A PRODUCT RULE FOR TRIANGULAR NUMBERS

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Abstract. We prove that there are exactly five sequences, including the triangular numbers, that satisfy the product rule $T(mn) = T(m)T(n) + T(m-1)T(n-1)$ for all $m, n \geq 1$.

1. Introduction

The nth triangular number is

$$T(n) = 1 + 2 + \ldots + n = \frac{1}{2}n(n+1).$$

It represents the number of dots in a triangular arrangement as shown in Figure 1, with 1 dot in the first row, 2 dots in the second row, and so on.

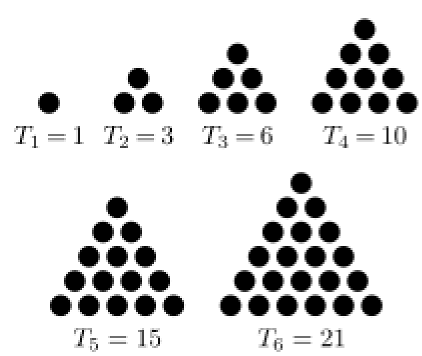
The triangular numbers satisfy many interesting properties, including a *product rule*:

$$T(mn) = T(m)T(n) + T(m-1)T(n-1).$$

This rule can be demonstrated visually by subdividing a triangle into smaller triangles. Figure 2 illustrates the case $T(20) = T(5)T(4) + T(4)T(3)$.

In this note, we determine all sequences of real numbers that satisfy the product rule for triangular numbers. Our main result is the following:

Figure 1. The first six triangular numbers.



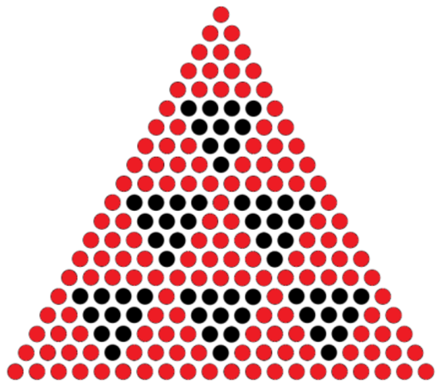


FIGURE 2. The 20th triangular number.

Theorem 1. *Each of the following sequences satisfies the product rule*

$$T(mn) = T(m)T(n) + T(m-1)T(n-1)$$

for all $m, n \geq 1$. *No other sequences satisfy this recurrence.*

(1) $T(n) = 0$ *for all* $n \geq 0$.
(2) $T(n) = \frac{1}{2}$ *for all* $n \geq 0$.
(3) $T(0) = 0$ *and* $T(2n) = T(2n-1) = n$ *for all* $n \geq 1$.
(4) $T(3n) = T(3n+2) = 0$ *and* $T(3n+1) = 1$ *for all* $n \geq 0$.
(5) $T(n) = \frac{1}{2}n(n+1)$ *for all* $n \geq 0$.

2. Verifying the solutions

In this section, we prove that all of the sequences listed in Theorem 1 satisfy the product rule for triangular numbers. Readers who prefer to verify the solutions for themselves should feel free to skip to the next section.

Lemma 1. *If* $T(n) = 0$ *for all* $n \geq 0$, *then* T *satisfies the product rule.*

Proof. If $m, n \geq 1$ then $T(mn) = 0$ and

$$T(m)T(n) + T(m-1)T(n-1) = 0 \cdot 0 + 0 \cdot 0 = 0.$$

□

Lemma 2. *If* $T(n) = \frac{1}{2}$ *for all* $n \geq 0$, *then* T *satisfies the product rule.*

Proof. If $m, n \geq 1$ then $T(mn) = \frac{1}{2}$ and

$$T(m)T(n) + T(m-1)T(n-1) = \tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{2}.$$

□

Lemma 3. *If* $T(0) = 0$ *and* $T(2n) = T(2n-1) = n$ *for all* $n \geq 1$*, then* T *satisfies the product rule.*

