Chapter 9

Fundamentals of Hypothesis Testing: One-Sample Tests

Learning Objectives

In this chapter, you learn:

- The basic principles of hypothesis testing
- How to use hypothesis testing to test a mean or proportion
- The assumptions of each hypothesis-testing procedure, how to evaluate them, and the consequences if they are seriously violated
- How to avoid the pitfalls involved in hypothesis testing
- Pitfalls & ethical issues involved in hypothesis testing

What is a Hypothesis?

DCOVA

- A hypothesis is a claim (assertion) about a population parameter:

  - population mean

Example: The mean monthly cell phone bill in this city is μ = $42

  - population proportion

Example: The proportion of adults in this city with cell phones is $\pi = 0.68$

The Null Hypothesis, H_0

DCOVA

- States the claim or assertion to be tested

Example: The mean diameter of a manufactured bolt is 30mm $(H_0 : \mu = 30)$

- Is always about a population parameter, not about a sample statistic

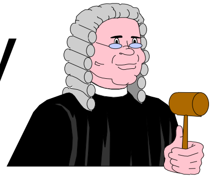
$H_0 : \mu = 30$

$\cancel{H_0 : \bar{X} = 30}$

The Null Hypothesis, H_0

DCOVA
(continued)

- Begin with the assumption that the null hypothesis is true
  - Similar to the notion of innocent until proven guilty



- Refers to the status quo or historical value
- Always contains "=", or "≤", or "≥" sign
- May or may not be rejected

The Alternative Hypothesis, H_1

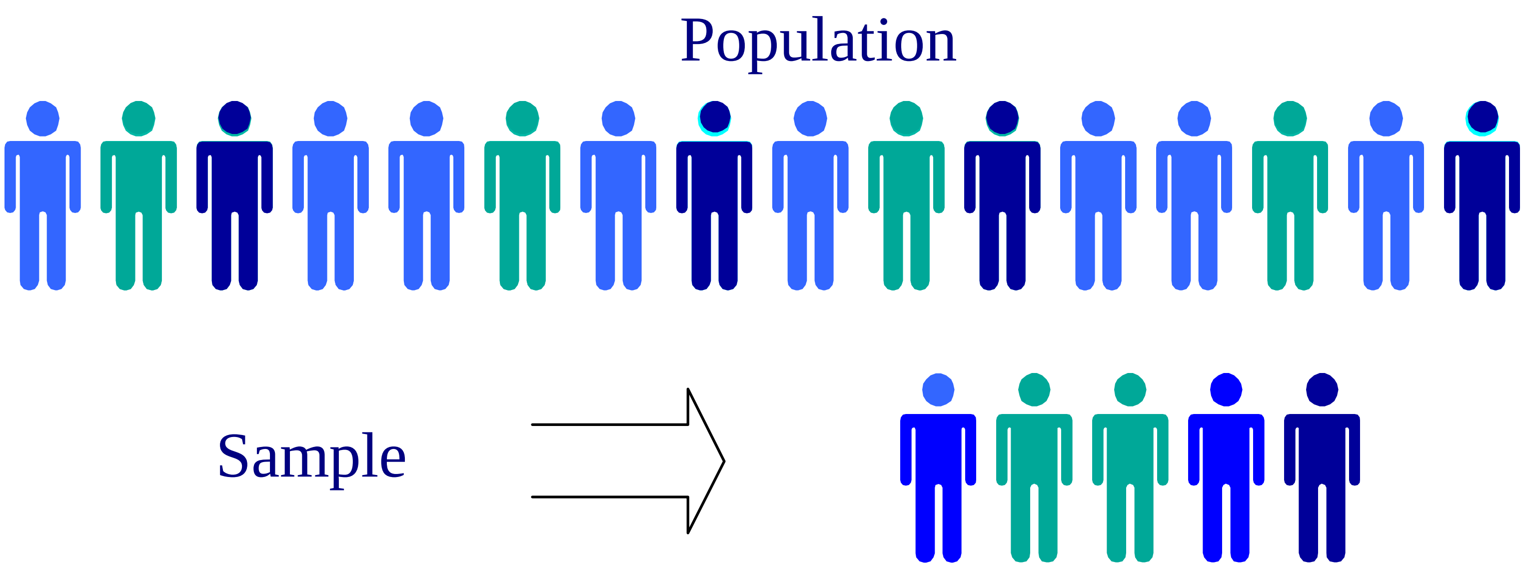
DCOVA

- Is the opposite of the null hypothesis
  - e.g., The average diameter of a manufactured bolt is not equal to 30mm (H_1: $\mu \neq 30$)
- Challenges the status quo
- Never contains the "=", or "≤", or "≥" sign
- May or may not be proven
- Is generally the hypothesis that the researcher is trying to prove

The Hypothesis Testing Process

DCOVA

- Claim: The population mean age is 50.
  - H_0: $\mu = 50$, H_1: $\mu \neq 50$
- Sample the population and find sample mean.



The Hypothesis Testing Process

DCOVA

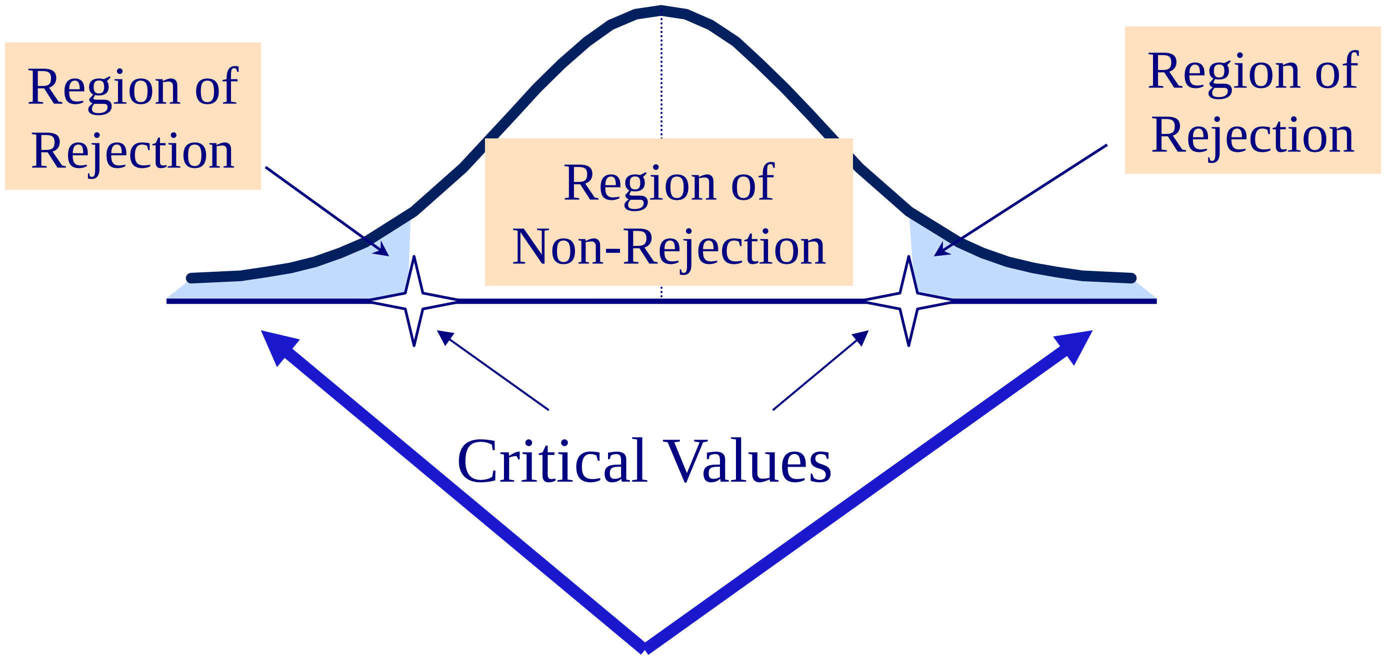
(continued)

- Suppose the sample mean age was $\bar{X}$ = 20.
- This is significantly lower than the claimed mean population age of 50.
- If the null hypothesis were true, the probability of getting such a different sample mean would be very small, so you reject the null hypothesis .
- In other words, getting a sample mean of 20 is so unlikely if the population mean was 50, you conclude that the population mean must not be 50.

The Test Statistic and Critical Values

DCOVA

Sampling Distribution of the test statistic



"Too Far Away" From Mean of Sampling Distribution

Possible Errors in Hypothesis Test Decision Making

DCOVA

(continued)

Possible Hypothesis Test Outcomes		
	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject H_0	No Error Probability 1 - α	Type II Error Probability β
Reject H_0	Type I Error Probability α	No Error Power 1 - β

Possible Errors in Hypothesis Test Decision Making

DCOVA

(continued)

- The confidence coefficient (1-α) is the probability of not rejecting H_0 when it is true.
- The confidence level of a hypothesis test is (1-α)*100%.
- The power of a statistical test (1-β) is the probability of rejecting H_0 when it is false.

