Symlectic Geometry

Radim Čech

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Symplectic geometry is a branch of differential geometry that studies symplectic manifolds, which are smooth manifolds equipped with a closed, non-degenerate 2-form called a symplectic form. It originated from classical mechanics.

Definition 1 (Symlectic manifold)**.** Let M be a smooth manifold of even dimension $2m$ and let $\omega \in \Omega^2(M)$ be a closed non degenerate 2-form i.e.

$$d\omega = 0 \text{ and } \omega^m = \omega \wedge \omega \wedge \cdots \wedge \omega \neq 0,$$

Then ω is called a *simplectic form* and the pair (M, ω) is called a *simplectic manifold*.

ekvivalentni definice nedegenerovanosti.
Narozdil od riemannovske geometrie nelze pouzit partitions of unity na konstrukci metriky.
napsat poznamku o koncenci se psanim dimenze manifoldu :D

Example 2 (Canonical symplectic structure)**.** Let $M = \mathbb{R}^{2m}$ with the global coordinates $q_1, \ldots, q_m, p_1, \ldots, p_m$. and let ω be a form s.t.,

$$\omega = \sum_{i=1}^{m} dp_i \wedge dq_i.$$

Then

$$\omega^m = m! \cdot (-1)^{m(m-1)/2} \cdot dp_1 \wedge \cdots \wedge dp_m \wedge dq_1 \wedge \cdots \wedge dq_m.$$

We call R^2m with the form ω the canonical symplectic structure.

Example 3 (Cotangent bundle is a symplectic manifold.)**.** Let Q be a manifold, and consider the manifold $M = T^*Q$. Then there is a canonical 1-form $\theta \in \Omega^1(M)$ given by

$$\theta(X) = \langle \pi_{T^*Q}(X), T(\pi_Q)(X) \rangle, \quad X \in T(T^*Q), \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between tangent and cotangent spaces and the projections are the following:

$$\begin{tikzcd}
 & T(T^*Q) \arrow[dl, "\pi_{T^*Q}"'] \arrow[dr, "T(\pi_Q)"] & \\
T^*Q \arrow[dr, "\pi_Q"'] & & TQ \arrow[dl, "\pi_Q"] \\
 & Q &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (q,p;\xi,\eta) \arrow[dl, "\pi_{T^*Q}"'] \arrow[dr, "T(\pi_Q)"] & \\
(q,p) \arrow[dr, "\pi_Q"'] & & (q,\xi) \arrow[dl, "\pi_Q"] \\
 & q &
\end{tikzcd}$$

Let $q = (q^1, \ldots, q^n) : U \to \mathbb{R}^n$ be a chart on Q, the we have the induced chart $T^*q : T^*U \to \mathbb{R}^n \times \mathbb{R}^n$, where $T^*_x q = (T_x q^{-1})^*$, we put $p_i := \langle e_i, T^*q(\cdot) \rangle : T^*U \to \mathbb{R}$. Then $(q^1, \ldots, q^n, p_1, \ldots, p_n) : T^*U \to \mathbb{R}^n \times (\mathbb{R}^n)^*$ are the induced coordinates and locally in these coordinates

$$\theta(q,p) = \sum_{i=1}^{n} \left(\theta\left(\frac{\partial}{\partial q^i}\right) dq^i + \theta\left(\frac{\partial}{\partial p_i}\right) dp_i \right) = \sum_{i=1}^{n} p_i \, dq^i + 0, \tag{2}$$

since $\theta\left(\frac{\partial}{\partial q^i}\right) = \theta_{R^n}\left((q, p; e_i, 0)\right) = \langle p, e_i \rangle = p_i$.

Now we define the 2-form $\omega \in \Omega^2(T^*Q)$ by

$$\omega := -d\theta \overset{\text{locally}}{=} \sum_{i=1}^{n} dq^i \wedge dp_i. \tag{3}$$

We see that the 2-form ω is non-degenerate.

Definition 4. The form $\theta \in \Omega^1(M)$ from (1), locally given by (2), is called the *tautological 1-form* on T^*Q. The induced 2-form ω from (3) is called the *canonical symplectic structure* on T^*Q.

dukaz ze je neni degen?