Proof. If $m = 2a$ and $n = 2b$ then $T(mn) = T(4ab) = 2ab$ and

$$T(m)T(n) + T(m-1)T(n-1) = ab + ab = 2ab.$$

If $m = 2a$ and $n = 2b-1$ then $T(mn) = T(2a(2b-1)) = a(2b-1)$ and

$$T(m)T(n) + T(m-1)T(n-1) = ab + a(b-1) = a(2b-1).$$

If $m = 2a-1$ and $n = 2b-1$, then

$$\begin{aligned} T(mn) = T(4ab - 2a - 2b + 1) &= 2ab - a - b + 1 \text{ and} \\ T(m)T(n) + T(m-1)T(n-1) &= ab + (a-1)(b-1) \\ &= 2ab - a - b + 1. \end{aligned}$$

□

Lemma 4. *If* $T(3n) = T(3n+2) = 0$ *and* $T(3n+1) = 1$ *for all* $n \geq 0$*, then* T *satisfies the product rule.*

Proof. Let $m, n \geq 1$. If $m \equiv 0 \pmod 3$, then $T(mn) = 0$ and

$$T(m)T(n) + T(m-1)T(n-1) = 0 \cdot T(n) + 0 \cdot T(n-1) = 0.$$

The case where $n \equiv 0 \pmod 3$ is similar.
If $m \equiv n \equiv 1 \pmod 3$, then $T(mn) = 1$ and

$$T(m)T(n) + T(m-1)T(n-1) = 1 \cdot 1 + 0 \cdot 0 = 1.$$

If $m \equiv n \equiv 2 \pmod 3$, then $T(mn) = 1$ and

$$T(m)T(n) + T(m-1)T(n-1) = 0 \cdot 0 + 1 \cdot 1 = 1.$$

If $m \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$, then $T(mn) = 0$ and

$$T(m)T(n) + T(m-1)T(n-1) = 1 \cdot 0 + 0 \cdot 1 = 0.$$

The case where $m \equiv 2 \pmod 3$ and $n \equiv 1 \pmod 3$ is similar.

□

Lemma 5. *If* $T(n) = \frac{1}{2}n(n+1)$ *for all* $n \geq 0$ *then* T *satisfies the product rule.*

Proof. Let $m, n \geq 1$. Then $T(mn) = \frac{1}{2}mn(mn+1)$ and

$$\begin{aligned}
&T(m)T(n) + T(m-1)T(n-1) \\
&= \frac{1}{2}m(m+1)\frac{1}{2}n(n+1) + \frac{1}{2}(m-1)m\frac{1}{2}(n-1)n \\
&= \frac{mn}{4}\left((m+1)(n+1) + (m-1)(n-1)\right) \\
&= \frac{mn}{4}(2mn+2) \\
&= \frac{1}{2}mn(mn+1).
\end{aligned}$$

□

3. Proof of completeness

We will prove that the list of solutions from the previous solutions is complete. Let T be any sequence that satisfies the product rule

$$T(mn) = T(m)T(n) + T(m-1)T(n-1)$$

for all $m, n \geq 1$. For the sake of brevity, we will set $a = T(0)$, $b = T(1)$, $c = T(2)$, and $d = T(3)$.

Lemma 6. *The equation* $b = b^2 + a^2$ *holds. In particular, if* $a = 0$ *then* $b = 0$ *or* $b = 1$.

Proof. Substituting $m = 1$ and $n = 1$ into the product rule gives $T(1) = T(1)T(1) + T(0)T(0)$, or $b = b^2 + a^2$. If $a = 0$ then $b = b^2$, which implies that $b = 0$ or $b = 1$. □

Lemma 7. *The identity* $T(n) = bT(n) + aT(n-1)$ *holds for all* $n \geq 0$. *If* $a = 0$ *and* $b = 0$ *then* $T(n) = 0$ *for all* $n \geq 0$.

