

Estimating the Stability of Recurrent Spiking Networks

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1. Introduction

Stability is an important property of a neural network – it determines the character of the network dynamics and the influence of external inputs. Recent studies have demonstrated that recurrent neural networks that operate at the critical point are the most optimal to be used in computation [1]. Additionally, there is a growing body of evidence in the field of neurophysiology that the spontaneous activity of biological neural networks forms clusters in space and time – neuronal avalanches. The statistics of the avalanches follow a power law, suggesting that brain operates at a phase transition between order and chaos [2]. It would be of a high interest to study the properties of critical networks with biologically plausible model constraints. However, one first needs a reliable estimate of the stability of the network. The goal of this project is to numerically estimate the stability of a recurrent spiking network by the means of the largest Lyapunov exponent. This work extends a project from the previous year's *Theorie und Simulation Neuronaler Netze* course, where the stability of echo state networks with firing rate model neurons was assessed [11].

2. Methods

2.1 Network model

The network architecture was chosen to resemble a cortical column and is based on the models proposed in [3,4]. The network consists of 135 neurons arranged in a 3x3x15 grid (Fig. 1). 20% of the neurons are inhibitory. Connectivity probabilities and connection strengths between different types of neurons reflect the network parameters found biological neural networks. Probability of a connection between neuron i and neuron j is given by

$$p_{i,j} = C(i,j) e^{\frac{-d_{i,j}^2}{\lambda^2}}$$

where the connectivity weights C for the connections between excitatory (E) and inhibitory (I) neurons are $C(E,E) = 0.3$, $C(E,I) = 0.2$, $C(I,E) = 0.4$, $C(I,I) = 0.1$. d is the Euclidean distance between neurons in the grid, λ controls the number and typical length of the connections.

Synaptic weights are obtained by drawing from a uniform distribution over $[0,1]$ and then multiplying by strength weights $w(E,E) = 0.3$, $w(E,I) = 0.6$, $w(I,E) = 0.19$, $w(I,I) = 0.19$. To test the network stability as a function of coupling strength, the weights were scaled by a parameter W_{scale} .

In this report values of $\lambda = 1$ and $\lambda = 2$ were considered. They correspond to connectivity of about 1% (≈ 1 synapse per neuron), 4% (5 synapses) which is reminiscent of the sparse connectivity in the brain.