Definition 5. Let $X : J \times M \to TM$ be a smooth mapping such that $\pi_M \circ X = pr_2$, where J is open. Then we call X a *time dependent vector field* on a manifold M.

There is an associated vector field $\bar{X} \in \mathfrak{X}(J \times M)$, given by $\bar{X}(t,x) = (\frac{\partial}{\partial t}, X(t,x)) \in T_t\mathbb{R} \times T_x M$.

Definition 6. Let X be a time dependent vector field on a manifold M and let $\Phi^X : J \times J \times M \to M$ be a map defined on a maximal neighborhood of $\Delta_J \times M$ satisfying the differential equation

$$\begin{aligned} \frac{d}{dt}\Phi^X(t,s,x) &= X\left(t, \Phi^X(t,s,x)\right) \\ \Phi^X(s,s,x) &= x \end{aligned} \tag{4}$$

Definition 6 is equivalent with

$$(t, \Phi^X(t,s,x)) = Fl^{\bar{X}}(t-s,(s,x)),$$

so the evolution operator exits and is unique on a maximal integral curve and satisfies

$$\Phi^X_{t,s} = \Phi^X_{t,r} \circ \Phi^X_{r,s}, \text{ where } \Phi^X_{t,r}(x) = \Phi(t,s,x).$$

Lemma 7. *Let f_t be a curve of diffeomorphisms on a manifold M locally defined for each t such that $f_0 = Id$. Defined two time dependent vector fields*

$$\xi_t(x) := (T_x f_t)^{-1} \frac{\partial}{\partial t} f_t(x), \;\; \eta_t(x) := \left(\frac{\partial}{\partial t} f_t\right)\left(f_t^{-1}(x)\right) \tag{5}$$

Then $T(f_t) \cdot \xi_t = \frac{\partial}{\partial t} f_t = \eta_t \circ f_t$, so ξ_t and η_t are f_t-related. Let $\omega \in \Omega^k(M)$. Then

$$i_{\xi_t} f_t^* \omega = f_t^* i_{\eta_t} \omega, \tag{6}$$

$$\frac{\partial}{\partial t} f_t^* \omega = f_t^* \mathcal{L}_{\eta_t} \omega = \mathcal{L}_{\xi_t} f_t^* \omega. \tag{7}$$

Proof.

$$\begin{aligned} \left(i_{\xi_t} f_t^* \omega\right)_x (X_2, \dots, X_k) &= \left(f_t^* \omega\right)_x (\xi_t(x), X_2, \dots, X_k) \\ &= \omega_{f_t(x)} \left(T_x f_t \cdot \xi_t(x), T_x f_t \cdot X_2, \dots, T_x f_t \cdot X_k\right) \\ &= \omega_{f_t(x)} \left(\eta_t\left(f_t(x)\right), T_x f_t \cdot X_2, \dots, T_x f_t \cdot X_k\right) \\ &= \left(f_t^* i_{\xi_t} \omega\right)_x (X_2, \dots, X_k) \end{aligned}$$

This proves (6). Now consider $\bar{\eta} \in \mathfrak{X}(\mathbb{R} \times M), \bar{\eta}(t,x) = (\partial_t, \eta_t(x))$ and let $\Phi^\eta : \mathbb{R} \times \mathbb{R} \times M \to M$ be the evolution operator, i.e.

$$\frac{\partial}{\partial t}\Phi^\eta_{t,s}(x) = \eta_t\left(\Phi^\eta_{t,s}(x)\right), \quad \Phi^\eta_{s,s}(x) = x,$$

such that

$$\left(t, \Phi^\eta_{t,s}(x)\right) = \mathrm{Fl}^{\bar{\eta}}_{t-s}(s,x), \;\; \Phi^\eta_{t,s} = \Phi^\eta_{t,r} \circ \Phi^\eta_{r,s}(x).$$

