Chapter 10

Two-Sample Tests

Learning Objectives

In this chapter, you learn:

- How to use hypothesis testing for comparing the difference between
  - The means of two independent populations
  - The means of two related populations
  - The proportions of two independent populations
  - The variances of two independent populations

Two-Sample Tests

DCOVA

Two-Sample Tests

- Population Means, Independent Samples
- Population Means, Related Samples
- Population Proportions
- Population Variances

Examples:

- Group 1 vs. Group 2
- Same group before vs. after treatment
- Proportion 1 vs. Proportion 2
- Variance 1 vs. Variance 2

Difference Between Two Means

DCOVA

Population means, independent samples *

- σ_1 and σ_2 unknown, assumed equal
- σ_1 and σ_2 unknown, not assumed equal

Goal: Test hypothesis or form a confidence interval for the difference between two population means, $\mu_1 - \mu_2$

The point estimate for the difference is

$$\bar{X}_1 - \bar{X}_2$$

Difference Between Two Means: Independent Samples DCOVA

- Different data sources
  - Unrelated
  - Independent
    - Sample selected from one population has no effect on the sample selected from the other population

Population means, independent samples *

σ_1 and σ_2 unknown, assumed equal → Use S_p to estimate unknown σ. Use a **Pooled-Variance t test.**

σ_1 and σ_2 unknown, not assumed equal → Use S_1 and S_2 to estimate unknown σ_1 and σ_2. Use a **Separate-variance t test**

Hypothesis Tests for Two Population Means

DCOVA

Two Population Means, Independent Samples

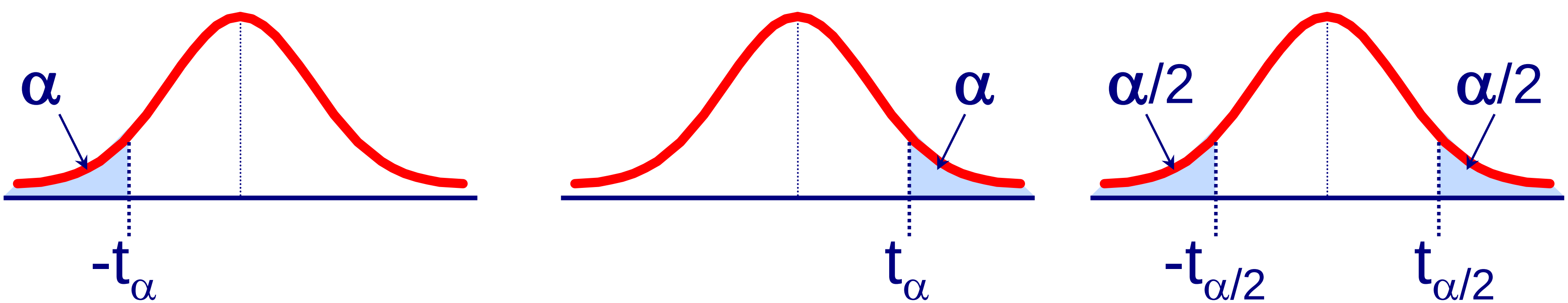
Lower-tail test:	Upper-tail test:	Two-tail test:
$H_0: \mu_1 \geq \mu_2$ $H_1: \mu_1 < \mu_2$	$H_0: \mu_1 \leq \mu_2$ $H_1: \mu_1 > \mu_2$	$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 \neq \mu_2$
i.e.,	i.e.,	i.e.,
$H_0: \mu_1 - \mu_2 \geq 0$ $H_1: \mu_1 - \mu_2 < 0$	$H_0: \mu_1 - \mu_2 \leq 0$ $H_1: \mu_1 - \mu_2 > 0$	$H_0: \mu_1 - \mu_2 = 0$ $H_1: \mu_1 - \mu_2 \neq 0$

Hypothesis tests for $\mu_1 - \mu_2$

DCOVA

Two Population Means, Independent Samples

Lower-tail test:	Upper-tail test:	Two-tail test:
$H_0: \mu_1 - \mu_2 \geq 0$ $H_1: \mu_1 - \mu_2 < 0$	$H_0: \mu_1 - \mu_2 \leq 0$ $H_1: \mu_1 - \mu_2 > 0$	$H_0: \mu_1 - \mu_2 = 0$ $H_1: \mu_1 - \mu_2 \neq 0$



Reject H_0 if $t_{STAT} < -t_\alpha$	Reject H_0 if $t_{STAT} > t_\alpha$	Reject H_0 if $t_{STAT} < -t_{\alpha/2}$ or $t_{STAT} > t_{\alpha/2}$