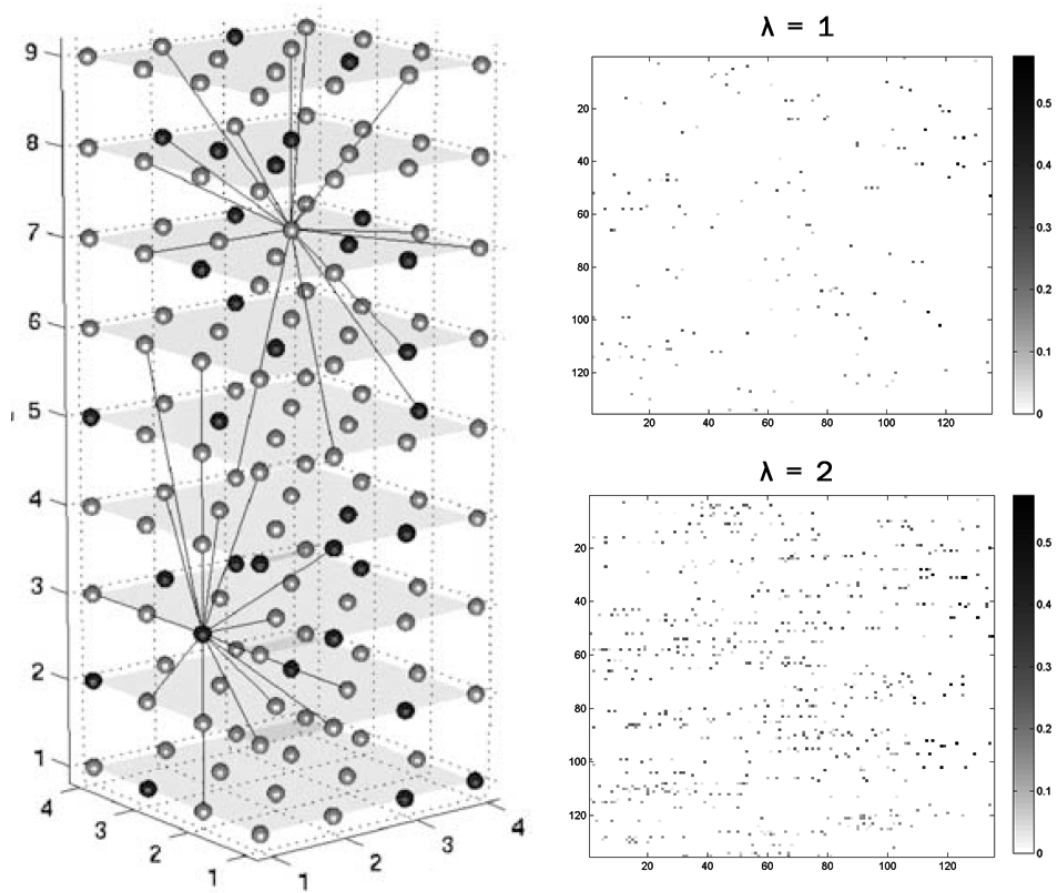


Figure 1. Left panel: an example of a 3-dimensional network architecture and preferred local connectivity between neurons (adapted from [5]). Right panel: weight matrices for $\lambda = 1$ and $\lambda = 2$.

2.2 Neuron model

Neuron model used in this project is the two-dimensional Rulkov map [6]. [7] showed that the map can model similar behaviour to Hodgkin-Huxley model neurons and is computationally very inexpensive. The map can be written in the form:

$$x_{n+1} = f_\alpha(x_n, u) = \begin{cases} \frac{\alpha}{1-x} + u, & x \leq 0 \\ \alpha + u, & 0 < x < \alpha + u \\ -1, & x \geq \alpha + u \end{cases}$$

$$y_{n+1} = y_n - \omega(x_n + 1) + \omega\sigma + \omega\sigma_e I_n$$

where x_n is the fast and y_n the slow dynamical variable. u is the input to the fast map and it's role is to shift the fast map vertically and change the amplitude of x_n . It is composed of the slow variable y_n and synaptic and external influences I_n scaled by β_e

$$u = y_n + \beta_e \cdot I_n$$

The slow dynamics of y_n is due to the very small parameter ω (in the order of 10^{-3}). σ controls the intrinsic properties of the neuron while I_n reflects the influence of synaptic and external inputs scaled by σ_e .

The synaptic currents are described as

$$I_{n+1} = I_{ext} + \gamma I_n - \begin{cases} w(x_n - E), & \text{presynaptic spike} \\ 0, & \text{otherwise} \end{cases}$$

where I_{ext} is the external online input, γ controls the relaxation rate of the synaptic current, (time constant of the current $\tau \approx (1 - \gamma)^{-1}$), w is the synaptic coupling and E is the reversal potential which defines whether the synapse is inhibitory or excitatory. All the parameters used in simulations are given in Table 1.

Network received online input – random 20 Hz Poisson spike trains, different for every neuron (the frequency of the Poisson train was calculated assuming that one iteration of the Rulkov map is approximately 0.5 ms [7]). The input amplitude of a spike was $I_{ext} = 1$.

Table 1. The parameters of Rulkov map and synapses.

α	3.6	E_E	0.0
β_e	0.133	γ_E	0.75
σ	0.04	E_I	-1.1
σ_e	1.0	γ_I	0.75
ω	0.0005		

2.3 Estimation of the largest Lyapunov exponent

A popular method for determining whether a dynamical system is chaotic (and how chaotic) is to estimate the system's sensitivity to initial conditions. If trajectories that start out very close to each other converge toward the same orbit then the system is stable. However, in the case of a chaotic system the trajectories separate exponentially with time, and in a finite time their separation attains the size of the accessible state space [10].

The trajectories can be written as

$$x(t) = f^t(x_0)$$
$$x(t) + \delta x(t) = f^t(x_0 + \delta x_0)$$

The sensitivity of initial conditions can be quantified as

$$\|\delta x(t)\| = \|\delta x_0\| e^{\mu t}$$

where μ is the mean rate of separation of trajectories of the system, also called the largest Lyapunov exponent (LLE). If $\mu > 0$ the trajectories will exponentially diverge, if $\mu < 0$ the trajectories will converge to a stable point or orbit, and $\mu = 0$ describes marginal stability where the initial difference neither explodes nor collapses (criticality). From the equation above follows

$$\log \left(\frac{\|\delta x(t)\|}{\|\delta x_0\|} \right) = \mu t$$

Therefore the LLE will be the slope of the log plot of the time evolution of the difference between trajectories.

