تمرین سری 8 درس BSP - رادین خیام . 99101579

سؤال 1-

$$P(x|\omega_1) = \frac{1}{\sqrt{2\pi}\sqrt{0.5}} e^{\frac{-x^2}{2\times 0.5}} = \frac{1}{\sqrt{\pi}} e^{-x^2}$$

$$P(x|\omega_2) = \frac{1}{\sqrt{2\pi}\sqrt{0.25}} e^{\frac{-(x-1)^2}{2\times 0.25}} = \frac{1}{\sqrt{\frac{\pi}{2}}} e^{\frac{-(x-1)^2}{0.5}} = \frac{\sqrt{2}}{\sqrt{\pi}} e^{-2(x-1)^2}$$

$$\frac{P(x|\omega_1)}{P(x|\omega_2)} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} \frac{\lambda_{21}}{\lambda_{12}} \frac{P(\omega_2)}{P(\omega_1)} \xrightarrow[\text{مرز}]{\text{برای پیدا کردن}} \lambda_{12} P(x|\omega_1) P(\omega_1) = \lambda_{21} P(x|\omega_2) P(\omega_2)$$

الف)

$$P(\omega_1) = P(\omega_2) \quad , \quad \lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P(x|\omega_1) = P(x|\omega_2) \longrightarrow \frac{1}{\cancel{\sqrt{\pi}}} e^{-x^2} = \frac{\sqrt{2}}{\cancel{\sqrt{\pi}}} e^{-2(x-1)^2}$$

$$\longrightarrow 2(x-1)^2 - x^2 = \ln\sqrt{2} \longrightarrow x_1 = 2 \pm \sqrt{2 + \ln\sqrt{2}}$$

$$\longrightarrow \begin{cases} x \in R_1 , & \text{خارج آن بازه} \\ x \in R_2 , & 2 - \sqrt{2+\ln\sqrt{2}} < x < 2 + \sqrt{2+\ln\sqrt{2}} \end{cases}$$

ب)

$$P(\omega_1) = P(\omega_2) \quad , \quad \lambda = \begin{bmatrix} 0 & 0.25 \\ 1 & 0 \end{bmatrix}$$

$$\frac{1}{4} P(x|\omega_1) = P(x|\omega_2) \longrightarrow \frac{1}{4} e^{-x^2} = \sqrt{2} e^{-2(x-1)^2}$$

$$\longrightarrow 2(x-1)^2 - x^2 = \ln(4\sqrt{2}) \longrightarrow x = 2 \pm \sqrt{2 + \ln 4\sqrt{2}}$$

$$\longrightarrow \begin{cases} x \in R_1 , & \text{خارج آن بازه} \\ x \in R_2 , & 2 - \sqrt{2+\ln 4\sqrt{2}} < x < 2 + \sqrt{2+\ln 4\sqrt{2}} \end{cases}$$

پ)

$P(\omega_1) = \frac{3}{4} P(\omega_2)$, $\lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\to \frac{3}{4} P(x|\omega_1) = P(x|\omega_2) \to 2(x-1)^2 - x^2 = \ln\left(\frac{4}{3}\sqrt{2}\right)$

$\to x = 2 \pm \sqrt{2 + \ln\left(\frac{4\sqrt{2}}{3}\right)} \to \begin{cases} x \in R_1 & \text{خارج آن بازه} \\ x \in R_2 & 2 - \sqrt{2+\ln\left(\frac{4\sqrt{2}}{3}\right)} < x < 2 + \sqrt{2+\ln\left(\frac{4\sqrt{2}}{3}\right)} \end{cases}$

ب)

$P(\omega_1) = 3P(\omega_2)$, $\lambda = \begin{bmatrix} 0 & 0.25 \\ 1 & 0 \end{bmatrix}$

$3P(x|\omega_1) \times \frac{1}{4} = P(x|\omega_2) \longrightarrow$ جواب مانند قسمت پ

سوال 2 -

$\mu_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mu_2 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, $\mu_3 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

الف)

$\Sigma = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $g_{ij}(\underline{x}) = \left(\Sigma^{-1}(\underline{\mu}_i - \underline{\mu}_j)\right)^T \underline{x} + \ln\frac{P(\omega_i)}{P(\omega_j)} - \frac{1}{2}\mu_i^T \Sigma^{-1} \mu_i + \frac{1}{2}\mu_j^T \Sigma^{-1}\mu_j$

$\to g_{12} = \left(\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -6 \\ 0 \end{bmatrix}\right)^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 6 & 0 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$= -4x_1 + 2x_2 + \frac{1}{6} \times 72 = -4x_1 + 2x_2 + 12 \to \boxed{-2x_1 + x_2 + 6 = 0}$

