تمرین سری اول در مقدمه ای بر یادگیری ماشین - رادین خیام - ۹۹۱۰۱۳۷۹

سوال ۱ -

$$f_X(x) = \begin{cases} 1-x & |x| < 0.5 \\ 0 & |x| > 0.5 \end{cases}$$

$$\sum_{k=0}^{\infty} E[X^k] = \sum_{k=0}^{\infty} \int_{-\infty}^{+\infty} x^k f_X(x)\,dx = \sum_{k=0}^{\infty} \int_{-0.5}^{0.5} x^k (1-x)\,dx$$

$$= \sum_{k=0}^{\infty} \left(\frac{x^{k+1}}{k+1} - \frac{x^{k+2}}{k+2} \right) \Bigg|_{-0.5}^{0.5} = \sum_{k=0}^{\infty} \left(\frac{(\frac{1}{2})^{k+1}}{k+1} - \frac{(-\frac{1}{2})^{k+1}}{k+1} - \frac{(\frac{1}{2})^{k+2}}{k+2} + \frac{(-\frac{1}{2})^{k+2}}{k+2} \right)$$

if k even : $\dfrac{(\frac{1}{2})^k}{k+1}$ if k odd : $-\dfrac{(\frac{1}{2})^{k+1}}{k+2}$

$$\rightarrow \sum_{k=0}^{\infty} E[X^k] = \frac{(\frac{1}{2})^0}{1} - \cancel{\frac{(\frac{1}{2})^2}{3}} + \cancel{\frac{(\frac{1}{2})^2}{3}} - \cancel{\frac{(\frac{1}{2})^4}{5}} + \cancel{\frac{(\frac{1}{2})^4}{5}} + \cdots = \boxed{1}$$

سؤال 2 -

الف)

$$f_{X,Y}(x,y) = \begin{cases} kxy & 0<x\leq y,\ 0<y<1 \\ 0 & o.w \end{cases}$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1 \longrightarrow \int_0^1\int_0^y kxy\,dx\,dy = 1$$

$$\longrightarrow \int_0^1 ky\,\frac{x^2}{2}\Big|_0^y dy = 1 \longrightarrow \frac{k}{2}\int_0^1 y^3\,dy = 1 \longrightarrow \frac{k}{8}\,y^4\Big|_0^1 = 1$$

$$\longrightarrow \frac{k}{8} = 1 \longrightarrow \boxed{k=8}$$

ب)

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_x^1 8xy\,dy = 8x\times\frac{y^2}{2}\Big|_x^1 = 4x(1-x^2)$$

$$\longrightarrow \boxed{f_X(x) = \begin{cases} 4x(1-x^2), & 0\leq x\leq 1 \\ 0, & o.w \end{cases}}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \int_0^y 8xy\,dx = 8y\,\frac{x^2}{2}\Big|_0^y = 4y^3$$

$$\longrightarrow \boxed{f_Y(y) = \begin{cases} 4y^3, & 0\leq y\leq 1 \\ 0, & o.w \end{cases}}$$

ب)

$$\overline{x} = \int_0^1\int_0^y 8x^2y\,dx\,dy = 8\times\frac{1}{3}\times\frac{y^5}{5}\Big|_0^1 = \boxed{\frac{8}{15}}$$

$$\overline{x^2} = \int_0^1\int_0^y 8x^3y\,dx\,dy = 8\times\frac{1}{4}\times\frac{y^6}{6}\Big|_0^1 = \frac{1}{3}$$

$$\sigma_x^2 = \overline{x^2} - \overline{x}^2 = \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \boxed{0.049}$$

$$\overline{y} = \int_0^1\int_0^y 8xy^2\,dx\,dy = 8\times\frac{1}{2}\times\frac{y^3}{5}\Big|_0^1 = \boxed{\frac{4}{5}}$$

$$\overline{y^2} = \int_0^1\int_0^y 8xy^3\,dx\,dy = 8\times\frac{1}{2}\times\frac{y^6}{6}\Big|_0^1 = \frac{2}{3}$$