The algorithm for estimating LLE is adapted from [8] and [11]. Initial simulation of 7000 iterations was run. Afterwards it was re-run with the same initial conditions but the state vector $\mathbf{x}(n)$ which contains the Rulkov map variables x_n of the network neurons was perturbed at the 1000th iteration. The perturbation was the addition of a uniform random noise vector of magnitude 10^{-8} . In this case the equation for LLE can be written as

$$\log \left(\frac{d(n)}{d_0} \right) = \mu n$$

where $d(n)$ is the Euclidian distance between the original state vector $\mathbf{x}(n)$ and the state vector of the perturbed trajectory, d_0 is the perturbation. This was repeated for 50 times for

each set of network parameters, each time with a different random perturbation. All simulations and analysis were performed in *Matlab*.

3. Results

3.1 Estimation of LLE for a single trial

The estimation of the LLE from the slope of the log plot was not entirely trivial. The slope was never constant and featured sudden downward and upward jumps. In Fig. 2 a typical plot can be seen. It has four distinct regions with a downward slope and the average of the slopes was estimated in order to obtain LLE. Notice that in the beginning as well as at the very end of the graph there are small positive trends. It is not entirely clear whether to include or exclude them from the analysis. The approach used in this work was to include the trends that followed the global trend of the graph. A possible explanation of the jumps in the plot is given in the Discussion.

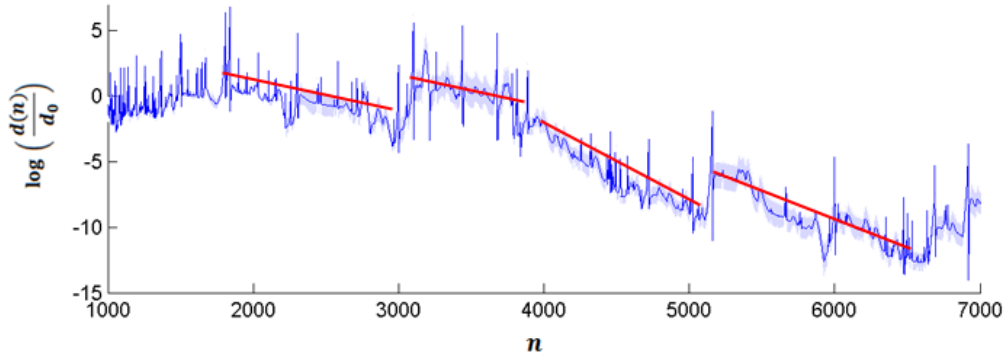


Figure 2. Time evolution of the log of Euclidean distance between original and perturbed network state vector trajectories. Network parameters: $\lambda = 1$, $W_{scale} = 0.4$.

3.2 Relationship between the network parameters and LLE

LLE was estimated for $\lambda = 1$ and $\lambda = 2$ over various coupling strengths (Fig. 3-5). The network undergoes a phase transition from being stable to chaotic as the LLE increases with stronger coupling strengths. Higher connectivity ($\lambda = 2$) gives higher absolute LLE values and the phase transition is less gradual. There is a considerable standard deviation for some of the values – most likely due to the aforementioned uncertainty in the estimation of the slopes.

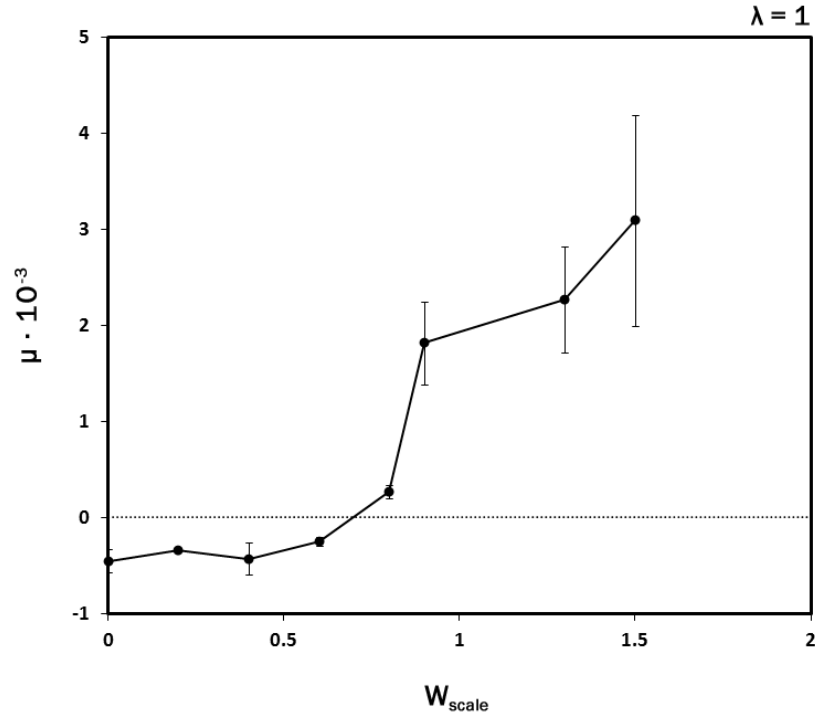


Figure 3. Largest Lyapunov exponent as a function of the scaling factor of synaptic coupling strengths W_{scale} for $\lambda = 1$.

