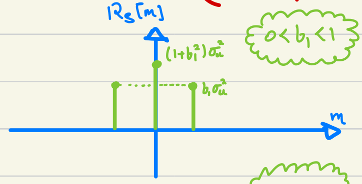
تمرین سری ۴ درس BSP - رادین خیال - 99101579

سوال 1 -

الف)

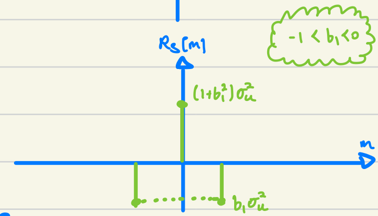
$$S[n] = U[n] + b_1 U[n-1]$$

$$R_S[m] = \begin{cases} \sigma_u^2 \sum_{k=0}^{1-|m|} b_k b_{k+|m|} & 0 \leqslant |m| \leqslant 1 \\ 0 & |m| > 1 \end{cases}$$



$$\rightarrow R_S[0] = \sigma_u^2 \sum_{k=0}^{1} b_k b_k = (1+b_1^2)\sigma_u^2$$

$$R_S[1] = R_S[-1] = \sigma_u^2 \sum_{k=1}^{0} b_k b_{k+1} = b_1 \sigma_u^2$$



$$\rightarrow R_S[m] = b_1\sigma_u^2 \left(\delta[m-1] + \delta[m+1]\right) + (1+b_1^2)\sigma_u^2\, \delta[m]$$

ب)

$$S[n] = U[n] + b U[n-1] + b_2 U[n-2]$$

$$R_S[m] = \begin{cases} \sigma_u^2 \sum_{k=0}^{2-|m|} b_k b_{k+|m|} & 0 \leqslant |m| \leqslant 1 \\ 0 & |m| > 1 \end{cases}$$

$$\rightarrow R_S[0] = \sigma_u^2 \sum_{k=0}^{1} b_k b_k = (1+b_1^2+b_2^2)\sigma_u^2$$

$$R_S[1] = R_S[-1] = \sigma_u^2 \sum_{k=0}^{1} b_k b_{k+1} = (b_1 + b_1 b_2)\sigma_u^2$$

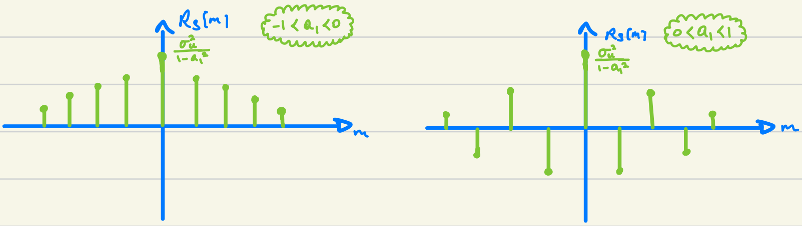
$$R_S[2] = R_S[-2] = \sigma_u^2 \sum_{k=0}^{0} b_k b_{k+2} = b_2 \sigma_u^2$$

$$\rightarrow \boxed{R_S[m] = b_2\sigma_u^2 \left(\delta[m-2] + \delta[m+2]\right) + (b_1+b_1 b_2)\sigma_u^2 \left(\delta[m-1] + \delta[m+1]\right) + (1+b_1^2+b_2^2)\sigma_u^2\, \delta[m]}$$

پ)

$$S[n] = U[n] - a_1 S[n-1]$$

$$S(z) = U(z) - a_1 z^{-1} S(z) \rightarrow \frac{S(z)}{U(z)} = \frac{1}{1+a_1 z^{-1}} \rightarrow R_S[m] = \frac{\sigma_u^2}{1-a_1^2}(-a_1)^{|m|}$$



ت)

$$S[n] = U[n] + (P_1 + P_2) S[n-1] - (P_1 P_2) S[n-2]$$

$$S(z) = U(z) + (P_1 + P_2) z^{-1} S(z) - (P_1 P_2) z^{-2} S(z)$$

$$\rightarrow H(z) = \frac{S(z)}{U(z)} = \frac{1}{1-(P_1+P_2)z^{-1} + P_1 P_2 z^{-2}} = \frac{A}{1-P_1 z^{-1}} + \frac{B}{1-P_2 z^{-1}}$$

$$\begin{cases} A+B=1 \\ -AP_2 - BP_1 = 0 \end{cases} \rightarrow \boxed{A = \frac{P_1}{P_1 - P_2}} \quad , \quad \boxed{B = \frac{-P_2}{P_1 - P_2}}$$

