MATHEMATICAL QUESTIONS

Question 1

A vestigial-sideband modulation system is depicted in Fig. 1. The bandwidth of the message signal $m(t)$ is W, and the transfer function of the bandpass filter is shown in the figure.

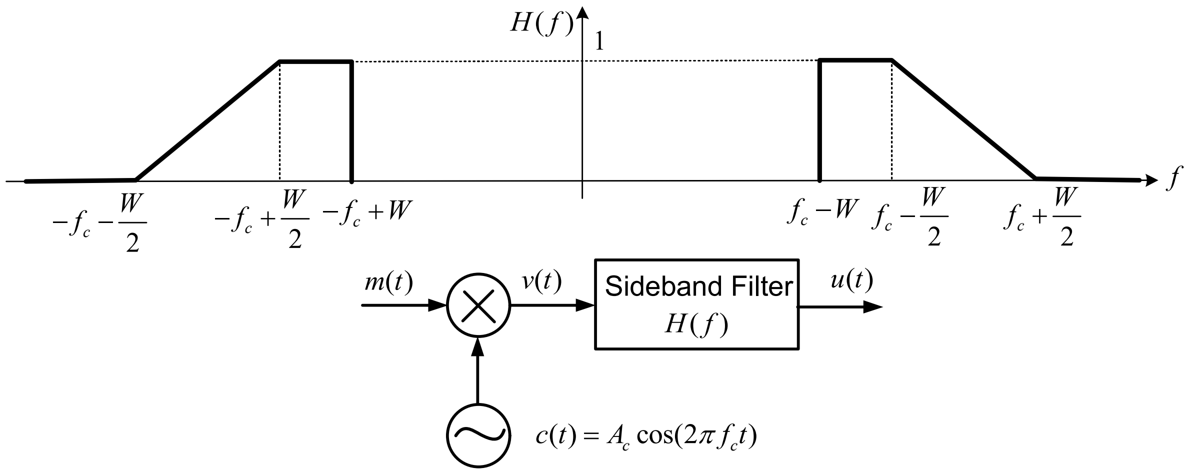

Figure 1: A VSB modulation system.

(a) Determine $h_l(t)$, the lowpass equivalent of $h(t)$, where $h(t)$ represents the impulse response of the bandpass filter.

$$H_l(f) = 2H(f+f_c)u(f+f_c) \rightarrow H_l(f) = \begin{cases} \frac{-2}{W}f + 1, & |f| \leq \frac{W}{2} \\ 2, & -W \leq f < \frac{-W}{2} \end{cases}$$

$$h(t) = \mathcal{F}^{-1}\{H_l(f)\} = \int_{-\infty}^{+\infty} H_l(f)e^{j2\pi ft}\,df = \int_{-\frac{W}{2}}^{+\frac{W}{2}} (\frac{-2}{W}f+1)e^{j2\pi ft}\,df + \int_{-W}^{\frac{-W}{2}} 2e^{j2\pi ft}\,df$$

$$= \frac{-2}{W}\left(\frac{1}{j2\pi t}fe^{j2\pi ft} - \frac{1}{(j2\pi t)^2}e^{j2\pi ft}\right)\Bigg|_{\frac{-W}{2}}^{\frac{W}{2}} + \left(\frac{1}{j2\pi t}e^{j2\pi ft}\right)\Bigg|_{\frac{-W}{2}}^{\frac{W}{2}} + \left(\frac{2}{j2\pi t}e^{j2\pi ft}\right)\Bigg|_{-W}^{\frac{-W}{2}}$$

$$= \left(\frac{-1}{j2\pi t}\left(e^{j\pi Wt} + e^{-j\pi Wt}\right) + \frac{-2}{4\pi^2 t^2 W}\left(e^{j\pi Wt} - e^{-j\pi Wt}\right)\right) + \frac{1}{j2\pi t}\left(e^{j\pi Wt} - e^{-j\pi Wt}\right)$$

$$+\frac{1}{j\pi t}\left(e^{-j\pi Wt} - e^{-j2\pi Wt}\right)$$

$$= \frac{1}{j\pi t}e^{-j2\pi Wt} - \frac{j}{\pi^2 t^2 W}\sin(\pi Wt) = \boxed{\frac{1}{j\pi t}\left(e^{-j2\pi Wt} + \text{sinc}(Wt)\right)}$$

Question 1 continued on next page. . .

