Symlectic Geometry

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Symplectic geometry is a branch of differential geometry that studies symplectic manifolds, which are smooth manifolds equipped with a closed, non-degenerate 2-form called a symplectic form. It originated from classical mechanics.

Definition 1 (Symlectic manifold)**.** Let M be a smooth manifold of even dimension $2m$ and let $\omega \in \Omega^2(M)$ be a closed non degenerate 2-form i.e.

$$d\omega = 0 \text{ and } \omega^m = \omega \wedge \omega \wedge \cdots \wedge \omega \neq 0,$$

Then ω is called a *simplectic form* and the pair (M, ω) is called a *simplectic manifold.*

ekvivalentni definice nedegenerovanosti.
Narozdil od riemannovske geometrie nelze pouzit partitions of unity na konstrukci metriky.
napsat poznamku o koncenci se psanim dimenze manifoldu :D

Example 2 (Canonical symplectic structure)**.** Let $M = \mathbb{R}^{2m}$ with the global coordinates $q_1, \dots, q_m, p_1, \dots, p_m$. and let ω be a form s.t.,

$$\omega = \sum_{i=1}^{m} dp_i \wedge dq_i.$$

Then

$$\omega^m = m! \cdot (-1)^{m(m-1)/2} \cdot dp_1 \wedge \cdots \wedge dp_m \wedge dq_1 \wedge \cdots \wedge dq_m.$$

We call R^2m with the form ω the canonical symplectic structure.

Example 3 (Cotangent bundle is a symplectic manifold.)**.** Let Q be a manifold, and consider the manifold $M = T^*Q$. Then there is a canonical 1-form $\theta \in \Omega^1(M)$ given by

$$\theta(X) = \langle \pi_{T^*Q}(X), T(\pi_Q)(X) \rangle, \quad X \in T(T^*Q), \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between tangent and cotangent spaces and the projections are the following:

$$\begin{tikzcd} & T(T^*Q) \arrow[dl, "\pi_{T^*Q}"'] \arrow[dr, "T(\pi_Q)"] & \\ T^*Q \arrow[dr, "\pi_Q"'] & & TQ \arrow[dl, "\pi_Q"] \\ & Q & \end{tikzcd} \qquad \begin{tikzcd} & (q,p;\xi,\eta) \arrow[dl, "\pi_{T^*Q}"'] \arrow[dr, "T(\pi_Q)"] & \\ (q,p) \arrow[dr, "\pi_Q"'] & & (q,\xi) \arrow[dl, "\pi_Q"] \\ & q & \end{tikzcd}$$

Let $q = (q^1, \dots, q^n) : U \to \mathbb{R}^n$ be a chart on Q, the we have the induced chart $T^*q : T^*U \to \mathbb{R}^n \times \mathbb{R}^n$, where $T_x^*q = (T_x q^{-1})^*$, we put $p_i := \langle e_i, T^*q(\cdot) \rangle : T^*U \to \mathbb{R}$. Then $(q^1, \dots, q^n, p_1, \dots, p_n) : T^*U \to \mathbb{R}^n \times (\mathbb{R}^n)^*$ are the induced coordinates and locally in these coordinates

$$\theta(q,p) = \sum_{i=1}^{n} \left(\theta\left(\frac{\partial}{\partial q^i}\right) dq^i + \theta\left(\frac{\partial}{\partial p_i}\right) dp_i \right) = \sum_{i=1}^{n} p_i \, dq^i + 0, \tag{2}$$

since $\theta\left(\frac{\partial}{\partial q^i}\right) = \theta_{R^n}\left((q,p;e_i,0)\right) = \langle p, e_i \rangle = p_i$.

Now we define the *canonical symplectic structure* $\omega \in \Omega^2(T^*Q)$ by

$$\omega := -d\theta \overset{\text{locally}}{=} \sum_{i=1}^{n} dq^i \wedge dp_i. \tag{3}$$

We see that the 2-form ω is non-degenerate.

dukaz ze je neni degen?