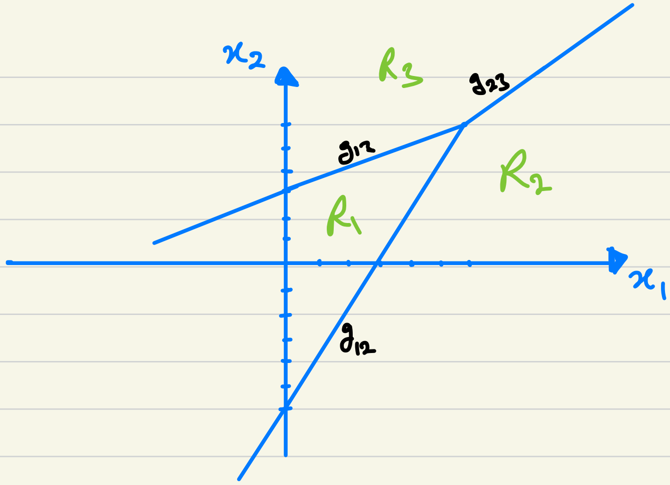
$g_{13} = \left(\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -6 \end{bmatrix}\right)^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 6 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

$= 2x_1 - 4x_2 + 12 \to \boxed{x_1 - 2x_2 + 6 = 0}$

$g_{23} = \left(\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -6 \end{bmatrix}\right)^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 6 & 0 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 6 \\ 0 \end{bmatrix} + \frac{1}{2} \times \begin{bmatrix} 0 & 6 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

$= 6x_1 - 6x_2 \to \boxed{x_1 - x_2 = 0}$

ادامه سؤال 2 الف)



ب)

اگر $b=0$ باشد ماتریس کواریانس قطری و با درایه‌های برابر می‌شود، بنابراین می‌توانیم تابع تفکیک را بدین شکل تعیین کنیم.

$$g_i(\underline{x}) = \left(\frac{1}{\sigma^2}\underline{\mu}_i\right)^T \underline{x} + \ln P(\omega_i) - \frac{1}{2\sigma^2}\underline{\mu}_i^T \underline{\mu}_i$$

$$\rightarrow g_1(\underline{x}) = \ln\left(\frac{1}{3}\right)$$

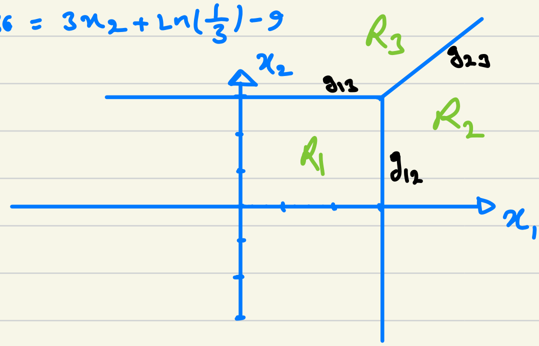
$$g_2(\underline{x}) = \frac{1}{2}[6\ 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \ln\left(\frac{1}{3}\right) - \frac{1}{4}\times 36 = 3x_1 + \ln\left(\frac{1}{3}\right) - 9$$

$$g_3(\underline{x}) = \frac{1}{2}[0\ 6]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \ln\left(\frac{1}{3}\right) - \frac{1}{4}\times 36 = 3x_2 + \ln\left(\frac{1}{3}\right) - 9$$

$$\rightarrow g_1 - g_2 = 0 \rightarrow \boxed{x_1 = 3}$$

$$g_1 - g_3 = 0 \rightarrow \boxed{x_2 = 3}$$

$$g_2 - g_3 = 0 \rightarrow \boxed{x_1 = x_2}$$



سوال 3 -

$$\mu_1=\begin{bmatrix}0\\0\end{bmatrix},\ \mu_2=\begin{bmatrix}-1\\+1\end{bmatrix},\ \Sigma=\begin{bmatrix}4&3\\3&4\end{bmatrix},\quad P(\omega_1)=3P(\omega_2)$$

