Math Riddle of the Day: If 9999 = 4, 8888 = 8, 1816 = 6, 1212 = 0, then what does 1919 equal?

1 Theory

What is Induction? Induction is a method to prove statements that are usually true for all natural numbers. Induction works by first, proving that P(1) is true, and then proving the statement, "If P(n) is true, P(n+1) is also true."
To understand why this works, we can use the analogy of dominoes. To prove that a line of dominoes will all fall when we push the first one, we just have to prove that:

1. The first domino falls down (base case)
2. The dominoes are close enough that each domino will knock over the next one when it falls (step)

2 Examples:

Example 1: Using induction, prove that $1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}$

Solution: Prove that P(1) is true. $1 = 1(2)/2 = 1$
Therefore, P(1) holds.

Prove that if P(n) is true, then P(n+1) is true.
If $1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}$, then $1 + 2 + 3 + 4 + ... + n + (n + 1) = \frac{(n+1)(n+2)}{2}$. Plugging the righthand side of P(n) into the lefthand side of P(n+1), we get $\frac{n(n+1)}{2} + n + 1 = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = n(n + 1) + \frac{2(n+1))}{2} = \frac{(n+2)(n+1)}{2}$, which is equal to the righthand side of P(n+1), so P(n+1) holds.

Example 2: Given n circles in a plane, prove that we can color the regions determined by them with two colors s.t. any two neighboring regions have different colors.

Solution: The case of one circle is trivial. Assume that we can color n circles. With $n + 1$ circle, we can flip all of the colors inside the new circle.

Example 3: Prove Fermat's Little Theorem, or $a^p \equiv a \mod p$. Prove the base case for a = 1: $1^p \equiv 1 \mod p$ is true because $1^d = 1$ for any positive integer d.

Solution: Assume $a^p \equiv a \mod p$. We can induct on a. We are trying to prove $(a + 1)^p \equiv a + 1 \mod p$.
$(a + 1)^p \equiv \binom{a}{0}a^p + \binom{a}{1}a^{p-1}...\binom{a}{a} + 1 \mod p$. Any prime choose a number for 0 to p-1 is always divisible by the prime. So, this is congruent to $a^p + 1 \equiv a \mod p$.

Example 4 In the game Survivor, people have pebbles. They can add either 1,2,3, or 4 pebbles gto the pile. The person places the 21st pebble in the pile loses. Prove by induction that all multiples of 5 are P-positions. *A P-position is a position in which the previous player will win (who moved to that position) and a N-position is a position where the next player will win (who moves away from that position.

Solution: First, notice that 20 is a P-position. Thus, we can do backward's induction, which is our base case. Since we want to prove the P positions are all 0 (mod 5), we want to show that if $5n$ is a P-position, $5n - 5$ is also a P-position. This is the same thing as proving that with $5n$ pebbles, you can obtain $5n + 5$ pebbles. Assume that the player plays x pebbles. Then, you can play (-x (mod 5)) pebbles (since it is gauranteed that it is legal to play (-x (mod 5)) marbles. Thus, all multiples of 5 are P-positions.

3 Problems:

Problem 1: Using induction, prove that $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{(n)(n+1)(2n+1)}{6}$ for all natural numbers n.

Solution: Base: n=1. 1=1*(1+1)/2. Step: n=k. 1+2+3...+k=$\frac{k(k+1)}{2}$. 1+2+3...+k+k+1=$\frac{k(k+1)}{2}$+k+1=$\frac{k(k+1)+2*(k+1)}{2}$ =$\frac{(k+1)(k+2)}{2}$

Problem 2: $5^n - 1$ is divisible by 4.

Solution: If $n = 1$, we know that 4 is divisible by 4. If $n = k$, we want to prove $n = k + 1$ to be true. Since we know $5^k - 1$ is divisible by 4, we know $5^{k+1} - 5$ is divisible by 4. Since we know that this is divisible by four, $5^{k+1} - 5 + 4$ is also divisible by 4, and this, $n = k + 1$ works.

Problem 3: Show that an isosceles triangle triangle with one angle of 120 can be partitioned into n triangles similar to it.

Solution: Draw diagrams for $n = 4, 5, 6$. Then, we can always fraw an isosceles triange in an isosceles triangle going to state n to $n + 3$.