Definition 4. Let $X : J \times M \to TM$ be a smooth mapping such that $\pi_M \circ X = pr_2$, where J is open. Then we call X a *time dependent vector field* on a manifold M.

There is an associated vector field $\bar{X} \in \mathfrak{X}(J \times M)$, given by $\bar{X}(t,x) = (\frac{\partial}{\partial t}, X(t,x)) \in T_t\mathbb{R} \times T_xM$.

Definition 5. Let X be a time dependent vector field on a manifold M and let $\Phi^X : J \times J \times M \to M$ be a map defined on a maximal neighborhood of $\Delta_J \times M$ satisfying the differential equation

$$\begin{aligned} \frac{d}{dt}\Phi^X(t,s,x) &= X\left(t, \Phi^X(t,s,x)\right) \\ \Phi^X(s,s,x) &= x \end{aligned} \tag{4}$$

Definition 5 is equivalent with

$$(t, \Phi^X(t,s,x)) = Fl^{\bar{X}}(t-s,(s,x)),$$

so the evolution operator exits and is unique on a maximal integral curve and satisfies

$$\Phi^X_{t,s} = \Phi^X_{t,r} \circ \Phi^X_{r,s}, \text{ where } \Phi^X_{t,r}(x) = \Phi(t,s,x).$$

Lemma 6. *Let f_t be a curve of diffeomorphisms on a manifold M locally defined for each t such that $f_0 = Id$. Defined two time dependent vector fields*

$$\xi_t(x) := (T_xf_t)^{-1}\frac{\partial}{\partial t}f_t(x), \;\; \eta_t(x) := \left(\frac{\partial}{\partial t}f_t\right)\left(f_t^{-1}(x)\right) \tag{5}$$

Then $T(f_t) \cdot \xi_t = \frac{\partial}{\partial t}f_t = \eta_t \circ f_t$, so ξ_t and η_t are f_t-related. Let $\omega \in \Omega^k(M)$. Then

$$i_{\xi_t} f_t^*\omega = f_t^* i_{\eta_t}\omega, \tag{6}$$

$$\frac{\partial}{\partial t} f_t^*\omega = f_t^*\mathcal{L}_{\eta_t}\omega = \mathcal{L}_{\xi_t} f_t^*\omega. \tag{7}$$

Proof.

$$\begin{aligned} \left(i_{\xi_t} f_t^*\omega\right)_x(X_2,\dots,X_k) &= \left(f_t^*\omega\right)_x(\xi_t(x), X_2,\dots,X_k) = \\ &= \omega_{f_t(x)}(T_xf_t \cdot \xi_t(x), T_xf_t \cdot X_2, \dots, T_xf_t \cdot X_k) = \\ = \omega_{f_t(x)}\left(\eta_t\left(f_t(x)\right), T_xf_t \cdot X_2,\dots,T_xf_t\cdot X_k\right) &= \left(f_t^* i_{\xi_t}\omega\right)_x(X_2,\dots,X_k) \end{aligned}$$

This proves (6). Now consider $\bar{\eta} \in \mathfrak{X}(\mathbb{R} \times M), \bar{\eta}(t,x) = (\partial_t, \eta_t(x))$ and let $\Phi^\eta : \mathbb{R} \times \mathbb{R} \times M \to M$ be the evolution operator, i.e.

$$\frac{\partial}{\partial t}\Phi^\eta_{t,s}(x) = \eta_t\left(\Phi^\eta_{t,s}(x)\right), \quad \Phi^\eta_{s,s}(x) = x,$$

such that

$$\left(t, \Phi^\eta_{t,s}(x)\right) = \mathrm{Fl}^{\bar{\eta}}_{t-s}(s,x), \;\; \Phi^\eta_{t,s} = \Phi^\eta_{t,r} \circ \Phi^\eta_{r,s}(x).$$