Type I & II Error Relationship

DCOVA

- Type I and Type II errors cannot happen at the same time
  - A Type I error can only occur if H_0 is true
  - A Type II error can only occur if H_0 is false

If Type I error probability (α) $\Uparrow$, then

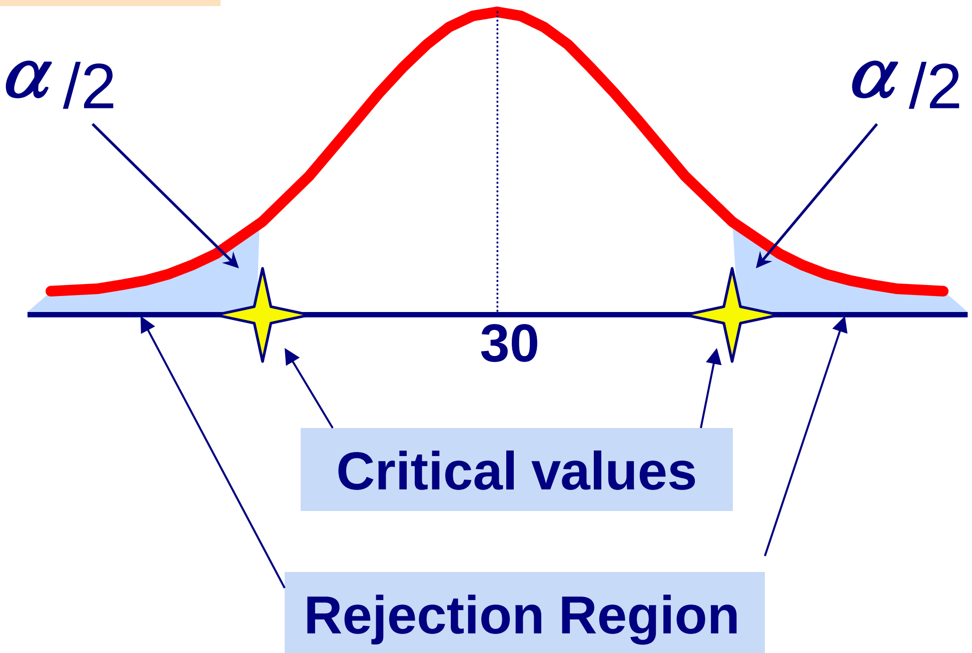
Type II error probability (β) $\Downarrow$

Level of Significance and the Rejection Region

DCOVA

H_0: μ = 30
H_1: μ ≠ 30

Level of significance = α



This is a two-tail test because there is a rejection region in both tails

Hypothesis Tests for the Mean

DCOVA

Hypothesis Tests for μ

- σ Known (Z test)
- σ Unknown (t test)

Z Test of Hypothesis for the Mean (σ Known)

DCOVA

- Convert sample statistic ($\bar{X}$) to a Z_{STAT} test statistic

Hypothesis Tests for μ

σ Known (Z test)

σ Unknown (t test)

The test statistic is:

$$Z_{STAT} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Critical Value Approach to Testing

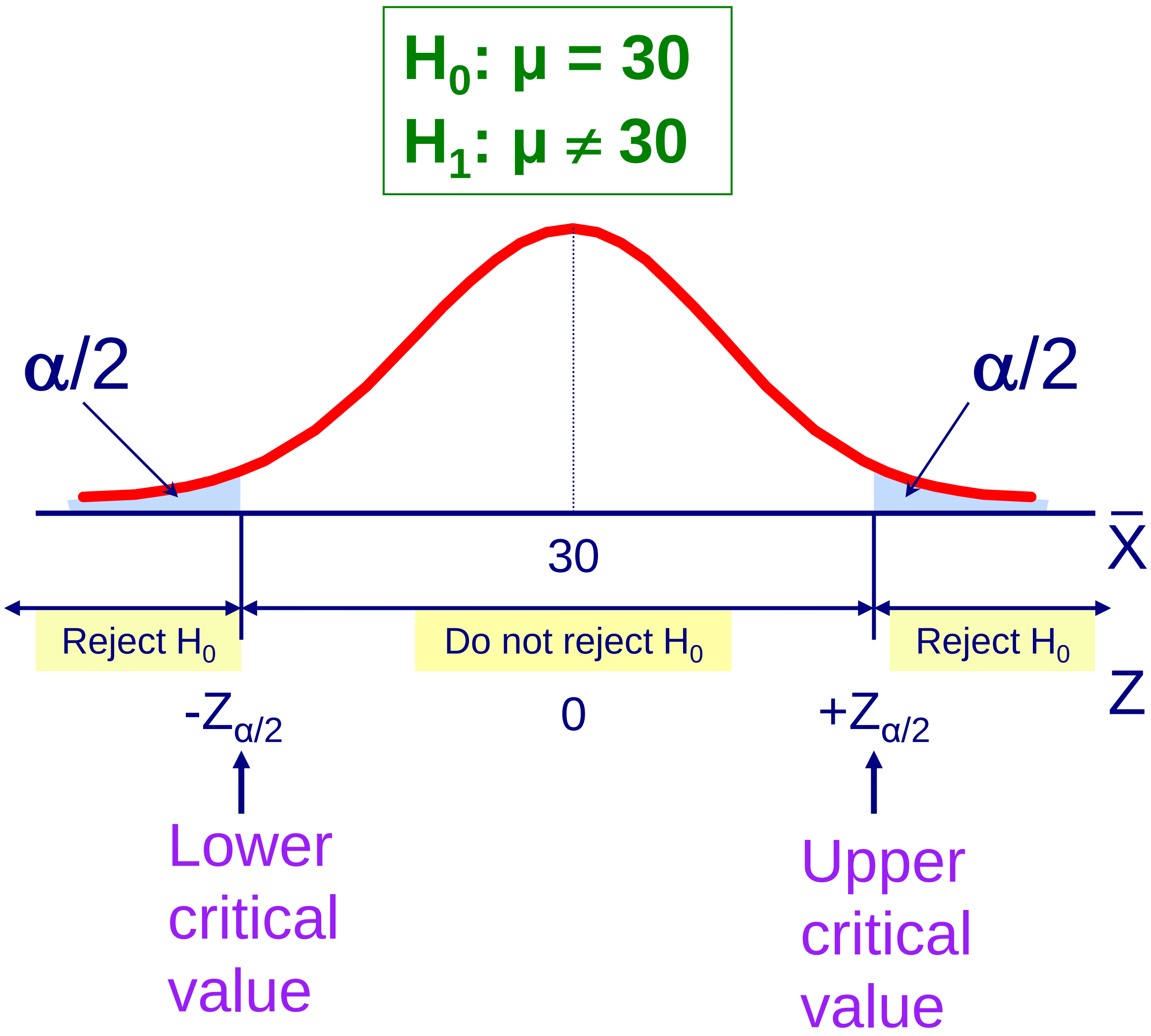
DCOVA

- For a two-tail test for the mean, σ known:
- Convert sample statistic ($\bar{X}$) to test statistic (Z_{STAT})
- Determine the critical Z values for a specified level of significance α from a table or computer
- Decision Rule: If the test statistic falls in the rejection region, reject H_0 ; otherwise do not reject H_0

Two-Tail Tests

DCOVA

- There are two cutoff values (critical values), defining the regions of rejection



Hypothesis Testing Example

DCOVA

Test the claim that the true mean diameter of a manufactured bolt is 30mm.
(Assume $\sigma = 0.8$)

1. State the appropriate null and alternative hypotheses
   - $H_0: \mu = 30$ $H_1: \mu \neq 30$ (This is a two-tail test)
2. Specify the desired level of significance and the sample size
   - Suppose that $\alpha = 0.05$ and $n = 100$ are chosen for this test

Hypothesis Testing Example

DCOVA

(continued)

3. Determine the appropriate technique
   - σ is assumed known so this is a Z test.
4. Determine the critical values
   - For α = 0.05 the critical Z values are ±1.96
5. Collect the data and compute the test statistic
   - Suppose the sample results are

     n = 100, $\bar{X}$ = 29.84 (σ = 0.8 is assumed known)

So the test statistic is:

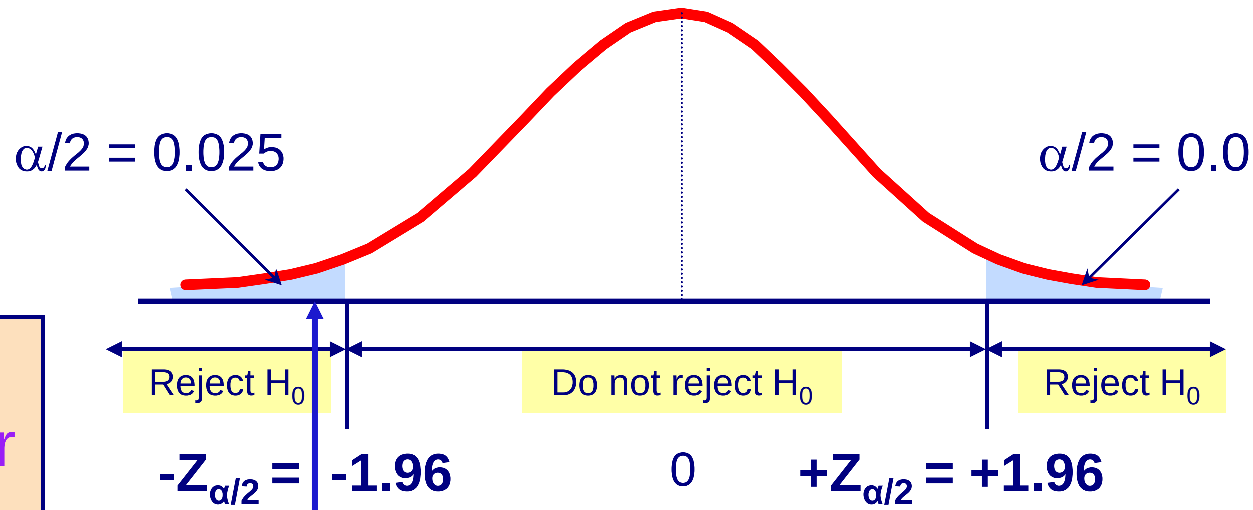
$$Z_{STAT} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{29.84 - 30}{\frac{0.8}{\sqrt{100}}} = \frac{-.16}{0.08} = -2.0$$

Hypothesis Testing Example

DCOVA

(continued)

- 6. Is the test statistic in the rejection region?