Proof. The identity is verified by substituting $m = 1$ into the product rule. By induction on n, it follows that if $a = b = 0$ then $T(n) = 0$ for all $n \geq 0$. □

Lemma 8. *If* $a \neq 0$ *then* $T(n) = 1/2$ *for all* $n \geq 0$.

Proof. Since $b = b^2 + a^2$, it follows that $b \neq 1$, so we can write the equation as

$$\frac{a}{1-b} = \frac{b}{a}.$$

The equation $T(n) = bT(n) + aT(n-1)$ from Lemma 7 implies that

$$\frac{T(n)}{T(n-1)} = \frac{a}{1-b} = \frac{b}{a}.$$

Therefore, $T(n) = ar^n$ for all $n \geq 0$, where $r = b/a$.

Substituting $m = 2$ into the product formula gives

$$\begin{aligned} T(2n) &= T(2)T(n) + T(1)T(n-1) \\ ar^{2n} &= ar^2 \cdot ar^n + ar \cdot ar^{n-1} \\ r^n &= a(r^2 + 1) \end{aligned}$$

Since the right side is independent of n, it follows that $r = 1$, hence $T(n) = a$ for all $n \geq 0$. But if $b = b^2 + a^2$ and $a = b \neq 0$, then $a = \frac{1}{2}$. Therefore, $T(n) = \frac{1}{2}$ for all $n \geq 0$. □

Lemma 9. *If $a = 0$ and $b = 1$, then $T(n)$ can be computed recursively for all $n \geq 3$ by means of the identities $T(2n) = cT(n) + T(n-1)$ and $T(2n-1) = T(n) + (d-c)T(n-1)$. In particular, $T(n)$ is a function of c and d for all $n \geq 3$.*

Proof.

$$\begin{aligned} T(2n) &= T(2)T(n) + T(1)T(n-1) = cT(n) + T(n-1) \\ T(4) &= T(2)T(2) + T(1)T(1) = c^2 + 1 \\ T(4n) &= T(4)T(n) + T(3)T(n-1) = (c^2+1)T(n) + dT(n-1) \\ T(4n) &= T(2)T(2n) + T(1)T(2n-1) = cT(2n) + T(2n-1). \end{aligned}$$

Combining these equations yields

$$\begin{aligned} T(2n-1) &= T(4n) - cT(2n) \\ &= (c^2+1)T(n) + dT(n-1) - c^2T(n) - cT(n-1) \\ &= T(n) + (d-c)T(n-1). \end{aligned}$$

Since $T(2n)$ and $T(2n-1)$ are linear combinations of previous terms for all $n \geq 2$, and the coefficients are functions of c and d, it follows that $T(n)$ is a function of c and d for all $n \geq 3$. □

Lemma 10. *If $a = 0$ and $b = 1$, then*

$$d = \frac{3c^3 + c}{c^2 + 2c - 1}.$$

Consequently, $T(n)$ is uniquely determined for each $n \geq 2$ by the value of c alone.

Proof. Using the formulas from Lemma 9 we calculate as follows:

$$\begin{aligned}
T(4) &= c^2 + 1 \\
T(5) &= T(3) + (d-c)T(2) = d + (d-c)c = d + cd - c^2 \\
T(6) &= cT(3) + T(2) = cd + c \\
T(8) &= cT(4) + T(3) = c(c^2+1) + d = c^3 + c + d \\
T(9) &= T(3)T(3) + T(2)T(2) = d^2 + c^2 \\
T(18) &= T(3)T(6) + T(2)T(5) = d(cd+c) + c(d + cd - c^2) \\
T(18) &= cT(9) + T(8) = c(d^2+c^2) + (c^3 + c + d)
\end{aligned}$$