Hypothesis tests for μ_1 - μ_2 with σ_1 and σ_2 unknown and assumed equal

DCOVA

Population means, independent samples

σ_1 and σ_2 unknown, assumed equal *

σ_1 and σ_2 unknown, not assumed equal

Assumptions:

- Samples are randomly and independently drawn
- Populations are normally distributed or both sample sizes are at least 30
- Population variances are unknown but assumed equal

Hypothesis tests for μ_1 - μ_2 with σ_1 and σ_2 unknown and assumed equal

(continued)

DCOVA

Population means, independent samples

- σ_1 and σ_2 unknown, assumed equal *
- σ_1 and σ_2 unknown, not assumed equal

- The pooled variance is:

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{(n_1 - 1) + (n_2 - 1)}$$

- The test statistic is:

$$t_{STAT} = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

- Where t_{STAT} has d.f. = $(n_1 + n_2 - 2)$

Confidence interval for μ_1 - μ_2 with σ_1 and σ_2 unknown and assumed equal

DCOVA

Population means, independent samples

σ_1 and σ_2 unknown, assumed equal *

σ_1 and σ_2 unknown, not assumed equal

The confidence interval for $\mu_1 - \mu_2$ is:

$$\left(\bar{X}_1 - \bar{X}_2\right) \pm t_{\alpha/2} \sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

Where $t_{\alpha/2}$ has d.f. = $n_1 + n_2 - 2$

Pooled-Variance t Test Example

DCOVA

You are a financial analyst for a brokerage firm. Is there a difference in dividend yield between stocks listed on the NYSE & NASDAQ? You collect the following data:

	NYSE	NASDAQ
Number	21	25
Sample mean	3.27	2.53
Sample std dev	1.30	1.16

Assuming both populations are approximately normal with equal variances, is there a difference in mean yield (α = 0.05)?

Pooled-Variance t Test Example: Calculating the Test Statistic

(continued)

DCOVA

H0: $\mu_1 - \mu_2 = 0$ i.e. $(\mu_1 = \mu_2)$
H1: $\mu_1 - \mu_2 \neq 0$ i.e. $(\mu_1 \neq \mu_2)$

The test statistic is:

$$t = \frac{\left(\bar{X}_1 - \bar{X}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{\left(3.27 - 2.53\right) - 0}{\sqrt{1.5021\left(\frac{1}{21} + \frac{1}{25}\right)}} = \boxed{2.040}$$

$$S_p^2 = \frac{(n_1 - 1)S_1^{\ 2} + (n_2 - 1)S_2^{\ 2}}{(n_1 - 1) + (n_2 - 1)} = \frac{(21-1)1.30^2 + (25-1)1.16^2}{(21\text{-}1) + (25-1)} = 1.5021$$

Pooled-Variance t Test Example: Hypothesis Test Solution

DCOVA

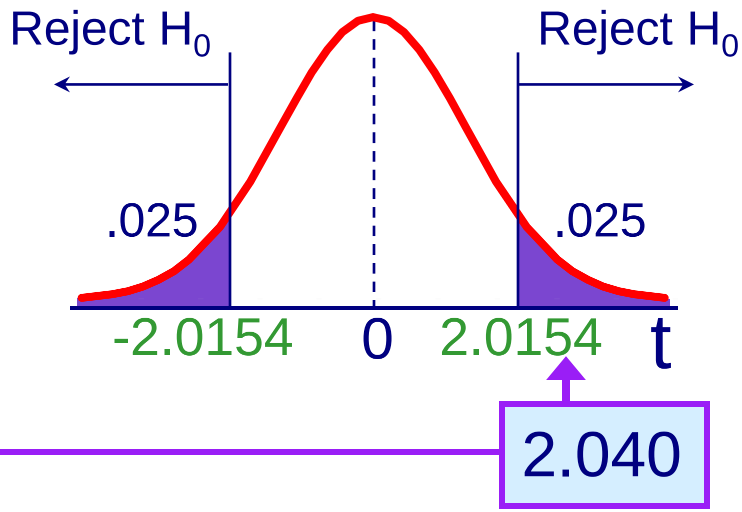
H_0: $\mu_1 - \mu_2 = 0$ i.e. $(\mu_1 = \mu_2)$

H_1: $\mu_1 - \mu_2 \neq 0$ i.e. $(\mu_1 \neq \mu_2)$

$\alpha = 0.05$

df = 21 + 25 - 2 = 44

Critical Values: t = ± 2.0154



Test Statistic:

$$t = \frac{3.27 - 2.53}{\sqrt{1.5021\left(\frac{1}{21} + \frac{1}{25}\right)}} = \boxed{2.040}$$

Decision:

Reject H_0 at $\alpha = 0.05$

Conclusion:

There is evidence of a difference in means.