(b) Derive an expression for the modulated signal $u(t)$.

$$u(t) = A_c m(t)\cos(2\pi f_c t) * h(t) = A_c m(t)\mathcal{R}\{e^{j2\pi f_c t}\} * h(t)$$
$$h(t) = \mathcal{R}\{x_l(t)e^{j2\pi f_c t}\} \rightarrow u(t) = A_c m(t)\mathcal{R}\{e^{j2\pi f_c t}\} * \mathcal{R}\{e^{j2\pi f_c t}\}$$
$$\rightarrow u(t) = \mathcal{R}\{A_c m(t) * h_l(t)\}e^{j2\pi f_c t}$$
$$\rightarrow u(t) = \mathcal{R}\{A_c m(t) * \frac{1}{j\pi t}(e^{-j2\pi W t} + \mathrm{sinc}(Wt))\}e^{j2\pi f_c t}$$
$$= \mathcal{R}\{(A_c m(t) * \frac{1}{j\pi t}\mathrm{sinc}(Wt))e^{j2\pi f_c t} - (A_c m(t) * \frac{1}{j\pi t}e^{-j2\pi W t})e^{j2\pi f_c t}\}$$
$$\mathcal{F}\{m(t) * \frac{1}{j\pi t}e^{-j2\pi W t}\} = -M(f)sgn(f+W) = -M(f) \rightarrow m(t) * \frac{1}{j\pi t}e^{-j2\pi W t} = -m(t)$$
$$\rightarrow u(t) = \mathcal{R}\{(A_c m(t) * (\frac{1}{j\pi t}\mathrm{sinc}(Wt)))e^{j2\pi f_c t} - A_c m(t)e^{-j2\pi f_c t}\}$$
$$\rightarrow \boxed{u(t) = -A_c m(t)cos(2\pi f_c t) - A_c(m(t) * \frac{1}{\pi t}\mathrm{sinc}(Wt))\sin(2\pi f_c t)}$$

Question 2

Follow the steps below to show the power of the FM signal $u(t) = A_c\cos(2\pi f_c t + \phi(t))$ is $\frac{A_c^2}{2}$.

(a) Write the power expression for the FM signal and show that the power equals $P = \frac{A_c^2}{2} + I$, where

$$I = \lim_{T\to\infty}\frac{A_c^2}{2T}\int_{-T/2}^{T/2}\cos(4\pi f_c t + 2\phi(t))dt$$

$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}u^2(t)dt = \frac{1}{T}\int_{-T/2}^{T/2}A_c^2\cos^2(2\pi f_c t + \phi(t))dt$$
$$= \frac{A_c^2}{2} + \lim_{T\to\infty}\frac{A_c^2}{2T}\int_{-T/2}^{T/2}\cos(4\pi f_c t + 2\phi(t))dt = \frac{A_c^2}{2} + I$$
$$\rightarrow \boxed{I = \lim_{T\to\infty}\frac{A_c^2}{2T}\int_{-T/2}^{T/2}\cos(4\pi f_c t + 2\phi(t))dt}$$

(b) Show that

$$I_\infty = \int_{-\infty}^{\infty}\cos(4\pi f_c t + 2\phi(t))dt$$

relates to the Fourier transforms $\mathcal{F}\{e^{j2\phi(t)}\}$ and $\mathcal{F}\{e^{-j2\phi(t)}\}$ at the frequencies $-2f_c$ and $2f_c$, respectively.

$$I_\infty = \int_{-\infty}^{\infty}\cos(4\pi f_c t + 2\phi(t))dt = \int_{-\infty}^{\infty}\frac{1}{2}(e^{(j4\pi f_c t + 2\phi(t))} + e^{(-j4\pi f_c t + 2\phi(t))})dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{2}(e^{4\pi f_c t} \times e^{2j\phi(t)}) + \int_{-\infty}^{\infty} \frac{1}{2}(e^{-4\pi f_c t} \times e^{-2j\phi(t)})$$

$$= \boxed{\frac{1}{2}\mathcal{F}\{e^{j2\phi(t)}\}|_{f=-2f_c} + \frac{1}{2}\mathcal{F}\{e^{-j2\phi(t)}\}|_{f=2f_c}}$$

(c) Use Taylor series expansion to show that I_∞ depends to the Fourier transforms $\mathcal{F}\{\phi^n(t)\}, n \in \mathbb{W}$ at the frequency $\pm 2f_c$.