الف)

$$g_{ij}=\left(\Sigma^{-1}(\underline{\mu}_i-\underline{\mu}_j)\right)^T\underline{x}+\ln\left(\frac{P(\omega_i)}{P(\omega_j)}\right)-\frac{1}{2}\underline{\mu}_i^T\Sigma^{-1}\underline{\mu}_i+\frac{1}{2}\underline{\mu}_j^T\Sigma^{-1}\underline{\mu}_j$$

$$g_{12}=\left(\frac{1}{7}\begin{bmatrix}4&-3\\-3&4\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\right)^T\begin{bmatrix}x_1\\x_2\end{bmatrix}+\ln(3)+\frac{1}{2}\begin{bmatrix}-1&1\end{bmatrix}\times\frac{1}{7}\begin{bmatrix}4&-3\\-3&4\end{bmatrix}\times\begin{bmatrix}-1\\+1\end{bmatrix}$$

$$=x_1-x_2+\ln(3)+1\xrightarrow{g_{12}=0}\boxed{x_2=x_1+1+\ln(3)}$$

ب)

$$|\Sigma-\lambda I|=\begin{vmatrix}4-\lambda&3\\3&4-\lambda\end{vmatrix}=16+\lambda^2-8\lambda-9=\lambda^2-8\lambda+7=0$$

$$\to\ \begin{matrix}\lambda_1=7\\\lambda_2=1\end{matrix}\ \to\ \begin{matrix}\underline{u}_1=\begin{bmatrix}1\\1\end{bmatrix}\\\underline{u}_2=\begin{bmatrix}-1\\1\end{bmatrix}\end{matrix}$$

← باید بردار ویژه متناظر با کوچکترین مقدار ویژه یعنی بردار $\underline{u}_2$ را انتخاب کنیم:

$$\underline{y}=\underline{u}_2\,\underline{x}\ \to\ \underline{\mu}_{y_1}=\begin{bmatrix}-1&1\end{bmatrix}\underline{\mu}_1=0$$

$$\underline{\mu}_{y_2}=\begin{bmatrix}-1&1\end{bmatrix}\underline{\mu}_2=2$$

← تفکیک پذیری حفظ شده است چون میانگین ها روی هم نیفتاده است.

$$\Sigma_y=\underline{u}_2^T\Sigma\underline{u}_2=\begin{bmatrix}-1&1\end{bmatrix}\begin{bmatrix}4&3\\3&4\end{bmatrix}\begin{bmatrix}-1\\+1\end{bmatrix}=2$$

$$\to g_{12}=\frac{1}{2}\times-2\times y+\ln(3)+\frac{1}{2}\times\frac{1}{2}\times4=-y+\ln(3)+1$$

$$\xrightarrow{g_{12}=0}\boxed{y=1+\ln(3)}\to$$ مرز تصمیم گیری

پ) در این قسمت باید بردار متناظر با بزرگترین مقدار ویژه یعنی بردار اول را انتخاب کنیم.

$$\underline{y} = \underline{u}_1 \underline{x} \to \begin{cases} \mu_{y_1} = \underline{u}_1 \underline{\mu}_1 = [1 \ 1]\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \\ \mu_{y_2} = \underline{u}_1 \underline{\mu}_2 = [1 \ 1]\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0 \end{cases}$$

تفکیک پذیری از بین رفته است چون میانگین هر دو کلاس روی هم افتاده است.

$$\Sigma_y = \underline{u}_1^T \Sigma \underline{u}_1 = [1 \ 1]\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 14$$

$$g_{12} = \frac{1}{4} \times 0 \times y + \ln(3) + 0 = \ln(3) \to g_{12} = 0 \ \times$$

به معنای تن دهد اصن چون تفکیک پذیری حفظ نشده است.

می شود عدد دیگری هم باشد.

د) $$\underline{w} = \alpha \Sigma^{-1}(\underline{\mu}_1 - \underline{\mu}_2) = \frac{\alpha}{7}\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xrightarrow{\alpha=1} \underline{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\to \mu_{y_1} = 0, \ \mu_{y_2} = -2, \ \Sigma_y = [1 \ -1]\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2$$

$$\to g_{12} = \frac{1}{2} \times 2 \times y + \ln(3) + \frac{1}{2} \times \frac{1}{2} \times 4 = y + \ln(3) + 1$$

$$\xrightarrow{g_{12}=0} \boxed{y = -1 - \ln(3)} \to$$ مرز تصمیم گیری

هم باید تفکیک پذیری حفظ شده است.

ث)

$$FDR = J = \frac{(\mu_1 - \mu_2)^2}{\sigma_1^2 + \sigma_2^2} \begin{cases} \to x_1 : \dfrac{(0+1)^2}{2+2} = \dfrac{1}{4} \\ \to x_2 : \dfrac{(0-1)^2}{2+2} = \dfrac{1}{4} \end{cases}$$

* این معیار، برای هر 2 ویژگی یکسان هست بنابراین نمی‌توانیم هیچکدام را برتر بدانیم.