Pooled-Variance t Test Example: Confidence Interval for $\mu_1 - \mu_2$

DCOVA

Since we rejected H_0 can we be 95% confident that $\mu_{NYSE} > \mu_{NASDAQ}$?

95% Confidence Interval for $\mu_{NYSE} - \mu_{NASDAQ}$

$$\left(\overline{X}_1 - \overline{X}_2\right) \pm t_{\alpha/2}\sqrt{S_p^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)} = \mathbf{0.74 \pm 2.0154 \times 0.3628 = (0.009,\ 1.471)}$$

Since 0 is less than the entire interval, we can be 95% confident that $\mu_{NYSE} > \mu_{NASDAQ}$

Hypothesis tests for $\mu_1 - \mu_2$ with σ_1 and σ_2 unknown, not assumed equal

DCOVA

Population means, independent samples

σ_1 and σ_2 unknown, assumed equal

σ_1 and σ_2 unknown, not assumed equal *

Assumptions:

- Samples are randomly and independently drawn
- Populations are normally distributed or both sample sizes are at least 30
- Population variances are unknown and cannot be assumed to be equal

Hypothesis tests for μ_1 - μ_2 with σ_1 and σ_2 unknown and not assumed equal

(continued)

DCOVA

Population means, independent samples

- σ_1 and σ_2 unknown, assumed equal
- σ_1 and σ_2 unknown, not assumed equal *

The test statistic is:

$$t_{STAT} = \frac{\left(\bar{X}_1 - \bar{X}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

t_{STAT} has d.f. ν =

$$\nu = \frac{\left(\frac{S_1^{\,2}}{n_1} + \frac{S_2^{\,2}}{n_2}\right)^2}{\frac{\left(\frac{S_1^{\,2}}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{S_2^{\,2}}{n_2}\right)^2}{n_2 - 1}}$$

Separate-Variance t Test Example

DCOVA

You are a financial analyst for a brokerage firm. Is there a difference in dividend yield between stocks listed on the NYSE & NASDAQ? You collect the following data:

	NYSE	NASDAQ
Number	21	25
Sample mean	3.27	2.53
Sample std dev	1.30	1.16

Assuming both populations are approximately normal with unequal variances, is there a difference in mean yield (α = 0.05)?

Separate-Variance t Test Example: Calculating the Test Statistic

(continued)

H0: $\mu_1 - \mu_2 = 0$ i.e. ($\mu_1 = \mu_2$)
H1: $\mu_1 - \mu_2 \neq 0$ i.e. ($\mu_1 \neq \mu_2$)

DCOVA

The test statistic is:

$$t = \frac{\left(\bar{X}_1 - \bar{X}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)}} = \frac{\left(3.27 - 2.53\right) - 0}{\sqrt{\left(\frac{1.30^2}{21} + \frac{1.16^2}{25}\right)}} = \boxed{2.019}$$

$$\nu = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{\left(\frac{S_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{S_2^2}{n_2}\right)^2}{n_2 - 1}} = \frac{\left(\frac{1.30^2}{21} + \frac{1.16^2}{25}\right)^2}{\frac{\left(\frac{1.30^2}{21}\right)^2}{20} + \frac{\left(\frac{1.16^2}{25}\right)^2}{24}} = 40.57$$

Use degrees of freedom = 40

Separate-Variance t Test Example: Hypothesis Test Solution

DCOVA

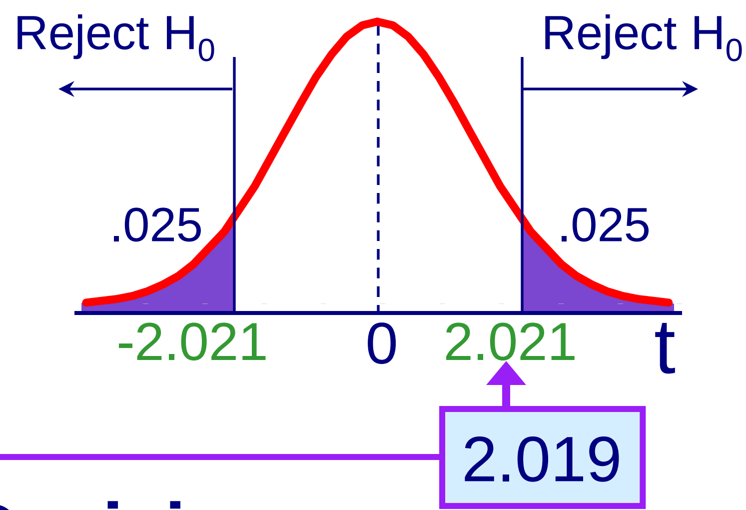
H_0: $\mu_1 - \mu_2 = 0$ i.e. $(\mu_1 = \mu_2)$