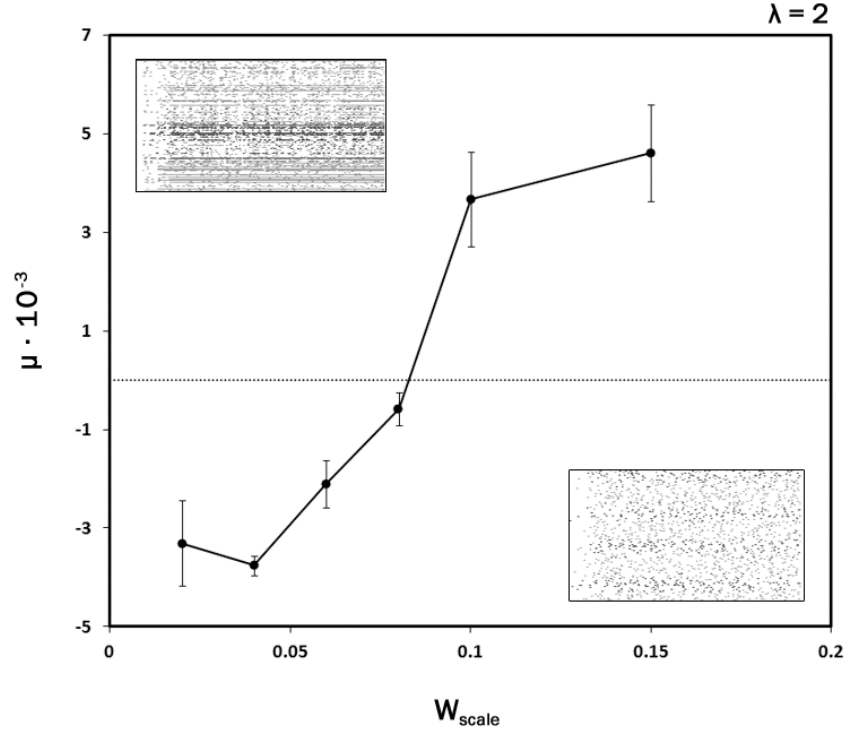


Figure 4. Largest Lyapunov exponent as a function of the scaling factor of synaptic coupling strengths W_{scale} for $\lambda = 2$. Notice the different scales of x-axis in Fig. 3 and Fig. 4. Raster plots of unstable and stable network activity are shown in the upper left and bottom right corners respectively.

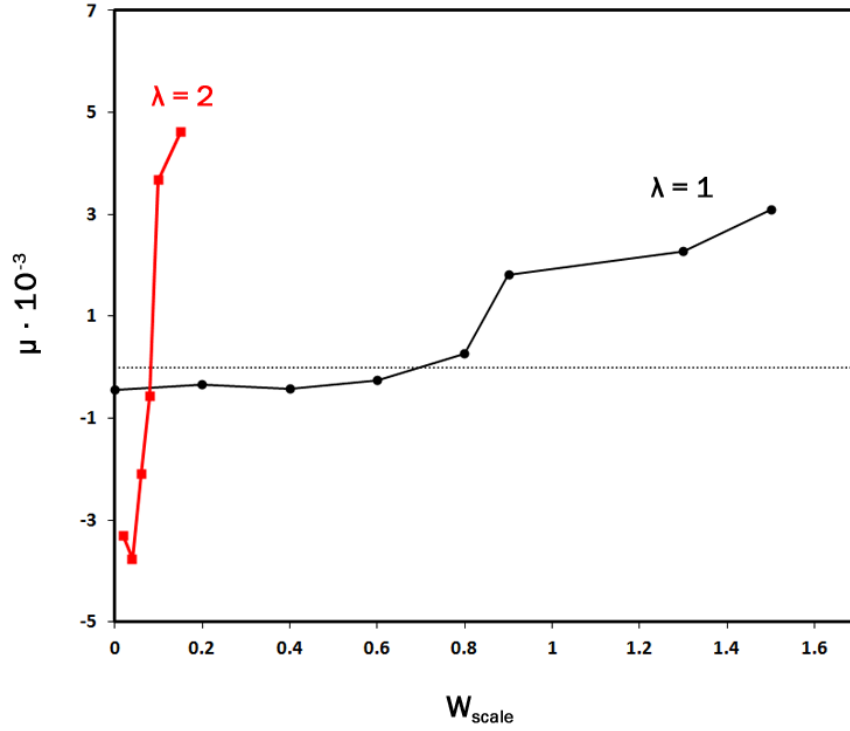


Figure 5. A comparison of both phase transitions. The effects of higher connectivity ($\lambda=2$) can be seen – a steeper transition and higher absolute LLE values for the same coupling strength.

