RIGIDITY OF RIGHT-ANGLED COXETER GROUPS

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Abstract. If S and S' are two finite sets of Coxeter generators for a right-angled Coxeter group W, then the Coxeter systems (W,S) and (W,S') are equivalent.

Introduction

A Coxeter group is a group having a presentation of the form

$$W = \langle S \mid (st)^{m(s,t)} = 1 \ (s,t \in S) \rangle$$

where S is a finite set of generators of W, $m(s,s) = 1$ for all $s \in S$, and $m(s,t) = m(t,s) \in \{2,3,4,...,\infty\}$ for all $s,t \in S$ with $s \neq t$. If $m(s,t) = \infty$ then the corresponding relation is omitted. It can be shown that $m(s,t)$ is the order of st in W [H, p. 110].

The pair (W,S) is called a Coxeter system, and S is a set of Coxeter generators for W. Two Coxeter systems (W,S) and (W',S') are equivalent if there is an isomorphism $\phi : W \to W'$ so that $\phi(S) = S'$.

If $m(s,t) \in \{2,\infty\}$ for all $s \neq t$ then (W,S) is called a right-angled Coxeter system. I will show that if S and S' are two sets of Coxeter generators for W, and (W,S) is a right-angled Coxeter system, then (W,S') is also a right-angled Coxeter system. We call W a right-angled Coxeter group in case (W,S) is a right-angled Coxeter system for some (and hence for any) set S of Coxeter generators for W.

The principal result of this article is the following.

Theorem. *If S and S' are two sets of Coxeter generators for a right-angled Coxeter group W, then the Coxeter systems (W,S) and (W,S') are equivalent.*

The hypothesis that (W,S) is right-angled cannot be omitted. For example, $\langle a,b \mid a^2 = b^2 = (ab)^6 = 1\rangle$ and $\langle r,s,t \mid r^2 = s^2 = t^2 = (rs)^3 = (rt)^2 = (st)^2 = 1\rangle$ are inequivalent presentations of the symmetry group of a regular hexagon.

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The Nerve of a Coxeter System

Let (W,S) be a Coxeter system. Let W_A denote the subgroup of W generated by a subset A of S. By convention, $W_\varnothing$ is the trivial subgroup. It can be shown ([H, p. 113]) that (W_A, A) is a Coxeter system. Furthermore, $W_A \cap W_B = W_{A\cap B}$ [H, p. 114]. A subgroup of the form W_A is called a standard subgroup of (W,S). A parabolic subgroup of (W,S) is a conjugate of a standard subgroup.

The nerve $N(W,S)$ of a Coxeter system is the set of all nonempty $A \subseteq S$ such that W_A is finite. It will also prove useful to consider the subset $N^*(W,S)$ of all maximal elements of $N(W,S)$. Explicitly,

$$N^*(W,S) = \{A \in N(W,S) \mid A \subseteq B \in N(W,S) \implies A = B\}.$$

If W is right-angled then $N(W,S)$ is a flag complex, meaning that $A \in N(W,S)$ if and only if A is a nonempty subset of S and $\{s,t\} \in N(W,S)$ for all $s,t \in A$.

The Davis-Vinberg Complex

A Coxeter system (W,S) determines a simplicial complex $\Sigma(W,S)$, called the Davis-Vinberg complex. It may be defined as follows: Let $\mathcal{C}$ be the set of all finite left cosets of the form wW_T. Then

$$\Sigma(W,S) = \{F \subset \mathcal{C} \mid F \neq \varnothing,\ A \subseteq B \text{ or } B \subset A\ (\forall A, B \in F)\}.$$

W acts on $\Sigma(W,S)$ as a group of simplicial automorphisms via left multiplication: $w \cdot F = \{w \cdot A \mid A \in F\}$ [D].

Let $X \equiv |\Sigma(W,S)|$ denote the geometric realization of $\Sigma(W,S)$. There is a complete W-invariant metric on X such that X is a Hadamard space (i.e. a complete simply-connected geodesic space of non-positive curvature in the sense of Alexandrov) [M],[D]. The theory of Hadamard spaces is developed in [B] and [BH]. We use only one property of such spaces: If G is a finite group of isometries of a Hadamard space, then the fixed set of G is nonempty [BH, Cor. II.2.8].

We can now characterize the finite subgroups of W. If G is a finite subgroup of W, then G fixes a point of X. Therefore G must fix a vertex, since $g \in W$ stabilizes a simplex of $\Sigma(W,S)$ if and only if it fixes each vertex of that simplex. Thus G is contained in the isotropy group of some vertex wW_A, and so G is contained in wW_Aw^{-1}, a finite parabolic subgroup of W. Therefore, the maximal finite subgroups of W are exactly the maximal finite parabolic subgroups, i.e. subgroups of the form wW_Aw^{-1} where $A \in N^*(W,S)$. An alternative proof of this characterization is outlined in [Bo, p. 130].