این دو ترم را هم داریم

$$H(z) = \underbrace{\frac{A}{1-P_1 z^{-1}}}_{H_1(z)} + \underbrace{\frac{B}{1-P_2 z^{-1}}}_{H_2(z)} \rightarrow H(z)H\left(\frac{1}{z}\right) = \overbrace{H_1(z)H_1\left(\frac{1}{z}\right) + H_2(z)H_2\left(\frac{1}{z}\right)}$$

$$+ H_1(z)H_2\left(\frac{1}{z}\right) + H_1\left(\frac{1}{z}\right)H_2(z)$$

ادامه سوال قسمت الف

$$H_1(z)H_2(\tfrac{1}{z}) = \frac{A}{1-P_1z^{-1}} \times \frac{B}{1-P_2z} = AB \times \frac{-\frac{1}{P_2}z^{-1}}{(1-P_1z^{-1})(1-\frac{1}{P_2}z^{-1})}$$

$$= AB \times \left(\frac{C}{1-P_1z^{-1}} + \frac{D}{1-\frac{1}{P_2}z^{-1}}\right) \to \begin{array}{l} C+D=0 \\ P_1D+\frac{1}{P_2}C=\frac{1}{P_2} \end{array} \to \left(\frac{1}{P_2}-P_1\right)C = \frac{1}{P_2} \to \boxed{C=\frac{1}{1-P_1P_2}}$$

$$\boxed{D=-C}$$

$$H_1(\tfrac{1}{z})H_2(z) = AB \times \frac{-\frac{1}{P_1}z^{-1}}{(1-\frac{1}{P_1}z^{-1})(1-P_2z)} = AB\left(\frac{E}{1-\frac{1}{P_1}z^{-1}} + \frac{F}{1-P_2z}\right) \to \begin{array}{l} E+F=0 \\ EP_2+\frac{F}{P_1}=\frac{1}{P_1} \end{array}$$

$$\to E\left(P_2-\frac{1}{P_1}\right) = \frac{1}{P_1} \to \boxed{E=-C} \;,\; \boxed{F=C}$$

$$S_s(z) = H(z)H(\tfrac{1}{z})\sigma_u^2 = \Big(H_1(z)H_1(\tfrac{1}{z}) + H_2(z)H_2(\tfrac{1}{z})$$

$$+ AB\left(\frac{C}{1-P_1z^{-1}} - \frac{C}{1-\frac{1}{P_2}z^{-1}}\right) + AB\left(\frac{-C}{1-\frac{1}{P_1}z^{-1}} + \frac{C}{1-P_2z}\right)\Big)\sigma_u^2$$

$$\to R_s[m] = \frac{A^2\sigma_u^2}{1-P_1^2}P_1^{|m|} + \frac{B^2\sigma_u^2}{1-P_2^2}P_2^{|m|} + ABC\Big(P_1^{m}u[m] + P_2^{-m}u[-m-1]$$
$$+ P_1^{-m}u[-m-1] + P_2^{m}u[m]\Big)\sigma_u^2$$

$$\to R_s[m] = \frac{A^2\sigma_u^2}{1-P_1^2}P_1^{|m|} + \frac{B^2\sigma_u^2}{1-P_2^2}P_2^{|m|} + ABC\left(P_1^{|m|}+P_2^{|m|}\right)\sigma_u^2$$

$$\to \boxed{R_s[m] = \left(\frac{A^2}{1-P_1^2}+ABC\right)\sigma_u^2 P_1^{|m|} + \left(\frac{B^2}{1-P_2^2}+ABC\right)\sigma_u^2 P_2^{|m|}}$$

$$\boxed{A=\frac{P_1}{P_1-P_2} \;,\; B=\frac{-P_2}{P_1-P_2} \;,\; C=\frac{1}{1-P_1P_2}}$$

باید توانستیم به آن فرم بنویسیم:

$$\boxed{\alpha=P_1} \;,\; \boxed{\beta=P_2}$$

$$\boxed{K_1=\left(\frac{A^2}{1-P_1^2}+ABC\right)\sigma_u^2} \;,\; \boxed{K_2=\left(\frac{B^2}{1-P_2^2}+ABC\right)\sigma_u^2}$$

سوال 2-

الف)

