تمرین سری 7 درس BSP - رادین خیام - 9901579

سؤال 1-

الف)

$$\overset{(*)}{s_1[n] - \frac{3}{4} s_1[n] = u_1[n]}, \quad \overset{(**)}{y_1[n] = s_1[n] + u_2[n]}$$

$$\longrightarrow \begin{cases} z_k = y_1[n] \\ x_k = s_1[n] \\ w_k = u_1[n] \\ v_k = u_2[n] \end{cases} \xrightarrow{(*),(**)} \begin{matrix} x_k = \frac{3}{4} x_{k-1} + w_k \\ z_k = x_k + v_k \end{matrix} \longrightarrow \begin{matrix} F_k = \frac{3}{4} \\ H_k = 1 \\ Q_k = \sigma_1^2 = 70 \\ R_k = \sigma_2^2 = 224 \end{matrix}$$

$$\text{Prediction} \begin{cases} \hat{x}_k^- = \frac{3}{4} \hat{x}_{k-1} \\ P_k^- = \frac{9}{16} P_{k-1} + \sigma_1^2 \end{cases}, \quad \text{initial Condition} \begin{cases} \hat{x}_0 = E[x_0] = 0 \\ \hat{P}_0 = E[x_0^2] = \sigma_{s_1}^2 = \frac{\sigma_1^2}{1-\frac{9}{16}} = 160 \end{cases}$$

ب)

$$\begin{cases} P_\infty = P_\infty^- - G_\infty P_\infty^- \\ G_\infty = P_\infty^- (P_\infty^- + \sigma_2^2)^{-1} \\ P_\infty^- = \frac{9}{16} P_\infty + \sigma_1^2 \end{cases} \longrightarrow \begin{matrix} P_\infty = P_\infty^- - \frac{P_\infty^{-2}}{P_\infty^- + \sigma_2^2} = P_\infty^- \frac{\sigma_2^2}{P_\infty^- + \sigma_2^2} \\ P_\infty^- = \frac{9}{16} P_\infty + \sigma_1^2 \quad (*) \end{matrix}$$

$$\longrightarrow P_\infty (P_\infty^- + \sigma_2^2) = P_\infty^- \sigma_2^2 \xrightarrow{(*)} \frac{16}{9} (P_\infty^- - \sigma_1^2)(P_\infty^- + \sigma_2^2) = P_\infty^- \sigma_2^2$$

$$\longrightarrow P_\infty^{-2} + \left(\sigma_2^2 - \frac{9}{16}\sigma_2^2 - \sigma_1^2\right) P_\infty^- - \sigma_1^2 \sigma_2^2 = 0$$

$$\longrightarrow P_\infty^- = \cancel{-140}, \; 112 \quad \longrightarrow P_\infty = -210 \times \frac{16}{9} \simeq \cancel{-373.3}, \; 42 \times \frac{16}{9} \simeq 74.6$$

$$G_\infty = \cancel{\frac{-140}{84} \simeq -1.66}, \; \frac{+112}{336} \simeq 0.33 \quad \longrightarrow \begin{matrix} P_\infty^- = 112 \\ P_\infty = 74.6 \\ G_\infty = 0.33 \end{matrix}$$

نتایج با استفاده از متلب چک شدند.

Made with Goodnotes

پ)

$$\hat{X}_k = \hat{X}_k^- + G_k(Z_k - \hat{X}_k^-) \rightarrow \hat{X}_k = \frac{3}{4}\hat{X}_{k-1} + \overset{\frac{1}{3}}{\cancel{G_k}}\left(Z_k - \frac{3}{4}\hat{X}_{k-1}\right)$$

$$\rightarrow \hat{X}_k = \frac{1}{2}\hat{X}_{k-1} + \frac{1}{3}Z_k \rightarrow \hat{X}_k - \frac{1}{2}\hat{X}_{k-1} = \frac{1}{3}Z_k$$

$$Z_k \longrightarrow \boxed{H_c(z) = \frac{0.33}{1-\frac{1}{2}z^{-1}}} \longrightarrow \hat{X}_k$$

* جواب فیلتر کالمن به جواب فیلتر وینر IIR سببی میل می‌کند.