$$\sigma_y^2 = \overline{y^2} - \overline{y}^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \boxed{0.027}$$

$$\overline{xy} = \int_0^1\int_0^y 8x^2y^2\,dx\,dy = 8\times\frac{1}{3}\times\frac{y^6}{6}\Big|_0^1 = \boxed{\frac{4}{9}}$$

$$\sigma_{xy} = \overline{xy} - \overline{x}\,\overline{y} = \frac{4}{9} - \frac{8}{15}\times\frac{4}{5} = \boxed{0.018}$$

$$r = \frac{\sigma_{xy}}{\sigma_x\sigma_y} = \frac{0.018}{0.221\times 0.164} = \boxed{0.497}$$

ت)

$$f_X(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{8xy}{4y^3} = \frac{2x}{y^2}$$

$$\longrightarrow \boxed{f_{X|Y}(x|y) = \begin{cases} \frac{2x}{y^2}, & 0<x<y \\ 0, & \text{o.w} \end{cases}}$$

$$f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{8xy}{4x(1-x^2)} = \frac{2y}{1-x^2}$$

$$\longrightarrow \boxed{f_{Y|X}(y|x) = \begin{cases} \frac{2y}{1-x^2}, & x<y<1 \\ 0, & \text{o.w} \end{cases}}$$

$$m_{x|y} = \int_{-\infty}^{+\infty} x\, f_{X|Y}(x|y)\, dx = \int_0^y \frac{2x^2}{y^2}\, dx = \frac{2}{y^2}\, \frac{x^3}{3}\Big|_0^y = \boxed{\frac{2y}{3}}$$

$$m_{x^2|y} = \int_0^y x^2 f_{X|Y}(x|y)\, dx = \int_0^y \frac{2x^3}{y^2}\, dx = \frac{2}{y^2} \times \frac{x^4}{4}\Big|_0^y = \frac{y^2}{2}$$

$$\sigma^2_{x|y} = m_{x^2|y} - m^2_{x|y} = \frac{y^2}{2} - \frac{4y^2}{9} = \boxed{\frac{y^2}{18}}$$

ث)

$\hat{x} = \arg\max_{x} f_X(x)$, $\frac{d f_X(x)}{dx} = \frac{d}{dx}\left[4x(1-x^2)\right] = \frac{d}{dx}(4x-4x^3)$

$= 4 - 12x^2 = 0 \rightarrow x = \pm\sqrt{\frac{1}{3}}$

$x>0 \longrightarrow \boxed{\hat{x} = +\sqrt{\frac{1}{3}}}$

ج)

$\hat{x} = \bar{x} = \boxed{\frac{8}{15}}$, $\overline{e^2} = \sigma_x^2 = \boxed{0.049}$

چ)

$0<x<y$

$\hat{x} = \arg\max_{x} f_X(x|y)$, $\frac{d}{dx} f_X(x|y) = \frac{d}{dx}\left(\frac{2x}{y^2}\right) = \frac{2}{y^2}$

$\boxed{\hat{x} = y} \longleftarrow$ همواره مثبت

ح)

$\hat{x} = m_{x|y} = \boxed{\frac{2y}{3}}$, $\overline{e_2} = \sigma_{x|y}^2 = \boxed{\frac{y^2}{18}}$

3-

$$A = \begin{bmatrix} X \\ Y \end{bmatrix} \sim N(\mu, \Sigma), \quad \mu = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{2\pi\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(A-\mu)^T \Sigma^{-1} (A-\mu)\right)}{\frac{1}{\sigma_y\sqrt{2\pi}} \exp\left(-\frac{1}{2} \times \left(\frac{y-\bar{y}}{\sigma_y}\right)^2\right)}$$