برای فرآیند اول :

$$s[n] = u[n] + \frac{3}{4}u[n-1] + \frac{1}{4}u[n-2]$$

$$\to S(z) = U(z) + \frac{3}{4}z^{-1}U(z) + \frac{1}{4}z^{-2}U(z) \to H(z) = \frac{S(z)}{U(z)} = 1+\frac{3}{4}z^{-1}+\frac{1}{4}z^{-2}$$

$$\to 1+\frac{3}{4}z^{-1}+\frac{1}{4}z^{-2} = \frac{1}{\sum_{k=0}^{\infty} a_k z^{-k}} \to \left(1+\frac{3}{4}z^{-1}+\frac{1}{4}z^{-2}\right)\left(a_0+a_1z^{-1}+a_2z^{-2}+\dots\right)=1$$

$$z^{-0}: \boxed{a_0=1}$$

$$z^{-1}: a_1+\frac{3}{4}=0 \to \boxed{a_1=-\frac{3}{4}} \qquad \xrightarrow{a_k,\ k>2} \boxed{a_k = -\frac{3}{4}a_{k-1} - \frac{1}{4}a_{k-2}}$$

$$z^{-2}: a_2+\frac{1}{4}+\frac{3}{4}a_1=0 \to \boxed{a_2=\frac{-5}{16}}$$

برای فرآیند دوم :

$$s[n] = u[n] + 0.81\,s[n-2] \to S(z) = U(z)+0.81\,z^{-2}S(z) \to H(z)=\frac{S(z)}{U(z)}=\frac{1}{1-0.81z^{-2}}$$

$$\sum_{k=0}^{p} b_k z^{-k} = \frac{1}{1-0.81z^{-2}} \to \left(1-0.81z^{-2}\right)\left(b_0+b_1z^{-1}+b_2z^{-2}+\dots\right)=1$$

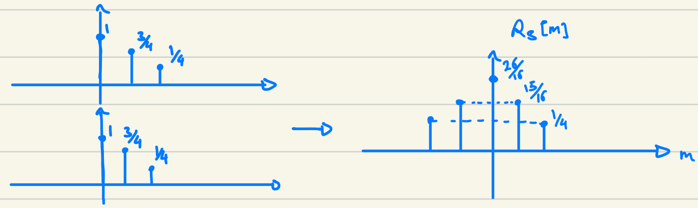
$$z^{-0}: \boxed{b_0=1} \qquad z^{-1}: \boxed{b_1=0} \qquad \xrightarrow{b_k,\ k>2} \boxed{b_k=0.81\,b_{k-2}} \to b_k = \begin{cases}(0.81)^{m/2}, & m=2k\\ 0, & m=2k+1\end{cases}$$

ب)

برای فرآیند اول:

$$H(z) = 1 + \frac{3}{4}z^{-1} + \frac{1}{4}z^{-2} \rightarrow h[m] = \delta[m] + \frac{3}{4}\delta[m-1] + \frac{1}{4}\delta[m-2]$$

$$R_s[m] = h[m] * h^*[-m]$$



$$\rightarrow \boxed{R_s[0] = \frac{26}{16}},\ \boxed{R_s[1] = \frac{15}{16}},\ \boxed{R_s[2] = \frac{1}{4}}$$

برای فرآیند دوم:

$$H(Z) = \frac{1}{1-0.81Z^{-2}} = \frac{A}{1-0.9Z^{-1}} + \frac{B}{1+0.9Z^{-1}} \rightarrow \begin{matrix} A+B=1 \\ -0.9B+0.9A=0 \end{matrix} \rightarrow A=B=0.5$$

$$\rightarrow H(z) = \underbrace{\frac{0.5}{1-0.9z^{-1}}}_{H_1(z)} + \underbrace{\frac{0.5}{1+0.9z^{-1}}}_{H_2(z)}$$

$$H(z)H(\tfrac{1}{z}) = \underbrace{H_1(z)H_1(\tfrac{1}{z})}_{} + \underbrace{H_2(z)H_2(\tfrac{1}{z})}_{} + H_1(z)H_2(\tfrac{1}{z}) + H_2(z)H_1(\tfrac{1}{z})$$

