Symplectic Geometry

Radim Čech

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Symplectic geometry is a branch of differential geometry that studies symplectic manifolds, which are smooth manifolds equipped with a closed, non-degenerate 2-form called a symplectic form. It originated from Hamiltonian mechanics.

1 Symplectic manifolds

Definition 1.1 (Symlectic manifold)**.** Let M be a smooth manifold of even dimension $2m$ and let $\omega \in \Omega^2(M)$ be a closed non degenerate 2-form i.e.

$$d\omega = 0 \text{ and } \omega^m = \omega \wedge \omega \wedge \cdots \wedge \omega \neq 0,$$

Then ω is called a *simplectic form* and the pair (M, ω) is called a *simplectic manifold.*

Example 1.2 (Canonical symplectic structure)**.** Let $M = \mathbb{R}^{2m}$ with the global coordinates $q_1, \dots, q_m, p_1, \dots, p_m$. and let ω be a form s.t.,

$$\omega = \sum_{i=1}^{m} dp_i \wedge dq_i.$$

Then

$$\omega^m = m! \cdot (-1)^{m(m-1)/2} \cdot dp_1 \wedge \cdots \wedge dp_m \wedge dq_1 \wedge \cdots \wedge dq_m. \tag{1}$$

We call R^2m with the form ω the canonical symplectic structure.

Example 1.3 (Cotangent bundle is a symplectic manifold.)**.** Let Q be a manifold, and consider the manifold $M = T^*Q$. Then there is a canonical 1-form $\theta \in \Omega^1(M)$ given by

$$\theta(X) = \langle \pi_{T^*Q}(X), T(\pi_Q)(X) \rangle, \quad X \in T(T^*Q), \tag{2}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between tangent and cotangent spaces and the projections are the following:

$$\begin{tikzcd} & T(T^*Q) \arrow[dl, "\pi_{T^*Q}"'] \arrow[dr, "T(\pi_Q)"] & \\ T^*Q \arrow[dr, "\pi_Q"'] & & TQ \arrow[dl, "\pi_Q"] \\ & Q & \end{tikzcd} \qquad \begin{tikzcd} & (q,p;\xi,\eta) \arrow[dl, "\pi_{T^*Q}"'] \arrow[dr, "T(\pi_Q)"] & \\ (q,p) \arrow[dr, "\pi_Q"'] & & (q,\xi) \arrow[dl, "\pi_Q"] \\ & q & \end{tikzcd}$$

Let $q = (q^1, \dots, q^n) : U \to \mathbb{R}^n$ be a chart on Q, the we have the induced chart $T^*q : T^*U \to \mathbb{R}^n \times \mathbb{R}^n$, where $T^*_x q = (T_x q^{-1})^*$, we put $p_i := \langle e_i, T^*q(\cdot) \rangle : T^*U \to \mathbb{R}$. Then $(q^1, \dots, q^n, p_1, \dots, p_n) : T^*U \to \mathbb{R}^n \times (\mathbb{R}^n)^*$ are the induced coordinates and locally in these coordinates

$$\theta(q,p) = \sum_{i=1}^{n} \left(\theta\left(\frac{\partial}{\partial q^i}\right) dq^i + \theta\left(\frac{\partial}{\partial p_i}\right) dp_i \right) = \sum_{i=1}^{n} p_i \, dq^i + 0, \tag{3}$$

since $\theta\left(\frac{\partial}{\partial q^i}\right) = \theta_{R^n}\left((q,p;e_i,0)\right) = \langle p, e_i \rangle = p_i$.

Now we define the 2-form $\omega \in \Omega^2(T^*Q)$ by

$$\omega := -d\theta \overset{\text{locally}}{=} \sum_{i=1}^{n} dq^i \wedge dp_i. \tag{4}$$

By the same argument as in (1) the 2-form ω is non-degenerate.

Definition 1.4. The form $\theta \in \Omega^1(M)$ from (2), locally given by (3), is called the *tautological 1-form* on T^*Q. The induced 2-form ω from (4) is called the *canonical symplectic structure* on T^*Q.

Definition 1.5. Let $X : J \times M \to TM$ be a smooth mapping such that $\pi_M \circ X = pr_2$, where J is open. Then we call X a *time dependent vector field* on a manifold M.

