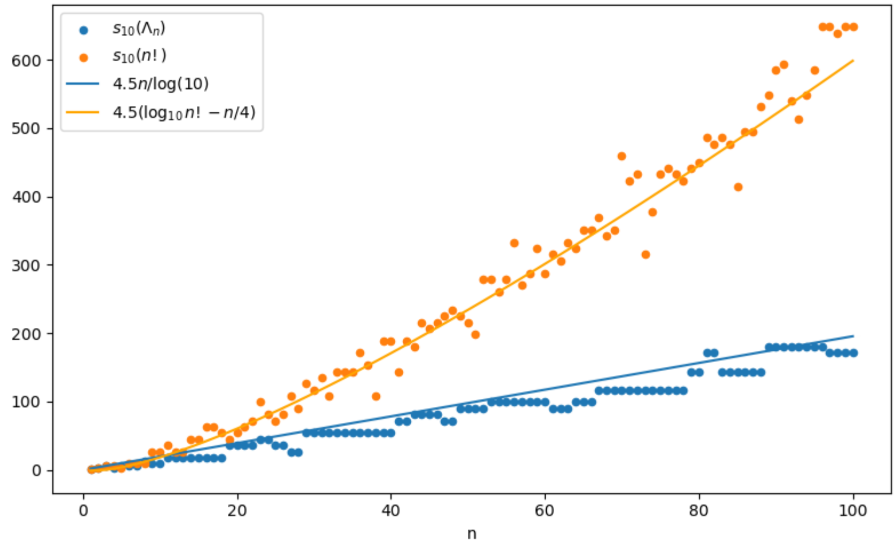

FIGURE 3. Scatter plot of the digital sums of $n!$ and Λ_n for $n \leq 100$, together with their conjectured approximations.

Lemma 3. *Let $a, b \geq 2$ be integers with $\log(a)/\log(b)$ irrational. Then there exist C and C', depending only on a and b, such that whenever*

$$a^n = b^r t,$$

we have

$$Cn \leq \log t \leq C'n.$$

In other words, $\log t \asymp n$.

Proof. By Lemma 1, there exists a prime divisor p of a such that

$$\nu_p(t) > C_1 n$$

holds for all n, where $C_1 > 0$ depends only on a and b.

Therefore, $t > p^{C_1 n}$, which implies that

$$\log t > C_1 n \log p.$$

On the other hand, $t \leq a^n$, hence $\log t \leq n \log a$.

Therefore,

$$Cn \leq \log t \leq C'n$$

holds for all n, where $C = C_1 \log p$ and $C' = \log a$. $\square$

Our proof relies on a version of Baker's theorem on lower bounds for linear forms in logarithms, which we state without proof. See [5] for an explicit lower bound.

Theorem 5. [2, p. 23] *Let $\alpha_1, \ldots, \alpha_n$ be nonzero algebraic numbers with degrees at most d. Suppose that $\alpha_1, \ldots, \alpha_{n-1}$ and α_n have heights at most $A' \geq 4$ and $A \geq 4$ respectively. Let $\beta_1, \ldots, \beta_n$ be integers of absolute value at most $B \geq 2$. If*

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \alpha_n \neq 0,$$

there exists $C > 1$, depending only on n, d, and A', such that

$$|\Lambda| > C^{-\log A \log B}.$$

Recall that a nonzero algebraic number $\alpha \in \mathbb{C}$ is a root of a unique irreducible integer polynomial P with positive leading coefficient and coprime coefficients. The *degree* of α is equal to the degree of P, and the *height* of α is the maximum of the absolute values of the cofficients of P. A rational integer has degree 1 and height equal to its absolute value. In our application of Baker's theorem, $\alpha_1, \ldots, \alpha_n$ are rational integers.

We now prove the main theorem.

Theorem 6. *Let $a, b \geq 2$ be integers with $a < b$, and suppose that $\log(a)/\log(b)$ is irrational. Then there exists $C > 0$, depending only on a and b, such that*

$$c_b(a^n) > \frac{\log n}{\log \log n + C}$$

holds for all sufficiently large n.

Proof. Let

$$a^n = b^m \left(d_1 b^{-m_1} + \cdots + d_k b^{-m_k} \right),$$

where $m = \lceil \log_b a^n \rceil$, $k = c_b(a^n)$, $d_i \in \{1, \ldots, b-1\}$, and

$$1 = m_1 < \cdots < m_k \leq m.$$

That is, $d_1, \ldots, d_k$ are the nonzero digits of a^n in base b, and $m_1, \ldots, m_k$ are their positions when the digits are numbered from left (most significant) to right (least significant). We may assume that $k \geq 2$.

Fix $i \in \{1, \ldots, k-1\}$. Our goal is to show that large gaps between digit positions cannot occur, by showing that

$$\frac{m_{i+1}}{m_i} < C \log n$$

for some $C > 0$. This will imply that the number of gaps, and hence the number of nonzero digits, cannot be too small.

