Optimized chaos control with simple limiters

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We present an elementary derivation of chaos control with simple limiters using the logistic map and the Henon map as examples. This derivation provides conditions for optimal stabilization of unstable periodic orbits of a chaotic attractor.

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Recently, Corron *et al.* [1] proposed a new efficient experimental chaos control method. The original Ott, Grebogi, and Yorke (OGY) method [2] and their variants [3,4] require the measurement of the system state, the generation of the control signal, and its application to a system parameter. The time it takes to accomplish these tasks is the latency of the controller that limits the frequency range of chaos control. By simplifying the control scheme using limiters, the new method reduces the latency, which allows for speculations of chaos control in the GHz range. Here we discuss the mathematical foundation of controlling chaos with simple limiters. Consideration of these principles implies that the algorithm can be optimized.

Corron *et al.* [1] presented two experiments where the system is controlled on unstable periodic orbits (UPO) using a limiter based algorithm. In the case of the chaotic driven pendulum, an additional repulsive momentum is applied to the system if the angle of the pendulum exceeds a given threshold. The second example deals with the double scroll oscillator, where a diode acts as a limiter as soon as the amplitude of the voltage exceeds the threshold. In both examples, the correction is proportional to the difference between the value of the state variable without limiter and the threshold. Using the logistic map as chaotic system the implementation of the control scheme yields

$$x_{n+1}=\begin{cases} rx_n(1-x_n), & rx_n(1-x_n)\leqslant x_{\mathrm{th}},\\ (1-\alpha)rx_n(1-x_n)+\alpha x_{\mathrm{th}}, & rx_n(1-x_n)>x_{\mathrm{th}}. \end{cases} \tag{1}$$

Here, x_n and x_{n+1} denote the state variable at time n and $n+1$, respectively, α is the proportionality factor of the perturbation, and x_{th} represents the limiter, which defines the threshold of the state variable above which the correction is applied. For maps, period-k UPOs are determined by the fixed points of the k-fold iterated map f^k, with their stability being given by the derivative of this map at the fixed points. If the absolute value of the derivative of the control map is less than unity, the system can be stabilized on the UPO. The aim of the present work is to show that by use of this condition, the control mechanism can be optimized. Figure 1 shows the modified map [Eq. (1)] for different values of α,

