

# Simplicial Geometry

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Ahoj, já mám malý penis.

**Definition 1** (Symplectic manifold). Let  $M$  be a smooth manifold of even dimension  $2m$  and let  $\omega \in \Omega^2(M)$  be a closed non degenerate 2-form i.e.

$$d\omega = 0 \text{ and } \omega^m = \omega \wedge \omega \wedge \cdots \wedge \omega \neq 0,$$

Then  $\omega$  is called a *symplectic form* and the pair  $(M, \omega)$  is called a *symplectic manifold*.

ekvivalentni definice nede degenerovanosti.

Narozdil od riemannovske geometrie nelze pouzit partitions of unity na konstrukci metriky.

napsat poznamku o koncenci se psanim dimenze manifoldu :D

**Example 2** (Canonical symplectic structure). Let  $M = \mathbb{R}^{2m}$  with the global coordinates  $q_1, \dots, q_m, p_1, \dots, p_m$ . and let  $\omega$  be a form s.t.,

$$\omega = \sum_{i=1}^m dp_i \wedge dq_i.$$

Then

$$\omega^m = m! \cdot (-1)^{m(m-1)/2} \cdot dp_1 \wedge \cdots \wedge dp_m \wedge dq_1 \wedge \cdots \wedge dq_m$$

We call  $\mathbb{R}^{2m}$  with the form  $\omega$  the canonical symplectic structure.

**Example 3** (Cotangent bundle is a symplectic manifold.). Let  $M$  be a manifold of dimension  $m$ , let  $\eta \in T^*M$  be a tangent covector and  $\nu \in T_\eta(T^*M)$  be a tangent vector at  $\eta$ . Represent  $\nu$  as a curve  $\nu : (-\epsilon, \epsilon) \rightarrow T^*M$  s.t.

$$\nu(0) = \eta, \quad \nu'(0) = \nu$$

Project this curve by the projection  $\pi : T^*M \rightarrow M$  and apply  $\eta$  to the tangent vector of the projected curve

$$\theta(\nu) := \eta \left( \frac{d}{dt} (\pi \circ \nu(t))|_{t=0} \right) \quad (1)$$

Then  $\omega = d\theta$  is a symplectic form on  $T^*M$ . Any system of coordinates  $\{q_1, \dots, q_m\}$  in  $M$  determines coordinates  $\{q_1, \dots, q_m, p_1, \dots, p_m\}$  in  $T^*M$  by the relation  $\eta = \sum p_i \cdot dq_i$ . From (1) we have

TOHLE SPOCITAT!!!!!!!!!!!!!!

$$\theta = \sum_{i=1}^m p_i \cdot dq_i$$

and the 2-form

$$\omega = d\theta = \sum_{i=1}^m p_i \wedge dq_i$$

is non-degenerate.

**Theorem 4** (Darboux theorem). Let  $M^{2m}, \omega$  be a symplectic manifold. Then for all  $x \in M$  exists a chart  $(U, \phi)$  such that  $\omega$  is the pullback of the usual symplectic form  $\omega = \phi^* \left( \sum_{i=1}^m dp_i \cdot dq_i \right)$

*Proof.* In  $T_x^*M$  choose a basis  $\sigma_1, \dots, \sigma_m, \mu_1, \dots, \mu_m$  such that  $\omega$  is in normal form i.e.

$$\omega(x) = \sum_m^{i=1} \sigma_i \wedge \mu_i.$$

Now consider a chart  $\phi : V \rightarrow \mathbb{R}^{2m}$  around  $x$  such that

$$\phi(x) = 0 \text{ and } \omega(x) = \phi^* \left( \sum_m^{i=1} \sigma_i \wedge \mu_i(0) \right)$$

Denote the corresponding symplectic form on  $V$  by

$$\omega_1 = \phi^* \left( \sum_m^{i=1} \sigma_i \wedge \mu_i(0) \right).$$

Then there exists a neighbourhood  $U \subset V$  such that for all  $t \in [0, 1]$  the form

$$\omega_t = (1 - t)\omega + t \cdot \omega_1$$

is a symplectic form on  $U$ . Also  $d\omega_t = 0$  and since  $\omega_t(x) = \omega(x)$  for all  $t$ . By compactness, all  $\omega_t$  do not degenerate at the same time in a neighbourhood of  $x$ .

**TADY POINCARÉ LEMMA**

Since  $d(w_1 - w_0) = 0$ , Poincaré lemma shows the existence of a 1-form  $\alpha$  such that  $\omega_1 - \omega_0 = d\alpha$ . By subtracting locally (if necessary) a 1-form with constant coefficients from  $\alpha$ , we may assume, that  $\alpha$  vanishes at the point  $x$ ,  $\alpha(x) = 0$ . By dualizing  $\alpha$  by using the symplectic forms  $\omega_t$ , we get a family  $\Omega_t$  of vector fields on  $U$  parametrized by  $t$

$$\omega_t(\nu, \mathcal{W}_t) = \alpha(\nu).$$

Let  $\varphi(y, t) \in M$  be a solution of the non-autonomous differential equation

$$\varphi'(t) = \mathcal{W}_t(\varphi(t)), \quad \varphi(0) = y$$

All the vector fields  $\mathcal{W}_t(x) = 0$  vanish at  $x$ , so the solution corresponding to  $x$  is constant  $\varphi(x, t) = x$ . So there exists a neighbourhood of  $x$   $U_1 \subset U$  such that for every initial condition  $y \in U_1$ ,  $\varphi(y, t)$  is defined on  $[0, 1]$ . Let  $\varphi_t : U_1 \rightarrow M$  be the corresponding map. Then the formula for TADY CHYBI NEJAKA VETA :d implies that

$$\begin{aligned} \frac{d}{dt} \varphi_t^*(\omega_t) &= \varphi_t^* \left( \frac{d\omega_t}{dt} \right) + \varphi_t^* (\mathcal{L}_{\partial\varphi_t/\partial t}(\omega_t)) = \varphi_t^*(\omega_t - \omega) + \varphi_t^*(\mathcal{L}_{W_t}(\omega_t)) \\ &= \varphi_t^*(\omega_t - \omega + d(W_t \lrcorner \omega_t) + W_t \lrcorner d\omega_t) = \varphi_t^*(\omega_t - \omega - d\alpha) = 0. \end{aligned}$$

Thus  $(\varphi_t^*(\omega_1) = \varphi_0^*(\omega) = \omega)$ , and  $(\Phi \circ \varphi_1)$  is the chart we were looking for,

$$(\Phi \circ \varphi_1)^* \left( \sum_{i=1}^m dp_i \wedge dy_i \right) = \omega.$$

□

**Lemma 5** (Lemmatko).  $2 + 2 = 4 - 1 = 3$  quick maffs.

A ted si rekneme dulezitou vetu.

**Theorem 6** (Hlavni veta o gaystvi). *Jsi gay.*

**Corollary 7.** *Vlastne dusledek tohoto kratkeho textu je, ze bych se mel jit zabít. Jdu na to!*