Since f_t satisfies $\frac{\partial}{\partial t} f_t = \eta_t \circ f_t$ and $f_0 = Id_M$, either $f_t = \Phi^{\eta}_{t,0}$, or $(t, f_t(x)) = Fl^{\bar{\eta}}_t(0, x)$, so $f_t = pr_2 \circ Fl^{\bar{\eta}}_t \circ ins_0$. Thus

$$\frac{\partial}{\partial t} f_t^* \omega = \frac{\partial}{\partial t} \left(\mathrm{pr}_2 \circ Fl^{\bar{\eta}}_t \circ ins_0\right)^* \omega = ins_0^* \frac{\partial}{\partial t} (Fl^{\bar{\eta}}_t)^* pr_2^* \omega = ins_0^* (Fl^{\bar{\eta}}_t)^* \mathbb{L}_{\bar{\eta}} pr_2^* \omega.$$

For time dependant vector fields X_i(tady mozna nejaka vlastnost lie derivative!!!) we have

$$\begin{aligned}
\left(\mathcal{L}_{\bar{\eta}}\, \mathrm{pr}_2^* \omega\right)(0 \times X_1, \ldots, 0 \times X_k)|_{(t,x)} &= \bar{\eta}((\mathrm{pr}_2^* \omega)(0 \times X_1, \ldots))|_{(t,x)} \\
&\quad - \sum_i (\mathrm{pr}_2^* \omega)(0 \times X_1, \ldots, [\bar{\eta}, 0 \times X_i], \ldots, 0 \times X_k)|_{(t,x)} \\
&= (\partial_t, \eta_t(x))\left(\omega\left(X_1, \ldots, X_k\right)\right) - \sum_i \omega\left(X_1, \ldots, \left[\eta_t, X_i\right], \ldots, X_k\right)|_x \\
&= \left(\mathcal{L}_{\eta_t} \omega\right)_x \left(X_1, \ldots, X_k\right).
\end{aligned}$$

For $X_i \in T_x M$, this implies

$$\begin{aligned}
\left(\frac{\partial}{\partial t} f_t^* \omega\right)_x (X_1, \ldots, X_k) &= \left(\mathrm{ins}^* \left(\mathrm{Fl}^{\eta}_t\right)^* \mathcal{L}_{\eta}\, \mathrm{pr}_2^* \omega\right)_x (X_1, \ldots, X_k) \\
&= \left(\left(\mathrm{Fl}^{\eta}_t\right)^* \mathcal{L}_{\eta}\, \mathrm{pr}_2^* \omega\right)_{(0,x)} (0 \times X_1, \ldots, 0 \times X_k) \\
&= \left(\mathcal{L}_{\eta}\, \mathrm{pr}_2^* \omega\right)_{(t, f_t(x))} (0_t \times T_x f_t \cdot X_1, \ldots, 0_t \times T_x f_t \cdot X_k) \\
&= \left(\mathcal{L}_{\eta_t} \omega\right)_{f_t(x)} (T_x f_t \cdot X_1, \ldots, T_x f_t \cdot X_k) \\
&= \left(f_t^* \mathcal{L}_{\eta_t} \omega\right)_x (X_1, \ldots, X_k),
\end{aligned} \tag{8}$$

We have proven the first part of (7), the second part follow from (6)

$$\begin{aligned}
\frac{\partial}{\partial t} f_t^* \omega &= f_t^* \mathcal{L}_{\eta_t} \omega \\
&= f_t^* \left(d i_{\eta_t} + i_{\eta_t} d\right) \omega \\
&= d f_t^* i_{\eta_t} \omega + f_t^* i_{\eta_t} d\omega \\
&= d i_{\xi_t} f_t^* \omega + i_{\xi_t} f_t^* d\omega \\
&= d i_{\xi_t} f_t^* \omega + i_{\xi_t} d f_t^* \omega \\
&= \mathcal{L}_{\xi_t} f_t^* \omega.
\end{aligned} \tag{9}$$

□

dopsat co je ins a pr2?

Theorem 8 (Darboux)**.** *Let (M, ω) be a symplectic manifold of dimension $2n$. Then for all points $x \in M$ exists a chart (U, u) centered at x such that $\omega|_U = \sum_{i=1}^n du^i \wedge du^{n+i}$.*

Proof. Take a chart (U, u) centered at x and choose coordinates such that $\omega_x = \sum_{i=1}^n du^i \wedge du^{n+i}$ at x. Then $\omega_0 = \omega|_U$ and $\omega_1 = \sum_{i=1}^n du^i \wedge du^{n+i}$ are two symplectic forms that are equal at x. Now interpolate $\omega_t = \omega_0 + t(\omega_1 + \omega_0)$. Then ω_t is a symplectic form on a possibly smaller neighbourhood of x for all $t \in [0, 1]$.

