Simplicial Geometry

Radim Čech

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Ahoj, já mám malý penis.

Definition 1 (Simplectic manifold)**.** Let M be a smooth manifold of even dimension $2m$ and let $\omega \in \Omega^2(M)$ be a closed non degenerate 2-form i.e.

$$d\omega = 0 \text{ and } \omega^m = \omega \wedge \omega \wedge \cdots \wedge \omega \neq 0,$$

Then ω is called a *simplectic form* and the pair (M, ω) is called a *simplectic manifold.*

ekvivalentni definice nedegenerovanosti.
Narozdil od riemannovske geometrie nelze pouzit partitions of unity na konstrukci metriky.
napsat poznamku o koncenci se psanim dimenze manifoldu :D

Example 2 (Canonical symplectic structure)**.** Let $M = \mathbb{R}^2 m$ with the global coordinates $q_1, \ldots, q_m, p_1, \ldots, p_m$. and let ω be a form s.t.,

$$\omega = \sum_{i=1}^{m} dp_i \wedge dq_i.$$

Then

$$\omega^m = m! \cdot (-1)^{m(m-1)/2} \cdot dp_1 \wedge \cdots \wedge dp_m \wedge dq_1 \wedge \cdots \wedge dq_m$$

We call $R^2 m$ with the form ω the canonical symplectic structure.

Example 3 (Cotangent bundle is a symplectic manifold.)**.** Let M be a manifold of dimension m, let $\eta \in T^*M$ be a tangent covector and $\nu \in T_\eta(T^*M)$ be a tangent vector at η. Represent ν as a curve $\nu : (-\epsilon, \epsilon) \to T^*M$ s.t.

$$\nu(0) = \eta, \;\; \nu'(0) = \nu$$

Project this curve by the projection $\pi : T^*M \to M$ and apply η to the tangent vector of the projected curve

$$\theta(\nu) := \eta\Big(\frac{d}{dt}\big(\pi \circ \nu(t)|_{t=0}\big)\Big) \tag{1}$$

Then $\omega = d\theta$ is a simlectic form on T^*M. Any system of coordinates $\{q_1, \ldots, q_m\}$ in M determines coordinates $\{q_1, \ldots, q_m, p_1, \ldots, p_m\}$ in T^*M by the relation $\eta = \sum p_i \cdot dq_i$. From (1) we have

TOHLE SPOCITAT!!!!!!!!!!!!!!

$$\theta = \sum_{i=1}^{m} p_i \cdot dq_i$$

and the 2-form

$$\omega = d\theta = \sum_{i=1}^{m} p_i \wedge dq_i$$

is non-degenerate.

tady nejak rozepsat predpoklady, lcordner je iota jakoby :ddddd

Lemma 4. *The Lie derivative of a differential form is expressed in terms of the exterior derivative and the inner product as*

$$\mathcal{L}_V(\omega^i) = d(V \lrcorner\, \omega^i) + V \lrcorner\, (d\omega^i).$$

Lemma 5 (Poincare's Lemma)**.** *Let U be a star shaped open set in R^n, then for each closed k-form $\omega \in \Omega^k(U)$ there exists a (k-1)-form $\eta \in \Omega^{k-1}(U)$ such that $d\eta = \omega$.*

Theorem 6 (Darboux theorem)**.** *Let M^{2m}, ω be a symplectic manifold. Then for all $x \in M$ exists a chart (U, ϕ) such that ω is the pullback of the usual symplectic form $\omega = \phi * \left(\sum_m^{i=1} dp_i \cdot dq_i \right)$*

Proof. In $T_x^* M$ choose a basis $\sigma_1, \ldots, \sigma_m, \mu_1, \ldots, \mu_m$ such that ω is in normal form i.e.

$$\omega(x) = \sum_m^{i=1} \sigma_i \wedge \mu_i.$$

Now consider a chart $\phi : V \to \mathbb{R}^2 m$ around x such that

$$\phi(x) = 0 \text{ and } \omega(x) = \phi * \Big(\sum_m^{i=1} \sigma_i \wedge \mu_i(0) \Big)$$

