CS5691-Assignment 1

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1 Motivation

There are many methods of matrix factorization among which Eigen Value Decomposition and Singular Value Decomposition are prominent.These methods are used in many fields and are particularly popular in Machine learning for **data compression**(here, image compression) and for **denoising data** before working on it and so on. These decompositions result in diagonal matrices,and **operations on diagonal matrices** are less computationally expensive, hence making these methods valuable.In this assignment, EVD and SVD have been applied to a gray scale 256x256 image and later on to an rgb 256x256 image and experiments have been performed on them which result in a better understanding of the same.

2 Eigen value decomposition

For an eigenvector, multiplication with the transformation matrix A, is equivalent to multiplying with a simple scalar λ .We expand this idea from vectors to matrices, and most square matrices can be decomposed into a matrix of column eigenvectors X and a diagonal matrix V that is filled with eigenvalues on the main diagonal and X^{-1}.

$$A = XVX^{-1}$$

What we are doing is effectively a change of basis using the matrix X^{-1} and we get back the actual value by scaling with values obtained from multiplication of the other 2 matrices.Basically,the columns of A can be decomposed in terms of its eigenvectors.

3 Singular value decomposition

In SVD, the matrix A is decomposed as follows:

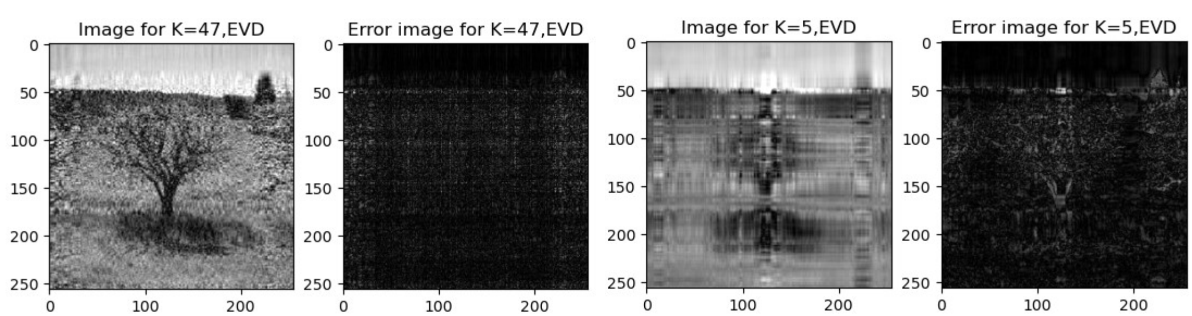
$$A = USV^*$$

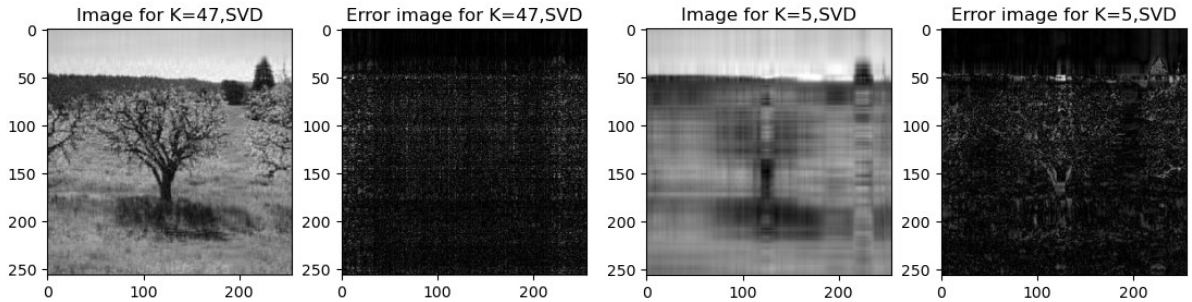
Here, U and V are complex unitary matrices formed by using the eigenvectors of AA^T(Columns of U) and A^TA(Columns of V).These are unitary since they are eigenvectors of symmetric positive semidefinite matrices.The matrix S is a diagonal matrix with diagonal elements as non negative root of the eigenvalues of these symmetric matrices.SVD is a change of basis too and can be thought of as a rotations and possibly dilations and reflections.