H_1: $\mu_1 - \mu_2 \neq 0$ i.e. $(\mu_1 \neq \mu_2)$

$\alpha = 0.05$

df = 40

Critical Values: t = ± 2.021



Test Statistic:

$t = \boxed{2.019}$

Decision:

Fail To Reject H_0 at $\alpha = 0.05$

Conclusion:

There is no evidence of a difference in means.

Related Populations
The Paired Difference Test

DCOVA

Related samples

Tests Means of 2 Related Populations

- Paired or matched samples
- Repeated measures (before/after)
- Use difference between paired values:

$$D_i = X_{1i} - X_{2i}$$

- Eliminates Variation Among Subjects
- Assumptions:
  - Both Populations Are Normally Distributed
  - Or, if not Normal, use large samples

Related Populations
The Paired Difference Test

DCOVA

(continued)

Related samples

The i^{th} paired difference is D_i , where

$$D_i = X_{1i} - X_{2i}$$

The point estimate for the paired difference population mean μ_D is $\bar{D}$:

$$\bar{D} = \frac{\sum_{i=1}^{n} D_i}{n}$$

The sample standard deviation is S_D

$$S_D = \sqrt{\frac{\sum_{i=1}^{n} (D_i - \bar{D})^2}{n-1}}$$

n is the number of pairs in the paired sample

The Paired Difference Test: Finding t_{STAT}

DCOVA

Paired samples

- The test statistic for μ_D is:

$$t_{STAT} = \frac{\bar{D} - \mu_D}{\frac{S_D}{\sqrt{n}}}$$

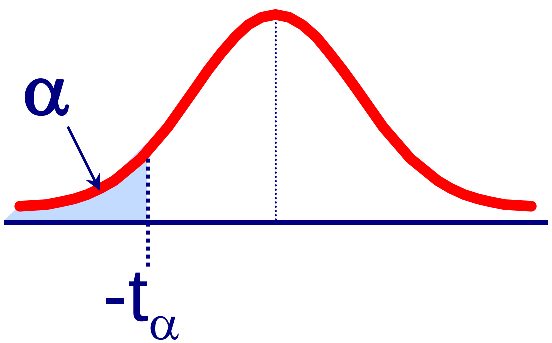
- Where t_{STAT} has $n - 1$ d.f.

The Paired Difference Test: Possible Hypotheses

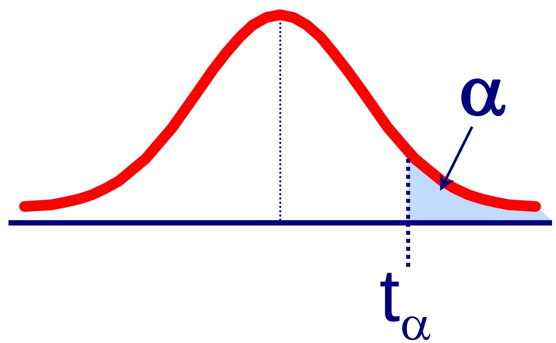
DCOVA

Paired Samples

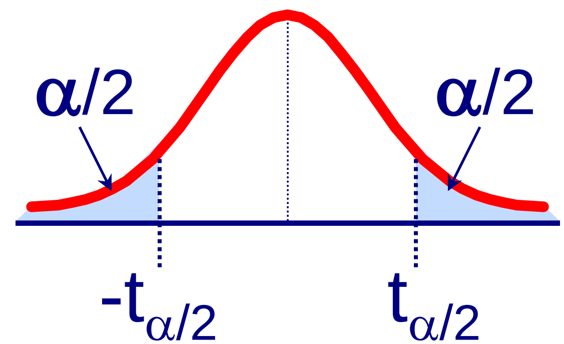
Lower-tail test:	Upper-tail test:	Two-tail test:
H_0: $\mu_D \geq 0$ H_1: $\mu_D < 0$	H_0: $\mu_D \leq 0$ H_1: $\mu_D > 0$	H_0: $\mu_D = 0$ H_1: $\mu_D \neq 0$



Reject H_0 if $t_{STAT} < -t_\alpha$



Reject H_0 if $t_{STAT} > t_\alpha$



Reject H_0 if $t_{STAT} < -t_{\alpha/2}$ or $t_{STAT} > t_{\alpha/2}$

Where t_{STAT} has $n - 1$ d.f.