این دو تا ترم رو می دونیم

$$H_1(z)H_2(\tfrac{1}{z}) = \frac{1}{4}\,\frac{1}{1-0.9z^{-1}} \times \frac{1}{1+0.9z} = \frac{1}{4} \times \frac{\frac{10}{9}z^{-1}}{(1-0.9z^{-1})(1+\frac{10}{9}z^{-1})} = \frac{1}{4}\times\left(\frac{A}{1-\frac{9}{10}z^{-1}} + \frac{B}{1+\frac{10}{9}z^{-1}}\right)$$

$$\rightarrow \begin{matrix} A+B=0 \\ -\frac{9}{10}B + \frac{10}{9}A = \frac{10}{9} \end{matrix} \rightarrow \frac{9}{10}A + \frac{10}{9}A = \frac{10}{9} \rightarrow A = \frac{1}{1.81} \simeq 0.55 \rightarrow B = -0.55$$

$$\rightarrow H_1(z)H_2(\tfrac{1}{z}) = \frac{1}{4}\times\left(\frac{0.55}{1-\frac{9}{10}z^{-1}} - \frac{0.55}{1+\frac{10}{9}z^{-1}}\right)$$

ادامه سؤال 2 قسمت ب)

$$H_1(\tfrac{1}{2})H_2(z) = \frac{1}{4}\times\frac{1}{1-0.9z}\times\frac{1}{1+0.9z^{-1}} = \frac{1}{4}\times\frac{-\frac{10}{9}z^{-1}}{(1-\frac{10}{9}z^{-1})(1+0.9z^{-1})}$$

$$= \frac{1}{4}\times\left(\frac{A}{1-\frac{10}{9}z^{-1}}+\frac{B}{1+0.9z^{-1}}\right) \rightarrow \begin{cases} A+B=0 \\ -\frac{10}{9}B+\frac{9}{10}A=-\frac{10}{9}\end{cases} \rightarrow \frac{10}{9}A+\frac{9}{10}A=-\frac{10}{9}$$

$$\rightarrow A=-0.55, \quad B=0.55$$

$$\rightarrow H_1(\tfrac{1}{2})H_2(z) = \frac{1}{4}\times\left(\frac{-0.55}{1-\frac{10}{9}z^{-1}}+\frac{0.55}{1+0.9z^{-1}}\right)$$

$$\rightarrow R_s[m] = \frac{1}{4}\Big(\frac{1}{1-0.81}(0.9)^{|m|}+\frac{1}{1-0.81}(-0.9)^{|m|}$$
$$+0.55\times(\tfrac{9}{10})^m u[m]+0.55\times(-\tfrac{10}{9})^m u[-m-1]$$
$$+0.55\times(\tfrac{10}{9})^m u[-m-1]+0.55\times(-\tfrac{9}{10})^m u[m]\Big)$$

$$\rightarrow R_s[0] = \frac{1}{4}\times\left(\frac{100}{19}+\frac{100}{19}+0.55+0.55\right) \simeq \boxed{2.91}$$

$$s[n] = u[n]+0.81s[n-2] \rightarrow a_1=0,\ a_2=-0.81$$

$$R_s[m]+\sum_{k=1}^{2}a_kR_s[m-k]=0$$: با استفاده از Yule-walker *

$$\rightarrow R_s[1]+a_2R_s[-1]=0 \rightarrow R_s[1]=0$$

$$R_s[2]+a_2R_s[0]=0 \rightarrow R_s[2]=+\frac{81}{100}\times 2.91 = +2.36$$

$$\rightarrow \boxed{\begin{aligned} R_s[0]&=2.91\\ R_s[1]&=0\\ R_s[2]&=2.36\end{aligned}}$$

ادامه سؤال 2 قسمت ب)

یک راه ساده‌تر:

$$H(Z) = \frac{0.5}{1-0.9Z^{-1}} + \frac{0.5}{1+0.9Z^{-1}} \longrightarrow h[m] = \frac{1}{2}(0.9)^m u[m] + \frac{1}{2}(-0.9)^m u[m]$$

$$\longrightarrow h[m] = \begin{cases} (0.9)^m & m \geq 0,\ m \text{ even} \\ 0 & m \geq 0,\ m \text{ odd} \\ 0 & m < 0 \end{cases}$$

$$R_S[m] = h[m] * h^*[-m] \longrightarrow R_S[0] = 1 + \left(\frac{9}{10}\right)^2\left(\frac{9}{10}\right)^2 + \left(\frac{9}{10}\right)^4\left(\frac{9}{10}\right)^4 + \dots$$

$$\longrightarrow R_S[0] = 1 + \left(\frac{9}{10}\right)^4 + \left(\frac{9}{10}\right)^8 + \dots$$

$$- \quad \left(\frac{9}{10}\right)^4 R_S[0] = \left(\frac{9}{10}\right)^4 + \left(\frac{9}{10}\right)^8 + \dots$$

$$1 - \left(\frac{9}{10}\right)^4 R_S[0] = 1 \longrightarrow R_S[0] = \frac{1}{1-\left(\frac{9}{10}\right)^4} \simeq \boxed{2.91}$$

$$\longrightarrow \boxed{\begin{aligned} R_S[0] &= 2.91 \\ R_S[1] &= 0 \\ R_S[2] &= 0.81\, R_S[0] = 2.36 \end{aligned}}$$