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سؤال 2 -

$$H_k = \begin{bmatrix}1\\1\end{bmatrix}, \quad R_k = \begin{bmatrix}\overset{224}{\cancel{\sigma_2^2}} & b \\ b & \underset{a}{\cancel{\sigma_3^2}}\end{bmatrix}$$

الف)

$$s_1[n] - \frac{3}{4}s_1[n-1] = u_1[n]$$

$$y_1[n] = s_1[n] + u_2[n]$$

$$y_2[n] = s_1[n] + u_3[n]$$

$$\longrightarrow Z_k = \begin{bmatrix}y_1[n]\\y_2[n]\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}\overset{s_1[n]}{X_k} + \begin{bmatrix}u_2[n]\\u_3[n]\end{bmatrix}$$

$$X_k = \frac{3}{4}X_{k-1} + W_k \rightarrow u_1[n]$$

Prediction
$$\begin{cases}\hat{X}_k^- = \frac{3}{4}\hat{X}_{k-1}^- \\ P_k^- = \frac{9}{16}P_{k-1} + \sigma_1^2\end{cases}$$

initial Condition
$$\begin{cases}\hat{x}_0 = E[x_0] = 0 \\ \hat{P}_0 = E[x_0^2] = \sigma_{s_1}^2 = \frac{\sigma_1^2}{1-\frac{9}{16}} = 160\end{cases}$$

Update
$$\begin{cases}\hat{x}_k = \hat{x}_k^- + G_k\left(Z_k - \begin{bmatrix}1\\1\end{bmatrix}\hat{x}_k^-\right) \\ G_k = P_k^-[1,1]\left(\begin{bmatrix}1\\1\end{bmatrix}P_k^-[1\ 1] + \begin{bmatrix}\sigma_2^2 & b\\ b & \sigma_3^2\end{bmatrix}\right)^{-1} \\ P_k = P_k^- - G_k\begin{bmatrix}1\\1\end{bmatrix}P_k^-\end{cases}$$

ادامه سؤال 2-

ب) خیر لازم نیست که حتماً $b=0$ باشد (برخلاف فیلتر وفقی) چون الگوریتم با صفر نبودن $b$ هم به مشکل نخواهد خورد.

پ) اگر صرفاً همین دو مشاهده را داشته باشیم نمی‌توانیم به جواب مسئله 1 سری 6 برسیم.

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سؤال 3-

الف)

$$s_1[n] = \frac{3}{4} s_1[n-1] + u_1[n] \quad , \quad y_1[n] = s_1[n] + u_2[n]$$

$$s_2[n] = -0.3 s_2[n-1] + u_3[n] \quad , \quad y_2[n] = s_1[n] + s_2[n]$$

$$X_k = \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} \quad , \quad Z_k = \begin{bmatrix} y_1[n] \\ y_2[n] \end{bmatrix}$$

$$X_k = \begin{bmatrix} 3/4 & 0 \\ 0 & -0.3 \end{bmatrix} X_{k-1} + \begin{bmatrix} u_1[n] \\ u_3[n] \end{bmatrix} \quad , \quad Z_k = \overset{H_k}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} X_k + \begin{bmatrix} u_2[n] \\ 0 \end{bmatrix}$$

$$Q_k = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_3^2 \end{bmatrix} \quad , \quad R_k = \begin{bmatrix} \sigma_2^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Prediction} \begin{cases} \hat{X}_k^- = \begin{bmatrix} 3/4 & 0 \\ 0 & -0.3 \end{bmatrix} \hat{X}_{k-1} \\ P_k^- = \begin{bmatrix} 3/4 & 0 \\ 0 & -0.3 \end{bmatrix} P_{k-1} \begin{bmatrix} 3/4 & 0 \\ 0 & -0.3 \end{bmatrix} + \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_3^2 \end{bmatrix} \end{cases} \quad , \quad \text{Update} \begin{cases} \hat{X}_k = \hat{X}_k^- + G_k (Z_k - H_k \hat{X}_k^-) \\ G_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1} \\ P_k = P_k^- - G_k H_k P_k^- \end{cases}$$

$$\text{initial} \rightarrow \hat{X}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad , \quad \hat{P}_0 = \begin{bmatrix} \frac{\sigma_1^2}{1-\frac{9}{16}} & 0 \\ 0 & \frac{\sigma_3^2}{1-\frac{9}{100}} \end{bmatrix}$$

سوال 4 -

روابط به شکل رو برو در میایند .

$$\text{Prediction}\begin{cases} \hat{X}_k^- = F_k \hat{X}_{k-1} + B_k U_k \\ P_k^- = F_k P_{k-1} F_k^T + C_k Q_k C_k^T \end{cases}$$

$$\text{Update}\begin{cases} \hat{X}_k = \hat{X}_k^- + G_k (Z_k - H_k \hat{X}_k^-) \\ G_k = P_k^- H_k^T (H_k P_k^- H_k^T + D_k R_k D_k^T)^{-1} \\ P_k = P_k^- - G_k H_k P_k^- \end{cases}$$