Reject H_0 if $Z_{STAT} < -1.96$ or $Z_{STAT} > 1.96$; otherwise do not reject H_0

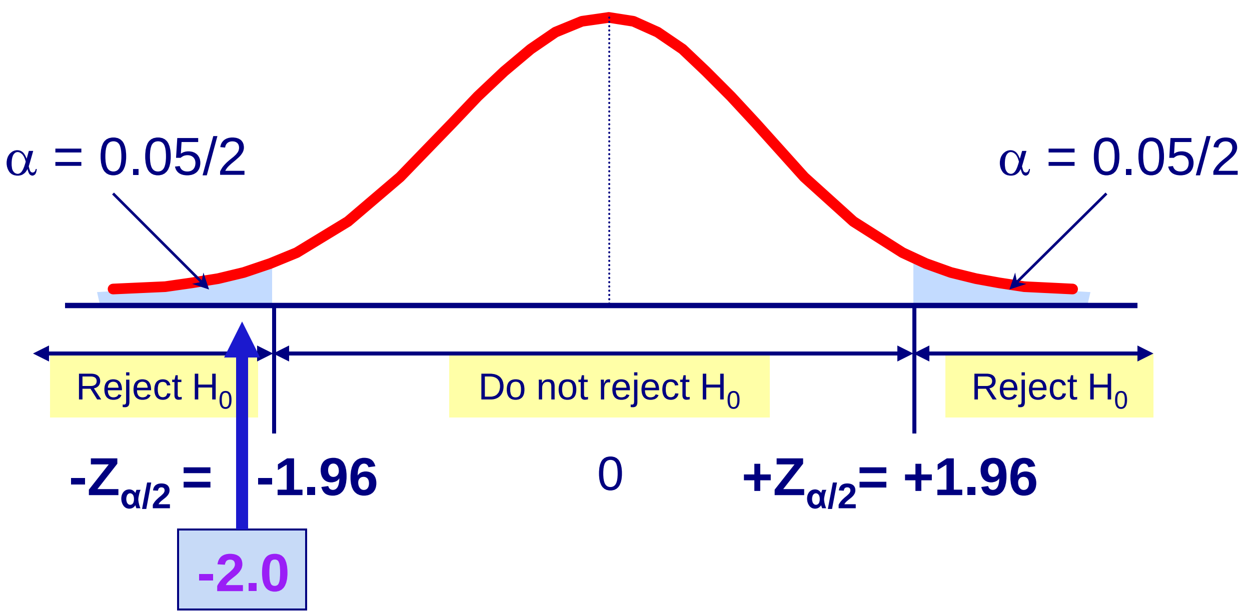
Here, $Z_{STAT} = -2.0 < -1.96$, so the test statistic is in the rejection region

Hypothesis Testing Example

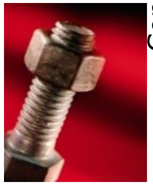
DCOVA

(continued)

6 (continued). Reach a decision and interpret the result



Since Z_{STAT} = -2.0 < -1.96, <u>reject the null hypothesis</u> and conclude there is sufficient evidence that the mean diameter of a manufactured bolt is not equal to 30



p-Value Approach to Testing

DCOVA

- p-value: Probability of obtaining a test statistic equal to or more extreme than the observed sample value given H_0 is true
  - The p-value is also called the observed level of significance
  - It is the smallest value of α for which H_0 can be rejected

p-Value Approach to Testing: Interpreting the p-value

DCOVA

- Compare the p-value with α
  - If p-value $< \alpha$, reject H_0
  - If p-value $\geq \alpha$, do not reject H_0

- Remember
  - If the p-value is low then H_0 must go

The 5 Step p-value approach to Hypothesis Testing DCOVA

1. State the null hypothesis, H_0 and the alternative hypothesis, H_1
2. Choose the level of significance, α, and the sample size, n
3. Determine the appropriate test statistic and sampling distribution
4. Collect data and compute the value of the test statistic and the p-value
5. Make the statistical decision and state the managerial conclusion. If the p-value is < α then reject H_0, otherwise do not reject H_0. State the managerial conclusion in the context of the problem

p-value Hypothesis Testing Example

DCOVA

Test the claim that the true mean diameter of a manufactured bolt is 30mm. (Assume σ = 0.8)

1. State the appropriate null and alternative hypotheses
   - H_0: μ = 30 H_1: μ ≠ 30 (This is a two-tail test)
2. Specify the desired level of significance and the sample size
   - Suppose that α = 0.05 and n = 100 are chosen for this test

p-value Hypothesis Testing Example

DCOVA

(continued)

3. Determine the appropriate technique
   - σ is assumed known so this is a Z test.
4. Collect the data, compute the test statistic and the p-value
   - Suppose the sample results are

     $n = 100, \quad \bar{X} = 29.84 \quad (\sigma = 0.8 \text{ is assumed known})$

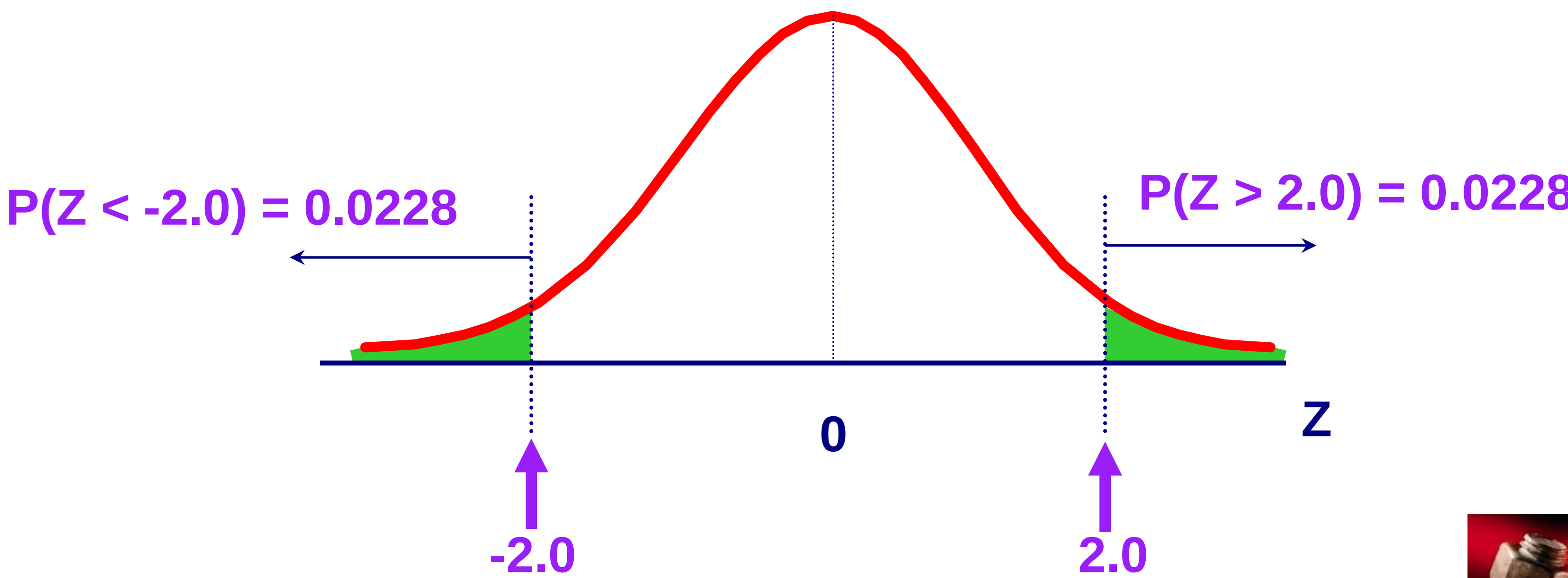
So the test statistic is:

$$Z_{STAT} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{29.84 - 30}{\frac{0.8}{\sqrt{100}}} = \frac{-0.16}{0.08} = -2.0$$

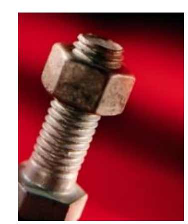
p-Value Hypothesis Testing Example: Calculating the p-value

DCOVA

4. (continued) Calculate the p-value.
   - How likely is it to get a Z_{STAT} of -2 (or something further from the mean (0), in either direction) if H_0 is true?



p-value = 0.0228 + 0.0228 = 0.0456



p-value Hypothesis Testing Example

DCOVA

(continued)

- 5. Is the p-value < α?
  - Since p-value = 0.0456 < α = 0.05 Reject H_0
- 5. (continued) State the managerial conclusion in the context of the situation.
  - There is sufficient evidence to conclude the average diameter of a manufactured bolt is not equal to 30mm.

