1 Review of Bijections:

Consider set A and set B.

In a surjective (onto) function, the collective maps from A hit all elements in the set B. Formally, $\forall y \in B \exists x \in A s.t. f(x) = y$, or for all y, there exists a value x such that f(x)=y.

In an injective function, every value of A has exactly one corresponding element in B. Formally, $\forall x, y \in A, f(x) = f(y) \Rightarrow x = y$, or for all x and y, $f(x) = f(y)$ implies that $x = y$.
Surjective is sometimes called "onto" whereas injective is sometimes caled "one-to-one". Bijective functions combine injective and surjective. If we can find a inverse function $g(x)$ such that $g(f(a)) = a$ and $f(g(a)) = a$, the sets can be proven to be bijective. Also, all values $f(x)$ for all x in A in a map from set A to set B have to lie in set B and all values $g(x)$ for all x in B needs to map to an element in A. Inverses are usually found by intuition and guessing and checking. Due to this definitions, bijective functions are also called "invertible."

In simpler terms, if a bijective function exists between set A and set B, every element in A can be matched with a unique element in B, and every element in B can be matched with a unique element in A. In doing this, we establish a one-to-one correspondence between A and B. Also, the elements in set A have to map to all fo the elements in set B, or the "onto" component. Sets that have a bijective function have the same cardinality, or in this case, size.

2 Prerequisites

Problem 1: Find a injective, but not surjective, function that maps 0,1,2 to 3,4,5,6.

Problem 2: Prove $|[0,1]| = |[0,2]$.

Problem 3: Prove $|\mathbb{N}| = |\mathbb{Z}|$

Solution: This is basically mapping a countably infinite number numbers to another countably infinite number of numbers. Logically, we can look to Hilbert's Hotel. If you want to do it mathematically, you can use the solution below.

We need to find a function that maps every natural number to every integer and vice versa. To find one, first list out the natural numbers as $\{1, 2, 3, 4 \dots\}$ and the integers as $\{\dots -2, -1, 0, 1, 2 \dots\}$. We can design a function as follows: match 1 to 0, match 2 to 1, match 3 to -1, match 4 to 2, match 5 to -2, and so on. Thus, $f(x) = \begin{cases} \frac{x}{2} & x \equiv 0 \mod 2 \\ \frac{1-x}{2} & x \equiv 1 \mod 2 \end{cases}$. Since finding an inverse is difficult, let's check one-to-one and onto. This function is onto since it clearly hits all negative and positive integers and 0 by a loop. This is also one-to-one since a unique value of x is needed for a unique integer in the second set. This can be seen because all negative integers in the second set and 0 corresponds to an integer that is odd in the first set and all positive integers corresponds to a even value in the first set. Also, after you detrmine which "path" (top or bottom) you use for the piecewise, there is only one unique solution since the function boils down to a linear one.

3 Combinatorial Set Theory

Theorem 1: (Cantor-Bernstein-Schroeder) If there exists injective functions $f : A \to B$ and $g : B \to A$, the sets A and B have the same cardinality.

Definition 1: (von Neumann) If $n \in \mathbb{N}$, define set $\mathbf{n}$ to be $\{a \in \mathbb{N} | a < n\}$

Definition 2: Functions can be parts of sets too! If A and B are sets, define A^B to be the set of functions $f : B \to A$.

Definition 3: If A is a set, Define the power set $P(A)$ to be the set of all subsets of A.

Example 1: $|A| = n$, then $|\mathbf{2}^A| = 2^n$.

Solution: Create a bijection bewteem A and $\mathbf{n}$. Thus, we simply need to prove $|\mathbf{2^n}| = 2^n$. This is simple, for we can either take or leave elements in the set $\mathbf{n}$.

Example 2: Prove for any set A, $|P(A)| = |\mathbf{2}^A|$.

Solution: There is an abvious bijection with the set A of length n and $\mathbf{n}$. Thus, we simply need to prove $|P(\mathbf{n})| = |\mathbf{2^n}|$. The desired bijection between the left and rights set is that whenever we map an element x to 0, we don't include it in our subset and whenever we map it to 1, we include it.

Example 3: (Cantor's Theorem) Prove that $|\mathbb{A}| < |2^{\mathbb{A}}|$.

Solution: Use Cantor's diagonal trick.

Example 4: Prove that $|\mathbb{Q}| < |\mathbb{R}|$

Solution: The previous proof shows $n < 2^n$. Since this covers all rational n (that are not integers), we have proven this fact. Note, this does not imply that there are more rational numbers than real numbers per se. We can only eliminate the cardinality signs during equality or finite sets.

Definition 4: If A is a set and n is a natural number, define $\binom{A}{n}$ to be the set of all n-element subsets of A. If m, n are natural numbers, we define the natural number $\binom{m}{n}$ to be the cardinality $\binom{\mathbf{m}}{n}$.

Example 5: Show that if $|A| = n$ and $0 \leq k \leq n$, then $\left|\binom{A}{k}\right| = |\binom{A}{n-k}|$.

Solution: Interpretting this problem, we want to prove that the number of n element subsets of A is the same as the number of $n - k$ element subsets of A. We do this by noticing all subsets of A that do not include each subset of $|\binom{A}{k}$ gives us what we desire.

Example 6: Show that if $|A| = n$ for a natural number n, then $|\mathbf{2}^A| = \left|\binom{A}{0} + \binom{A}{1} + \ldots + \binom{A}{n}\right|$.