Define

$$q = b^{m_i}(d_1 b^{-m_1} + \cdots + d_i b^{-m_i}),$$
$$r = b^m(d_{i+1} b^{-m_{i+1}} + \cdots + d_k b^{-m_k}),$$

so that

$$a^n = b^{m-m_i} q + r.$$

In other words, the digits up to the i-th nonzero digit form an integer q, and the digits from the $(i+1)$-st nonzero digit onward form an integer r.

From the base-b expansion we obtain the bounds

$$b^{m-1} < a^n < b^m,$$
$$b^{m_i-1} \le q < b^{m_i},$$
$$b^{m-m_{i+1}} \le r < b^{m-m_{i+1}+1}.$$

Consequently,

$$b^{-m_{i+1}} < a^{-n} r < b^{-m_{i+1}+2}, \tag{6}$$

which implies that

$$\frac{m_{i+1}-2}{m_i} < \frac{-\log(a^{-n}r)}{\log q} < \frac{m_{i+1}}{m_i - 1}$$

provided that $q \ge 2$ and $m_i \ge 2$.

If $m_i \ge 3$ (and $m_{i+1} \ge 4$) then

$$\frac{m_{i+1}-2}{m_i} \ge \frac{1}{2}\,\frac{m_{i+1}}{m_i}, \qquad \frac{m_{i+1}}{m_i - 1} \le \frac{3}{2}\,\frac{m_{i+1}}{m_i},$$

hence

$$\frac{1}{2}\,\frac{m_{i+1}}{m_i} < \frac{-\log(a^{-n}r)}{\log q} < \frac{3}{2}\,\frac{m_{i+1}}{m_i}. \tag{7}$$

Now set

$$\Lambda = \log(a^{-n} b^{m-m_i} q) = -n \log a + (m - m_i)\log b + \log q. \tag{8}$$

Then (since $a^{-n} r < 1/2$)

$$|\Lambda| = -\log(1 - a^{-n} r) < 2a^{-n} r. \tag{9}$$

We apply Baker's theorem to obtain a lower bound on $|\Lambda|$. This will imply an upper bound on the gap m_{i+1}/m_i, as shown in (11) below. There are two cases, depending on the size of q.

Case 1: $q < b^2$, or equivalently, $m_i \leq 2$.

In this case, we can apply Baker's theorem to (8) with $A \leftarrow b^2$ and $B \leftarrow n$, giving

$$|\Lambda| > C_1^{-\log n}$$

for some $C_1 > 1$ depending only on a and b. Thus,

$$2a^{-n} r > n^{-C_1},$$

and hence, for n sufficiently large,

$$2b^{-m_{i+1}+2} > n^{-C_1},$$

which implies that

$$m_{i+1} < C_2 \log n$$

for some $C_2 > 0$ depending only on a and b.

Since $m_i \geq 1$, we also have

$$\frac{m_{i+1}}{m_i} < C_2 \log n.$$

Case 2: $q > b^2$, or equivalently, $m_i \geq 3$.

In this case, we can apply Baker's theorem to (8) with $A \leftarrow q$ and $B \leftarrow n$, giving

$$|\Lambda| > C_3^{-\log q \log n}$$

for some $C_3 > 1$ depending only on a and b. Thus,

$$2a^{-n} r > C_3^{-\log q \log n},$$

and hence, for n sufficiently large,

$$\frac{-\log(a^{-n} r)}{\log(q)} < C_4 \log n \tag{10}$$

for some $C_4 > 0$ depending only on a and b.

Combining (7) and (10) gives

$$\frac{m_{i+1}}{m_i} < C_5 \log n, \quad C_5 = 2C_4.$$

In either case, we have

$$\frac{m_{i+1}}{m_i} < C_6 \log n, \quad C_6 = \max(C_2, C_5), \tag{11}$$

and thus

$$\log\left(\frac{m_{i+1}}{m_i}\right) < \log\log n + C_7, \quad C_7 = \log C_6. \tag{12}$$

Summing the logarithms of these ratios,

$$\log m_k = \sum_{i=1}^{k-1} \log\left(\frac{m_{i+1}}{m_i}\right)$$

which yields

$$\log m_k < (k-1)(\log\log n + C_7). \tag{13}$$

Write a^n as $b^{m-m_k}u$, where

$$b^{m_k-1} < u < b^{m_k}.$$

Thus $m_k \asymp \log u$, and $\log u \asymp n$ by Lemma 3, so

$$\log m_k = \log n + O(1). \tag{14}$$

Therefore, (13) and (14) imply that

$$k > \frac{\log n}{\log\log n + C}$$

for all sufficiently large n, as required. □

This is a long and complicated proof, so let's review the main steps.

(1) Approximate a^n by truncating to the i-th nonzero digit.

(2) Estimate the digit gap m_{i+1}/m_i in terms of the truncation error $a^{-n}r$.

(3) Use Baker's theorem to obtain a lower bound on the truncation error.

(4) Deduce thereby an upper bound on the digit gap.

(5) Compute a lower bound on the number of gaps, and hence the number of nonzero digits.

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