4. Discussion

4.1 Comments on the methods

Similar findings on the stability of spiking recurrent networks have been reported by [1]. Interestingly, they don't find chaotic behaviour for $\lambda=1$. However, the results are cannot be directly compared because [1] uses a different neuron model (integrate-and-fire). In this project it was found that the character of the network immensely depends on the character of the neurons – for certain neuron parameter values (such as $\alpha \approx 3$ or negative σ) the network would be stable for all network parameter values, while for others (e.g., $\alpha = 3.8$ and positive σ which results in tonic spiking) it would be always unstable. For this project the neuron parameters were chosen that corresponded to the traces of a regular spiking neuron from electrophysiological recordings [7] and further tuned in order to see the whole phase transition. It remains an open question which set of parameters of the Rulkov map permit both stable and chaotic network states.

In this project the perturbation was a very small change in the Rulkov map's x_n values of the neurons. In the literature another type of perturbation can be found – shifting of a random spike in the online input [1]. In this way both dynamical variables x_n and y_n are perturbed, but the perturbation would be influenced by the input scaling parameters β_e and σ_e . In the initial stages of this project it was found that this kind of perturbation causes very high variance of the estimated LLE. That might be caused by the fact that only a single neuron is perturbed. For instance, the change in the network state will depend on the number of connections the perturbed neuron has. Furthermore, the resulting initial difference due to the input spike was found to be relatively very large (in the order of 10^{-1}). Taking all this into account, the perturbation of the state vector was deemed to be the better choice.

4.2 Effects of the online input

It must be noted that the network under consideration was receiving external online input. There were two reasons for this. Firstly, online input was necessary to maintain the network activity. An unceasing network activity can be maintained only with a very limited set of parameters of the network and most likely requires some form of plasticity [9]. The design and analysis of such kind of network is beyond the scope of this work. Secondly, neural networks in the brain receive unceasing, time-varying inputs from the sensory systems as well as from other areas. Also in the case of artificial spiking neural networks that perform computations (e.g., Liquid State Machine) there is an online input. Therefore, by having the external input, a more realistic situation was modelled. However, the external input definitely has an effect on the network activity. To get a flavour of what the effect could be, a small experiment was performed where the network activity was maintained by setting high λ and coupling strength which caused high frequency firing for a finite length of iterations. The stability was compared in no input versus online input cases and, while both networks were chaotic, the network receiving online input had a higher LLE (Fig. 6). However, this is just one example and more effort should be made in order to understand the dynamics of the combined input–recurrent network system. Another effect of the input is described further in the next section.