There is an associated vector field $\bar{X} \in \mathfrak{X}(J \times M)$ to each time dependant vector field X given by $\bar{X}(t,x) = (\frac{\partial}{\partial t}, X(t,x)) \in T_t\mathbb{R} \times T_x M$.

Definition 1.6. Let X be a time dependent vector field on a manifold M and let $\Phi^X : J \times J \times M \to M$ be a map defined on a maximal neighborhood of $\Delta_J \times M$ satisfying the differential equation

$$\begin{aligned} \frac{d}{dt}\Phi^X(t,s,x) &= X\left(t, \Phi^X(t,s,x)\right) \\ \Phi^X(s,s,x) &= x \end{aligned} \tag{5}$$

Definition 1.6 is equivalent with

$$(t, \Phi^X(t,s,x)) = Fl^{\bar{X}}(t-s,(s,x)),$$

so the evolution operator exits and is unique on a maximal integral curve and satisfies

$$\Phi^X_{t,s} = \Phi^X_{t,r} \circ \Phi^X_{r,s}, \text{ where } \Phi^X_{t,r}(x) = \Phi(t,s,x).$$

Lemma 1.7. *Let f_t be a curve of diffeomorphisms on a manifold M locally defined for each t such that $f_0 = Id$. Defined two time dependent vector fields*

$$\xi_t(x) := (T_x f_t)^{-1} \frac{\partial}{\partial t} f_t(x), \;\; \eta_t(x) := \left(\frac{\partial}{\partial t} f_t\right)\left(f_t^{-1}(x)\right) \tag{6}$$

Then $T(f_t) \cdot \xi_t = \frac{\partial}{\partial t} f_t = \eta_t \circ f_t$, so ξ_t and η_t are f_t-related. Let $\omega \in \Omega^k(M)$. Then

$$i_{\xi_t} f_t^* \omega = f_t^* i_{\eta_t}\omega, \tag{7}$$

$$\frac{\partial}{\partial t} f_t^*\omega = f_t^* \mathcal{L}_{\eta_t}\omega = \mathcal{L}_{\xi_t} f_t^*\omega. \tag{8}$$

Proof.

$$\begin{aligned} \left(i_{\xi_t} f_t^*\omega\right)_x (X_2, \dots, X_k) &= \left(f_t^*\omega\right)_x (\xi_t(x), X_2, \dots, X_k) \\ &= \omega_{f_t(x)} \left(T_x f_t \cdot \xi_t(x), T_x f_t \cdot X_2, \dots, T_x f_t \cdot X_k\right) \\ &= \omega_{f_t(x)} \left(\eta_t\left(f_t(x)\right), T_x f_t \cdot X_2, \dots, T_x f_t \cdot X_k\right) \\ &= \left(f_t^* i_{\eta_t}\omega\right)_x (X_2, \dots, X_k) \end{aligned}$$

This proves (7). Now consider $\bar{\eta} \in \mathfrak{X}(\mathbb{R} \times M), \bar{\eta}(t,x) = (\partial_t, \eta_t(x))$ and let $\Phi^\eta : \mathbb{R} \times \mathbb{R} \times M \to M$ be the evolution operator, i.e.

$$\frac{\partial}{\partial t}\Phi^\eta_{t,s}(x) = \eta_t\left(\Phi^\eta_{t,s}(x)\right), \quad \Phi^\eta_{s,s}(x) = x,$$

such that

$$\left(t, \Phi^\eta_{t,s}(x)\right) = \mathrm{Fl}^{\bar{\eta}}_{t-s}(s,x), \;\; \Phi^\eta_{t,s} = \Phi^\eta_{t,r} \circ \Phi^\eta_{r,s}(x).$$

Since f_t satisfies $\frac{\partial}{\partial t} f_t = \eta_t \circ f_t$ and $f_0 = Id_M$, either $f_t = \Phi^\eta_{t,0}$, or $(t, f_t(x)) = Fl^{\bar{\eta}}_t(0,x)$, so $f_t = pr_2 \circ Fl^{\bar{\eta}}_t \circ ins_0$. Thus

$$\frac{\partial}{\partial t} f_t^*\omega = \frac{\partial}{\partial t}\left(\mathrm{pr}_2 \circ Fl^{\bar{\eta}}_t \circ ins_0\right)^*\omega = ins_0^* \frac{\partial}{\partial t}(Fl^{\bar{\eta}}_t)^* pr_2^*\omega = ins_0^*(Fl^{\bar{\eta}}_t)^* \mathbb{L}_{\bar{\eta}} pr_2^*\omega,$$

where pr_2 is a projection to the second component and ins_0 is the map $ins_0 : M \to \mathbb{R}\times M, \quad x \to (0,x)$.

