There are infinite numbers between 0 and 1. Theres .1 and .12 and .112 and an infinite collection of others. Of course, there is a bigger infinite set of numbers between 0 and 2, or between 0 and a million. Some infinities are bigger than other infinities.

-Hazel Grace from John Green's *The Fault in Our Stars*

1 Theory

A set is a collection of "things" which often have a property in common. To denote a set, we use curly brackets, $\{\}$, and list the elements inside. We can also use bold letters. For instance, we have $\mathbf{n} = \{4, 10, 99, \ldots\}$ or $\{pink, purple, yellow\}$. The first set, $\mathbf{n}$, has ellipses, so it is an infinite set, but the second one does not, so it is a finite set.

List of Common Number Sets

$\mathbb{P}$=Primes	$\mathbb{W}$=Whole Numbers	$\mathbb{N}$=Natural Numbers
$\mathbb{Z}$=Integers	$\mathbb{I}$=Irrational Numbers	$\mathbb{Q}$=Rational Numbers
$\mathbb{R}$=Real Numbers	$\mathbb{C}$=Complex Numbers	

2 Infinity

2.1 Countable Vs. Uncountable Infinities

Countable Infinity is a infinite set that can be put into a bijection with $\mathbb{N}$. In other words, you can count all elements in this set one at a time, though you may never finish. These sets are denoted by the symbol $\aleph_0$ or "aleph-zero." Uncountably infinite sets are sets that cannot be counted one at a time. In other words, it has too many elements to count. Spend a couple minutes trying to think of a uncountable set.

Cantor's Diagonal Trick: The first uncountable sets were discovered by mathematian Georg Cantor. Cantor's diagonal trick is one of the first uncountable sets discovered. Cantor proved that the set of binary sets, or $\mathbf{T}$, is uncountable. He does this by displaying the set $\mathbf{T}$ as the following:

$$\mathbf{0}000000\ldots$$
$$0\mathbf{1}01010\ldots$$
$$11\mathbf{1}0001\ldots$$
$$011\mathbf{0}101\ldots$$
$$1111\mathbf{1}01\ldots$$
$$11101\mathbf{0}1\ldots$$
$$100000\mathbf{0}\ldots$$

Define a new set $\mathbf{n}$ to be a set such that the nth element of set $\mathbf{T}$ is flipped (as in changine from 1 to 0 or 0 to 1), and then included in $\mathbf{n}$. For the example, $\mathbf{n} = \{1, 0, 0, 1, 0, 1, 1\}$. Notice that $\mathbf{n}$ is not part of any set above, since at least one element is different. Assume $\mathbf{T}$ is coutnable. Clearly, since all binary strings are listed, $\mathbf{n} \varepsilon \mathbf{T}$. Thus, even though you list out all the possible elements of $\mathbf{T}$, you will be missing at least one set from it. Contradiction.

2.2 Hilbert's Great Hotel: Thinking about infinity

Imagine a hotel with a countably infinite number of rooms. Each room has one guest, so there are countably infinite guests at the hotel as well.
Now, suppose that one day another guest would like to check into this hotel. Is there space? Of course! The hotel has infinite rooms. However, how will the front desk make space in an orderly fashion? Move guest 1 from room 1 to room 2, guest 2 from room 2 to room 3, and so on, infinitely. In fact, this will work for any finite number of new guests - if a party of 10 trillion guests showed up at the same time, there would still be space.

Example 1: Is it possible to add countably infinite guests to the hotel? If so, find a way to do so.

Solution: There will still be space. How? Well, move the guest in room 1 to room 2, room 2 to room 4, room 3 to room 6, room 4 to room 8, room n to room 2n for every single room. Since there are infinite rooms, there's enough space. And since all the odd numbers are now vacant, there are a countably infinite number of vacant rooms. Thus, there's enough space for countably infinite new guests.

Example 2: Let's take it up another notch. What if we have countably infinite coachloads of countably infinite people, all arriving at the hotel, who all want to check into this strange hotel. How can we move the existing guests to fit these new guests in?

Solution: There are a few ways to do this. Most of them involve primes. 1. Empty all the odd numbered rooms (how do we do this?). Then, put the first coach load into all rooms 3^n, then 5^n, then p^n. 2. Prime factorization - for each coach c and seat n, put new guests into room $2^n 3^c$. (Guests who are already at the hotel have $c = 0$.) Since each number has a different prime factorization, each guest will go into a different room. 3. For each passenger, compare the lengths of n and c as written in decimal. If either number is shorter, add leading zeroes to it until both values have the same number of digits. The hotel (coach 0) guest in room number 1729 moves to room 01070209. The passenger on seat 1234 of coach 789 goes to room 01728394. This one fills the hotel completely.

Example 3: One more! Level three of infinity. An infinite number of boats with infinite coaches with infinite passengers want to check in. How?