Since f_t satisfies $\frac{\partial}{\partial t}f_t = \eta_t \circ f_t$ and $f_0 = Id_M$, either $f_t = \Phi^\eta_{t,0}$, or $(t, f_t(x)) = Fl^{\bar{\eta}}_t(0,x)$, so $f_t = pr_2 \circ Fl^{\bar{\eta}}_t \circ ins_0$. Thus

$$\frac{\partial}{\partial t} f_t^*\omega = \frac{\partial}{\partial t}\left(\mathrm{pr}_2 \circ Fl^{\bar{\eta}}_t \circ ins_0\right)^*\omega = ins_0^*\frac{\partial}{\partial t}(Fl^{\bar{\eta}}_t)^* pr_2^*\omega = ins_0^*(Fl^{\bar{\eta}}_t)^*\mathbb{L}_{\bar{\eta}} pr_2^*\omega.$$

For time dependant vector fields X_i(tady mozna nejaka vlastnost lie derivative!!!) we have

$$\begin{aligned}
\left(\mathcal{L}_{\bar\eta}\,\mathrm{pr}_2^*\,\omega\right)(0\times X_1,\dots,0\times X_k)|_{(t,x)} &= \bar\eta((\mathrm{pr}_2^*\,\omega)(0\times X_1,\dots))|_{(t,x)} \\
&\quad -\sum_i (\mathrm{pr}_2^*\,\omega)(0\times X_1,\dots,[\bar\eta,0\times X_i],\dots,0\times X_k)|_{(t,x)} \\
= (\partial_t,\eta_t(x))\left(\omega\left(X_1,\dots,X_k\right)\right) &- \sum_i \omega\left(X_1,\dots,\left[\eta_t,X_i\right],\dots,X_k\right)|_x \\
&= \left(\mathcal{L}_{\eta_t}\omega\right)_x\left(X_1,\dots,X_k\right).
\end{aligned}$$

For $X_i \in T_xM$, this implies

$$\begin{aligned}
\left(\frac{\partial}{\partial t} f_t^*\omega\right)_x (X_1,\dots,X_k) &= \left(\mathrm{ins}^*\left(\mathrm{Fl}_t^\eta\right)^*\mathcal{L}_\eta\,\mathrm{pr}_2^*\omega\right)_x (X_1,\dots,X_k) \\
&= \left(\left(\mathrm{Fl}_t^\eta\right)^*\mathcal{L}_\eta\,\mathrm{pr}_2^*\omega\right)_{(0,x)}(0\times X_1,\dots,0\times X_k) \\
&= \left(\mathcal{L}_\eta\,\mathrm{pr}_2^*\omega\right)_{(t,f_t(x))}(0_t\times T_xf_t\cdot X_1,\dots,0_t\times T_xf_t\cdot X_k) \\
&= \left(\mathcal{L}_{\eta_t}\omega\right)_{f_t(x)}(T_xf_t\cdot X_1,\dots,T_xf_t\cdot X_k) \\
&= \left(f_t^*\mathcal{L}_{\eta_t}\omega\right)_x (X_1,\dots,X_k),
\end{aligned} \tag{8}$$

We have proven the first part of (7), the second part follow from (6)

$$\begin{aligned}
\frac{\partial}{\partial t} f_t^*\omega &= f_t^*\mathcal{L}_{\eta_t}\omega \\
&= f_t^*\left(di_{\eta_t} + i_{\eta_t}d\right)\omega \\
&= df_t^* i_{\eta_t}\omega + f_t^* i_{\eta_t} d\omega \\
&= di_{\xi_t} f_t^*\omega + i_{\xi_t} f_t^* d\omega \\
&= di_{\xi_t} f_t^*\omega + i_{\xi_t} df_t^*\omega \\
&= \mathcal{L}_{\xi_t} f_t^*\omega.
\end{aligned} \tag{9}$$

□

dopsat co je ins a pr2?