We want to find a curve of diffeomorphisms f_t near x such that $f_0 = id$, $f_t(x) = x$ and such that the pullback condition $f_t^* \omega_t = \omega_0$ is satisfied. Assume that U is contractible, then the second cohomology group $H^2(U) = 0$ and every closed 2-form is exact, so $d(\omega_1 - \omega_0) = 0$ implies $\omega_1 - \omega_0 = d\psi$ for some $\psi \in \Omega^1(U)$. By adding a constant we may assume that $\psi_x = 0$. Now by using Lemma 7, (7), we get a time dependant vector field $\eta_t = \frac{\partial}{\partial t} f_t \circ f_t^{-1}$, then by differentiating with respect to t, (cartan formula!!)

$$0 = \frac{\partial}{\partial t} f_t^* \omega_t = f_t^* \left(\mathcal{L}_{\eta_t} \omega_t + \frac{\partial}{\partial t} \omega_t\right) = f_t^* \left(d i_{\eta_t} \omega_t + i_{\eta_t} d\omega_t + \omega_1 - \omega_0\right) = f_t^* d\left(i_{\eta_t} \omega_t + \psi\right)$$

Since ω_t is non-degenerate, the equation $i_{\eta_t} \omega_t = -\psi$ prescribes the vector field η_t uniquely. Also $\eta_t(x) = 0$ sine $\psi_x = 0$. On some neighbourhood of x the left evolution operator f_t of η_t exists for all $t \in [0, 1]$ and $\frac{\partial}{\partial t}(f_t^* \omega_t) = 0$, so $f_t^* \omega_t = \omega_0$ for all $t \in [0, 1]$. □

NOW lets study symplectomorphisms.

mozna dopsat definici lagrangian manifold

Definition 9 (Lagrangian submanifold)**.** Let (M^{2m}, ω) be a symplectic manifold. We call a submanifold Y of M lagrangian, if at each $p \in Y$, T_pY is a lagrangian subspace of T_pM , that is, $\omega_p|_{T_pY} \equiv 0$ and $\dim T_pY = \frac{1}{2}\dim T_pM$.

Equivalently, if $i : Y \to M$ is the inclusion map, then Y is lagrangian if and only if $i * \omega = 0$ and $\dim Y = \frac{1}{2}\dim M$

Example 10 (The zero section of T^*M)**.** Let M^m be a manifold and consider its cotangent bundle T^*M with the local coordinates $x_1, \dots, x_m, \xi_1, \dots, \xi_m$ on T^*U. Then the *zero section* of T^*M is the set

$$M_0 = (x, \xi) \in T^*M|\xi = 0inT^*_xM$$

is an m-dimensional submanifold of T^*M whose intersection with $T * U$ is given by the equations $\xi_1 = \ldots = \xi_n = 0$. Then clearly the tautological 1-form $\theta(x, \xi)$ vanishes on $M_0 \cap T^*U$. Let $i_0 : M_0 \to T^*M$ be the inclusion map, then $i_0^*\theta = 0$. Hence $i_0^*\omega = i_0^*d\theta = 0$ and so M_0 is lagrangian.

Now let $\mu \in \Omega^1(M)$ be a 1-form and consider the set

$$M_\mu = \{(x, \mu_x) \mid x \in X, \mu_x \in T^*_xX\} \tag{10}$$

Then M_μ is a submanifold of T^*M. When is M_μ lagrangian?

