1 Theory

Double Counting is a advanced technique to prove things that would be hard to prove algebraically (or inductively). The technique is classified under combinatorics, hence the name "counting," and comes from set theory. The main concept of set theory that enables double counting is called a bijection, which can be used to relate two mathmatical formulas by proving an inverse function exists between the set of their outputs (this will be discussed in a later PMC lecture). To understand many interesting double counting proofs, one first has to understand combinations.

1.1 Combinations

A combination is the number of ways to count how many ways you can choose k items from n items. For example, we can use combinations to count the number of ways to form a team of three from five people. We have five ways to choose the first person, four ways to choose the second, and then three ways to choose the third. But if we do 5*4*3, this means we've put them in a specific order, meaning instead of a team of three, we have member 1, member 2, and member 3. This is actually a permutation, not a combination. Since we can put any set of three people into 3*2 = 6 different orders, we divide by six.

Example 1: What does $\binom{9}{3}$ represent?

Example 2: April is giving out cookies in Pleasanton Math Circle. She gives one cookie each to five people. If there are 25 people in PMC, how many different ways can April give out cookies?

Solution 2: $\binom{25}{5}$

Example 3: This time, April is giving out a cookie, a brownie, and a dollar such that three students will get an item. If there are still 25 students in PMC, how many ways can April give out these items?

Solution 3: Permutation 25 3 what is the format there is no format screw you latex

Make sure you understand the difference between the two previous questions! If you don't, be sure to flag one of us down and get help.

1.2 Sigma Notation

Sigma notation is a handy way to denote sums, rather than writing out the entire sum. We'll explain this on the board.

Example 4: Evaluate $\sum_{k=1}^{7}(2k+1)$

Example 5: Write 2 + 4 + 6 + 8 + 10 + 12 in sigma notation.

Example 6: Write 1 + 4 + 10 + 28 + 82 + 244 + 730 in sigma notation.

1.3 How to Count Twice

The concept of double counting can be best explored through an example.

Example 7: Prove $\sum_{k=0}^{n}\binom{n}{k}=2^n$.

Consider a committee of size n. We realize that 2^n describes the number of different committees we can have that is a subset of n (everyone can say "yes" or "no" when asked to join a committee). Thus, we want another way to count this. Let's consider what $\binom{n}{k}$ counts. It counts the numbers of committes of size k that is a subset of n. Thus, we realize that summing up all of these counts the number of committees size 0 to n. Thus, $\sum_{k=1}^{n}\binom{n}{k}=2^n$.

Example 8: (Hockey Stick) Prove, using double counting, $\binom{a}{a}+\binom{a+1}{a}+\ldots+\binom{a+b}{a}=\binom{a+b+1}{a+1}$

Solution: Interpreting the right hand side as the number of subsets of 1,2,...,a+b+1 of size a+1, suppose a+k+1 is the position of the last element of such a subset. Then the number of ways to choose the rest of the subset is $\binom{a+k}{a}$.

Example 9: How many ways are there to cut a convex octagon into six triangles using straight lines?

Solution: Well find a bijection between triangulations of a n + 2-gon and full binary trees (a tree in which every node other than the leaves has two children) with n + 1 leaves. Pick you favorite edge of the polygon, and draw a root node inside the triangle adjacent to it. The children of the root node will be inside the two triangles connected to the triangle containing it, and similarly for any triangle there will be a node having children corresponding to the two triangles adjacent to it which dont correspond to the parent of the node. If a triangle is adjacent to an edge which is not your favorite edge, we consider the outside of that edge to be a leaf node.

Example 10: Prove $\sum_{k=0}^{floor(\frac{n}{2})} \binom{n-k}{k} = f_n$ Where fn is the nth fibonacci number.

Solution: Count the number of ways to tile a 1 by n grid with tiles of size 1 by 1 and 1 by 2 in two different ways.

2 Problems

Problem 1: Prove $\sum_{k=d}^{n} \binom{n}{k}\binom{k}{d} = 2^{n-d}\binom{n}{d}$

Solution: Using $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, we can simplify the lhs.

Problem 2: Prove that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ using double counting.

