The Infected Checkerboard

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The infected checkerboard

The following problem is from Mathematical Puzzles: A Connoisseur's Collection by Peter Winkler.

> *An infection spreads among the squares of an $n \times n$ checkerboard in the following manner. If a square has two or more infected neighbors, it becomes infected itself. Prove that you cannot infect the whole board if you begin with fewer than n infected squares.*

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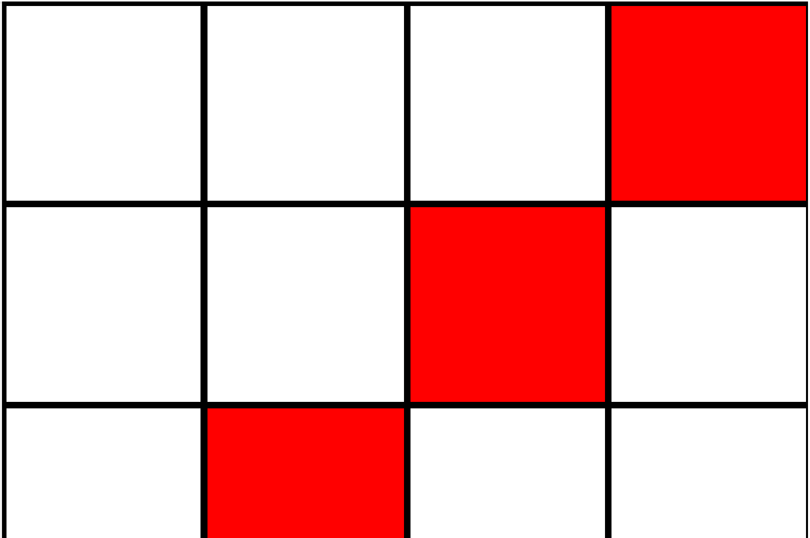

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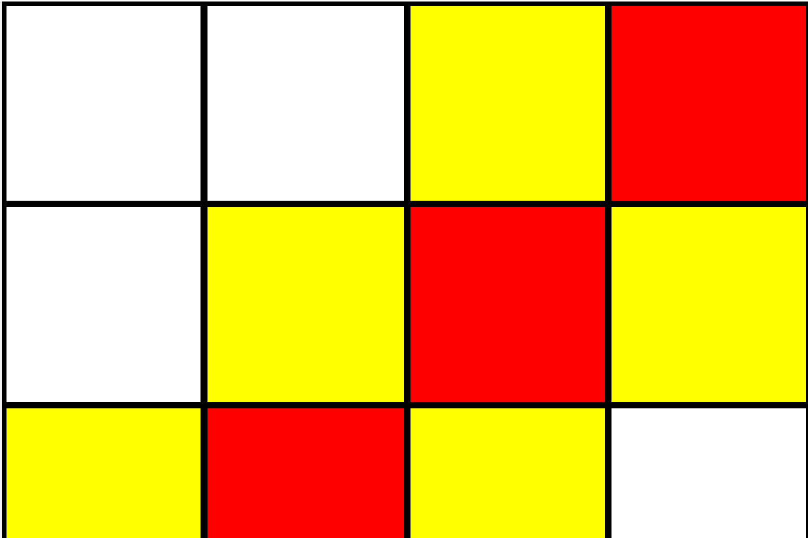

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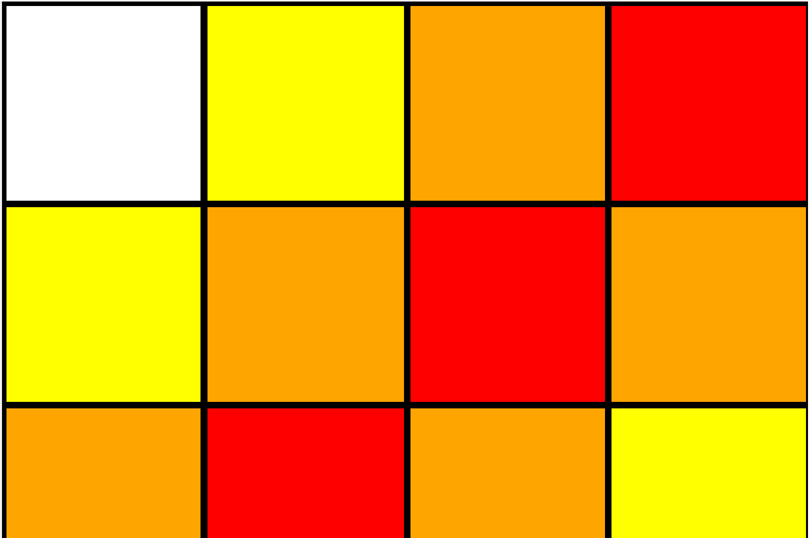

