1 Theory

The basic pigeonhole principle states that if n items, or pigeons, are put into m pigeonholes with $n > m$, then at least one pigeonhole must contain more than one pigeon. If we draw a quick chart, we can see how this works.

Exercise 1: I have red, white, and blue socks in a drawer. What is the minimum number of socks I need to take out of the drawer in order to guarantee that I have at least one pair?

Solution 1: The answer is 4.

Exercise 2: Show that in a list of five numbers, there is at least one pair of numbers whose difference is divisible by four.

Solution 2: When you divide a number by four, you can have four different remainders: 0, 1, 2, and 3. We have five numbers, so the pigeons are four remainders and the holes are the five numbers that are divided. This proves that at least two numbers have the same remainder, and when they have the same remainder, their difference is divisible by four.

Exercise 3: Using the previous exercise, prove that in a list of n numbers, there is at least one pair of numbers whose difference is divisible by n - 1.

Solution: The holes are all possible remainders $\mod n-1$.Since there are n numbers, two of them are the same $\mod n-1$. Therefore, by subtracting them, they are congruent to $0 \mod n-1$

Exercise 4: If there are twelve different 2 digit numbers, show that one can choose two of them so that their difference is a two-digit number whose first and second digits are the same.

Solution 4: Apply the answer to exercise 3.

Exercise 5: Suppose that 5 points lie on a sphere. Prove that there exists a closed semi-sphere (half a sphere including boundary), which contains 4 of the points.

Solution 5: Consider the great circle through any two of the points. This partitions the sphere into two hemispheres. By the pigeonhole principle, 2 of the remaining 3 points must lie in one of the hemispheres. These two points, along with the original two points, lie in a closed semi-sphere.

The general pigeonhole principle states that with n pigeons and k pigeonholes, with $n > k$, there must be at least n/k pigeons in at least one of the pigeonholes.

Exercise 6: Prove the general pigeonhole principle.

Solution 6: We will prove by contradiction. Prove that in k holes, there are less than n/k pigeons. Go from there.

2 Problems

Problem 1: If each square of a 3-by-7 chessboard is colored either black or white, then the board must contain a rectangle consisting of at least four squares whose corner squares are either all white or all black. Show that if the grid is only 3-by-6, there are colorings for which the conclusion fails.

Solution 1: We examine the columns of the grid. If one column is all white, then well be done if any other column has at least two white squares. But there are only four possible columns with one or no white squares. Therefore, either we must include a column that has at least two white squares, or we have to duplicate a column. If the same pattern is duplicated, then these two columns have two squares each on the same color. If there is an all black column, the reasoning can be duplicated. Therefore we need only consider the case when there is no entirely black column and no entirely white column. In this case, there are just six possible patterns, BBW, BWB, WBB, BWW, WBW, and WWB. But there are seven columns to color, so one of these patterns must be duplicated. We saw above that when a pattern is duplicated, a rectangle can be found all of whose are the same color.

Problem 2: Prove that among any set of 51 positive integers less than 100, there is a pair whose sum is 100.

Solution 2: Let $a_1, a_2, ..., a_{51}$ denote these 51 positive integers, Let $S_1 = (1, 99), S_2 = (2, 98), S_3 = (3, 97), ..., S_k = (k, 100k), ..., S_{49} = (49, 51)$, and $S_{50} = (50)$. Let $S_1, S_2, ..., S_{50}$ denote the pigeonholes, and $a_1, a_2, ..., a_{51}$ the pigeons. By PHP, we must have at least one S_i with two numbers in it. Their sum is a 100.

Problem 3: Given 8 integers between 1 and 100, prove that it is possible to choose two whose ratio lies between 1 and 2 (including 2).

Solution 3: Prove by contradiction, so that there exist 8 integers, and no pair of them has a ratio between 1 and 2. We try to make a sequence where the ratios are always greater than 2, but this quickly fails because $2^7 > 100$, leaving only 7 holes below 100. There are 8 integers (pigeons), so there will be at least 2 integers in at least one range that gives them a ratio between 1 and 2.

Problem 4: Prove that if you choose more than n numbers from 1 to $2n$, at least one is a multiple of another.

Solution 4: In the set $1, 2, ..., 2n$ there are n even and n odd integers. Let A consist of at least $n + 1$ numbers from that set: $|A| > n$. Every integer is a product of a power of 2 and an odd integer. Remove the powers of 2 from the members of A. The resulting set B consists of odd integers and, in addition, $|B| = |A| > n$. The terms of B are among the n odd members of the set $1, 2, ..., 2n$, meaning that some two of them must be equal. Of the corresponding terms of A, the smaller divides the larger, for the two are in the form $2^k b$, $2^m b$, with b odd and $k \neq m$.

Problem 5: Assume that 101 distinct points are placed in a 10 x 10 square. Find upper bound for the smallest distance between a given point with any other point.

Solution: Divide the rectangle into 100 1x1 squares. Using pigeonhole, there must be a square that has two points. Thus, the maximum distance is when the two points are on diagonals.

Problem 6: Assume that 101 distinct points are placed in a 10 x 10 square such that no three of them lie on a line. Prove that we can choose three of the given points that form a triangle whose area is at most 1.

Solution 1: Pigeonhole: Divide the large square into 50 2*1 rectangles. By pigeonhole, one rectangle has 3 points. We can easily see that any three points in this rectangle will form a triangle of maximum area one (since rectangle is 2 by 1).

Solution 2: Trigonometry The area of a triangle is $\frac{1}{2}ab\sin C$. This has to be less than or equal to 1. since $\sin C$ is maximized at 1, set it equal to one. You now have $ab \leq 1$. You know one of a or b must be $\sqrt{2}$ from the previous problem. Also, you assumed that this happens when two points are on diagonals of a unit square. Thus, since there is one point per square, there must be another point at max $\sqrt{2}$ away. This shows the maximum of ab is $\sqrt{2} * \sqrt{2}$ or 2, which satistfies our condition.

Problem 7: Forty-one Rooks are placed on a 10 x 10 chessboard. Prove that there must exist five rooks, none of which attack each other. (Note: Rooks move vertically and horizontally on the board)

Solution: We start by noting that there are 10 rows on a chessboard. With 41 rooks, we see that there must be one row with at least$\lceil 41/10 \rceil$, or 5 rooks. Since this row may have anywhere from 5 to 10 rooks, we move on to analyzing the next 31 rooks. With 9 rows left, and at least 31 rooks, there must be some second row with $\lceil 31/9 \rceil$, or 4 rooks. Again, since this row may have up to 10 rooks, we continue to analyze the remaining 21 (at least) rooks. With 8 rows and 21 rooks, there must be a third row with $\lceil 21/8 \rceil$, or 3 rooks. Continuing in this fashion, we find there must be a fourth row with at least 2 rooks, and a fifth row with at least 1 rook.

Compare the row with 1 rook and the row with 2 rooks. There must be a rook in the second row which doesn't share a column with the sole rook in the first row. Thus, these two rooks cannot attack each other. Keep these two rooks in mind. Moving on to the third row, which has 3 rooks, there must be a rook in this row which doesn't share a column with either of the two rooks we previously discussed, the sole rook in the first row and the rook in the second row. As a result, these three rooks cannot attack each other. We use this same tactic on the fourth row, which has 4 rooks, to show that there must also be such a rook in this row, and our group of pacifist rooks has grown to 4. Finally, there must be a rook in the fifth row (5 rooks in this one), which doesn't share a column with any of the rooks in our group of 4 pacifist rooks. Thus, we have shown that there must be 5 rooks that do not attack each other in any arrangement of 41 rooks on a 10x10 chessboard.