ادامه سؤال 2 -

پ)

$$s[n] = u[n] + \frac{3}{4}u[n-1] + \frac{1}{4}u[n-2]$$

$$R_s[0] = \frac{26}{16}, \quad R_s[1] = \frac{15}{16}, \quad R_s[2] = \frac{1}{4}$$

روش بازگشتی:

$$AR(1): \quad s[n] + a_1 s[n-1] = u[n]$$

step 0: $E_0 = R_s[0] = \frac{26}{16}$

step 1: $K_1 = \frac{-R_s[1]}{E_0} = \frac{-\frac{15}{16}}{\frac{26}{16}} = \frac{-15}{26}$

$$a_1^{(1)} = K_1 = \frac{-15}{26} \simeq -0.577$$

$$\longrightarrow \boxed{s[n] - 0.577\, s[n-1] = u[n]}$$

روش یول واکر:

$$R_s[1] + a_1 R_s[0] = 0 \longrightarrow a_1 = \frac{-R_s[1]}{R_s[0]} = \frac{-15}{26}$$

AR(2):

ادامه سؤال 2 قسمت پ)

$$\text{step}_0: \; E_0 = R_s[0] = \frac{26}{16}$$

$$\text{step 1}: \; K_1 = \frac{-R_s[1]}{E_0} = \frac{-15}{26}$$

$$a_1^{(1)} = \frac{-15}{26} \quad , \quad E_1 = (1-K_1^2)E_0 = \frac{451}{676} \times \frac{26}{16} = 1.08$$

$$\text{step 2}: \; K_2 = -\frac{R_s[2] + a_1^{(1)} R_s[1]}{E_1} = -\frac{\frac{1}{4} - \frac{15}{26}\times\frac{15}{16}}{1.08} = -\frac{0.25 - 0.541}{1.08} = +0.268$$

$$a_1^{(2)} = a_1^{(1)} + K_2 a_1^{(1)} = -0.577 - 0.269 \times 0.577 = -0.732$$

$$\boxed{S[n] - 0.732\, S[n-1] + 0.268\, S[n-2] = U[n]}$$

به روش یول واکر :

$$\begin{bmatrix} R_s[0] & R_s[-1] \\ R_s[1] & R_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} R_s[1] \\ R_s[2] \end{bmatrix}$$

$$\begin{bmatrix} \frac{26}{16} & \frac{15}{16} \\ \frac{15}{16} & \frac{26}{16} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -\frac{15}{16} \\ -\frac{1}{4} \end{bmatrix} \rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{(\frac{26}{16})^2 - (\frac{15}{16})^2} \begin{bmatrix} \frac{26}{16} & -\frac{15}{16} \\ -\frac{15}{16} & \frac{26}{16} \end{bmatrix} \begin{bmatrix} -\frac{15}{16} \\ -\frac{1}{4} \end{bmatrix}$$

$$\rightarrow a_1 = \frac{\frac{26}{16} \times -\frac{15}{16} + \frac{15}{16} \times \frac{1}{4}}{(\frac{26}{16})^2 - (\frac{15}{16})^2} = \frac{26 \times -15 + 15 \times 4}{26^2 - 15^2} = \frac{-330}{451} = -0.732$$

$$a_2 = \frac{(\frac{15}{16})^2 - \frac{26}{16} \times \frac{1}{4}}{(\frac{26}{16})^2 - (\frac{15}{16})^2} = \frac{225 - 26 \times 4}{26^2 - 15^2} = \frac{121}{451} = 0.268$$

$$\rightarrow \boxed{S[n] - 0.732\, S[n-1] + 0.268\, S[n-2] = U[n]}$$

* در قسمت الف: $a_1 = -0.75$, $a_2 = -0.3125$

تو حالتی که با AR(1) تخمین زدیم ضریب a_1 خیلی تفاوت داشت درحالتی که با AR(2) تخمین زدیم ضریب a_1 تقریباً مشابه بود ول a_2 خیلی فرق داشت. انتظار می‌رود که با افزایش مرتبه AR ضرایب مشابه با قسمت الف بشود.

