DISJOINT FIGURE EIGHTS IN THE PLANE

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We show that any set of disjoint figure eights in the plane must be countable. By a “figure eight” we mean the union of two simple closed curves C_1 and C_2 whose intersection $C_1 \cap C_2$ is a single point.

A *rational point* in the plane is a point (x, y) so that x and y are rational numbers. The set of all rational points in the plane is denoted $\mathbb{Q}^2$. It is not hard to show that $\mathbb{Q}^2$ is a countable set, and it is dense in the plane (i.e. every nonempty open subset of the plane contains a point in $\mathbb{Q}^2$).

Let $E = C_1 \cup C_2$ be a figure eight. By the Jordan Curve Theorem, the complement of E has three components:

(1) An open set U_1, bounded by the simple closed curve C_1.
(2) An open set U_2, bounded by the simple closed curve C_2.
(3) An unbounded open set U_3, lying outside of E.

Since U_1 and U_2 are nonempty open sets, we may choose a rational point in each set: $p \in U_1 \cap \mathbb{Q}^2$ and $q \in U_2 \cap \mathbb{Q}^2$.

Now let $E' = C_1' \cup C_2'$ be a figure eight that is disjoint from E. As before, we may choose rational points p' and q', one from each bounded complementary component of E'. But note that E' is wholly contained in U_1, U_2, or U_3, since E' is connected. Therefore, $(p', q') \neq (p, q)$.

This observation leads to a proof of our claim. Let $\mathbb{X}$ be any set of disjoint figure eights in the plane. For each $E \in \mathbb{X}$ we may choose two rational points p and q, one in each loop of E. This determines a function from $\mathbb{X}$ to $\mathbb{Q}^2 \times \mathbb{Q}^2$. The function is injective by the argument of the previous paragraph, and $\mathbb{Q}^2 \times \mathbb{Q}^2$ is countable. Therefore, $\mathbb{X}$ is countable.

Date: March 16, 2017.