1 Barycentric Coordinates

2 Mass Points

Mass points is a specific form of barycentric coordinates. It is relatively easy, yet powerful technique that simplifies calculations for finding ratios in a triangle. It is a simple as trying to balance a seesaw, the seesaw being the side lengths of the triangle.

2.1 A Little Physics...

The idea of Mass points uses basic principles of physics and equilibrium. TO understand how mass poitns works, we will use a simple lever. Consider an ant and an elephant. They are trying to sit on a seesaw such that the seesaw remains horizontal. Where would the ant and the elephant have to sit to make this possible? Let's learn some physics!
Force: A force is simply a push or pull. It is used to speed up objects or slow them down. The SI unit of force is Newtons (N).

Torque: Torque is the counterpart of force that deals with rotation. Torque speeds up or slows down the rotation. The formula for Torque is $\tau = FL$ Where L is the perpendicular distance from the pivot point (the fulcrum in this case) and the force.
For a lever to be balanced, you need the torques to cancel out.

Example 1: In order for an elephant weighing 1000N to balance a ant weighing 1N, where do the ant and the elephant have to sit? Assume there is no friction between the fulcrum and the lever and assume the lever is massless.

Example 2: You are given a lever with an elephant and an ant which are 20 and 10000 feet away from the fulcrum respectively. What are the weights of these animals?

2.2 Applying the Physics to Triangles

Let's try to apply this common knowledge to a triangle. Consider the triangle ABC. Let P be a point on side AB. If side length $AP = 2$ and $PB = 6$ is a lever and there are weights on points A and B, what is the ratio of these weights?

As it turns out, the ratio of masses of $A : B{=}BP/AP$. Notice that given the weights, we can solve for the ratios easily.

The problem with applying this approach continuously is that you need to relate these ratios to other ratios throughout the triangle. Consider triangle ABC with points P on AB and Q on AC. Let $AP = 2$, $BP = 6$, $AQ = 2$ and $CQ = 10$. What is a logical mass A should have?

Definition 1: Mass Point Define a mass point to be a pair consisting of a "mass" and a point. We denote this as (m, P) or mP.

Definition 2: Addition Define addition of two mass points mP and nQ to be $(n + m)R$ where R is on PQ and $PR : RQ = n : m$.

Example 3: (2016 AMC 12A) n $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Point D lies on $\overline{BC}$, and $\overline{AD}$ bisects $\angle BAC$. Point E lies on $\overline{AC}$, and $\overline{BE}$ bisects $\angle ABC$. The bisectors intersect at F. What is the ratio $AF : FD$?

2.3 Split Masses

At this point, we can easily solve problems where there are just cevians, or lines from the vertex to a point on the opposite side. What if a line does not pass through the vertex? Consider a triangle ABC. Let Q be on BC, X on AB, and Y on BC.
Definition: If A_1 is the mass assinged of A when considering the ratio of $XB : XA$ and A_2 is the mass assinged of A when considering the ratio of $XC : XA$, $A = A_1 + A_2$.

3 Problems

Problem 1: In triangle ABC, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?

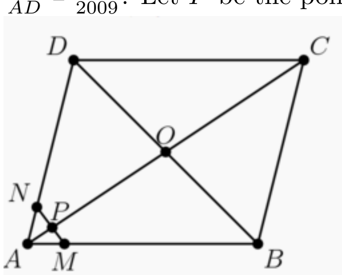
Solution: Let us use mass points: Assign B mass 1. Thus, because E is the midpoint of AB, A also has a mass of 1. Similarly, C has a mass of 1. D and E each have a mass of 2 because they are between B and C and A and B respectively. Note that the mass of D is twice the mass of A, so AP must be twice as long as PD. PD has length 2, so AP has length 4 and AD has length 6. Similarly, CP is twice PE and $PE = 1.5$, so $CP = 3$ and $CE = 4.5$. Now note that triangle PED is a $3-4-5$ right triangle with the right angle DPE. The area of a kite is half the product of the diagonals, AD and CE. Recall that they are 6 and 4.5 respectively, so the area of $AEDC$ is $6 * 4.5/2 = \boxed{\textbf{(B)}13.5}$

Problem 2: In $\triangle ABC$ points D and E lie on BC and AC, respectively. If AD and BE intersect at T so that $AT/DT = 3$ and $BT/ET = 4$, what is CD/BD?