Lemma 11. *Let M_μ be of the form (10). Denote by $s_\mu : M \to T^*M, x \to (x, \mu_x)$ the 1-form μ regarded as a map. Let θ be the tautological 1-form on T^*M. Then*

$$s_\mu^*\theta = \mu$$

Proof. By definition of θ, $\theta_p = (d\pi)^*\xi$ at $p = (x, \xi) \in M$. For $p = s_\mu(x) = (x, \mu_x)$, we have $\theta_p = (d\pi_p)^*\mu_x$. Then

$$(s_\mu^*\theta)_x = (ds_\mu)_x^*\theta_p = (ds_\mu)_x^*(d\pi_p)^*\mu_x = (d(\pi \circ s_\mu))_x^*\,\mu_x = \mu_x.$$

□

Lemma 12. *Let M_μ be of the form (10). Then M_μ is lagrangian iff. μ is closed.*

Proof. The map $s_\mu : M \to T^*M, x \to (x, \mu_x)$ is an embedding with image M_μ. Then there is a diffeomorphism $\tau : M \to M_\mu$, $\tau(x) := (x, \mu_x)$, such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "s_\mu"] \arrow[dr, "\tau"'] & T^*X \\ & X_\mu \arrow[u, hook, "i"'] \end{tikzcd}$$

We want to express the condition of M_μ being Lagrangian in terms of the form μ:

$$\begin{aligned} M_\mu \text{ is Lagrangian} &\iff i^*d\theta = 0 \\ &\iff \tau^*i^*d\theta = 0 \\ &\iff (i \circ \tau)^*d\theta = 0 \\ &\iff s_\mu^*d\theta = 0 \\ &\iff ds_\mu^*\theta = 0 \\ &\iff d\mu = 0 \\ &\iff \mu \text{ is closed.} \end{aligned} \tag{11}$$

Therefore, there is a one-to-one correspondence between the set of Lagrangian submanifolds of T^*M of the form (10) and the set of closed 1-forms on M. □

There are other lagrangian submanifolds of T^*M. Lets study the conormal bundles.

Definition 13. Let S^k be submanifold of M^m, then the *conormal space* at x is the set

$$N_x^* S = \{\xi \in T_x^* M \mid \xi(v) = 0 \; , \; v \in T_x M\} \, .$$

The *conormal bundle* of S is

$$N^* S = \{(x, \xi) \in T^* M \mid x \in S, \xi \in N_x^* S\} \, .$$

Example 14. Let $S \subset X$ be a submanifold, then the conormal bundle $N^* S$ is a lagrangian submanifold of $T^* x$.

Lemma 15. *The conormal bundle $N^* S$ is an n-dimensional submanifold of $T^* M$.*

Lemma 16. *Let $i : N^* S \hookrightarrow T^* M$ be the inclusion and θ the tautological 1-form on $T^* M$. Then*

$$i^* \theta = 0.$$

Proof. Let $(U, x_1, \ldots, x_n)$ be a coordinate system on M centered at $x \in S$ and adapted to M, so that $U \cap S$ is described by

$$x_{k+1} = \ldots = x_n = 0. \text{ and } \xi_1 = \ldots = \xi_k = 0$$

Let $(U, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ be the associated cotangent coordinate system. The submanifold $N^* S \cap T^* U$ is then described by

$$x_{k+1} = \ldots = x_n = 0 \text{ and } \xi_1 = \ldots = \xi_k = 0. \tag{12}$$

Since $\theta = \sum \xi_i dx_i$ on $T^* U$, we conclude that, at $p \in N^* S$,

$$(i^* \theta)_p = \theta_p |_{T_p(N^* S)} = \sum_{i > k} \xi_i dx_i \Bigg|_{\text{span}\left\{\frac{\partial}{\partial x_i}, i \leq k\right\}} = 0. \tag{13}$$

□

Corollary 17. *For any submanifold S of M, the conormal bundle $N^* S$ is a lagrangian submanifold of $T^* M$.*

Notice that taking $S = \{x\}$ to be a point, then the conormal bundle $N^* S = T_x^* M$ is a cotangent fiber. Taking $S = X$, the conormal bundle is the zero section M_0.

Lemma 18 (Lemmatko)**.** *$2 + 2 = 4 - 1 = 3$ quick maffs.*

A ted si rekneme dulezitou vetu.

Theorem 19 (Hlavni veta o gaystvi)**.** *Jsi gay.*

Corollary 20. *Vlastne dusledek tohoto kratkeho textu je, ze bych se mel jit zabit. Jdu na to!*