For time dependant vector fields X_i we have

$$\begin{aligned}(\mathcal{L}_{\bar\eta}\,\mathrm{pr}_2^*\,\omega)\,(0\times X_1,\ldots,0\times X_k)|_{(t,x)} &= \bar\eta((\mathrm{pr}_2^*\,\omega)(0\times X_1,\ldots))|_{(t,x)} \\ &\quad - \sum_i (\mathrm{pr}_2^*\,\omega)(0\times X_1,\ldots,[\bar\eta, 0\times X_i],\ldots,0\times X_k)|_{(t,x)} \\ &= (\partial_t, \eta_t(x))\,(\omega\,(X_1,\ldots,X_k)) - \sum_i \omega\,(X_1,\ldots,[\eta_t, X_i],\ldots,X_k)\,|_x \\ &= (\mathcal{L}_{\eta_t}\omega)_x\,(X_1,\ldots,X_k)\,.\end{aligned}$$

For $X_i \in T_xM$, this implies

$$\begin{aligned}\left(\frac{\partial}{\partial t} f_t^*\omega\right)_x (X_1,\ldots,X_k) &= \left(\mathrm{ins}^*\left(\mathrm{Fl}_t^\eta\right)^*\mathcal{L}_\eta\,\mathrm{pr}_2^*\omega\right)_x (X_1,\ldots,X_k) \\ &= \left(\left(\mathrm{Fl}_t^\eta\right)^*\mathcal{L}_\eta\,\mathrm{pr}_2^*\omega\right)_{(0,x)} (0\times X_1,\ldots,0\times X_k) \\ &= \left(\mathcal{L}_\eta\,\mathrm{pr}_2^*\omega\right)_{(t,f_t(x))} (0_t\times T_xf_t\cdot X_1,\ldots,0_t\times T_xf_t\cdot X_k) \\ &= \left(\mathcal{L}_{\eta_t}\omega\right)_{f_t(x)} (T_xf_t\cdot X_1,\ldots,T_xf_t\cdot X_k) \\ &= \left(f_t^*\mathcal{L}_{\eta_t}\omega\right)_x (X_1,\ldots,X_k),\end{aligned} \tag{9}$$

We have proven the first part of (8), the second part follow from (7)

$$\begin{aligned}\frac{\partial}{\partial t} f_t^*\omega &= f_t^*\mathcal{L}_{\eta_t}\omega \\ &= f_t^*\left(di_{\eta_t} + i_{\eta_t}d\right)\omega \\ &= df_t^* i_{\eta_t}\omega + f_t^* i_{\eta_t} d\omega \\ &= di_{\xi_t} f_t^*\omega + i_{\xi_t} f_t^* d\omega \\ &= di_{\xi_t} f_t^*\omega + i_{\xi_t} df_t^*\omega \\ &= \mathcal{L}_{\xi_t} f_t^*\omega.\end{aligned} \tag{10}$$

□

2 The Darboux theorem

The following theorem provides a canonical local form for symplectic manifolds.

Theorem 2.1 (Darboux). *Let (M,ω) be a symplectic manifold of dimension $2n$. Then for all points $x \in M$ exists a chart (U,u) centered at x such that $\omega|_U = \sum_{i=1}^n du^i \wedge du^{n+i}$.*

Proof. Consider a chart (U,u) centered at x and choose coordinates such that the symplectic form is expressed as $\omega_x = \sum_{i=1}^n du^i \wedge du^{n+i}$ at x. Then $\omega_0 = \omega|_U$ and $\omega_1 = \sum_{i=1}^n du^i \wedge du^{n+i}$ are two symplectic forms that are equal at x. Now interpolate $\omega_t = \omega_0 + t(\omega_1 + \omega_0)$. Then ω_t is a symplectic form on a possibly smaller neighbourhood of x for all $t \in [0,1]$.