ادامه سؤال 2 -

ت)

$$S[n] = U[n] + 0.75\,U[n-1] + 0.25\,U[n-2]$$

$$R_S[0] = 2.91 \quad , \quad R_S[1] = 0 \quad , \quad R_S[2] = 2.36$$

MA(1) :

$$R_S[m] = \sum_{K=0}^{1-m} b_K b_{K+m}$$

$$\longrightarrow R_S[0] = b_0^2 + b_1^2 = 2.91$$

$$R_S[1] = b_0 b_1 = 0 \quad \longrightarrow \quad b_0 = 0, \quad b_1 = \pm\sqrt{2.91} \simeq \pm 1.71$$

$$\longrightarrow S[n] = 1.71\,U[n-1]$$

$$S[n] = -1.71\,U[n-1]$$

MA(2) :

$$R_S[0] = b_0^2 + b_1^2 + b_2^2 = 2.91$$

$$\left.\begin{array}{l} R_S[1] = b_0 b_1 + b_1 b_2 = 0 \longrightarrow b_1(b_0 + b_2) = 0 \\ R_S[2] = b_0 b_2 = 2.36 \longrightarrow \text{هر دو یا مثبت یا منفی} \end{array}\right\} \longrightarrow \boxed{b_1 = 0}$$

$$\longrightarrow \begin{array}{l} b_0^2 + b_2^2 = 2.91 \\ b_0 b_2 = 2.36 \end{array} \longrightarrow b_0^2 + \frac{2.36^2}{b_0^2} = 2.91 \longrightarrow b_0^4 - 2.91\,b_0^2 + 5.57 = 0$$

* چهار جواب مختلط دارد. یکی از جواب ها:

$$b_0^2 = 1.455 + j1.86 \longrightarrow b_2^2 = 1.455 - j1.86$$

$$b_0^2 = 1.455 - j1.86 \longrightarrow b_2^2 = 1.455 - j1.86$$

سؤال 3 -

الف)

$$\begin{bmatrix} R_s[0] & R_s[-1] \\ R_s[1] & R_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -R_s[1] \\ -R_s[2] \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 64 & -16 \\ -16 & 64 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{64^2 - 16^2} \begin{bmatrix} 64 & 16 \\ 16 & 64 \end{bmatrix} \begin{bmatrix} 16 \\ 2 \end{bmatrix}$$

$$\rightarrow a_1 = \frac{-64 \times 16 - 2 \times 16}{64^2 - 16^2} \simeq \frac{-1056}{3840} = \boxed{-0.275}$$

$$a_2 = \frac{-16 \times 16 - 64 \times 2}{64^2 - 16^2} \simeq \frac{-384}{3840} = \boxed{-0.1}$$

$$\rightarrow R_s[3] + a_1 R_s[2] + a_2 R_s[1] = 0$$

$$\rightarrow R_s[3] = 0.275 \times 2 + 16 \times 0.1 = \boxed{2.15}$$

ب)

$$ARMA(2,1) \rightarrow \begin{matrix} p = 2 \\ q = 1 \end{matrix}$$

$$\begin{bmatrix} R_s[1] & R_s[0] \\ R_s[2] & R_s[1] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -R_s[2] \\ -R_s[3] \end{bmatrix} \rightarrow \begin{bmatrix} -16 & 64 \\ -2 & -16 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{16^2 + 128} \begin{bmatrix} -16 & -64 \\ +2 & -16 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} \rightarrow a_1 = \frac{-32 + 320}{384} = \boxed{0.75}$$

$$a_2 = \frac{-4 + 80}{384} = \boxed{0.22}$$

ادامه سؤال 3 قسمت ب)

$$R_V[m] = (\overbrace{1+a_1^2+a_2^2}^{1.61}) R_S[m] + (\overbrace{a_1+a_1 a_2}^{0.92}) (R_S[m-1]+R_S[m+1]) + \overbrace{a_2}^{0.22} (R_S[m-2]+R_S[m+2])$$

$$\rightarrow R_V[0] = 1.61 R_S[0] + 0.92 (R_S[1]+R_S[1]) + 0.22 (R_S[2]+R_S[2])$$