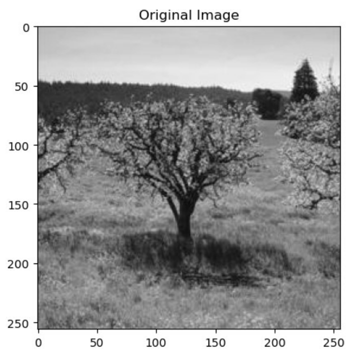
4 Experiments

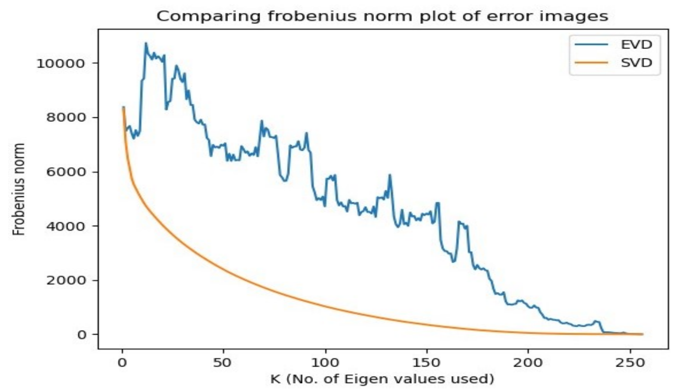
4.1 Expt 1 and Expt 2

EVD and SVD was performed on the gray scale image provided to us and the eigenvalues and corresponding eigenvectors were sorted in descending order.We then viewed the reconstructed image and error image for a few values of K to get a good idea visually.The frobenius norm of the error images for all K from 1 to 256 was calculated and has been plotted.We get a **smooth exponentially reducing** curve for SVD but a **noisy reducing)** curve for EVD(Reason in inference).









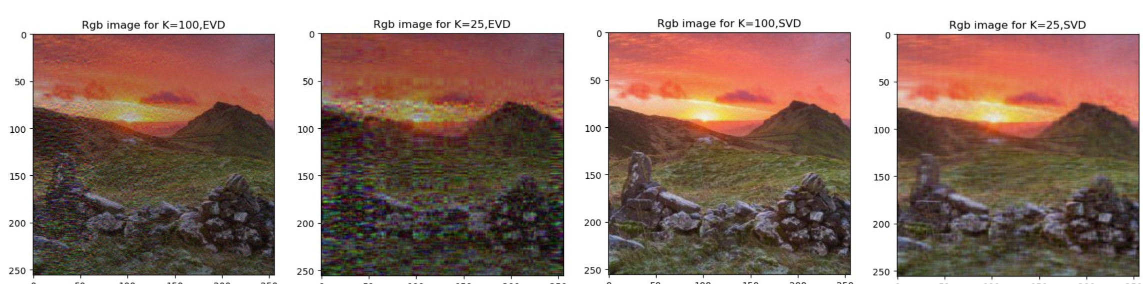
In the error images for K=5, we observe that it looks like the tree in the original image is being formed.The reason for this is that the gradients at the edges of objects is large and this is were most of the error occurs as small values of K are not enough to accommodate for these large changes in intensities and colors.

