A FUNCTION THAT IS SURJECTIVE ON EVERY INTERVAL

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ABSTRACT. We exhibit a real function that is surjective when restricted to any nonempty open interval.

Every calculus student learns the intermediate value theorem, which states that if f is a continuous real-valued function on the closed interval $[a, b]$, and if c is any real number between $f(a)$ and $f(b)$, then there exists $x \in [a, b]$ such that $f(x) = c$. A function that satisfies the conclusion of this theorem is called a Darboux function [4]. Although every continuous function is a Darboux function, it is not true that every Darboux function is continuous.

Perhaps surprisingly, there exist functions which are surjective on every nonempty open interval. Such functions are well known, and go back to Lebesgue [2, p. 90]. Any function with this property is necessarily discontinuous everywhere. Such functions can be defined in terms of the decimal (or base N) expansion of x. Conway's base 13 function [3] is an example of this approach. Halperin [1] used the existence of a Hamel basis to "construct" an example that is non-measurable on every set of positive measure.

In this note, we will present a simple example of a real function that is surjective when restricted to any nonempty open interval. Let $f(x) = \lim_{n\to\infty} \tan(n!\,\pi x)$, provided that the limit exists. If the limit does not exist, then we (somewhat arbitrarily) define $f(x) = 0$. We show that f has the following properties:

(1) if $x \in \mathbb{R}$ and $q \in \mathbb{Q}$ then $f(x + q) = f(x)$,
(2) f is surjective: For all $y \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $f(x) = y$,
(3) f is surjective on all nonempty open intervals: If $a, b \in \mathbb{R}$ and $a < b$ then $\{f(x) \colon a < x < b\} = \mathbb{R}$.

Proof of (1):

Let $x \in \mathbb{R}$ and $q \in \mathbb{Q}$ be given. There exist $r, s \in \mathbb{Z}$ with $s > 0$ such that $q = r/s$. If $n \geq s$, then $n!\,q$ is an integer, so $n!\,\pi x$ and $n!\,\pi(x + q)$

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differ by an integral multiple of π. It follows that $\tan(n!\,\pi(x+q)) = \tan(n!\,\pi x)$ for all $n \geq s$, hence $f(x+q) = f(x)$. (Either the limits are equal, or both limits fail to exist. In the latter case, $f(x+q) = f(x) = 0$.)

Proof of (2):

Let $y \in \mathbb{R}$ be given, and choose $r \in [0,1)$ such that $\tan(\pi r) = y$. Define $x \in \mathbb{R}$ by the following formula:

$$x = \sum_{n=0}^{\infty} \frac{\lfloor rn \rfloor}{n!}.$$

It remains for us to show that $f(x) = y$. To that end, let x_n be the nth partial sum and let ϵ_n be the remainder term. Hence,

$$x_n = \sum_{k=0}^{n} \frac{\lfloor rk \rfloor}{k!} \quad \text{and}$$
$$\epsilon_n = \sum_{k=n+1}^{\infty} \frac{\lfloor rk \rfloor}{k!} = x - x_n.$$

Note that $n!\,x_n \in \mathbb{Z}$ for all n, hence $\tan(n!\,\pi x) = \tan(n!\,\pi\epsilon_n)$ for all n. But

$$n!\,\epsilon_n = \frac{\lfloor r(n+1) \rfloor}{n+1} + n! \sum_{k=n+2}^{\infty} \frac{\lfloor rk \rfloor}{k!},$$

and the reader can verify that

$$\lim_{n\to\infty} \frac{\lfloor r(n+1) \rfloor}{n+1} = r$$

and

$$\lim_{n\to\infty} n! \sum_{k=n+2}^{\infty} \frac{\lfloor rk \rfloor}{k!} = 0.$$

Therefore,

$$f(x) = \lim_{n\to\infty} \tan(n!\,\pi x) = \lim_{n\to\infty} \tan(n!\,\pi\epsilon_n) = \tan(\pi r) = y.$$

Proof of (3):

Let $a, b, y \in \mathbb{R}$ be given with $a < b$. By (2) there exists $u \in \mathbb{R}$ such that $f(u) = y$, and by (1), $f(u+q) = y$ for all $q \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ such that $a < u+q < b$. Let $x = u+q$. Then $a < x < b$ and $f(x) = y$. Since y is an arbitrary real number, it follows that $\{f(x) \colon a < x < b\} = \mathbb{R}$.

References

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