Solution 2: Consider a triangle of dots with n+1 rows. The first n rows have a total of 1 + 2 + ... + n dots. Also, from each dot, travel down the diagonals until it reaches the n+1 row. Notice that the dots on the last row on the two diagonals form a pair. Also, notice that all pairs are injective and surjective. This gives us with a total of $\binom{n+1}{2}$.

Problem 3: (Handshaking Lemma) Prove every undirected graph (a graph where every edge points both ways) contains an even number of vertices of odd degree.

Solution: Same as $\sum_v d(v) = 2e$. Count by considering each graph contributes 2 degrees, or summing individual degrees.

Problem 4: In a competition, there are m contestants and n judges, where $n \geq 3$ is an odd integer. Each judge rates each contestant as either pass or fail. Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that $\frac{k}{m} \geq \frac{n-1}{2n}$.

Solution: Create a table in which the columns represent the contestants and the rows represent the score the judge awards to each contestant. If a contestant is passed by a judge they will be awarded a 1; otherwise, the contestant will be awarded a 0. Let us call an equal pair of entries a "cool" pair. The number of cool pairs in any 2 rows is equal to k when the total number of cool pairs is at most $\binom{n}{2} = \frac{kn(n-1)}{2}$.In any given column if there are x zeroes and $y = n - x$ then the total number of cool pairs is $\binom{x}{2} + \binom{y}{2}$. Since n is odd, we can write $n = 2i + 1$ for some integer i. Since either $i - x \geq 0$ or $i - y \geq 0$, we can write $\binom{x}{2} + \binom{y}{2} - i^2 = (i-x)^2 + (i-x) = (i-y)^2 + (i-y) \geq 0$ Thus the total number of cool pairs is at least $mi^2 = \frac{m(n-1)^2}{4}$. Therefore $\frac{m(n-1)^2}{4} \leq \frac{kn(n-1)}{2}$ Simplyfing this gives us our desired expression of $\frac{k}{m} \geq \frac{n-1}{2n}$.

Problem 5: Prove $\sum_{k=1}^{n} = \frac{n(n+1)(2n+1)}{6}$.

Solution: Let's say I have a group of n+1 people who want to see a show. I have three tickets into the the-

atre: a backstage pass and two regular (but distinguishable) tickets. I have to give out the tickets according to the following two rules:
1. The backstage pass must go to the oldest person who gets a ticket. (All the people in line are of different ages.)
2. The person who gets the backstage pass can't get either of the other two tickets, but the two normal tickets can both go to the same person.
How many ways can I give away the tickets? There are two ways to count it. For the first way, I can give the backstage pass to the youngest person and then give the other two tickets in a total of 00 ways each. Or I can give it to the second-youngest and give the other two tickets in a total of 11 ways ... or I can give it to the oldest person and give the other two tickets in a total of nn ways. So the first way of counting gives $\sum_{i=1}^{n} i^2$.
For the second way, either I give tickets to three people or I give tickets to two people. If I give them to three people, then the oldest person must get the backstage pass, but there are two ways to distribute the remaining tickets (remember, they're distinguishable). If I give them to two people, then there are no degrees of freedom: the older one gets the backstage pass and the younger one gets the other two. So this way of counting gives $2\binom{n+1}{3}\binom{n+1}{2}$.

Problem 6: Let $A = a_{1,j}$ be a $r * c$ boolean (0,1)-matrix with column sums C_j. Suppose that for every two rows, there exist exactly t columns that contain 1s from both rows. Then, prove $t\binom{r}{2} = \sum_{j=1}^{c}\binom{C_j}{2}$

Solution: Let T denote the set of all unordered pairs of 1s that lie in the same column. Let us count the elements of T in two dierent ways. Counting by rows: For any two rows, there are t pairs of 1s among these rows that belong to T, so $|T| = t\binom{r}{2}$. Counting by columns: In the jth column, there are C_j 1s, and thus $\binom{C_j}{2}$ pairs. Counting over all the columns gives $|T| = \sum_{j=1}^{c}\binom{C_j}{2}$. The result follows by equating the above two expressions.