The Paired Difference Confidence Interval

DCOVA

Paired samples

The confidence interval for μ_D is

$$\bar{D} \pm t_{\alpha/2} \frac{S_D}{\sqrt{n}}$$

where $$S_D = \sqrt{\frac{\sum_{i=1}^{n}(D_i - \bar{D})^2}{n-1}}$$

Paired Difference Test: Example

DCOVA

- Assume you send your salespeople to a "customer service" training workshop. Has the training made a difference in the number of complaints? You collect the following data:

Salesperson	Number of Complaints: Before (1)	After (2)	(2) - (1) Difference, D_i
C.B.	6	4	- 2
T.F.	20	6	-14
M.H.	3	2	- 1
R.K.	0	0	0
M.O.	4	0	- 4
			-21

$$\bar{D} = \frac{\sum D_i}{n}$$

$= -4.2$

$$S_D = \sqrt{\frac{\sum (D_i - \bar{D})^2}{n-1}}$$

$= 5.67$

Paired Difference Test: Solution

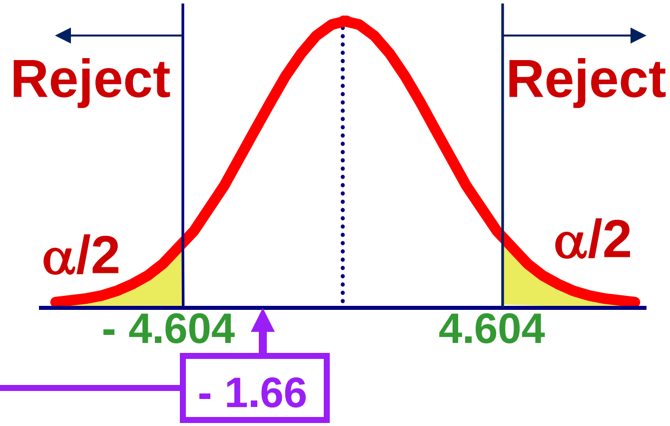
DCOVA

- Has the training made a difference in the number of complaints (at the 0.01 level)?

H_0: $\mu_D = 0$
H_1: $\mu_D \neq 0$

$\alpha = .01 \qquad \bar{D} = -4.2$

$t_{0.005} = \pm 4.604$



Test Statistic:

$$t_{STAT} = \frac{\bar{D} - \mu_D}{S_D/\sqrt{n}} = \frac{-4.2 - 0}{5.67/\sqrt{5}} = \boxed{-1.66}$$

Decision: Do not reject H_0
(t_{stat} is not in the reject region)

Conclusion: There is not a significant change in the number of complaints.

Two Population Proportions

DCOVA

Population proportions

Goal: test a hypothesis or form a confidence interval for the difference between two population proportions,

$$\pi_1 - \pi_2$$

Assumptions:

$n_1 \pi_1 \geq 5$, $n_1(1- \pi_1) \geq 5$

$n_2 \pi_2 \geq 5$, $n_2(1- \pi_2) \geq 5$

The point estimate for the difference is $p_1 - p_2$

Two Population Proportions

DCOVA

Population proportions

In the null hypothesis we assume the null hypothesis is true, so we assume $\pi_1 = \pi_2$ and pool the two sample estimates

The pooled estimate for the overall proportion is:

$$\bar{p} = \frac{X_1 + X_2}{n_1 + n_2}$$

where X_1 and X_2 are the number of items of interest in samples 1 and 2

Two Population Proportions

(continued)

DCOVA

Population proportions

The test statistic for $\pi_1 - \pi_2$ is a Z statistic:

$$Z_{STAT} = \frac{(p_1 - p_2) - (\pi_1 - \pi_2)}{\sqrt{\bar{p}(1-\bar{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where $\bar{p} = \frac{X_1 + X_2}{n_1 + n_2}$, $p_1 = \frac{X_1}{n_1}$, $p_2 = \frac{X_2}{n_2}$

Hypothesis Tests for Two Population Proportions

DCOVA

Population proportions

Lower-tail test:

H_0: $\pi_1 \geq \pi_2$
H_1: $\pi_1 < \pi_2$

i.e.,

H_0: $\pi_1 - \pi_2 \geq 0$
H_1: $\pi_1 - \pi_2 < 0$

Upper-tail test:

H_0: $\pi_1 \leq \pi_2$
H_1: $\pi_1 > \pi_2$

i.e.,

H_0: $\pi_1 - \pi_2 \leq 0$
H_1: $\pi_1 - \pi_2 > 0$

Two-tail test:

H_0: $\pi_1 = \pi_2$
H_1: $\pi_1 \neq \pi_2$

i.e.,

H_0: $\pi_1 - \pi_2 = 0$
H_1: $\pi_1 - \pi_2 \neq 0$

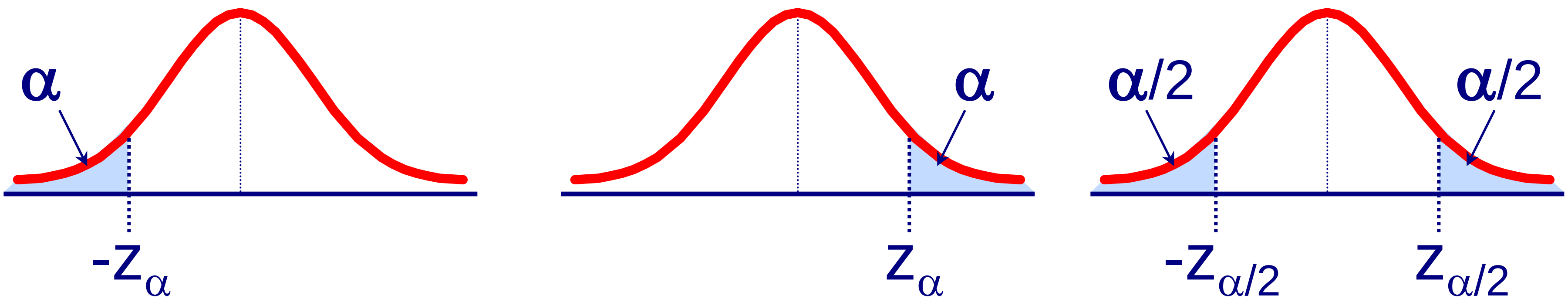
Hypothesis Tests for Two Population Proportions

(continued)

DCOVA

Population proportions

Lower-tail test:	Upper-tail test:	Two-tail test:
$H_0: \pi_1 - \pi_2 \geq 0$ $H_1: \pi_1 - \pi_2 < 0$	$H_0: \pi_1 - \pi_2 \leq 0$ $H_1: \pi_1 - \pi_2 > 0$	$H_0: \pi_1 - \pi_2 = 0$ $H_1: \pi_1 - \pi_2 \neq 0$



Reject H_0 if $Z_{STAT} < -Z_{\alpha}$

Reject H_0 if $Z_{STAT} > Z_{\alpha}$

Reject H_0 if $Z_{STAT} < -Z_{\alpha/2}$ or $Z_{STAT} > Z_{\alpha/2}$

Hypothesis Test Example: Two population Proportions

DCOVA

Is there a significant difference between the proportion of men and the proportion of women who will vote Yes on Proposition A?

- In a random sample, 36 of 72 men and 35 of 50 women indicated they would vote Yes

- Test at the .05 level of significance

Hypothesis Test Example: Two population Proportions

(continued)

DCOVA

- The hypothesis test is:

H_0: $\pi_1 - \pi_2 = 0$ (the two proportions are equal)

H_1: $\pi_1 - \pi_2 \neq 0$ (there is a significant difference between proportions)

- The sample proportions are:
  - Men: $p_1 = 36/72 = 0.50$
  - Women: $p_2 = 35/50 = 0.70$
- The pooled estimate for the overall proportion is:

$$\bar{p} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{36 + 35}{72 + 50} = \frac{71}{122} = .582$$