Connection Between Two Tail Tests and Confidence Intervals

DCOVA

- For $\bar{X} = 29.84$, $\sigma = 0.8$ and $n = 100$, the 95% confidence interval is:

$$29.84 - (1.96)\frac{0.8}{\sqrt{100}} \quad \text{to} \quad 29.84 + (1.96)\frac{0.8}{\sqrt{100}}$$

$$\boxed{29.6832 \le \mu \le 29.9968}$$

- Since this interval does not contain the hypothesized mean (30), we reject the null hypothesis at $\alpha = 0.05$

Do You Ever Truly Know σ?

DCOVA

- Probably not!
- In virtually all real world business situations, σ is not known.
- If there is a situation where σ is known then μ is also known (since to calculate σ you need to know μ.)
- If you truly know μ there would be no need to gather a sample to estimate it.

Hypothesis Testing: σ Unknown

DCOVA

- If the population standard deviation is unknown, you instead use the sample standard deviation S.
- Because of this change, you use the t distribution instead of the Z distribution to test the null hypothesis about the mean.
- When using the t distribution you must assume the population you are sampling from follows a normal distribution.
- All other steps, concepts, and conclusions are the same.

t Test of Hypothesis for the Mean (σ Unknown)

DCOVA

- Convert sample statistic ($\bar{X}$) to a t_{STAT} test statistic

Hypothesis Tests for μ

- **σ Known (Z test)**
- **σ Unknown (t test)**

The test statistic is:

$$t_{STAT} = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}}$$

Example: Two-Tail Test (σ Unknown)

DCOVA

The average cost of a hotel room in New York is said to be $168 per night. To determine if this is true, a random sample of 25 hotels is taken and resulted in an $\bar{X}$ of $172.50 and an S of $15.40. Test the appropriate hypotheses at $\alpha = 0.05$.

(Assume the population distribution is normal)

$H_0: \mu = 168$

$H_1: \mu \neq 168$

Example Solution: Two-Tail t Test

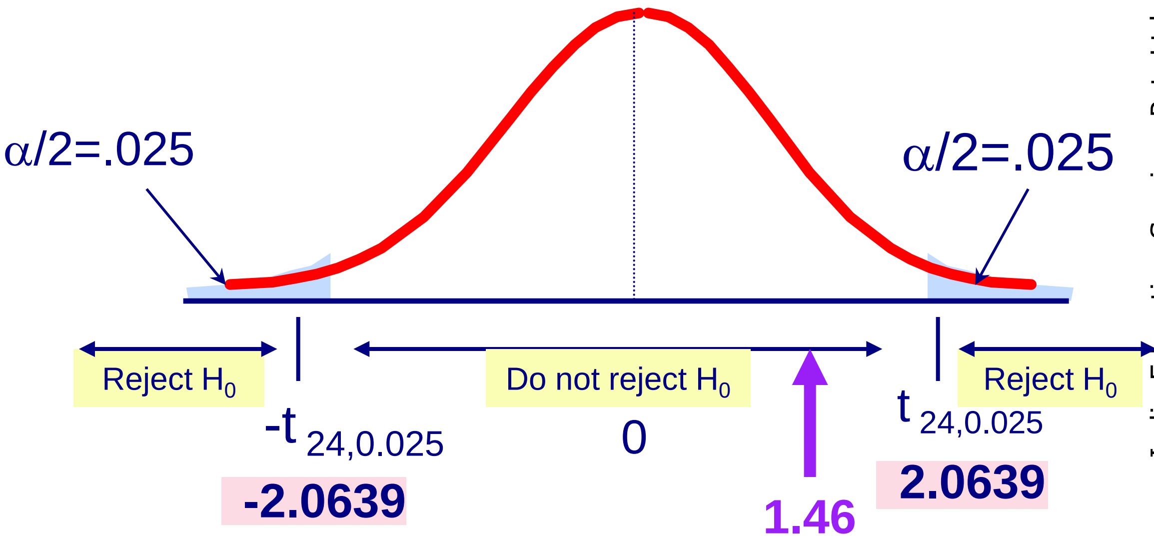
DCOVA

$H_0: \mu = 168$

$H_1: \mu \neq 168$

- $\alpha = 0.05$
- $n = 25$, df = 25-1=24
- **σ is unknown, so use a t statistic**
- **Critical Value:**

$\pm t_{24,0.025} = \pm 2.0639$



$$t_{STAT} = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} = \frac{172.50 - 168}{\frac{15.40}{\sqrt{25}}} = 1.46$$

Do not reject H_0: insufficient evidence that true mean cost is different from $168

Example Two-Tail t Test Using A p-value from Excel

DCOVA

- Since this is a t-test we cannot calculate the p-value without some calculation aid.
- The Excel output below does this:

t Test for the Hypothesis of the Mean

Data			
Null Hypothesis	**μ=**	**$**	**168.00**
Level of Significance			**0.05**
Sample Size			**25**
Sample Mean		**$**	**172.50**
Sample Standard Deviation		**$**	**15.40**

Intermediate Calculations			
Standard Error of the Mean	$	3.08	=B8/SQRT(B6)
Degrees of Freedom		24	=B6-1
t test statistic		**1.46**	=(B7-B4)/B11

Two-Tail Test		
Lower Critical Value	-2.0639	=-TINV(B5,B12)
Upper Critical Value	2.0639	=TINV(B5,B12)
p-value	0.157	=TDIST(ABS(B13),B12,2)
Do Not Reject Null Hypothesis		=IF(B18<B5, "Reject null hypothesis", "Do not reject null hypothesis")

p-value > α
So do not reject H_0

Connection of Two Tail Tests to Confidence Intervals

DCOVA

- For $\bar{X} = 172.5$, $S = 15.40$ and $n = 25$, the 95% confidence interval for μ is:

$$\boxed{172.5 - (2.0639)\ 15.4/\sqrt{25}} \quad \text{to} \quad \boxed{172.5 + (2.0639)\ 15.4/\sqrt{25}}$$

$$\boxed{166.14 \leq \mu \leq 178.86}$$

- Since this interval contains the Hypothesized mean (168), we do not reject the null hypothesis at $\alpha = 0.05$

One-Tail Tests

DCOVA

- In many cases, the alternative hypothesis focuses on a particular direction

H_0: $\mu \geq 3$
H_1: $\mu < 3$

This is a lower-tail test since the alternative hypothesis is focused on the lower tail below the mean of 3

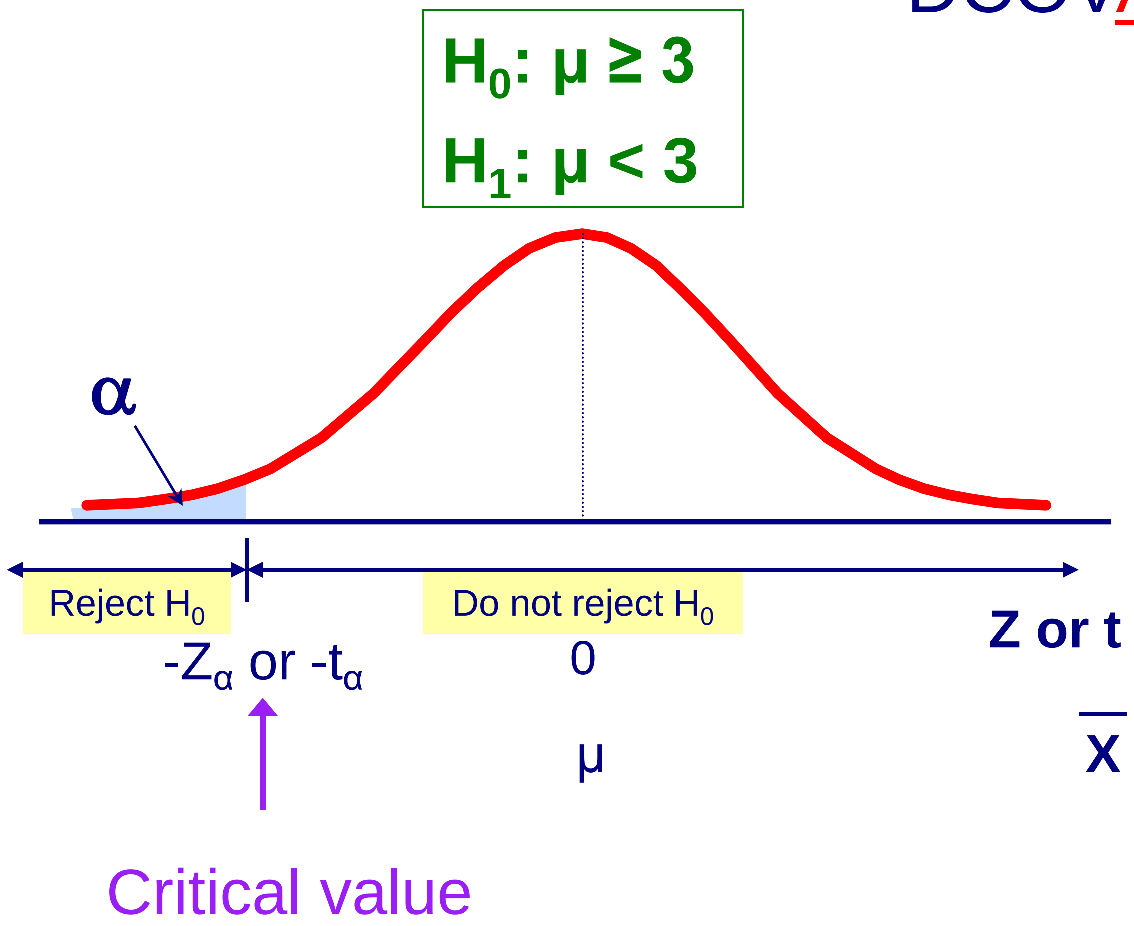
H_0: $\mu \leq 3$
H_1: $\mu > 3$

This is an upper-tail test since the alternative hypothesis is focused on the upper tail above the mean of 3