4.2 Expt 3

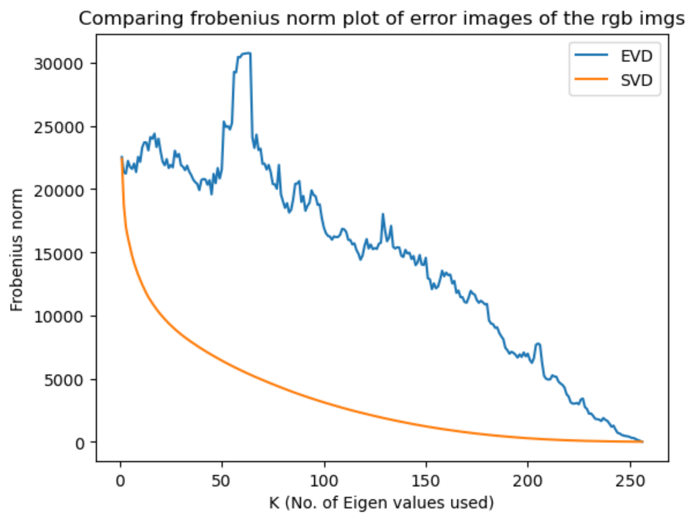
Condition number for the matrices were calculated using the np.linalg.cond() function but can be found manually too by dividing λmax/λmin.We use **The Bauer–Fike Theorem** to understand which method(EVD or SVD) is better.Informally speaking, what it says is that the sensitivity of the eigenvalues is estimated by the condition number of the matrix of eigenvectors.Assume A=XVX^{-1},then the **error/sensitivity** in the obtained eigenvalues of A is proportional to the **condition number of the matrix X** which is a matrix of eigen vectors.So we find the condition number of X in EVD and U in SVD where X and U have been previously defined.The values obtained are as follows for the gray scale image:
The condition number for the matrix in EVD = **568.7393539755072**
The condition number for the matrix in SVD = **1.000014508371775**
The closer the condition number is to 1, the better is the decomposition as it means less error in eigenvalues.Clearly here, SVD has a much better condition number than EVD.This is also the main reason why SVD has a smoothly reducing error curve whereas EVD has a noisy curve since the eigenvalues found are not very accurate.

4.3 Expt 4 and Expt 5

These 2 experiments are the same as Expt 1 and 2 except that now we are performing EVD and SVD on a colored(rgb) image of dimension 256x256x3.



- We can see that SVD is performing much much better than EVD through the above images and this is clear in the plot below too as the error difference between the two is massive.
- From the below plot,we can also conclude that the frobenius norm trends for EVD and SVD are similar to the norms in expt 1 and 2 respectively.
- SVD also clearly has the advantage of reducing noise in the image.



5 Inferences

The similarities and differences between EVD and SVD are as follows:

- Both of them are methods for decomposition resulting in a change in basis but EVD is not a simple rotation and is for quadratic form whereas SVD can be thought of as a rotation along with stretching/dilation and reflections and is for a linear form.
- The eigenvalues in EVD can be complex and negative whereas in SVD we get only non negative eigen values.In general,the eigen vectors in EVD are not orthogonal whereas SVD gives orthonormal eigenvectors(U and V are unitary).
- SVD exists for all rectangular and square matrices whereas EVD exists for only some square matrices(not all square matrices)

Inferences about the reconstructed images for varying K are as follows:

- We can say that SVD is better than EVD as it clearly has lower frobenius norm and hence lower error.
- From the images shown, we can see that other than constructing a better image, EVD also reduces noise in the image.This is another application of EVD.
- We see that frobenius norms of both EVD and SVD have a reducing trend but EVD is not monotonic and has several spikes(overall linearly decreasing) whereas SVD has monotonic and exponentially decreasing norm.
- In rgb images,SVD performs even more better relative to EVD than in gray scale as visible from the plot and this is especially true at low values of K.For example, if we take K=75, for gray scale image, EVD is approximately 4500 error units more than SVD whereas for rgb image,EVD is approximately 18000 error units more than SVD.This shows that SVD is far superior than EVD in rgb images.
- One of the important points is that if X and X^{-1} were orthogonal in EVD, which also means that if the input image was symmetric,then EVD and SVD would perform the same for this image.The curves for the frobenius norm would be smooth for both and will overlap.