Solution: Prime factorization again.

Example 4: What if we go to infinite levels of infinity? Infinite transformers with infinite platters of infinite flying saucers of infinite spaceships of infinite planets of ....?

Solution: This is now uncountably infinite.

3 Bijection

A bijection is basically condensing a problem into something that we already know how to count. Usually, a proof involving a bijection between two sets **A** and **B** should explain the following:

1. How to obtain an element of **B** from any element of **A**.
2. How to recover the element of **A** from any element of **B**.
3. Why the above two constructions are inverses of each other.

Example 5 How many ways are there to go from (0,0) to (4,5) moving only right and up?

Solution: To get to (4,5), we need to move right 4 times and up 5 times. This means a total of 9 moves, with 4 R's and 5 U's. To choose which of the 9 moves to be right, we do 9C4. We see how these orderings of R's and U's will result in a different path every time.

3.1 Surjective, Injective: Proofs with Bijection

Consider set A and set B.
In a surjective function, every element in B has at least one matching element in A.
In an injective function, every value of A has exactly one corresponding element in B.
Surjective is sometimes called "onto" whereas injective is sometimes caled "one-to-one". Bijective functions combine injective and surjective. If we can find a inverse function $g(x)$ such that $g(f(a)) = a$ and $f(g(a)) = a$, the sets can be proven to be bijective. Also, all values $f(x)$ for all x in A in a map from set A to set B have to lie in set B and all values $g(x)$ for all x in B needs to map to an element in A. Inverses are usually found by intuition and guessing and checking. In simpler terms, if a bijective function exists between set A and set B, every element in A can be matched with a unique element in B, and every element in B can be matched with a unique element in A. In doing this, we establish a one-to-one correspondence between A and B.

*Note: Bring up last week's solution for sum of first n numbers

Example 6: Give an example of a function that maps $|\{0,1,2\}| = |\{a,b,c\}|$ that is neither injective nor surjective.

Example 7: Give an example of a function that maps $|\{0,1,2\}| = |\{a,b,c\}|$ that is injective but not surjective.

Example 8: Give an example of a function that maps $|\{0,1,2\}| = |\{a,b,c\}|$ that is surjective but not injective.

Example 9: Give an example of a function that maps $|\{0,1,2\}| = |\{a,b,c\}|$ that is bijective.

3.2 Some Infinities are Bigger than Other Infinities

Define the $sizeof$ function, denoted by $|\mathbf{n}|$, to be the number of elements in the set $\mathbf{n}$. Define cardinality to be the comparison of the $sizeof$ two sets. For example, $|\mathbf{a}| \leq |\mathbf{b}|$ means that the cardinality of $\mathbf{a}$ is at most the cardinality of $\mathbf{b}$. If $|\mathbf{a}| = |\mathbf{b}|$, there exists a bijective function between the two sets and if a function is bijective, the two sets that the function links has the same cardinality. Note: If the cardinality of one set is bigger than another, it doesn't mean that set has more elements. In fact, for infinite sets, it is difficult to compare the number of elements, other than proving that the sets are of the same size.

Example 10: Prove $|\{0,1,2\}| = |\{a,b,c\}|$.

Solution: We can check injective and surjective. However, we can also find a function g. $g : 0 \mapsto a, 1 \mapsto b, 2 \mapsto c$. We can check for all values one by one that this inverse works. Thus, the number of elements of the left set is equal to the number of elements of the right set.

Example 11: Prove John Green wrong by proving $|[0,1]| = |[0,2]| = |[0,1000000]|$ where $[a,b]$ denotes a set that includes all rational numbers from a to b.

Solution: First, let's find a function that maps the left set to the right set. This is simply $f(x) = 2x$. Also, it is apparent that $f(x)$ will map all elements in the left set to a element in the right set (by considering the bounds). Let's try to find an inverse. This is simply $f^{-1}(x) = \frac{1}{2}x$ since $f(g(x)) = g(f(x))$. Also, all elements in the right set map to an element in the left set (also considering the bounds), so we are done. This can be repeated for the million part by defining $f(x) = 1000000x$ and $g(x) = \frac{1}{1000000}x$.

Example 12: Prove $|\mathbb{Z}| = |\mathbb{Q}|$.

Solution: Write a infinite rectangle where the first row has denomiator 1 and the second denominator two by alternating signs. For example, the first row will be $\frac{1}{1}$, $\frac{-1}{1}$, $\frac{2}{1}$, $\frac{-2}{1}\ldots$. Then, cross out the repeats. Finally, go through the diagonals and number them. The funcion is onto since we are looping through all possible perimeters and diagonals. It is also one-to-one for we crossed out the repeat sets. We are done.