ج)

$\beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

الف) $g_{12} = 0 - 1 + \ln(3) + 1 = \ln(3) > 0 \longrightarrow \omega_1$

ب) $\underline{u}_2^T \beta = [-1 \quad 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \to g_{12} = -1 + \ln(3) + 1 = \ln(3) > 0 \to \omega_1$

پ) معنای نمی‌دهد چون تفکیک پذیری نداریم.

ت) $\underline{w}^T \beta = [1 \quad -1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1 \to g_{12} = -1 + \ln(3) + 1 = \ln(3) > 0 \to \omega_1$

با تمام موارد به جز مورد پ در کلاس ω_1 طبقه بندی شده است.

سؤال 4-

الف)

$$\text{sensitivity} = \frac{TP}{TP+FN} = 1 \rightarrow FN = 0$$

$$\text{Accuracy} = \frac{TP+TN}{TP+TN+FP+\cancel{FN}_{0}}$$

$$ACC = \frac{TP+TN}{TP+TN+FP} = 1 - \frac{FP}{TP+TN+FP} \rightarrow \boxed{0 < ACC < 1}$$

ب)

$$\text{spec} = \frac{TN}{TN+FP} = 1 \rightarrow FP = 0$$

$$ACC = \frac{TP+TN}{TP+TN+FN+\cancel{FP}_{0}} = 1 - \frac{FN}{TP+TN+FN} \rightarrow \boxed{0 < ACC < 1}$$

پ) مثل ب هست.

$$PP = \frac{TP}{TP+FP} = 1 \rightarrow FP = 0 \rightarrow \boxed{0 < ACC < 1}$$

ت)

1 $\text{spec}=1,\ \text{sens}=1 \rightarrow FP=0,\ FN=0 \rightarrow ACC = \frac{TP+TN}{TP+TN} = \boxed{1}$

2 $\text{spec}=1,\ PP=1 \rightarrow FP=0 \rightarrow \boxed{0 < ACC < 1} \rightarrow$ تو این حالت یک نمی‌شود.

3 $\text{sens}=1,\ PP=1 \rightarrow FP=0,\ FN=0 \rightarrow \boxed{ACC=1}$

سوال 5 -

(a

$$g_i(\underline{x}) = -\frac{1}{2}(\underline{x}-\underline{\mu}_i)\Sigma_i^{-1}(\underline{x}-\underline{\mu}_i) + \ln(P(\omega_i)) - \frac{1}{2}\ln(|\Sigma_i|)$$

$$\longrightarrow g_1(\underline{x}) = -\frac{1}{2}\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \ln P(\omega_1) - \frac{1}{2}\cancelto{0}{\ln(1)}$$

$$= -\left(\frac{x_1^2+x_2^2}{2}\right) + \ln P(\omega_1)$$

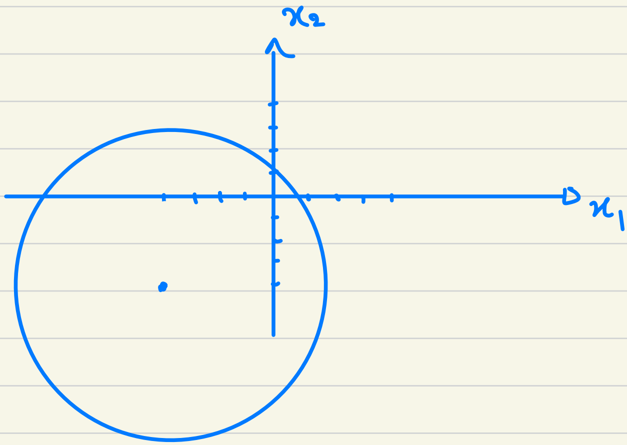
$$g_2(\underline{x}) = -\frac{1}{2}\begin{bmatrix}x_1-4 & x_2-4\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}x_1-4\\ x_2-4\end{bmatrix} + \ln P(\omega_2) - \frac{1}{2}\ln(4)$$

$$= -\frac{1}{4}\left((x_1-4)^2 + (x_2-4)^2\right) + \ln P(\omega_2) - \frac{1}{2}\ln(4)$$

$$g_{12}(\underline{x}) = g_1(\underline{x}) - g_2(\underline{x}) = \underbrace{-(x_1+4)^2 - (x_2+4)^2 + 64 + 4\ln(2) = 0}$$

یک دایره به شعاع $\sqrt{64+4\ln 2}$

و به مرکز $\begin{bmatrix}-4\\-4\end{bmatrix}$ می باشد.