Problem 4: Prove that every positive integer can be written in infinitely many ways in the form:
$n = \pm 1^2 \pm 2^2 \pm ... \pm m^2$

Solution: We induct on the number of terms. For the base case, we use the number 1, 2, 3, and 4. $1^2 = 1$. $-1^2 - 2^2 - 3^2 + 4^2 = 2$. $-1^2 + 2^2 = 3$. $-1^2 - 2^2 + 3^2 = 4. Let$ f(x) $=$ (m+1)$^2 - m^2 = 2m + 1$ for all n. Then for any a, $f(a + 2) - f(a) = 2(a + 2) - 2(a) = 4$. So, $f(a + 6) - f(a + 4) - f(a + 2) + f(a) = 0$. We can always add $f(a + 6) - f(a + 4) - f(a + 2) + f(a)$ to 1, 2, 3, and 4 for a values that are multiples of 8, resulting in infinity ways. Now, we can always add $f(a + 2) - f(a) = 4$ to the 4 base cases, completing the induction.

Problem 5: $2n$ dots are placed around the outside of the circle. n of them are colored red and the remaining n are colored blue. Going around the circle clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots, you consider it a successful trip around the circle. Prove that no matter how the dots are colored red and blue, it is possible to have a successful trip around the circle if you start at the correct point.

Solution: Use the base case of $n = 1$. We can start on the red dot. Then, number of red dots is at least the number of blue dots. Use backwards induction. Assume that the case with $2n$ dots works. There are two possible pairs of adjacent colors: red red, blue blue, red blue, or blue red.If the pair is red blue, we can take it out, transitioning from the n case to the $n - 1$ case. We just have to prove that there exists a red blue pair. Note that for every red blue pair, there is a blue red pair. Also, there is at least one red blue or blue red pair because there are two different colors. As a result, there is at least one red blue pair, which completes our induction.

Problem 6: On a circular route, there are n identical cars. Together, they have enough gas to make a complete tour. Prove that there is a car that can make a complete tour by taking the gas from all the cars that it encounters.

Solution: If $n = 1$, there is a single car with all of the gas, so we are done.

Assume the statement of the problem is true for $n = k$ cars, and consider a situation with $k + 1$ cars on the loop. If we can find a particular car $C1$ that has enough gas to make it to the next car $C2$ then the situation is equivalent to transferring all of $C2$s gas to $C1$ and eliminating $C2$ from the track. With one car eliminated, we know that there is a car that can complete the circuit from the induction hypothesis.

Assume, for contradiction, that no car has enough gas to make it to the next one. Then if each car drove forward as far as it could, there would be a gap in front of each one, so the total amount of gas would not be sufficient for one car to make it all the way around the track. Thus there must be a car that can make it to the next, and the proof is complete.

Problem 7: Prove, by Mathematical Induction, that n(n + 1)(n + 2)(n + 3) is divisible by 24, for all natural numbers n.

Solution: We can split this problem into 3 parts.
Firstly, let's prove that n(n + 1) is divisible by 2! =2. Base case: 12 = 2 is divisible by 2. Step: (k + 1)(k + 2) = k(k + 1) + 2(k + 1) = 2a + 2(k + 1) , by (1)= 2 [a + k + 1], which is divisible by 2.
Next, Let's prove that n(n + 1)(n + 2) is divisible by 3! = 6. Base: 123 = 6 is divisible by 3. Step: (k + 1)(k + 2)(k + 3) = k(k + 1)(k + 2) + 3(k + 1)(k + 2) = 6b + 3 2[a + k + 1] , by (2), (4) = 6 [b + a + k + 1] , which is divisible by 6.
Finally, let's prove that n(n + 1)(n + 2)(n + 3) is divisible by 4! = 24. Base: 1234 = 24 is divisible by 24. (k + 1)(k + 2)(k + 3)(k + 4) = k(k + 1)(k + 2)(k + 3) + 4(k + 1)(k + 2)(k + 3) = 24c + 4 6[b + a + k + 1] , by (3) , (5) = 24 [c + b + a + k + 1] , which is divisible by 24 .
We are done!

4 Challenge Problems

Problem 1: Prove, by Mathematical Induction, that n(n + 1)(n + 2)(n + 3) (n + r 1) is divisible by r! , for all natural numbers n, where r = 1, 2, . . .

Problem 2: A circular loop of wire has a radius of 0.025 m and a resistance of 3.0 . It is placed in a 1.6 T magnetic field which is directed in through the loop as shown and then turned off uniformly over a period of 0.10 s. What is the current in the wire during the time that the magnetic field changes from 1.6 T to zero? (Note: this requires a special type of induction: magnetic induction)