$$= \underbrace{\frac{\sigma_y}{\sqrt{|\Sigma|}} \times \frac{1}{\sqrt{2\pi}}}_{I} \exp\left(-\frac{1}{2} \underbrace{\left((A-\mu)^T \Sigma^{-1} (A-\mu) - \left(\frac{y-\bar{y}}{\sigma_y}\right)^2\right)}_{II}\right)$$

$$I = \frac{\sigma_y}{\sqrt{\sigma_x^2\sigma_y^2 - \sigma_{xy}^2}} \times \frac{1}{\sqrt{2\pi}} = \frac{\sigma_y^2}{\sigma_x^2\sigma_y^2 - \sigma_{xy}^2} \times \frac{1}{\sqrt{2\pi}} = \frac{1}{\underbrace{\sigma_x^2 - \frac{\sigma_{xy}^2}{\sigma_y^2}}_{\sigma_{\bar{v}}^2}} \times \frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi\sigma_{\bar{v}}^2}}$$

$$II = \begin{bmatrix} x-\bar{x} & y-\bar{y} \end{bmatrix} \times \frac{1}{\sqrt{\sigma_x^2\sigma_y^2 - \sigma_{xy}^2}} \times \begin{bmatrix} \sigma_y^2 & -\sigma_{xy} \\ -\sigma_{xy} & \sigma_x^2 \end{bmatrix} \times \begin{bmatrix} x-\bar{x} \\ y-\bar{y} \end{bmatrix} - \left(\frac{y-\bar{y}}{\sigma_y}\right)^2$$

$$= \frac{1}{\sigma_x^2\sigma_y^2 - \sigma_{xy}^2}\left((x-\bar{x})^2\sigma_y^2 + (y-\bar{y})^2\sigma_x^2 - 2(x-\bar{x})(y-\bar{y})\sigma_{xy}\right) - \frac{(y-y)^2}{\sigma_y^2}$$

$$= \frac{\left((x-\bar{x})^2\sigma_y^4 + (y-\bar{y})^2\sigma_y^2\sigma_x^2 - 2(x-\bar{x})(y-\bar{y})\sigma_y^2\sigma_{xy}\right) - (y-\bar{y})^2(\sigma_x^2\sigma_y^2 - \sigma_{xy}^2)}{\sigma_x^2\sigma_y^4 - \sigma_{xy}^2\sigma_y^2}$$

$$= \frac{(x-\bar{x})^2\sigma_y^4 + (y-\bar{y})^2\sigma_{xy}^2 - 2(x-\bar{x})(y-\bar{y})\sigma_y^2\sigma_{xy}}{\sigma_x^2\sigma_y^4 - \sigma_{xy}^2\sigma_y^2}$$

$$= \frac{(x-\bar{x})^2 + (y-\bar{y})^2\frac{\sigma_{xy}}{\sigma_y^4} - 2(x-\bar{x})(y-\bar{y})\frac{\sigma_{xy}}{\sigma_y^2}}{\sigma_{\bar{v}}^2}$$

Made with Goodnotes

$$= \frac{\left(x - \bar{x} - \frac{\sigma_{xy}}{\sigma_y^2}(y - \bar{y})\right)^2}{\sigma_v^2} \longrightarrow \bar{v} = \bar{x} + \frac{\sigma_{xy}}{\sigma_y^2}(y - \bar{y})$$

$$\longrightarrow f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi\sigma_v^2}} e^{-\frac{(x-\bar{v})^2}{2\sigma_v^2}}$$

سوال ۴-

$$X \sim Uni[-1,1] \longrightarrow f_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & o.w \end{cases}$$

$$F_X(x) = \frac{x+1}{2}, \quad -1 \leq x \leq 1$$

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(|X| \leq \sqrt{y}) = P(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$\longrightarrow F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{\sqrt{y}+1}{2} - \frac{-\sqrt{y}+1}{2} = \sqrt{y}$$

$$\longrightarrow f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 \leq y \leq 1 \\ 0 & o.w \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} \delta & y = x^2 \\ 0 & o.w \end{cases} \longrightarrow f_{Y|X}(y|x) \neq f_Y(y) \longrightarrow \boxed{\text{مستقل نیستند}}$$