Lower-Tail Tests

DCOVA

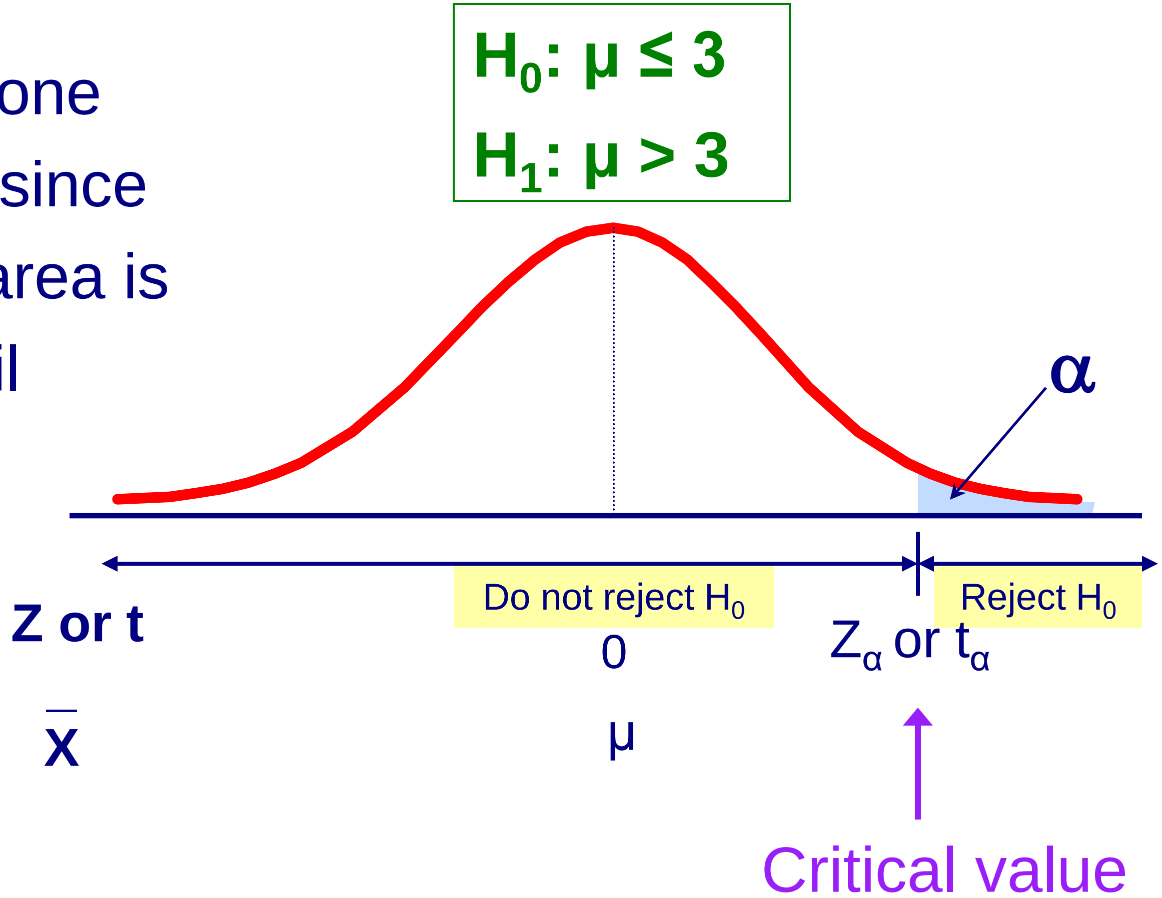
- There is only one critical value, since the rejection area is in only one tail



Upper-Tail Tests

DCOVA

- There is only one critical value, since the rejection area is in only one tail



Example: Upper-Tail t Test for Mean (σ unknown)

DCOVA

A phone industry manager thinks that customer monthly cell phone bills have increased, and now average over \$52 per month. The company wishes to test this claim. (Assume a normal population)

Form hypothesis test:

H_0: $\mu \le 52$ the average is not over \$52 per month

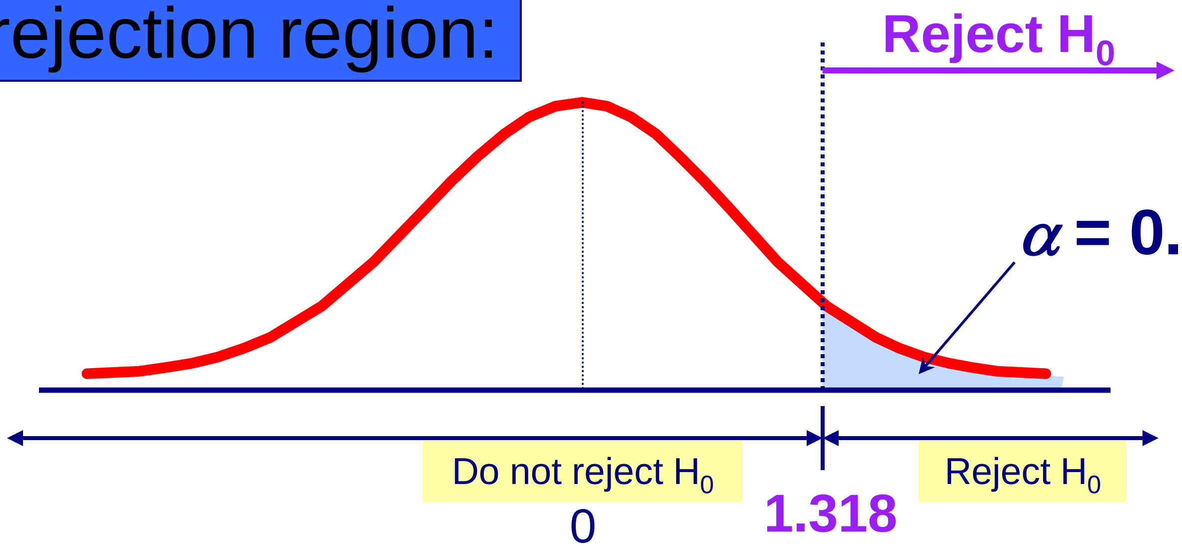
H_1: $\mu > 52$ the average **is** greater than \$52 per month (i.e., sufficient evidence exists to support the manager's claim)

Example: Find Rejection Region

DCOVA
(continued)

- Suppose that α = 0.10 is chosen for this test and n = 25.

Find the rejection region:



Reject H_0 if $t_{STAT} > 1.318$

Example: Test Statistic

DCOVA

(continued)

Obtain sample and compute the test statistic

Suppose a sample is taken with the following results: $n = 25$, $\bar{X} = 53.1$, and $S = 10$

- Then the test statistic is:

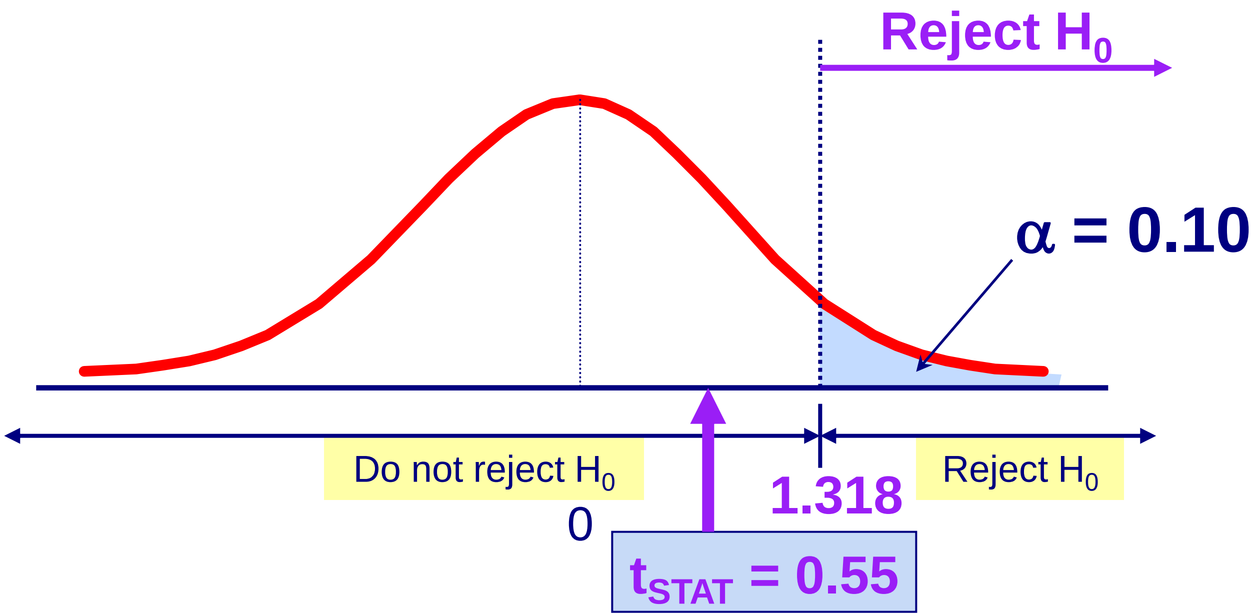
$$t_{STAT} = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} = \frac{53.1 - 52}{\frac{10}{\sqrt{25}}} = 0.55$$