Now construct a curve of diffeomorphisms f_t near x such that $f_0 = id$, $f_t(x) = x$ and such that the pullback condition $f_t^*\omega_t = \omega_0$ is satisfied. Assume that U is contractible, then the second cohomology group $H^2(U) = 0$ and every closed 2-form is exact, so $d(\omega_1 - \omega_0) = 0$ and by the Poincare lemma $\omega_1 - \omega_0 = d\psi$ for some $\psi \in \Omega^1(U)$. By adding a constant we may assume that $\psi_x = 0$. Now by using Lemma 1.7, (8), we get a time dependant vector field $\eta_t = \frac{\partial}{\partial t} f_t \circ f_t^{-1}$, then by differentiating with respect to t,

$$0 = \frac{\partial}{\partial t} f_t^*\omega_t = f_t^*\left(\mathcal{L}_{\eta_t}\omega_t + \frac{\partial}{\partial t}\omega_t\right) = f_t^*\left(di_{\eta_t}\omega_t + i_{\eta_t}d\omega_t + \omega_1 - \omega_0\right) = f_t^* d\left(i_{\eta_t}\omega_t + \psi\right)$$

Since ω_t is non-degenerate, the equation $i_{\eta_t}\omega_t = -\psi$ uniquely determines the vector field η_t. Also $\eta_t(x) = 0$ since $\psi_x = 0$. On some neighbourhood of x the left evolution operator f_t of η_t exists for all $t \in [0,1]$ and $\frac{\partial}{\partial t}(f_t^*\omega_t) = 0$, so $f_t^*\omega_t = \omega_0$ for all $t \in [0,1]$. □

3 Lagrangian submanifolds

Now we shift our focus on lagrangian submanifolds, which are submanifolds of half dimension on which the symplectic form vanishes.

Definition 3.1 (Lagrangian submanifold)**.** Let (M^{2m}, ω) be a symplectic manifold. We call a submanifold Y of M lagrangian, if at each $p \in Y$, T_pY is a lagrangian subspace of T_pM , that is, $\omega_p|_{T_pY} \equiv 0$ and $\dim T_pY = \frac{1}{2} \dim T_pM$.

Equivalently, if $i : Y \to M$ is the inclusion map, then Y is lagrangian if and only if $i * \omega = 0$ and $\dim Y = \frac{1}{2} \dim M$

Example 3.2 (The zero section of T^*M)**.** Let M^m be a manifold and consider its cotangent bundle T^*M with the local coordinates $x_1, \ldots, x_m, \xi_1, \ldots, \xi_m$ on T^*U. Then the *zero section* of T^*M is the set

$$M_0 = \{(x, \xi) \in T^*M \mid \xi = 0 \text{ in } T^*_xM\}$$

is an m-dimensional submanifold of T^*M whose intersection with $T * U$ is given by the equations $\xi_1 = \ldots = \xi_n = 0$. Then clearly the tautological 1-form $\theta(x, \xi)$ vanishes on $M_0 \cap T^*U$. Let $i_0 : M_0 \to T^*M$ be the inclusion map, then $i_0^*\theta = 0$. Hence $i_0^*\omega = i_0^*d\theta = 0$ and so M_0 is lagrangian.

Now let $\mu \in \Omega^1(M)$ be a 1-form and consider the set

$$M_\mu = \{(x, \mu_x) \mid x \in X, \mu_x \in T^*_xX\} \tag{11}$$

Then M_μ is a submanifold of T^*M. When is M_μ lagrangian?

Lemma 3.3. *Let M_μ be of the form (11). Denote by $s_\mu : M \to T^*M, x \to (x, \mu_x)$ the 1-form μ regarded as a map. Let θ be the tautological 1-form on T^*M. Then*

$$s_\mu^*\theta = \mu$$

Proof. By definition of θ, $\theta_p = (d\pi)^*\xi$ at $p = (x, \xi) \in M$. For $p = s_\mu(x) = (x, \mu_x)$, we have $\theta_p = (d\pi_p)^*\mu_x$. Then

$$(s_\mu^*\theta)_x = (ds_\mu)_x^*\theta_p = (ds_\mu)_x^*(d\pi_p)^*\mu_x = (d(\pi \circ s_\mu))_x^*\,\mu_x = \mu_x.$$