Equating the last two expressions for $T(18)$ yields

$$\begin{aligned}
&cd^2 + c^2d + 2cd - c^3 = cd^2 + 2c^3 + c + d \\
&c^2d + 2cd - d = 3c^3 + c \\
&(c^2 + 2c - 1)d = 3c^3 + c
\end{aligned}$$

If $c^2 + 2c - 1 = 0$ then $3c^3 + c = 0$ as well. But the polynomials have no roots in common, which is a contradiction. Therefore, we may divide by $c^2 + 2c - 1$, yielding

$$d = \frac{3c^3 + c}{c^2 + 2c - 1}.$$

This implies that the value of $T(n)$ is determined by c alone. □

Lemma 11. *If* $a = 0$ *and* $b = 1$ *then* $c \in \{0, 1, 3\}$.

Proof. By our previous results, we may compute $T(n)$ recursively by the following formulas.

$$\begin{aligned}
&T(0) = 0; T(1) = 1; T(2) = c \\
&T(3) = d = \frac{3c^3 + c}{c^2 + 2c - 1} \\
&T(2n) = cT(n) + T(n-1) \text{ for } n \geq 2 \\
&T(2n-1) = T(n) + (d-c)T(n-1) \text{ for } n \geq 3
\end{aligned}$$

These formulas are implemented by the Python script in Figure 3. This script uses SymPy[1], a Python library for symbolic mathematics. The last two lines of the script should evaluate to 0, so we obtain the following

```
from sympy import Symbol, factor, simplify
c = Symbol('c')
d = (3*c**3 + c) / (c**2 + 2*c - 1)
T = [0, 1, c, d] + [0]*12
for n in range(4, 16):
    if n % 2 == 0:
        T[n] = simplify(c*T[n/2] + T[n/2-1])
    else:
        T[n] = simplify(T[n/2+1] + (d-c) * T[n/2])
print (factor(T[9] - T[3]*T[3] - T[2]*T[2]))
print (factor(T[15] - T[3]*T[5] - T[2]*T[4]))
```

FIGURE 3. Python script to compute $T(n)$ in terms of c.

equations which c must satisfy.

$$\begin{aligned} 0 &= T(9) - T(3)T(3) - T(2)T(2) \\ &= \frac{c(c-3)(c-1)(c+1)(2c^3+c-1)}{(c^2+2c-1)^2}, \text{ and} \\ 0 &= T(15) - T(3)T(5) - T(2)T(4) \\ &= \frac{c(c-3)(c-1)(8c^6-c^5+7c^4-4c^3+4c^2-3c+1)}{(c^2+2c-1)^3}. \end{aligned}$$

The only solutions that are common to both equations are $c = 0, 1, 3$. □

Proof of Theorem 1. We know from Section 2 that all of the sequences listed in Theorem 1 satisfy the product rule for triangular numbers.

Let T be any sequence that satisfies the product rule for triangular numbers. If $T(0) \neq 0$, then $T(n) = \frac{1}{2}$ for all $n \geq 0$, by Lemma 8. If $T(0) = 0$, then either $T(1) = 0$ or $T(1) = 1$, by Theorem 6. In the first case, $T(n) = 0$ for all n, by Lemma 7.

In the second case, $T(2) = 0$, 1, or 3 by Lemma 11, and this value determines the value of $T(n)$ for all $n \geq 3$, by Lemma 10. If $T(2) = 0$, then $T(3n) = T(3n+2) = 0$ and $T(3n) = 1$ for all $n \geq 0$, by Lemma 4. If $T(2) = 1$, then $T(2n) = T(2n-1) = n$ for all $n \geq 0$, by Lemma 3. If $T(2) = 3$, then $T(n) = \frac{1}{2}n(n+1)$ for all $n \geq 0$, by Lemma 5. These cases are mutually exclusive and collectively exhaustive, so the proof is complete. □

REFERENCES

[1] SymPy Development Team. *SymPy: Python library for symbolic mathematics*, 2016.