Example: Decision

DCOVA

(continued)

Reach a decision and interpret the result:



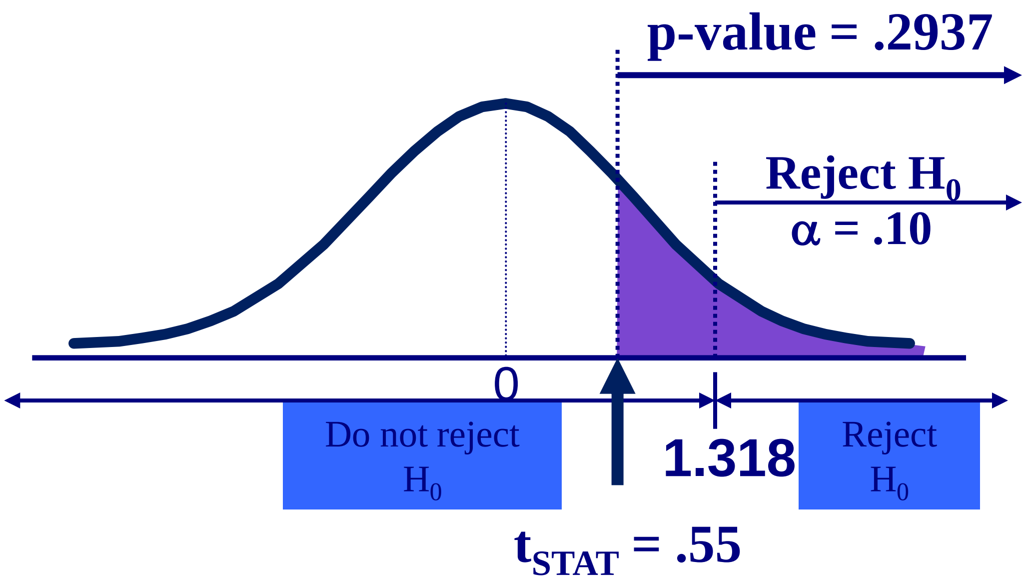
Do not reject H_0 since $t_{STAT} = 0.55 \leq 1.318$

there is not sufficient evidence that the mean bill is over $52

Example: Utilizing The p-value for The Test

DCOVA

- Calculate the p-value and compare to α (p-value below calculated using excel spreadsheet on next page)



Do not reject H_0 since p-value = .2937 > α = .10

Excel Spreadsheet Calculating The p-value for An Upper Tail t Test

DCOVA

	A	B	
1	t Test for the Hypothesis of the Mean		
2			
3	Data		
4	Null Hypothesis μ=	184.2	
5	Level of Significance	0.05	
6	Sample Size	25	
7	Sample Mean	170.8	
8	Sample Standard Deviation	21.3	
9			
10	Intermediate Calculations		
11	Standard Error of the Mean	4.2600	=B8/SQRT(B6)
12	Degrees of Freedom	24	=B6 - 1
13	*t* Test Statistic	-3.1455	=(B7 - B4)/B11
14			
15	Lower-Tail Test		
16	Lower Critical Value	-1.7109	=-T.INV.2T(2 * B5, B12)
17	*p*-Value	0.0022	=IF(B13 < 0, E11, E12)
18	Reject the null hypothesis		=IF(B17 < B5,"Reject the null hypothesis", "Do not reject the null hypothesis")

	D	E	
10	One-Tail Calculations		
11	T.DIST.RT value	0.0022	=T.DIST.RT(ABS(B13), B12)
12	1-T.DIST.RT value	0.9978	=1 - E11

Hypothesis Tests for Proportions

DCOVA

- Involves categorical variables
- Two possible outcomes
  - Possesses characteristic of interest
  - Does not possess characteristic of interest
- Fraction or proportion of the population in the category of interest is denoted by π

Proportions

DCOVA

(continued)

- Sample proportion in the category of interest is denoted by p

$$p = \frac{X}{n} = \frac{\text{number in category of interest in sample}}{\text{sample size}}$$

- When both $n\pi$ and $n(1-\pi)$ are at least 5, p can be approximated by a normal distribution with mean and standard deviation
  - $\mu_p = \pi$ $\quad\quad$ $\sigma_p = \sqrt{\frac{\pi(1-\pi)}{n}}$

Hypothesis Tests for Proportions

DCOVA

- The sampling distribution of p is approximately normal, so the test statistic is a Z_{STAT} value:

$$Z_{STAT} = \frac{p - \pi}{\sqrt{\dfrac{\pi(1-\pi)}{n}}}$$

Hypothesis Tests for p

$n\pi \geq 5$ and $n(1-\pi) \geq 5$

$n\pi < 5$ or $n(1-\pi) < 5$

Not discussed in this chapter

Z Test for Proportion in Terms of Number in Category of Interest

DCOVA

- An equivalent form to the last slide, but in terms of the number in the category of interest, X:

$$Z_{STAT} = \frac{X - n\pi}{\sqrt{n\pi(1-\pi)}}$$

Hypothesis Tests for X

$X \geq 5$ and $n-X \geq 5$

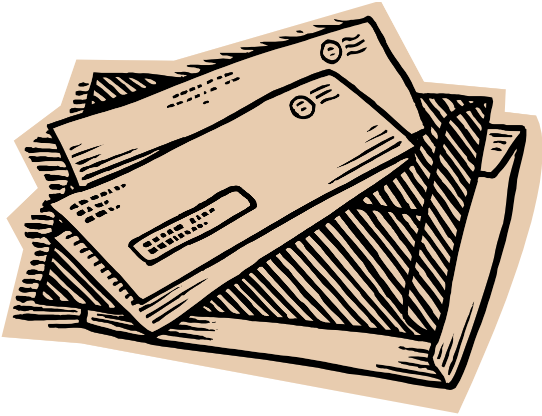
$X < 5$ or $n-X < 5$

Not discussed in this chapter

Example: Z Test for Proportion

DCOVA

A marketing company claims that it receives 8% responses from its mailing. To test this claim, a random sample of 500 were surveyed with 25 responses. Test at the $\alpha = 0.05$ significance level.



Check:

$$n\pi = (500)(.08) = 40$$

$$n(1-\pi) = (500)(.92) = 460$$

✓

Z Test for Proportion: Solution

DCOVA

H_0: $\pi = 0.08$
H_1: $\pi \neq 0.08$

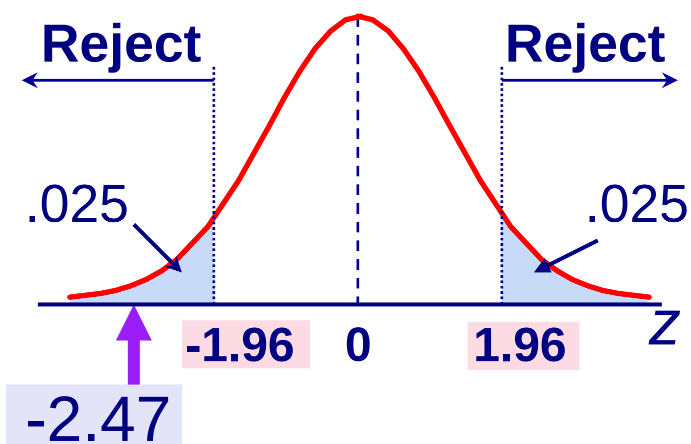
$\alpha = 0.05$

$n = 500$, $p = 0.05$

Test Statistic:

$$Z_{STAT} = \frac{p - \pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} = \frac{.05 - .08}{\sqrt{\frac{.08(1-.08)}{500}}} = -2.47$$

Critical Values: ± 1.96



Decision:

Reject H_0 at $\alpha = 0.05$

Conclusion:

There is sufficient evidence to reject the company's claim of 8% response rate.