Hypothesis Test Example: Two population Proportions

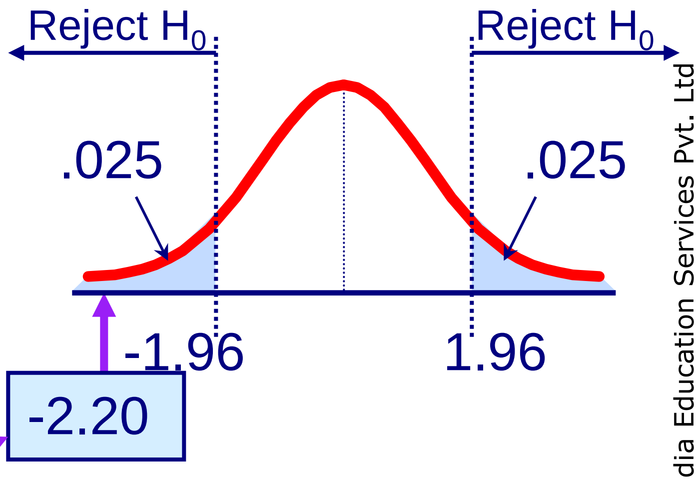
DCOVA

(continued)

The test statistic for $\pi_1 - \pi_2$ is:

$$z_{STAT} = \frac{(p_1 - p_2) - (\pi_1 - \pi_2)}{\sqrt{\bar{p}(1-\bar{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{(.50 - .70) - (0)}{\sqrt{.582(1-.582)\left(\frac{1}{72} + \frac{1}{50}\right)}} = -2.20$$



Critical Values = ±1.96
For α = .05

Decision: Do not reject H_0

Conclusion: There is not significant evidence of a difference in proportions who will vote yes between men and women.

Confidence Interval for Two Population Proportions

DCOVA

Population proportions

The confidence interval for $\pi_1 - \pi_2$ is:

$$\left(p_1 - p_2\right) \pm Z_{\alpha/2} \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

Testing for the Ratio Of Two Population Variances

DCOVA

Tests for Two Population Variances *

F test statistic

Hypotheses	F_{STAT}
$H_0: \sigma_1^2 = \sigma_2^2$ $H_1: \sigma_1^2 \neq \sigma_2^2$	S_1^2 / S_2^2
$H_0: \sigma_1^2 \leq \sigma_2^2$ $H_1: \sigma_1^2 > \sigma_2^2$	

Where:

S_1^2 = Variance of sample 1 (the larger sample variance)

n_1 = sample size of sample 1

S_2^2 = Variance of sample 2 (the smaller sample variance)

n_2 = sample size of sample 2

$n_1 - 1$ = numerator degrees of freedom

$n_2 - 1$ = denominator degrees of freedom

The F Distribution

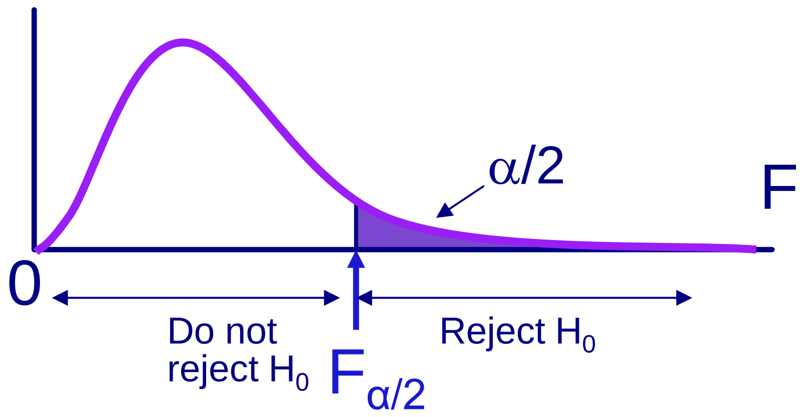
DCOVA

- The F critical value is found from the F table
- There are two degrees of freedom required: numerator and denominator
- The larger sample variance is always the numerator
- When $\boxed{F_{STAT} = \frac{S_1^2}{S_2^2}}$ $df_1 = n_1 - 1$; $df_2 = n_2 - 1$
- In the F table,
  - numerator degrees of freedom determine the column
  - denominator degrees of freedom determine the row