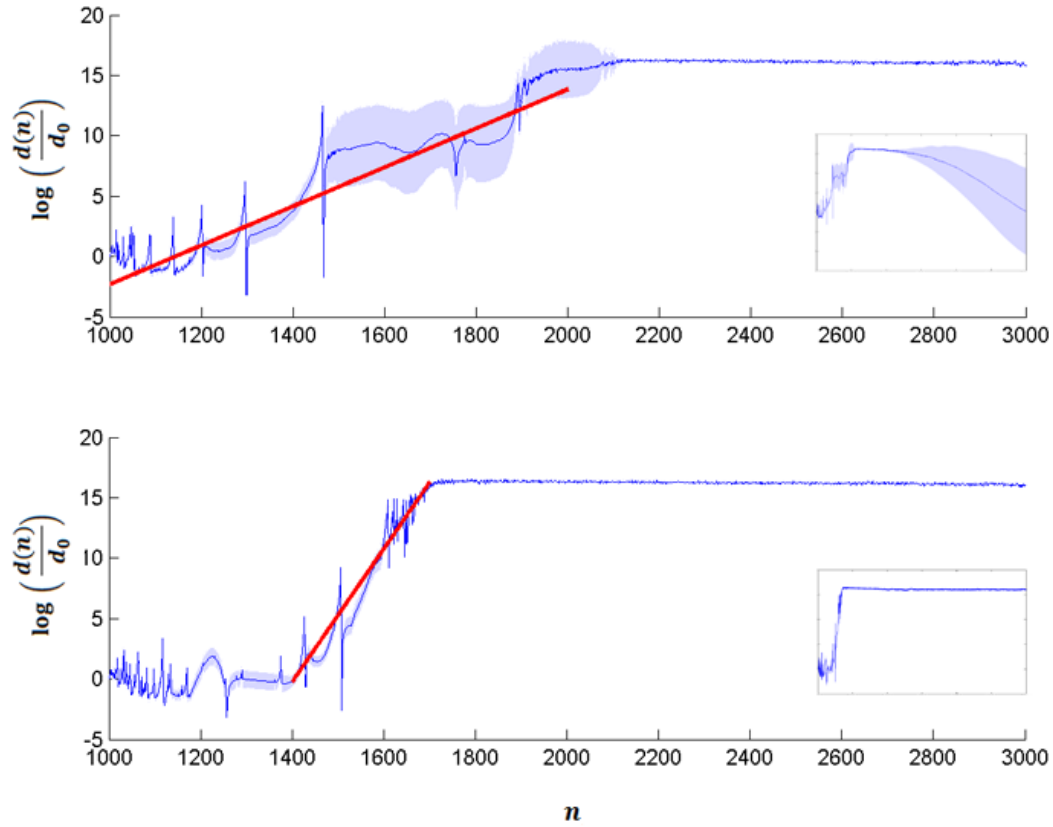


Figure 6. Time evolution of the log of Euclidean distance between original and perturbed network state vector trajectories. Top panel – no external input; bottom panel – external online input. In the insets the log plot for the whole duration of the simulation is shown – notice that the activity of the network with no input dies out and network becomes stable. Network parameters: $\lambda = 2$, $W_{scale} = 0.3$.

4.3 The erratic excursions of the perturbed trajectory

In previous examples one can very well see the erratic behaviour of the time evolution of the distance between the original and perturbed trajectories. In Fig. 2 a jump to a higher or lower value is followed by a change in the separation rate (i.e. change in slope). In Fig. 6 a monotonic increase in the distance is interrupted by spike-like jumps. To better understand the dynamics of the perturbed network, a much simpler case – a single, chaotic Rulkov map – was analyzed using the same LLE estimation algorithm (Fig. 7). Parameters corresponding to a continuous tonic spiking behaviour were chosen, the map was perturbed and the LLE of the map was estimated. Similar spike-like jumps that were observed in Fig. 6 can also be seen in this plot, suggesting that they are directly caused by the dynamics of the Rulkov map.

Furthermore, external online input can be added to the map (Fig. 8). Now in addition to the spike-like jumps there is also a higher amplitude jump which is followed by a change in slope. This indicates that the erratic behaviour of the perturbed network state trajectory can be due to either synaptic or external inputs. Conveniently the average LLE estimated from Fig. 8 is $\mu = 9.7 \cdot 10^{-3}$ which is close to the case of no input $\mu = 9.6 \cdot 10^{-3}$ (Fig. 7). To further validate this observation another small experiment was done with the fully developed parabola map where exactly the same procedure was done (Fig. 9-10). A reset rule was added to the parabola map so that the perturbations would not end up exploding to infinity. Again jumps are observed but the mean LLE is close to the analytically obtained Lyapunov exponent of fully developed parabola $\mu = \log(2)$. This gives credibility to the LLE algorithm used in this project. However, the situation of the erratic excursions is more severe in the whole network case. Perhaps it would be beneficial to find some better estimate for the stability of a dynamical system that would not be so sensitive to the excursions. [10] is especially critical on using Lyapunov exponents to assess stability and suggests using stability exponents and covariant vectors instead.

4. Conclusion

Numerical estimation of the largest Lyapunov exponent of a spiking recurrent neural network with biologically plausible constraints was performed and phase transition diagrams of the network were obtained. Although the results are comparable to the data found in literature, there is a considerable variance in the estimates of the largest Lyapunov exponent due to erratic excursions of the perturbed trajectory. The underlying causes for the excursions were identified to be the intrinsic map dynamics of the neuron model and external inputs. As both of these things are not escapable when dealing with neural networks a better estimate for LLE might be necessary.

