1 Definitions

When dealing with expected value, we define a variable that randomly takes on values as a **random variable**. Random variables are usually denoted by capital letters (X, Y, Z, etc.). For example, define D_6 to be a random variable that is the outcome of a dice roll. Also, define $\mathbb{P}$, or probability, to be $\frac{SuccessfulOutcomes}{TotalOutcomes}$.
Example 1: What is the probability that exactly 5 coin tosses yields 3 heads?
Example 2: What is the probability that a number leaves a remainder of 2, 3, or 8 when divided by 7?

We then define expected value to be a "weighted average" value. The formula (for now) is

$$E[AEvent] = \mathbb{P}(SomethingHappeningInThatEvent) \cdot WeightofEvent$$

Let's do examples to illustrate this.
Example 3: What is the expected value of a dice roll?

Solution: $\mathbb{E}[D_6] = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 \ldots + \frac{1}{6} \cdot 6 = 3.5$.

2 Properties of Expected Value

2.1 A Classical Example

At the annual Planetary Mathematical Conference (PMC), a very smart mathmatician notices that there willbe precisely n people going to the meeting, where n is an integer. He thus makes n nametags with the names of all people attending the conference. Of course, since he is book smart, this mathematician is not very street smart and accidentally mixes the name tags up. He, thus, randomly hands the nametags to the participants of the conference as they walk in through the main door. Find the expected value of the number of nametags that go to the right person.

Solution: Define S to be the random variable representing the number of people who receive their own name tag and call those who received their own tag as "fixed points." Thus, S is the number of fixed points. We want to find $\mathbb{E}[\mathbb{S}]$. Illustrating for 4 people (W,X,Y,Z), we see the figrue to the right. Looking at the rightmost column, there doesn't seem to be a pattern. However, summing colums, we see that there are 6 fixed points in each column! This can be seen as (n-1)!. Thus,

$$\mathbb{E}[\mathbb{S}] = \frac{1}{n!}((n-1)! + (n-1)! + \ldots + (n-1)!(ntimes)) = \frac{1}{n!}n(n-1)! = 1$$

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	W	X	Y	Z	Σ
1	**W**	**X**	**Y**	**Z**	4
2	**W**	**X**	Z	Y	2
3	**W**	Y	X	**Z**	2
4	**W**	Y	Z	X	1
5	**W**	Z	X	Y	1
6	**W**	Z	**Y**	X	2
7	X	W	**Y**	**Z**	2
8	X	W	Z	Y	0
9	X	Y	W	**Z**	1
10	X	Y	Z	W	0
11	X	Z	W	Y	0
12	X	Z	**Y**	W	1
13	Y	W	X	**Z**	1
14	Y	W	Z	X	0
15	Y	**X**	W	**Z**	2
16	Y	**X**	Z	W	1
17	Y	Z	W	X	0
18	Y	Z	X	W	0
19	Z	W	X	Y	0
20	Z	W	**Y**	X	1
21	Z	**X**	W	Y	1
22	Z	**X**	**Y**	W	2
23	Z	Y	W	X	0
24	Z	Y	X	W	0
Σ	6	6	6	6	24

2.2 Linearity of Expectation

Linearity of expectation basically says that the expected value of a sum of random variables is equal to the sum of the individual expectations. This can be neatly summarized in the following expression:
$E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$
This theorem is obvious if the $X_1 + X_2 + \cdots + X_n$ are independent of each other if I roll 100 dice, I expect an average of 350. The cool thing is this remains true even if the variables are not independent. The proof for this invoves double counting; the idea is that even if the variables depend on each other, if you look only at the expected value, you can still add just by columns.

(We will not discuss the proof today due to time constraints but
feel free to look it up at home.)

Example 1: At a nursery, 2006 babies sit in a circle. Suddenly, each baby randomly pokes either the baby to its left or to its right. What is the expected value of the number of unpoked babies?

Solution 1: We want to find $E[X_1+X_2+\cdots+X_{2006}]$ which we know to be equal to $E[X_1]+E[X_2]+\cdots+E[X_{2006}]$. The $E[X_i]$ for any number i is $(\frac{1}{2})^2 = \frac{1}{4}$. We now see that the answer is simply $2006 \times \frac{1}{4} = \frac{1003}{2}$.

3 The Probability Mass Function

The range of a random variable X is any of the values that X can be equal to. For example, the range of a die roll is 1-6.
The function $f_x(k) = P(X = k)$ is called the probability mass function (PMF). It basically calculates the probability that a random variable X will equal a number k. FOr example, we know that

$$\mathbb{P}(D_6 = 1) = \mathbb{P}(D_6 = 2)... = \mathbb{P}(D_6 = 6) = \frac{1}{6}$$

Thus, redefining expected value, we have

$$\mathbb{E}[\mathbb{X}] = \sum_x \mathbb{P}(X = x) \cdot x$$

Example 2: I flip a coin 10 times. Find the range of X. What is the PMF of this example?

Example 3: Xavier and Yuri each pick a number from 1, 2, 3 at random (they could both pick the same number). Let X = Xaviers number and Y = Yuris number. Find the PMF of X and the PMF of Y.