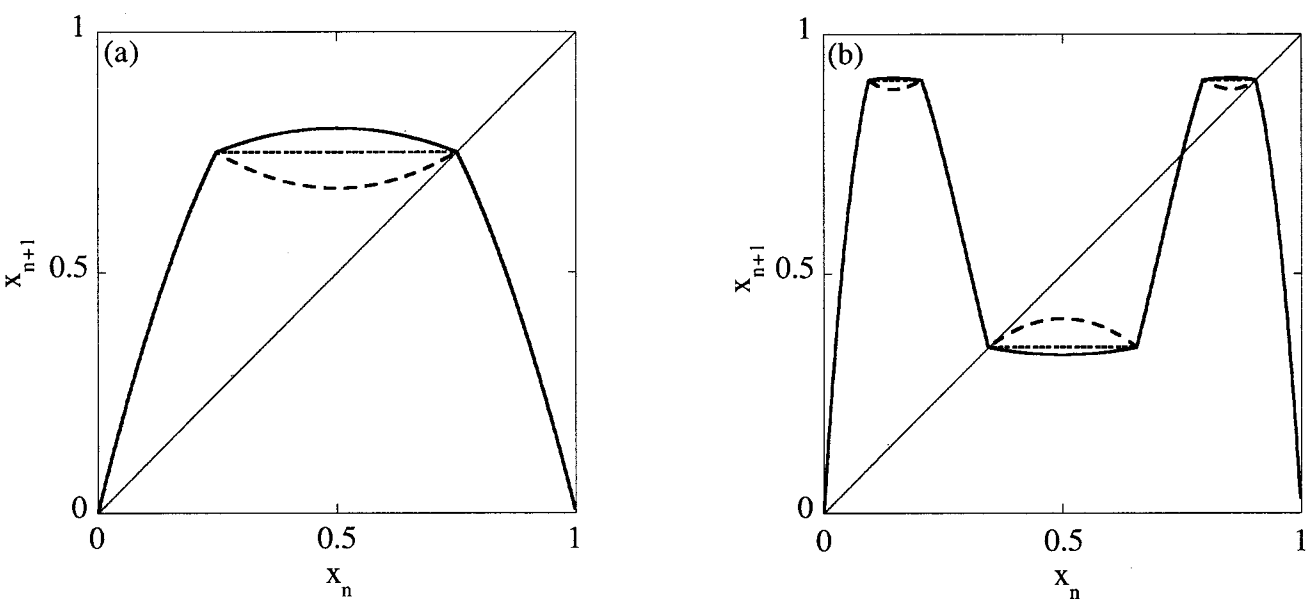

FIG. 1. Modified maps used for limiter-control of the period-1 orbit (a) and the period-2 orbit (b) of the logistic map, see Eq. (1). (a) Full line, $\alpha_{\min}=0.75$; dashed line, $\alpha_{\max}=1.5$. Superstable orbits are obtained for $\alpha=1$ (dotted line). (b) Full line, $\alpha_{\min}=0.937\,50$; dashed line, $\alpha_{\max}=1.249\,99$, dotted line, $\alpha=1$.

where, without loss of generality, we have set the system parameter to $r=4$ (fully developed chaos). There are two limiting values for α within which the map can be controlled: a maximal value $\alpha_{\mathrm{max}}>1$ (dashed line) and a minimal value $\alpha_{\mathrm{min}}<1$ (full line). The special case $\alpha=1$ (dotted line) provides superstable UPOs; this case already has been analyzed by L. Glass *et al.* [5]. As the simplest example for our theoretical analysis, we focus on the unstable period 1 orbit. The derivative of the unperturbed system is $r(1-2x^*)=-2$ (where $x^*=0.75$ denotes the fixed point; the absolute slope larger than 1 refers to an unstable orbit). First we consider the case where $\alpha<1$ (Fig. 1, full line). Driving the system slightly out of the fixed point x^*, the trajectory alternates between the two branches of the map which meet at the fixed point. In order to make the fixed point attractive, the absolute value of the product of the derivatives of the two branches at the fixed point must be smaller than 1. This leads to a condition for the minimal value of α,

$$|(1-\alpha)(r(1-2x^*))^2|<1. \tag{2}$$

Thus, the lower threshold becomes $\alpha_{\mathrm{min}}=0.75$. In the case of $\alpha>1$ (1, dashed line) the trajectory propagates only along the perturbed branch after pushing the system out of the fixed point. Therefore the corresponding condition is

$$|(1-\alpha)r(1-2x^*)|<1, \tag{3}$$

which yields the upper limit $\alpha_{\mathrm{max}}=1.5$. The results can readily be generalized to stabilize orbits of higher periodicity.

This immediately leads to a suggestion on how to improve the first experiment in Ref. [1]. For $\alpha=1$ the derivative becomes zero at the fixed point and the periodic orbits are superstable (Fig. 1, dotted line). Experimentally, this corresponds to limiting weights that cannot be lifted anymore. At this point, the periodic orbits become optimally stable and therefore less sensitive to noise. As a matter of fact, the diode used as limiter in the second experiment of Ref. [1], approximates this case. Finally, the procedure of stabilizing UPOs can be adapted for two-dimensional maps. As an example we determined the range of a for the Henon map given by

$$x_{n+1}=\begin{cases} a+by_n-x_n^2, & a+by_n-x_n^2\leq x_{\mathrm{th}},\\ (1-\alpha)(a+by_n-x_n^2)+\alpha x_{\mathrm{th}}, & a+by_n-x_n^2>x_{\mathrm{th}},\end{cases}$$

$$y_{n+1}=x_n,$$

with parameters $a=1.4$ and $b=0.3$. For the period 1 and the period 2 orbit we obtained $0.790\,11<\alpha<1.681\,29$ and $0.800\,90<\alpha<1.204\,08$, respectively.

Provided that the control scheme is optimized ($\alpha=1$) we believe that the limiter-based approach indeed represents a progress in chaos control, since the cutoff algorithm requires no computational effort in experiments. A minor drawback of the presented control method is that the perturbations may not be small, as during the initial transient, they may be comparable to the system size.

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