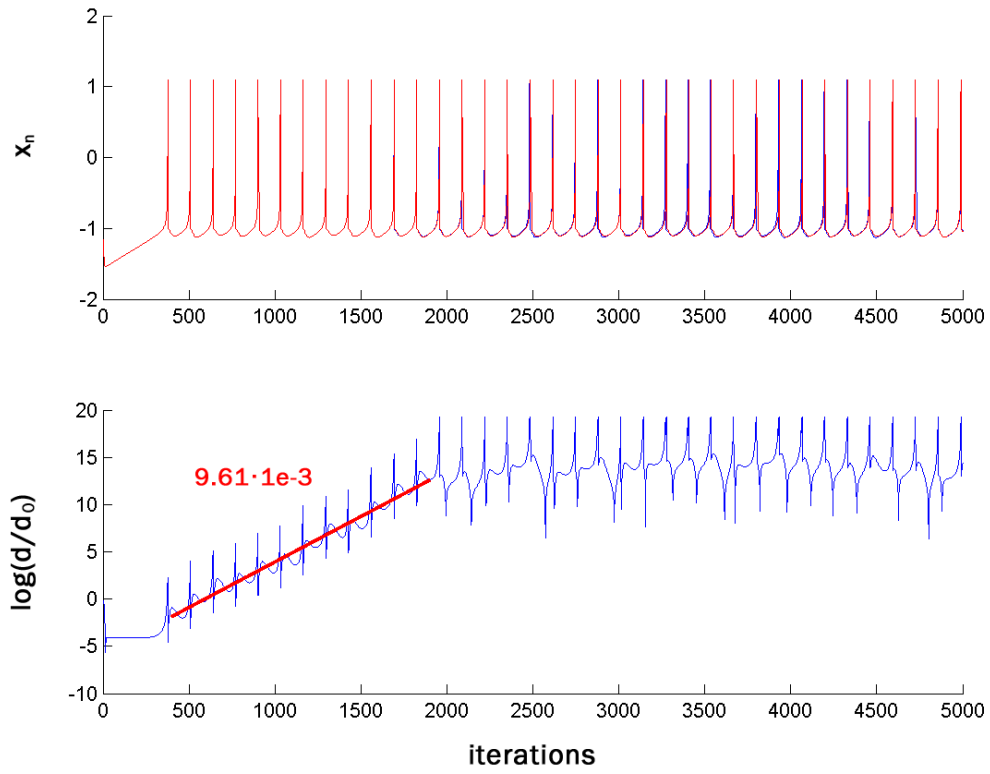


Figure 7. Estimation of the LLE of a single Rulkov map. $\alpha = 4.2$.

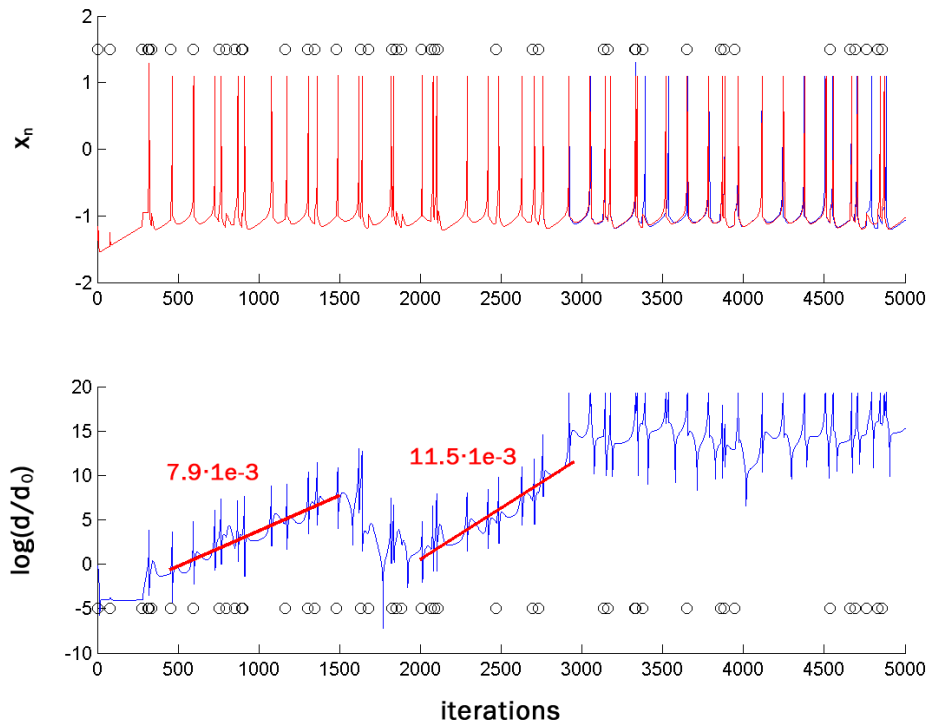


Figure 8. Estimation of the LLE of a single Rulkov map with online input. Input timing is indicated by the black circles. Notice that the average value of both slopes is close to the slope in Fig. 6.