Solution: As per the problem, we assign a mass of 1 to point A, and a mass of 3 to D. Then, to balance A and D on T, T has a mass of 4.
Now, were we to assign a mass of 1 to B and a mass of 4 to E, we'd have $5T$. Scaling this down by 4/5 (to get $4T$, which puts B and E in terms of the masses of A and D), we assign a mass of $\frac{4}{5}$ to B and a mass of $\frac{16}{5}$ to E.
Now, to balance A and C on E, we must give C a mass of $\frac{16}{5} - 1 = \frac{11}{5}$.

Finally, the ratio of CD to BD is given by the ratio of the mass of B to the mass of C, which is $\frac{4}{5} \cdot \frac{5}{11} = \boxed{\textbf{(D)}\ \frac{4}{11}}$.

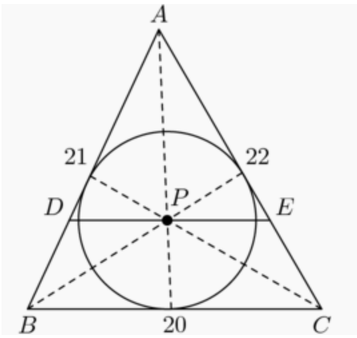
Problem 3: In parallelogram $ABCD$, point M is on $\overline{AB}$ so that $\frac{AM}{AB} = \frac{17}{1000}$ and point N is on $\overline{AD}$ so that $\frac{AN}{AD} = \frac{17}{2009}$. Let P be the point of intersection of $\overline{AC}$ and $\overline{MN}$. Find $\frac{AC}{AP}$.



Solution: We approach the problem using mass points on triangle ABD as displayed below. But as MN does not protrude from a vertex, we will have to "split the mass" at point A. First, we know that DO is congruent to BO because diagonals of parallelograms bisect each other. Therefore, we can assign equal masses to points B and D. In this case, we assign B and D a mass of 17 each. Now we split the mass at A, so we balance segments AB and AD separately, and then the mass of A is the sum of those masses. A mass of 983 is required to balance segment AB, while a mass of 1992 is required to balance segment AD. Therefore, A has a mass of $1992 + 983 = 2975$. Also, O has a mass of 34. Therefore, $\frac{AO}{AP} = \frac{2975+34}{34} = \frac{3009}{34}$, so $\frac{AC}{AP} = \frac{2(3009)}{34} = 177$.

Problem 4: Triangle ABC has $AB = 21$, $AC = 22$ and $BC = 20$. Points D and E are located on $\overline{AB}$ and

$\overline{AC}$, respectively, such that $\overline{DE}$ is parallel to $\overline{BC}$ and contains the center of the inscribed circle of triangle ABC. Then $DE = m/n$, where m and n are relatively prime positive integers. Find $m + n$.



Solution: Let P be the incircle; then it is be the intersection of all three angle bisectors. Draw the bisector AP to where it intersects BC, and name the intersection F.
Using the angle bisector theorem, we know the ratio $BF : CF$ is $21 : 22$, thus we shall assign a weight of 22 to point B and a weight of 21 to point C, giving F a weight of 43. In the same manner, using another bisector, we find that A has a weight of 20. So, now we know P has a weight of 63, and the ratio of $FP : PA$ is $20 : 43$. Therefore, the smaller similar triangle ADE is 43/63 the height of the original triangle ABC. So, DE is 43/63 the size of BC. Multiplying this ratio by the length of BC, we find DE is $860/63 = m/n$. Therefore, $m + n = \boxed{923}$.

Problem 5: Let P be an interior point of triangle ABC and extend lines from the vertices through P to the opposite sides. Let a, b, c, and d denote the lengths of the segments indicated in the figure. Find the product abc if $a + b + c = 43$ and $d = 3$.