$$I_\infty = \frac{1}{2}\mathcal{F}\{1 + 2j\phi(t) + \frac{(2j\phi(t))^2}{2!} + \frac{(2j\phi(t))^3}{3!} + ...\}|_{f=-2f_c}$$

$$+\frac{1}{2}\mathcal{F}\{1 - 2j\phi(t) + \frac{(2j\phi(t))^2}{2!} - \frac{(2j\phi(t))^3}{3!} + ...\}|_{f=2f_c}$$

$$= \boxed{\frac{1}{2}\sum_{n=0}^{\infty}\frac{(2j)^n}{n!}\mathcal{F}\{\phi^n(t)\}|_{f=-2f_c} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-2j)^n}{n!}\mathcal{F}\{\phi^n(t)\}|_{f=2f_c}}$$

(d) Show that if $f_c \gg W$, where W is the bandwidth of the message-related phase $\phi(t)$, $I_\infty \approx 0$.

(e) Show that the power is approximately equal to $\frac{A_c^2}{2}$.

Question 3

Find the spectrum of the narrowband FM signal

$$u(t) = A_c \cos(2\pi f_c t) - A_c\big[2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\big] \sin(2\pi f_c t)$$

and narrowband PM signal

$$u(t) = A_c \cos(2\pi f_c t) - A_c k_p m(t) \sin(2\pi f_c t)$$

in terms of the message spectrum $M(f)$.

$$u(t) = A_c \cos(2\pi f_c t) - A_c\big[2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\big] \sin(2\pi f_c t)$$

$$\to U(f) = \mathcal{F}\{A_c \cos(2\pi f_c t)\} - \mathcal{F}\{2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\} * \mathcal{F}\{A_c \sin(2\pi f_c t)\}$$

$$\mathcal{F}\left\{\int_{-\infty}^{t} m(\tau)d\tau\right\} = \frac{M(f)}{2j\pi f} + \frac{1}{2}M(0)\delta(f)$$

$$\to U(f) = \frac{A_c}{2}[\delta(f-f_c) + \delta(f+f_c)] - \frac{2A_c\pi k_f}{2j}[\delta(f-f_c) - \delta(f+f_c)] * [\frac{M(f)}{2j\pi f} + \frac{M(0)\delta(f)}{2}]$$

$$= \frac{A_c}{2}[\delta(f-f_c) + \delta(f+f_c)] + 2A_c\pi k_f[\frac{M(f-f_c)}{4\pi(f-f_c)} - \frac{M(f+f_c)}{4\pi(f+f_c)} + \frac{jM(0)\delta(f-f_c)}{4}$$

$$-\frac{jM(0)\delta(f+f_c)}{4}]$$

if assume M(0)=0,

$$\to \boxed{U_f = \frac{A_c}{2}[\delta(f-f_c) + \delta(f+f_c)] + \frac{A_c k f}{2}[\frac{M(f-f_c)}{(f-f_c)} - \frac{M(f+f_c)}{(f+f_c)}]}$$

Question 4

The cross-correlation of the power signals $w(t)$ and $v(t)$ is defined as $R_{vw}(\tau) = \langle v(t)w^*(t-\tau)\rangle$, where the time average operator

$$\langle x(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt$$

.

(a) *Show that $R_{vw}(\tau) = R^*_{wv}(-\tau)$.*

we can obtain that,

$$\langle x(t)\rangle^* = [\lim_{T\to\infty} \int_{-T/2}^{T/2} x(t)dt]^* = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^*(t)dt = \langle x^*(t)\rangle$$

so we will have,

$$R^*_{wv}(-\tau) = [\langle w(t)v^*(t+\tau)\rangle]^* = \langle w^*(t)v(t+\tau)\rangle = \lim_{T\to\infty} \int_{-T/2}^{T/2} w^*(t)v(t+\tau)dt$$

$$t = t - \tau \rightarrow R_{wv}^*(-\tau) = \lim_{T\rightarrow\infty} \int_{-T/2-\tau}^{T/2-\tau} w^*(t-\tau)v(t)dt = \lim_{T\rightarrow\infty} \int_{-T/2}^{T/2} w^*(t-\tau)v(t)dt$$

$$\rightarrow \boxed{R_{vw}(\tau) = R_{wv}^*(-\tau)}$$

(b) Prove that $|R_{vw}(\tau)|^2 \leqslant P_v P_w$.