□

Lemma 3.4. *Let M_μ be of the form (11). Then M_μ is lagrangian iff. μ is closed.*

Proof. The map $s_\mu : M \to T^*M, x \to (x, \mu_x)$ is an embedding with image M_μ. Then there is a diffeomorphism $\tau : M \to M_\mu$, $\tau(x) := (x, \mu_x)$, such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "s_\mu"] \arrow[dr, "\tau"'] & T^*X \\ & X_\mu \arrow[u, hook, "i"'] \end{tikzcd}$$

We want to express the condition of M_μ being Lagrangian in terms of the form μ:

$$\begin{aligned} M_\mu \text{ is Lagrangian} &\iff i^*d\theta = 0 \\ &\iff \tau^*i^*d\theta = 0 \\ &\iff (i \circ \tau)^*d\theta = 0 \\ &\iff s_\mu^*d\theta = 0 \\ &\iff ds_\mu^*\theta = 0 \\ &\iff d\mu = 0 \\ &\iff \mu \text{ is closed.} \end{aligned} \tag{12}$$

Therefore, there is a one-to-one correspondence between the set of Lagrangian submanifolds of T^*M of the form (11) and the set of closed 1-forms on M. □

Not all Lagrangian submanifolds of T^*M are of the form M_μ. Another important class is the conormal bundles.

Definition 3.5. Let S^k be submanifold of M^m, then the *conormal space* at x is the set

$$N_x^*S = \{\xi \in T_x^*M \mid \xi(v) = 0\ ,\ v \in T_xM\}\,.$$

The *conormal bundle* of S is

$$N^*S = \{(x, \xi) \in T^*M \mid x \in S, \xi \in N_x^*S\}\,.$$

Example 3.6. Let $S \subset X$ be a submanifold, then the conormal bundle N^*S is a lagrangian submanifold of T^*x.

Lemma 3.7. *The conormal bundle N^*S is an n-dimensional submanifold of T^*M.*

Proof. Choose coordinates on X adapted to S, that is

$$S = \{x_{k_1} = \ldots = x_n = 0\}.$$

In these coordinates, a basis of T_pS at a point $p\ inS$ is given by

$$\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}\}$$

and the cotangent space T_p^*X has a dual basis $(dx_1, \ldots, dx_n)$. The conormal space N_p^*S consists of coverctors of the form

$$\xi = \sum_{i=1}^n \xi_i dx_i$$

such that ξ annihilates T_pS, that is $\xi_i = \xi(\frac{\partial}{\partial x^i}) = 0$ for $i \in 0, \ldots, k$. Then the elements of N^*S in these coordinates are given by

$$(x_1, \ldots, x_k, 0, \ldots, 0, \xi_{k+1}, \ldots, \xi_n)$$

and we see that the dimension of N^*S is $k + n - k = n$. ☐

Lemma 3.8. *Let $i : N^*S \hookrightarrow T^*M$ be the inclusion and θ the tautological 1-form on T^*M. Then*

$$i^*\theta = 0.$$

Proof. Let $(U, x_1, \ldots, x_n)$ be a coordinate system on M centered at $x \in S$ and adapted to M, so that $U \cap S$ is described by

$$x_{k+1} = \ldots = x_n = 0. \text{ and } \xi_1 = \ldots = \xi_k = 0$$

Let $(U, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ be the associated cotangent coordinate system. The submanifold $N^*S \cap T^*U$ is then described by

$$x_{k+1} = \ldots = x_n = 0 \text{ and } \xi_1 = \ldots = \xi_k = 0. \tag{13}$$

Since $\theta = \sum \xi_i dx_i$ on T^*U, we conclude that, at $p \in N^*S$,

$$(i^*\theta)_p = \theta_p|_{T_p(N^*S)} = \sum_{i>k} \xi_i dx_i \bigg|_{\text{span}\left\{\frac{\partial}{\partial x_i}, i \leq k\right\}} = 0. \tag{14}$$

☐

Corollary 3.9. *For any submanifold S of M, the conormal bundle N^*S is a lagrangian submanifold of T^*M.*

Notice that taking $S = \{x\}$ to be a point, then the conormal bundle $N^*S = T_x^*M$ is a cotangent fiber. Taking $S = X$, the conormal bundle is the zero section M_0.