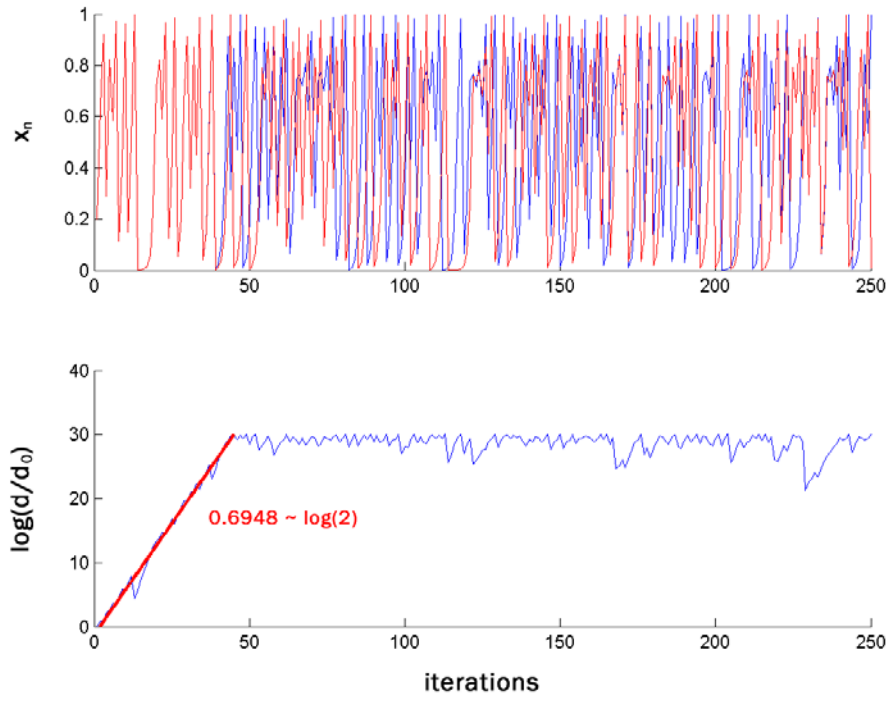


Figure 9. Estimation of the LLE of the fully developed parabola. The slope is close to the analytically estimated Lyapunov exponent $\mu = \log(2)$.

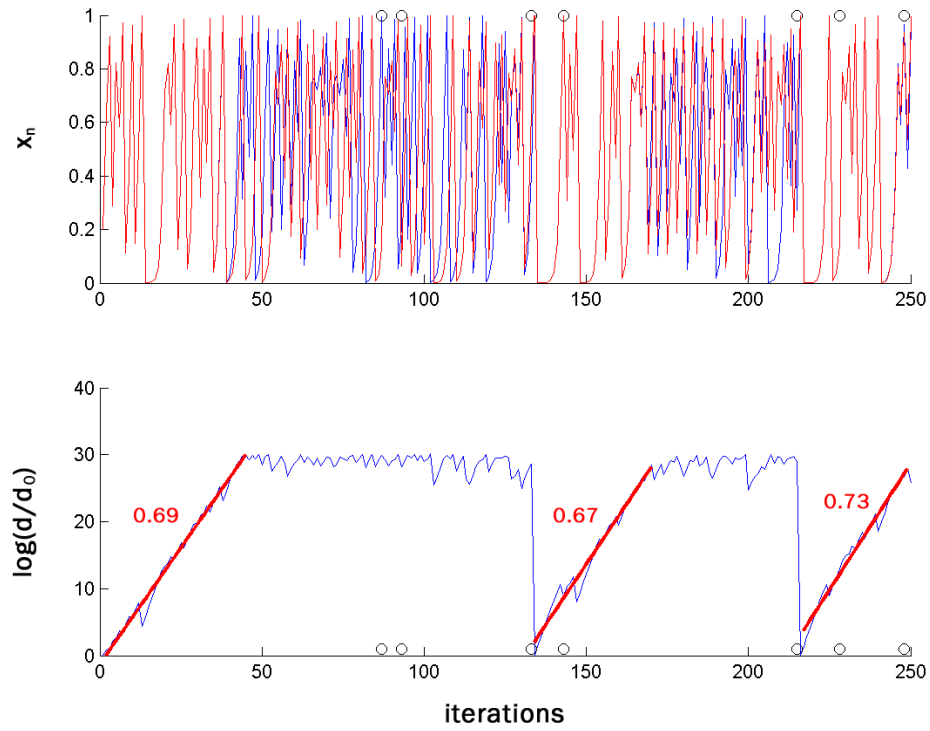


Figure 10. Estimation of the LLE of the fully developed parabola with online input. Input timing is indicated by the black circles. The average slope is close to the analytically estimated Lyapunov exponent.

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