Finding the Rejection Region

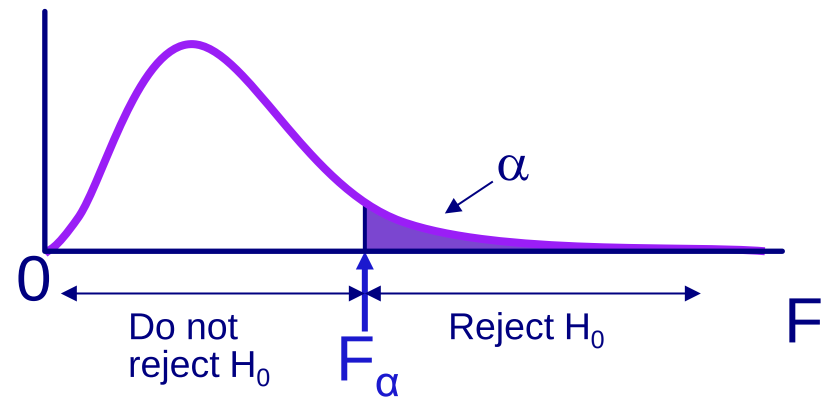
DCOVA

$H_0: \sigma_1^2 = \sigma_2^2$
$H_1: \sigma_1^2 \neq \sigma_2^2$



Reject H_0 if $F_{STAT} > F_{\alpha/2}$

$H_0: \sigma_1^2 \leq \sigma_2^2$
$H_1: \sigma_1^2 > \sigma_2^2$



Reject H_0 if $F_{STAT} > F_{\alpha}$

F Test: An Example

DCOVA

You are a financial analyst for a brokerage firm. You want to compare dividend yields between stocks listed on the NYSE & NASDAQ. You collect the following data:

	NYSE	NASDAQ
Number	**21**	**25**
Mean	**3.27**	**2.53**
Std dev	**1.30**	**1.16**

Is there a difference in the variances between the NYSE & NASDAQ at the $\alpha = 0.05$ level?

F Test: Example Solution

DCOVA

- Form the hypothesis test:

H_0: $\sigma^2_1 = \sigma^2_2$ (there is no difference between variances)

H_1: $\sigma^2_1 \neq \sigma^2_2$ (there is a difference between variances)

- Find the F critical value for $\alpha = 0.05$:

- Numerator d.f. = $n_1 - 1 = 21 - 1 = 20$

- Denominator d.f. = n2 – 1 = 25 –1 = 24

- $F_{\alpha/2} = F_{.025,\ 20,\ 24} = 2.33$

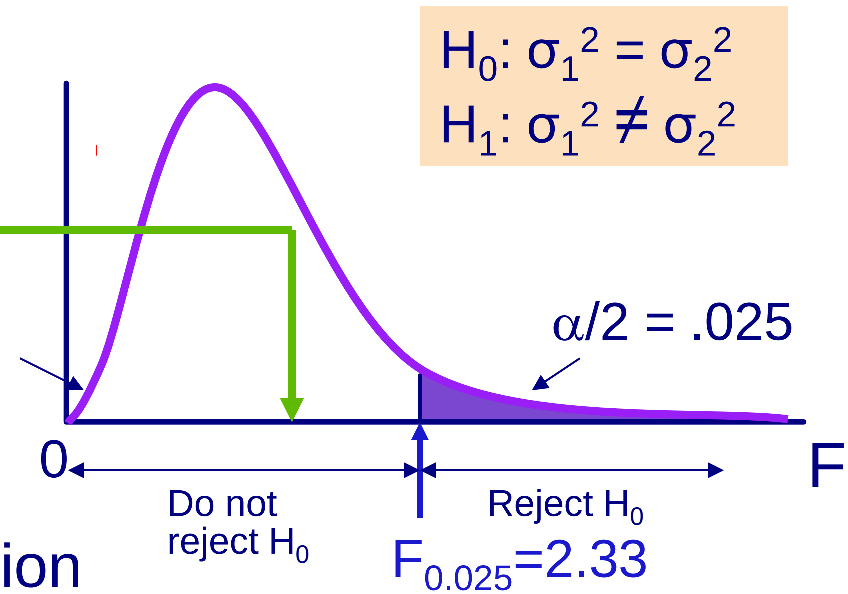
F Test: Example Solution

DCOVA

(continued)

- The test statistic is:

$$F_{STAT} = \frac{S_1^2}{S_2^2} = \frac{1.30^2}{1.16^2} = \boxed{1.256}$$



- F_{STAT} = 1.256 is not in the rejection region, so we do not reject H_0
- Conclusion: There is not sufficient evidence of a difference in variances at α = .05

Chapter Summary

In this chapter we discussed

- Comparing two independent samples
  - Performed pooled-variance t test for the difference in two means
  - Performed separate-variance t test for difference in two means
  - Formed confidence intervals for the difference between two means
- Comparing two related samples (paired samples)
  - Performed paired t test for the mean difference
  - Formed confidence intervals for the mean difference

Chapter Summary

(continued)

- Comparing two population proportions
  - Performed Z-test for two population proportions
  - Formed confidence intervals for the difference between two population proportions
- Performing an F test for the ratio of two population variances