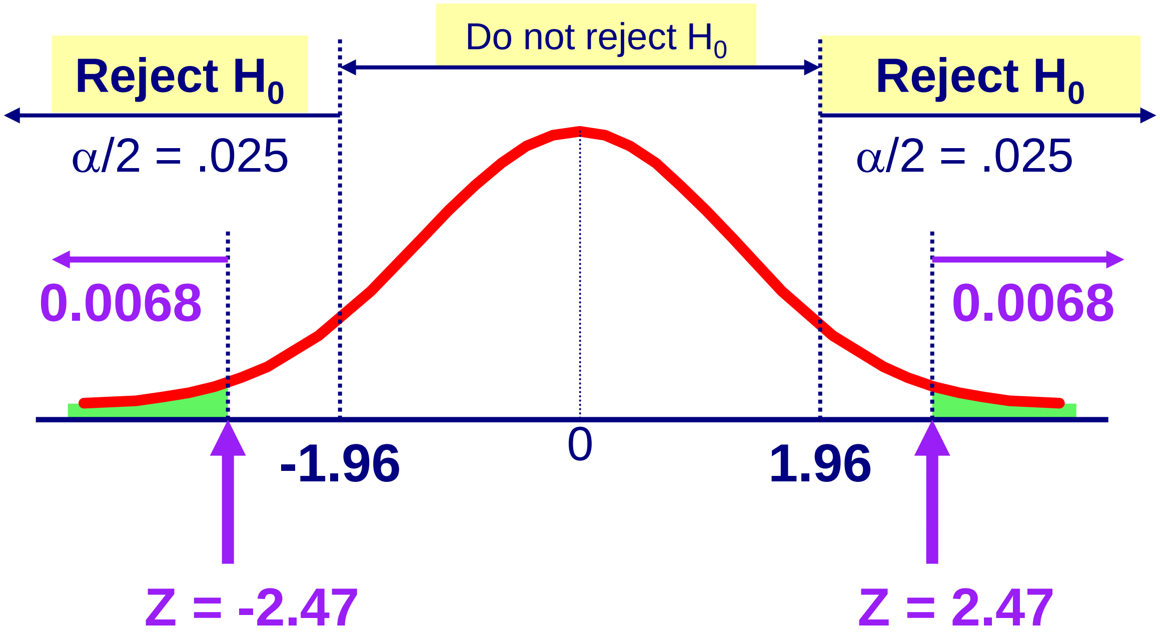
p-Value Solution

DCOVA

(continued)

Calculate the p-value and compare to α

(For a two-tail test the p-value is always two-tail)



p-value = 0.0136:

$$P(Z \le -2.47) + P(Z \ge 2.47)$$

$$= 2(0.0068) = 0.0136$$

Reject H_0 since p-value = 0.0136 < α = 0.05

The Power Of A Test Is An Important Part Of Planning

- The power of a hypothesis test is included as an on-line topic

Online Topic

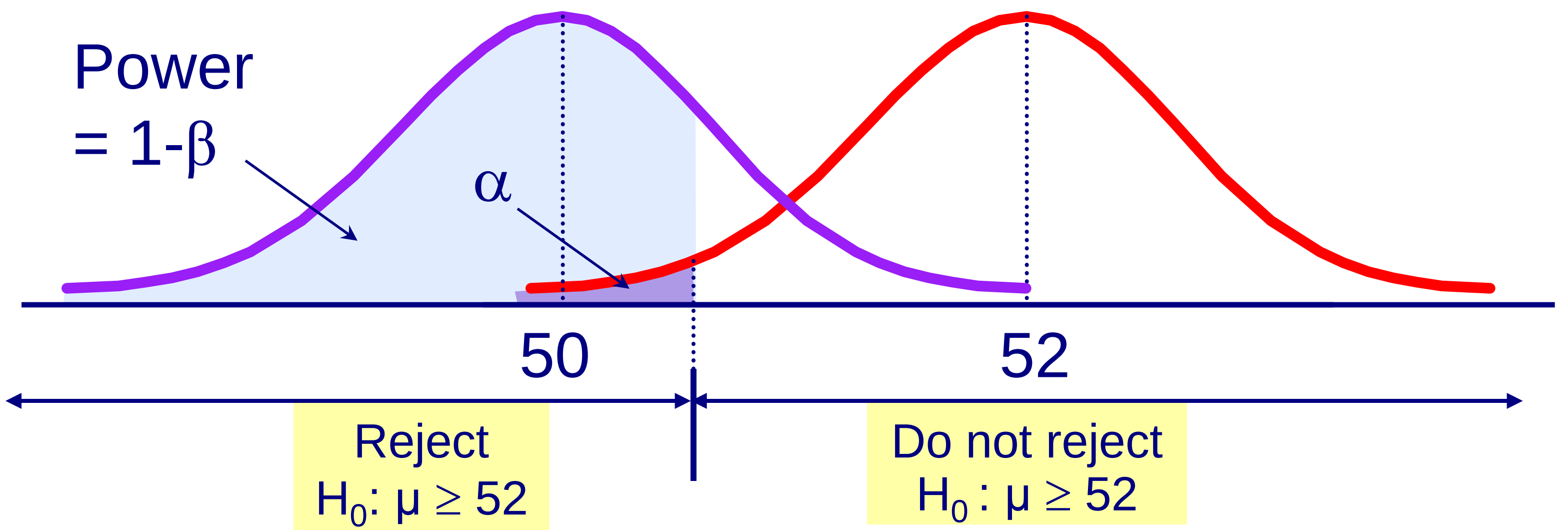
Power of a Test

The Power of a Test

DCOVA

- **The power of the test is the probability of correctly rejecting a false H_0**

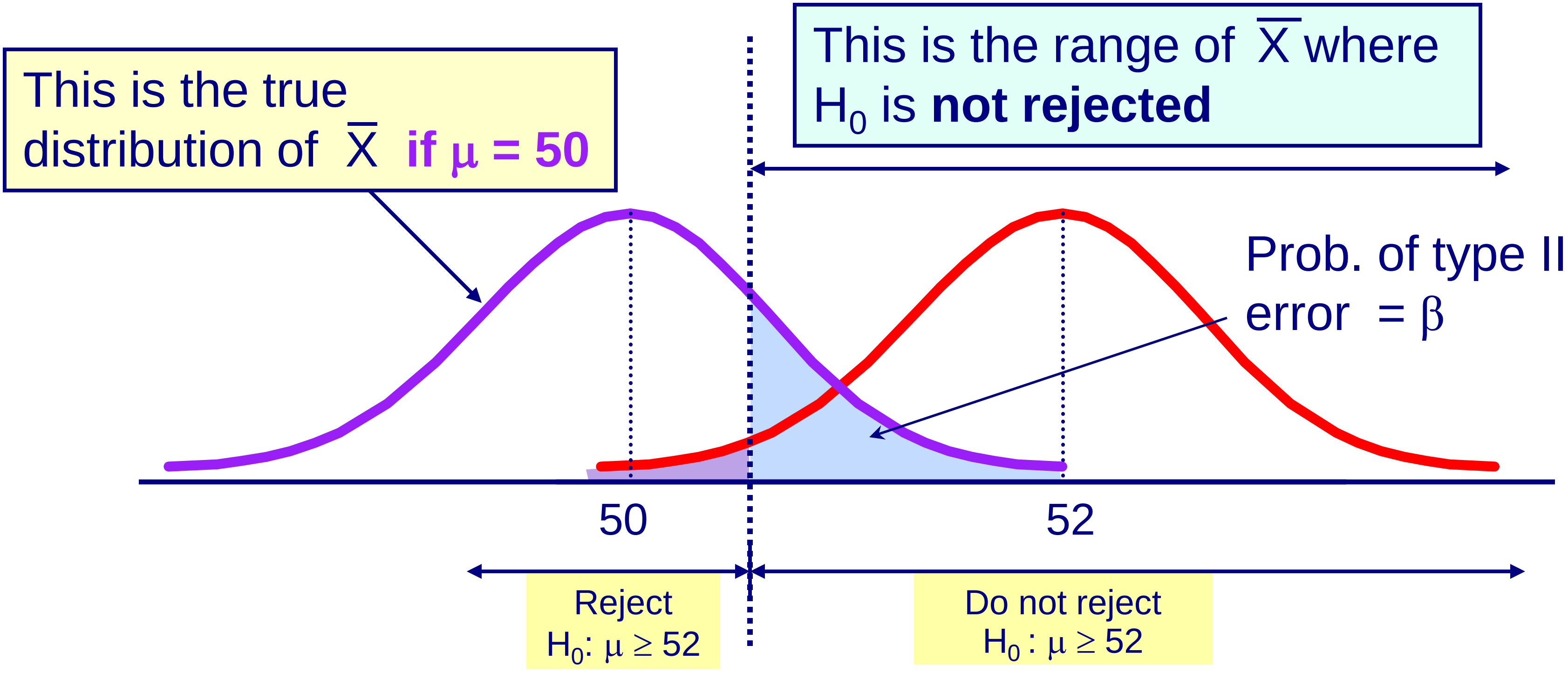
Suppose we correctly reject **H_0: μ ≥ 52** when in fact the true mean is **μ = 50**



Type II Error

DCOVA

- Suppose we do not reject H_0: $\mu \geq 52$ when in fact the true mean is $\mu = 50$