Example 13: Prove $|\mathbb{N}| = |\mathbb{Z}|$

Solution: This is basically mapping a countably infinite number numbers to another countably infinite number of numbers. Logically, we can look to Hilbert's Hotel. If you want to do it mathematically, you can use the solution below.

We need to find a function that maps every natural number to every integer and vice versa. To find one, first list out the natural numbers as $\{1, 2, 3, 4\dots\}$ and the integers as $\{\dots-2, -1, 0, 1, 2\dots\}$. We can design a function as follows: match 1 to 0, match 2 to 1, match 3 to -1, match 4 to 2, match 5 to -2, and so on. Thus, $f(x) = \begin{cases} \frac{x}{2} & x \equiv 0 \mod 2 \\ \frac{1-x}{2} & x \equiv 1 \mod 2 \end{cases}$. Since finding an inverse is difficult, let's check one-to-one and onto. This function is onto since it clearly hits all negative and positive integers and 0 by a loop. This is also one-to-one since a unique value of x is needed for a unique integer in the second set. This can be seen because all negative integers in the second set and 0 corresponds to an integer that is odd in the first set and all positive integers corresponds to a even value in the first set. Also, after you detrmine which "oath" (top or bottom) you use for the piecewise, there is only one unique solution since the function boils down to a linear one.

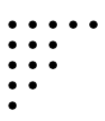
4 Fun with Bijections

Problem 1:
How many ways can a set of rooks be arranged on a chess board so that none of the rooks attack eachother, and no rook is below and to the left of another rook?

Solution: No rook can be above and to the right of another rook either, thus to go from a higher rook to a lower rook, only turning once, we must always travel downwards and rightwards. Thus we can make a bijection between the number of paths from a hypothetical square to the left of the top left corner, to a hypothetical square below the bottom right corner of the chessboard, only traveling down and right, and the number of such rook placements; simply take any path, and whenever there is a corner facing the top right, we place a rook there. We need these hypothetical squares to dictate when the path begins, because otherwise there would always exist a rook on the top row; these hypothetical squares allow us to travel down the side of the chessboard before we move onto it. Of course, the number of such paths is $\binom{16}{8}$. However we must account for the case where our path travels directly down to the hypothetical bottom left corner, and then straight across to the hypothetical bottom right square, in which case no rooks are actually on the board, since no corners are facing upwards and rightwards. Thus the final answer is $\binom{16}{8} - 1 = 12869$.

Problem 2: Let n and k be positive integers. Define a "partition" as splitting natural numbers into natural numbers in a way such that order doesn't matter. (For example, we can split 5 into $2 + 3$ or $1 + 1 + 1 + 2$, but order doesn't matter.) Show that the number of partitions of n with exactly k parts equals the number of partitions of n whose largest part is exactly k.

Solution: To solve this problem, we consider a graphical representation of a partition. For each partition $n = a_1 + a_2 + \ + a_r$, where $a_1 \geq a_2 \geq \ \geq a_r > 0$, we consider a diagram with ai dots on the i-th row, all left aligned. This is known as the Ferrar diagram of the partition. For instance, the partition $15 = 5 + 3 + 3 + 2 + 1 + 1$ corresponds to the following diagram:



We flip it over the main diagonal, flipping rows and columns. Since switching it twice undos the action and a switch in one set lies in other set, we have a bijection.

Problem 3: A triangular grid is obtained by tiling an equilateral triangle of side length n by n^2 equilateral triangles of side length 1. Determine the number of parallelograms bounded by line segments of the grid.

Solution: Extend the triangular grid by one extra row at the bottom. The key (and clever) observation is that starting from any such parallelogram in the original grid, we can extend its sides to meet the lines to meet the bottom edge of the new row in the large triangular grid, and there would be four distinct intersection points.Conversely, starting from any four distinct grid points in new bottom edge, we can extend 60 lines from the first two points and 120 lines from last two points to obtain a parallologram in the original grid. This gives us a bijection between the set of parallelograms in the original grid with no horizontal sides with set of four distinct points in the new bottom edge, and hence there must be $\binom{n+2}{4}$ of them. Accounting for all 4 rotations gives $4\binom{n+2}{4}$.

Problem 4: Tram tickets have six-digit numbers (from 000000 to 999999). A ticket is called lucky is the sum of its first digits is equal to that of its last three digits. A ticket is called medium if the sum of all its digits is 27. Let A and B denote the numbers of lucky tickets and medium tickets, respectively. Find $A - B$.

Solution: For any ticket with number abcdef let pair it with abcd'e'f' where d+d' = e+e' = f +f' = 9. Now one can easily observe that every lucky tickets pair is medium and every medium tickets pair is lucky and finally every ticket is the pair of its pair. So by these arguments we can observe the bijection between lucky and medium tickets therefore $A = B$ must hold and $A - B = 0$.