4 The Poisson bracket

Now lets study the Poisson bracket, which defines a Lie bracket structure on on smooth functions.

Definition 4.1 (Hamiltonian vector field)**.** Let (M, ω) be a symplectic manifold and $f \in C^\infty(M)$. Then the *Hamiltonian vector field* or *symplectic gradient* of f $H_f = grad^\omega(f) \in \mathfrak{X}(M)$ is defined by the following equivalent prescriptions:

$$\begin{aligned} i(H_f)\omega &= df, \\ H_f &= \omega^{-1} df, \\ \omega(H_f, X) &= X(f) \quad \text{for } X \in TM. \end{aligned}$$

Definition 4.2. For two functions $f, g \in C^\infty(M)$, we define their *Poisson bracket* $\{f, g\}$ by

$$\{f, g\} := i(H_f)i(H_g)\omega = \omega(H_g, H_f) = H_f(g) = \mathcal{L}_{H_f} g = dg(H_f) \in C^\infty(M). \tag{15}$$

Also define

$$\mathfrak{X}(M, \omega) := \{X \in \mathfrak{X}(M) : \mathcal{L}_X \omega = 0\} \tag{16}$$

and call this the space of *locally Hamiltonian vector fields* or *ω-respecting vector fields.*

Theorem 4.3. *Let (M, ω) be a symplectic manifold. Then $(C^\infty(M), \{\ ,\ \})$ is a Lie algebra which also satisfies*

$$\{f, gh\} = \{f, g\}h + g\{f, h\},$$

i.e., $\mathrm{ad}_f = \{f, \cdot\}$ *is a derivation of* $(C^\infty(M), \cdot)$*.*

Moreover, there is an exact sequence of Lie algebras and Lie algebra homomorphisms

$$0 \longrightarrow H^0(M) \xrightarrow{\alpha} \{C^\infty(M), \{\cdot,\cdot\}\} \xrightarrow{H = \mathrm{grad}^\omega} \{\mathfrak{X}(M, \omega), [\cdot,\cdot]\} \xrightarrow{\gamma} H^1(M) \longrightarrow 0,$$

where α is the embedding of the space of all locally constant functions, and where

$$\gamma(X) := [i_X \omega] \in H^1(M).$$

The sequence behaves invariantly under the pullback by symplectomorphisms $\varphi : M \to M$: For example

$$\varphi^*\{f, g\} = \{\varphi^* f, \varphi^* g\}, \quad \varphi^*(H_f) = H_{\varphi^* f}, \quad \textit{and} \quad \varphi^*\gamma(X) = \gamma(\varphi^* X).$$

Consequently, for $X \in \mathfrak{X}(M, \omega)$ we have

$$\mathcal{L}_X\{f, g\} = \{\mathcal{L}_X f, g\} + \{f, \mathcal{L}_X g\}, \quad \textit{and} \quad \gamma(\mathcal{L}_X Y) = 0.$$

Proof. The operator H takes values in $\mathfrak{X}(M, \omega)$ since

$$\mathcal{L}_{H_f}\omega = i_{H_f} d\omega + d i_{H_f}\omega = 0 + ddf = 0.$$

$H(\{f, g\}) = [H_f, H_g]$ since

$$\begin{aligned} i_{H(\{f,g\})}\omega &= d\{f, g\} \\ &= d\mathcal{L}_{H_f} g \\ &= \mathcal{L}_{H_f} dg - 0 \\ &= \mathcal{L}_{H_f} i_{H_g}\omega - i_{H_g}\mathcal{L}_{H_f}\omega \\ &= [\mathcal{L}_{H_f}, i_{H_g}]\omega \\ &= i_{[H_f, H_g]}\omega. \end{aligned} \tag{17}$$

The sequence is exact at $H^0(M)$ since the embedding α of the locally constant functions is injective.

The sequence is exact at $C^\infty(M)$: For a locally constant function c we have $H_c = \tilde{\omega}^{-1} dc = \tilde{\omega}^{-1} 0 = 0$. If $H_f = \tilde{\omega}^{-1} df = 0$ for $f \in C^\infty(M)$ then $df = 0$, so f is locally constant.