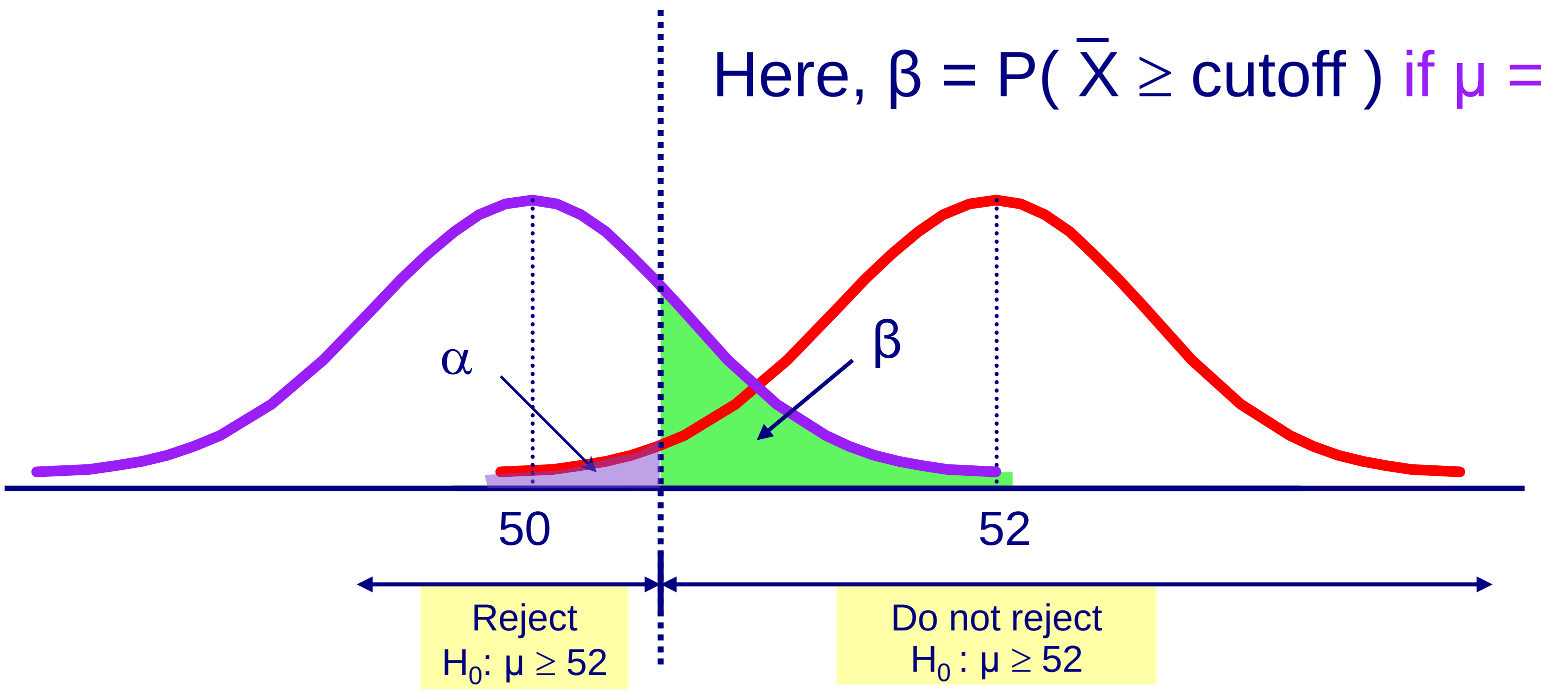
Type II Error

DCOVA

(continued)

- Suppose we do not reject H_0: $\mu \geq 52$ when in fact the true mean is $\mu = 50$

Here, $\beta = P(\bar{X} \geq \text{cutoff})$ if $\mu = 50$



Calculating β

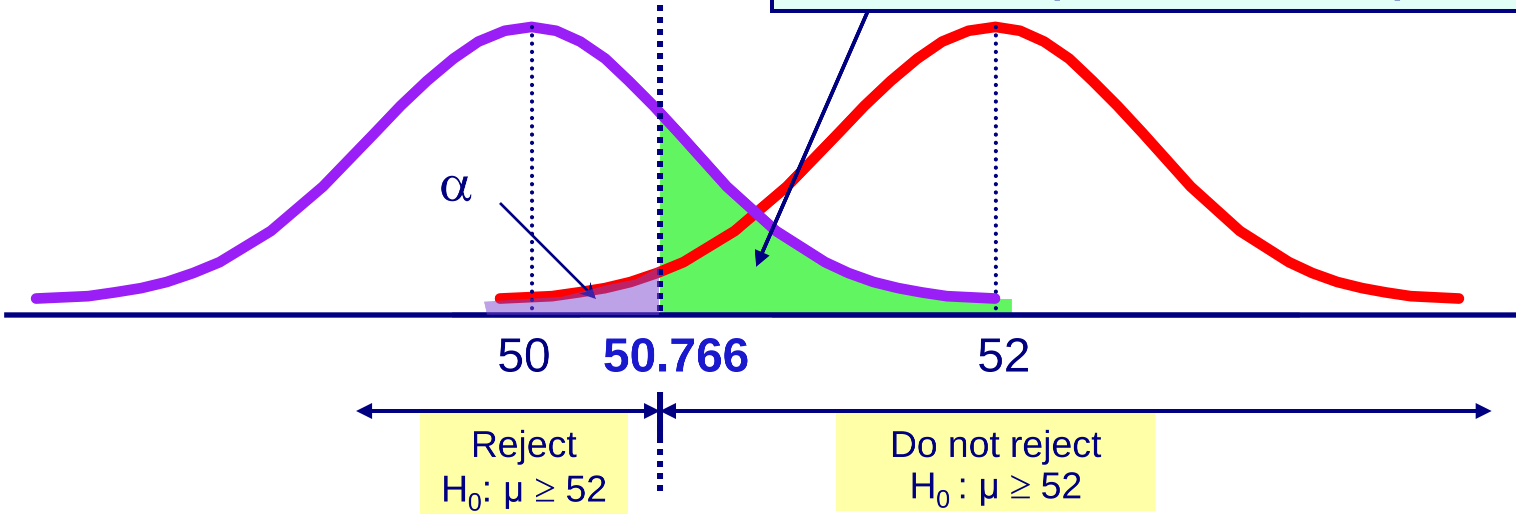
DCOVA

- Suppose n = 64 , σ = 6 , and α = .05

$$\text{cutoff} = \bar{X}_\alpha = \mu - Z_\alpha \frac{\sigma}{\sqrt{n}} = 52 - 1.645\frac{6}{\sqrt{64}} = \boxed{50.766}$$

(for $H_0 : \mu \geq 52$)

So $\beta = P(\bar{x} \geq 50.766)$ if $\mu = 50$



Calculating β and Power of the test

DCOVA

(continued)

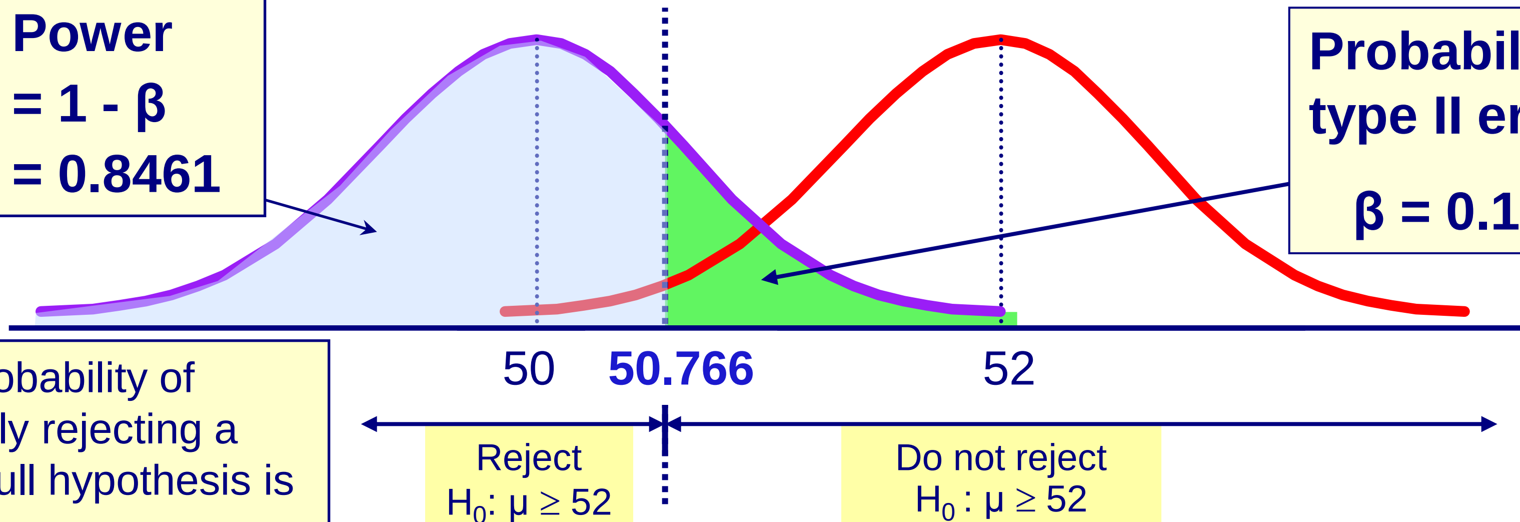
- Suppose n = 64 , σ = 6 , and α = 0.05

$$P(\bar{X} \geq 50.766 \mid \mu = 50) = P\left(Z \geq \frac{50.766 - 50}{6/\sqrt{64}} \right) = P(Z \geq 1.02) = 1.0 - 0.8461 \boxed{= 0.1539}$$

Power
= 1 - β
= 0.8461

Probability of type II error:

β = 0.1539



The probability of correctly rejecting a false null hypothesis is 0.8641

Power of the Test

- Conclusions regarding the power of the test:

  - A one-tail test is more powerful than a two-tail test
  - An increase in the level of significance (α) results in an increase in power
  - An increase in the sample size results in an increase in power