Example 4: Using the last problem, define new random variables Z = X + Y and W = —X Y—. What do Z and W represent? Find the PMF of Z and of W.

Example 5: A restaurant sells meals priced 4, 6, and 8 dollars. They sell the 4 dollar meal twice as often as the 8 dollar meal and 3 times as often as the 6 dollar meal. What is the expected value of any given meal?

Example 6: What is the expected number of face cards (jack, queen, or king) in a three card hand drawn at random from a standard deck of cards?

Example 7: Suppose you have a coin with P (heads) = p. Flip the coin n times and let X = the number of heads. What is the PMF of X? What is the expected value of X?

Solution: Let Xi =the number of heads on the ith flip. Then X = X1 ++Xn so EX = EX1 ++EXn =p+p=np.

4 Problems

Problem 1: Suppose you have a coin with P (heads) = p. Flip the coin until you get a heads. Let X =the number of flips.
a) Find the PMF of X.
b) Find E(x)
Solution 1:
Problem 2: Johann has 64 fair coins. He flips all the coins. Any coin that lands on tails is tossed again. Coins

that land on tails on the second toss are tossed a third time. What is the expected number of coins that are now heads? (Source: AMC 10)
Solution 2: The only coins which won't be heads at the end of the process are the ones that came up tails all three times. This happens with probability $\frac{1}{8}$, and on average we expect to see $\frac{64}{8} = 8$ of these. Therefore our answer is $64 - 8 = 56$.

Problem 3: Suppose I have a bag with 12 slips of paper in it. Some of the slips have a 2 on them, and the rest have 7 on them. If the expected value of the number shown on a randomly drawn slip is 3.25, then how many slips have a 2.
Solution 3: 9 slips

Problem 4: You start with a full deck of cards, which have been shuffled. You draw cards from the deck, without replacement, until you get a card other than an ace. What is the expected value of the number of cards drawn?
Solution 4: So let W be the number of draws before the first Ace, not including the draw that got us the Ace. We want E(W).
Define random variable Xi by Xi=1 if the card with label i was drawn before any Ace, and let Xi=0 otherwise. Then $W = X1 + X2 + \cdots + X48$ By the linearity of expectation, which holds even when the random variables are not independent, we have E(W)=E(X1+X2++X48)=E(X1)+E(X2)++E(X48). By symmetry, all the Xi have the same distribution. We find, for example, the probability that X1=1. So we want the probability that card with label 1 is drawn before any Ace. Consider the 5-card collection consisting of the 4 Aces and the card labelled 1. All orders of these cards in the deck are equally likely. It follows that the probability that card with label 1 is in front of the 4 Aces is 15. Thus E(X1)=15 We conclude that E(W)=48/5 .

Problem 5: Suppose that 7 boys and 13 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row $GBBGGGBGBGGGBGBGGBGG$ we have $S = 12$. Find the expected value of S.
Solution 5: In a given pair, the probability that there is a boy next to a girl is 7/20 * 13/19 * 2. Then, because there are 19 pairs of kids next to each other in a row of 20 people, we multiply by 19 and get 91/10 = 9.1.

Problem 6: Two integers, x and y, are chosen uniformly between 0 and 1. WHat is the expected value of their difference?
Solution: Let the first chosen number be p. Then, the expected value for the positive difference betweem the first and second numbers is $p^2/2 + (1-p)^2/2$ where the first formula comes from the second number is less than p and the second number comes from the second number is greater than p. From $0 \le p \le 1$, p^2 and $(1-p)^2$ are symmetrical, so this is the equivaklent to finding the average value of p^2 as p ranges from 0 to $1 = \frac{1}{3}$.
Solution 2: Consider geometric probability, the expected value is equal to the volume of a pyramid with base 1/2 (the probability that the second number is greater than the first) and height 1. Thus, the volume is 1/6, and the expected avlue is 1/6*2=1/3.

Problem 7: Two integers x and y (possibly equal) are chosen at random such that $1 \le x \le 5$ and $1 \le y \le 5$. What is the expected value of the area of the triangle with vertices $(0,0)$,$(x,0)$ and $(0,y)$.
Solution 7: The area of the triangle will be equal to xy. Thus, since all values between 1 and 5 are equally likely for x and y, the expected value of both x and y is 3, so the expected value of the area is 9/2.

Problem 8: A treasure of gold worth $\$100,000$ gets stolen by the Pleasantly Magnanimous Criminals (PMC). In order to find the missing gold, Preetham, a wealthy tycoon, commissions helicopters to search for the treasure. If each helicoper costs $\$1,000$ and has a 90% of finding the treasure, how many helicopters should Preetham hire?
Keep in mind that Preetham is very money-minded and will only hire the number of helicopters that theoretically maximizes his profit margin.
Solution 8: The expected profit with one helicopter is .9*100,000 - 1000 = 89,000. With two helicopters, .9*100,000 + .1*.9*100,000 - 2000 = 97,000. With three, we realize that .1*.1*.9*100,000 is less than 1000 (the expense of the additional helicopter) so the profit is no longer going up (it's now at 96,900). Therefore, Preetham should hire 2 helicopters.