1 Introduction

Probability theory is the branch of mathematics that deals with that deals with the analysis of random events. Most of us have seen probability before and are quite familiar with it. As a recap, we will go over the following definitions and formulas.

- The probability that an event X, or P(X) is given by $\frac{\text{The number of successful outcomes}}{\text{The number of total outcomes}}$
- Given two events, A and B, $P(A \cup B) = P(A) + P(A \cap B) + P(B)$. For mutually exclusive events, $P(A \cap B) = 0$, so $P(A \cup B) = P(A) + P(B)$.
- Conditional Probability: "The probability that A occurs given B occurs is the probability that both A and B occur over the probability that B occurs" This can be written as $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- Expected value is defined as the weighted average of all possible values. It is defined as $E[X] = x_1p_1 + x_2p_2 + \cdots + x_np_n$ where p_n is the probability that event n occurs.

As intuitive as probability seems, there are many times when we get a result that is completely different from what is expected. We will examine different problems and use probability to explain the phenomenon.

2 Probability

Problem 1: You flip two coins, a nickel and a quarter. The nickel lands heads. What is the probability that both coins land heads?

Solution If the quarter lands heads, then both coins land heads. Therefore, the probability is $\frac{1}{2}$

Problem 2: You roll two dice. One die shows a 1, but the other die rolls under the table and you cannot see it. What is the probability that both die show 1?

Solution We define event A as both dice showing a 1 and event B as at least one die showing a 1. $P(B) = \frac{11}{36}$. $P(A \cap B) = 1/36$. Therefore, $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{11}$.

Problem 3: Suppose there are three cards. One card is red on both sides, one is green on both sides, and one has a side of each color. Suppose one card is chosen at random (i.e. blindly out of a bag). You are able to see that one side of the card is red. What is the probability that the other side of the card is also red?

Solution 1/2

Problem 4: Two dice appear to be standard dice with their faces numbered from 1-6, but each dice is weighted so that the probability of rolling the number k is directly proportional to k (meaning that if $P(1) = x$, then $P(k) = kx$ for $1 \le k \le 6$). Find the probability of rolling a 7 with this pair of dice. (Source: 2016 AIME I #2)

Solution We first note that if $P(1) = x$, then $P(1) + P(2) + ... + P(6) = 21x = 1$, implying that $x = \frac{1}{21}$ and $P(k) = \frac{k}{21}$ for all $1 \le k \le 6$. Therefore, the probability of rolling a 7 is $\frac{2\times1\times6+2\times2\times5+2\times3\times4}{21^2} = \frac{56}{441} = \boxed{\frac{8}{63}}$.

Problem 5: In Poker, a full house is defined as three cards of one rank (or number) and two cards of another rank. What is the probability, when drawing 5 cards at random from a full deck (without jokers), that a full house is drawn?

Solution There are $\binom{4}{1} \times 13 = 52$ ways to choose the three cards of one rank and $\binom{4}{2} \times 12 = 72$ ways to choose the two cards of the other rank. Therefore, our probability is $\frac{52\times72}{\binom{52}{5}} = \frac{52\times72\times120}{52\times51\times50\times49\times48} = \boxed{\frac{6}{4165}}$.

Problem 6: : A coin that comes up heads with probability $p > 0$ and tails with probability $1 - p > 0$ independently on each flip is flipped eight times. Suppose the probability of three heads and five tails is equal to $\frac{1}{25}$ of the probability of five heads and three tails. Find p.

Solution: The probability of three heads and five tails is $\binom{8}{3}p^3(1-p)^5$ and the probability of five heads and three tails is $\binom{8}{3}p^5(1-p)^3$. $p = \frac{5}{6}$.

Problem 7: Dave rolls a fair six-sided die until a six appears for the first time. Independently, Linda rolls a fair six-sided die until a six appears for the first time. Let m and n be relatively prime positive integers such that $\frac{m}{n}$ is the probability that the number of times Dave rolls his die is equal to or within one of the number of times Linda rolls her die. Find $m+n$.
Solution: Let p be the probability that the number of times Dave rolls his die is equal to or within one of the number of times Linda rolls her die. (We will call this event "a win", and the opposite event will be "a loss".)