به صورت کلی می‌توان ثابت کرد X و X^2 تنها زمانی مستقل هستند که توزیع X یک توزیع $X=\alpha$ باشد.

$$\mu_X = 0, \quad \mu_Y = \int y f_Y(y)\, dy = \int_0^1 y \times \frac{1}{2\sqrt{y}} = \frac{1}{3}$$

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = E\left[X\left(X^2 - \tfrac{1}{3}\right)\right] = E\left[X^3 - \tfrac{X}{3}\right]$$

$$= \int_{-1}^{1} \left(x^3 - \frac{x}{3}\right) \times \frac{1}{2}\, dx = \frac{1}{2} \times \left(\frac{x^4}{4} - \frac{x^2}{6}\right)\Big|_{-1}^{1} = \frac{1}{2} \times 0 = \boxed{0} \longrightarrow \boxed{\text{ناهم‌بسته هستند.}}$$

سوال 5 -

الف) مو چون X, Y تواماً نرمال هستند برای همین U و V هم که ترکیب خطی آن‌ها است متغیرهای نرمال می‌باشند.

$$X \sim N(0,\sigma^2),\ Y \sim N(0,\sigma^2) \quad , \quad U = X - Y \quad , \quad V = X + Y$$

$$\longrightarrow U \sim N(m_U, \sigma_U^2),\quad V \sim N(m_V, \sigma_V^2)$$

$$m_U = m_V = 0 \quad , \quad \sigma_U^2 = 2\sigma^2 - 2\sigma_{XY} \quad , \quad \sigma_V^2 = 2\sigma^2 + 2\sigma_{XY}$$

$$Z = U + V = 2X$$

$$\left.\begin{array}{l} m_Z = 0 \quad , \quad \sigma_Z^2 = 4\sigma^2 \\ \sigma_Z^2 = \sigma_U^2 + \sigma_V^2 + 2\sigma_{UV} = 4\sigma^2 + 2\sigma_{UV} \end{array}\right\} \longrightarrow \boxed{\sigma_{UV} = 0}$$

چون U و V متغیرهای نرمال و ناهم‌بسته‌ای هستند نتیجه می‌شود که مستقل می‌باشند.

ب)

$$E[X^3 - Y^3 \mid X - Y = x - y] = E[(X - Y)((X - Y)^2 + 3XY) \mid X - Y = x - y]$$

$$X - Y = U, \quad X + Y = V \quad \longrightarrow \quad X = \frac{V+U}{2}, \quad Y = \frac{V-U}{2}$$

$$\longrightarrow = E[U(U^2 + 3XY) \mid U = x - y] = E[U(U^2 + \frac{3}{4}(U+V)(U-V) \mid U = x - y]$$

$$= E[U(U^2 + \frac{3}{4}(U^2 - V^2)) \mid U = x - y] = \underbrace{\frac{7}{4} E[U^3 \mid U = x - y]}_{\frac{7}{4}(x-y)^3} - \frac{3}{4} \underbrace{E[UV^2 \mid U = x - y]}_{(*)}$$

و در قسمت الف اثبات کردیم که U و V مستقل هستند از همین موضوع استفاده می کنیم

تا عبارت $(*)$ را ساده کنیم:

$$E[UV^2 \mid U = x-y] = E[U \mid U = x-y]\, E[V^2 \mid U = x-y]$$

$$= (x-y)\, E[V^2] = (x-y)\left(Var(V) + \cancelto{0}{E[V]}^2\right)$$

$$Var(V) = \sigma_V^2 = 2\sigma^2 + 2\sigma_{XY} = 2\sigma^2(1+\rho)$$

$$\rightarrow E[X^3 - Y^3 \mid X - Y = x-y] = \boxed{\frac{7}{4}(x-y)^3 - \frac{3\sigma^2}{2}(x-y)(1+\rho)}$$

سوال 6 -

ماتریس R_X

$$\begin{bmatrix} 6 & 10 & 4 \\ 10 & 21 & 11 \\ 4 & 11 & 7 \end{bmatrix}$$

جمع ستون اول و سوم برابر ستون دوم

هست پس دترمینان این ماتریس صفر هست

پس درایه‌های این بردار تصادفی وابسته بوده اند.