(b

$$g_1(\underline{x}) = -\frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \ln P(\omega_1) - \frac{1}{2}\ln(2)$$

$$= -\frac{1}{2}\left(x_1^2 + \frac{x_2^2}{2}\right) + \ln P(\omega_1) - \frac{1}{2}\ln(2)$$

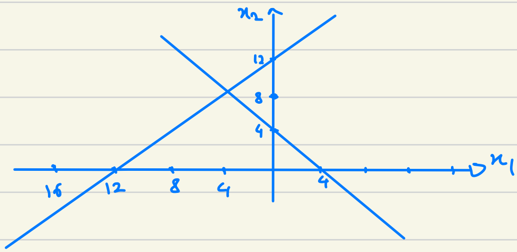
$$g_2(\underline{x}) = -\frac{1}{2}\begin{bmatrix} x_1-4 & x_2-4 \end{bmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}\begin{bmatrix} x_1-4 \\ x_2-4 \end{bmatrix} + \ln P(\omega_2) - \frac{1}{2}\ln(2)$$

$$= -\frac{1}{2}\left(\frac{(x_1-4)^2}{2} + (x_2-4)^2\right) + \ln P(\omega_2) - \frac{1}{2}\ln(2)$$

$$\to g_1(\underline{x}) - g_2(\underline{x}) = (x_1+4)^2 - (x_2-8)^2 = 0$$

$\to x_1+4 = x_2-8 \to x_2 - x_1 = 12$

$\to x_1+4 = -x_2+8 \to x_2 + x_1 = 4$



(c

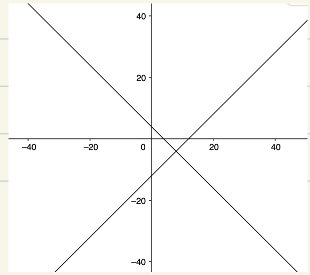
$$g_1(\underline{x}) = -\frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \ln P(\omega_1) - \frac{1}{2}\ln(2)$$

$$= -\frac{1}{2}\left(x_1^2 + \frac{x_2^2}{2}\right) + \ln P(\omega_1) - \frac{1}{2}\ln(2)$$

$$g_2(\underline{x}) = -\frac{1}{2}\begin{bmatrix} x_1-4 & x_2-4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1-4 \\ x_2-4 \end{bmatrix} + \ln P(\omega_2) - \frac{1}{2}\ln\left(\frac{1}{2}\right)$$

$$= -\frac{1}{2}\left(2(x_1-4)^2 + (x_2-4)^2\right) + \ln P(\omega_2) - \frac{1}{2}\ln\left(\frac{1}{2}\right)$$

$$g_1 - g_2 = (x_1-8)^2 - (x_2+4)^2 = \ln(4)$$



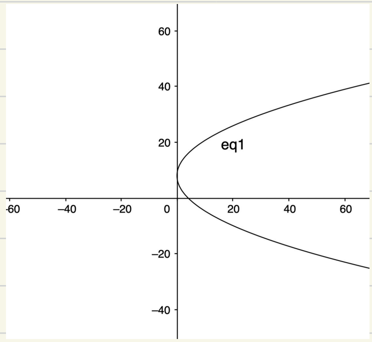
(d

$$g_1(\underline{x}) = -\frac{1}{2}\left(x_1^2 + \frac{x_2^2}{2}\right) + \ln P(\omega_1) - \frac{1}{2}\ln 2$$

$$g_2(\underline{x}) = -\frac{1}{2}\begin{bmatrix} x_1-4 & x_2-4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_2-4 \\ x_2-4 \end{bmatrix} + \ln P(\omega_2) - \frac{1}{2}\overset{0}{\cancel{\ln(1)}}$$

$$= -\frac{1}{2}\left((x_1-4)^2+(x_2-4)^2\right) + \ln P(\omega_2)$$

$$\rightarrow g_1 - g_2 = (x_2-8)^2 - 16x_1 = 2\ln(2)$$



سوال ۶ -

$$P(\omega_1) = P(\omega_2)$$

$$P(x|\omega_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad , \quad P(x|\omega_2) = \begin{cases} \frac{1}{25} & -5 < x < 20 \\ 0 & \text{o.w} \end{cases}$$

$$P(x|\omega_1) = P(x|\omega_2) \rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{25}$$

$$\rightarrow e^{-\frac{x^2}{2}} = \frac{\sqrt{2\pi}}{25} \rightarrow x^2 = -2 \times \overset{-2.3}{\cancel{\ln\left(\frac{\sqrt{2\pi}}{25}\right)}} \simeq 4.6$$

$$\rightarrow \boxed{x = \pm 2.145}$$

$$\rightarrow \begin{cases} x \in R_1 & -2.145 < x < 2.145 \\ x \in R_2 & \text{o.w} \end{cases}$$

Made with Goodnotes