Right-angled Coxeter Groups

Now we restrict our attention to right-angled Coxeter groups. Let (W,S) be a right-angled Coxeter system, and let S' be another set of Coxeter generators for W.

From the results of the previous section, we know that each finite subgroup of W is contained in a finite parabolic subgroup. But each finite parabolic subgroup is isomorphic to a direct sum of cyclic groups of order 2. Therefore, every element of W has order 1, 2, or ∞. It follows that (W, S') is a right-angled Coxeter system.

Let W' be the commutator subgroup of W. Then W/W' is isomorphic to $(Z/2Z)^S$, hence S and S' have the same cardinality. Let $q : W \to W/W'$ be the canonical quotient map.

Let V be the union of all finite standard subgroups of (W, S). Then $V = \{e\} \cup \{s_1 s_2 \cdots s_k \mid \{s_1, \dots, s_k\} \in N(W, S),\ s_i \neq s_j\ (\forall\ i < j)\}$. Note that the restriction of q to V is one-to-one. Indeed, if we identify W/W' with $(Z/2Z)^S$, then $q(s_1 \cdots s_k)(t) = 1$ if $t \in \{s_1, \dots, s_k\}$, and 0 otherwise.

Proof of the Main Theorem

The proof relies on the following results.

Theorem. *For each $A \in N^*(W, S)$ there exists a unique $A^* \in N^*(W, S')$ so that $q(W_A) = q(W_{A^*})$. Furthermore, the correspondence $A \mapsto A^*$ is a bijection from $N^*(W, S)$ to $N^*(W, S')$.*

Proof. Let $A \in N^*(W, S)$. Then W_A is a maximal finite subgroup of W, so it is conjugate to a subgroup W_{A^*} for some $A^* \in N^*(W, S')$. Since the image of q is abelian, it follows that $q(W_A) = q(W_{A^*})$.

Now A^* is unique because q is one-to-one on V'. Likewise, for each $A \in N^*(W, S')$ there is a unique $A^* \in N^*(W, S)$ so that $q(W_A) = q(W_{A^*})$. Then $A^{**} = A$ for all A in either $N^*(W, S)$ or $N^*(W, S')$. Therefore the correspondence is bijective.

Theorem. *If $A_1, \dots, A_r \in N^*(W, S)$, then*

$$\left|\bigcap A_i\right| = \left|\bigcap A_i^*\right|$$

where $|A|$ is the cardinality of A.

Proof. $|W_{\cap A_i}| = |\bigcap W_{A_i}| = |\bigcap q(W_{A_i})|$, since q is one-to-one on V. But $q(W_A) = q(W_{A^*})$, thus $|\bigcap q(W_{A_i})| = |\bigcap q(W_{A_i^*})| = |\bigcap W_{A_i^*}| = |W_{\cap A_i^*}|$.

Note that $|W_A| = 2^{|A|}\ \forall A \in N(W, S) \cup \{\varnothing\}$, since $W_A \cong (Z/2Z)^A$. Therefore, $|\bigcap A_i| = |\bigcap A_i^*|$.

Corollary. *If $A_1, \dots, A_r, B_1 \dots, B_s \in N^*(W, S)$, then*

$$\left|\bigcap A_i - \bigcup B_j\right| = \left|\bigcap A_i^* - \bigcup B_j^*\right|.$$

Proof. We have already proved this statement for the case $s = 0$. If the equation holds for $s = t$, then

$$\left|\bigcap^{r} A_i - \bigcup^{t+1} B_j\right| = \left|\bigcap^{r} A_i - \bigcup^{t} B_j\right| - \left|\bigcap^{r} A_i \cap B_{t+1} - \bigcup^{t} B_j\right| =$$

$$\left|\bigcap^{r} A_i^* - \bigcup^{t} B_j^*\right| - \left|\bigcap^{r} A_i^* \cap B_{t+1}^* - \bigcup^{t} B_j^*\right| = \left|\bigcap^{r} A_i^* - \bigcup^{t+1} B_j^*\right|.$$

Thus the equation holds for $s = t + 1$. Therefore it is true for all s by induction.

Proof of the Main Theorem.

By the above corollary, there exists a bijection $\phi : S \to S'$ so that, for every $A \in N^*(W, S)$ and every $s \in S$,

$$s \in A \iff \phi(s) \in A^*.$$

Let $s, t \in S$ with $s \neq t$. If $m(s,t) = 2$ then $\{s,t\} \subseteq A$ for some $A \in N^*(W,S)$, so $\{\phi(s), \phi(t)\} \in A^* \in N^*(W, S')$, thus $m(\phi(s), \phi(t)) = 2$.

Similarly, if $m(s,t) = \infty$ then $\{s,t\} \not\subseteq A$ for all $A \in N^*(W,S)$, so $\{\phi(s), \phi(t)\} \not\subseteq A^*$ for all $A^* \in N^*(W, S')$, thus $m(\phi(s), \phi(t)) = \infty$.

Therefore $m(s,t) = m(\phi(s), \phi(t)$ for all $s, t \in S$, so ϕ extends to an automorphism of W.

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