The sequence is exact at $\mathfrak{X}(M, \omega)$: For $X \in \mathfrak{X}(M, \omega)$ we have

$$d i_X \omega = i_X d\omega + i_X d\omega = \mathcal{L}_X \omega = 0,$$

thus $\gamma(X) = [i_X\omega] \in H^1(M)$ is well-defined. For $f \in C^\infty(M)$ we have

$$\gamma(H_f) = [i_{H_f}\omega] = [df] = 0 \in H^1(M).$$

If $X \in \mathfrak{X}(M, \omega)$ with $\gamma(X) = [i_X\omega] = 0 \in H^1(M)$ then $i_X\omega = df$ for some $f \in \Omega^0(M) = C^\infty(M)$, but then $X = H_f$.

The sequence is exact at $H^1(M)$: The mapping γ is surjective since for $\varphi \in \Omega^1(M)$ with $d\varphi = 0$ we may consider

$$X := \tilde{\omega}^{-1}\varphi \in \mathfrak{X}(M)$$

which satisfies

$$\mathcal{L}_X\omega = i_X d\omega + d i_X\omega = 0 + d\varphi = 0$$

and

$$\gamma(X) = [i_X\omega] = [\varphi] \in H^1(M).$$

The Poisson bracket $\{\ ,\ \}$ is a Lie bracket and

$$\{f, gh\} = \{f, g\}h + g\{f, h\} :$$

$$\begin{aligned}
\{f, g\} &= \omega(H_g, H_f) = -\omega(H_f, H_g) = \{g, f\}, && (18)\\
\{f, \{g, h\}\} &= \mathcal{L}_{H_f}\mathcal{L}_{H_g}h = [\mathcal{L}_{H_f}, \mathcal{L}_{H_g}]h + \mathcal{L}_{H_g}\mathcal{L}_{H_f}h && (19)\\
&= [\mathcal{L}_{H_f}, \mathcal{L}_{H_g}]h + \{g, \{f, h\}\} = \mathcal{L}_{H_{\{f,g\}}}h + \{g, \{f, h\}\} && (20)\\
&= \{\{f, g\}, h\} + \{g, \{f, h\}\}, && (21)\\
\{f, gh\} &= \mathcal{L}_{H_f}(gh) = \mathcal{L}_{H_f}(g)h + g\mathcal{L}_{H_f}(h) = \{f, g\}h + g\{f, h\}. && (22)
\end{aligned}$$

All mappings in the sequence are Lie algebra homomorphisms: For local constants $\{c_1, c_2\}$ we have $H_{c_1c_2} = 0$. For H we already checked. For $X, Y \in \mathfrak{X}(M, \omega)$ we have

$$i_{[X,Y]}\omega = [\mathcal{L}_X, i_Y]\omega = \mathcal{L}_X i_Y\omega - i_Y\mathcal{L}_X\omega = d i_X i_Y\omega + i_X d i_Y\omega - 0 = d i_X i_Y\omega,$$

thus

$$\gamma([X, Y]) = [i_{[X,Y]}\omega] = 0 \in H^1(M).$$

Let us now transform the situation by a symplectomorphism $\varphi : M \to M$ via pullback. Then

$$\varphi^*\omega = \omega \quad \Leftrightarrow \quad (T\varphi)^* \circ \tilde{\omega} \circ T\varphi = \tilde{\omega}$$

$$\Rightarrow H_{\varphi^*f} = \tilde{\omega}^{-1}d(\varphi^*f) = \tilde{\omega}^{-1}(\varphi^*df) = (T\varphi^{-1} \circ \tilde{\omega}^{-1} \circ (T\varphi)^*)(df \circ \varphi) = \varphi^*(H_f)$$
$$\varphi^*\{f, g\} = \varphi^*(dg(H_f)) = (\varphi^*dg)(\varphi^*H_f) = d(\varphi^*g)(H_{\varphi^*f}) = \{\varphi^*f, \varphi^*g\}.$$

The operator $X \mapsto \mathcal{L}_X$ is the Lie derivative followed by applying

$$\mathcal{L}_X = \frac{\partial}{\partial t}\bigg|_{t=0}(F_t^X)^*.$$

□

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