Denote the corresponding symplectic form on V by

$$\omega_1 = \phi * \Big(\sum_m^{i=1} \sigma_i \wedge \mu_i(0) \Big).$$

Then there exists a neighbourhood $U \subset V$ such that for all $t \in [0, 1]$ the form

$$\omega_t = (1 - t) omega + t \cdot omega_1$$

is a symplectic form on U. Also $d\omega_t = 0$ and since $\omega_t(x) = \omega(x)$ for all t. By compactness, all ω_t do not degenerate at the same time in a neighbourhood of x. Since $d(w_1 - w_0) = 0$, Lemma 5 shows the existence of a 1-form α such that $\omega_1 - \omega_0 = d\alpha$. By substracting locally (if necessary) a 1-form with constant coefficients from α, we may assume, that α vanishes at the point x, $\alpha(x) = 0$. By dualizing α by using the symplectic forms ω_t, we get a family Ω_t of vector fields on U parametrized by t

$$\omega_t(\nu, \mathcal{W}_t) = \alpha(\nu).$$

Let $\varphi(y, t) \in M$ be a solution of the non-autonomous differential equation

$$\varphi'(t) = \mathcal{W}_t(\varphi(t)), \;\; \varphi(0) = y$$

All the vector fields $\mathcal{W}_t(x) = 0$ vanish at x, so the solution corresponding to x is constant $\varphi(x, t) = x$. So there exists a neighbourhood of x $U_1 \in U$ such that for every initial condition $y \in U_1, \varphi(y, t)$ is defined on $[0, 1]$. Let $\varphi_t : U_1 \to M$ be the corresponding map. Then by application of Lemma 4,

$$\begin{aligned} \frac{d}{dt} \varphi_t^*(\omega_t) = \varphi_t^* \left(\frac{d\omega_t}{dt} \right) + \varphi_t^* \left(\mathcal{L}_{\partial\varphi_t/\partial t}(\omega_t) \right) = \varphi_t^*(\omega_t - \omega) + \varphi_t^* \left(\mathcal{L}_{W_t}(\omega_t) \right) \\ = \varphi_t^*(\omega_t - \omega + d(W_t \lrcorner \omega_t) + W_t \lrcorner d\omega_t) = \varphi_t^*(\omega_t - \omega - d\alpha) = 0. \end{aligned}$$

Thus $(\varphi_t^*(\omega_1) = \varphi_0^*(\omega) = \omega)$, and $(\Phi \circ \varphi_1)$ is the chart we were looking for,

$$(\Phi \circ \varphi_1)^* \left(\sum_{i=1}^{m} dp_i \wedge dy_i \right) = \omega.$$

□

tohle je v lsg.pdf
lagrangian subspace? submanifold? embeddings? asi spis ne....

Definition 7 (Lagrangian submanifold)**.** Let (M^{2m}, ω) be a symplectic manifold. We call a submanifold Y of M lagrangian, if at each $p \in Y$, $T_p Y$ is a lagrangian subspace of $T_p M$, i.e., $\omega_p|_{T_p Y} \equiv 0$ and $dim T_p Y = \frac{1}{2} dim T_p M$. Equivalently, if $i : Y \to M$ is the inclusion map, then Y is lagrangian if and only if $i * \omega = 0$ and $dim Y = \frac{1}{2} dim M$

dukazem tehle sracky se napise i priklad ktery chceme napsat

Theorem 8. *the image of a section of T^*M is a lagrangian submanifold iff nejaka forma je close...lsg strana 17.*

lsg dalsi section

Example 9. Let $S \subset X$ be a submanifold, then the conormal bundle N^*S is a lagrangian submanifold of T^*x.

Lemma 10 (Lemmatko)**.** $2 + 2 = 4 - 1 = 3$ *quick maffs.*

A ted si rekneme dulezitou vetu.

Theorem 11 (Hlavni veta o gaystvi)**.** *Jsi gay.*

Corollary 12. *Vlastne dusledek tohoto kratkeho textu je, ze bych se mel jit zabit. Jdu na to!*