Let both players roll their first die.

With probability $\frac{1}{36}$, both throw a six and we win.

With probability $\frac{10}{36}$ exactly one of them throws a six. In this case, we win iff the remaining player throws a six in their next throw, which happens with probability $\frac{1}{6}$.

Finally, with probability $\frac{25}{36}$ none of them throws a six. Now comes the crucial observation: At this moment, we are in exactly the same situation as in the beginning. Hence in this case we will win with probability p.

We just derived the following linear equation:

$$p = \frac{1}{36} + \frac{10}{36} \cdot \frac{1}{6} + \frac{25}{36} \cdot p$$

Solving for p, we get $p = \frac{8}{33}$, hence the answer is $8 + 33 = \boxed{041}$.

3 Birthday Problem

We all had the moment in class when we realized that one person has the exact same birthday as another. What a coincidence...or is it? Given there are 50 people in a room, what is the probability that two of them share a birthday? How about n people?

Solution: Instead of calculating the odds that two people have the same birthday, we can calculate the odds that no two people share a birthday instead. We can keep assigning one day from the remaining days to each person. Note that if $n > 365$, there is a 100 percent probability that two people will share the same birthday (pigeonhole of proof by contradiction). Or else, we can calculate the probability that no two people share the same birthday as $P(A') = \frac{365}{365}\frac{364}{365}\frac{363}{365}...\frac{365-n+1}{365}$. Therefore, the probability P(A) that two people share the same birthday is 1-P(A').

Number of people	Probability
5	0.027
10	0.117
30	0.706
50	0.97
100	0.9999997

4 100 Prisoners Problem

There are 100 prisoners, each of which have assigned numbers. Each prisoner is led to a different room with a cupboard containing 100 drawers with 100 numbers. Each prisoner is allowed to open 50 of the cabinets, one at a time. After the prisoner leaves, the drawers are closed, and the prisoners cannot communicate in any way with other prisoners once the drawer-opening starts. If he finds his own number, he is freed.

Problem 1: Given each prisoner randomly opens the 50 drawers, what is the probability that all of them will survive?

Problem 2: Is there a way to increase this tiny probability? Devise a strategy that increases the odds of every single prisoner surviving and compute the probability.

5 Two Envelopes Problem

You are playing a game for money. There are two envelopes on a table. You know one envelope contains X dollars, while another one contains $2X$ dollars. A person is able to choose one envelope at the start, but can later choose to switch (he can choose to switch many times). A certain mathematician reasons this:

Assume the envelope that the mathematician chooses contains Y dollars.
1. There is a 1/2 chance that the mathematician chooses the smaller envelope and a 1/2 chance that he chooses the bigger one.
2. If the mathematician chooses the smaller envelope, the other one contains $2Y$ dollars.
3. If the mathematician chooses the bigger envelope, the other one contains $\frac{Y}{2}$ dollars.
4. Therefore, the expected value of the amount of money in the other envelope is $\frac{1}{2}(2Y + \frac{Y}{2}) = \frac{1}{2}(\frac{5Y}{2}) = \frac{5}{4}Y$. 5. By switching, the mathematician gains $\frac{Y}{4}$ dollars on average, and thus, should switch indefinitely.

What is wrong with the reasoning?

Solution: Although many solutions were proposed, no solution was definitive. We can notice that the contents of the first envelope provides us with useful information (if it did not, it does not matter if we stick or switch).

This is the most common resolution of the problem: The flaw if the Y, in the first case, is the value of the money in the first envelope, given that it is less than the second. Meanwhile, Y, in the second case, is the expected value of the money in the first envelope, given that it is greater than the second. Therefore, we are using two conflicting meanings of Y to calculate the expected value.

The correct calculation should be:
Expected value in B = 1/2 (Expected value in A (given A is larger than B) + Expected value in A (given A is smaller than B))
Notice how this is slightly different than the calculation the mathematician's calculation.