$$= 1.61 \times 64 + 0.92 \times 2 \times -16 + 0.22 \times 2 \times -2 = \underline{72.72}$$

$$R_V[1] = 1.61 R_S[1] + 0.92 (R_S[0]+R_S[2]) + 0.22 (R_S[1]+R_S[3])$$

$$= 1.61 \times -16 + 0.92 (64-2) + 0.22 (-16+5) = \underline{\underline{28.86}}$$

$$\rightarrow \begin{cases} b_0^2 + b_1^2 = 72.72 \\ b_0 b_1 = 28.86 \end{cases} \rightarrow b_0^2 + \frac{(28.86)^2}{b_0^2} = 72.72$$

$$\rightarrow b_0^4 - 72.72 b_0^2 + (28.86)^2 = 0 \begin{cases} \rightarrow b_0 = \pm 3.77, \quad b_1 = \pm 7.64 \\ \rightarrow b_0 = \pm 7.64, \quad b_1 = \pm 3.77 \end{cases}$$

$$R_S[4] = -(a_1 R_S[3] + a_2 R_S[2])$$

$$= -0.75 \times 5 + 0.22 \times 2 = \boxed{-3.31}$$

ادامه سؤال 3 -

پ)

$AR(1,1) \longrightarrow \begin{matrix} p=1 \\ q=1 \end{matrix} \quad m = p+q = 2$

$$\begin{bmatrix} R_S[0] & R_S[1] \\ R_S[1] & R_S[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -R_S[1] \\ -R_S[2] \end{bmatrix}$$

$$\begin{bmatrix} 64 & -16 \\ -16 & 64 \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \end{bmatrix} \longrightarrow \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{bmatrix} = \frac{1}{64^2-16^2} \begin{bmatrix} 64 & 16 \\ 16 & 64 \end{bmatrix} \begin{bmatrix} 16 \\ 2 \end{bmatrix}$$

$$\longrightarrow \quad \tilde{a}_1 = \frac{64\times16+16\times2}{64^2-16^2} = \frac{1056}{3840} = \underline{0.275}$$

$$\tilde{a}_2 = \frac{16\times16+64\times2}{64^2-16^2} = \frac{384}{3840} = \underline{0.1}$$

$$\sigma_u^2 = R_S[0] + \tilde{a}_1 \hat{R}_S[-1] + \tilde{a}_2 \hat{R}_S[-2] = 64 - 0.275\times16 - 0.1\times2$$
$$= 59.4$$

$$\hat{R}_S[3] = -\tilde{a}_1 \hat{R}_S[2] - \tilde{a}_2 \hat{R}_S[1] = +0.275\times2 + 0.1\times16 = \boxed{2.15}$$

$$(1+\hat{b}_1 z^{-1})(1+0.275z^{-1}+0.1z^{-2}) \simeq (1+\hat{a}_1 z^{-1})$$

$$\begin{matrix} \hat{b}_1 + 0.275 = \hat{a}_1 \\ 0.275\hat{b}_1 + 0.1 = 0 \end{matrix} \longrightarrow \boxed{\begin{matrix} \hat{b}_1 = -0.364 \\ \hat{a}_1 = -0.089 \end{matrix}}$$

ت) ضرایب قسمت الف و ب باهم برابرند چون هر دو AR(2) هستند.

ARMA(1,1) متفاوت است ولی.

ب)

$$\hat{a}_1 = -\frac{\hat{R}_s[2]}{\hat{R}_s[1]} = \frac{2}{-16} = \boxed{-\frac{1}{8}}$$

$$\hat{R}_v[0] = (1+\hat{a}_1^2)\hat{R}_s[0] + \hat{a}_1(\hat{R}_s[-1]+\hat{R}_s[1]) = 69$$

$$\hat{R}_v[1] = (1+\hat{a}_1^2)\hat{R}_s[1] + \hat{a}_1[\hat{R}_s[0]+\hat{R}_s[2]] = -\frac{65}{4} - \frac{62}{8} = -24$$

$$\rightarrow \begin{cases} \hat{b}_0^2 + \hat{b}_1^2 = 69 \\ \hat{b}_0 \hat{b}_1 = 20 \end{cases} \rightarrow \boxed{\hat{b}_0 = \pm 3.12} \quad \boxed{\hat{b}_1 = \mp 7.7}$$

$$R_s[3] = -\hat{a}_1 R_s[2] = \frac{1}{8} \times -2 = \boxed{-0.25}$$

پاسخ بدست آمده متفاوت است.