Question 5

Fig. 2 shows the block diagram of an FM to AM demodulator and the schematic of its practical implementation, which is called balanced FM demodulator or FM discriminator.

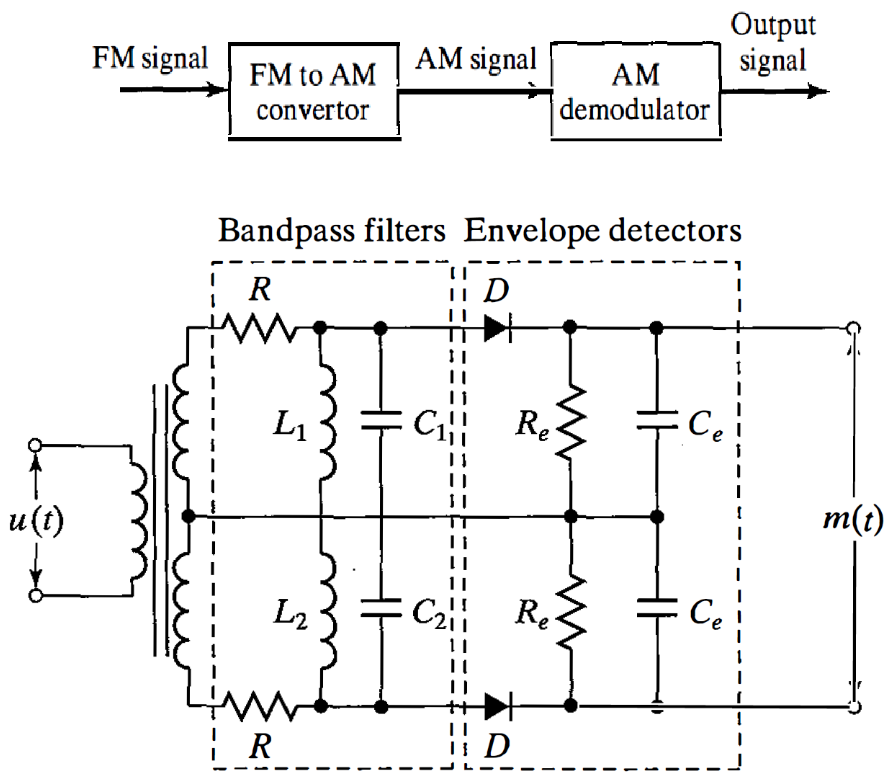

Figure 2: Balanced FM demodulator.

(a) Find the output of the FM to AM converter block if its frequency response is $H(f) = j[V_0 + k(f - f_c), \quad |f - f_c| < 0.5B_c$, where B_c denotes the bandwidth of the input FM modulated signal $u(t) = A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau)$. Note that the FM to AM impulse response is real and therefore, $H(-f) = H^*(f)$.

(b) Explain how the shown schematic implements the FM to AM demodulator? Why are there two filters and two envelope detectors in the schematic?

SOFTWARE QUESTIONS

Question 6

Develop a MATLAB/Python code that plots the cross-correlation $R_{vw}(\tau)$ of two power signals $v(t)$ and $w(t)$. Illustrate the output of the code for sample power signals.

BONUS QUESTIONS

Question 7

Two power signals of $v(t)$ and $w(t)$ are called uncorrelated if their cross-correlation $R_{vw}(\tau) = 0, \forall\tau$. Show that for these uncorrelated signals, the power P_z of $z(t) = v(t) + w(t)$ equals $P_z = P_v + P_w$.

$$P_z = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |z|^2 dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} zz^* dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} (v+w)(v^*+w^*) dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |v|^2 dt + \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |w|^2 dt + \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} v(t)w^*(t) dt$$

$$+ \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} v^*(t)w(t) dt$$

$$= \boxed{P_v + P_w + \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} v(t)w^*(t) dt + \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} v^*(t)w(t) dt}$$

$$R_{vw}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} v(t)w^*(t-\tau) dt = 0$$

$$\to R_{vw}(0) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} v(t)w^*(t) = 0$$

$$\to R^*_{vw}(\tau) = 0 \to \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} v^*(t)w(t) = 0$$

$$\boxed{\to P_z = P_v + P_w}$$

Question 8

Return your answers by filling the LaTeXtemplate of the assignment.