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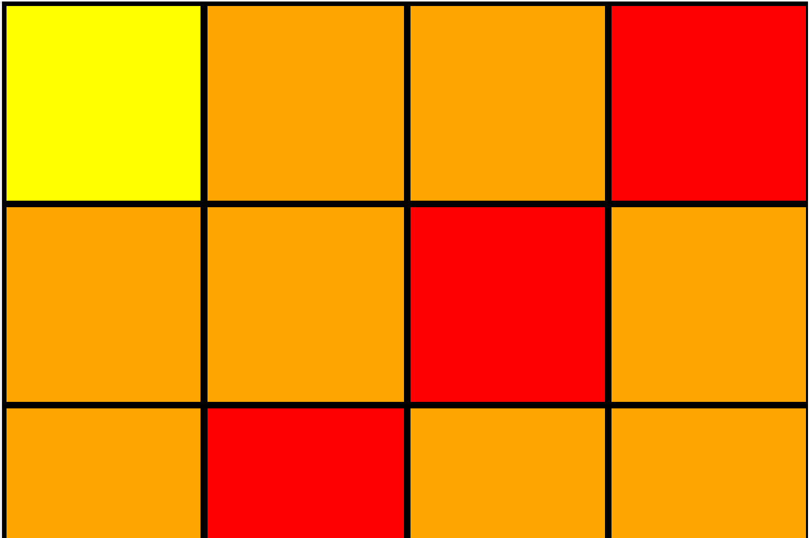

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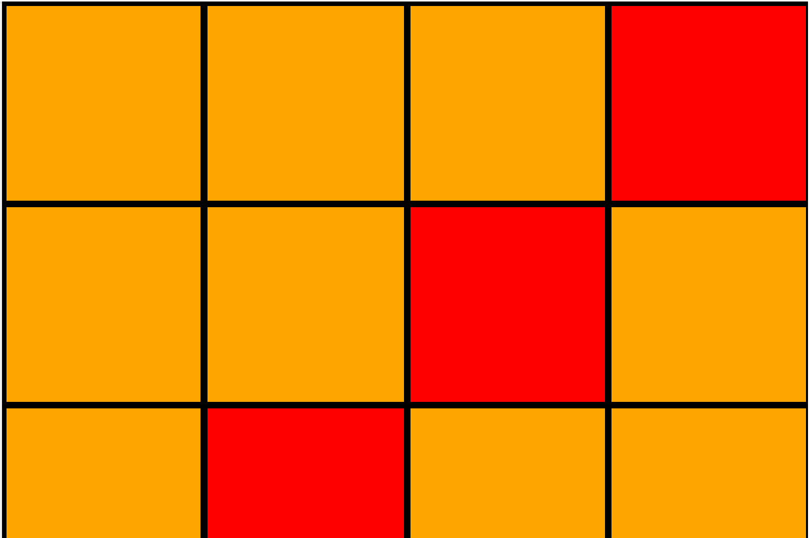

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- If there are fewer than n infected squares, the perimeter of the infected region is less than the perimeter of the whole board.
- When a new square is infected, the perimeter of the infected region decreases or stays the same.
- So the infected region can never cover the entire board.

Bootstrap percolation

- This process is called *2-neighbor bootstrap percolation*.
- It's an example of a cellular automaton.
- The process can be defined on any undirected graph.

NEW QUESTION

Suppose that a square needs three or more infected neighbors to become infected itself. What is the minimum number of squares that must be initially infected in order to infect an $m \times n$ grid?

The pandemic formula

Suppose that an initial set of s squares will infect an $m \times n$ grid. Then

$$s = \frac{mn + m + n + a + c}{3}$$

where

- a is the number of adjacent pairs of infected squares, and
- c is the number of "islands" of uninfected squares.

Perfect initial sets

Suppose that an initial set of s squares will infect an $m \times n$ grid. Then

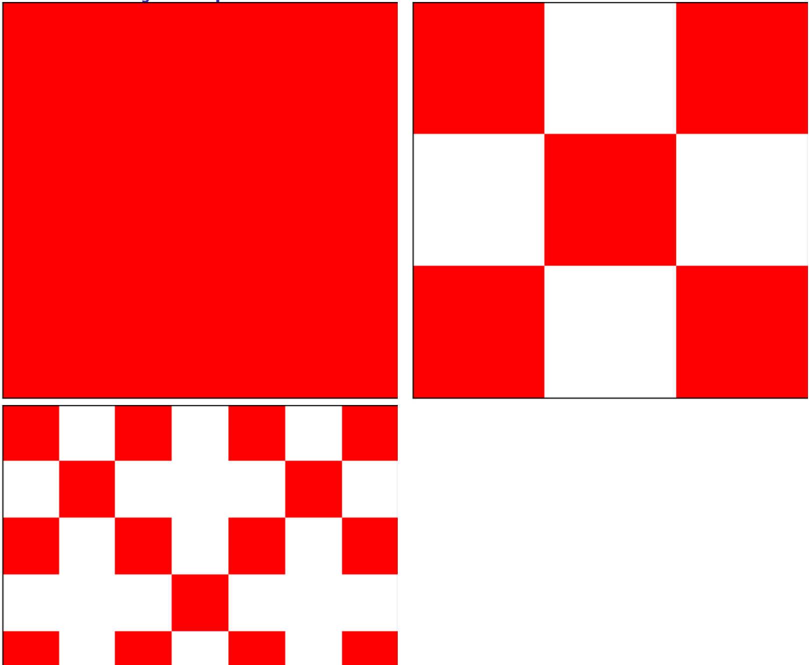
$$s \geq \frac{mn + m + n}{3}.$$

If

$$s = \frac{mn + m + n}{3}$$

then we say that the initial set is **perfect**.

Infinite family of perfect initial sets



Main results

- The only perfect initial sets are the infinite families described previously.
- If $m, n \geq 4$, the minimum number of squares s needed to infect an $m \times n$ grid satisfies

$$\frac{mn+m+n}{3} \leq s \leq \frac{mn+m+n+6}{3}.$$

- In many cases I have found tight lower bounds.