ج)

$$\hat{R}_s[0] = b_0^2 + b_1^2 + b_2^2 = 64$$

$$\hat{R}_s[1] = b_0 b_1 + b_1 b_2 = -16$$

5 جواب دارد و ضرایب حقیقی هستند.

$$\hat{R}_s[2] = b_0 b_2 = -2$$

$R_s[3] = 0$

$$\boxed{b_0 = -0.26 \,,\; b_1 = -2.15 \,,\; b_2 = 7.7}$$

* یکی از جواب ها

سؤال 4 -

روش دستگاه معادلات:

```python
from sympy import symbols, Eq, solve, Matrix
from sympy.solvers import solve

R = [30,20,11,4]
b0 = Symbol('b0')
b1 = Symbol('b1')
b2 = Symbol('b2')
b3 = Symbol('b3')

eq1 = R[0]-b0**2-b1**2-b2**2-b3**2
eq2 = R[1]-b0*b1 - b1*b2 - b2*b3
eq3 = R[2]- b0*b2 - b1*b3
eq4 = R[3]- b0*b3

solve([eq1, eq2, eq3, eq4],b0, b1, b2, b3)

print('b0 = ', b0)
print('b1 = ', b1)
print('b2 = ', b2)
print('b3 = ', b3)
```

```
b0 =  4.0
b1 =  3.0
b2 =  2.0
b3 =  1.0
----------------
```

روش بازگشتی :

```python
import numpy as np


R = np.array([30, 20, 11, 4])
b = np.ones(4)
for i in range(100):
    b[0] = np.sqrt(R[0] - np.sum(b[1:]**2))
    b[1] = 1 / b[0] * (R[1] - np.sum(b[2:] * b[1:-1]))
    b[2] = 1 / b[0] * (R[2] - b[1]*b[3])
    b[3] = 1 / b[0] * (R[3])

print('b0 = ', b[0])
print('b1 = ', b[1])
print('b2 = ', b[2])
print('b3 = ', b[3])
print('-----------------')
```

```
b0 =  4.0
b1 =  3.0
b2 =  2.0
b3 =  1.0
-----------------
```

هر دو روش به پاسخ یکسان رسیدند، اما روش بازگشتی سریعتر بود.

همچنین روش دستگاه معادلات چندین جواب دیگر هم داشت.

سوال 5 -

الف)

$$X[n] = U[n] + \alpha \left(\sum_{k=1}^{\infty} U[n-k] \right)$$

$$\rightarrow E[X[n]] = E[U[n]] + \alpha \sum_{k=1}^{\infty} E[U[n-k]] = \boxed{0}$$

$$R_X[n_1,n_2] = E[X[n_1]X[n_2]] = E[U[n_1]U[n_2]] + \alpha E\left[\sum_{k=1}^{\infty} U[n_1]U[n_2-k]\right]$$

$$+ \alpha E\left[\sum_{k=1}^{\infty} U[n_2]U[n_1-k]\right] + \alpha^2 E\left[\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} U[n_1-k]U[n_2-l]\right]$$

$$= \sigma^2 \delta[n_1-n_2] + \alpha \sum_{k=1}^{\infty} \sigma^2 \delta[n_1-n_2+k] + \alpha \sum_{k=1}^{\infty} \sigma^2 \delta[n_2-n_1+k] + \alpha^2 \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \sigma^2 \delta[n_1-n_2-k+l]$$

* فرآیند ایستا هست زیرا میانگین عدد ثابت صفر است و کورلیشن هم تنها تابعی از اختلاف در لحظه است.

ب)

$$Y[n] = X[n] - X[n-1] \rightarrow Y[n] = U[n] - U[n-1] + \alpha\left(\sum_{k=1}^{\infty} U[n-k] - \sum_{k=1}^{\infty} U[n-k-1] \right)$$

$$\rightarrow Y[n] = U[n] - U[n-1] + \alpha \sum_{k=1}^{\infty} U[n-k] - U[n-k-1]$$

$$\rightarrow Y[n] = U[n] - U[n-1] + \alpha\left(U[n-1] - \cancel{U[n-2]} + \cancel{U[n-2]} - \cancel{U[n-3]} + \cdots \right)$$

$$\rightarrow \boxed{Y[n] = U[n] + (\alpha-1)U[n-1]} \rightarrow \boxed{H(z) = 1 + (\alpha-1)z^{-1}}$$

فرآیند MA(1) می باشد.