Theorem 7 (Darboux)**.** *Let (M,ω) be a symplectic manifold of dimension $2n$. Then for all points $x \in M$ exists a chart (U,u) centered at x such that $\omega|_U = \sum_{i=1}^n du^i \wedge du^{n+i}$.*

Proof. Take a chart (U,u) centered at x and choose coordinates such that $\omega_x = \sum_{i=1}^n du^i \wedge du^{n+i}$ at x. Then $\omega_0 = \omega|_U$ and $\omega_1 = \sum_{i=1}^n du^i \wedge du^{n+i}$ are two symplectic forms that are equal at x. Now interpolate $\omega_t = \omega_0 + t(\omega_1 + \omega_0)$. Then ω_t is a symplectic form on a possibly smaller neighbourhood of x for all $t \in [0,1]$.

We want to find a curve of diffeomorphisms f_t near x such that $f_0 = id$, $f_t(x) = x$ and such that the pullback condition $f_t^*\omega_t = \omega_0$ is satisfied. Assume that U is contractible, then the second cohomology group $H^2(U) = 0$ and every closed 2-form is exact, so $d(\omega_1 - \omega_0) = 0$ implies $\omega_1 - \omega_0 = d\psi$ for some $\psi \in \Omega^1(U)$. By adding a constant we may assume that $\psi_x = 0$. Now by using Lemma 6, (7), we get a time dependant vector field $\eta_t = \frac{\partial}{\partial t} f_t \circ f_t^{-1}$, then by differentiating with respect to t, (cartan formula!!)

$$0 = \frac{\partial}{\partial t} f_t^*\omega_t = f_t^*\left(\mathcal{L}_{\eta_t}\omega_t + \frac{\partial}{\partial t}\omega_t\right) = f_t^*\left(di_{\eta_t}\omega_t + i_{\eta_t}d\omega_t + \omega_1 - \omega_0\right) = f_t^* d\left(i_{\eta_t}\omega_t + \psi\right)$$

Since ω_t is non-degenerate, the equation $i_{\eta_t}\omega_t = -\psi$ prescribes the vector field η_t uniquely. Also $\eta_t(x) = 0$ sine $\psi_x = 0$. On some neighbourhood of x the left evolution operator f_t of η_t exists for all $t \in [0,1]$ and $\frac{\partial}{\partial t}(f_t^*\omega_t) = 0$, so $f_t^*\omega_t = \omega_0$ for all $t \in [0,1]$. □

tohle je v lsg.pdf

lagrangian subspace? submanifold? embeddings? asi spis ne....

Definition 8 (Lagrangian submanifold)**.** Let (M^{2m}, ω) be a symplectic manifold. We call a submanifold Y of M lagrangian, if at each $p \in Y$, T_pY is a lagrangian subspace of T_pM , i.e., $\omega_p|_{T_pY} \equiv 0$ and $dimT_pY = \frac{1}{2}dimT_pM$. Equivalently, if $i : Y \to M$ is the inclusion map, then Y is lagrangian if and only if $i * \omega = 0$ and $dimY = \frac{1}{2}dimM$

dukazem tehle sracky se napise i priklad ktery chceme napsat

Theorem 9. *the image of a section of T^*M is a lagrangian submanifold iff nejaka forma je close...lsg strana 17.*

lsg dalsi section

Example 10. Let $S \subset X$ be a submanifold, then the conormal bundle N^*S is a lagrangian submanifold of T^*x.

Lemma 11 (Lemmatko)**.** $2 + 2 = 4 - 1 = 3$ *quick maffs.*

A ted si rekneme dulezitou vetu.

Theorem 12 (Hlavni veta o gaystvi)**.** *Jsi gay.*

Corollary 13. *Vlastne dusledek tohoto kratkeho textu je, ze bych se mel jit zabit. Jdu na to!*