Solution: Let the labels A, B, C be the weights of the vertices. First off, replace d with 3. We see that the weights of the feet of the cevians are $A + B, B + C, C + A$. By mass points, we have that:

$$\frac{a}{3} = \frac{B + C}{A}$$
$$\frac{b}{3} = \frac{C + A}{B}$$
$$\frac{c}{3} = \frac{A + B}{C}$$

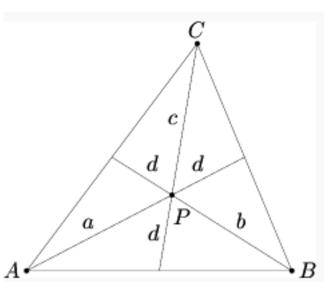
If we add the equations together, we get

$$\frac{a + b + c}{3} = \frac{A^2B + A^2C + B^2A + B^2C + C^2A + C^2B}{ABC} = \frac{43}{3}$$

If we multiply them together, we get

$$\frac{abc}{27} = \frac{A^2B + A^2C + B^2A + B^2C + C^2A + C^2B + 2ABC}{ABC} = \frac{43}{3} + 2 = \frac{49}{3}$$

Multiplying both sides by 27, we get that $abc = 27 \cdot \frac{49}{3} = \boxed{441}$.

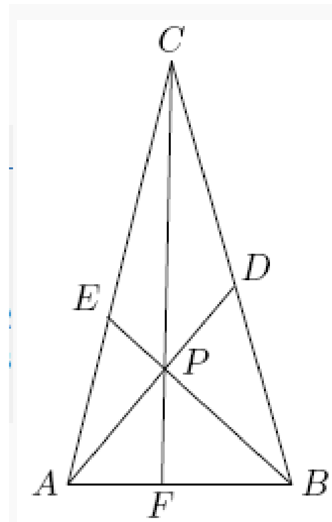


Problem 6: In triangle ABC, A', B', and C' are on the sides BC, AC, and AB, respectively. Given that

AA', BB', and CC' are concurrent at the point O, and that $\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92$, find $\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}$.

Solution: Using mass points, let the weights of A, B, and C be a, b, and c respectively.
Then, the weights of A', B', and C' are $b+c$, $c+a$, and $a+b$ respectively.
Thus, $\frac{AO}{OA'} = \frac{b+c}{a}$, $\frac{BO}{OB'} = \frac{c+a}{b}$, and $\frac{CO}{OC'} = \frac{a+b}{c}$.
Therefore: $\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'} = \frac{b+c}{a} \cdot \frac{c+a}{b} \cdot \frac{a+b}{c} = \frac{2abc+b^2c+bc^2+c^2a+ca^2+a^2b+ab^2}{abc} =$
$2 + \frac{bc(b+c)}{abc} + \frac{ca(c+a)}{abc} + \frac{ab(a+b)}{abc} = 2 + \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} = 2 + \frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 2 + 92 = \boxed{094}$.

Problem 7: Point P is inside $\triangle ABC$. Line segments APD, BPE, and CPF are drawn with D on BC, E on AC, and F on AB (see the figure below). Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of $\triangle ABC$.



Solution: Because we're given three concurrent cevians and their lengths, it seems very tempting to apply Mass points. We immediately see that $w_E = 3$, $w_B = 1$, and $w_A = w_D = 2$. Now, we recall that the masses on the three sides of the triangle must be balanced out, so $w_C = 1$ and $w_F = 3$. Thus, $CP = 15$ and $PF = 5$.
Recalling that $w_C = w_B = 1$, we see that $DC = DB$ and DP is a median to BC in $\triangle BCP$. Applying Stewart's Theorem, $BC^2 + 12^2 = 2(15^2 + 9^2)$, and $BC = 6\sqrt{13}$. Now notice that $2[BCP] = [ABC]$, because both triangles share the same base and the $h_{\triangle ABC} = 2h_{\triangle BCP}$. Applying Heron's formula on triangle BCP with sides 15, 9, and $6\sqrt{13}$, $[BCP] = 54$ and $[ABC] = \boxed{108}$.