Solution: $|\mathbf{2}^A| = \left|\binom{A}{0}\right| + \left|\binom{A}{1}\right| + \ldots + \left|\binom{A}{n}\right|$. Notice we are finding the number of 0-element subsets, 1 element subsets, etc....

Definition 5: If A is a set, define a permutation, define σ to be a bijection such that $\sigma : A \to A$. Define the set $A!$ to be all possible permutations of A. If n is a natural number, define $n!$ to be the cardinality of $\mathbf{n}!$.

Example 7: If n and k are natural numbers, prove $|\mathbf{n}!| = |\mathbf{k}! \cdot \mathbf{n\text{-}k}! \cdot \binom{\mathbf{n}}{k}|$.

Solution: Notice the right, for each subset in $\mathbf{n}$, reorders the first n elements of the susbet, and then reorders the last $n - k$.

4 Fun with Bijections

*Note: Problems are not necessarily in order of difficulty.

Problem 1:
How many ways can a set of rooks be arranged on a chess board so that none of the rooks attack eachother, and no rook is below and to the left of another rook?

Solution: No rook can be above and to the right of another rook either, thus to go from a higher rook to a lower rook, only turning once, we must always travel downwards and rightwards. Thus we can make a bijection between the number of paths from a hypothetical square to the left of the top left corner, to a hypothetical square below the bottom right corner of the chessboard, only traveling down and right, and the number of such rook placements; simply take any path, and whenever there is a corner facing the top right, we place a rook there. We need these hypothetical squares to dictate when the path begins, because otherwise there would always exist a rook on the top row; these hypothetical squares allow us to travel down the side of the chessboard before we move onto it. Of course, the number of such paths is $\binom{16}{8}$. However we must account for the case where our path travels directly down to the hypothetical bottom left corner, and then straight across to the hypothetical bottom right square, in which case no rooks are actually on the board, since no corners are facing upwards and rightwards. Thus the final answer is $\binom{16}{8} - 1 = 12869$.

Problem 2: Let n and k be positive integers. Define a "partition" as splitting natural numbers into natural numbers in a way such that order doesn't matter. (For example, we can split 5 into $2+3$ or $1+1+1+2$, but order doesn't matter.) Show that the number of partitions of n with exactly k parts equals the number of partitions of n whose largest part is exactly k.

Solution: To solve this problem, we consider a graphical representation of a partition. For each partition $n = a_1 + a_2 + \ + a_r$, where $a_1 \geq a_2 \geq \ \geq a_r > 0$, we consider a diagram with ai dots on the i-th row, all left aligned. This is known as the Ferrar diagram of the partition. For instance, the partition $15 = 5+3+3+2+1+1$ corresponds to the following diagram:
We flip it over the main diagonal, flipping rows and columns. Since switching it twice undos the action and a switch in one set lies in other set, we have a bijection.

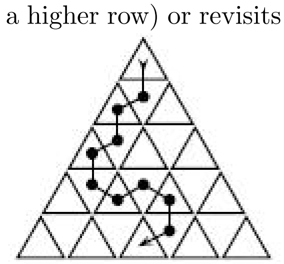
Problem 3: A triangular grid is obtained by tiling an equilateral triangle of side length n by n^2 equilateral triangles of side length 1. Determine the number of parallelograms bounded by line segments of the grid.

Solution: Extend the triangular grid by one extra row at the bottom. The key (and clever) observation is that starting from any such parallelogram in the original grid, we can extend its sides to meet the lines to meet the bottom edge of the new row in the large triangular grid, and there would be four distinct intersection points.Conversely, starting from any four distinct grid points in new bottom edge, we can extend 60 lines from the first two points and 120 lines from last two points to obtain a parallologram in the original grid. This gives us a bijection between the set of parallelograms in the original grid with no horizontal sides with set of four distinct points in the new bottom edge, and hence there must be $\begin{pmatrix} n+2 \\ 4 \end{pmatrix}$ of them. Accounting for all 4 rotations gives $4\begin{pmatrix} n+2 \\ 4 \end{pmatrix}$.

Problem 4: Tram tickets have six-digit numbers (from 000000 to 999999). A ticket is called lucky is the sum of its first digits is equal to that of its last three digits. A ticket is called medium if the sum of all its digits is 27. Let A and B denote the numbers of lucky tickets and medium tickets, respectively. Find $A - B$.

Solution: For any ticket with number abcdef let pair it with abcd'e'f' where d+d' = e+e' = f +f' = 9. Now one can easily observe that every lucky tickets pair is medium and every medium tickets pair is lucky and finally every ticket is the pair of its pair. So by these arguments we can observe the bijection between lucky and medium tickets therefore $A = B$ must hold and $A - B = 0$.

Problem 5: An equilateral triangle of side length n is divided into unit triangles. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in a path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle.



An example is shown on the picture for $n = 5$. Determine the value of $f(2005)$.

Solution 5: Define a last triangle of a row as the triangle in the row that the path visits last. From the last triangle in row k, the path must move down and then directly across to the last triangle in row $k+1$. Therefore, there is exactly one path through any given set of last triangles. For $1 \le m \le n-1$, there are m possible last triangles for the mth row. The last triangle of the last row is always in the center. Therefore, $f(n) = (n-1)!$, and $f(2005) = 2004!$.