وقتی که دترمینان برابر صفر هست یعنی یک برداری مثل u وجود دارد که:

$$R_X u = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & 10 & 4 \\ 10 & 21 & 11 \\ 4 & 11 & 7 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow u = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

همچنین می‌توانیم نشان دهیم که:

$$u^T R_X u = 0$$

حال تعریف ماتریس همبستگی را می‌نویسیم:

$$R_X = E[XX^T] \rightarrow u^T R_X u = u^T E[XX^T] u = E[\underbrace{u^T X}_{\text{اسکالر}} X^T u] = 0$$

که چون بردار ثابت هست u

$$\rightarrow u^T X = 0 \rightarrow \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0$$

$$\rightarrow \boxed{X_1 - X_2 + X_3 = 0}$$

سوال 7 -

الف)

$$C_x = \begin{bmatrix} 4.1 & 1.2 \\ 1.2 & 3.4 \end{bmatrix} \rightarrow C_x u = \lambda u$$

$$\rightarrow \det(C_x - \lambda I) = 0 \rightarrow \begin{vmatrix} 4.1-\lambda & 1.2 \\ 1.2 & 3.4-\lambda \end{vmatrix} = 0$$

$$\rightarrow (4.1-\lambda)(3.4-\lambda) = 1.2^2 \rightarrow \begin{array}{l} \lambda_1 = 5 \\ \lambda_2 = 2.5 \end{array}$$

$$\lambda_1 = 5 \rightarrow \begin{bmatrix} -0.9 & 1.2 \\ 1.2 & -1.6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \rightarrow \begin{array}{l} x = 4/3 \\ y = 1 \end{array} \rightarrow u_1 = \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 2.5 \rightarrow \begin{bmatrix} 1.6 & 1.2 \\ 1.2 & 0.9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \rightarrow \begin{array}{l} x = -\frac{3}{4} \\ y = 1 \end{array} \rightarrow u_2 = \begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix}$$

ب)

$$Y = AX \rightarrow C_Y = A C_X A^T \quad , \quad C_Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \; C_x = \begin{bmatrix} 4.1 & 1.2 \\ 1.2 & 3.4 \end{bmatrix}$$

$$U = [u_1 \; u_2] \quad , \quad Z = U^T X \rightarrow C_Z = U^T C_X U$$

$$\rightarrow C_Z = \begin{bmatrix} 4/3 & 1 \\ -3/4 & 1 \end{bmatrix}\begin{bmatrix} 4.1 & 1.2 \\ 1.2 & 3.4 \end{bmatrix}\begin{bmatrix} 4/3 & -3/4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 125/8 & 0 \\ 0 & 125/32 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 125/8 & 0 \\ 0 & 125/32 \end{bmatrix}$$

برای اینکه ماتریس کوواریانس I شود کافیست $\Lambda^{-\frac{1}{2}}$ را هم ضرب کنیم.

$$\rightarrow Y = \Lambda^{-\frac{1}{2}} U^T X \rightarrow \boxed{A = \Lambda^{-\frac{1}{2}} U^T}$$

سؤال 8 -

$$AA^H = \begin{bmatrix} 5 & 1 & 5 \\ 1 & 10 & 1 \\ 5 & 1 & 5 \end{bmatrix}$$

$$A^H A = \begin{bmatrix} 11 & 1 \\ 1 & 9 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 10-\sqrt{2} & 0 \\ 0 & 10+\sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$A = U \Lambda^{\frac{1}{2}} V^T$$

$$V = \begin{bmatrix} -\sqrt{2}+1 & \sqrt{2}+1 \\ 1 & 1 \end{bmatrix}$$